Hamiltonian dynamics and persistent homology

by

Jun Zhang

(Under the Direction of Michael Usher)

Abstract

In this dissertation, we first develop an alternative method of persistent homology from filtered chain complex level, called Floer-Novikov persistent theory. Then we apply it to study a concrete dynamics problem on quantitatively measuring how far a Hamiltonian flow is from being autonomous. The main results in this thesis are divided into two parts. The first part consists of many results on our Floer-Novikov persistent theory which are analogous to those in classical persistent theory. This includes the important Structure Theorem on the decomposition of a Floer-type complex and Stability Theorem. The main tool we use to develop this theory is non-Archimedean orthogonality and singular value decomposition. The second part consists of the main result that for symplectic manifold in the form of $\Sigma_g \times M$ (where surface $\Sigma_g$ has genus $g \geq 4$ and $M$ is *any* symplectic manifold), the subset of non-autonomous Hamiltonian diffeomorphisms can be arbitrarily far away in Hofer's metric from the group of autonomous Hamiltonian diffeomorphisms. This generalizes Polterovich-Shelukhin's result.

Index words: Hamiltonian dynamics, Hamiltonian Floer theory, persistent homology, non-Archimedean orthogonality, singular value decomposition, barcode, generalized boundary depth, egg-beater model.

# HAMILTONIAN DYNAMICS AND PERSISTENT HOMOLOGY

by

JUN ZHANG

M.A., University of Georgia, 2012

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2016

Hamiltonian dynamics and persistent homology

by

Jun Zhang

Approved:

Professor: Michael Usher

Committee: David Gay
William H. Kazez
Gordana Matic

Electronic Version Approved:

Suzanne Barbour
Dean of the Graduate School
The University of Georgia
August 2016

# Acknowledgement

First of all I would like to thank my supervisor Michael Usher for his great guidance during my Ph.D program. While writing this thesis, I am greatly indebted to very useful conversations with him. His patience and helpful suggestions have shaped this thesis more than what I ever expected.

Second, I would also like to thank professor Leonid Polterovich for his hospitality while I was visiting Tel Aviv University. Moreover, communication with him considerably improves my understanding of symplectic topology. Meanwhile, his academic suggestions provide many interesting and potential problems for my future career.

Third, I am grateful to the active research environment of Department of Mathematics in the University of Georgia (UGA). My academic growth benefits from numerous opportunities in UGA where I can communicate with different professors, make academic friends and also present my study and work. Meanwhile, in general, I am also grateful to many discussions with professor William Kazez, Dr. David Gay, professor Theodore Shifrin, Kenneth Jacobs, Daniel Rosen, Jean Gutt and Ergo Shelukhin.

Last but not least, without strong support from my parents, I can't go this far. So I thank them from the bottom of my heart.

# Contents

# Chapter 1

# Introduction

## 1.1 Main problem and main results

The Hofer distance between the set of autonomous Hamiltonian diffeomorphisms and a time-dependent Hamiltonian diffeomorphism has been recently studied in [PS14]. We first give several definitions. Denote by $Ham(X, \omega)$ the set of all the Hamiltonian diffeomorphisms. Not only can we prove $Ham(X, \omega)$ is a group, but also we can associate a bi-invariant metric on this group which is the well-known Hofer's metric, denoted as $d_H$. It will be defined in (4.1) in Section 4.2.

*Definition* 1.1.1. For a closed symplectic manifold $(X, \omega)$, define

$$\mathrm{Aut}(X) = \{\phi \in Ham(X, \omega) \,|\, \phi = \phi_H^1 \text{ where } H(t, x) \text{ is independent of } t\}.$$

*Definition* 1.1.2. Let $k \geq 2$. For a symplectic manifold $X$, define

$$\mathrm{aut}(X) = \sup_{\phi \in Ham(X,\omega)} d_H(\phi, \mathrm{Aut}(X)),$$

where $d_H$ is Hofer's metric defined by (4.1).

A special feature of time independent (or usually called autonomous) Hamiltonian diffeomorphism compared with time-dependent Hamiltonian comes from the following easy observation. If a time-dependent function $F(t,x)$, as a Hamiltonian, generates $\psi$, then

$$H(t,x) = pF(pt,x) \quad \text{generates} \quad \phi = \psi^p. \tag{1.1}$$

If, in particular, $\phi = \phi_H^1 \in \mathrm{Aut}(X)$, then for *any* prime $p$, we can simply take generating function $F(x) = \frac{1}{p}H(x)$ which generates a $p$-th root of $\phi$. This motivates another more delicate definition as follows.

*Definition* 1.1.3. Let prime $p \geq 2$. For a symplectic manifold $X$, define

$$Power_p(X) = \{\phi = \psi^p \,|\, \psi \in Ham(X)\}.$$

*Definition* 1.1.4. Let prime $p \geq 2$. For a symplectic manifold $X$, define

$$power_p(X) = \sup_{\phi \in Ham(X)} d_H(\phi, Power_p(X))$$

where $d_H$ is Hofer's metric defined by (4.1). [1]

As we have noticed earlier, $\mathrm{Aut}(X) \subset \bigcap_{p \text{ is prime}} Power_p(X)$. With the notations above, we can state the following main theorem in [PS14],

**Theorem 1.1.5.** *[Theorem 1.3 in [PS14]] Let $\Sigma_g$ be a fixed closed oriented surface with genus $g \geq 4$. For any symplectically aspherical closed manifold $M$ and for any $p \geq 2$. We have*

$$power_p(\Sigma_g \times M) = +\infty.$$

Note that Theorem 1.1.5 (taking the limit along all the prime number $p$) immediately implies, under the same hypothesis of Theorem 1.1.5,

$$aut(\Sigma_g \times M) = +\infty.$$

[1] The original definition in [PS14] is defined for any integer $k \geq 2$. But since

$$d_H(\phi, Power_p(X)) \leq d_H(\phi, Power_k(X)) \text{ when } p \,|\, k,$$

we will only consider prime number $p$ here.

This is actually another theorem stated in [PS14] (Theorem 1.2), where its original proof comes from different (easier) process than the original proof of Theorem 1.1.5. Last but not least, we emphasize that results mentioned above are within the effort to prove or understand the following general conjecture,

**Conjecture 1.1.6.** *For any closed symplectic manifold $X$,* $\mathrm{aut}(X) = +\infty$*.*

Here is our main theorems in this paper.

**Theorem 1.1.7.** *Let $\Sigma_g$ be a fixed closed oriented surface with genus $g \geq 4$. For any closed symplectic manifold $M$, for any prime $p > \sum_{0 \leq i \leq 2n} b_i(M)$,*

$$power_p(\Sigma_g \times M) = +\infty.$$

This immediately implies

**Theorem 1.1.8.** *Let $\Sigma_g$ be a fixed closed oriented surface with genus $g \geq 4$. For any closed symplectic manifold $M$,*

$$aut(\Sigma_g \times M) = +\infty.$$

So we are closer to the Conjecture 1.1.6 compared with results in [PS14]. Moreover, the method used in this paper, as a combination of Hamiltonian Floer theory and persistent homology theory, provides on the one hand, a sophisticated application of the non-Archimedean version of persistent homology developed in [UZ15] and on the other hand, a potentially useful scheme (see Section 4.1) to attack other Hamiltonian dynamics problems.

## 1.2 Review of Poltervoch-Shelukhin's method

Now back to Theorem 1.1.5, from the title of [PS14], its proof is a successful combination of Hamiltonian Floer theory (whose background will be given in Chapter 2) and persistent homology theory (whose background will be given in Chapter 3). For any given prime $p$, define an operator $R_p$ on the component of the loop space $\mathcal{L}_\alpha X$ consisting of all loops representing some homotopy class $\alpha$ by rotation

$$R_p(x(t)) = x\left(t + \frac{1}{p}\right).$$

Given a time-dependent Hamiltonian function $H(t,x)$ generating $\phi$, denote $H^{(p)}(t,x) = pH(pt,x)$ generating $\phi^p$ by (1.1). Notice if $x(t)$ is a Hamiltonian 1-periodic orbit of Hamiltonian $H^{(p)}$, then $R_p(x(t))$ is also a Hamiltonian 1-periodic orbit from $H^{(p)}$ because $p(t+1/p) = pt+1 = pt$ on $\mathbb{R}/\mathbb{Z}$. This induces a filtered chain isomorphism between two Floer chain complexes for each degree $k \in \mathbb{Z}$,

$$R_p : CF_k(H^{(p)}, J_t)_\alpha \to CF_k(H^{(p)}, (R_p)_* J_t)_\alpha = CF_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha. \tag{1.2}$$

After passing to the homology, taking advantage of the fact that Floer homology is independent of almost complex structure, we get a pair

$$(\mathbb{H}_k(\phi), T) \tag{1.3}$$

where components are

- $\mathbb{H}_k(\phi) = (\{HF_k^{(-\infty,s)}(\phi^p)_\alpha)\}_{s\in\mathbb{R}}; \phi_{s,t})$ is a persistence module (see Definition 3.2.1 below), where transition function $\phi_{s,t}$ is induced by inclusion for any $s < t$;
- $[(R_p)_*] = T$ is a filtered isomorphism (or 0-interleaving) giving a $\mathbb{Z}_p$ action on $\mathbb{H}_k(\phi)$, that is, $T^p = \mathbb{I}$.

A numerical measurement $\mu_p(\phi)$ is defined (on the top of page 40 in [PS14]) by using combinatorics data (barcode) from the associated persistence module (1.3) satisfying Lipschitz continuity with respect to Hofer's metric. Therefore, by *the* most important theorem in this theory — *stability theorem*, this proposition translates combinatorics information from barcode to analysis information between (Floer) chain complexes which is captured partially by Hofer's metric. This turns out to be the key step in the proof of Theorem 1.1.5. These Lipschitz type continuity will be carefully explain in Chapter 4.

To generalize this process, we will make efforts in two directions. First, we will use more sophisticated and powerful persistence module theory that is developed in the paper [UZ15] to rewrite the set-up of this problem in the (Floer) chain complex level. Second, following the idea above, we will also define some numerical measurement (in fact proved to be a symplectic invariant) which satisfies Lipschitz continuity with respect to Hofer's metric. We point out no effort will be made in this paper improving the product structure as in the result of Theorem 1.1.5.

*Remark* 1.2.1. Through out the paper [PS14], the field of scalars $\mathcal{K}$, should satisfies the following important restriction. As we also assume this condition in this paper, we state it separately here.
*Irreducible condition:*

- $char(\mathcal{K}) \neq p$ and $\mathcal{K}$ contains all $p$-th roots of unity;
- For any primitive $p$-th root of unity $\xi_p$, there is no solution of the following equation $x^p = \xi_p^q$ unless $p \,|\, q$.

Note that this condition gives a strong restriction on the dimension of invariant subspace. Specifically, if $V$ is a $\mathcal{T}$-invariant subspace with $\mathcal{T}^p = \xi_p^q \cdot \mathbb{I}$ for some $p$-th root of unity $\xi_p^q$ where $1 \leq q \leq p-1$, then $p \,|\, \dim(V)$. (Lemma 4.15 in [PS14]).

