Stochastic Approximation Methods And Applications in Financial Optimization Problems

by

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(Under the direction of Qing Zhang)

Abstract

Optimizations play an increasingly indispensable role in financial decisions and financial models. Many problems in mathematical finance, such as asset allocation, trading strategy, and derivative pricing, are now routinely and efficiently approached using optimization. Not until recently have stochastic approximation methods been applied to solve optimization problems in finance.

This dissertation is concerned with stochastic approximation algorithms and their applications in financial optimization problems. The first part of this dissertation concerns trading a mean-reverting asset. The strategy is to determine a low price to buy and a high price to sell so that the expected return is maximized. Slippage cost is imposed on each transaction. Our effort is devoted to developing a recursive stochastic approximation type algorithm to estimate the desired selling and buying prices. In the second part of this dissertation we consider the trailing stop strategy. Trailing stops are often used in stock trading to limit the maximum of a possible loss and to lock in a profit. We develop stochastic approximation algorithms to estimate the optimal trailing stop percentage. A modification using projection is developed to ensure that the approximation sequence constructed stays in a reasonable range. In both parts, we also study the convergence and the rate of convergence. Simulations and real market data are used to demonstrate the performance of the proposed algorithms. The advantage of using stochastic approximation in stock trading is that the underlying asset is model free. Only observed stock prices are required, so it can be performed on line to provide guidelines for stock trading. Other than in stock trading, stochastic approximation methods can also be used in parameter estimations. In the last part of this dissertation, we consider a regime switching option pricing model. The underlying stock price evolves according to two geometric Brownian motions coupled by a continuous-time finite state Markov chain. Recursive stochastic approximation algorithms are developed to estimate the implied volatility. Convergence of the algorithm is obtained and the rate of convergence is also ascertained. Then real market data are used to compare our algorithms with other schemes.

Index words: Stochastic Approximation, Stochastic Optimization, Financial Optimization, Trading Strategy, Parameter Estimation, Option Pricing, Trailing Stop, Mean-reverting

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Dedication

To my parents,

Yuzhen Zhang and Chenzhao Zhuang,

and in memory of my grandfather,

Gongwu Zhuang,

whose support and encouragement made this possible.

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Chapter 1

Introduction

Stochastic approximation (SA) concerns recursive estimations of quantities based on noise-corrupted observations. The basic stochastic approximation algorithms were introduced in the early 1950s by Robbins and Monro and by Kiefer and Wolfowitz. Robbins and Monro [32] proposed procedures to find roots of certain unknown functions observed in situations with noises. Kiefer and Wolfowitz [25] provide algorithms that aim to find extrema of such functions. Since then, stochastic approximation methods have been the subject of an enormous literature, both theoretical and applied.

The basic paradigm is a stochastic recursive algorithm such as $\theta_{n+1} = \theta_n + \varepsilon_n X_n$, where θ_n takes its value in some Euclidean space, X_n is a random variable, and ε_n is known as a step size and might go to 0 as $n \to \infty$. In this simple form, if θ denotes an exact parameter of a system, X_n is the noise-corrupted observed value of an unknown function when the parameter θ is set to θ_n. One recursively adjusts the value of θ_n so that some goal is achieved asymptotically. This dissertation focuses on the applications of such recursive algorithms in the field of finance and their qualitative and asymptotic properties.

The original work in Robbins and Monro [32] was motivated by the problem of finding a root of a continuous unknown function $f(\theta)$. The value of the function $f(\theta)$ can be observed with noise at any desired value of θ. If f is known and continuously differentiable, then the problem becomes a classical problem in numerical analysis and Newton's method can be used to find the root. The values $f(\theta)$ are not known, but the noise-corrupted observations can be measured at any given values of θ. Due to the observation noise, Newton's method cannot be used. Robbins and Monro [32] propose the following algorithm to recursively find the root of the function $f(\theta)$:

$$\theta_{n+1} = \theta_n + \varepsilon_n Y_n, \tag{1.0.1}$$

where ε_n is the appropriate step size and Y_n is the noisy estimate of the value of $f(\theta_n)$. A major insight of Robbins and Monro is that, if the step sizes in the above algorithm go to 0 in an appropriate way as $n \to \infty$, then there is an implicit averaging that eliminates the noisy effects in the long run. With some requirements on the function f, the convergence of the algorithm (1.0.1) can be obtained. The asymptotic behavior of the above algorithm can be approximated by the asymptotic behavior of the solution to an ODE.

Other than root finding, in application, one often faces the optimization problem, i.e., to find extrema of an unknown function $F(\cdot)$. It is well known that extrema of F occur at the root set of its gradient DF, although it may be only in the local sense. If the gradient DF can be observed with or without noise, then this optimization problem can be solved by Newton's method or the Robbins and Monro algorithm mentioned above. Kiefer and Wolfowitz [25] propose an algorithm to solve the optimization problem for the case where the noise-corrupted observations of the function F itself rather than its gradient $DF(\cdot)$ can be taken. The classical Kiefer-Wolfowitz algorithm [25] uses the finite differences as the estimates of the partial derivatives. To be precise, let θ_n be the n^{th} estimate for an extremum of $F(\theta)$ and define

$$Y_n^{\pm} = \widetilde{F}(\theta_n \pm \delta_n, \xi_n), \tag{1.0.2}$$

where δ_n is the finite difference sequence satisfying $\delta_n \to 0$ as $n \to \infty$, ξ_n is the sequence of collective noise and $\widetilde{F}(\theta_n)$ is the observed value of $F(\theta_n)$ with noise ξ_n. Then the n^{th} gradient estimate is given by

$$D\widetilde{F}(\theta_n, \xi_n) = (Y_n^+ - Y_n^-)/(2\delta_n). \tag{1.0.3}$$

With the above definitions, the algorithm

$$\theta_{n+1} = \theta_n + \varepsilon_n D\widetilde{F}_n(\theta_n, \xi_n) \tag{1.0.4}$$

is called the Kiefer-Wolfowitz algorithm. With certain conditions on the function $F(\cdot)$ and the noise, the convergence of the algorithm (1.0.4) is obtained in [25].

Since the introduction of stochastic approximation algorithms in the early 1950s, they have found applications in many diverse areas, such as signal processing, communications, and adaptive control. New challenges have arisen in applications to the finance area. To the author's best knowledge, the first application of stochastic approximation in finance was introduced by Yin, Liu and Zhang [37], where they use a class of recursive algorithms to determine the timing of stock liquidation. Their trading strategy is to sell a stock if its price drops below a predetermined low price to stop loss or to sell it if its price rises up to a high price to lock in the profit. The analytic solution of this problem was obtained in Zhang [39] where the stock price is subject to two geometric Brownian motions (GBM) coupled by a two-state Markov chain. However, it is hard to implement the results in [39] in practice due to the difficulty of determining the values of parameters used in the stock price model in [39]. Using the stochastic approximation method, [37] proposes a class of recursive algorithms to calculate the threshold values. Further numerical and asymptotic properties are obtained in Yin et al. [35].

Chapter 2 considers the application of stochastic approximation to stock trading. Unlike [37], we now take the timing of buying stocks into account. Although we do not assume any model on stock price to implement the algorithm in practice, we assume that the stock prices are subject to mean-reverting processes. That means the stock price will eventually converge to the equilibrium level. The strategy used is called the buy-low-and-sell-high strategy (see Zhang and Zhang [40]), in that one buys a stock when its price hits the preset low price and sells it when its price exceeds the preset high price. The objective is to maximize the discounted reward function. Zhang and Zhang [40] provide a closed-form solution to this problem. However, their solution is difficult to use in practice because many parameters need to be determined. We propose a stochastic approximation algorithm to compute the optimal buying and selling price. The asymptotic properties are analyzed. Simulations and real market data are also used to demonstrate the efficiency of the proposed algorithm. One distinct feature of this method is that it only requires the historical stock price to yield sound estimations. And due to its simple recursive form, it can be easily used for online stock trading.

Chapter 3 offers a study on a different stock trading tool, trailing stop. Trailing stop maintains a stop-loss order at a precise percentage below the market price. As the market price advances, so does the stop price. Should the market price drop, the stop price does not change, and one decides to sell the stock whenever the stop price is reached. This order is used to limit the maximal possible loss and to lock in a profit. Deciding the value of the trailing stop percentage is crucial in the successful application. However, there was not a rigorous mathematical analysis on the choice of the trailing stop percentage until Glynn and Iglehart [17] first provided mathematical discussion on this topic based on an unrealistic model, in which stock price is allowed to be negative. It is difficult to extend their results to a reasonable market model and to use their results in practice. In this chapter, we design a stochastic recursive algorithm to compute the optimal trailing stop percentage. Unlike the algorithm used in Chapter 2, we use a projection to constrain the iterates to remain in a bounded region. It is clear that the trailing stop percentage should be in the interval $[0, 1]$. Therefore, if the iterate is less then 0, we force it to be 0. Likewise, if the iterate exceeds 1, we force it back to be 1. The convergence of this projection algorithm is also obtained.

Another technique used in Chapter 3 is interval estimation, which provides a stopping rule for the stochastic recursive algorithms. The classical stop criterion for the stochastic approximation algorithm is when the number of iterations reaches a predetermined upper bound. However, in practice it may be difficult to decide this upper bound. The interval estimates can be used to provide a stopping criterion. The algorithm stops when the difference in the estimates from current and previous iterations is small enough, i.e., when $|\theta_{n+1} - \theta_n| < \alpha$, where α is a preset accuracy level. We show that with large probability (probability close to 1), a sequence of scaled and centered estimates and a stopped sequence both converge weakly to diffusion processes. Based on this result, we then can build confidence intervals for the iterates. Results of simulations and numerical experiments are also reported. Our algorithm is a sound procedure for estimating optimal trailing stop percentages and systematically yields useful guidelines for stock trading. One major advantage of this approach is its model free feature.

Other than in stock trading, stochastic approximation algorithms can also be used in option trading. There is a large body of literature studying the pricing problem of options. Among them, Yao, Zhang and Zhou [34] use a switching GBM model to price European options. A closed-form solution is obtained assuming that the underlying Markov chain jumps at most once. To use their result in practice, it is necessary to estimate various parameters. One of the parameters to be estimated is the implied volatility. A standard approach is to employ the least squares method using a large number of observations. However, in view of real-time trading, the least squares method is too slow and requires too many observations to meet the practical needs. Alternatively, in Chapter 4, we develop a stochastic approximation algorithm to estimate implied volatility first. Then we use the estimated implied volatility to price options. Since the volatility generally lies in a bounded interval, we also use a projection algorithm to force all iterates to remain in a reasonable bounded set. The convergence is obtained and the rate of convergence is ascertained under certain conditions. As demonstrated by using real market data, this algorithm provides good estimates for implied volatilities in real time by using only 20 to 30 observations. Together with the regime switching model in [34], the proposed procedure provides more accurate prices than that of the traditional Black-Scholes model and it can be easily implemented for online option trading due to its simple recursive form.

Chapter 2

Mean-Reverting Asset Trading

2.1 Introduction

This chapter is concerned with developing a systematic numerical procedure for trading a mean-reverting asset. The trading strategy consists of two ingredients, buy and sell. One wishes to buy low and sell high. Nevertheless, it is challenging to be able to correctly identify these low and high prices in practice. The purpose of this chapter is to develop and implement an easy systematical procedure to determine the buying and selling prices when the underlying asset price is subject to a mean-reverting process.

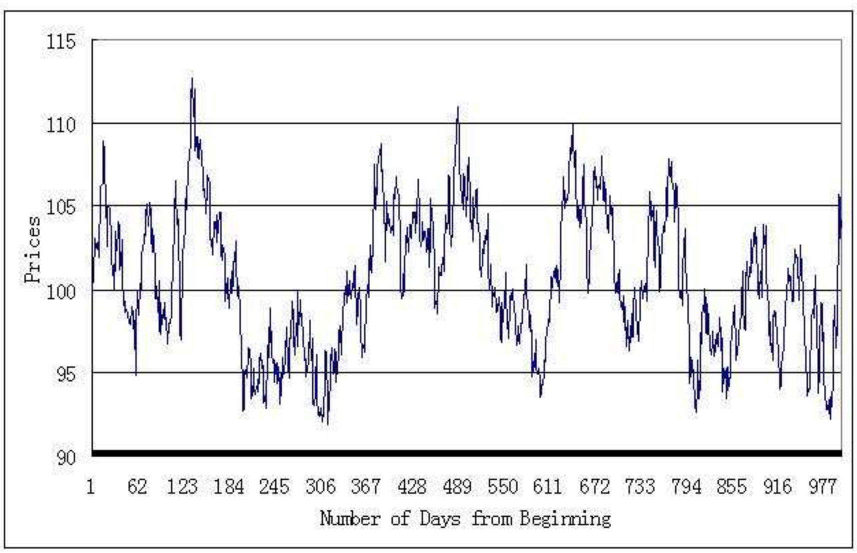

Figure 2.1: Demonstration of Mean-Reverting Type Stock Price

A mean-reverting model is often used in financial and energy markets to capture the price movements that will eventually move back to an "equilibrium" level. Figure 2.1 shows a sample path of mean-reverting stock prices. Empirical studies on mean-reverting stock prices can be traced back to the 1930s (see Cowles and Johns [12]). The research was carried further in many studies. Among them, Fama and French [16] and Poterba and Summer [31] were the first to provide direct empirical evidence that mean reversion occurs in U.S. stock market over the long horizon. Balvers, Wu and Gilliland [2] provided international evidence to support mean-reverting stock prices in 18 countries during the period 1969 to 1996. Conrad and Kaul [9] found mean reversion in short-horizon expected returns. Other than stock prices, mean-reverting models are also used to characterize stochastic volatility (Hafner and Herwartz [19]) and asset price in the energy market (Blanco and Soronow [1]).

There is a large body of literature studying trading rule in financial markets for many years, especially on the sell side. For example, Zhang [39] studies a selling rule when the stock price evolves according to a series of geometric Brownian motions (GBM) coupled with a continuous-time finite state Markov chain. An investor makes a selling decision whenever the stock price exceeds the target price or hit the stop-loss price. The objective is to determine these threshold prices by maximizing a discounted expected reward function, and the optimal threshold values are obtained by solving a set of two-point boundary value problems. In [17], Glynn and Iglehart considered the trailing stop strategy for two models for stock prices: a discrete-time random walk and continuous-time Brownian motion. A trailing stock order maintains a stop price at a precise percentage below the market price. For both models,they discuss the question of optimizing the percentage from the current price to the stop. In Guo and Zhang [18], the optimal selling rule was considered for stock price under a switching GBM model and the optimal stopping problem is solved by using a smooth-fit technique. Pemy, Yin and Zhang [30] considered the liquidation problem of a large block of stocks. Other than these analytical results, various numerical methods have been developed to compute these threshold. In Yin, Liu and Zhang [37], a stochastic approximation technique was used to obtain the optimal selling rule. Further numerical and asymptotic results were obtained in Yin et al. [35]. In addition, a liner programming approach was developed in Helems [20] and fast Fourier transformation was used in Liu, Zhang and Yin [28]. Furthermore, capital gain taxes and transaction costs in stock selling were considered in Cadenillas and Pliska [6], Constantinides [11] and Dammon and Spatt [13], among others.

On the other hand, most work on the buying side of trading is qualitative. For example, contrarian and momentum strategies were studied in Bondt and Thaler [4], Conrad and Kaul [10], and Jagadeesh and Titman [22, 23, 24]. Not until recently was a rigorous mathematical analysis on the buying side provided in Zhang and Zhang [40], in which they considered buying and selling for assets governed by mean-reverting processes. The objective is to buy and sell the underlying asset sequentially to maximize the discounted reward function when the slippage cost is taken into account. [Slippage cost usually refers to the difference between the estimated price and the actual price paid.] In [40], the optimal buying and selling prices were obtained using a dynamic programming approach and the associated HJB equations for the value functions. One makes a buy decision when the market price hits the buying price and makes a sell decision when the market price exceeds the selling price. In order to implement the strategy in [40], one needs to know the values of parameters in the mean-reverting processes in oder to compute the optimal threshold values. In practice, it is difficulty to determine those values. Taking this point into consideration, in [33], Song, Yin, and Zhang proposed a stochastic approximation algorithm to solve the problem for buying and selling the asset once. Instead of solving two quasi-algebraic equation, the problem is formulated as a stochastic optimization procedure. The algorithm is model free and uses observed stock prices only. In this chapter, we further develop the algorithm that allows buying and selling to take place a multiple number of times. The essential feature of our approach is the use of stochastic approximation methods (see Kushner and Yin [27] and Chen [8] for up-to-date development of stochastic approximation algorithms). The proposed stochastic approximation algorithm allows us to deal with the general model free case and use only observed stock prices to determine the optimal buying and selling prices. Therefore the proposed method can be easily implemented in practice.

The rest of the chapter is arranged as follows. Section 2.2 offers a precise formulation of the problem and the description of algorithm. Section 2.3 proceeds with the study of asymptotic properties of the underling algorithm; convergence is obtained. The rate of convergence is ascertained in Section 2.4. To demonstrate the feasibility and efficiency of the algorithm, numerical experiments using simulations and real market data are given in Section 2.5. We demonstrate that the proposed algorithm provides sound estimated optimal threshold values; they can be easily implemented in real time and provide guidelines for stock trading. We conclude this chapter with some further remarks in Section 2.6.

2.2 Problem Formulation

In Zhang and Zhang [40], they assume that $X(t) \in \mathbb{R}$ is a mean-reverting process governed by

$$dX(t) = a(b - X(t))dt + \sigma dW(t), \quad X(0) = x, \tag{2.2.1}$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ is the volatility, and $W(t)$ is a standard Brownian motion. Then the asset price is given by

$$S(t) = \exp(X(t)). \tag{2.2.2}$$

Implied by the mean-reverting process (2.2.1), when $X(t) > b$, the drift term $a(b - X(t))dt$ is negative, pulling $X(t)$ back down toward the equilibrium level. When $X(t) < b$, the drift term is positive, resulting in a pull back up to the equilibrium level. The rate of reversion a determines the reversion speed of $X(t)$. The greater the value of a, the fast the $X(t)$ converges to the equilibrium value. And due to the stochastic term $\sigma dW(t)$, the value of $X(t)$ tends to oscillate around the equilibrium level.

In our formulation, we do not require the asset price $S(t)$ be any specific stochastic process or follow any specific distribution. We only assume that the asset price can be observed. Based on the observed stock price, two sequences of stopping times $\tau^{\{b_i\}}$ and $\tau^{\{s_i\}}$ with

$$0 \le \tau^{\{b_1\}} \le \tau^{\{s_1\}} \le \tau^{\{b_2\}} \le \tau^{\{s_2\}} \le \cdots$$

are considered. One makes a buying decision at time $\tau^{\{b_i\}}$ and makes a selling decision at time $\tau^{\{s_i\}}$, with $i = 1, 2, ...$ Suppose that $0 < K < 1$ is the percentage of slippage per transaction and $\rho > 0$ is the discount factor. We aim to find the optimal buying and selling prices that maximize a suitable reward function. Thus the formulation is

$$\text{Problem } \mathcal{P}: \begin{cases} \text{Find argmax } \Phi(\theta) = E[J(\theta)], \text{ where} \\ \theta = (\theta^1, \theta^2)' \in (0,\infty) \times (0,\infty) \text{ is a column vector,} \\ J(\theta) = \displaystyle\sum_{i=1}^{\infty} [\exp(-\rho\tau^{\{s_i\}})S(\tau^{\{s_i\}})(1-K) - \exp(-\rho\tau^{\{b_i\}})S(\tau^{\{b_i\}})(1+K)], \end{cases} \tag{2.2.3}$$

where

$$\tau^{\{b_1\}} = \inf\{t > 0, S(t) \leq \exp(\theta^1)\},$$

$$\tau^{\{b_i\}} = \inf\{t > \tau^{\{s_{i-1}\}}, S(t) \leq \exp(\theta^1)\}, \text{ for } i \geq 2,$$

$$\tau^{\{s_i\}} = \inf\{t > \tau^{\{b_i\}}, S(t) \geq \exp(\theta^2)\}, \text{ for } i \geq 1.$$

Note that $\tau^{\{b_i\}}$ and $\tau^{\{s_i\}}$ denote the stopping times for buying and selling respectively, θ^1 and θ^2 denote the buying and selling threshold values respectively, and $S(t)$ is the stock price at time t.

The analytic solution is obtained in Zhang and Zhang [40] when $S(t)$ is governed by (2.2.2). However, the solution depends on the values of a and b in (2.2.1), which are difficult to determine in practice. Our contribution is to devise a optimization procedure that estimates the optimal threshold value θ and only requires observed stock prices. We will use a stochastic approximation procedure (SA) to resolve the problem by constructing a sequence of estimates of the optimal threshold value θ, using

$$\theta_{n+1} = \theta_n + \{\text{step size}\}\{\text{gradient estimate of } \Phi(\theta)\} \tag{2.2.4}$$

Let us begin with a simple noisy finite difference scheme. The only provision is that $S(t)$ can be observed. Associated with the iteration number n, denote the threshold value by θ_n. Let us begin with an arbitrary initial guess θ_0, we construct a sequence of estimates $\{\theta_n\}$ recursively as follows. Then we can determine stopping times $\tau_n^{\{b_i\}}$ and $\tau_n^{\{s_i\}}$, buying times and selling times as

$$\tau_n^{\{b_1\}} = \inf\{t > 0, S(t) \le \exp(\theta_n^1)\},$$

$$\tau_n^{\{b_i\}} = \inf\{t > \tau_n^{\{s_{i-1}\}}, S(t) \le \exp(\theta_n^1)\}, \text{ for } i \ge 2,$$

$$\tau_n^{\{s_i\}} = \inf\{t > \tau_n^{\{b_i\}}, S(t) \ge \exp(\theta_n^2)\}, \text{ for } i \ge 1.$$

Define a combined process ξ_n that includes the random effect from $S(t)$ and the stopping times $\tau_n^{\{b_i\}}$ and $\tau_n^{\{s_i\}}$ as

$$\xi_n = (S(\tau_n^{\{b_1\}}), S(\tau_n^{\{s_1\}}), S(\tau_n^{\{b_2\}}), S(\tau_n^{\{s_2\}}), ..., \tau_n^{\{b_1\}}, \tau_n^{\{s_1\}}, \tau_n^{\{b_2\}}, \tau_n^{\{s_2\}}, ...)'. \tag{2.2.5}$$

We call ξ_n the sequence of collective noise. Let $\widetilde{\Phi}(\theta, \xi)$ be the observed value of the objective function $\Phi(\theta)$ with collective noise ξ. When the threshold value is set at θ, take random samples of size n_0 with sequence $\{\xi_{n,l}^{\pm}\}_{l=1}^{n_0}$ such that

$$\widehat{\Phi}(\theta, \xi_n^{\pm}) \stackrel{\text{def}}{=} \frac{\widetilde{\Phi}(\theta, \xi_{n,1}^{\pm}) + \cdots + \widetilde{\Phi}(\theta, \xi_{n,n_0}^{\pm})}{n_0}. \tag{2.2.6}$$

We assume that

$$E\widehat{\Phi}(\theta, \xi_n^{\pm}) = \Phi(\theta) \;\text{ for each } \theta. \tag{2.2.7}$$

Then for each θ, $\widehat{\Phi}(\theta, \xi_n^{\pm})$ is an estimate of $\Phi(\theta)$. In the simulation study, we can use independent random samples to estimate the expected value of $\Phi(\theta_n)$. The law of large numbers implies that $\widehat{\Phi}(\theta, \xi_n^{\pm})$ converges to $\Phi(\theta)$ w.p.1 as $n_0 \to \infty$. We will not assume independence in the proof of convergence theorem. In lieu of using (2.2.6) with $\widehat{\Phi}(\theta, \xi_{n,l}^{\pm})$, we will use the form $\widehat{\Phi}(\theta, \xi_n)$.