## 1.3 Outline of constructing obstruction

First, we give the recipe in this story to construct a numerical measurement (proved to be symplectic invariant).

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Floer chain complex}\\ (CF_*(H^{(p)}, J_t)_\alpha, T)\end{array}} \arrow[r, "(a)"] & \boxed{\begin{array}{c}\text{self-mapping cone}\\ (Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I})_*, \partial_{co})\end{array}} \arrow[r, "(b)"] & \boxed{\begin{array}{c}\text{barcode}\\ \{\beta_i(H)\}\end{array}} \arrow[dl, "(c)"] \\
 & \boxed{\text{numerical measurement } \mathfrak{o}_X(\phi_H)} \arrow[ul, "(d)"] &
\end{tikzcd} \tag{1.4}$$

We will explain each of the boxes followed by explanation of each of the arrows above. Meanwhile, we emphasize the speciality when $\phi = \phi_H \in Power_p(X)$ because eventually we will use the numerical measurement constructed above to form an obstruction to the condition that $\phi \in Power_p(X)$.

### 1.3.1 Floer chain complex

Rotation action (1.2) on the Floer chain complex can pushforward the almost complex structure that we start from, so once we are working on the Floer chain complex, the rotation action (1.2) does not behave as well as on the Floer homology in the sense that in order to work on the same chain complex, we need to use some continuation map $C$,

$$\begin{tikzcd} CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "R_p"] \arrow[rr, bend left, "T=C\circ R_p"] & CF_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \arrow[r, "C"] & CF_k(H^{(p)}, J_t)_\alpha \end{tikzcd}. \tag{1.5}$$

Note that in general, $T^p \neq \mathbb{I}$, which is the source of many difficulties when we are working on the chain complex level. Meanwhile, recall in the proof of Theorem 4.22 in [PS14], if $\phi \in Power_p(X)$, say $\phi = \psi^p$ for some $\psi \in Ham(X, \omega)$, then there exists a well-defined chain map for each degree $k \in \mathbb{Z}$,

$$R_{p^2} : CF_k(H^{(p)}, J_t)_\alpha \to CF_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha$$

where $H(t,x) = pF(pt,x)$ and $F$ is a Hamiltonian generating $\psi$. Again, in order to work on a single space itself, we also need to use some continuation map $C'$ to form the following composition

$$\begin{tikzcd} CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "R_{p^2}"] \arrow[rr, bend left, "S=C'\circ R_{p^2}"] & CF_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha \arrow[r, "C'"] & CF_k(H^{(p)}, J_t)_\alpha \end{tikzcd}. \tag{1.6}$$

Our first observation is

**Proposition 1.3.1.** *For any closed symplectic manifold $(X, \omega)$, if $\phi = \phi_H \in Power_p(X)$, then for any degree $k \in \mathbb{Z}$, there exists a continuation map $C : CF_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \to CF_k(H^{(p)}, J_t)_\alpha$ and a continuation map $C' : CF_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha \to CF_k(H^{(p)}, J_t)_\alpha$ such that, $T = S^p$, where $T$ and $S$ are compositions in (1.5) and (1.6).*

This result will be proved in Chapter 5, section 5.1.

*Notation* 1.3.2. For the rest of the paper, whenever we use $T$, it always means the composition defined in (1.5). If we need to emphasize the Hamiltonian $H$ of the corresponding system, we will denote it as $T^H$. Whenever we use $T_p$, it always means the resulting composition from Proposition 1.3.1 that has a $p$-th root.

### 1.3.2 Self-mapping cone

Define mapping cone of chain complex $CF_*(H^{(p)}, J_t)_\alpha$ with respect to map $T - \xi_p \cdot \mathbb{I}$. We call it a *self-mapping cone* of $(CF_*(H^{(p)}, J_t), \partial)$. Degree-$k$ piece is

$$(Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I}))_k = CF_k(H^{(p)}, J_t)_\alpha \oplus CF_{k-1}(H^{(p)}, J_t)_\alpha \tag{1.7}$$

and the boundary map $\partial_{co}$ is

$$\begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} \tag{1.8}$$

where $\partial$ is Floer boundary operator of $CF_*(H^{(p)}, J_t)$. Moreover,

*Definition* 1.3.3. For any element $(x_1, x_2) \in (Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I}))_*$, if $CF_*(H^{(p)}, J_t)_\alpha$ is acted by some map $A$, then define its *double map* $\mathcal{D}_A$ by

$$\mathcal{D}_A(x_1, x_2) = (Ax_1, Ax_2).$$

In particular, by (1.5) and (1.6), self-mapping cone is acted by double maps

$$\mathcal{D}_T = \mathcal{D}_{R_p} + C_T, \tag{1.9}$$

for some map $C_T$ who strictly lowers the filtration, and (if it exists) also

$$\mathcal{D}_S = \mathcal{D}_{R_{p^2}} + C_S, \tag{1.10}$$

for some map $C_S$ who also strictly lowers the filtration. Moreover, by Proposition 1.3.1, $\mathcal{D}_S^p = \mathcal{D}_T$. The following proposition shows our self-mapping cone is well-defined.

**Proposition 1.3.4.** *For any closed symplectic manifold $(X, \omega)$, up to a filtered isomorphism, construction of $(Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I}))_*, \partial_{co})$ is independent of choice of continuation map (to form* *map $T$). Moreover, $\mathcal{D}_T$ and $\mathcal{D}_S$ defined in (1.9) and (1.10) are chain maps on* $(Cone_{CF(H^{(p)},J_t)_\alpha}(T-\xi_p \cdot \mathbb{I}))_*, \partial_{co})$, i.e., *commute with boundary operator of mapping cone* $\partial_{co}$.

This will be proved in Chapter 5, section 5.2.

### 1.3.3 Barcode of self-mapping cone

From discussion in Section 3.4, we know these combinatorics data can reveal algebraic structures of the chain complex. For barcode of self-mapping cone defined above, a natural question is whether this "special" action $\mathcal{D}_S$ will shape its barcode in some way. In fact, we have the following important theorem.

**Theorem 1.3.5.** *For any degree $k \in \mathbb{Z}$ and boundary map $(\partial_{co})_{k+1} : (Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I}))_{k+1} \to \mathrm{Im}(\partial_{co})_{k+1}$ [2], each bar in the concise barcode of $(\partial_{co})_{k+1}$ (that is, degree-$k$ concise barcode) has its multiplicity divisible by $p$.*

This proposition is an analogue (but stronger) result with Proposition 4.18 in [PS14]. The proof of this theorem is the most time-consuming part of this paper. The entire Chapter 6 is devoted to its proof. But it should be easy to believe in that $\mathcal{D}_S$ is a strictly lower filtration perturbation of a group action $\mathcal{D}_{R_{p^2}}$ with order $p^2$, which makes each degree-$k$ piece $(Cone_{CF(H^{(p)},J_t)_\alpha}(T - \xi_p \cdot \mathbb{I}))_k$ a representation. This restricts the singular value decomposition (see Theorem 3.3.6) in a certain special form.

### 1.3.4 Numerical measurement

Based on Theorem 1.3.5, we can define some numerical measurement from this combinatorics data.

*Definition* 1.3.6. First take the collection of length of bar in degree-$k$ concise barcode of self-mapping cone constructed with respect to $\phi = \phi_H$, denoted as $\{\beta_i\}$ and by definition,

$$\beta_1(\phi) \geq \beta_2(\phi) \ldots \geq \beta_{m_k}(\phi) > 0$$

[2] In general, to compute (degree-$k$) barcode of $(\partial_{co})_{k+1}$, we need codomain to be $\ker(\partial_{co})_k$. But in this paper, we only consider Hamiltonian Floer chain complex of non-contractible loop, so it can be shown that homology of mapping cone vanishes. Therefore, $\ker(\partial_{co})_k = \mathrm{Im}(\partial_{co})_{k+1}$.

(so $m_k$ = multiplicity of degree-$k$ concise barcode). Then *degree-$k$ divisibility sensitive invariant* of $\phi = \phi_H$ is defined as

$$\mathfrak{o}_X(\phi)_k = \max_{s \in \mathbb{N}} \left(\beta_{sp+1}(\phi) - \beta_{(s+1)p}(\phi)\right)$$

and if $(s+1)p > m_k$, set $\beta_{(s+1)p}(\phi) = 0$. In general, divisibility sensitive invariant of $\phi = \phi_H$ is defined as

$$\mathfrak{o}_X(\phi) = \max_{\text{primitive } \xi_p} \sup_{k \in \mathbb{Z}} \mathfrak{o}_X(\phi)_k. \tag{1.11}$$

Likewise multiplicity sensitive spread $\mu(\phi)$ in [PS14], $\mathfrak{o}_X(\phi)$ is used to provide an obstruction to the condition $\phi \in Power_p(X)$. Specifically, we have the proposition,

**Proposition 1.3.7.** *If $\phi \in Power_p(X)$, $\mathfrak{o}_X(\phi) = 0$. If $p \nmid m_k$ for some degree $k$, then $\phi_X(\phi) \geq \beta_{m_k}(\phi)$.*

This will be proved in Chapter 5, section 5.3.

## 1.4 Lipschitz continuity

Now we move to the arrows in the diagram (1.4). All $(a)$, $(b)$ and $(c)$ are in the flavor of Lipschitz continuity (see Section 4.3). From now on, we will simply denote

$$Cone(H)_* := ((Cone_{CF(H^{(p)},J_t)_\alpha}(T^H - \xi_p \cdot \mathbb{I}))_*, \partial_{co,H}),$$

and

$$Cone(G)_* := ((Cone_{CF(G^{(p)},J_t)_\alpha}(T^G - \xi_p \cdot \mathbb{I}))_*, \partial_{co,G}),$$

the mapping cones constructed from different Hamiltonian functions $H$ and $G$. Then $(a)$ is corresponding to the following Lipschitz continuity between quasiequivalence distance and Hofer distance (see Section 4.2).

**Proposition 1.4.1.** *For any Hamiltonian $H$ and $G$, we have*

$$d_Q(Cone(H)_*, Cone(G)_*) \leq 3p \cdot ||H - G||_H.$$

Moreover, $(b)$ is corresponding to the following proposition which will be a direct application of Corollary 8.8 in [UZ15].

**Proposition 1.4.2.** *Denote $\beta_i(\phi_H)$ as the length of $i$-th bar in degree-$k$ verbose barcode of $Cone(H)_*$ and $\beta_i(\phi_G)$ as the length of $i$-th bar in degree-$k$ verbose barcode of $Cone(G)_*$. We have*

$$|\beta_i(\phi_H) - \beta_i(\phi_G)| \leq 4\, d_Q(Cone(H)_*, Cone(G)_*)$$

*for every $i \in \mathbb{Z}$.*

Note that $(a)$ and $(b)$ together imply the following proposition which corresponds to $(c)$,

**Proposition 1.4.3.** *For any closed symplectic manifold $(X, \omega)$ and $\phi, \psi \in Ham(X, \omega)$, we have*

$$|\mathfrak{o}_X(\phi) - \mathfrak{o}_X(\psi)| \leq 24p \cdot d_H(\phi, \psi).$$

*Remark* 1.4.4. Indeed, $\mathfrak{o}_X(\phi)$ defined here and $\mu(\phi)$ defined in [PS14] are similar but not completely related. On the one hand, we point out that $\mu(\phi)$ can also be defined in the Floer-Novikov persistent homology theory language developed in [UZ15] in the chain complex level. Unfortunately, (weak) stabilization proposition - Theorem 4.23 in [PS14] (with *corrected version*) can be modified to hold for general symplectic manifold $M$, but it can not be applied in the same way as in [PS14], especially when $c_1(M)$ is not zero. On the other hand, in some special cases (for instance, $\Gamma$ is dense), interested reader can verify there exist positive constants $C_1$ and $C_2$ both depending on $p$ such that $C_1\mu(\phi) \leq \mathfrak{o}_X(\phi) \leq C_2\mu(\phi)$.

All these Lipschitz type results will be proved in Chapter 7. Finally, $(d)$ combines all the results together giving the following intermediate theorem that reflects the essential part of the argument for the proof of our main theorem in Section 1.1.

**Theorem 1.4.5.** *Let $(X, \omega)$ be a closed symplectic manifold. Suppose there exists a Hamiltonian diffeomorphism $\phi = \phi_H$ such that for some $k \in \mathbb{Z}$, $p \nmid m_k$ where $m_k$ is the multiplicity of degree-$k$*

*concise barcode of self-mapping cone $Cone(H)_*$. We have*

$$power_p(X) \geq \frac{1}{24p}\beta_{m_k}(\phi_H).$$

*Proof.* For any given $\epsilon > 0$, we have

$$\begin{aligned}
power_p(X) + \epsilon &= \sup_{\phi \in Ham(X,\omega)} d_H(\phi, Power_p(X)) + \epsilon \\
&\geq d_H(\phi_H, Power_p(X)) + \epsilon && \text{by Definition 1.4} \\
&\geq d_H(\phi_H, \psi) && \text{for some } \psi \in Power_p(X) \\
&\geq \frac{1}{24p}|\mathfrak{o}_X(\phi_H) - \mathfrak{o}_X(\psi)| && \text{by Proposition 1.4.2} \\
&= \frac{1}{24p}\mathfrak{o}_X(\phi_H) && \text{by Proposition 1.3.7} \\
&\geq \frac{1}{24p}\beta_{m_k}(\phi_H). && \text{by Proposition 1.3.7}
\end{aligned}$$

Since $\epsilon$ is arbitrarily chosen, we get the conclusion. □

## 1.5 Egg-beater model and product structure

From the argument above, we notice that in order to prove Conjecture 1.1.6, for a given symplectic manifold $X$, we should be able to create the following two situations:

(i) find a family of $\phi_\lambda = \phi^1_{H_\lambda} \in Ham(X,\omega)$ such that $\beta_{m_k}(\phi_\lambda) \to \infty$ as $\lambda \to \infty$;

(ii) control the non-divisibility (by $p$) of multiplicity of concise barcode of $Cone(H_\lambda)_*$;

In general, this might be difficult, especially for condition (i). Our main theorem indicates we can do these on $X = \Sigma_g \times M$ for any closed symplectic manifold $M$, which is the key to succeed in generalizing results from [PS14]. Here we take advantage of a chaotic model called "egg-beater model" which has been carefully studied in [PS14]. A brief introduction of this is needed.

### 1.5.1 Egg-beater model

Egg-beater model $(\Sigma_g, \phi)$ with $g \geq 4$ is constructed to create large action gap. On $\Sigma_g$, we will focus on a pair of annuli intersecting each other in a way such that in each component separated by the annuli there exists some genus (see Figure 2 in [PS14]) . Moreover, our Hamiltonian dynamics comes from a family of shear flow $\phi_\lambda^t$ (highly degenerated) generated by a family of special Hamiltonian functions supported only on the union of these annuli (see Figure 3 in [PS14]). The upshot is that we have a well-defined Floer chain complex (for non-contractible loop), denoted as

$$CF_*(\Sigma_g, \phi_\lambda)_\alpha \tag{1.12}$$

where $\alpha$ represents a homotopy class of a non-contractible loop and $\phi_\lambda$ is a family of Hamiltonian diffeomorphism parametrized by sufficiently large $\lambda$ (with family of generating Hamiltonians denoted as $H_\lambda$). The generator of this chain complex is non-contractible Hamiltonian orbits which can be identified with fixed point and interestingly, by the construction of $\phi_\lambda$, there are exactly $2^{2p}$-many generators coming in $p$-tuple in the sense that if $z$ is a non-degenerate fixed point, then each one from the following cyclic permutation

$$\{z, \phi_\lambda z, ..., \phi_\lambda^{p-1} z\} \tag{1.13}$$

is also a non-degenerate fixed point. Moreover, actions and indices on $\phi_\lambda^j z$ are the same for all $j \in \{0, .., p-1\}$. Rotation action $R_p$ acts on generators as $R_p(\phi_\lambda^j z) = \phi_\lambda^{j+1 \bmod p} z$.

This model itself provides an example that we can carry on explicit computation concerning (i) and (ii) mentioned above. On the one hand, Proposition 5.1 in [PS14] confirms the asymptotic (to infinity) behavior of action gap required in (i). On the other hand, we have the following proposition handling the other issue on multiplicity (and so divisibiity) in (ii),

**Proposition 1.5.1.** *For any given prime number $p \geq 3$, total multiplicity of concise barcode of self-mapping cone of $CF_*(\Sigma_g, \phi_\lambda)$ is $2^{2p}$. In particular, there exists some degree $k$ such that $p \nmid m_k$ where $m_k$ is the multiplicity of degree-$k$ concise barcode.*

This will be proved in Chapter 8, section 8.3.

### 1.5.2 Product structure

First, for the corresponding product Floer chain complex,

$$CF_*(\Sigma_g \times M, \phi_\lambda \times \mathbb{I})_{\alpha \times \{pt\}} = CF_*(\Sigma_g, \phi_\lambda)_\alpha \otimes CF_*(M, \mathbb{I})_{\{pt\}}$$

by the recipe (1.4) above, we will consider degree-1 concise barcode of self-mapping cone

$$Cone_\otimes(H_\lambda)_* := (Cone_{CF_*(\Sigma_g \times M, \phi_\lambda \times \mathbb{I})_{\alpha \times \{pt\}}}(T^{H_\lambda} \times \mathbb{I} - \xi_p \cdot \mathbb{I})_*, \partial_{co}) \tag{1.14}$$

Study of barcode under product structure implies the following two propositions, in Section 9.2.

**Proposition 1.5.2.** *For any $i \in \mathbb{Z}$, length of $i$-th bar in degree-1 concise barcode of $Cone_\otimes(H_\lambda)_*$ satisfies*

$$\beta_i(\phi_\lambda \times \mathbb{I}) \to \infty \quad as \quad \lambda \to \infty$$

*for any $i \leq m_1$ where $m_1$ is the multiplicity of degree-1 concise barcode.*

By referring to CZ-index formula of generator of the egg-beater model (see Theorem 5.2 in [AKKKPRRSSZ15]) and the following definition,

*Definition* 1.5.3. Define *$k$-th quantum Betti number* of symplectic manifold $X$ as $qb_k(X)$ by

$$qb_k(X) = \sum_{s \in \mathbb{Z}} b_{k+2Ns}(X)$$

where $b_{k+2Ns}(X)$ is the classical $(k+2Ns)$-th Betti number of $M$ and $N$ is minimal Chern number of $M$. Note when $N$ is sufficiently large, for instance, $c_1(TX) = 0$, $qb_k(X) = b_k(X)$.

we can show,

**Proposition 1.5.4.** *Let $X = \Sigma_g \times M$. Denote $m_1$ as multiplicity of degree-1 concise barcode of $Cone_\otimes(H_\lambda)_*$, if*

$$p \nmid (qb_p(X) + 2qb_0(X) + qb_{-p}(X)) \tag{1.15}$$

*then $p \nmid m_1$. In particular, if $p > \sum_{i=0}^{\dim(X)} b_i(X)$, then (1.15) is always satisfied.*

# Chapter 2

# Background of Floer theory

## 2.1 Overview

For a closed symplectic manifold $(X, \omega)$, a smooth Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times X \to \mathbb{R}$ gives rise to a Hamiltonian flow $\{\phi_H^t\}_{0 \leq t \leq 1}$ and the fixed points of the time-1 map $\phi_H^1$ (called a Hamiltonian diffeomorphism) can be used to construct a chain complex called the Floer chain complex, denoted as $CF_*(H, J)$. Floer theory is motivated to resolve Arnold's conjecture on the minimal number of the fixed point of a Hamiltonian diffeomorphism, which has been fully developed (see, [Flo89], [HS95], [FO99] and [Par13]). By now, various applications of Floer theory together with its extended version involving Lagrangians (see [FOOO09]) have shaped people's fundamental understanding of symplectic or contact structures. In particular, many symplectic invariants constructed from $CF_*(H, J)$ or its homology $HF_*(X)$, such as spectral invariant $\rho(a, H)$ (see, [Vit92], [Sch00], [Oh05] and [Ush08]), boundary depth $\beta(\phi_H)$ (see, [Ush13]), symplectic quasi-state $\zeta_a(H)$ (see, [EP08]), etc., have been successfully used to solve many problems on Hamiltonian dynamics (see, [Sey13], [HLS15]) as well as some rigidity type topological problem (see, [EP09], [Ush10]).

## 2.2 Construction of Hamiltonian Floer chain complex

Suppose $(X, \omega)$ is a close connected symplectic manifold. Given any smooth Hamiltonian function $H : \mathbb{R}/\mathbb{Z} \times X \to \mathbb{R}$, it induces a time-dependent Hamiltonian vector field $X_H$ by taking advantage of non-degeneracy of $\omega$, i.e.,

$$\omega(\cdot, X_H) = d(H(t, \cdot)).$$

This vector field $X_H$ induces a flow denoted as $\phi_H^t$. Fixed point of time-1 map $\phi_H^1$ is corresponding to a loop $\gamma : \mathbb{R}/\mathbb{Z} \to X$ such that $\gamma(t) = \phi_H^t(\gamma(0))$. We say that $H$ is *non-degenerate* if for each such loop $\gamma$,

$$(d\phi_H^1)_{\gamma(0)} : T_{\gamma(0)}X \to T_{\gamma(0)}X$$

has all eigenvalues distinct from 1. This will guarantee that there are only finitely many fixed point of $\phi_H^1$. In terms of constructing Floer chain complex, we will not use all the loops. There are two different versions. One is to use all the contractible loops. Denote the collection of all such loops by $\mathcal{L}(X)$ (or $\mathcal{L}_{\{pt\}}(X)$ if necessary). The other is to use non-contractible loops in a fixed homotopy class $\alpha \in \pi_1(M)$. Denote the collection of all such loops by $\mathcal{L}_\alpha(X)$.