To obtain the desired estimate, we construct a stochastic approximation procedure with finite difference gradient estimates. Define $Y_n^{\pm} = (Y_n^{\pm,1}, Y_n^{\pm,2})$ as

$$Y_n^{\pm,\iota}(\theta, \xi_n^{\pm}) = \widehat{\Phi}(\theta \pm \delta_n e_\iota, \xi_n^{\pm}), \text{ for, } \iota = 1, 2, \tag{2.2.8}$$

where e_ι is the standard unit vector with $e_1 = (1,0)'$ and $e_2 = (0,1)'$, $\xi_n^{\pm}$ are two different collective noises taken at threshold values $\theta \pm \delta_n e_\iota$, respectively, and δ_n is the a difference sequence satisfying $\delta_n \to 0$ as $n \to \infty$. We write $Y_n^{\pm} = Y_n^{\pm}(\theta, \xi_n^{\pm})$. For simplicity, henceforth, we often use ξ_n to represent ξ_n^+ and ξ_n^- if there is no confusion. The gradient estimate at iteration n is given by

$$D\widehat{\Phi}(\theta_n, \xi_n) \stackrel{\text{def}}{=} (Y_n^+ - Y_n^-)/(2\delta_n). \tag{2.2.9}$$

Then the recursive algorithm is

$$\theta_{n+1} = \theta_n + \varepsilon_n D\widehat{\Phi}(\theta_n, \xi_n), \tag{2.2.10}$$

where ε_n is a sequence of real numbers known as step size. A frequently used choice of step size and finite difference sequences is $\varepsilon_n = O(1/n)$ and $\delta_n = O(1/n^{1/6})$. Throughout this chapter, this is our default choice of step size and finite difference sequences.

To proceed, we define

$$\begin{aligned}
&\lambda_n = (Y_n^+ - Y_n^-) - E_n(Y_n^+ - Y_n^-),\\
&\eta_n^\iota = [E_n Y_n^{+,\iota} - \Phi(\theta_n + \delta_n e_\iota)] - [E_n Y_n^{-,\iota} - \Phi(\theta_n - \delta_n e_\iota)], \quad \iota = 1, 2,\\
&\beta_n^\iota = \frac{\Phi(\theta_n + \delta_n e_\iota) - \Phi(\theta_n - \delta_n e_\iota)}{2\delta_n} - \Phi_{\theta^\iota}(\theta_n), \quad \iota = 1, 2,
\end{aligned} \tag{2.2.11}$$

where E_n denotes the conditional expectation with respect to $\mathcal{F}_n$, the σ-algebra generated by $\{\theta_1, \xi_j^{\pm} : j < n\}$, $\Phi_{\theta^\iota}(\theta) = (\partial/\partial\theta^\iota)\Phi(\theta)$, and $\Phi_\theta = (\Phi_{\theta^1}(\cdot), \Phi_{\theta^2}(\cdot))'$ denotes the gradient of $\Phi(\cdot)$. In the above, η_n^ι and β_n^ι for $\iota = 1, 2$ represent the noise and bias, and λ_n is a martingale difference sequence. It is reasonable to assume that after taking the conditional expectations, the resulting function is smooth. Thus we separate the noise into two parts, uncorrelated noise λ_n and correlated noise η_n. In what follows, we write $\eta_n = (\eta_n^1, \eta_n^2)'$ and $\beta_n = (\beta_n^1, \beta_n^2)'$, and note that $\eta_n = \eta_n(\theta_n, \xi_n)$ With the above definitions, algorithm (2.2.10) becomes

$$\theta_{n+1} = \theta_n + \varepsilon_n \Phi_\theta(\theta_n) + \varepsilon_n \frac{\lambda_n}{2\delta_n} + \varepsilon_n \beta_n + \varepsilon_n \frac{\eta_n(\theta_n, \xi_n)}{2\delta_n}. \tag{2.2.12}$$

2.3 Convergence

This section studies the convergence of the recursive algorithm. We will show that θ_n defined in (2.2.10) is closely related to an ordinary differential equation (ODE). The stationary points of the ODE are the optimal buying and selling prices that we are seeking.

To carry out the study of convergence, we define the following:

$$\begin{cases} t_n = \sum_{i=1}^{n-1} \varepsilon_i, \quad m(t) = \max\{n : t_n \le t\}, \\ \\ \theta^0(t) = \theta_n \quad \text{for} \quad t \in [t_n, t_{n+1}), \ \theta^n(t) = \theta^0(t + t_n), \\ \\ N_n = \min\{i : t_{n+i} - t_n \ge T\}, \quad \text{for an arbitrary} \quad T > 0. \end{cases} \tag{2.3.1}$$

Note that $\theta^0(\cdot)$ is a piecewise constant process and $\theta^n(\cdot)$ is its shift. With the above definition, the interpolated process $\theta^n(\cdot)$ becomes

$$\theta^n(t) = \theta_n + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j \Phi_\theta(\theta_j) + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j \frac{\lambda_j}{2\delta_j} + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j \beta_j + \sum_{j=n}^{m(t_n+t)-1} \frac{\varepsilon_j}{2\delta_j} \eta_j(\theta_j, \xi_j). \tag{2.3.2}$$

We need the following conditions:

(A2.1) The second derivative $\Phi_{\theta\theta}(\cdot)$ is continuous.

(A2.2) For each compact set G,

(a) $\sup_n E|Y_n^{\pm} I_{\{\theta_n \in G\}}|^2 < \infty$.

(b) For each θ belonging to a bounded set,

$$\sup_n \sum_{j=n}^{n+N_n-1} E^{\frac{1}{2}} \left|E_n \eta_j(\theta, \xi_j)\right|^2 < \infty, \quad \lim_n \sup_{0 \le i < N_n} E|\widetilde{\gamma}_i^n| = 0, \tag{2.3.3}$$

$$\text{where } \widetilde{\gamma}_i^n = \frac{1}{\varepsilon_{n+i}} \sum_{j=n+i}^{n+N_n-1} \frac{\varepsilon_j}{2\delta_j} E_{n+i}[\eta_j(\theta_{n+i+1}, \xi_j) - \eta_j(\theta_{n+i}, \xi_j)], \ i \le N_n - 1.$$

Remark 2.3.1. Our default choice of step size and finite difference sequences is $\varepsilon_n = O(1/n)$ and $\delta_n = O(1/n^{1/6})$. It follows that the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ satisfy $0 < \varepsilon_n \to 0$, $\sum_n \varepsilon_n = \infty$, $0 < \delta_n \to 0$, and $\varepsilon_n/\delta_n^2 \to 0$ as $n \to \infty$. Moreover,

$$\limsup_n \sup_{0 \le i \le N_n - 1} \frac{\varepsilon_{n+i}}{\varepsilon_n} < \infty, \qquad \limsup_n \frac{\delta_{n+i}}{\delta_n} < \infty,$$

$$\limsup_n \left[\frac{(\varepsilon_{n+i}/\delta_{n+i}^2)}{(\varepsilon_n/\delta_n^2)}\right] < \infty.$$

For simplicity, we use a mixing condition. Assume that $\xi_n^{\pm} = g_0(\zeta_n^{\pm})$ where $g_0(\cdot)$ is a real-valued function, $\{\zeta_{n,\ell}^{\pm}\}$ are homogeneous finite-state Markov chains whose transition matrices are irreducible and aperiodic. Thus the noise is bounded since the Markov chain takes only finite values. Then $\xi_{n,\ell}^{\pm}$ are ϕ-mixing sequences with exponential mixing rates ([3, p.167]), i.e., $\varpi(j) = c_0 \varpi^j$ for some $c_0 > 0$ and some $0 < \varpi < 1$. Using the exponential mixing rates, conditions (A2.2)(a) and (A2.2)(b) are easily verified.

To obtain convergence, we first prove the tightness of $\theta^n(\cdot)$ (2.3.2) and then extract its weak limit(see Kushner [26] and Kushner and Yin [27]). We will use a truncation device. Let b be a fixed but otherwise arbitrary positive real number, and $\psi_b(\cdot)$ be a smooth function with compact support satisfying $\psi_b(h) = 1$ when $|\theta| \le b$, and $\psi_b(\theta) = 0$ when $|\theta| \ge b + 1$. Corresponding to (2.2.12), $\{\theta_n^b\}$ is defined recursively by $\theta_1^b = \theta_1$ and

$$\theta_{n+1}^b = \theta_n^b + \left[\varepsilon_n \Phi_\theta(\theta_n^b) + \frac{\varepsilon_n}{2\delta_n}\lambda_n + \varepsilon_n \beta_n + \varepsilon_n \frac{\eta_n(\theta_n^b, \xi_n)}{2\delta_n}\right] \psi_b(\theta_n^b), \quad n \ge 1. \tag{2.3.4}$$

Similar as $\theta^0(t)$ and $\theta^n(t)$, we define the interpolation as $\theta^{0,b}(t) = \theta_n^b$ for $t \in [t_n, t_{n+1})$ and $\theta^{n,b}(t) = \theta^{0,b}(t_n + t)$. Note that $\theta^{n,b}(t) = \theta^n(t)$ until the first exit from the b-sphere

$S_b = \{\theta \in \mathbb{R}^2 : \ |\theta| \le b\}$. Thus, $\theta^{n,b}(\cdot)$ is a b-truncation of $\theta^n(\cdot)$ (see [26, p. 43] and [27, p. 268]).

In light of (A2.1), the continuity of $\Phi_{\theta\theta}(\cdot)$ implies the boundedness of $\Phi_{\theta\theta}(\theta)$ for θ in a bounded set. Thus, for each $\iota = 1, 2$,

$$\begin{aligned} \beta_n^\iota \psi_b(\theta_n^b) &= \left[\frac{\Phi(\theta_n + \delta_n e_\iota) - \Phi(\theta_n - \delta_n e_\iota)}{2\delta_n e_\iota} - \Phi_{\theta^\iota}(\theta_n) \right] \psi_b(\theta_n^b) \\ &= O\left(\frac{|\Phi_{\theta\theta}(\theta_n^+)|\delta_n^2}{2\delta_n} \right) = O(\delta_n), \end{aligned} \tag{2.3.5}$$

where θ_n^+ is on the line segment connecting $\theta_n^b - \delta_n e_\iota$ and $\theta_n^b + \delta_n e_\iota$.

In light of (2.3.2) and (2.3.4), the truncated process $\{\theta^{n,b}(\cdot)\}$ can be rewritten as:

$$\theta^{n,b}(t) = \theta_n^b + \sum_{j=n}^{m(t_n+t)-1} \left(\varepsilon_j \Phi_\theta(\theta_j^b) + \frac{\varepsilon_j}{2\delta_j} \lambda_j + \varepsilon_j \beta_j + \frac{\varepsilon_j}{2\delta_j} \eta_j(\theta_j^b, \xi_j) \right) \psi_b(\theta_j^b).$$

In the process of averaging, $\sum_{j=n}^{m(t_n+t)-1} (\varepsilon_j/(2\delta_j))\eta_j(\theta_j^b, \xi_j)\psi_b(\theta_j^b)$ is difficult to deal with. We claim that this term has weak limit 0. To proceed, define

$$\Delta_{n,i} = \sum_{j=n}^{n+i} \frac{\varepsilon_j}{2\delta_j} \eta_j(\theta_j^b, \xi_j)\psi_b(\theta_j^b), \quad i \le N_n - 1,$$

$$\Delta^n(t) = \Delta_{n,i} \ \text{ for } \ t \in [t_{n+i}, t_{n+i+1}),$$

$$\Gamma_i^n = \sum_{j=n+i}^{n+N_n-1} \frac{\varepsilon_j}{2\delta_j} E_{n+i} \eta_j(\theta_{n+i}^b, \xi_j)\psi_b(\theta_{n+i}^b), \quad i \le N_n - 1,$$

$$\gamma_i^n = \frac{1}{\varepsilon_{n+i}} \sum_{j=n+i}^{n+N_n-1} \frac{\varepsilon_j}{2\delta_j} E_{n+i} \left[\eta_j(\theta_{n+i+1}^b, \xi_j)\psi_b(\theta_{n+i+1}^b) - \eta_j(\theta_{n+i}^b, \xi_j)\psi_b(\theta_{n+i}^b) \right], \quad i \le N_n - 1. \tag{2.3.6}$$

Note that Γ_i^n and γ_i^n are introduced to add some perturbations so as to eliminate certain un-wanted terms. This follows from the use of perturbed test function methods, which have been successfully used in stochastic systems after introduction to treat problems arising in partial differential equations. (see Kushner and Yin [27]).

Lemma 2.3.2 *Under* (A2.1)–(A2.2), $\Delta^n(\cdot)$ *converges weakly to* 0.

Proof. Define $l_n^\kappa = n + \min\{i : |\Delta_{n,i}| > \kappa\}$, for each $\kappa > 0$. We first obtain the weak limit of the truncated sequence $\{\Delta^{n,\kappa}(\cdot)\}$, which is defined by

$$\Delta^{n,\kappa}(t) = \sum_{j=n}^{(m(t_n+t)-1)\wedge l_n^\kappa} \frac{\varepsilon_j}{2\delta_j} \eta_j(\theta_j^b, \xi_j)\psi_b(\theta_j^b),$$

where $(x \wedge y) = \min(x, y)$.

Define

$$\Delta_{n,i}^\kappa = \sum_{j=n}^{(n+i)\wedge l_\kappa^n} \frac{\varepsilon_j}{2\delta_j} \eta_j(\theta_j^b, \xi_j)\psi_b(\theta_j^b), \quad i \le N_n - 1. \tag{2.3.7}$$

Then

$$\sup_{0\le i\le N_n-1} |\Delta_{n,i}^\kappa| \le \kappa + \sup_{1\le j\le N_n-1} \frac{\varepsilon_{n+j}}{2\delta_{n+j}} |\eta_{n+j}(\theta_{n+j}^b, \xi_{n+j})\psi_b(\theta_{n+j}^b)|. \tag{2.3.8}$$

By virtue of Remark 2.3.1, applying Chebyshev's inequality yields that for $0 \le j \le N_n - 1$, and for any $\mu > 0$,

$$\begin{aligned} &P\left(\sup_{0\le j<N_n} \frac{\varepsilon_{n+j}}{2\delta_{n+j}} |\eta_{n+j}(\theta_{n+j}^b, \xi_{n+j})\psi_b(\theta_{n+j}^b)| \ge \mu\right) \\ &\le \sum_{j=0}^{N_n-1} P\left(\frac{\varepsilon_{n+j}}{2\delta_{n+j}} |\eta_{n+j}(\theta_{n+j}^b, \xi_{n+j})\psi_b(\theta_{n+j}^b)| \ge \mu\right) \\ &\le \frac{KT}{\mu^2} O(\frac{\varepsilon_n}{\delta_n^2}) \sum_{j=0}^{N_n-1} \varepsilon_{n+j} \limsup_n \frac{(\varepsilon_{n+j}/\delta_{n+j}^2)}{(\varepsilon_n/\delta_n^2)} \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{2.3.9}$$

Thus for each κ, $\{\Delta^{n,\kappa}(\cdot)\}$ is bounded in probability in view of of (2.3.8) and (2.3.9).

Next we apply the perturbed test function method of Kushner and Yin [27, Theorem 7.4.3]. Let $\pi(\cdot) \in C_0^2$ (the space of real-valued C^2 functions with compact support). Note that by definition (2.3.7), $E_{n+i}\left[\pi(\Delta_{n,i+1}^\kappa) - \pi(\Delta_{n,i}^\kappa)\right] = 0$ for $n + i \ge l_\kappa^n$. Thus we need only consider $n + i < l_\kappa^n$ in what follows. For $n + i < l_\kappa^n$,

$$\begin{aligned} E_{n+i}\left[\pi(\Delta_{n,i+1}^\kappa) - \pi(\Delta_{n,i}^\kappa)\right] &= \pi_h'(\Delta_{n,i}^\kappa)\frac{\varepsilon_{n+i}}{2\delta_{n+i}} E_{n+i}\eta_{n+i}(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b) \\ &\quad + O\left(\frac{\varepsilon_{n+i}^2}{\delta_{n+i}^2}\right) \left|E_{n+i}\eta_{n+i}(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b)\right|^2. \end{aligned}$$

Define a perturbed test function by

$$\pi_i^n = \pi(\Delta_{n,i}^{\kappa}) + \pi_\theta'(\Delta_{n,i}^{\kappa})\Gamma_i^n,$$

where Γ_i^n is defined by (2.3.6). Note that

$$E_{n+i}\left[\pi_\theta'(\Delta_{n,i}^{\kappa})\Gamma_{i+1}^n - \pi_\theta'(\Delta_{n,i}^{\kappa})\Gamma_i^n)\right]$$

$$= E_{n+i}\left[\pi_\theta'(\Delta_{n,i+1}^{\kappa}) - \pi_\theta'(\Delta_{n,i}^{\kappa})\right]\Gamma_{i+1}^n + E_{n+i}\pi_\theta'(\Delta_{n,i}^{\kappa})[\Gamma_{i+1}^n - \Gamma_i^n],$$

since

$$E_{n+i}[\pi_\theta(\Delta_{n,i+1}^{\kappa}) - \pi_\theta'(\Delta_{n,i}^{\kappa})]\Gamma_{n+i}^n \leq KE_{n+i}|(\Delta_{n,i+1}^{\kappa} - \Delta_{n,i}^{\kappa})\Gamma_{i+1}^n|$$

$$\leq K\frac{\varepsilon_{n+i}}{2\delta_{n+i}}E_{n+i}|\eta_{n+i}(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b)||\Gamma_{i+1}^n|,$$

and

$$E_{n+i}[\Gamma_{i+1}^n - \Gamma_i^n] = \varepsilon_{n+i}\pi_\theta'(\Delta_{n,i}^{\kappa})\gamma_i^n - \frac{\varepsilon_{n+i}}{2\delta_{n+i}}E_{n+i}\eta_j(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b).$$

In addition $\sup_{i<N_n}|\Gamma_i^n| \to 0$ in probability by virtue of (A2.1)–(A2.2). We can write

$$\begin{aligned}
&E_{n+i}[\pi_{i+1}^n - \pi_i^n]\\
&= \varepsilon_{n+i}O\left(\frac{\varepsilon_{n+i}}{\delta_{n+i}^2}|\eta_{n+i}(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b)|^2\right)\\
&\quad+\frac{\varepsilon_{n+i}}{\delta_{n+i}}O\left(\left|E_{n+i}\eta_{n+i}(\theta_{n+i}^b, \xi_{n+i})\psi_b(\theta_{n+i}^b)\right|\left|\sum_{j=n+i+1}^{n+N_n}\frac{\varepsilon_j}{2\delta_j}E_{n+i+1}\eta_j(\theta_{n+i+1}^b, \xi_j)\psi_b(\theta_{n+i+1}^b)\right|\right)\\
&\quad+\varepsilon_{n+i}\pi_\theta'(\Delta_{n,i}^{\kappa})\gamma_i^n\psi_b(\theta_{n+i}^b).
\end{aligned} \tag{2.3.10}$$

By virtue of (A2.2), use of the truncation function $\psi_b(\cdot)$ yields that $\{|\eta_n(\theta_n^b, \xi_n)|\psi_b(\theta_n^b)\}$ is uniformly integrable. Together with $\varepsilon_n/\delta_n^2 \to 0$, this implies that the term on the second line of (2.3.10) goes to 0 in mean uniformly in $0 \le i \le N_n - 1$. Again, using (A2.2), the term on the third line also tends to 0 in mean uniformly in $0 \le i \le N_n - 1$ and the expectation of the last term is bounded by $O(\varepsilon_n/\delta_n^2) \to 0$ uniformly in $0 \le i \le N_n - 1$. Thus, in light of Kushner and Yin [27, Theorem 7.4.3], the weak limit of κ-truncated sequence $\{\Delta^{n,\kappa}(\cdot)\}$ is a zero process. Finally, [27, Theorem 7.3.6] implies that the original un-truncated sequence $\{\Delta^n(\cdot)\}$ also converges to the zero process. $\square$

Theorem 2.3.3. *Assume that* (A2.1)–(A2.2) *and that* $\{\theta_n\}$ *is tight in* $\mathbb{R}^2$. *Then* $\theta^n(\cdot)$ *converges weakly to* $\theta(\cdot)$, *the solution to the differential equation*

$$\dot{\theta} = \Phi_\theta(\theta), \tag{2.3.11}$$

which has a unique solution for each initial condition.

Proof. The proof is divided into three steps. We will first prove that $\{\theta^{n,b}(\cdot)\}$ is tight and thus derive its weak convergence. Using this truncated process, we can obtain the convergence of $\theta^n(\cdot)$

Step 1: We proceed to obtain the tightness of $\{\theta^{n,b}(\cdot)\}$. For any $\nu > 0$, $t > 0$, and $0 \le s \le \nu$, using(2.3.4), it is straight forward to see that

$$\theta^{n,b}(t+s) = \theta^{n,b}(t) = \widetilde{\theta}^{n,b}(t+s) - \widetilde{\theta}^{n,b}(t) + o_n(1), \tag{2.3.12}$$

where

$$\widetilde{\theta}^{n,b}(t+s) - \widetilde{\theta}^{n,b}(t) = \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \left[\Phi_\theta(\theta_j^b) + \frac{\lambda_j}{2\delta_j} + \beta_j\right] \psi_b(\theta_j^b), \tag{2.3.13}$$

and $o_n(1)$ converges weakly (or in probability) to 0 as $n \to \infty$. By virtue of Kushner [26, p. 50, Lemma 5], to obtain the tightness of $\{\theta^{n,b}(\cdot)\}$, we just need to prove $\{\widetilde{\theta}^{n,b}(\cdot)\}$ is tight.

Denote by E_t^n the conditional expectation with respect to $\mathcal{F}_t^n = \sigma\{\theta^n(s) : s \le t\}$; then

$$
\begin{aligned}
& E_t^n \left| \widetilde{\theta}^{n,b}(t+s) - \widetilde{\theta}^{n,b}(t) \right|^2 \\
& \le K E_t^n \left| \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \Phi_\theta(\theta_j^b) \psi_b(\theta_j^b) \right|^2 + K E_t^n \left| \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \frac{\lambda_j}{2\delta_j} \psi_b(\theta_j^b) \right|^2 \\
& \quad + K E_t^n \left| \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \beta_j \psi_b(\theta_j^b) \right|^2 \\
& \le O\left(\left(\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \right)^2 + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{\varepsilon_j^2}{4\delta_j^2} \right)^2,
\end{aligned}
$$

Thus $\lim_{\nu \to 0} \limsup_n E E_t^n |\widetilde{\theta}^{n,b}(t+s) - \widetilde{\theta}^{n,b}(t)|^2 = 0$. The tightness criterion (see Ethier and Kurtz [15, Section 3.8, p. 132] and Kushner [26, p. 47]) implies that $\{\widetilde{\theta}^{n,b}(\cdot)\}$ is tight and so is $\{\theta^{n,b}(\cdot)\}$.

Step 2: We will extract the weak limit of $\{\theta^{n,b}(\cdot)\}$. Because of the tightness of $\{\theta^{n,b}(\cdot)\}$, we can extract a convergent subsequence, and for simplicity, still denote it by $\theta^{n,b}(\cdot)$.

Since $\{\lambda_n\}$ is a martingale difference sequence,

$$
E \left| \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{\varepsilon_j}{2\delta_j} \lambda_j \psi_b(\theta_j^b) \right|^2 = \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{\varepsilon_j^2}{4\delta_j^2} E|\lambda_j \psi_b(\theta_j^b)|^2 \to 0 \quad \text{as } n \to \infty.
$$

Hence, the second term in square brackets of (2.3.13) goes to 0 in probability uniformly in t as $n \to \infty$.

For $t, s > 0$, in view of(2.3.5), let $n \to \infty$,

$$
E \left| \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \beta_j \psi_b(\theta_j^b) \right| \le K \left(\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \right) O(\delta_{m(t_n+t)}) \to 0.
$$

Thus the last term in square brackets of (2.3.13) has limit 0.

To deal with the first term in square brackets of (2.3.13). Take a sequence of positive real numbers $\{\varsigma_n\}$ such that

$$\varsigma_n \to 0, \quad \text{and} \quad \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \to 1 \quad \text{as} \quad n \to \infty.$$

Note that

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \varepsilon_j \Phi_\theta(\theta_j^b)\psi_b(\theta_j^b) = \sum_l \varsigma_n \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta_j^b)\psi_j(\theta_j^b).$$

By the smoothness of $\Phi_\theta(\cdot)$ and $\psi_b(\cdot)$, the limit of $\frac{1}{\varsigma_n}\sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta_j^b)\psi_b(\theta_j^b)$ is the same as that of

$$\frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta^b_{m(t_n+t+l\varsigma_n)})\psi_b(\theta^b_{m(t_n+t+l\varsigma_n)})$$

as $n \to \infty$. Fix $\widetilde{u}$ and let $t_{m(t_n+t+l\varsigma_n)} \to \widetilde{u}$ as $n \to \infty$, we only need to verify that

$$\frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta^b_{m(t_n+t+l\varsigma_n)})\psi_b(\theta^b_{m(t_n+t+l\varsigma_n)}) \to \Phi_\theta(\theta^b(\widetilde{u}))\psi_b(\theta^b(\widetilde{u})) \tag{2.3.14}$$

in probability as $n \to \infty$. For each $\nu > 0$, select a finite number of disjoint sets A_ι^ν, $\iota = 1, \dots, r$ such that the range of $\{\theta_n^b\}$ is contained in $\cup_{\iota=1}^r A_\iota^\nu$ and

$$P(\theta^{n,b}(\widetilde{u}) \in \partial A_\iota^\nu) = 0, \quad \text{and} \quad \mathrm{diam}(A_\iota^\nu) \le \nu, \tag{2.3.15}$$

where ∂A_ι^ν denotes the boundary of A_ι^ν. Pick a point θ_ι^ν in A_ι^ν; then we can approximate the term on the left-hand side of (2.3.14) by using a small $\nu > 0$ via

$$\sum_{\iota=1}^r I_{\{\theta^{n,b}(\widetilde{u}) \in B_\iota^\nu\}} \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta_\iota^\nu)\psi_b(\theta_\iota^\nu).$$

In light of the choice of ς_n together with the interpolation, we have

$$\sum_l \varsigma_n \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \varepsilon_j \Phi_\theta(\theta_j^b)\psi_b(\theta_j^b) \to \int_t^{t+s} \Phi_\theta(\theta^b(\widetilde{u}))\psi_b(\theta^b(\widetilde{u}))d\widetilde{u}.$$

Therefore, we conclude that $\theta^{n,b}(\cdot)$ converges weakly to $\theta^b(\cdot)$ as $n \to \infty$ such that

$$\theta^b(t+s) - \theta^b(t) = \int_t^{t+s} \Phi_\theta(\theta^b(\widetilde{u}))\psi_b(\theta^b(\widetilde{u}))d\widetilde{u}.$$

Therefore mean ODE $\dot{\theta}^b(t) = \Phi_\theta(\theta^b(t))\psi_b(\theta^b(t))$ is obtained.

Step 3: We will prove the convergence of $\theta^n(\cdot)$.

Indeed, we only need to show that for each $T > 0$,

$$\limsup_{b\to\infty} \limsup_{n\to\infty} P(\theta^{n,b}(t) \neq \theta^n(t) \quad \text{for some} \quad t \leq T) = 0.$$

This follows immediately from the argument in [27, Chapter 8]. The details are not shown here. $\square$

Corollary 2.3.4. *Suppose that* (2.3.11) *has a unique stationary point θ_* that is globally asymptotically stable in the sense of Liapunov, and that $\{s_n\}$ is a sequence of real numbers such that $s_n \to \infty$. Then $\theta^n(s_n + \cdot)$ converges weakly to θ_* as $n \to \infty$.*

Proof. Let $T > 0$ and choose a convergent subsequence $\{(\theta^n(s_n + \cdot), \theta^n(s_n - T + \cdot))\}$ with limit $(\theta(\cdot), \theta_T(\cdot))$. It is straight forward to show that $\theta(0) = \theta_T(T)$. Even though the value of $\theta_T(0)$ may not be known, the collection of possible $\{\theta_T(0)\}$ over all T and all convergent subsequences belongs to a tight set. In view of the stability of the ODE, for any $\nu > 0$ there is a $0 < T_\nu < \infty$ such that for all $T > T_\nu$, $P(\theta_T(T) \in B(\theta_*, \nu)) \geq 1 - \nu$, where $B(\theta_*, \nu)$ is a neighborhood of θ_* with radius ν. Thus the desired result follows. $\square$

2.4 Rate of Convergence

This section is devoted to the rate of convergence of the algorithm (2.2.10). For simplification, we take $\varepsilon_n = 1/n$ and $\delta_n = \delta/n^{1/6}$. We assume all the conditions of Theorem 2.3.3 hold. Define $u_n = n^{1/3}(\theta_n - \theta_*)$. The rate of convergence study aims to exploit the asymptotic properties of this scaled sequence. We shall show that the weak limit of the interpolation of u_n is a diffusion process. To proceed, let us state conditions needed in what follows.

(A2.3) Assume $\theta_n \to \theta_*$ in probability, and $\Phi_{\theta\theta\theta}(\cdot)$ exists and is continuous in a neighborhood of θ_*. In addition, assume

(a) $\{u_n\}$ is tight;

(b) all eigenvalues of $\Phi_{\theta\theta}(\theta_*) + (1/3)I$ have negative real parts;

(c) for each θ,

$$\eta_n(\theta,\xi) = \eta_n(\theta_*,\xi) + \eta_{n,\theta}(\theta_*,\xi)(\theta-\theta_*)$$

$$+\left(\int_0^1 [\eta_{n,\theta}(\theta_* + (\theta_n-\theta_*)s,\xi) - \eta_{n,\theta}(\theta_*,\xi)]ds\right)(\theta-\theta_*);$$

(d) the sequence $\{\eta_n(\theta_*,\xi_n)\}$ is stationary ϕ-mixing such that $E|\eta_n(\theta_*,\xi_n)|^{2+\Delta} < \infty$ for some $\Delta > 0$ and $E\eta_n(\theta_*,\xi_n) = 0$ and that the mixing measure $\varpi(\cdot)$ is given by $\varpi(j) = \sup_{A\in\mathcal{F}^{n+j}} E^{(1+\Delta)/(2+\Delta)}|P(A|\mathcal{F}_n) - P(A)|^{(2+\Delta)/(1+\Delta)}$, satisfying $\sum_{j=1}^{\infty} (\varpi(j))^{\Delta/(1+\Delta)} < \infty$.

Remark 2.4.1. Applying perturbed Liapunov function methods, (A2.3)(a) can be verified. (see Kushner and Yin [27, Section 10.4]). The existence and continuity of $\Phi_{\theta\theta\theta}(\cdot)$ in a neighborhood of θ_* allows us to linearize $\Phi(\cdot)$ about θ_*. Conditions (A2.3) (c) and (d) concern the sequence $\eta_n(\theta,\xi)$. Let us examine these conditions in conjunction with (2.2.6) and using independent samples and noises $\{\xi^{\pm}_{n,l}\}$. It is important to note that due to the use of the stopping times $\tau^{\{b_i\}}$ and $\tau^{\{s_i\}}$, $\widehat{\Phi}(\theta,\xi)$ defined in (2.2.6) may not be continuous in θ. However, we can assume that its expectation is smooth.

Take for instance, $\widetilde{\Phi}(\theta,\xi) = \Phi(\theta) + f_0(\theta)\xi$, where $f_0(\theta)$ is a bounded and continuous function. Suppose that for a positive integer m_0, $\{\xi^{\pm}_{n,l}\}$ are m_0-dependent sequences (see Billingsley [3, p.167]). For example, $\xi^{\pm}_{n,l} = \sum_{j=0}^{m_0} c_j\zeta^{\pm}_{n-j,l}$, where $\{\zeta^{\pm}_{n,l}\}$ are martingale difference sequences satisfying $E|\zeta^{\pm}_{n,l}|^2 < \infty$. Then they are mixing processes and the mixing measures satisfy $\varpi(j) = 0$ for all $j > m_0$. For each θ belonging to a bounded set, it is easily verified that $E\left|\frac{1}{n_0}\sum_{l=1}^{n_0}[\widetilde{\Phi}(\theta+\delta_n e_\iota,\xi^{\pm}_{n,l})]\right|^2 < \infty$. Thus $E|Y_n^{\pm,\iota}|^2 < \infty$. In addition, for each $j > n$, $E_n[\eta_j(\theta,\xi_j)] = E_n\{[E_jY_j^{+,\iota}(\theta,\xi_j) - \Phi(\theta+\delta_n)] - [E_jY^{-,\iota}(\theta,\xi_j)_j - \Phi(\theta-\delta_n)]\} = 0$. It is easy to see that $\eta_n(\theta_*,\xi_n)$ is a zero mean sequence and is ϕ-mixing (in fact n_0-dependent). Thus (A2.3)(d) is verified.

Let

$$\lambda_n^{*,\iota} = \widehat{\Phi}(\theta_* + \delta_n e_\iota, \xi_n^+) - \widehat{\Phi}(\theta_* - \delta_n e_\iota, \xi_n^-)$$

and

$$\lambda_n^* = (\lambda_n^{*,1}, \lambda_n^{*,2})' \text{ be a column vector.}$$

Together with (2.2.8), the integrability and the convergence of θ_n to θ_* imply that

$$E|\lambda_n - \lambda_n^*|^2 \to 0 \text{ as } n \to \infty. \tag{2.4.1}$$

Lemma 2.4.2. *Assume* (A2.1)–(A2.3).

(a) *The following inequalities hold:*

$$\begin{aligned} &|E\eta_j(\theta_*, \xi_j)\eta_k(\theta_*, \xi_k)| \le K\left(\varpi(j)\right)^{\Delta/(1+\Delta)}, \\ &E\left|E(\eta_{n+j}(\theta_*, \xi_{n+j})\middle|\mathcal{F}_n)\right| \le K\left(\varpi(j)\right)^{\Delta/(1+\Delta)}; \end{aligned} \tag{2.4.2}$$

(b) *The weak limit of the sequence* $\sum_{j=n}^{m(t_n+t)-1}(\eta_j(\theta_*, \xi_j) + \lambda_j^*)/\sqrt{j}$ *is an* $\mathbb{R}^2$*-valued Brownian motion* $\widetilde{w}(\cdot)$ *with covariance* Σt,

Proof. Part (a) of the lemma follows that of Ethier and Kurtz [15, Propositions 7.2.2 and 7.2.4]; part (b) can be proved similarly to [15, Theorem 7.3.1]. We omit the details here. □

In what follows, we shall show that the interpolation of $u^n(\cdot)$ is tight and extract its weak limit.

Using (2.2.10), (A2.1), and $\delta_n = \delta/n^{1/6}$, we obtain

$$\begin{aligned} \theta_{n+1} - \theta_* = \theta_n - \theta_* &+ \frac{1}{n}\Phi_{\theta\theta}(\theta_*)(\theta_n - \theta_*) + \frac{1}{n^{\frac{5}{6}}}\frac{\lambda_n}{2\delta} + \frac{1}{n}\beta_n + \frac{1}{n^{\frac{5}{6}}}\frac{\eta_n(\theta_n, \xi_n)}{2\delta} \\ &+ \frac{1}{n}\left(\int_0^1 (\theta_n - \theta_*)'\Phi_{\theta\theta\theta}(\theta_* + (\theta_n - \theta_*)s)ds\right)(\theta_n - \theta_*). \end{aligned} \tag{2.4.3}$$

Without loss of generality, assume that $\{u_n\}$ is bounded, otherwise, we can use a truncation device as in the proof of the convergence of the algorithm in the previous section. Then we show that the truncated process converges, and finally conclude that the un-truncated process also converges. Since

$$(\frac{n+1}{n})^{\frac{1}{3}} = 1 + \frac{1}{3n} + O(\frac{1}{n^2}),$$

by virtue of (A2.3) we arrive at

$$\begin{aligned} u_{n+1} = u_n \, & + \frac{1}{n}(\Phi_{\theta\theta}(\theta_*) + (\frac{1}{3})I)u_n + \frac{1}{n}(n^{\frac{1}{3}}\beta_n) + \frac{1}{\sqrt{n}}\frac{1}{2\delta}(\eta_n(\theta_*, \xi_n) + \lambda_n^*) \\ & + \left(\frac{n+1}{n}\right)^{\frac{1}{3}} \frac{1}{n}\left(\int_0^1 u_n' \Phi_{\theta\theta\theta}(\theta_* + \frac{s}{n^{\frac{1}{3}}}u_n)ds\right) u_n \\ & + \left(\frac{n+1}{n}\right)^{\frac{1}{3}} \frac{1}{n^{\frac{7}{6}}}\left(\int_0^1 [\eta_{n,\theta}(\theta_* + \frac{s}{n^{\frac{1}{3}}}u_n), \xi_n) - \eta_{n,\theta}(\theta_*, \xi_n)]ds\right) u_n \\ & + \left(\frac{n+1}{n}\right)^{\frac{1}{3}} \frac{1}{\sqrt{n}}\frac{1}{2\delta}(\lambda_n - \lambda_n^*) + \frac{1}{n}o(1 + |u_n|). \end{aligned} \tag{2.4.4}$$

Define a piecewise constant interpolation

$$u^0(t) = u_n, \; t \in [t_n, t_{n+1}), \quad \text{and} \quad u^n(t) = u^0(t_n + t).$$

It can be demonstrated that using the definition of interpolation in (2.4.4), the following three terms

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \left(\frac{j+1}{j}\right)^{\frac{1}{3}} \frac{1}{j}\left(\int_0^1 u_j' \Phi_{\theta\theta\theta}(\theta_* + \frac{s}{j^{\frac{1}{3}}}u_j)ds\right) u_j,$$

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \left(\frac{j+1}{j}\right)^{\frac{1}{3}} \frac{1}{j^{\frac{7}{6}}}\left(\int_0^1 [\eta_{j,\theta}(\theta_* + \frac{s}{j^{\frac{1}{3}}}u_j), \xi_j) - \eta_{j,\theta}(\theta_*, \xi_j)]ds\right) u_j,$$

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j}o(1 + |u_j|)$$

are asymptotically unimportant and contribute a limit 0 in distribution. Furthermore, note that $\{\lambda_n - \lambda_n^*\}$ is a martingale difference sequence

$$E\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\left(\frac{n+1}{n}\right)^{\frac{1}{3}}\frac{1}{\sqrt{j}}(\lambda_j-\lambda_j^*)\right|^2$$
$$=\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\left(\frac{n+1}{n}\right)^{\frac{2}{3}}\frac{1}{j}E|\lambda_j-\lambda_j^*|^2 \to 0 \text{ as } n\to\infty$$

by (2.4.1). This leads to

$$\begin{aligned} u^n(t+s)-u^n(t) = & \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j}(\Phi_{\theta\theta}(\theta_*)+(\frac{1}{3}))u_j+\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j}(j^{\frac{1}{3}}\beta_j) \\ & +\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{\sqrt{j}}\frac{1}{2\delta}(\eta_j(\theta_*,\xi_j)+\lambda_j^*)+o(1), \end{aligned} \tag{2.4.5}$$

where $o(1)\to 0$ in probability uniformly in t.

To proceed, let us state the following lemma.

Lemma 2.4.3. *Under* (A2.1-A2.3),*the sequence* $u^n(\cdot)$ *is tight.*

Proof. Under the boundedness of $\{u_n\}$ for each $\nu > 0$, $t, s > 0$ with $s < \nu$, we obtain

$$\begin{aligned} & E_t^n|u^n(t+s)-u^n(t)|^2 \\ & \leq KE_t^n\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j}(\Phi_{\theta\theta}(\theta_*)+(\frac{1}{3})I)u_j\right|^2+KE_t^n\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j}(j^{\frac{1}{3}}\beta_j)\right|^2 \\ & \quad +KE_t^n\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{\sqrt{j}}(\eta_j(\theta_*,\xi_j)+\lambda_j^*)\right|^2+o(1) \\ & \leq Ks^2+K\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\sum_{k>j}\frac{1}{\sqrt{j}}\frac{1}{\sqrt{k}}|E_t^n[\eta_j(\theta_*,\xi_j)+\lambda_j^*][E_{j+1}(\eta_k(\theta_*,\xi_k)+\lambda_k^*)]|+o(1), \end{aligned} \tag{2.4.6}$$

where $o(1) \to 0$ as $n \to \infty$. Using (2.4.2) leads $\lim_{\nu \to 0} \limsup_{n \to \infty} E E_t^n |u^n(t+s) - u^n(t)|^2 = 0$. Hence the tightness follows. $\square$

Theorem 2.4.4. *Assume that* (A2.3) *holds. Then* $u^n(\cdot)$ *converges weakly to a diffusion process* $u(\cdot)$ *that is a solution of the stochastic differential equation*

$$du = \left\{ \left(\Phi_{\theta\theta}(\theta_*) + \frac{I}{3} \right) u + \frac{\delta^2}{3!} \begin{pmatrix} \Phi_{\theta^1,\theta^1,\theta^1}(\theta_*) \\ \\ \Phi_{\theta^2,\theta^2,\theta^2}(\theta_*) \end{pmatrix} \right\} dt + \frac{1}{2\delta} d\widetilde{w}, \tag{2.4.7}$$

where $\widetilde{w}(\cdot)$ *is the Brownian motion with covariance* $\Sigma^{1/2}(\Sigma^{1/2})'t = \Sigma t$ *given by Lemma* 2.4.2.

Proof. Since $u^n(\cdot)$ is tight, we proceed to characterize the limit process $u(\cdot)$. Consider the bias term β_n, by virtue of (A2.1) and (A2.3), a Taylor expansion of β_n yields

$$\begin{aligned} \beta_n &= \frac{\Phi(\theta_n + \delta_n e_\iota) - \Phi(\theta_n - \delta_n e_\iota)}{2\delta_n} - \Phi_{\theta^\iota}(\theta_n) \\ &= \frac{1}{3!} \Phi_{\theta^\iota\theta^\iota\theta^\iota}(\theta_*) + \frac{1}{3!} \left(\Phi_{\theta^\iota\theta^\iota\theta^\iota}(\theta_n) - \Phi_{\theta^\iota\theta^\iota\theta^\iota}(\theta_*) \right) + o(\delta_n^2), \end{aligned}$$

and the last term above goes to 0 in mean and hence in probability as $n \to \infty$. Since $\delta_n^2 = \delta^2/n^{1/3}$, then we have

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j} j^{1/3}\beta_j = \frac{\delta}{3!} \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j} \begin{pmatrix} \Phi_{\theta^1,\theta^1,\theta^1}(\theta_j) \\ \\ \Phi_{\theta^2,\theta^2,\theta^2}(\theta_j) \end{pmatrix} + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j} o(1)$$

$$= \frac{\delta}{3!} \sum_l \varsigma_n \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \frac{1}{j} \begin{pmatrix} \Phi_{\theta^1,\theta^1,\theta^1}(\theta_*) \\ \\ \Phi_{\theta^2,\theta^2,\theta^2}(\theta_*) \end{pmatrix} + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j} o(1)$$

$$+ \frac{\delta}{3!} \sum_l \varsigma_n \frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \frac{1}{j} O(|\Phi_{\theta\theta\theta}(\theta_n) - \Phi_{\theta\theta\theta}(\theta_*)|)$$

$$= \frac{\delta}{3!} \sum_l \begin{pmatrix} \Phi_{\theta^1,\theta^1,\theta^1}(\theta_*) \\ \\ \Phi_{\theta^2,\theta^2,\theta^2}(\theta_*) \end{pmatrix} \varsigma_n \left[\frac{1}{\varsigma_n} \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} \frac{1}{j} \right] + o(1),$$

where $o(1) \to 0$ in probability as $n \to \infty$ by virtue of $\theta_n \to \theta_*$ in probability and the continuity of $\Phi_{\theta\theta\theta}(\cdot)$ (in a neighborhood of θ_*). Moreover, since $(1/\varsigma_n) \sum_{j=m(t_n+t+l\varsigma_n)}^{m(t_n+t+(l+1)\varsigma_n)-1} (1/j) \to 1$ as $n \to \infty$, the limit of the next to the last term yields

$$\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j} j^{1/3}\beta_j \to \delta^2 \begin{pmatrix} \Phi_{\theta^1,\theta^1,\theta^1}(\theta_*) \\ \\ \Phi_{\theta^2,\theta^2,\theta^2}(\theta_*) \end{pmatrix} s.$$

Thus the desired result follows. $\square$

Remark 2.4.5. To easily handle real market data, we do not take a sample average as in (2.2.6). Instead of that, we set $n_0 = 1$ and use $\widehat{\Phi}(\theta, \xi_n^{\pm}) = \widetilde{\Phi}(\theta, \xi_n^{\pm})$ without using sample averages. The resulting estimate is not expected to be a smooth and the bias will be larger. Nevertheless, it does provide us a reasonable estimate.

Define $Y_n^{\pm}$ as in (2.2.8) and use algorithm (2.2.10). Since conditions concerning the noise given in Section 2.3 and 2.4 all refer to the function $\widehat{\Phi}(\theta, \xi_n)$, we obtain the convergence and rate of convergence of the algorithm just as in the previous sections. We summarize the above as the following proposition.

Proposition 2.4.6. *Let $n_0 = 1$, under the conditions of Theorem 2.3.3 and Theorem 2.4.4, the conclusions of these theorems continue to hold.*

2.5 Numerical Demonstration

In this section, we report our simulation results and numerical experiments. We first use Monte Carlo simulation and compare our algorithm with the analytic solution obtained in [40]. Then we test our algorithm using real market data.

2.5.1 Simulation Study

In this section, we compare the results of stochastic approximation algorithm with the closed-form solution in Zhang and Zhang [40]. We find that the proposed algorithm indeed provides good approximation results. Recall that the mean-reverting SDE follows

$$dX(t) = a(b - X(t))dt + \sigma dW(t), \quad X(0) = x, \tag{2.5.1}$$

and the stock price is given by $S(t) = \exp(X(t))$. First, we take $a = 0.8$, $b = 2$, $\sigma = 0.5$, $x = 0$, let the slippage rate $K = 0.01$, and the discount rate $\rho = 0.5$. In this case, the analytic solution in [40] gives $(\theta_*^1, \theta_*^2) = (1.331, 1.631)$.

We let $n_0 = 5000$, where n_0 is the number of random samples used in each iteration; see (2.2.6). Then we use (2.5.1) to simulate the stock prices and the recursive algorithm (2.2.10) is applied for 200 iterations. The sequence of ξ_n and δ_n are chosen to be $\xi_n = 1/(n + k_0)$ and $\delta_n = 1/(n^{1/6} + k_1)$, respectively, where k_0 and k_1 are some positive constants. The proposed algorithm yields the optimal estimation of $(\theta^1, \theta^2) = (1.3225, 1.6292)$. The (absolute) error $=$ $\sqrt{(\theta^1-\theta^1_*)^2+(\theta^2-\theta^2_*)^2} = 0.0087$, and the relative error $=$ error$/\sqrt{(\theta^1_*)^2+(\theta^2_*)^2} = 0.0041$.

Note that the analytic solutions depend on the knowledge of various parameters: a, b, σ, K and ρ. We next vary one of the parameters at a time and compare the approximation results with analytic results.

We first vary the values of b, the equilibrium levels of stock price. Then we compute the threshold values (θ^1, θ^2) associated with varying b. Intuitively, larger b would result in larger threshold values (θ^1, θ^2). Our approximation results confirm this. The detail results are reported in Table 2.1, where θ^1_* and θ^2_* are threshold values calculated by analytic solution and θ^1 and θ^2 are threshold values calculated by SA method; see (2.2.10).

b	1	1.5	2	2.5	3
θ^1_*	0.331	0.831	1.331	1.831	2.331
θ^2_*	0.631	1.131	1.631	2.131	2.631
θ^1	0.3303	0.8211	1.3225	1.8199	2.3127
θ^2	0.6504	1.1384	1.6292	2.1403	2.6407
error	0.0194	0.0124	0.0087	0.0145	0.0207
relative error	0.0272	0.0088	0.0041	0.0052	0.0059

Table 2.1: (θ^1, θ^2) with varying b: average error $= 0.01511$, average relative error $= 0.0102$

Next, we vary a. A larger a means fast reversion rate for X_t to reach the equilibrium level b and thus results in larger reward in short time. Consistently, Table 2.2 shows that the values of (θ^1, θ^2) increase in a.

a	0.6	0.7	0.8	0.9	1
θ^1_*	1.175	1.264	1.331	1.383	1.425
θ^2_*	1.375	1.564	1.631	1.683	1.725
θ^1	1.146	1.238	1.300	1.367	1.425
θ^2	1.497	1.559	1.621	1.685	1.731
error	0.1254	0.0265	0.0326	0.016	0.006
relative error	0.0693	0.0132	0.0155	0.0074	0.0027

Table 2.2: (θ^1, θ^2) with varying a: average error = 0.0413, average relative error = 0.0216

In Table 2.3, we vary the volatility σ. The larger the σ, the greater the range for the stock price $S_t = \exp(X_t)$. Table 2.3 shows the values (θ^1, θ^2) increase in σ.

In Table 2.4, we vary the discount rate ρ. A larger discount rate ρ implies smaller return and smaller threshold values (θ^1, θ^2). These are confirmed in Table 2.4.

Finally, we choose different slippage rates K. The results in Tables 2.5 suggest that θ_1 is decreasing slightly in K and θ_2 is almost flat. The possible explanation is that larger slippage cost discourages stock transactions and thus has to be compensated by smaller θ_1.

It is clear to see from the above tables that the stochastic approximation constructed in this chapter indeed provide sound estimates of optimal threshold values (θ^1, θ^2). Overall, the average error is 0.0440 and the average relative error is only 0.0220, or 2.20%.