We will explicitly explain the construction of the first one. The construction of the second one is similar. View $\gamma$ as a boundary of an embedded disk $D^2$ in $X$, i.e., there is a map $u : D^2 \to M$ and $u|_{S^1} = \gamma$. Now consider a covering space of $\mathcal{L}(X)$, denoted as $\widetilde{\mathcal{L}(X)}$ constructed by

$$\widetilde{\mathcal{L}(X)} = \left\{ \begin{array}{c|c} \text{equivalent class } [\gamma, u] & (\gamma, u) \text{ is equivalent to } (\tau, v) \Longleftrightarrow \\ \text{of pair } (\gamma, u) & \gamma(t) = \tau(t) \text{ and } [u\#(-v)] \in \ker([\omega]) \cap \ker(c_1) \end{array} \right\}.$$

For each $[\gamma, u] \in \widetilde{\mathcal{L}(X)}$, there are two functions associated to it. One is *action functional* $\mathcal{A}_H : \widetilde{\mathcal{L}(X)} \to \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma, u]) = -\int_{D^2} u^*\omega + \int_0^1 H(t, \gamma(t))dt.$$

The other one is Conley-Zehnder index $\mu_{CZ} : \widetilde{\mathcal{L}(X)} \to \mathbb{Z}$ defined by, roughly speaking, counting rotation of $d\phi_H^t$ on along $\gamma(t)$ with the help of trivialization induced by $u$. Its explicit definition can be referred to [RS93]. Because of the conditions in $\widetilde{\mathcal{L}(X)}$ above, action functional and Conley-Zehnder index (or CZ-index) of $[\gamma, u]$ are both well-defined. As a vector space over ground field $\mathcal{K}$, for any $k \in \mathbb{Z}$, degree-$k$ part of Floer chain complex $CF_k(H, J)$ (or $CF_k(H, J)_{\{pt\}}$ if necessary), as a vector space is defined as

$$\left\{ \sum_{\substack{[\gamma,u]\in\widetilde{\mathcal{L}(X)},\\ \mu_{CZ}([\gamma,u])=k}} a_{[\gamma,u]}[\gamma,u] \,\middle|\, a_{[\gamma,u]}\in\mathcal{K}, (\forall C\in\mathbb{R})(\#\{[\gamma,u]|a_{[\gamma,u]}\neq 0,\, \mathcal{A}_H([\gamma,u])>C\}<\infty) \right\}.$$

Now denote

$$s_0 = \left\{ w\colon\ S^2\to X \,|\, \langle c_1(TX), w_*[S^2]\rangle = 0 \right\}.$$

Note that if we change $[\gamma,u]$ by gluing some sphere $w\in s_0$ on the capping $u$, it will change the action functional by $-\int_{S^2} w^*\omega$, possibly not zero, but keep the degree the same. More importantly, if such action difference is non-zero, then $[\gamma,u]\neq[\gamma,u\#w]$ in $CF_k(H,J)$, which implies that as a vector space over $\mathcal{K}$, $CF_k(H,J)$ is in general infinitely dimensional. It is finite dimensional if $\omega$ vanishes on the image of Hurewicz map $i:\pi_2(X)\to H_2(X,\mathbb{Z})/\mathrm{Torsion}$, in which case $X$ is usually called *weakly exact* (a stronger condition called *symplectic aspherical* if both $\omega$ and $c_1$ vanishes).

In order to overcome this dimension issue, [HS95] suggests to consider a bigger coefficient field - Novikov field $\Lambda^{\mathcal{K},\Gamma}$ defined as

$$\Lambda^{\mathcal{K},\Gamma} = \left\{ \sum_{g\in\Gamma} a_g t^g \,|a_g\in\mathcal{K}, (\forall C\in\mathbb{R})(\#\{g|a_g\neq 0,\, g<C\}<\infty) \right\}$$

where $\Gamma = \left\{\int_{S^2} w^*\omega \,|\, w\in s_0\right\} \leq \mathbb{R}$ and $t$ is a formal variable. It is then easy to check that $CF_k(H,J)$ is now a finite dimensional vector space over $\Lambda^{\mathcal{K},\Gamma}$ and its dimension is equal to the number of $\gamma\in\mathcal{L}(X)$ such that there is $u:D^2\to X$ with $u|_{\partial D^2}=\gamma$ and $\mu_{CZ}([\gamma,u])=k$.

Next, graded vector space $CF_*(H,J)$ will become a chain complex once we define the (Floer) boundary operator $(\partial_{H,J})_*$. Degree-$k$ part of boundary operator $(\partial_{H,J})_k : CF_k(H,J)\to CF_{k-1}(H,J)$ is defined by counting the solution (modulo $\mathbb{R}$-translation) of the following partial differential equation (as a formal negative gradient flow of $\mathcal{A}_H$)

$$\frac{\partial u}{\partial s} + J_t(u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t,u(s,t))\right) = 0, \tag{2.1}$$

where $\{J_t\}_{0\leq t\leq 1}$ is a family of almost complex structure compatible with $\omega$ and $u : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to X$ such that

- $u$ has finite energy $E(u) = \int_{\mathbb{R}\times\mathbb{R}/\mathbb{Z}} \left|\frac{\partial u}{\partial s}\right|^2 dtds$;
- $u$ has asymptotic condition $u(s,\cdot) \to \gamma_\pm(\cdot)$ as $s \to \pm\infty$;
- $\mu_{CZ}([\gamma_-, w_-]) - \mu_{CZ}([\gamma_+, w_+]) = 1$ and $[\gamma_+, w_+] = [\gamma_+, w_-\#u]$.

By now, it is a deep but well-known and standard fact that

**Theorem 2.2.1** (See [HS95] for semi-positive $X$ or see [Par13] for general $X$)**.** $(\partial_{H,J})_*$ *is well-defined such that* $\partial_{H,J}\circ\partial_{H,J} = 0$.

## 2.3 Continuation map

The construction of Hamiltonian Floer chain complex $(CF_*(H,J), (\partial_{H,J})_*)$ clearly depends on the pair $(H,J)$. We will now recall the relation between two such chain complexes if they are constructed from different $(H,J)$. Specifically, if we have $(H_-,J_-)$ and $(H_+,J_+)$, the standard way is to form a (regular) homotopy $(\mathcal{H},\mathcal{J})$ (with homotopy parameter $s\in\mathbb{R}$) between them so that when $s \ll 0$, $(\mathcal{H}_s,\mathcal{J}_s) = (H_-,J_-)$ and when $s \gg 0$, $(\mathcal{H}_s,\mathcal{J}_s) = (H_+,J_+)$. For instance, we can use a cut-off function $\alpha(s)$, that is,

- $\alpha(s)$ is monotone increasing,
- $\alpha(s) = 0$ when $s \ll 0$ and $\alpha(s) = 1$ when $s \gg 0$,

to form the following homotopy,

$$\mathcal{H}_s(t,\cdot) = (1-\alpha(s))H_-(t,\cdot) + \alpha(s)(H_+(t,\cdot)),$$

and it is similar to consider a $\mathcal{J}_s$. The upshot is there exists a well-defined chain map (where it is a chain map by standard Floer gluing argument, see proof of Theorem 11.1.15 in [AD14]), usually called *continuation map* $\Phi_{(\mathcal{H},\mathcal{J})} : CF_*(H_-,J_-) \to CF_*(H_+,J_+)$ constructed (similar to (2.1)) by counting the solution of the following parametrized partial differential equation

$$\frac{\partial u}{\partial s} + (J_t)_s(u(s,t)) \left( \frac{\partial u}{\partial t} - X_{H_s}(t, u(s,t)) \right) = 0, \tag{2.2}$$

where again $u$ is required to satisfy certain conditions as above as for the defining properties of boundary operator except here we require $\mu_{CZ}([\gamma_-, u_-]) = \mu_{CZ}([\gamma_+, u_+])$. Here we emphasize that if we use another homotopy $(\mathcal{H}', \mathcal{J}')$, the same construction will give another chain map $\Phi_{(\mathcal{H}', \mathcal{J}')} : CF_*(H_-, J_-) \to CF_*(H_+, J_+)$. These two chain maps are actually chain homotopic to each other, induced by a "1-homotopy" (homotopy of a homotopy) between $(\mathcal{H}, \mathcal{J})$ and $(\mathcal{H}', \mathcal{J}')$. In other words, there exists a degree-1 map $K : CF_*(H_-, J_-) \to CF_{*+1}(H_+, J_+)$ such that

$$\Phi_{(\mathcal{H}, \mathcal{J})} - \Phi_{(\mathcal{H}', \mathcal{J}')} = \partial \circ K + K \circ \partial. \tag{2.3}$$

The explicit construction of $K$ is carried out in Lemma 6.3 in [Sal90]. Now using another homotopy $(\tilde{\mathcal{H}}, \tilde{\mathcal{J}})$ from $(H_+, J_+)$ to $(H_-, J_-)$ gives a well-defined chain map $\Phi_{(\tilde{\mathcal{H}}, \tilde{\mathcal{J}})} : CF_*(H_+, J_+) \to CF_*(H_-, J_-)$. Together, we have the following picture,

$$\begin{tikzcd} CF_*(H_-, J_-) \arrow[r, "\Phi_{(\mathcal{H}, \mathcal{J})}"] \arrow[rr, bend left, "\mathbb{I}"] & CF_*(H_+, J_+) \arrow[r, "\Phi_{(\tilde{\mathcal{H}}, \tilde{\mathcal{J}})}"] & CF_*(H_-, J_-) \end{tikzcd} \tag{2.4}$$

where identity map $\mathbb{I}$ can be regarded as the induced chain map by the obvious constant homotopy $(\mathcal{H}_{const}, \mathcal{J}_{const})$ between $(H_-, J_-)$ and itself. On the one hand, the well-known gluing argument (see Chapter 10 in [MS04] or B.10 in [Par13]) implies that

$$\Phi_{(\tilde{\mathcal{H}}, \tilde{\mathcal{J}})} \circ \Phi_{(\mathcal{H}, \mathcal{J})} = \Phi_{(\mathcal{H}_R, \mathcal{J}_R)} \tag{2.5}$$

where the right hand side is an induced chain map from a "gluing" homotopy $(\mathcal{H}_R, \mathcal{J}_R)$ (for some $R \gg 0$) from $(H_-, J_-)$ to itself constructed from $(\mathcal{H}, \mathcal{J})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{J}})$. On the other hand, it is also well-known (see explicit construction in [Ush11, p.14]) that the resulting $\Phi_{(\mathcal{H}_R, \mathcal{J}_R)}$ is chain homotopic to $\mathbb{I}$, that is, there exists a degree-1 map $K_- : CF_*(H_-, J_-) \to CF_{*+1}(H_-, J_-)$ such that

$$\Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})} \circ \Phi_{(\mathcal{H},\mathcal{J})} - \mathbb{I} = \partial_{H_-,J_-} \circ K_- + K_- \circ \partial_{H_-,J_-}. \tag{2.6}$$

Similarly, there exists a degree-1 map $K_+ : CF_*(H_+, J_+) \to CF_{*+1}(H_+, J_+)$ such that

$$\Phi_{(\mathcal{H},\mathcal{J})} \circ \Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})} - \mathbb{I} = \partial_{H_+,J_+} \circ K_+ + K_+ \circ \partial_{H_+,J_+}. \tag{2.7}$$