σ	0.3	0.4	0.5	0.6	0.7
θ^1_*	1.231	1.275	1.331	1.400	1.481
θ^2_*	1.531	1.575	1.631	1.700	1.781
θ^1	1.213	1.274	1.316	1.360	1.469
θ^2	1.496	1.564	1.624	1.732	1.831
error	0.0394	0.0110	0.0166	0.0512	0.0514
relative error	0.0200	0.0055	0.0079	0.0232	0.0222

Table 2.3: (θ^1, θ^2) with varying σ: average error = 0.0339, average relative error = 0.0158

2.5.2 Tests with Market Data

In this section, we study the performance of the algorithm using real market data. The proposed algorithm is employed as follows:

Step 1. We collect stock prices during the period January 2, 2002 to December 31, 2007;

Step 2. We divide the whole period into three sub-periods: Period 1, the first 500 trading days beginning at January 2, 2002; Period 2, the subsequent 500 trading day following Period 1; and Period 3, the next 500 trading days following Period 2;

Step 3. We run 600 iterations of stochastic approximation algorithm (2.2.10) on stock prices in the Period 1 and compute the threshold values (θ^1, θ^2);

Step 4. We use the threshold values (θ^1, θ^2) obtained in Step 3 to simulate trading the same stock in Period 2 and compute the overall dollar return. Recall that we buy stocks

ρ	0.3	0.4	0.5	0.6	0.7
θ^1_*	1.681	1.456	1.331	1.206	1.081
θ^2_*	1.781	1.756	1.631	1.506	1.281
θ^1	1.604	1.440	1.320	1.181	1.060
θ^2	1.901	1.782	1.627	1.521	1.384
error	0.1426	0.0305	0.0117	0.0292	0.1051
relative error	0.0582	0.0134	0.0056	0.0151	0.0627

Table 2.4: (θ^1, θ^2) with varying ρ: average error $= 0.0638$, average relative error $= 0.0310$

whenever stock price $S(t) < \exp(\theta^1)$ and sell stocks whenever stock price $S(t) > \exp(\theta^2)$;

Step 5. Again, we run 600 iterations of algorithm (2.2.10) on stock prices in the Period 2 and compute the threshold values (θ^1, θ^2);

Step 6. We use the threshold values (θ^1, θ^2) calculated in Step 5 to simulate trading the same stock in Period 3 and compute the overall dollar return.

In the above procedure, if a stock is bought in any period but the stock price doesn't exceed the selling price after buying, we will sell stock at the end of the period regardless of selling price. Now we choose different stocks to conduct above experiment. For example, Figure 2.2 is the graph of historical prices of Wal-Mart Stores Inc. during the period January 2nd, 2002 to December 31st, 2007.

Applying the stochastic approximation algorithm to Period 1, the computed buying price

K	0.001	0.005	0.01	0.015	0.02
θ^1_*	1.431	1.431	1.331	1.331	1.231
θ^2_*	1.631	1.531	1.631	1.531	1.631
θ^1	1.413	1.351	1.316	1.264	.1252
θ^2	1.585	1.619	1.624	1.633	1.634
error	0.0494	0.1189	0.0166	0.1220	0.0212
relative error	0.0228	0.0568	0.0079	0.0602	0.0104

Table 2.5: (θ^1, θ^2) with varying K: average error = 0.0656, average relative error = 0.0316

is \$44.6088 and the selling price is \$53.3786. Using these threshold values in Period 2, we buy stock at \$44.46 on Aug 23, 2005 and sell it at \$47.12 at the end of Period 2. The dollar return is \$1.7442 in Period 2. Using stock prices during Period 2, the calculated buying price is \$42.2969 and the selling price is \$47.8489. Applying these threshold values in Period 3, we buy stock at \$41.73 on Jul 14, 2006 and sell it at \$47.92 on Sep 26, 2006, and we buy it again at \$42.06 on Sep 5, 2007 and finally sell it at \$48.45 on Dec 5, 2007. The total dollar return in Period 3 is \$10.7784.

We apply the same procedure to Home Depot's stock; see Figure 2.3 for the daily stock price during the period January 2, 2002 to December 31, 2007.

Based on the prices of Home Depot in Period 1, after 600 iteration of (2.2.10), the calculated buying price is \$34.0484 and selling price is \$38.9126. Using the threshold values above, we trade Home Depot in Period 2. We first buy it at \$33.29 on Sep 2, 2004 and sell it at \$39.45 on Nov 10, 2004. We buy it again at \$33.96 on May 10, 2005. However, before the end of Period 2, the price doesn't rise above the selling price \$38.9126 again. Thus we have

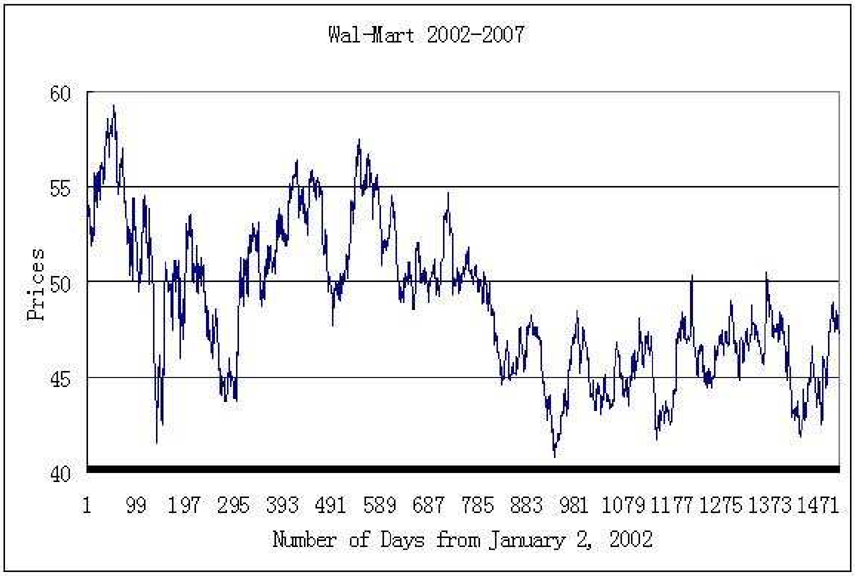

Figure 2.2: The Prices of Wal-Mart from Jan 2, 2002 to Dec 31, 2007

to sell it on last trading day in Period 2 for \$32.77. The total dollar return in Period 2 is \$3.5753. Using the prices in Period 2, the computed selling price is \$32.7166 and \$38.7353. Therefore, in Period 3, we buy it at \$32.61 on Dec 27, 2005 and sell it at \$39.87 on Aug 9, 2007, resulting a total profit of \$6.5352.

The same procedure are also applied to other stocks. The detail trading results are shown in Table 2.6, where buying and selling prices are computed by proposed algorithm. As can be seen, using the threshold values computed by the proposed algorithm does not necessarily trigger transactions in every period. However, overall, the proposed algorithm in this chapter may provide trading guidelines in practice.

2.6 Further Remarks

A stochastic approximation algorithm has been developed for buying low and selling high strategy in stock trading. Compared with the analytic solution, the simulation results indicate that this algorithm provides sound estimates for optimal buying and selling prices. One

Figure 2.3: The Prices of Home Depot from Jan 2, 2002 to Dec 31, 2007

advantage of the proposed method is its simple recursive form. In addition, this method only requires the observed stock prices. Thus this method can be easily implemented in practice. As demonstrated by using real market data, the proposed algorithm can provide useful guidelines for stock trading.

A note of caution is in order. The approximation only works for stocks under mean reversion. If the stock prices do not revert to an equilibrium level, then the threshold values provided by the proposed algorithm may make no sense in practice. Thus, before using the stochastic approximation methods, one needs to check if the mean reversion occurs in stock prices. Finally, we note that developing a rigorous procedure to identify mean-reverting assets is a challenging problem and may be added to current literature.

Stock	Period 2			Period 3		
	Buying Price	Selling Price	Total Profit	Buying Price	Selling Price	Total Profit
AIG	$52.1486	$63.6487	$13.9529	$52.0795	$59.9433	$9.04
Du Pont	$30.3011	$39.50	$0.00	$36.2255	$46.00	$9.0689
Home Depot	$34.0484	$38.9126	$3.5753	$32.7166	$38.7353	$6.5352
IBM	$54.2993	$82.1306	$0.00	$71.8155	$833755	$10.9796
Intel	$12.6282	$28.1449	$0.00	$19.4474	$26.9935	$7.5134
Microsoft	$19.8149	$22.2436	$0.00	$21.0327	$26.9364	$5.8964
3M	$46.989	$69.2105	$0.00	$69.8979	$80.0914	$20.0621
Verizon	$25.6557	$31.5786	$0.0715	$28.8343	$34.4958	$7.6625
Wal-Mart	$44.6088	$53.3786	$1.7442	$42.2969	$47.8489	$10.7784

Table 2.6: Trading Results

Chapter 3

Recursive Algorithms for Trailing Stop

3.1 Introduction

Decision in selling a stock is crucial in successfully investing in equity markets. The selling strategy can be determined by either a target level or a stop-loss limit. In this chapter, we focus on the stop-loss side of the equation. We refer to O'Neil [29] for some empirical rules.

In equity trading, a stop-loss order is an order placed with a broker to sell once the stock drops to a certain price. A stop loss is designed to limit the investor's loss on a security position. The advantage of a stop-loss order is that one need not monitor the market constantly on how the a stock is performing. A disadvantage is that the stop-loss order cannot help the investor to lock in his profits after a substantial rise in price. A key in successful trading is to "cut the losses and let the profit run." Such needs give rise to the so-called trailing stop. The trailing stop maintains a stop-loss order at a precise percentage below the market price. The stop-loss order is adjusted continually based on fluctuations in the market price, always maintaining the same percentage below the market price. The trader is then "guaranteed" to know the exact minimum profit that his or her position will garner.

As the market price rises, the stop price also advances. If the price drops, the stop price does not change, and the position is closed whenever the stop price is reached. For example, assume an investor buy the Coca Cola stock (KO) on January 2, 1998 at the price of \$57.31. If he/she sets the trailing stop at 15% below the market price, then the initial stop price is \$48.71 (15% below the market price). If the price rises above \$57.31, he or she maintains the stop price at 15% below the highest price. Otherwise, the stop price would not change. On August 27, 1998, several days after reaching its maximum value, the market price of KO drops

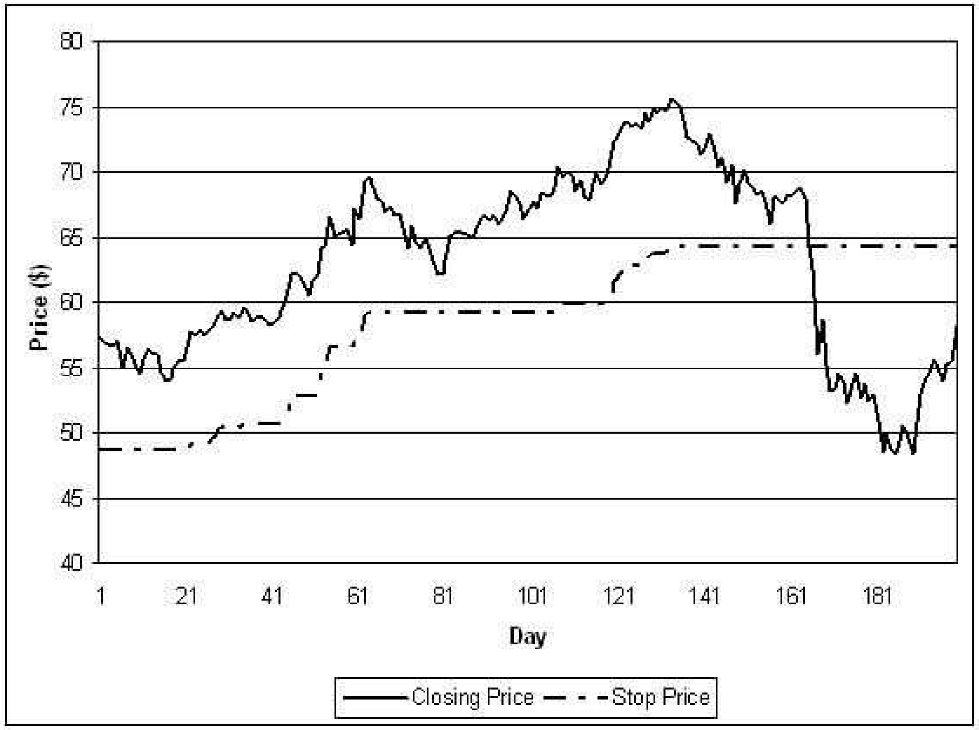

Figure 3.1: The 200 Days Prices of Coca Cola from Jan 12, 1998, to October 15, 1998

to \$64.24, which is below the pre-set stop price. The investor should close his/her position, resulting in a raw return of 12.09%. The daily closing price of KO and the corresponding trailing stop levels are given in Figure 3.1. Clearly, trailing stop is an effective tool helping the investor to lock in the profit when the market moves against his or her position.

Traditionally, a trailing stop percentage is determined based on the trader's predilection toward aggressive or conservative trading. In stock investing, deciding what constitutes appropriate profits (or acceptable losses) is perhaps the most difficult aspect of establishing a trailing-stop system for your disciplined trading decisions.

Research using mathematical models on trailing stops is scarce in the literature. Glynn and Iglehart [17] studied the problem in both discrete and continuous time. In their continuous-time model, they considered a diffusion model and show that the optimal strategy is not to sell at all, i.e., $h_* = 100\%$. This corresponds to the so-called buy and hold strategy. A drawback is: the stock price can become negative in their model. It appears difficult to extend their results to a reasonable market model such as the geometric Brownian motion model.

It is the purpose of this chapter to study optimal trailing stops. Here, the main issue is to determine the optimal stop percentage h_*. We develop a stochastic approximation (SA) approach. It provides a systematic way to compute the optimal trailing stop percentage h_*. The SA approach is effective in real time because of its recursive form. A main advantage of the SA approach is that there is no price model needed and one only needs the stock prices to come up with desired percentage.

To some extent, the iterates obtained using the recursive algorithm can be thought of as a point estimator of the true optimal trailing percentage. It would be nice if we also provide the quality of the estimation sequence. We do this by considering a confidence interval estimate. For related work on stopping rules for Robbins-Monro type stochastic approximation algorithms for root finding, we refer the reader to Yin [36]. Here the purpose of our interval estimates are two folds. (i) It provides a practically useful range of estimations, and ensures that the limit confidence level is the desired one. (ii) It gives an implementable stopping criterion for the iterates; with large probability, the iterates will be terminated when the criterion is met. Crucial to the development of the confidence estimate is asymptotic distribution of a scaled sequence of estimation errors. Furthermore, instead of examining the discrete iterates directly, we focus on continuous-time interpolations leading to diffusion limits. Among other things, a random time change argument is used to deduce the result when the deterministic iteration number is replaced by a random variable.

For comparison purposes, we use a Monte Carlo method to obtain optimal percentage rates. In addition, we demonstrate our results using real market data.

The rest of this chapter is arranged as follows. Section 3.2 begins with the precise formulation of the problem. We also provide the recursive algorithm and its variations. Section

3.3 studies the convergence of the underlying algorithm. Section 3.4 concentrates on interval estimates. To demonstrate the feasibility and efficiency of the algorithms, numerical experiments using simulations and real market data are given in Section 3.5. We show that these algorithms provide sound estimates of optimal trailing stop percentage; they can be easily implemented in real time. We conclude this chapter with some further remarks in Section 3.6.

3.2 Formulation

In our formulation, we shall not require that the stock price $S(t)$ be any specific stochastic process or follow any specific distributions; we only assume the stock price is observable. Based on the observed stock price, for a given time t, we define the stop price at a trailing stop percentage h with $0 < h < 1$ as

$$T_h(t) = (1-h)S_{\max}(t), \tag{3.2.1}$$

where

$$S_{\max}(t) = \max\{S(u) : 0 \le u \le t\}. \tag{3.2.2}$$

Let

$$\tau = \inf\{t > 0 : S(t) \le T_h(t)\}. \tag{3.2.3}$$

Then τ is the first time the stock price reaches the stop price and depends on h. We aim to find the optimal trailing stop percentage $h_* \in [a, 1]$ with $a > 0$ that maximize a suitable objective function. Thus the problem is

$$\text{Problem } \mathcal{P} : \text{Find argmax } J(h) = E[\Phi(S(\tau)) \exp(-\rho\tau)], \quad h \in [a, 1]. \tag{3.2.4}$$

Here $a > 0$ is a reasonable lower bound for the trailing stop percentage, $\rho > 0$ is an appropriate discount rate, and the reward function

$$\Phi(S) = \frac{S - S_0}{S_0}.$$

Please note that τ depends on the trailing stop percentage h. Therefore J is a function of h.

In general, an analytic solution is difficult to obtain even if $S(t)$ is a specific process, e.g., a geometric Brownian motion. Our contribution is to devise a numerical approximation procedure that estimates the optimal trailing stop percentage h. We will use a stochastic approximation procedure to resolve the problem by constructing a sequence of estimates of the optimal trailing stop percentage h, using

$$h_{n+1} = h_n + \{\text{step size}\}\{\text{gradient estimate of } J(h)\}.$$

Moreover, in accordance with (3.2.4), we need to make sure the iterate $h_n \in [a, 1]$.

3.2.1 Recursive Algorithm

Let us begin with a simple noisy finite difference scheme. The only provision is that $S(t)$ can be observed. Associated with the iteration number n, denote the trailing stop percentage by h_n. Beginning at an arbitrary initial guess, we construct a sequence of estimates $\{h_n\}$ recursively as follows. We figure out τ_n, the first time when the stock price declines under the stop price as

$$\tau_n = \inf\{t > 0 : S(t) \le T_{h_n}(t)\}. \tag{3.2.5}$$

Define a combined process ξ_n that includes the random effect from $S(t)$ and the stopping time τ_n as

$$\xi_n = (S(\tau_n), \tau_n)', \tag{3.2.6}$$

where $S(\tau_n)$ denotes the stock price process $S(t)$ stopped at stopping time τ_n. Henceforth, we call $\{\xi_n\}$ the sequence of collective noise. Let $\widetilde{J}(h, \xi)$ be the observed value of the objective function $J(h)$ with collective noise ξ. With the values $h \pm \delta_n$, define $Y_n^{\pm}$ as

$$Y_n^{\pm}(h, \xi_n^{\pm}) = \widetilde{J}(h \pm \delta_n, \xi_n^{\pm}). \tag{3.2.7}$$

$\xi_n^{\pm}$ being the two different collective noises taken at the trailing stop percentages $h \pm \delta_n$, where δ_n is the finite difference sequence satisfying $\delta_n \to 0$ as $n \to \infty$. We shall write

$Y_n^{\pm} = Y_n^{\pm}(h, \xi_n^{\pm})$. For simplicity, in what follows, we often use ξ_n to represent both ξ_n^+ and ξ_n^- if there is no confusion. The gradient estimate at iteration n is given by

$$D\widehat{J}(h_n, \xi_n) \stackrel{\text{def}}{=} (Y_n^+ - Y_n^-)/(2\delta_n). \tag{3.2.8}$$

Then the recursive algorithm is

$$h_{n+1} = h_n + \varepsilon_n D\widehat{J}(h_n, \xi_n), \tag{3.2.9}$$

where ε_n is a sequence of real numbers known as step sizes. A frequently used choice of step size and finite difference sequences is $\varepsilon_n = O(1/n)$ and $\delta_n = O(1/n^{1/6})$. Recall that this is also our default choice of step size and finite difference sequences in Chapter 2.

To proceed, define

$$\begin{aligned}
\rho_n &= (Y_n^+ - Y_n^-) - E_n(Y_n^+ - Y_n^-), \\
\eta_n &= [E_n Y_n^+ - J(h_n + \delta_n)] - [E_n Y_n^- - J(h_n - \delta_n)], \\
\beta_n &= \frac{J(h_n + \delta_n) - J(h_n - \delta_n)}{2\delta_n} - J_h(h_n),
\end{aligned} \tag{3.2.10}$$

where E_n denotes the conditional expectation with respect to $\mathcal{F}_n$, the σ-algebra generated by $\{h_1, \xi_j^{\pm} : j < n\}$, $J_h(h_n) = (\partial/\partial h)J(h_n)$. In the above, η_n and β_n represent the noise and bias, and $\{\rho_n\}$ is a martingale difference sequence. We separate the noise into two parts, uncorrelated noise ρ_n and correlated noise η_n. It is reasonable to assume that after taking the conditional expectations, the resulting function is smooth. With the above definitions, algorithm (3.2.9) can be rewritten as

$$h_{n+1} = h_n + \varepsilon_n J_h(h_n) + \varepsilon_n \frac{\rho_n}{2\delta_n} + \varepsilon_n \beta_n + \varepsilon_n \frac{\eta_n(h_n, \xi_n)}{2\delta_n}. \tag{3.2.11}$$

3.2.2 Projection Algorithms

The use of projections in the algorithms stems from two reasons. First, for the purpose of computations, it is more convenient if one uses projections to force the iterates to remain in a bounded region. In addition, the problems under consideration may well be constrained so that the iterates will be in a given set. Current problem under consideration is such an example (the iterates need to stay in the interval $[a, 1]$). For example, one might choose a lowest trailing stop percentage of 10% to ensure the holding position will not be closed due to the normal fluctuations of daily stock price. Obviously, there is a upper bound for the optimal trailing stop percentage, 100%. To solve the Problem (3.2.4) with constrains, we construct the following stochastic approximation algorithm with a projection

$$h_{n+1} = \Pi[h_n + \varepsilon_n D\widetilde{J}(h_n, \xi_n)], \tag{3.2.12}$$

where $\varepsilon_n = 1/n$, $\delta_n = \delta/(n^{1/6})$ and $\Pi[x]$ is a projection given by

$$\Pi[h] = \begin{cases} a, & \text{if } h < a, \\ 1, & \text{if } h > 1, \\ h, & \text{otherwise.} \end{cases}$$

As explained in Kushner and Yin [27], The projection algorithm (3.2.12) can be rewritten as

$$h_{n+1} = h_n + \varepsilon_n D\widehat{J}(h_n, \xi_n) + \varepsilon_n r_n. \tag{3.2.13}$$

where $\varepsilon_n r_n = h_{n+1} - h_n - \varepsilon_n D\widehat{J}(h_n, \xi_n)$ is the real number with the shortest distance needed to bring $h_n + \varepsilon_n D\widehat{J}(h_n, \xi_n)$ back to the constraint set $[a, 1]$ if it is outside this set.

3.3 Convergence

This section is devoted to the study of convergence of the recursive algorithm. We will show that h_n defined in (3.2.12) is closely related to an ordinary differential equation (ODE). The stationary points of ODE are the optimal trailing stop percentage that we are seeking. The details asymptotic analysis can be worked out by virtue of the approaches given in Chapter 2 and Yin, Liu and Zhang [37]; see also Kushner and Yin [27, Chapters 5 and 8]. Chapter 4 in this dissertation will also provide an approach to prove the convergence for similar problem. Thus we shall be brief and only summarize the results via the following proposition.

To carry out the study of convergence, we define the following:

$$\begin{cases} t_n = \displaystyle\sum_{i=1}^{n-1} \varepsilon_i, \quad m(t) = \max\{n : t_n \le t\}, \\ h^0(t) = h_n \ \text{ for } \ t \in [t_n, t_{n+1}), \ h^n(t) = h^0(t + t_n), \\ \widetilde{r}^0(t) = \displaystyle\sum_{j}^{m(t)-1} \varepsilon_j r_j \ \text{ and } \ \widetilde{r}^n(t) = \widetilde{r}^0(t+t_n) - \widetilde{r}^0(t_n). \end{cases} \tag{3.3.1}$$

Note that $h^0(\cdot)$ is a piecewise constant process and $h^n(\cdot)$ is its shift. Then the interpolated process $h^n(\cdot)$ can be rewritten as

$$\begin{aligned} h^n(t) = h_n &+ \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j J_h(h_j) + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j \frac{\rho_j}{2\delta_j} + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j \beta_j \\ &+ \sum_{j=n}^{m(t_n+t)-1} \frac{\varepsilon_j}{2\delta_j} \eta_j(h_j, \xi_j) + \sum_{j=n}^{m(t_n+t)-1} \varepsilon_j r_j. \end{aligned} \tag{3.3.2}$$

To proceed, we will use the following assumptions.

(A3.1) The second derivative $J_{hh}(\cdot)$ is continuous.

(A3.2) For each h belongs to a bounded set, $E|Y_n^{\pm}|^2 < \infty$, and the sequence $\{\eta_j(h, \xi_j)\}$ is a bounded ϕ-mixing sequence with mixing rate $\widetilde{\phi}_k$ such that $\sum_k \widetilde{\phi}_k^{1/2} < \infty$.

Remark 3.3.1. For our default choice $\varepsilon_n = O(1/n)$ and $\delta_n = O(1/n^{1/6})$,

$$\limsup_n \sup_{0 \le i \le N_n - 1} \frac{\varepsilon_{n+i}}{\varepsilon_n} < \infty, \ \limsup_n \frac{\delta_{n+i}}{\delta_n} < \infty, \ \limsup_n \left[\frac{(\varepsilon_{n+i}/\delta_{n+i}^2)}{(\varepsilon_n/\delta_n^2)} \right] < \infty.$$

Concerning the noise, for the weak convergence alone, we only need a law of large numbers type result holds. That is, as $n \to \infty$,

$$\sum_{j=n}^{m(t_n+t)-1} \frac{\varepsilon_j}{2\delta_j} E_n \eta_j(h, \xi_j) \to 0 \ \text{ in probability.} \tag{3.3.3}$$

We used a mixing condition for simplicity. Suppose that $\xi^{\pm}_{n,\ell} = g_0(\zeta^{\pm}_{n,\ell})$ where $g_0(\cdot)$ is a real-valued function, $\{\zeta^{\pm}_{n,\ell}\}$ are homogeneous finite-state Markov chains whose transition matrices are irreducible and aperiodic. Then, the noise is bounded since the Markov chain takes only finite values. Then $\xi^{\pm}_{n,\ell}$ are ϕ-mixing sequences with exponential mixing rates ([3, p.167]), i.e., $\varpi(j) = c_0\varpi^j$ for some $c_0 > 0$ and some $0 < \varpi < 1$. Using the exponential mixing rates, condition (A3.2)(b) is verified.

As far as the noise is concerned, we are dealing with bounded mixing type noises. We could add an unbounded noise of martingale difference type as follows. Assume that

$$Y^{\pm}_n(h, \xi^{\pm}_n) = \widetilde{J}(h \pm \delta_n, \xi^{\pm}_n) \pm \check{\xi}^{\pm}_n,$$

where $\{\check{\xi}^{\pm}_n\}$ are sequences of martingale difference noise satisfying $\sup_n E|\check{\xi}^{\pm}_n|^{2+\Delta_0} < \infty$ for some $\Delta_0 > 0$ and $\widetilde{J}(h, \xi^{\pm}_n)$ satisfy the conditions stated in (A3.2). It is then easily verified that $\sup_n E|Y^{\pm}_n(h, \xi^{\pm})|^2 < \infty$ for each h belongs to a bounded set. Then all subsequent development goes through. Here for simplicity of notation, we choose a relatively simpler setting.

The following theorem and its corollary can be proved as in Yin, Liu and Zhang [37]. We thus omit the details.

Theorem 3.3.2. *Assume* (A3.1)–(A3.2)*. Suppose the differential equation*

$$\dot{h} = J_h(h) + r(t) \tag{3.3.4}$$

has a unique solution for each initial condition. Then $(h^n(\cdot), \widetilde{r}^n(\cdot))$ *converges weakly to* $(h(\cdot), \widetilde{r}(\cdot))$*, the solution to* (3.3.4) *with*

$$\widetilde{r}(t) = \int_0^t r(s)ds,$$

and $r(t) = 0$ *when* $h(t) \in [a, 1]$*.*

Corollary 3.3.3. *Suppose that* (3.3.4) *has a unique stationary point* $h_* \in (a, 1)$ *being globally asymptotically stable in the sense of Liapunov, and that* $\{s_n\}$ *is a sequence of real numbers such that* $s_n \to \infty$*. Then the weak limit of* $h^n(s_n + \cdot)$ *is* h_**.*

Under current setting, we can also obtain the with probability one convergence of algorithm (3.2.13). We use the technique developed in Kushner and Yin [27, Chapters 5 and 6]. We will keep the discussion brief.

Note that $\{\rho_n\}$ is a martingale difference. The martingale inequality together with the Tchbyshev's inequality implies that for any $\kappa > 0$,

$$
\begin{aligned}
&P\left(\max_{n\le j\le m}\left|\sum_{i=n}^{j-1}\frac{1}{i^{5/6}}\rho_i\right|\ge\kappa\right)\\
&\le \frac{1}{\kappa^2}E\max_{n\le j\le m}\left|\sum_{i=n}^{j-1}\frac{1}{i^{5/6}}\rho_i\right|^2\\
&\le \frac{1}{\kappa^2}E\left|\sum_{i=n}^{m-1}\frac{1}{i^{5/6}}\rho_i\right|^2\\
&\le \frac{1}{\kappa^2}\sum_{i=n}^{m-1}\frac{1}{i^{10/6}}E\rho_i^2 \le O(n^{-2/3}) \to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Thus the asymptotic rate of change (for a definition, see [27, p. 137]) of

$$
A^0(t) = \sum_{i=1}^{m(t)-1}\frac{1}{i^{5/6}}\rho_i
$$

goes to 0 w.p.1.

Next for the bias term, we note that it can be shown that $|\beta_n| = O(n^{-1/3})$ w.p.1. Thus we have for any $m > n$,

$$
\left|\sum_{i=n}^{m-1}\frac{1}{i^{5/6}}\beta_i\right| \le K\sum_{i=n}^{m-1}\frac{1}{i^{5/6}}|\beta_i| \le O(n^{-1/6}) \text{ w.p.1 as } \ n\to\infty.
$$

So the asymptotic rate of change of $B^0(t) = \sum_{i=0}^{m(t)-1}\frac{1}{i^{5/6}}\beta_i$ goes to 0 w.p.1 as $n\to\infty$.

As for the correlated noise term, the mixing condition together with the Mensov-Rademacher moment estimate in Kushner and Yin [27, p. 172] imply that for some $K > 0$,

$$\begin{aligned} & E \max_{n \le m \le n+M} \left| \sum_{i=n}^{m} \frac{1}{i^{5/6}} \eta_i \right|^2 \\ & \le K(\log_2 4M) \sum_{i=n}^{M+n-1} \frac{1}{i^{10/6}}. \end{aligned}$$

It is easily seen that

$$\sum_{i=1}^{\infty} \left(\frac{1}{i^{5/6}} \log_2 i \right)^2 < \infty.$$

Therefore, the rate of change of

$$C^0(t) = \sum_{i=1}^{m(t)-1} \frac{1}{i^{5/6}} \eta_i$$

goes to 0 w.p.1. Combining the estimates obtained thus far and using the results in [27, Chapter 6], we obtain the w.p.1 convergence of the algorithm. We state the result below.

Theorem 3.3.4 *Under the conditions of Theorem* 3.3.2*, $h^n(\cdot)$ converges w.p.1 to $h(\cdot)$ that is the solution of* (3.3.4)*. Moreover, if* (3.3.4) *has a unique stationary point $h_* \in (a, 1)$ being globally asymptotically stable in the sense of Liapunov, and that $\{s_n\}$ is a sequence of real numbers such that $s_n \to \infty$. Then $h^n(s_n + \cdot) \to h_*$ w.p.1.*

3.4 Interval Estimates

This section is devoted to obtaining interval estimates as well as a practically useful stopping rule for the recursive computation. Roughly, with prescribed confidence level, we wish to show that with large probability (probability close to 1), a sequence of scaled and centered estimates and a stopped sequence converge weakly to a diffusion process. Based on this result, we will then be able to build confidence interval for the iterates.

To proceed, for simplicity of notation, we take $\varepsilon_n = 1/n$ and $\delta_n = \delta_0/n^{1/6}$. In the analysis to follow, for simplicity and without loss of generality, we take $\delta_0 = 1$. We assume all the conditions of Theorem 3.3.2 holds. To carry out the subsequent study, we also assume an additional condition.

(A3.3)

$$J_h(h) = J_{hh}(h_*)(h - h_*) + o(|h - h_*|^2),$$

where $J_{hh}(h_*) - (1/2) < 0$. In addition,

$$k^{2/3} E(h_k - h_*)^2 = O(1)$$

and the bound holds uniformly in k.

Remark 3.4.1 While the first condition in (A3.3) indicates that $J_h(h)$ is linearizable. The second condition is a moment estimate. Sufficient conditions guaranteeing this can be provided by means of perturbed Liapunov function methods; see for example in Yin et al. [38] for liquidation related issues and the more extensive discussion in Kushner and Yin [27] for general setting. For simplicity, here we assume this condition.

Define

$$\rho_n^* = [Y(h_*, \xi_n^+) - Y(h_*, \xi_n^-)] - E_n[Y(h_*, \xi_n^+) - Y(h_*, \xi_n^-)].$$

That is, ρ_n^* is ρ_n with the argument h_n replaced by h_*. The detailed development of the interval estimates can be outlined as follows. Suppose that we can show that $n^{1/3}(h_n - h_*)$ is asymptotically normal with mean zero and asymptotic variance σ^2. Choose α, such that $0 < \alpha < 1$ and $1 - \alpha$ is the desired confidence coefficient. Given $\varepsilon > 0$, then the asymptotic normality implies that

$$P\left(\frac{n^{1/3}|h_n - h_*|}{\sigma} \le z_{\alpha/2}\right) \to 1 - \alpha \quad \text{as} \quad n \to \infty. \tag{3.4.1}$$

This will lead to the desired confidence interval estimator. Then we require the length of the interval $|h_n - h_*|$ be small enough in that for any $\varepsilon > 0$, for sufficiently large n, we can make

$\sigma z_{\alpha/2}/n^{1/3} < \varepsilon$ or equivalently $n > \lfloor \sigma z_{\alpha/2}/\varepsilon \rfloor$. Define

$$M^n_{\varepsilon,\alpha} = \lfloor \frac{\sigma z_{\alpha/2}}{\varepsilon} \rfloor,$$
$$\mu_{\varepsilon,\alpha} = \inf\{n : M^n_{\varepsilon,\alpha} \leq n\}, \tag{3.4.2}$$

where $\lfloor z \rfloor$ denotes the greatest integer that is less than or equal to z. Then $\mu_{\varepsilon,\alpha}$ is a stopping rule for the iterating sequence $\{h_n\}$. Denote

$$I_{\mu_{\varepsilon,\alpha}} = \Big[h_{\mu_{\varepsilon,\alpha}} - \sigma \frac{z_{\alpha/2}}{n^{2/3}}, h_{\mu_{\varepsilon,\alpha}} + \sigma \frac{z_{\alpha/2}}{n^{2/3}}\Big].$$

We shall show that as the length of the interval shrinks, i.e., $\varepsilon \to 0$,

$$P\{h \in I_{\mu_{\varepsilon,\alpha}} \text{ and } |I_{\mu_{\varepsilon,\alpha}}| \leq \varepsilon\} \to 1 - \alpha,$$

where $|I_{\mu_{\varepsilon,\alpha}}|$ denotes the length of the interval $I_{\mu_{\varepsilon,\alpha}}$.

Remark 3.4.2 In view of definition (3.4.2), we obtain the following result. Note that as $\varepsilon \to 0$, $M_{\varepsilon,\alpha} \to \infty$ and $\mu_{\varepsilon,\alpha} \to \infty$ w.p.1. Moreover, the definitions of $M_{\varepsilon,\alpha}$ and $\mu_{\varepsilon,\alpha}$ implies that as $\varepsilon \to 0$, $\mu_{\varepsilon,\alpha}/M_{\varepsilon,\alpha} \to 1$ w.p.1. This will be used in what follows.

Since $h_* \in (a, 1)$, h_* is not on the boundary of the projection. Thus we drop the projection part or the reflection term $\varepsilon_n r_n$ in what follows for simplicity. We simply assume that $h_n \in (a, 1)$ for all n large enough.

To obtain the desired result, our plan is as follows. We first establish an asymptotic equivalence. Then we define a sequence of interpolated processes and show that the limit of the interpolation is a diffusion process, and further obtain the diffusion limit for a sequence involving the $\mu_{\varepsilon,\alpha}$ defined above. To proceed, define

$$v_n = n^{1/3}(h_n - h_*).$$

We first establish the following result.

Lemma 3.4.3 *Assume* (A3.1)–(A3.3).

$$v_{n+1} = \sum_{k=1}^{n} \frac{1}{2k^{1/2}} A_{n,k}(\rho_k^* + \eta_j(h_*)) + o(1), \tag{3.4.3}$$

$$v_{\mu_{\varepsilon,\alpha}+1} = \sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{2k^{1/2}} A_{\mu_{\varepsilon,\alpha},k}(\rho_k^* + \eta_j(h_*)) + o(1), \tag{3.4.4}$$

where $o(1) \to 0$ *in probability as* $\varepsilon \to 0$, *and*

$$A_{jk} = \begin{cases} \prod_{l=k+1}^{j}(I + \frac{J_{hh}(h_*)}{l}), & \text{if } j > k; \\ I, & \text{if } j = k. \end{cases}$$

Proof of Lemma 3.4.3**.** We will only prove (3.4.4). The proof of (3.4.3) is even simpler.

It follows from the recursion, for some $m > 1$,

$$h_{n+1} - h_* = h_n - h_* + \frac{J_{hh}(h_*)}{n}(h_n - h_*)$$

$$+ \frac{1}{n}[\Delta(h_n) + \beta_n] + \frac{1}{2n^{5/6}}[\rho_n + \eta_n],$$

where $\Delta(h) = O(|h - h_*|^2)$ owing to (A3.3). Thus

$$\begin{aligned} v_{n+1} &= v_n + \frac{J_{hh}(h_*)}{n} v_n + \frac{1}{n^{2/3}}[\Delta(h_n) + \beta_n] + \frac{1}{2n^{1/2}}[\rho_n + \eta_n] \\ &= A_{n,m-1} v_m + \sum_{k=m}^{n} \frac{1}{k^{2/3}} A_{nk}[\Delta(h_k) + \beta_k] + \sum_{k=m}^{n} \frac{1}{2k^{1/2}} A_{nk}[\rho_k + \eta_k]. \end{aligned}$$

The above expression in turn yields

$$\begin{aligned} v_{\mu_{\varepsilon,\alpha}+1} = A_{\mu_{\varepsilon,\alpha},m-1} v_m &+ \sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} A_{\mu_{\varepsilon,\alpha}k}[\Delta(h_k) + \beta_k] \\ &+ \sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{2k^{1/2}} A_{\mu_{\varepsilon,\alpha}k}[\rho_k + \eta_k]. \end{aligned} \tag{3.4.5}$$

We shall show that as $\varepsilon \to 0$, the first and the second terms on the right-hand side of (3.4.5) tend to 0 in probability.

It is easily seen that

$$|\exp\left(J_{hh}(h_*)t\right)| \leq K_1 \exp\left(-\lambda_0 t\right),$$

for some $\lambda_0 > 0$ such that $-\lambda_0 + \frac{1}{2} < 0$. Moreover,

$$A_{nk} = (I + r_{nk}) \exp\left(J_{hh}(h_*) \log\left(n/k\right)\right), \ \ k \leq n-1,$$

where $|r_{nk}| \xrightarrow{k} 0$ uniformly in $n > k$. Hence

$$|A_{\mu_{\varepsilon,\alpha},m-1}| \leq |I + r_{\mu_{\varepsilon,\alpha},m-1}| |\exp\left(J_{hh}(h_*) \log\left(\frac{\mu_{\varepsilon,\alpha}}{m-1}\right)\right)|$$

$$\leq K \exp\left(-\lambda_0 \log\left(\frac{\mu_{\varepsilon,\alpha}}{m-1}\right)\right).$$

Now,

$$|A_{\mu_{\varepsilon,\alpha},m-1} v_m| \leq K \left(\frac{\mu_{\varepsilon,\alpha}}{m-1}\right)^{-\lambda_0} |v_m| \xrightarrow{\varepsilon} 0 \text{ w.p.1.}$$

Therefore, the first term on the right-hand side of (3.4.5) tends to 0 in probability.

As for the second term, the w.p.1 convergence implies that for any $\nu > 0$,

$$\begin{aligned}
&P\left(|\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} A_{\mu_{\varepsilon,\alpha}k} \Delta(h_k)| > \nu\right) \\
&\leq P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} |A_{\mu_{\varepsilon,\alpha}k}| |\Delta(h_k)| > \nu\right) \\
&\leq P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} |A_{\mu_{\varepsilon,\alpha}k}| |\Delta(h_k)| > \nu, \ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \leq 1\right) \\
&\qquad + P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} |A_{\mu_{\varepsilon,\alpha}k}| |\Delta(h_k)| > \nu, \ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} > 1\right) \\
&\leq P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} |A_{\mu_{\varepsilon,\alpha}k}| |\Delta(h_k)| > \nu, \ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \leq 1\right) + P\left(\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} > 1\right).
\end{aligned} \tag{3.4.6}$$

By virtue of Remark 3.4.2,

$$P\left(\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} > 1\right) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

So the last term in (3.4.6) can be discarded.

For the $\nu > 0$ given above, there is a $\delta > 0$, such that $|\Delta(h)| \le \nu^2 |h - h_*|$ for $|h - h_*| < \delta$, and

$$P\left(\sup_{n \ge m} |h_n - h_*| < \delta\right) > 1 - \nu.$$

Consequently, for some $K > 0$,

$$\begin{aligned}
&P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} |A_{\mu_{\varepsilon,\alpha} k}||\Delta(h_k)| > \nu,\ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \le 1\right) \\
&\le P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} \left(\frac{k}{\mu_{\varepsilon,\alpha}}\right)^{\lambda_0} |\Delta(h_k)| > \nu,\ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \le 1\right) \\
&\le P\left(\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{\lambda_0} \left(\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}}\right)^{-\lambda_0} |\Delta(h_k)| > \nu,\ \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \le 1\right) \\
&\le P\left(\sum_{k=m}^{M_{\varepsilon,\alpha}} \frac{1}{k^{2/3}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{\lambda_0} |h_k - h_*| > \frac{K}{\nu},\ \sup_{k \ge m} |h_k - h_*| < \delta\right) + \nu \qquad (3.4.7) \\
&\le P\left(\frac{1}{M_{\varepsilon,\alpha}} \sum_{k=m}^{M_{\varepsilon,\alpha}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{\lambda_0 - 1} |v_k| > \frac{K}{\nu}\right) + \nu \\
&\le \frac{K\nu}{M_{\varepsilon,\alpha}} \sum_{k=m}^{M_{\varepsilon,\alpha}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{\lambda_0 - 1} E^{1/2} |v_k|^2 + \nu \\
&\le K\nu.
\end{aligned}$$

In the above, we used K as a generic positive constant whose value may change for different usage. From the next to the last line to the last line, we also used

$$\lim_{\varepsilon \to 0} \frac{1}{M_{\varepsilon,\alpha}} \sum_{k=m}^{M_{\varepsilon,\alpha}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{\lambda_0 - 1} = \int_0^1 u^{\lambda_0 - 1} du = u^{\lambda_0}|_0^1 < \infty$$

since $0 < \lambda_0 < 1/2$.

By virtue of (3.2.7), the integrability and $n^{2/3}E|h_n - h_*|^2 = O(1)$, the smoothness of $J(\cdot,\xi)$ implies that

$$\begin{aligned} &E|\widehat{J}(h_n \pm \delta_n, \xi_n^{\pm}) - \widehat{J}(h_* + \delta_n, \xi_n^{\pm})| \\ &\quad \le K|\widehat{J}_{hh}(h_n^{\pm}, \xi_n^{\pm})| \\ &\quad \le \frac{K}{n^{2/3}}[n^{2/3}E|h_n - h_*|^2] \\ &\quad \le \frac{K}{n^{2/3}}. \end{aligned} \tag{3.4.8}$$

As a result, in view of the definition of ρ_n and ρ_n^*,

$$\begin{aligned} &E\left|\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{2k^{1/2}} A_{\mu_{\varepsilon,\alpha},k}[\rho_k - \rho_k^*]\right| \\ &\le K\frac{1}{(M_{\varepsilon,\alpha})^{1/6}} \frac{1}{M_{\varepsilon,\alpha}} \sum_{k=m}^{M_{\varepsilon,\alpha}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{-7/6} |A_{\mu_{\varepsilon,\alpha},k}| \\ &\le K\frac{1}{(M_{\varepsilon,\alpha})^{1/6}} \frac{1}{M_{\varepsilon,\alpha}} \sum_{k=m}^{M_{\varepsilon,\alpha}} \left(\frac{k}{M_{\varepsilon,\alpha}}\right)^{-7/6} |A_{M_{\varepsilon,\alpha},k}| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus ρ_k can be replaced by ρ_k^*. The proof of the lemma is concluded. □

By virtue of the above lemma, to get the desired asymptotic distribution, we need only work with the following expressions

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{2k^{1/2}} A_{n,k}(\rho_k^* + \eta_k(h_*)), \\ &\sum_{k=m}^{\mu_{\varepsilon,\alpha}} \frac{1}{2k^{1/2}} A_{\mu_{\varepsilon,\alpha},k}[\rho_k^* + \eta_k(h_*)]. \end{aligned}$$

Instead of working with the discrete expression directly, we shall first examine interpolations of appropriate sequences.

Let $\widetilde{W}_n(\cdot)$ and $W_n(\cdot)$ be defined by

$$\begin{aligned} \widetilde{W}_n(t) &= \sum_{k=1}^{\lfloor nt \rfloor} \frac{1}{2k^{1/2}} A_{\lfloor nt \rfloor k}[\rho_k^* + \eta_k(h_*)], \quad \text{for } t \in [0,1], \\ W_n(t) &= \frac{1}{2}\widetilde{W}_n(t), \end{aligned} \tag{3.4.9}$$

where $\lfloor z \rfloor$ denotes the greatest integer which is less than or equal to z. Note that $\widetilde{W}_n(\cdot) \in D[0,1]$ and so is $W_n(\cdot)$, where the $D[0,1]$ is the space of functions that are right continuous and have left hand limits endowed with the Skorohod topology (see Kushner and Yin [27, page 238]). For definitions and general notion of weak convergence, see Ethier and Kurtz [15] and [27].

We complete the proof by employing the idea of random change of time. As a result, $W_{\mu_{\varepsilon,\alpha}}(\cdot)$ converges weakly to $W(\cdot)$ is established.

Define

$$B_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{1}{j^{1/2}}[\rho_j^* + \eta_j(h_*)].$$

In view of (3.4.9), a summation by parts yields

$$\begin{aligned} \widetilde{W}_n(t) &= B_n(t) + \sum_{k=1}^{\lfloor nt \rfloor - 1} \left(A_{\lfloor nt \rfloor k} - A_{\lfloor nt \rfloor\,(k+1)}\right) B_n\left(\frac{k}{n}\right) \\ &= B_n(t) + j_{hh}(h_*) \sum_{k=1}^{\lfloor nt \rfloor - 1} \frac{1}{(k+1)} A_{\lfloor nt \rfloor\,(k+1)} B_n\left(\frac{k}{n}\right). \end{aligned} \tag{3.4.10}$$

(A3.4) $B_n(\cdot)$ converges weakly to $B(\cdot)$, a Brownian motion with covariance $t\sigma_B^2$

Remark 3.4.4 Again, for simplicity, we have assumed this condition, the proof of such a convergence is available; see for example [27, Chapter 10]. One may also find relevant information in [15].

Theorem 3.4.5 *Under assumptions* (A3.1)–(A3.4), $W_n(\cdot)$ *converges weakly to* $W(\cdot)$, *a diffusion process given by*

$$W(t) = \frac{1}{2}\int_0^t \exp\left(-J_{hh}(h_*)(\log u - \log t)\right) dB(u),$$

where $B(\cdot)$ *is the Brownian motion with covariance* $t\sigma_B^2$.