## 2.4 Hamiltonian Floer homology

In terms of algebraic structure, different choices of pair $(H, J)$ induce the same (up to isomorphism) Floer homology, therefore, we are allowed to define

$$HF_*(X) := H_*(CF(H, J), \partial_{H,J}).$$

Therefore, we can compute $HF_*(X)$ by choosing a preferred Hamiltonian function $H$. In most cases, we will choose a $C^2$-small $H$ so the Hamiltonian orbits of $H$ will be degenerated to critical points, which makes the corresponding analysis much easier. Moreover, we have

**Theorem 2.4.1** (See Theorem 6.1 in [HS95] for semi-positive $X$ or see Theorem 10.7.1 in [Par13] for general $X$)**.** *For any degree $k \in \mathbb{Z}$,*

$$HF_k(X) \simeq \bigoplus_{j=k \bmod 2N} H_j(X, \mathcal{K}) \otimes_{\mathcal{K}} \Lambda^{\mathcal{K},\Gamma}.$$

*In particular, when $k = 0$ or $\dim(X)$, $HF_k(X) \neq 0$.*

We will close this section by put a remark on the Hamiltonian Floer theory with respect to non-contractible loop (represented by homotopy class $\alpha$). We will denote the corresponding Floer chain complex as $CF_*(H, J)_\alpha$ and Floer homology as $HF_*(X)_\alpha$. Almost all the ingredients above can be defined and constructed in a parallel way by starting from a covering space of $\mathcal{L}_\alpha(X)$ once we fixed a reference non-contractible loop in the homotopy class $\alpha$. The general construction has been carried out explicitly in Section 5 in [Ush13]. What we want to emphasize is that Theorem 2.4.1 is not true for $HF_*(X)_\alpha$. In fact, we can readily show

**Theorem 2.4.2.** *$HF_*(X)_\alpha = 0$ if $\alpha$ is non-contractible.*

Indeed, since $HF_*(X)$ is independent of Hamiltonian $H$. A $C^2$-small $H$ will only provide critical points (as constant periodic orbits, so never in class $\alpha$), which makes $HF_*(X)_\alpha$ has no generators.

## 2.5 Non-Archimedean normed vector space

One perspective that makes Hamiltonian Floer chain complex distinguished from a general chain complex is that with the help of action functional $\mathcal{A}_H$, for each degree $k \in \mathbb{Z}$, we can turn each degree-$k$ piece $CF_k(H, J)$ into a (finite dimensional) non-Archimedean normed vector space over $\Lambda^{\mathcal{K},\Gamma}$. In general, a non-Archimedean normed vector space is defined as follows. First, recall

*Definition* 2.5.1. A *valuation* $\nu$ on a field $\mathcal{F}$ is a function $\nu : \mathcal{F} \to \mathbb{R} \cup \{\infty\}$ such that

(V1) $\nu(x) = \infty$ if and only if $x = 0$;

(V2) For any $x, y \in \mathcal{F}$, $\nu(xy) = \nu(x) + \nu(y)$;

(V3) For any $x, y \in \mathcal{F}$, $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ with equality when $\nu(x) = \nu(y)$.

Moreover, we call a valuation $\nu$ *trivial* if $\nu(x) = 0$ for $x \neq 0$ and (still) $\nu(x) = \infty$ precisely when $x = 0$.

In particular, for $\mathcal{F} = \Lambda^{\mathcal{K},\Gamma}$, we can associate a valuation simply by

$$\nu\left(\textstyle\sum_{g\in\Gamma} a_g t^g\right) = \min\{g \mid a_g \neq 0\}$$

where we use the standard convention that the minimum of the empty set is $\infty$. It is easy to see that this $\nu$ satisfies conditions (V1), (V2) and (V3). Note that the finiteness condition in the definition of Novikov field ensures that the minimum exists. If $\Gamma = \{0\}$, then the valuation $\nu$ is trivial.

*Definition* 2.5.2. A *non-Archimedean normed vector space* over $\mathcal{F}$ with filtration $\nu$ is a pair $(C, \ell)$ where $C$ is a $\mathcal{F}$-vector space endowed with a filtration function $\ell : C \to \mathbb{R} \cup \{-\infty\}$ satisfying the following axioms:

(F1) $\ell(x) = -\infty$ if and only if $x = 0$;

(F2) For any $\lambda \in \mathcal{F}$ and $x \in C$, $\ell(\lambda x) = \ell(x) - \nu(\lambda)$;

(F3) For any $x, y \in C$, $\ell(x + y) \leq \max\{\ell(x), \ell(y)\}$.

In terms of Definition 2.5.2, the standard convention would be that the norm on a non-Archimedean normed vector space $(C, \ell)$ is $e^{-\ell}$, not $\ell$. For the entire paper, we will only focus on the function $\ell$, not on the norm $e^{\ell}$.

**Example 2.5.3.** *For each $k \in \mathbb{Z}$, $(CF_k(H, J), \ell_H)$ is a non-Archimedean normed vector space with associated filtration function*

$$\ell_H \left( \sum a_{[\gamma,u]}[\gamma, u] \right) = \max\{\mathcal{A}_H([\gamma, u]) \mid a_{[\gamma,u]} \neq 0\}.$$

*Note that $\ell_H$ is defined for all degree $k \in \mathbb{Z}$.*

Both boundary operator $\partial_{H,J}$ defined in Section 2.2 and continuation map $\Phi_{H,J}$ described in Section 2.3 have relations with $\ell_H$ by the following two theorems. For boundary operator,

**Theorem 2.5.4.** *There exists some $\hbar > 0$ (coming from Gromov-Floer compactness theorem) such that for each $c \in CF_*(H, J)$, $\ell_H(\partial_{H,J}\, c) \leq \ell_H(c) - \hbar$.*

and for continuation map,

**Theorem 2.5.5.** *Suppose $\Phi_{(\mathcal{H},\mathcal{J})} : CF_*(H_-, J_-) \to CF_*(H_+, J_+)$ is a continuation map constructed in Section 2.3. For any $c \in CF_*(H_-, J_-)$, we have*

$$\ell_{H_+}(\Phi_{(\mathcal{H},\mathcal{J})}\, c) \leq \ell_{H_-}(c) + \int_0^1 \max_X (H_+(t, \cdot) - H_-(t, \cdot))dt. \tag{2.8}$$

There are a couple of results directly from Theorem 2.5.5.

(a) There exists a symmetric inequality coming from the converse continuation map $\Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})}$, that is, for any $c \in CF_*(H_+, J_+)$,

$$\ell_{H_-}(\Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})}\, c) \leq \ell_{H_+}(c) + \int_0^1 -\min_X (H_+(t, \cdot) - H_-(t, \cdot))dt. \tag{2.9}$$

(b) By (2.8) and (2.9), we have

$$\ell_{H_-}((\Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})} \circ \Phi_{(\mathcal{H},\mathcal{J})})c) \leq \ell_{H_-}(c) + \int_0^1 \max_X(H_+(t,\cdot) - H_-(t,\cdot)) - \min_X(H_+(t,\cdot) - H_-(t,\cdot))dt.$$

Meanwhile, by (2.6),

$$\ell_{H_-}((\Phi_{(\tilde{\mathcal{H}},\tilde{\mathcal{J}})} \circ \Phi_{(\mathcal{H},\mathcal{J})})c) = \ell_{H_-}(c + (\partial_{H_-,J_-} \circ K_- + K_- \circ \partial_{H_-,J_-})(c)).$$

So $K_-$ will shift filtration at most by the value

$$\int_0^1 \max_X(H_+(t,\cdot) - H_-(t,\cdot)) - \min_X(H_+(t,\cdot) - H_-(t,\cdot))dt. \tag{2.10}$$

The same conclusion holds for $K_+ : CF_*(H_+, J_+) \to CF_{*+1}(H_+, J_+)$.

## 2.6 Filtered Floer homology

By Theorem 2.5.4, for any $\lambda \in \mathbb{R}$, if denote

$$CF_*^{\lambda}(H,J) := CF_*^{(-\infty,\lambda)}(H,J) = \{c \in CF_*(H,J) \,|\, \ell_H(c) \leq \lambda\},$$

then $(CF_*^{\lambda}(H,J), \partial_{H,J})$ is a subcomplex of $CF_*(H,J)$. Therefore, we can define a *filtered Floer homology* (which can be proved that it is independent of almost complex structure $J$),

$$HF_*^{\lambda}(H) := \text{homology of } (CF_*^{\lambda}(H,J), \partial_{H,J}). \tag{2.11}$$

As this is defined for every $\lambda \in \mathbb{R}$, we get a $\mathbb{R}$-family of vector spaces $\{HF^{\lambda}(H)\}_{\lambda\in\mathbb{R}}$. More importantly, this $\mathbb{R}$-family of vector spaces is endowed with an additional structure. Note that for any $\lambda < \eta \in \mathbb{R}$, there exists an inclusion $\iota_{\lambda,\eta} : CF_*^{\lambda}(H,J) \hookrightarrow CF_*^{\eta}(H,J)$, therefore, it induces a morphism on the corresponding homologies, that is

$$(\iota_{\lambda,\eta})_* : HF_*^{\lambda}(H) \to HF_*^{\eta}(H) \tag{2.12}$$

where collection of maps $\{(\iota_{\lambda,\eta})_*\}_{\lambda,\eta\in\mathbb{R}}$ satisfies

- $(\iota_{\lambda,\lambda})_* = \mathbb{I}_{HF_*^\lambda(H)}$,
- $(\iota_{\lambda,\eta})_* \circ (\iota_{\rho,\lambda})_* = (\iota_{\rho,\eta})_*$.

*Remark* 2.6.1. In genera, a filtered Floer homology $HF_*^\lambda(H)$ is a $\mathcal{K}$-vector space, not a $\Lambda^{\mathcal{K},\Gamma}$-vector space (or at best a $\Lambda^{\mathcal{K},\Gamma_{\geq 0}}$-module) because the action of Novikov field does not preserve filtrations. Therefore, $HF_*^\lambda(H)$ is in general an infinite dimensional $\mathcal{K}$-vector space unless we only work over $\mathcal{K} = \Lambda^{\mathcal{K},0}$ (that is $\Gamma = 0$), for instance, on an aspherical symplectic manifold.