Proof. The weak limit of the second term on the right hand side of (3.4.10) is the same as

$$\begin{aligned} &J_{hh}(h_*)\sum_{k=1}^{\lfloor nt\rfloor-1}\frac{1}{(k+1)}\exp\left(-J_{hh}(h_*)\log\left(\frac{k+1}{\lfloor nt\rfloor}\right)\right)B\left(\frac{k}{n}\right) \\ &= J_{hh}(h_*)\frac{1}{\lfloor nt\rfloor}\sum_{k=1}^{\lfloor nt\rfloor-1}\frac{1}{\frac{(k+1)}{\lfloor nt\rfloor}}\exp\left(-J_{hh}(h_*)\log\left(\frac{k+1}{\lfloor nt\rfloor}\right)\right)B\left(\frac{k}{n}\right) \\ &\to J_{hh}(h_*)\int_0^1\frac{1}{u}\exp\left(-J_{hh}(h_*)\log u\right)B(ut)du. \end{aligned} \tag{3.4.11}$$

By virtue of (3.4.11) and (A3.4), we have that $W_n(\cdot)$ converges weakly to $W(\cdot)$ given by

$$\begin{aligned} W(t) &= B(t) + J_{hh}(h_*)\int_0^1 \exp\left(-(1+J_{hh}(h_*))\log u\right)B(ut)du \\ &= \int_0^1 \exp\left(-J_{hh}(h_*)\log u\right)dB(ut). \end{aligned} \tag{3.4.12}$$

In (3.4.12), make a change of variable $u \sim ut$, the desired result follows. $\square$

Remark 3.4.6 Note that Theorem 3.4.5 allows us to have a handle on the estimation errors. Note that it follows from Theorem 3.4.5, setting $t = 1$, we have

$$(n+1)^{1/3}(h_{n+1} - h_*) \sim N(0, \sigma^2),$$

where σ^2 is given by

$$\begin{aligned}
\sigma^2 = E[W(1)]^2 &= \frac{\sigma_B^2}{4}\int_0^1 \exp\left(-J_{hh}(h_*)\log u\right)\exp\left(-J_{hh}(h_*)\log u\right)du \\
&= \frac{\sigma_B^2}{4}\int_0^\infty \exp\left(-u\right)\exp\left(J_{hh}(h_*))u\right)\exp\left(J_{hh}(h_*))^u\right)du \\
&= \frac{\sigma_B^2}{4}\int_0^\infty \exp\left(-2Hu\right)du \\
&= \frac{\sigma_B^2}{8H},
\end{aligned} \tag{3.4.13}$$

where

$$H = \frac{1}{2} - J_{hh}(h_*).$$

The asymptotic variance σ^2 together with the scaling factor $n^{1/3}$ provides us with a rate of convergence result. We shall show that the weak convergence result still holds if n is replaced by $\mu_{\varepsilon,\alpha}$.

Theorem 3.4.7 *If the conditions of Theorem* 3.4.5 *are satisfied, then*

$$W_{\mu_{\varepsilon,\alpha}}(t) = \frac{1}{4}\sum_{k=1}^{[\mu_{\varepsilon,\alpha}t]}\frac{1}{k^{1/2}}A_{[\mu_{\varepsilon,\alpha}t]k}[\rho_k^* + \eta_k(h_*)] \quad \textit{converges weakly to} \quad W(t). \tag{3.4.14}$$

Proof. Without loss of generality, we may assume that $M_{\varepsilon,\alpha}$ is an integer. Recall that as $\varepsilon \to 0$, $M_{\varepsilon,\alpha} \to \infty$.

Define

$$\Psi_{M_{\varepsilon,\alpha}}(t) = \begin{cases} t\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}}, & if\,\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \le 1; \\ t, & otherwise. \end{cases}$$

Thus

$$\sup_t |\Psi_{M_{\varepsilon,\alpha}}(t) - t| \le |\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} - 1| \to 0.$$

Then $\Psi_{M_{\varepsilon,\alpha}}(t)$ converges in probability to $\Psi(t) = t$.

Recall that

$$W_{M_{\varepsilon,\alpha}}(t) = \frac{1}{4}\sum_{k=1}^{[M_{\varepsilon,\alpha}t]} \frac{1}{k^{1/2}} A_{[M_{\varepsilon,\alpha}t]k}[\rho_k^* + \eta_k(h_*)].$$

Then Billingsley [3, Theorem 4.4] leads to $(W_{M_{\varepsilon,\alpha}}(\cdot), \Psi_{M_{\varepsilon,\alpha}}(\cdot))$ converges weakly to $(W(\cdot), \Psi(\cdot))$, and both $W(\cdot)$ and $\Psi(\cdot)$ have continuous sample paths. This leads to that the convergence in each case is uniform.

$$\sup_{t\in[0,1]} |W_{M_{\varepsilon,\alpha}}(\Psi_{M_{\varepsilon,\alpha}}(t)) - W(\Psi(t))|$$

$$\leq \sup_{t\in[0,1]} |W_{M_{\varepsilon,\alpha}}(t) - W(t)| + \sup_{t\in[0,1]} |W(\Psi_{M_{\varepsilon,\alpha}}(t)) - W(\Psi(t))|.$$

Therefore,

$$W_{M_{\varepsilon,\alpha}}(\Psi_{M_{\varepsilon,\alpha}}(\cdot)) \ \text{ converges to } \ W(\Psi(\cdot)) \ \text{ uniformly.}$$

Since $W(\Psi(t)) = W(t)$ for any $t \in [0,1]$, we conclude that $W_{M_{\varepsilon,\alpha}}(\Psi_{M_{\varepsilon,\alpha}}(\cdot))$ converges weakly to $W(\Psi(\cdot)) = W(\cdot)$.

By using the definition of $\Psi_{M_{\varepsilon,\alpha}}$,

$$W_{M_{\varepsilon,\alpha}}(\Psi_{M_{\varepsilon,\alpha}}(\cdot)) = W_{\mu_{\varepsilon,\alpha}}(\cdot), \text{ if } \frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} \leq 1.$$

Moreover,

$$P\{\frac{\mu_{\varepsilon,\alpha}}{M_{\varepsilon,\alpha}} > 1\} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, $W_{\mu_{\varepsilon,\alpha}}(\cdot)$ converges weakly to $W(\Psi(\cdot)) = W(\cdot)$. Thus, both $W_{\mu_{\varepsilon,\alpha}}(\cdot)$ and $W_{M_{\varepsilon,\alpha}}(\cdot)$ have the same weak limit,

$$W_{\mu_{\varepsilon,\alpha}}(\cdot) \text{ converges weakly to } \ W(\cdot) \ \text{ as } \ \varepsilon \to 0.$$

The theorem has been proved. $\square$

Thus, $W_{\mu_{\varepsilon,\alpha}}(1)$ converges in distribution to $N(0,\sigma^2)$ as $\varepsilon \to 0$, and hence,

$$(\mu_{\varepsilon,\alpha})^{1/3}(h_{\mu_{\varepsilon,\alpha}+1} - h_*) \ \text{ converges in distribution to } \ N(0,\sigma^2),$$

as $\varepsilon \to 0$, where σ^2 is given (3.4.13).

Remark 3.4.8 In the process of constructing the desired confidence interval, we used a sequence $\{\sigma_n\}$, and we assumed that $\sigma_n^2 \to \sigma^2$, the asymptotic variance. Here, we illustrate how a sequence of consistent estimates $\{\sigma_n^2\}$ can be constructed. In view of the form of the asymptotic variance, to obtain a sequence of consistent estimates $\{\sigma_n^2\}$, we need only construct two consistent sequences, one for estimating $J_{hh}(h_*)$, the other one for estimating σ_B^2.

A consistent sequence of estimates for $J_{hh}(h_*)$ can be constructed by means of two-sided finite difference scheme similar to the estimate for $D\widehat{J}(h_n, \xi_n)$. That is, we construct a finite difference estimate of the derivative of $D\widehat{J}(h, \xi)$ with respect to h. Let assumptions (A3.1)–(A3.4) be satisfied. Then, a sequence of estimate $\{D_n\}$ can be constructed, and it is a sequence of consistent estimates of $J_{hh}(h_*)$.

In view of the form of σ_B^2, define

$$\zeta_{n,i} = \frac{1}{n}\sum_{k=1}^{n} Y_k Y_{k+i} \; i \geq 0 \text{ and } \zeta_n = \zeta_{n,0} + 2\sum_{i=1}^{n} \zeta_{n,i}.$$

Recall that if a process is ϕ-mixing, then it is ergodic. By this ergodicity, noting the noise involves a martingale difference sequence and a mixing sequence, it can be shown that $\zeta_n \to \sigma_B^2$ as $n \to \infty$. Moreover, the implementation can be made recursive. Finally, let $A_n = \frac{1}{2} - D_n$, with the constructions of D_n and ζ_n, we can define σ_n^2 as $\sigma_n^2 = \frac{\zeta_n}{8A_n}$.

3.5 Numerical results

In this section, we report our simulation and numerical experiment results. We first compare our algorithm with the Monte Carlo simulations. Then we test our algorithm using real market data and compare our results to those using a moving average crossing method, which is well studied in the literature.

3.5.1 Simulation study

Due to the absence of an analytic solution to Problem (3.2.4), we use Monte Carlo method to generate optimal trailing stop percentage h. By comparing the results of stochastic approximation algorithm, we demonstrate that the algorithm constructed indeed provides good approximation results. Assume that the stock price follows a geometric Brownian motion (GBM) given by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dw(t), \quad S(0) = S_0 \text{ initial price,} \tag{3.5.1}$$

where μ and σ represent the expected rate of return and volatility, respectively. Equation (3.5.1) also refers to Black-Scholes model (see [5]), which is widely used in derivative pricing to model stock prices because a process that follows a GBM may only take strictly positive value. The drift term μdt implies that the stock price will eventually grow up and the stochastic term $\sigma dw(t)$ captures the daily stock price fluctuations.

Using equation (3.5.1), we generate random samples of $S(t)$ for given values of μ, σ, and S_0. Then we compute the optimal trailing stop percentage h. We take $S_0 = 100$. As shown in Table 3.1, one can see the optimal values of h increase in σ. For example, when μ is fixed at 10%, h rises from 8.00% to 16.75% as σ increases from 10% to 20%. Intuitively, one should set a higher h to avoid being stopped out (or forced to sell) from a position due to normal price fluctuations when σ is larger.

On the other hand, the dependence of h on μ is not obvious. For instance, with a fixed σ at 20%, h varies in the range from 16.00% to 16.75%. These relations are also shown in Figures 3.2 and 3.3.

$\sigma \setminus \mu$	10.00%	11.00%	12.00%	13.00%	14.00%	15.00%	16.00%	17.00%
10.00%	8.00%	7.75%	7.50%	7.50%	7.00%	7.00%	6.75%	6.75%
15.00%	12.25%	12.00%	12.25%	12.00%	11.75%	11.75%	11.50%	11.50%
20.00%	16.75%	17.25%	16.75%	16.50%	16.00%	15.75%	16.25%	16.00%
30.00%	25.25%	25.00%	24.75%	24.50%	24.50%	25.25%	25.00%	25.25%

Table 3.1: Optimal trailing stop percentage using Monte Carlo Method for given expected rate of return μ and volatility σ.

We now can draw the graph of objective function $J(h)$ for fixed μ and σ. For example, the graph of $J(h)$ with $\mu = 20\%$ and $\sigma = 35\%$ is shown in Figure 3.4. Figure 3.5 is the graph of $J(h)$ with $\mu = 12\%$ and $\sigma = 25\%$. From these two figures, one can see the smoothness of $J(h)$ and hence our assumption of A3.1 is reasonable.

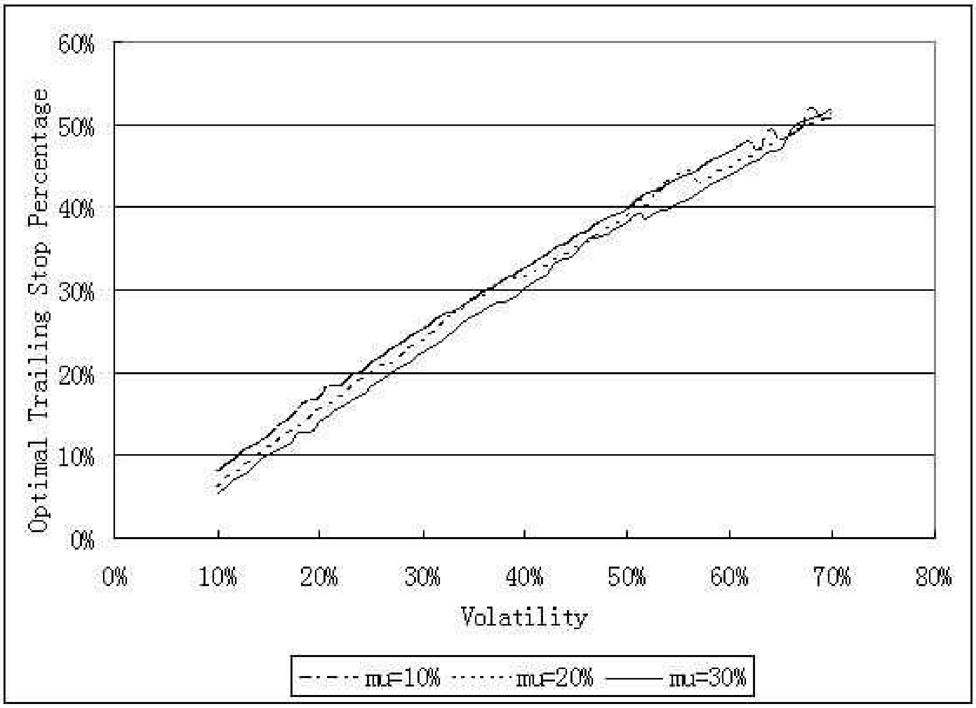

Figure 3.2: Optimal trailing stop percentage using Monte Carlo Method against stock price volatility given different expected rate of return μ.

We use the stochastic approximation to compute the optimal values of h. In the following approach, the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ are chosen to be $\varepsilon_n = 1/(n + k_0)$ and $\delta_n = 1/(n^{1/6} + k_1)$, respectively, where k_0 and k_1 are some positive integers. We choose $k_0 = 1$, $k_1 = 10$, $\rho = 0.04$, and the lower bound $a = 5\%$. When the trailing stop percentage is set at h, instead of the finite difference approximation of the gradient given in the algorithm, we may take random samples of size n_0 with random noise sequence $\{\xi^{\pm}_{n,l}\}_{l=1}^{n_0}$ such that

$$\widehat{J}(h, \xi^{\pm}_n) \stackrel{\text{def}}{=} \frac{\widetilde{J}(h, \xi^{\pm}_{n,1}) + \cdots + \widetilde{J}(h, \xi^{\pm}_{n,n_0})}{n_0}. \tag{3.5.2}$$

We assume that

$$E\widehat{J}(h, \xi^{\pm}_n) = J(h) \quad \text{for each } h. \tag{3.5.3}$$

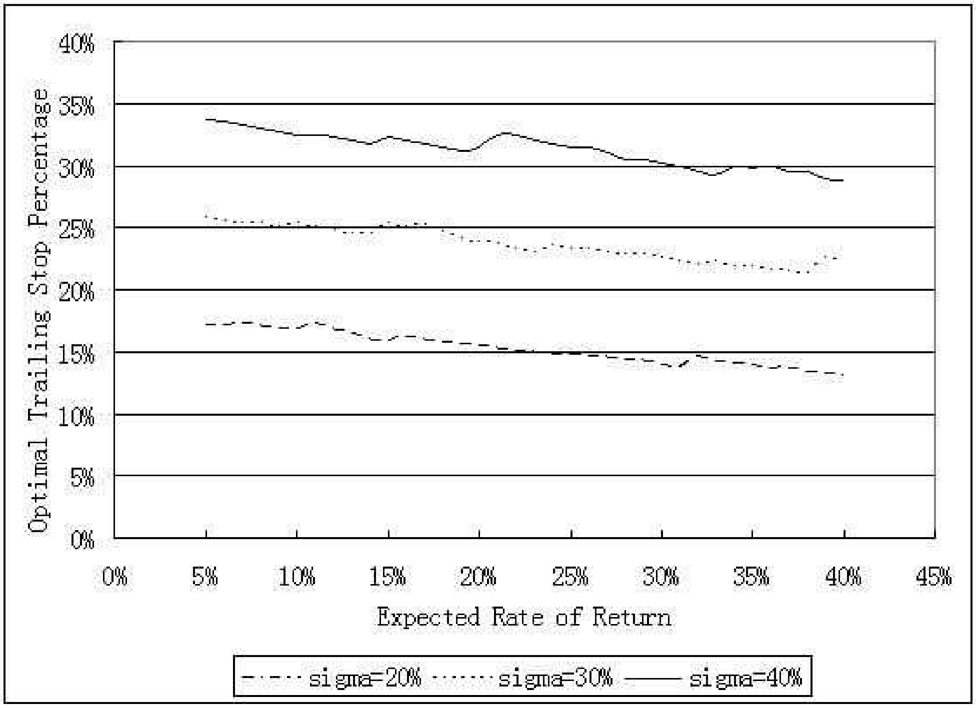

Figure 3.3: Optimal trailing stop percentage using Monte Carlo Method against expected rate of return given different stock price volatilities σ.

Then for each h, $\widehat{J}(h, \xi_n^{\pm})$ is an estimate of $J(h)$. In the simulation study, we can use independent random samples to estimate the expected value of $\Phi(S(\tau_n)) \exp(-\rho\tau_n)$. The law of large numbers implies that $\widehat{J}(h, \xi_n^{\pm})$ converges to $J(h)$ w.p.1 as $n_0 \to \infty$. Recall that n_0 is the number of random samples used in each iteration (see equation (3.5.2)). The iterates stop whenever $|h_{n+1} - h_n| < 0.0005$. Several different initial guesses are used. We take $n_0 = 1000$. Table 3.2 shows the results for $\mu = 10\%$ and $\sigma = 20\%$. The optimal trailing stop percentage calculated by Monte Carlo method is MC = 16.75%. In Table 3.2, SA1 is the optimal trailing stop percentage calculated by stochastic approximation with averages of samples.

It can be seen from Table 3.2 that the estimates are insensitive to the initial guesses, the algorithm leads to accurate estimation of the optimal value. Indeed, for $\sigma \in [10\%, 70\%]$ and $\mu \in [5\%, 40\%]$, the average value of $|\mathrm{MC} - \mathrm{SA1}|$ is only 1.34%.

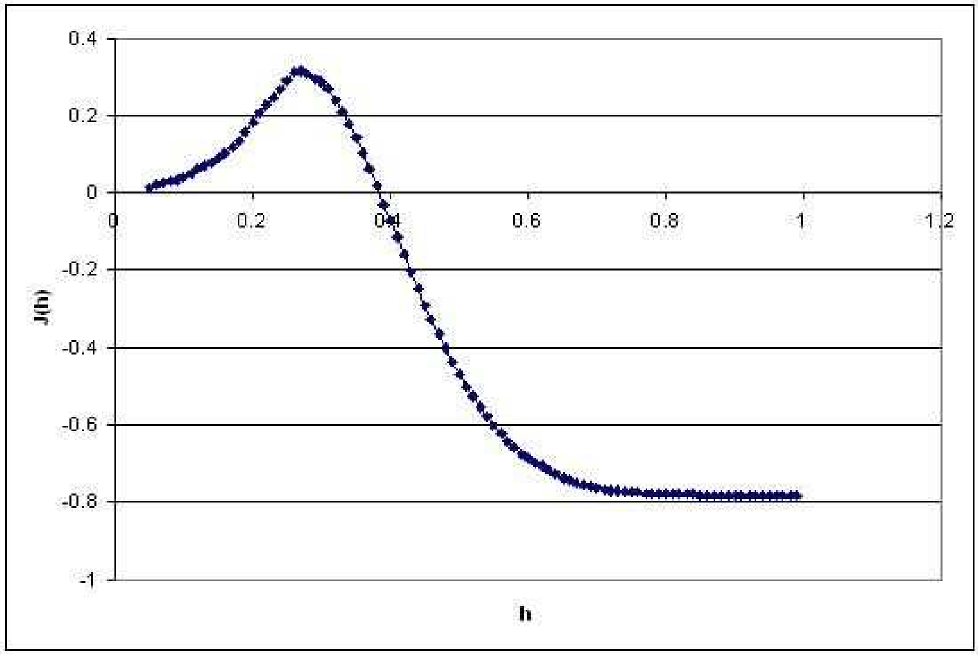

Figure 3.4: The objective function $J(h)$ against the trailing stop percentage h for fixed expected rate of return and volatility at $\mu = 20\%$ and $\sigma = 35\%$.

Next we use the similar method without averages of samples. In this case, $n_0 = 1$, all other parameters remain unchanged. The results are shown in Table 3.3, where SA2 denotes the optimal trailing stop percentage calculated by stochastic approximation without averages of samples. Again, the estimates are insensitive to the initial guesses. However, the bias |MC-SA2| is larger. For $\sigma \in [10\%, 70\%]$ and $\mu \in [5\%, 40\%]$, the average value of |MC-SA2| is 3.10%.

Compared to the Monte Carlo method, the SA methods take much less time to calculate the optimal trailing stop percentage. We run this algorithm on a Sun Fire 880 serve with 8GB memory, generally, it takes about 30 to 60 seconds to obtain the estimated optimal trailing stop percentage. With the Monte Carlo method, it takes at least 20 minutes for the corresponding computation.

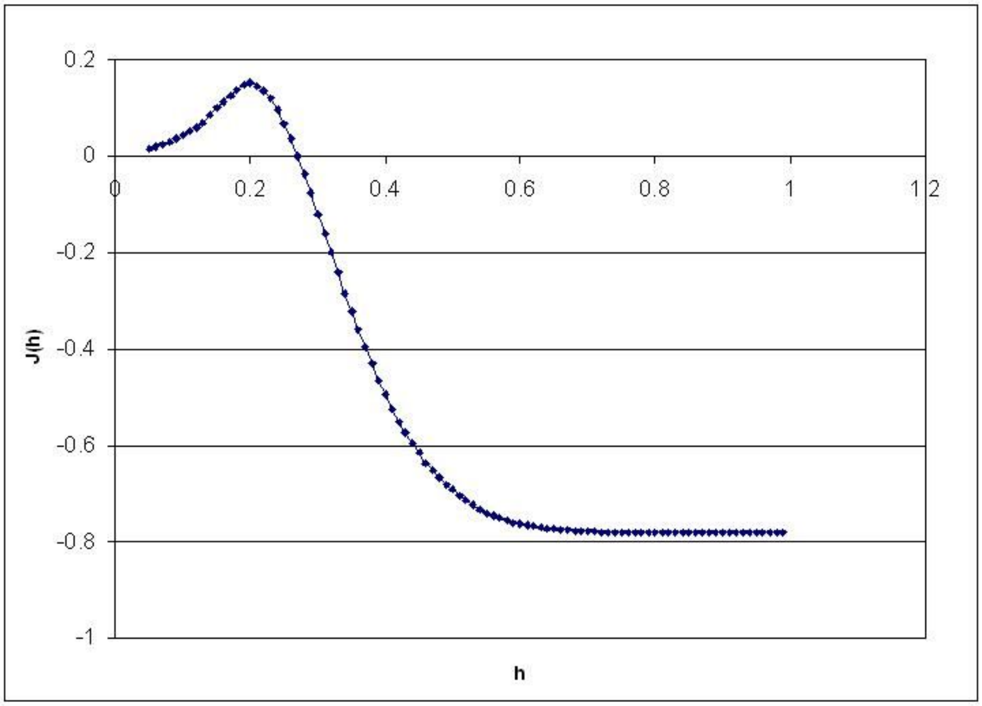

Figure 3.5: The objective function $J(h)$ against the trailing stop percentage h for fixed expected rate of return and volatility at $\mu = 12\%$ and $\sigma = 25\%$.

3.5.2 Using real market data

In what follows, we use the SA to compute the trailing stops. We use NASDAQ-100 components during the period from January 2,1995 - December 31, 2001. Here we consider two trading strategies. Since we need the 50-day moving averages of prices, we start our trading strategies on the fiftieth trading day after January 2, 1995.

> Strategy 1. If the stock price on the fiftieth trading day after January 2, 1995 is above the 50-day moving average, buy the stock. Otherwise, buy the stock when price is up-crossing 50-day moving averages. And sell stock when price is down-crossing 50-day moving averages. If the latter doesn't happen, then sell the stock on the last day of the period, December 31, 2001.

Initial guess	5.00%	17.00%	28.00%	40.00%	50.00%
SA1	16.82%	16.95%	16.94%	16.94%	16.94%
|MC-SA1|	0.07%	0.20%	0.19%	0.19%	0.19%

Table 3.2: Estimates using stochastic approximation with averages of samples (SA1) for fixed expected rate of return and volatility at $\mu = 10\%$ and $\sigma = 20\%$, where MC is the optimal trailing stop percentage calculated by Monte Carlo Method.

Initial guess	5.00%	17.00%	28.00%	40.00%	50.00%
SA2	18.02%	18.02%	18.02%	18.02%	18.02%
|MC-SA2|	1.27%	1.27%	1.27%	1.27%	1.27%

Table 3.3: Estimates using stochastic approximation without averages of samples (SA2) for fixed expected rate of return and volatility at $\mu = 10\%$ and $\sigma = 20\%$, where MC is the optimal trailing stop percentage calculated by Monte Carlo Method.

Strategy 2. The entry point is exactly the same as described in Strategy 1. Then use trailing stop technique with the percentage calculated via the stochastic approximation method. If price doesn't hit the stop price before December 31, 2001, sell stock on that day.

For example, let us assume we started collecting stock prices for Cadence Design Systems Inc (CDNS) on January 2, 1995. Then March 15, 1995 is the first day we have the 50-day moving average. It happens the closing price on that day is greater than the 50-day moving average, therefore we buy the CDNS for the price \$5.75. On June 2, 1995, the closing price of CDNS is \$7.33, which is less than the 50-day moving average. Therefore, Strategy 1 suggests to sell CDNS at $7.33, resulting a raw return of 27.48%. However, using the trailing stop technique, Strategy 2 suggest to hold till July 10, 1996. The closing price on that day is $15.69, resulting a raw return of 172.88%. The daily closing prices, their 50 day moving average, and the trailing stop curve are plotted in Figure 3.6.

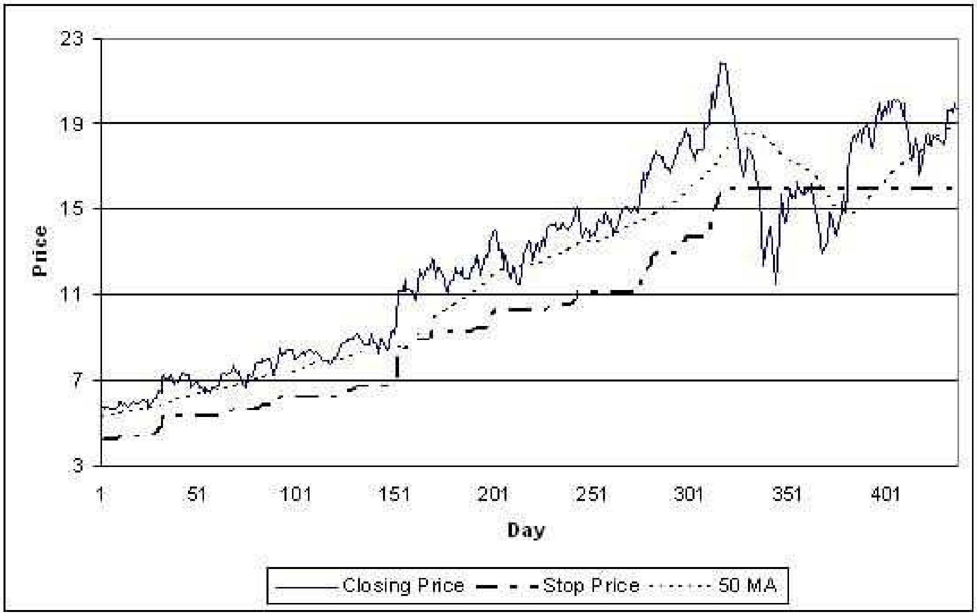

Figure 3.6: The prices of Cadence Design Systems Inc from March 15, 1995, to December 2, 1996

We perform the same experiments for NASDAQ-100 components if prices are available. Table 3.3 reports the average rate of returns from Strategies 1 and 2. The average rate of return from Strategy 1 is 11.45% while the average rate of return from Strategy 2 is 71.45%. It is easy to see that the Strategy 2 outperforms the Strategy 1 on average.

Return from strategy using moving average	Return from strategy using trailing stop
11.45%	71.45%

Table 3.4: Average Rate of Returns from Different Trading Strategies

3.6 Further Remarks

A class of stochastic approximation algorithms has been developed for finding the optimal trailing stop percentage in stock trading. Due to the absence of the closed-form solutions, we compare the proposed algorithm to Monte Carlo method and find that our algorithm is a sound procedure for estimating optimal trailing stop percentage. The approach developed is simple and systematic yielding useful guidelines for stock trading.

Note that one of the distinct features of the method in this chapter is that no specific stock price model needs to assumed. Only observed stock price is required in the SA calculation.

Chapter 4

Parameter Estimation in Option Pricing with Regime Switching

4.1 Introduction

This chapter is concerned with parameter estimations in option pricing. In the finance literature, the celebrated Black-Scholes model is widely used in analysis of option and portfolio management. Traditionally a geometric Brownian motion (GBM) is used to capture the dynamics of the stock market by using a stochastic differential equation with a deterministic expected return and a nonrandom volatility. It gives a reasonably good description of the market in a short time period. However, in the long run, it fails to describe the behavior of the stock price. This is mainly due to insensitivity to random parameter changes. In fact, it is well understood that the stock prices are far away from being a "random walk" in a longer time horizon. Recognizing the needs, various modifications of the models have been made. One of the ideas is to use a secondary stochastic differential equation to model the stochastic volatility.

One of the main factors that affects the option prices in a marketplace is the trend of the volatility. It is necessary to incorporate such trends in modeling to capture detailed stock price movements. In a recent paper of Zhang [39], a hybrid switching GBM model, involving a number of GBMs modulated by a finite-state Markov chain, is proposed. Such switching processes can be used to represent market trends or the trends of an individual stock.

The above switching GBM model has been used in Yao, Zhang, and Zhou [34] for pricing European options; a closed-form solution is obtained assuming the underlying Markov chain jumps only once. To use the result in practice, it is necessary to estimate various parameters. To accomplish this, it is natural to use market option prices to carry out estimation tasks.

One of the parameters to be estimated is the implied volatility. Normally, a large number of observations are needed to derive a meaningful estimator of the parameter. More observations typically bring better estimates. On the other hand, using more observations will take longer time to reach the desired result. Thus the time needed for successfully estimating the parameter is a major concern in applications.

A standard approach is to use the least squares method, which gives a bench-mark for comparison purposes. However, in view of real-time trading, the least squares method is too slow to meet the practical needs. In this chapter, we aim to find an alternative feasible algorithm that can be easily implemented for pricing options in real time. Inspired by the approach initiated in Yin, Liu, and Zhang [37] for treating stock liquidation, we develop stochastic approximation algorithms to estimate implied volatility first. Then we use the estimated parameter to price options.

The rest of the chapter is arranged as follows. Section 4.2 begins with the precise formulation of the problem. The model is given and then the recursive algorithm is proposed. The advantages include the simple form and systematic nature of the algorithms. In particular, it is useful for on-line computation. Another nice feature of the proposed algorithm is that only a few observations are needed. Section 4.3 then proceeds with the study of the asymptotic properties of the underlying algorithm; the convergence of the algorithm is obtained. The rate of convergence is ascertained in section 4.4. To demonstrate the feasibility and efficiency of the algorithms, numerical experiments using real market data are given in Section 4.5. The experiments indicate the stochastic approximation algorithms indeed provide good estimates of the desired parameter with a reasonably fast convergence speed and predict more accurately the option price than that of the traditional Black-Scholes model. Finally, we close this chapter with some further remarks in Section 4.6.

4.2 Formulation

4.2.1 Hybrid GBM Model

In a regime switching model, one typically uses a finite-state Markov chain $\alpha(\cdot) = \{\alpha(t) : t \geq 0\}$ to capture changes in the rate of return and the volatility of a stock to reflect the market modes. Let $X(t)$ denote the price of the stock at time t, which is governed by the following stochastic differential equation,

$$dX(t) = X(t)[\mu(\alpha(t))dt + \sigma(\alpha(t))dw(t)],\ 0 \leq t \leq T, \tag{4.2.1}$$

where $X(0) = X_0$ is the initial stock price at $t = 0$; $\mu(i)$ and $\sigma(i)$ for each $i \in \mathcal{M}$ with $\mathcal{M} = \{1, 2\}$, represent the expected rate of return and volatility of the stock price at regime i; $w(\cdot)$ is a standard one-dimensional Brownian motion independent of the Markov chain $\alpha(\cdot)$. Equation (4.2.1) is also called a hybrid GBM model. The randomness of stock price is characterized by the pair of $(\alpha(t), w(t))$, where $w(t)$ is the usual noise involved in the classical GBM model and $\alpha(t)$ captures the changes in market trend. The above model implies that the $X(t)$ randomly switches between two GBMs. In practice, this means that the expected rate of return and volatility of a stock could change in different market regimes. For instance, the expected rate of return μ in economic expansion is higher than that in the recession period. Or on the other hand, the volatility of stock price tends to be much higher during a sharp market downturn. Thus the regime switching model (4.2.1) better captures the dynamic of stock price.

In this chapter, we consider the case that the Markov chain jumps at most once on $[0, T]$. The corresponding generator is $Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$. That is, state 2 is an absorbing state. If we take $\alpha(0) = 1$, then there exits a stopping time τ such that τ is exponentially distributed with parameter λ and

$$\alpha(t) = \begin{cases} 1, & \text{if } \ t < \tau, \\ 2, & \text{if } \ t \geq \tau. \end{cases} \tag{4.2.2}$$

Therefore, the volatility process $\sigma(\alpha(t))$ jumps at most once at time $t = \tau$. Its jump size is $(\sigma(2) - \sigma(1))$, and the average sojourn time in state 1 (before jumping to state 2) is $1/\lambda$. A closed-form solution for European call option price has been obtained in Yao, Zhang, and Zhou [34]. In practice, we may already know the market prices for some options. Thus we may assume that $\sigma(1)$ is the implied volatility given by Black-Scholes model and λ is a known constant. We need only estimate the value of $\sigma(2)$ to calculate the price of a call option. Our goal here is to estimate the true value of $\sigma(2)$ given some real option prices. For convenience, we use σ instead of $\sigma(2)$ throughout the rest of this chapter if there is no confusion.

4.2.2 Stochastic Recursive Algorithm

In what follows, we formulate the task of finding the optimal value of σ as a stochastic optimization problem. We aim to find the optimal value $\sigma \in (0, \infty)$ so that a suitable objective function (the expected error) is minimized. The problem can be written as:

$$\text{Problem } \mathcal{P} : \text{Find argmin } \varphi(\sigma) = E[c(\sigma) - c_n]^2/2, \ \sigma \in (0, \infty), \tag{4.2.3}$$

where $c(\sigma)$ is the option price using model (4.2.1) and c_n is the corresponding market price observed at time n.

To obtain the desired estimate, we construct a recursive procedure

$$\sigma_{n+1} = \Pi\big[\sigma_n - \varepsilon_n[c(\sigma_n) - c_n]c_\sigma(\sigma_n)\big], \tag{4.2.4}$$

where $\{\varepsilon_n\}$ is a sequence of nonnegative decreasing step sizes satisfying $\varepsilon_n \to 0$ as $n \to \infty$, and $\sum_n \varepsilon_n = \infty$, $c_\sigma(\cdot)$ denotes the derivative of $c(\cdot)$, and $\Pi = \Pi_{[0,M]}$ for some $M > 0$ is a projection operator given by

$$\Pi[\sigma] = \begin{cases} 0, & \text{if } \sigma < 0, \\ M, & \text{if } \ \sigma > M, \\ \sigma, & \text{otherwise.} \end{cases}$$

We have used a projection device to make sure that the estimates are nonnegative. Moreover, we choose M to be a sufficiently large number so the estimate of the volatility stays bounded. As explained in Kushner and Yin [27], the projection algorithm may be written in an expanded form as

$$\sigma_{n+1} = \sigma_n - \varepsilon_n[c(\sigma_n) - c_n]c_\sigma(\sigma_n) + \varepsilon_n R_n, \tag{4.2.5}$$

where $\varepsilon_n R_n = \sigma_{n+1} - \sigma_n + \varepsilon_n[c(\sigma_n) - c_n]c_\sigma(\sigma_n)$ is the real number with the shortest distance needed to bring $\sigma_n - \varepsilon_n[c(\sigma_n) - c_n]c_\sigma(\sigma_n)$ back to the interval $[0, M]$ if it ever escapes from there.

4.3 Convergence

We proceed to prove the convergence of the algorithm. For simplicity, the step size $\{\varepsilon_n\}$ is assumed to be of the form $\varepsilon_n = O(1/n)$ henceforth. We will show that σ_n defined in (4.2.4) is closely related to an ordinary differential equation. First, let us state a couple of conditions needed in what follows.

(A4.1) The derivative $c_\sigma(\cdot)$ is continuous.

(A4.2) There is a $\overline{c}$ such that

$$\varepsilon_n \sum_{k=0}^{n-1} c_k \to \overline{c} \ \text{ w.p.1 as } \ n \to \infty. \tag{4.3.1}$$

Remark 4.3.1. Our objective here is to minimize the function $\varphi(\sigma)$ given in (4.2.3). Condition (A4.1) is satisfied in the context of typical option problems. Next we show that (A4.1) is satisfied in the case that the Markov chain has only two states for the European call option.

In particular, we show that for the European option $c(\sigma)$ is twice continuously differentiable. As in Yao, Zhang, and Zhou [34], given the risk-free rate r, the current stock price x, the maturity T, the strike price K, and the volatility vector $(\sigma_0, \sigma, \lambda)$, the call option price can be given as follows. Suppose that the Markov chain initially takes the value $\alpha(0) = 1$. Then

$$\begin{aligned} c(\sigma) = x \int_0^T e^{-rT} (\int_{-\infty}^{\infty} & e^u \psi^0(t, u + \log x, 2) \\ & \times \phi(u, (r - \sigma_0/2)t, \sigma_0 t) du) \lambda e^{-\lambda t} dt + x e^{-\lambda T} \psi^0(0, \log x, 1), \end{aligned} \tag{4.3.2}$$

where

$$\psi^0(s, y, 1) = e^{-y - r(T-s)} \int_{-\infty}^{\infty} h(e^{y+u}) \phi(u, (r - \sigma_0/2)(T - s), \sigma_0(T - s)) du,$$

$$\psi^0(s, y, 2) = e^{-y - r(T-s)} \int_{-\infty}^{\infty} h(e^{y+u}) \phi(u, (r - \sigma/2)(T - s), \sigma(T - s)) du,$$

$\phi(u, m, \Sigma)$ is the Gaussian density function with mean m and variance Σ,

$$h(x) = (x - K)^+.$$

It is then easy to see the twice continuously differentiability of $c(\sigma)$ with respect to σ.

Condition (A4.2) is essentially a law of large numbers condition. Take for instance, $\varepsilon_n = 1/n$. Then it is precisely the usual ergodicity of the sequence $\{c_n\}$. Suppose that $\{c_n\}$ is a stationary ϕ-mixing sequence. Then it is strongly ergodic. In such a case, (4.3.1) is readily verified.

To analyze the algorithm, we take a piecewise constant interpolation and work with a sequence of functions instead of the discrete iterates. To this end, define

$$t_0 = 0, \quad t_n = \sum_{k=0}^{n-1} \varepsilon_k,$$

$$m(t) = \begin{cases} n, & t_n \le t < t_{n+1}, \quad \text{for } \ t \ge 0, \\ 0, & \text{for } \ t < 0. \end{cases}$$

Define the continuous-time interpolation $\sigma^0(\cdot)$ on $(-\infty,\infty)$ by

$$\sigma^0(t) = \begin{cases} \sigma_0, & \text{for } \ t < 0, \\ \sigma_n & \text{for } \ t \geq 0 \ \text{ and } \ t_n \leq t < t_{n+1}. \end{cases}$$

Define a shifted process $\sigma^n(\cdot)$ by

$$\sigma^n(t) = \sigma^0(t_n + t), \quad t \in (-\infty,\infty).$$

Moreover, define

$$R_n = 0 \ \text{ and } \ [c(\sigma_n) - c_n]c_\sigma(\sigma_n) = 0, \quad \text{for } \ n < 0,$$

$$R^0(t) = \sum_{k=0}^{m(t)-1} \varepsilon_k R_k \ \text{ for } \ t \geq 0,$$

$$R^n(t) = \begin{cases} R^0(t_n + t) - R^0(t_n) = \displaystyle\sum_{k=n}^{m(t_n+t)-1} \varepsilon_k R_k, & t \geq 0, \\ -\displaystyle\sum_{k=m(t_n+t)}^{n-1} \varepsilon_k R_k, & t < 0. \end{cases}$$

As in Kushner and Yin [27, Section 4.3], define a set $C(\sigma)$ as follows. For $\sigma \in (0, M)$, $C(\sigma)$ contains only the zero element; for $\sigma = 0$ or $\sigma = M$, $C(\sigma)$ is the infinite cone (interval $(-\infty, 0)$ or (M,∞)) pointing to the direction away from $[0, M]$.

Theorem 4.3.2. *Assume conditions* (A4.1) *and* (A4.2)*. Then w.p.1,* $(\sigma^n(\cdot), R^n(\cdot))$ *is equicontinuous in the extended sense* (see Kushner and Yin [27, p. 102])*. Let* $(\sigma(\cdot), R(\cdot))$ *be the limit of a convergent subsequence of* $(\sigma^n(\cdot), R^n(\cdot))$ (*still indexed by* n *for simplicity*)*. Then it satisfies the projected ordinary differential equation*

$$\dot\sigma(t) = -[c(\sigma(t)) - \overline{c}]c_\sigma(\sigma(t)) + r(t), \quad r(t) \in -C(\sigma(t)), \tag{4.3.3}$$

where r *is the minimal force needed to keep the solution in* $[0, M]$ *with*

$$R(t) = \int_0^t r(s)ds.$$

Proof. Note that (4.2.5) can be written as

$$\sigma_{n+1} = \sigma_n - \varepsilon_n[c(\sigma_n)c_\sigma(\sigma_n) - \overline{c}c_\sigma(\sigma_n)] + \varepsilon_n(c_n - \overline{c})c_\sigma(\sigma_n) + \varepsilon_n R_n. \tag{4.3.4}$$

Define

$$\widetilde{c}_n = -[c(\sigma_n) - \overline{c}]c_\sigma(\sigma_n), \quad \text{for} \quad n \geq 0, \quad \text{and} \quad \widetilde{c}_n = 0 \quad \text{for} \quad n < 0,$$

$$\widehat{c}_n = [c_n - \overline{c}]c_\sigma(\sigma_n) \quad \text{for} \quad n \geq 0, \quad \text{and} \quad \widehat{c}_n = 0 \quad \text{for} \quad n < 0.$$

Define the interpolations of $\widetilde{c}_n$ and $\widehat{c}_n$ as $\widetilde{c}^0(\cdot)$, $\widetilde{c}^n(\cdot)$, $\widehat{c}^0(\cdot)$, and $\widehat{c}^n(\cdot)$ similar to that of $R^0(\cdot)$ and $R^n(\cdot)$. Then

$$\sigma^n(t) = \sigma_n + \widetilde{c}^n(t) + \widehat{c}^n(t), \quad \text{for} \quad t \in (-\infty, \infty).$$

To proceed, we use Kushner and Yin [27, Theorem 6.1.1] to prove the theorem. In this process, we need only verify an asymptotic rate of change condition holds. For each σ, define

$$\Phi(t) \stackrel{\text{def}}{=} \sum_{k=0}^{m(t)-1} \varepsilon_k(c_k - \overline{c})c_\sigma(\sigma).$$

Recall that the asymptotic rate of change of $\Phi(t)$ is said to go to 0 with probability 1 if for some $T > 0$,

$$\lim_n \sup_{j \geq n} \max_{0 \leq t \leq T} |\Phi(jT + t) - \Phi(jT)| = 0 \quad \text{w.p.1.}$$

To proceed, define

$$D_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} [c_k - \overline{c}].$$

Note that by means of a partial summation,

$$\sum_{k=m}^{n} \varepsilon_k[c_k - \overline{c}]c_\sigma(\sigma) = \varepsilon_n[D_{n+1} - D_m]c_\sigma(\sigma) + \sum_{k=m}^{n-1}[D_{k+1} - D_m][\varepsilon_k - \varepsilon_{k+1}]c_\sigma(\sigma).$$

Taking $m = 0$, $n = m(t) - 1$, we obtain

$$\Phi(t) = \varepsilon_{m(t)-1}D_{m(t)}c_\sigma(\sigma) + \sum_{k=0}^{m(t)-2} D_{k+1}\frac{\varepsilon_k - \varepsilon_{k+1}}{\varepsilon_k}\varepsilon_k c_\sigma(\sigma).$$

Note that (A4.2) implies that $\varepsilon_{m(t)-1} D_{m(t)} c_\sigma(\sigma) \to 0$ as $m(t) \to \infty$ (or $n \to \infty$) and

$$\sum_{k=0}^{m(t)-2} D_{k+1} \frac{\varepsilon_k - \varepsilon_{k+1}}{\varepsilon_k} \varepsilon_k c_\sigma(\sigma) = \sum_{k=0}^{m(t)-2} D_{k+1} O(\varepsilon_k^2) c_\sigma(\sigma) \to 0.$$

Thus the asymptotic rate of changes of $\Phi(t)$ goes to 0 w.p.1. By using Kushner and Yin [27, Theorem 6.1.1], the desired result then follows. $\square$

Corollary 4.3.3. *In addition to the conditions in Theorem* 4.3.2*, suppose that* $c_\sigma(\sigma) \neq 0$ *for all* σ*, and that* σ_* *is the unique solution of* $c(\sigma) - \overline{c} = 0$ *with* $\sigma_* \in (0, M)$ *such that* σ_* *is in the set of locally asymptotic stable points of the projected ODE. Then* $\sigma_n \to \sigma_*$ *w.p.1.*

Remark 4.3.4. In view of the definition of the cost function, σ_* is the unique minimizer of φ defined in (4.2.3). Thus the result indicates that the algorithm that we constructed converges to the unique minimizer of the least squares cost function.

Proof. We merely indicate that by use of the asymptotic stability in the sense of Liapunov (see Kushner and Yin [27, p. 104]), for any $s_n \to \infty$ as $n \to \infty$, using the argument as in Theorem 4.3.2, $\sigma^n(\cdot + s_n) \to \sigma_*$ w.p.1. $\square$

4.4 Rate of Convergence

This section is devoted to the rate of convergence of algorithm (4.2.4). To further simplicity the matters, we take $\varepsilon_n = 1/(n+1)$. We assume that all the conditions of Corollary 4.3.3 hold. Since σ_* is strictly interior to the constrained set $[0, M]$, without loss of generality, we will drop the reflection term $\varepsilon_n R_n$ throughout this section, and assume that the sequence of iterates $\{\sigma_n\}$ is nonnegative and uniformly bounded by M.

Define $u_n = \sqrt{n+1}(\sigma_n - \sigma_*)$. The rate of convergence study aims to exploit the asymptotic properties of this scaled sequence. We shall show that the interpolation of u_n converges to a diffusion limit. One of the key points here is to use linearization and local analysis. First note that from (4.2.5) with the reflection term dropped, we can write

$$
\begin{aligned}
u_{n+1} = \sqrt{\frac{n+2}{n+1}} u_n - \frac{1}{n+1}\sqrt{\frac{n+2}{n+1}} c_\sigma^2(\sigma_*) u_n + \sqrt{\frac{n+2}{n+1}} \frac{1}{\sqrt{n+1}} (c_n - \overline{c}) c_\sigma(\sigma_*) \\
+ \sqrt{\frac{n+2}{n+1}} \frac{1}{n+1} (c_n - \overline{c}) g(u_n) + \frac{1}{n+1} o(|u_n|),
\end{aligned}
\tag{4.4.1}
$$

where $g(\cdot)$ is a continuous function and $g(u) = O(|u|)$. Since

$$
\sqrt{\frac{n+2}{n+1}} = 1 + \frac{1}{2(n+1)} + O\left(\frac{1}{(n+1)^2}\right), \tag{4.4.2}
$$

to study the asymptotics of u_n, we need only consider an auxiliary process v_n defined by

$$
\begin{aligned}
& v_0 = u_0, \\
& v_{n+1} = v_n - \frac{1}{n+1}\left(c_\sigma^2(\sigma_*) - \frac{1}{2}\right) v_n + \frac{1}{\sqrt{n+1}} (c_n - \overline{c}) c_\sigma(\sigma_*) \\
& \qquad + \frac{1}{n+1} (c_n - \overline{c}) g(v_n) + \frac{1}{n+1} o(|v_n|).
\end{aligned}
\tag{4.4.3}
$$

(A4.3) The $\{c_n - \overline{c}\}$ is a stationary ϕ-mixing sequence with 0 mean and mixing rate ϕ_k satisfying $\sum_{k=0}^{\infty} \phi_k^{1/2} < \infty$. In addition, suppose $c_\sigma^2(\sigma_*) > 1/2$.

Remark 4.4.1. Under (A4.3), it can be shown (see Kushner and Yin [27, Chapter 7]) that $\sum_{k=0}^{m(t)-1} \frac{1}{\sqrt{k+1}} (c_k - \overline{c})$ converges weakly to a real-valued standard Brownian motion with variance $\varsigma^2 t$, where

$$
\varsigma^2 = E(c_0 - \overline{c})^2 + 2\sum_{k=1}^{\infty} E(c_k - \overline{c})(c_0 - \overline{c}).
$$

Moreover, by using the well-known mixing inequality (see [3, p. 166]), we obtain

$$
E\left|\sum_{k=0}^{n-1} \frac{1}{\sqrt{k+1}} (c_k - \overline{c})\right|^2 \leq K
$$

for some $K > 0$.