# Chapter 3

# Background of persistent homology

## 3.1 Overview

Persistent homology theory was first introduced with the goal of topological data analysis (see, [Car09]). Since then, many different versions of its associated algebraic framework have been developed and generalized (see, [BD13], [CCGGO09]). Briefly speaking, the basic idea is, for a filtered chain complex $(C_*, \partial_*, \ell)$ where $\ell$ is a filtration function, to consider a family of its homology $\{H_*^{(-\infty,t)}(C)\}_{t\in\mathbb{R}}$ (called persistence module) and trace the "birth" and "death" time for each generator along the time parameter $t$. It provides much more information than the usual homology in the sense that it can also detect when and where a generator of $\mathrm{Im}(\partial_*)$ representing a non-zero homology class becomes trivial in *filtered/truncated* homologies. There are two beautiful related theoretic results. First, the persistence module forms the easiest model of quiver representation (see, [DW05]), therefore, Gabriel's theorem classifies and decomposes it into direct sum of irreducible quivers, which is called a barcode in persistent homology theory and is identified with a collection of intervals. Moreover, in this special case, this abstract classification process can also be realized in an algorithmic way (see, [CZ05]). Second, using algebraic chain maps, a metric called the interleaving distance $d_P$ can be defined between two persistence modules while, using combinatorics data, another metric called bottleneck distance $d_B$ (see, Section 4.2 in [CdSGO12]) can also be defined between two barcodes. The most important theorem so far in persistent homology theory is Stability Theorem (see, Theorem 4.10 in [CdSGO12]) linking these two concepts together, saying $d_P = d_B$.

Therefore, to some extent, algebraic topology problems can be transferred into a combinatorial problems.

## 3.2 Classical persistent homology

The topological idea of persistent homology comes from computing (Morse) homology of sublevel submanifold (instead of the entire manifold). It results in a special algebraic structure called *persistence module*. Discussion of filtered Floer homology in Section 2.6 provides a concrete example of a persistence module. In general, it is defined as follows.

*Definition* 3.2.1. Define a *persistence module* as

$$\mathbb{V} = (\{V_t\}_{t\in\mathbb{R}}, \{\phi_{s,t}\})$$

where each $V_t$ is a module (over some algebra) and for any $s \leq t \in \mathbb{R}$, $\phi_{s,t} : V_s \to V_t$ is a transition map is the sense that if $s = t$, then $\phi_{s,s} = \mathbb{I}_{V_s}$ and if $s < t < r$, then $\phi_{t,r} \circ \phi_{s,t} = \phi_{s,r}$.

Similar to classification of finitely generated abelian group, for any algebraic object, we want to make it as simple as possible and as unique as possible. In other words, we want to find building blocks for a possible decomposition (and therefore a classification). This is demonstrated by the following example and the structure theorem next to it.

**Example 3.2.2.** *For any interval $[a, b)$ (with possibly $b = \infty$), associate an "easiest persistence module" $\mathbb{I}_{[a,b)} = (I_t, \phi_{s,t})$ where*

$$I_t = \begin{cases} \mathcal{F} & a \leq t < b \\ 0 & \textit{otherwise} \end{cases}$$

*Transition map $\phi_{s,t}$ is identity if and only if $s, t \in [a, b)$ and zero map otherwise.*

**Theorem 3.2.3.** *[Theorem 3.1 in [CZ05]] Suppose $\mathbb{V}$ (over $\mathcal{F}$ with trivial valuation) satisfies $\dim(V_t) < \infty$ for every $t \in \mathbb{R}$, then it can be uniquely (up to permutation) decomposed into the following normal form*

$$\mathbb{V} = \bigoplus_{[a,b)} \mathbb{I}^m_{[a,b)} \tag{3.1}$$

*where $m$ is the multiplicity of* $\mathbb{I}_{[a,b)}$.

The (persistent homology) *barcode* of $\mathbb{V}$ is then by definition the multiset $(S, \mu)$ where

$$S = \{[a,b) \,|\, [a,b) \,\text{appears in (3.1)}\}$$

and

$$\mu([a,b)] = \left\{m \,|\, \text{the multiplicity of } \mathbb{I}_{[a,b)}\right\}.$$

As follows from the discussion at the end of the introduction in [Cr12], the barcode is a *complete* invariant of a finite dimenisonal persistence module. Moreover, in classical persistent homology, [CZ05] provides an algorithm computing the resulting barcode. In this case the intervals in the barcode are all half-open intervals $[a,b)$ (with possibly $b = \infty$). See, e.g., [Ghr08, Figure 4], [Car09, p. 278] for some nice illustrations of barcodes.

In the spirit of Remark 2.6.1, for most of the cases, especially related with Hamiltonian Floer theory (on a general symplectic manifold), the condition - "finite dimensional" is almost never satisfied. However, note for Floer chain complex itself, it is a finite dimensional over $\Lambda^{\mathcal{K},\Gamma}$. Therefore, for the rest of this section, we will mainly focus on how to construct barcode by directly working on the chain complex level. This needs some new work on the algebra that will be explain in the next section.

## 3.3 Non-Archimedean linear algebra

The keyword of this section is "orthogonality". We will use the standard notions of orthogonality in non-Archimedean normed vector space (e.g. [MS65]).

*Definition* 3.3.1. Let $(C, \ell)$ be a non-Archimedean normed vector space over a Novikov field $\Lambda = \Lambda^{\mathcal{K},\Gamma}$ defined in Definition 2.5.2.

- Two subspaces $V$ and $W$ are said to be *orthogonal* if for all $v \in V$ and $w \in W$, we have

$$\ell(v + w) = \max\{\ell(v), \ell(w)\}.$$

- A finite ordered collection $(w_1, \ldots, w_r)$ of elements of $C$ is said to be *orthogonal* if, for all $\lambda_1, \ldots, \lambda_r \in \Lambda$, we have

$$\ell\left(\sum_{i=1}^{r} \lambda_i w_i\right) = \max_{1 \le i \le r} \ell(\lambda_i w_i). \tag{3.2}$$

In particular a pair of elements of $C$, $\{v, w\}$, are orthogonal if and only if the spans $\langle v \rangle_\Lambda$ and $\langle w \rangle_\Lambda$ are orthogonal as subspaces of $C$. Of course, by (F2), the criterion (3.2) can equivalently be written as

$$\ell\left(\sum_{i=1}^{r} \lambda_i w_i\right) = \max_{1 \le i \le r} (\ell(w_i) - \nu(\lambda_i)). \tag{3.3}$$

**Example 3.3.2.** *For $V = \mathrm{span}_{\mathcal{K}} \langle a, b \rangle$ with $\ell(a) = 1$ and $\ell(b) = 3$. For subspace $U = \mathrm{span}_{\mathcal{K}} \langle a + b \rangle$, $V_1 = \mathrm{span}_{\mathcal{K}} \langle a \rangle$ is an orthogonal complement but $V_2 = \mathrm{span}_{\mathcal{K}} \langle b \rangle$ is not an orthogonal complement.*

*Definition* 3.3.3. *An orthogonalizable $\Lambda$-space* $(C, \ell)$ is a finite-dimensional non-Archimedean normed vector space over $\Lambda$ such that there exists an orthogonal basis for $C$.

**Example 3.3.4.** $(\Lambda^n, -\vec{\nu})$ *is an orthogonalizable $\Lambda$-space, where*

$$\vec{\nu}(\lambda_1, ..., \lambda_n) = \min_{1 \le i \le n} \nu(\lambda_i).$$

Non-Archimedean Gram-Schimdt process (Theorem 2.16 in [UZ15]) guarantees that we can always modify an arbitrary given basis into an orthogonal basis. This implies the following expected property.

**Proposition 3.3.5.** *[Corollary 2.19 in [UZ15]] Suppose that $(C, \ell)$ is an orthogonalizable $\Lambda$-space and $U \le C$. Then there exists a subspace $V$ such that $U \oplus V = C$ and $U$ and $V$ are orthogonal. (We call any such $V$ an orthogonal complement of $U$).*

Likewise in the Archimedean case, the proof of this proposition heavily depends on a "best approximation type" property, see Theorem 2.14 in [UZ15]. What we want to emphasize here is orthogonal complement is not unique. For instance, for $C = \mathrm{span}_{\mathcal{K}} \langle x, y \rangle$ with $\ell(x) = 0$ and $\ell(y) = 1$, subspace $U = \langle x \rangle_{\mathcal{K}}$ has $V_1 = \langle y \rangle_{\mathcal{K}}$ as an orthogonal complement. Meanwhile, $V_2 = \langle x + y \rangle_{\mathcal{K}}$ is also an orthogonal complement.

Now consider a $\Lambda$-linear map between two orthogonalizable $\Lambda$-spaces, we have the following important theorem which is in the same spirit of singular value decomposition of a linear transformation $A : \mathbb{C}^n \to \mathbb{C}^m$.

**Theorem 3.3.6.** *For any $\Lambda$-linear map $A : (C, \ell_C) \to (D, \ell_D)$ with $\mathrm{rank}(A) = r$, there exists a singular value decomposition of $A$ in the sense that there is a choice of orthogonal ordered bases $(y_1, ..., y_n)$ for $C$ and $(x_1, ..., x_m)$ for $D$ such that:*

*(i)* $(y_{r+1}, ..., y_n)$ *is an orthogonal ordered basis for* $\ker A$*;*

*(ii)* $(x_1, ..., x_r)$ *is an orthogonal ordered basis for* $\mathrm{Im}A$*;*

*(iii)* $Ay_i = x_i$ *for* $i \in \{1, ..., r\}$*;*

*(iv)* $\ell_C(y_1) - \ell_D(x_1) \geq \ldots \geq \ell_C(y_r) - \ell_D(x_r)$.

*Remark* 3.3.7. To simplify the notation, we will simply denote a singular value decomposition satisfying (i), (ii), (iii) and (iv) as above by

$$\begin{array}{ccc} (y_{r+1}, ..., y_n) & & \\ & & \\ (y_1, ..., y_r) & \xrightarrow{A} & (x_1, ..., x_r) \\ & & \\ & & (x_{r+1}, ..., x_m). \end{array}$$

The way that we prove Theorem 3.3.6 is by providing an algorithm, see Theorem 3.5 in [UZ15].

## 3.4 Barcode from Floer-type complex

Example 2.5.3 and Theorem 2.5.4 suggest the abstract algebraic object we are studying is the following one.

*Definition* 3.4.1. A *Floer-type complex* $(C_*, \partial_C, \ell_C)$ over a Novikov field $\Lambda = \Lambda^{\mathcal{K},\Gamma}$ is a chain complex $(C_* = \oplus_{k\in\mathbb{Z}} C_k, \partial_C)$ over $\Lambda$ together with a function $\ell_C\colon C_* \to \mathbb{R} \cup \{-\infty\}$ such that each $(C_k, \ell|_{C_k})$ is an orthogonalizable $\Lambda$-space, and for each $x \in C_k$ we have $\partial_C x \in C_{k-1}$ with $\ell_C(\partial_C x) \leq \ell_C(x)$.

Similar to Example 3.2.2, we have the "easiest" Floer-type complexes.

**Example 3.4.2.** *We list them as two types.*

- *Type I*

$$\mathcal{E}_1(y)_* = \ldots \to 0 \to \langle y \rangle_\Lambda \to \langle \partial y \rangle_\Lambda \to 0 \to \ldots;$$

- *Type II*

$$\mathcal{E}_2(x)_* = \ldots \to 0 \to \langle x \rangle_\Lambda \to 0 \to \ldots.$$

It turns out these two types "easiest" Floer-type complexes form the building blocks of any Floer-type complex similar to the structure theorem, Theorem 3.2.3, of persistence module above.