Lemma 4.4.2. *In addition to the conditions of Corollary* 4.3.3*, assume* (A4.3) *holds. Then* $\{v_n\}$ *is tight.*

Proof. We claim that $\sup_n E|v_n| < \infty$. To this end, define

$$A_{nk} = \begin{cases} \displaystyle\prod_{j=k+1}^{n} \left(1 - \frac{c_\sigma^2(\sigma_*) - \frac{1}{2}}{j+1}\right), & \text{if } \ k < n, \\ 1, & \text{if } \ k = n. \end{cases}$$

Then

$$v_{n+1} = A_{n0} v_0 + \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}} A_{nk} (c_k - \overline{c}) c_\sigma(\sigma_*)$$

$$+ \sum_{k=0}^{n} \frac{1}{k+1} A_{nk} (c_k - \overline{c}) g(v_k) + \sum_{k=0}^{n} \frac{1}{k+1} A_{nk} o(|v_k|).$$

Note that in view of (A4.3), $E|c_k - \overline{c}||g(v_k)| \leq K E|v_k|$. Thus, we obtain

$$E|v_{n+1}| \leq |A_{n0}| E|v_0| + E \left| \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}} A_{nk} (c_k - \overline{c}) c_\sigma(\sigma_*) \right| + K \sum_{k=0}^{n} \frac{1}{k+1} |A_{nk}| E|v_k|. \quad (4.4.4)$$

It is easily verified that by (A4.3),

$$\sum_{k=0}^{n} \frac{1}{k+1} |A_{nk}| = \sum_{k=0}^{n} \frac{1}{k+1} A_{nk} < \infty.$$

Using the mixing inequality [3], we have

$$
\begin{aligned}
& E\left|\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}}A_{nk}(c_k-\overline{c})c_\sigma(\sigma_*)\right| \\
& \le E^{\frac{1}{2}}\left|\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}}A_{nk}(c_k-\overline{c})c_\sigma(\sigma_*)\right|^2 \\
& = \left(\sum_{k=0}^{n}\sum_{j=0}^{n}\frac{1}{\sqrt{k+1}}\frac{1}{\sqrt{j+1}}A_{nk}A_{nj}(c_k-\overline{c})(c_j-\overline{c})c_\sigma^2(\sigma_*)\right)^{\frac{1}{2}} \\
& \le K\left(\sum_{j=0}^{n}\frac{1}{j+1}A_{nj}^2\sum_{k>j}^{n}E(c_k-\overline{c})(c_j-\overline{c})c_\sigma^2(\sigma_*)\right)^{\frac{1}{2}} \\
& \le K\left(\sum_{j=0}^{n}\frac{1}{j+1}A_{nj}^2\sum_{k>j}^{n}|E(c_k-\overline{c})(c_j-\overline{c})-E(c_k-\overline{c})E(c_j-\overline{c})|\right)^{\frac{1}{2}} \\
& \le K\left(\sum_{j=0}^{n}\frac{1}{j+1}A_{nj}^2\sum_{k<j}^{n}\phi_{k-j}\right)^{\frac{1}{2}} \\
& \le K<\infty.
\end{aligned}
$$

Recall that we use K as a generic positive constant, whose value may change for different appearances. Combining the above estimates, we arrive at

$$
E|v_{n+1}|\le K+K\sum_{k=0}^{n}\frac{1}{k+1}|A_{nk}|E|v_k|. \tag{4.4.5}
$$

An application of the Gronwall's inequality leads to

$$
E|v_{n+1}|\le K<\infty \quad \text{and} \quad \sup_n E|v_n|<\infty.
$$

The desired tightness then follows from the well-known Markov inequality

$$
P\left(|v_n|\ge\widetilde{K}\right)\le\frac{\sup_n E|v_n|}{\widetilde{K}}.
$$

The lemma is proved. $\square$

Lemma 4.4.3. *Under the conditions of Lemma* 4.4.2*,* $\lim_n E|v_n-u_n|=0$.

Proof. We merely use the expansions in (4.4.2), and the definitions of u_n and v_n in (4.4.1) and (4.4.3), respectively. Detailed calculation yields the desired result. $\square$

Consider the sequence $\{v_n\}$. Define a piecewise constant interpolation $v^0(t)$ and its shift $v^n(t)$ as in the previous section. We then have

$$\begin{aligned} v^n(t+s) - v^n(t) = &\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1}\left(c_\sigma^2(\sigma_*) - \frac{1}{2}\right) v_j \\ &+ \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}}(c_j - \overline{c})c_\sigma(\sigma_*) \\ &+ \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1}(c_j - \overline{c})g(v_j) \\ &+ \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1} h_j v_j. \end{aligned} \tag{4.4.6}$$

Our next result shows that the scaled sequence $v^n(\cdot)$ converges weakly to $v(\cdot)$, a diffusion process.

Theorem 4.4.4. *The sequence of interpolated estimation errors $\{v^n(\cdot)\}$ converges weakly to $v(\cdot)$, which is a solution of the stochastic differential equation*

$$dv = \left(c_\sigma^2(\sigma_*) - \frac{1}{2}\right) v dt + \varsigma c_\sigma(\sigma_*) dw, \tag{4.4.7}$$

where $w(\cdot)$ is a real-valued standard Brownian motion.

Proof. The proof is naturally divided into two steps. The first step establishes the tightness, whereas the second step characterizes the limit process.

Step 1): Show that the sequence $\{v^n(\cdot)\}$ is tight in $D[0,\infty)$ the space of functions that are right continuous, have left limits, endowed with the Skorohod topology (see Kushner and Yin [27, page 238]). To this end, we apply the tightness criterion in [27]. Without loss of generality and for notational simplicity, assume that $\{v_n\}$ is bounded (otherwise, we can use a truncation device as in [27]).

Then we obtain that for any $t > 0$, $\eta > 0$, and any $0 < s \le \eta$,

$$E|v^n(t+s) - v^n(t)|^2 \le I_1 + I_2 + I_3 + I_4, \tag{4.4.8}$$

where I_i for $i = 1, 2, 3, 4$ are four terms on the right-hand side of (4.4.6). By virtue of the boundedness of $\{v_k\}$,

$$\begin{aligned} I_1 &= E\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1}\left(c_\sigma^2(\sigma_*) - \frac{1}{2}\right) v_j\right|^2 \\ &\le K \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \sum_{k=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1}\frac{1}{k+1} \\ &\le Ks^2 \le K\eta^2. \end{aligned} \tag{4.4.9}$$

Thus, taking $\limsup_n$ followed by $\lim_{\eta\to 0}$, the limit is 0.

The mixing inequality implies that

$$\begin{aligned} I_2 &= E\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}}(c_j - \overline{c})c_\sigma(\sigma_*)\right|^2 \\ &\le Ks \le K\eta. \end{aligned} \tag{4.4.10}$$

Thus, the double limits of this term also goes to 0. Likewise,

$$I_3 + I_4 \le E\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}}(c_j - \overline{c})g(v_j)\right|^2 + E\left|\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}}h_j v_j\right|^2 \le K\eta. \tag{4.4.11}$$

Combining the estimates above, we arrive at

$$\lim_{\eta\to 0}\limsup_n E|v^n(t+s) - v^n(t)|^2 = 0. \tag{4.4.12}$$

Therefore, $\{v^n(\cdot)\}$ is tight.

Step 2) Characterization of the limit process. By Prohorov's theorem, we can extract a convergent subsequence and still denote it by $\{v^n(\cdot)\}$ for notational simplicity. Denote the limit by $v(\cdot)$. By the Skorohod representation, without changing notation, we may assume that the sequence $v^n(\cdot)$ converges to $v(\cdot)$ w.p.1 and the convergence is uniform in any bounded time interval. We proceed to establish that the limit is nothing but the desired diffusion process.

We shall show that $v(\cdot)$ is a solution of the martingale problem with operator

$$\mathcal{L}f(v) = \frac{1}{2}\varsigma^2 c_\sigma^2(\sigma_*)\frac{d^2 f(v)}{dv^2} + \left(c_\sigma^2(\sigma_*) - \frac{1}{2}\right) v\frac{df(v)}{dv}, \tag{4.4.13}$$

where $f(\cdot)$ is a C^2 function with compact support. To this end, we show that

$$f(v(t+s)) - f(v(t)) - \int_t^{t+s} \mathcal{L}f(v(\tau))d\tau \ \text{ is a martingale.}$$

To do so, for any bounded and continuous function $\rho(\cdot)$, any $t, s > 0$, any positive integer κ, and $0 \le t_1 \le t_2 \le \cdots \le t_\kappa \le t$, we will show

$$E\rho(v(t_i) : i \le \kappa)\Big[f(v(t+s)) - f(v(t)) - \int_t^{t+s} \mathcal{L}f(v(\tau))d\tau\Big] = 0.$$

First, by the weak convergence and the Skorohod representation, it is readily seen that

$$\begin{aligned} &E\rho(v^n(t_i) : i \le \kappa)[f(v^n(t+s)) - f(v^n(t))] \\ &\to \rho(v(t_i) : i \le \kappa)[f(v(t+s)) - f(v(t))] \ \text{ as } \ n \to \infty. \end{aligned} \tag{4.4.14}$$

Let δ_n be a sequence a positive real numbers satisfying $\delta_n \to 0$ and select an increasing sequence $\{m_\ell(n)\}$ such that $m(t_n + t) = m_1(n) < m_2(n) < \cdots \le m(t_n + t + s) - 1$, and that for $m(t_n + t) \le m_\ell \le m_{\ell+1} \le m(t_n + t + s) - 1$,

$$\frac{1}{\delta_n} \sum_{j=m_\ell(n)}^{m_{\ell+1}(n)-1} \frac{1}{j+1} \to 1 \ \text{ as } \ n \to \infty.$$

In what follows, for notational simplicity, we suppress the n dependence in $m_\ell(n)$ and write it as m_ℓ instead. Denote by I_m the index set satisfying $m(t_n+t) \le m_\ell \le m_{\ell+1} \le m(t_n+t+s)-1$.

Using the notation defined above, we have

$$
\begin{aligned}
f(v^n(t+s)) - f(v^n(t)) &= \sum_{\ell\in I_m}[f(v_{m_{\ell+1}}) - f(v_{m_\ell})] \\
&= \sum_{\ell\in I_m}\frac{df(v_{m_\ell})}{dv}\Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}(v_{j+1}-v_j)\Big] \qquad (4.4.15)\\
&\quad + \sum_{\ell\in I_m}\frac{1}{2}\frac{d^2f(v_{m_\ell})}{dv^2}\Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}(v_{j+1}-v_j)\Big]^2 + o(1),
\end{aligned}
$$

where $o(1)\to 0$ in probability uniformly in t.

It follows that

$$
\begin{aligned}
&E\rho(v^n(t_i): i\le\kappa)\sum_{\ell\in I_m}\frac{df(v_{m_\ell})}{dv}\Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}(v_{j+1}-v_j)\Big]\\
&= E\rho(v^n(t_i): i\le\kappa)\sum_{\ell\in I_m}\frac{df(v_{m_\ell})}{dv}E_{m_\ell}\Big[-\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j+1}(c_\sigma^2(\sigma_*)-\frac{1}{2})v_j\\
&\qquad + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{\sqrt{j+1}}(c_j-\overline{c})c_\sigma(\sigma_*) \qquad (4.4.16)\\
&\qquad + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j+1}(c_j-\overline{c})g(v_j)\\
&\qquad + \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j+1}h_jv_j\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_n E\rho(v^n(t_i): i\le\kappa)\sum_{\ell\in I_m}\frac{df(v_{m_\ell})}{dv}\Big[-\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j+1}(c_\sigma^2(\sigma_*)-\frac{1}{2})v_j\Big]\\
&= \lim_n E\rho(v^n(t_i): i\le\kappa)\sum_{\ell\in I_m}\frac{df(v_{m_\ell})}{dv}\Big[-\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1}\frac{1}{j+1}(c_\sigma^2(\sigma_*)-\frac{1}{2})v_{m_\ell}\Big] \qquad (4.4.17)\\
&= E\rho(v(t_i): i\le\kappa)\Big[-\int_t^{t+s}\frac{df(v(\tau))}{d\tau}(c_\sigma^2(\sigma_*)-\frac{1}{2})v(\tau)d\tau\Big].
\end{aligned}
$$

As for the next term, using the mixing property,

$$\lim_n E\rho(v^n(t_i) : i \leq \kappa) \sum_{\ell \in I_m} \frac{df(v_{m_\ell})}{dv} \Big[- \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}} (c_j - \overline{c}) c_\sigma(\sigma_*) \Big]$$

$$= \lim_n E\rho(v^n(t_i) : i \leq \kappa) \sum_{\ell \in I_m} \frac{df(v_{m_\ell})}{dv} \Big[- \sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{\sqrt{j+1}} E_{m_\ell} (c_j - \overline{c}) c_\sigma(\sigma_*) \Big] = 0. \tag{4.4.18}$$

Using the continuity of $g(\cdot)$, detailed calculation also shows that

$$\begin{aligned} &\lim_n E\rho(v^n(t_i) : i \leq \kappa) \sum_{\ell \in I_m} \frac{df(v_{m_\ell})}{dv} \Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1} (c_j - \overline{c}) g(v_j) \Big] = 0, \\ &\lim_n E\rho(v^n(t_i) : i \leq \kappa) \sum_{\ell \in I_m} \frac{df(v_{m_\ell})}{dv} \Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} \frac{1}{j+1} h_j v_j \Big] = 0. \end{aligned} \tag{4.4.19}$$

Likewise, similar estimates lead to

$$\begin{aligned} &\lim_n E\rho(v^n(t_i) : i \leq \kappa) \sum_{\ell \in I_m} \frac{1}{2} \frac{d^2 f(v_{m_\ell})}{dv^2} \Big[\sum_{j=m(t_n+t)}^{m(t_n+t+s)-1} (v_{j+1} - v_j) \Big]^2 \\ &= E\rho(v(t_i) : i \leq \kappa) \Big[\frac{1}{2} \int_t^{t+s} \frac{d^2 f(v(\tau))}{dv^2} \varsigma^2 c_\sigma^2(\sigma_*) d\tau \Big]; \end{aligned} \tag{4.4.20}$$

we omit the details for brevity. Thus the desired result follows. □

Remark 4.4.5. Likewise, we can also construct an SA algorithm using constant step size

$$\sigma_{n+1} = \Pi \big[\sigma_n - \varepsilon [c(\sigma_n) - c_n] c_\sigma(\sigma_n) \big]. \tag{4.4.21}$$

Convergence of corresponding algorithm can also be obtained.

4.5 Numerical Results

In this section, we report certain numerical experiment results. We first show that our algorithm is insensitive to the initial values. Then we test our algorithm using real market data and compare our results to that of the traditional Black-Scholes, where the dividend is the total dividend paid during the life of the option.

4.5.1 Data Description

Here our numerical experiments are done using the data derived from Berkeley Options Data Base. We use OEX (SP100) call option data during the period from January 4, 1988 to December 30, 1988. The set of data is listed in Table 4.1.

Strike Price	Transaction Time	Exp.	Days to Exp.	Option Price	Volume	Stock Price	Interest Rate	Dividend
\$285.00	1/4/88 13:01	1/16/88	12	\$0.13	50	\$247.55	6%	\$0.024319
\$255.00	1/4/88 13:03	1/16/88	12	\$3.13	15	\$248.20	6%	\$0.024315

Table 4.1: Samples of Market Data

Note that OEX call options are American type and (4.3.2) only gives us the price for European call option. Nevertheless, if a stock does not pay dividend, the American call option on this stock should have the same price as its European counterpart. When a company declares a dividend, it specifies a date when the dividend is payable to all stockholders, called the holder-of-record date. Two business day before that day is called the ex-dividend date. One must buy the stock by the ex-dividend data so that he/she can be recorded as stockholder by the holder-of-record date. The stock price tends to drop by the amount of the dividend on the ex-dividend date. Therefore, the call price drops as the stock goes to ex-dividend. However, the option price could never falls by more than the stock price changes, i.e., the dividend.

An American option holder may avoid this loss by early exercising the option immediately before the ex-dividend date. This is the only time the American call option should be early exercised. Note that there is a situation in which one should not take an early exercise action for an American call option even if a stock goes ex-dividend. If the present value of all the dividends over the life of this call option is less than $K(1 - e^{-rT})$, then the option should never be exercised early because the loss of interest from paying out the exercise price early

Initial σ	0.05	0.1	0.15	0.2	0.25	0.3
Optimal σ	0.237682	0.237681	0.237681	0.237681	0.237681	0.237681

Table 4.2: Initial values for σ.

cannot be covered by the dividends. In such a case, the American call option on the dividend-paying stock should have the same price as its European counterpart. We checked our data and it turns out that OEX data satisfy the above condition.

4.5.2 Numerical Testing

In what follows, the sequence $\{\varepsilon_n\}$ and $\{\delta_n\}$ are chosen to be $\varepsilon_n = 1/(n+1000)$ and $\delta_n = 1/(n+1000)^{1/6}$. After calculating the volatility for S&P100, we choose $[0.05, 0.5]$ as the boundary for σ. We show the result on the transaction date March 17, 1988. In this experiment we choose the first 25 transactions as our observations, thus the total number of iterations is 25. Several different initial values of σ are used. Table 4.2 shows that for different initial values, the iterates using the proposed algorithm always reach the same optimal value of σ. This means the estimates are insensitive to the initial value of σ. Figure 4.1 demonstrates the convergence of the algorithm for two starting points, one from below and the other one from above the optimal point.

Compared to the least squares method, the proposed method takes much less time to calculate the parameter estimates. We run this algorithm on a Sun Fire 880 server with 8GB memory, it takes about 35 seconds to obtain the estimated σ value. With the least squares method, one can obtain a similar value but consumes at least 2 minutes for the corresponding computation.

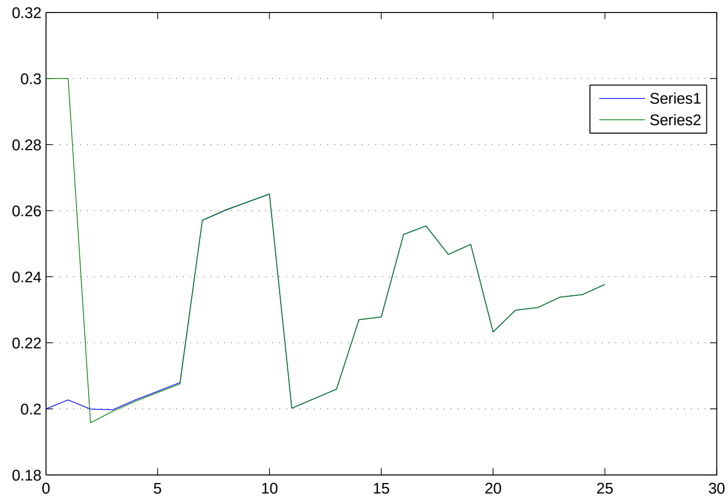

Figure 4.1: Estimates with different initial estimates. Horizontal axis represents the number of iterations; vertical axis represents the volatility σ

4.5.3 Comparing to Black-Scholes method

In this section, we use the proposed algorithm to calculate the optimal σ, and then use this optimal σ to price the OEX option with real market data. This method can be divided into three steps:

Step 1. We chose one day data, and use the market price to estimate the implied volatility by Black-Scholes model for each observation. Then use the average of all implied volatilities as our $\sigma(1)$.

Step 2. We divide daily data into 2 parts: One part is used for estimation of $\sigma = \sigma(2)$ and the other part reserved for testing. Generally, we use at least 20 observations for estimation.

Step 3. Using the estimated value of σ and (4.3.2), we compute the option prices. Then we compare the resulting price with the actual price from the testing data. Such error is referred to as RS error. To compare our result with the Black-Scholes formula, we also

Strike Price	Transac. Time	Days to Exp.	Volume	Market Price	RS Price	BS Price	RS Error	BS Error
\$250.00	2/24/88 13:01	18	5	\$8.88	\$8.3515	\$8.4186	5.95%	5.20%
\$260.00	2/24/88 13:01	18	5	\$3.5	\$3.6444	\$3.7112	4.13%	6.03%
\$260.00	2/24/88 13:03	18	3	\$3.63	\$3.6481	\$3.7149	0.50%	2.34%
						Mean	5.40%	6.09%
						Standard Deviation	4.52%	3.08%

Table 4.3: Mean and standard deviation of BS and RS.

compute the price obtained by plugging the implied volatility derived earlier into the Black-Scholes option pricing formula. We call the corresponding price the BS price. In addition, we term the difference between this price and the actual price from the testing data the BS error. Table 4.3 shows part of testing result for February 24, 1988.

From the above table, it is seen that our method performs better than that of the Black-Scholes method in estimating the option prices on the given date. Since we need at least 20 observations for each day, not all of the daily data satisfy this condition. We chose 20 days on which we have enough observations. Table 4.4 shows the testing result for total 411 testing options. It is easy to see that not only does the proposed method have a smaller average error but also it has a smaller deviation. We also plot the error distributions in Figure 4.2. It can be seen from Figure 4.2 that the RS errors are mostly concentrated near 0.03 while the BS errors are distributed more evenly along the horizontal axis.

	RS Error	BS Error
Mean	5.19%	6.55%
Standard Deviation	5.97%	7.86%

Table 4.4: Mean and standard deviation errors.

4.6 Further Remarks

A stochastic optimization algorithm has been developed for choosing optimal value of volatility in option pricing. As demonstrated by using real market data, this algorithm provides a sound procedure for estimating volatility in real time just by using twenty to thirty observations. A regime switching model is used here and it provides more accurate prices than that of the usual Black-Scholes model. The approach developed is simple and systematic. It can be used for on-line option pricing and provides useful guideline for option transactions.

The advantage of the proposed method is its simple recursive form. Compared with the least squares method, it does not require as many observations. The simple recursive algorithm often takes only seconds to reach the optimal value. Further effort may be devoted to improve the efficiency and to reduce the variance and bias.

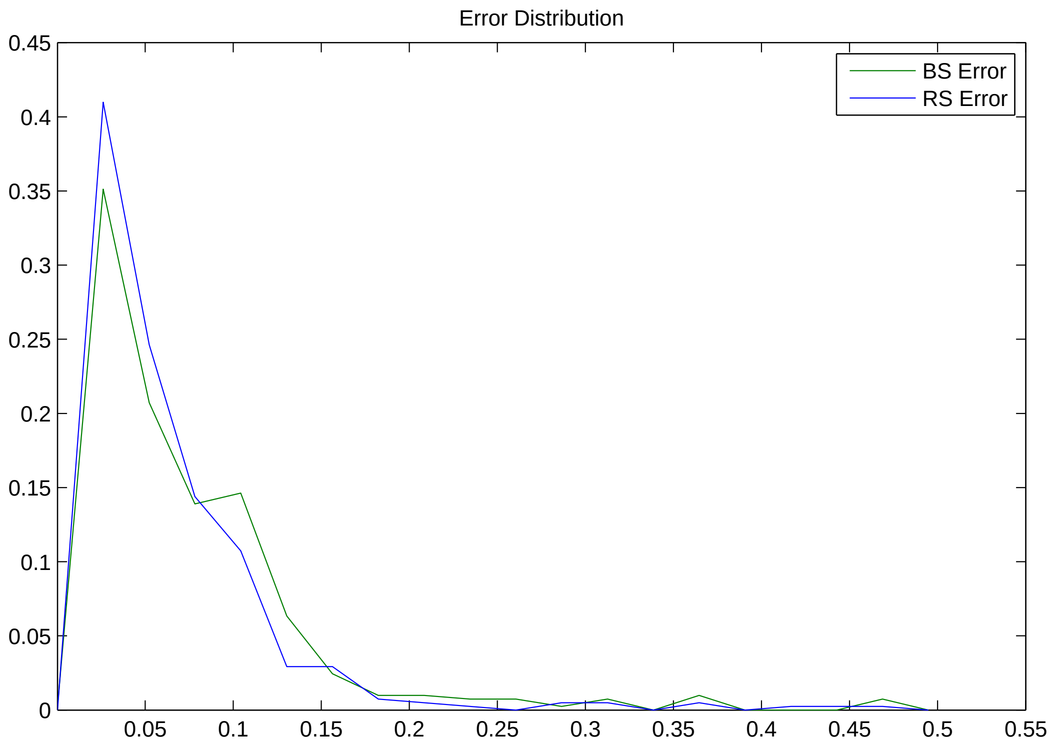

Figure 4.2: BS error and RS error distributions. Horizontal axis represents the estimation error; vertical axis represents the percentage of occurrence.

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