**Theorem 3.4.3.** *[Proposition 7.4 in [UZ15]] Any Floer-type complex $(C_*, \partial_C, \ell_C)$ over a Novikov field $\Lambda = \Lambda^{\mathcal{K},\Gamma}$ can be orthogonally decomposed as (a direct sum of chain complexes)*

$$(C_*, \partial_C, \ell_C) = \bigoplus_{x,y} (\mathcal{E}_1(y)_*)^{m_1(y)} \oplus (\mathcal{E}_2(x)_*)^{m_2(x)} \tag{3.4}$$

*where $m_1(y)$ and $m_2(x)$ are multiplicities.*

Here $x$'s and $y$'s can be obtained by a singular value decomposition of boundary map $\partial_*$ given by Theorem 3.3.6. Specifically, for degree $k \in \mathbb{Z}$, we can find a singular value decomposition of map $\partial_{k+1} : C_{k+1} \to \ker(\partial_k)$. This will decompose the following *two terms Floer-type complex*

$$\ldots \to 0 \to C_{k+1} \xrightarrow{\partial_{k+1}} \ker(\partial_k) \to 0 \to \ldots$$

into direct sum of some $E^1_*$ and $E^2_*$. Together all $k \in \mathbb{Z}$, we get the decomposition. Moreover, from this decomposition, for each $k \in \mathbb{Z}$, we can define degree-$k$ *verbose barcode* (of Floer-type complex) as a multiset $(S, \mu)$ where

$$S = \left\{ \begin{array}{c|c} [\ell_C(\partial y), \ell_C(y)), & \mathrm{ind}(y) = \mathrm{ind}(x) = k+1, \\ [\ell_C(x), \infty) & x, y \text{ appears in (3.4)} \end{array} \right\}$$

and

$$\mu([\ell_C(\partial y), \ell_C(y))) = \{m_1(y) \,|\, \text{the multiplicity of } \mathcal{E}_1(y)_*\}$$

$$\mu([\ell_C(x), \infty)) = \{m_2(x) \,|\, \text{the multiplicity of } \mathcal{E}_2(x)_*\}\,.$$

Note that it is possible $\ell_C(\partial y) = \ell_C(y)$, therefore, we define degree-$k$ *concise barcode* is a collection of all elements in $S$ which have positive lengths, that is $\ell_C(y) - \ell_C(\partial y) > 0$.

By classification theorems, Theorem A and Theorem B in [UZ15], verbose barcode is a *complete* invariant, up to filtered isomorphism (see Definition 4.4 in [UZ15]), of Floer-type complexes and concise barcode is a *complete* invariant, up to filtered homotopy equivalence (see Definition 4.5 in [UZ15]), of Floer-type complexes.

## 3.5 Example of computing barcode

So far, we have seen there are two approaches to generate barcode. One is from classical persistence module and the other is from Floer-type complex. By theorem 6.2 in [UZ15], reducing to the case that $\Lambda^{\mathcal{K},\Gamma} = \Lambda^{\mathcal{K},\{0\}} = \mathcal{K}$, these two approaches give the same barcode. We will demonstrate this by a concrete example.

**Example 3.5.1.** *This picture is borrowed from [Wei11] with our assigned initial data as follows.*

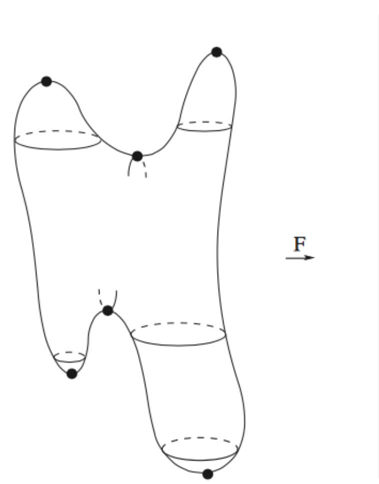


*Figure 1.* 2-dimensional manifold $X$ with height function $F$

*Associate a height function $F$ on it and the dots represent critical points, namely from bottom to top as $a, b, c, d, e, f$ and their heights are*

| *p/h* | *a* | *b* | *c* | *d* | *e* | *f* |
|---|---|---|---|---|---|---|
| *F* | *1* | *3* | *4* | *7* | *8* | *9* |

**Method one**: *By tracing the homology of truncated (by height function $F$) sublevel submanifolds, we can easily get the following table.*

| *level/homology* | *0* | *1* | *2* |
|---|---|---|---|
| $h < 1$ | *0* | *0* | *0* |
| $1 \leq h < 3$ | $\mathcal{K}$ | *0* | *0* |
| $3 \leq h < 4$ | $\mathcal{K} \oplus \mathcal{K}$ | *0* | *0* |
| $4 \leq h < 7$ | $\mathcal{K}$ | *0* | *0* |
| $7 \leq h < 8$ | $\mathcal{K}$ | $\mathcal{K}$ | *0* |
| $8 \leq h < 9$ | $\mathcal{K}$ | *0* | *0* |
| $9 \leq h$ | $\mathcal{K}$ | *0* | $\mathcal{K}$ |

*Therefore, we have our (persistent homology) barcode as*

- *degree-0 barcode* $= \{[1, \infty), [3, 4)\}$;
- *degree-1 barcode* $= \{[7, 8)\}$;
- *degree-2 barcode* $= \{[9, \infty)\}$.

**Method two**: *From critical points, we can form a Morse chain complex (which is a case of Floer-type complex). Namely,*

$$\ldots \to 0 \xrightarrow{\partial_3} CM_2(X, F) \xrightarrow{\partial_2} CM_1(X, F) \xrightarrow{\partial_1} CM_0(X, F) \xrightarrow{\partial_0} 0 \to \ldots,$$

*where by counting the corresponding indices,*

- $CM_2(X, F) = \mathrm{span}_{\mathcal{K}} \langle e, f \rangle$;
- $CM_1(X, F) = \mathrm{span}_{\mathcal{K}} \langle c, d \rangle$;

- $CM_0(X,F) = \mathrm{span}_{\mathcal{K}} \langle a, b \rangle$.

*Moreover, the boundary operators are*

- $\partial_1 c = a + b$ *and* $\partial_1 d = 0$;
- $\partial_2 e = \partial_2 f = d$.

*Finally, the filtration* $\ell = \ell_F$ *is induced by the height function in an obvious way. In order to get an orthogonal decomposition 3.4, we need to get a singular value decomposition of each boundary maps (or precise, the associated two terms Floer-type complex). For* $\partial_1 : CM_1(X,F) \to \ker(\partial_0) = CM_0(X,F)$,

$$\mathrm{Im}(\partial_1) = \mathrm{span}_{\mathcal{K}} \langle a+b \rangle \ \Rightarrow \ \textit{its orthogonal complement is } \ \mathrm{span}_{\mathcal{K}} \langle a \rangle .$$

*Note that orthogonality plays an important role in choosing orthogonal complement. From Example 3.3.2,* $\mathrm{span}_{\mathcal{K}} \langle b \rangle$ *is not an option for orthogonal complement. Therefore, we have a singular value decomposition of* $\partial_1$, *by notation from Remark 3.3.7,*

$$\begin{tikzcd} d & \\ c \arrow[r, "\partial_1"] & a+b \\ & a. \end{tikzcd} \tag{3.5}$$

*For* $\partial_2 : CM_2(X,F) \to \ker(\partial_1) = \mathrm{span}_{K} \langle d \rangle$. *Then*

$$\ker(\partial_2) = \mathrm{span}_{\mathcal{K}} \langle e - f \rangle \ \Rightarrow \ \textit{its orthogonal complement is } \ \mathrm{span}_{\mathcal{K}} \langle e \rangle .$$

*Again, by orthogonality* $\mathrm{span}_{\mathcal{K}} \langle f \rangle$ *is prohibited from being an orthogonal complement. So, a singular value decomposition of* $\partial_2$ *is*

$$\begin{tikzcd} e - f & \\ e \arrow[r, "\partial_2"] & d. \end{tikzcd} \tag{3.6}$$

*Therefore, if we put (3.5) and (3.6) together and complete them as complexes, we get* $(CM_*(X,F),\partial)$ *is orthogonally decomposed as a direct sum of*

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{span}_{\mathcal{K}}\langle e-f\rangle \longrightarrow 0 \longrightarrow \ldots \tag{3.7}$$

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{span}_{\mathcal{K}}\langle e\rangle \longrightarrow \mathrm{span}_{\mathcal{K}}\langle d\rangle \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{span}_{\mathcal{K}}\langle c\rangle \longrightarrow \mathrm{span}_{\mathcal{K}}\langle a+b\rangle \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{span}_{\mathcal{K}}\langle a\rangle \longrightarrow 0 \longrightarrow \ldots.$$

*Therefore, the verbose barcode (which is the same as concise barcode here) is*

- *degree-0* $= \{[\ell_F(a),\infty),[\ell_F(a+b),\ell_F(c))\} = \{[1,\infty),[3,4)\}$*;*
- *degree-1* $= \{[\ell_F(d),\ell_F(e))\} = \{[7,8)\}$*;*
- *degree-2* $= \{[\ell_F(e-f),\infty)\} = \{[9,\infty)\}$.

## 3.6 Relation to some symplectic invariants

As verbose or concise is a complete invariant for Floer-type complex, any invariant constructed from Floer-type chain complex (containing information up to filtered isomorphism or filtered homotopy equivalence) should be rewritten by some information from barcode. Here we give two relations.

### 3.6.1 Relation to spectral invariant

Following a construction that is found in [Sch00], [Oh05] in the context of Hamiltonian Floer theory (and which is closely related to classical minimax-type arguments in Morse theory), we may describe the *spectral invariant* associated to a Floer-type complex $(C_*,\partial,\ell)$: where $H_k(C_*)$ is the degree-$k$ homology of $C_*$, as a map $\rho\colon H_k(C_*)\to\mathbb{R}\cup\{-\infty\}$ defined by, for $\alpha\in H_k(C_*)$,

$$\rho(\alpha) = \inf\{\ell(c)|c\in C_k,\ [c]=\alpha\}$$

(where $[c]$ denotes the homology class of $c$). In a more general context the main result of [Ush08] shows that the infimum in the definition of $\rho(\alpha)$ is always attained.

The spectral invariants are reflected in the concise barcode in the following way.

**Proposition 3.6.1** (Proposition 6.4 in [UZ15]). *Let $\mathcal{B}_{C,k}$ denote the degree-$k$ part of the concise barcode of a Floer-type complex $(C_*, \partial, \ell)$, obtained from a singular value decomposition of $\partial_{k+1}\colon C_{k+1} \to \ker \partial_k$. Then:*

- *For each $\alpha \in C_k \setminus \{0\}$, the concise barcode $\mathcal{B}_{C,k}$ contains an element of the form $([\rho(\alpha)], \infty)$, where $[\rho(\alpha)]$ is the reduction of $\rho(\alpha)$ modulo $\Gamma$.*
- *There is a basis $\alpha_1, \ldots, \alpha_h$ for $H_k(C_*)$ such that the submultiset of $\mathcal{B}_{C,k}$ consisting of elements with second coordinate equal to $\infty$ is equal to $\{([\rho(\alpha_1)], \infty), \ldots, ([\rho(\alpha_h)], \infty)\}$.*

### 3.6.2 Relation to boundary depth

Recall in [Ush13], *boundary depth* of a two terms Floer-type complex $\partial : C_1 \to C_0$ is defined as

$$\beta(\partial) = \sup_{x \leq \mathrm{Im}\partial} \inf_{x \in V \setminus \{0\}} \{\ell(y) - \ell(x) \,|\, \partial y = x\}. \tag{3.8}$$

Actually, we can generalize this definition to the following one,

*Definition* 3.6.2. For any given $k \in \mathbb{Z}$, define the *generalized boundary depth* of a two terms Floer-type complex $\partial : C_1 \to C_0$ by

$$\beta_k(\partial) = \sup_{\substack{V \leq \mathrm{Im}\partial \\ \dim(V) = k}} \inf_{x \in V \setminus \{0\}} \{\ell(y) - \ell(x) \,|\, \partial y = x\}$$

and $\beta_k(\partial) = 0$ if $\partial$ is the zero map or if $k > \dim(\mathrm{Im}\partial)$.

When $k = 1$, this is exactly the definition of boundary depth in (3.8). Clearly one has

$$\beta_1(\partial) \geq \beta_2(\partial) \geq \cdots \beta_k(\partial) \geq 0$$

for all $k$. In terms of computation, the following theorem which relates the $\beta_k(\partial)$'s to singular value decompositions.

**Theorem 3.6.3** (Theorem 4.11 in [UZ15]). *Given a singular value decomposition* $((y_1, ..., y_n), (x_1, ..., x_m))$ *for a two-term chain complex* $\partial : C_1 \to C_0$*, the numbers* $\beta_k(\partial)$ *are given by*

$$\beta_k(\partial) = \begin{cases} \ell(y_k) - \ell(x_k) & 1 \leq k \leq r \\ 0 & k > r \end{cases}$$

*where* $r$ *is the rank of* $\partial$.

On the one hand, (3.6.2) indicates the values of $\beta_k(\partial)$ are independent of choice of singular value decomposition. On the other hand, Theorem 3.6.3 together with classification theorems implies that values of $\beta_k(\partial)$ are just lengths of finite-length bars in the barcode associated to the Floer-type complex. In particular, boundary depth in [Ush13] is the length of the longest bars (within all *finite* length bars).

# Chapter 4

# Application and distance comparison

## 4.1 Quantitative application

In this section, we will give a general scheme on how to combine Hamiltonian Floer theory and persistent homology to solve a dynamics problem. First, we give a logic picture as follows.

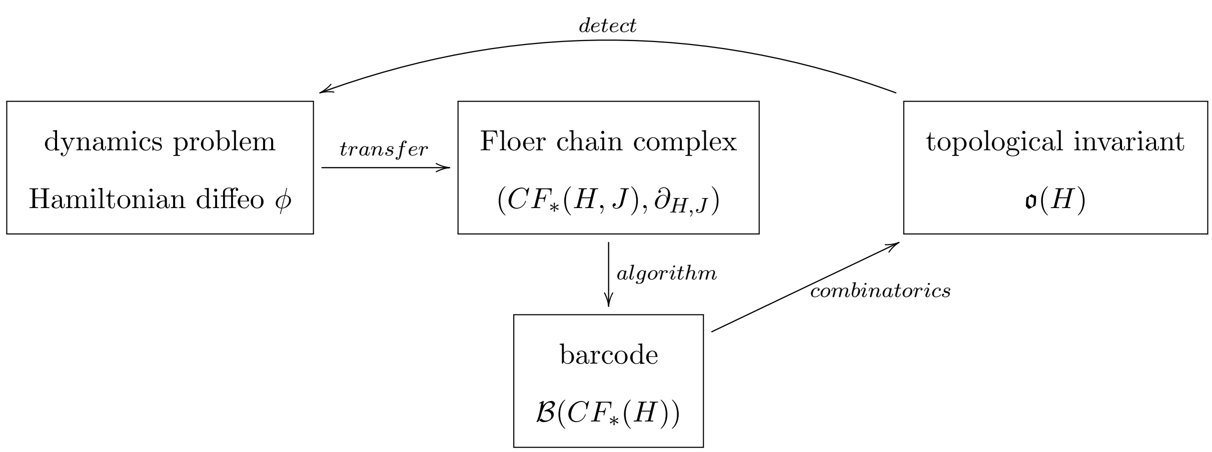


Figure 2. Logic picture of solving a dynamics problem

Second, a brief explanation goes as follows. Starting from a Hamiltonian dynamics problem that involves a given Hamiltonian function $H$, by its geometric or topological construction, we can use language of Floer theory to rewrite this problem so that it will be possible to formulate a Floer chain complex, as a special case of Floer-type complex defined in Chapter 3. As mentioned earlier, there is an algorithm to compute the barcode of Floer chain complex so that we obtain a rich resource to construct symplectic invariants. The concrete construction of invariants depends on the initial problem and possibly not unique. Some constructions, especially in this paper, rely on the observation from combinatorics. Eventually, we will use these invariants to detect or answer the initial dynamics problem.

Third, an important family of properties is that for each of categories above in Figure 2, we can associate a meaningful "distance". The most obvious one is for topological invariant. As they are numbers in $\mathbb{R}$, the obvious distance is the (absolute value of) difference of two numbers. The associated distances we will use later for the other three will be explicitly explained and defined in Section 4.2. What's more important is that all the distances are *Lipschitz continuous* in terms of the others, which will be explained in Section 4.3. Therefore, each step in the logic picture can be quantitatively detected and controlled.

## 4.2 Various distances

### 4.2.1 Hamiltonian diffeomorphism $\phi$

There are many different ways to compare two diffeomorphisms. However, as introduced in the Chapter 1, on Hamiltonian diffeomorphism group $Ham(X, \omega)$, the well-known Hofer's metric $d_H$ of $\phi = \phi_H^1$ is the one that we will mainly focus on in this paper. Hofer's metric is defined by the following two steps. First, for any $\phi \in Ham(X, \omega)$, define (the Hofer norm)

$$||\phi||_H = \inf \left\{ \int_0^1 \left( \max_X H(t, \cdot) - \min_X H(t, \cdot) \right) dt \,\middle|\, \phi = \phi_H^1 \right\}.$$

Then define Hofer's metric by, for $\phi, \psi \in Ham(X, \omega)$,

$$d_H(\phi, \psi) = ||\phi^{-1} \circ \psi||_H. \tag{4.1}$$

This is a bi-invariant metric on $Ham(X, \omega)$ which leads to a fast-developed subject called *Hofer geometry* partly because this metric is closely related with dynamics. For more details, please see a well-written book [Pol01].

### 4.2.2 Floer chain complex $(CF_*(H,J), \partial_{H,J})$

It might sound strange at the first sight that how we will measure the distance between two complexes, but by the special filtration shift property of Hamiltonian Floer chain complex, Theorem 2.5.5 and the discussion after it, we can give the following abstract definition.

*Definition* 4.2.1. Let $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ be two Floer-type complexes, and $\delta_+, \delta_- \geq 0$. A $(\delta_+, \delta_-)$*-quasiequivalence between* $C_*$ *and* $D_*$ *is a quadruple* $(\Phi, \Psi, K_C, K_D)$ *where:*

- $\Phi\colon C_* \to D_*$ *and* $\Psi\colon D_* \to C_*$ *are chain maps, with* $\ell_D(\Phi c) \leq \ell_C(c) + \delta_+$ *and* $\ell_C(\Psi d) \leq \ell_D(d) + \delta_-$ *for all* $c \in C_*$ *and* $d \in D_*$.
- $K_C\colon C_* \to C_{*+1}$ *and* $K_D\colon D_* \to D_{*+1}$ *obey the homotopy equations* $\Psi \circ \Phi - \mathbb{I}_{C_*} = \partial_C \circ K_C + K_C \circ \partial_C$ *and* $\Phi \circ \Psi - \mathbb{I}_{D_*} = \partial_D \circ K_D + K_D \circ \partial_D$*, and for all* $c \in C_*$ *and* $d \in D_*$ *we have* $\ell_C(K_C c) \leq \ell_C(c) + \delta_+ + \delta_-$ *and* $\ell_D(K_D d) \leq \ell_D(d) + \delta_+ + \delta_-$.

*The quasiequivalence distance between* $(C_*, \partial_C, \ell_C)$ *and* $(D_*, \partial_D, \ell_D)$ *is then defined to be*

$$d_Q(C_*, D_*) = \inf \left\{ \frac{\delta_+ + \delta_-}{2} \geq 0 \;\middle|\; \begin{array}{c} \text{There exists a } (\delta_+, \delta_-)\text{-quasiequivalence between} \\ (C_*, \partial_C, \ell_C) \text{ and } (D_*, \partial_D, \ell_D) \end{array} \right\}.$$

**Example 4.2.2.** *From (2.8), (2.9) and (2.10), using the notation introduced above, we can summarize the relation between* $(CF_*(H_-, J_-), (\partial_{H_-,J_-})_*, \ell_{H_-})$ *and* $(CF_*(H_+, J_+), (\partial_{H_+,J_+})_*, \ell_{H_+})$ [1] *that they are* $(\int_0^1 \max_X(H_+ - H_-)dt, \int_0^1 - \min_X(H_+ - H_-)dt)$*-quasiequivalent. Moreover, Proposition 5.1 in [Ush13] implies this still holds for* $CF_*(H,J)_\alpha$ *with non-contractible homotopy class* $\alpha$.

*Remark* 4.2.3. If passing to the corresponding homologies, for persistent homologies, people very often use *interleaving distance* (*e.g.*, in [CCGGO09]). The relation between quasiequivalence distance and interleaving distance has been studied carefully in Appendix A in [UZ15].

[1] Here we choose normalized Hamiltonians $H_+$ and $H_-$ in the sense that $\int_X H_+(t, \cdot) = \int_X H_-(t, \cdot) = 0$ for every $t \in [0, 1]$. Therefore, $\int_0^1 - \min_X(H_+ - H_-)dt \geq 0$.

### 4.2.3 Barcode $\mathcal{B}(CF_*(H,J))$

By definition, barcode is nothing but a collection of half closed and half open intervals. More precisely, we will denote each interval as $[[a],[b])$ because in general over $\Lambda^{\mathcal{K},\Gamma}$, because of action from $\Gamma$, the (left) end point of this interval is actually in coset $\mathbb{R}/\Gamma$. Therefore, we can regard $[[a],[b])$ as a pair $([a],L) \in \mathbb{R}/\Gamma \times (0,\infty]$ where $L = b - a$. In the spirit of Gromov-Hausdroff distance, we can define a distance between two concise barcodes in the following two steps (cf. [CdSGO12]). First,

*Definition* 4.2.4. Consider two concise barcodes (viewed as multisets of elements of $(\mathbb{R}/\Gamma) \times (0,\infty]$) $\mathcal{S}$ and $\mathcal{T}$. A *$\delta$-matching* between $\mathcal{S}$ and $\mathcal{T}$ consists of the following data:

(i) submultisets $\mathcal{S}_{short}$ and $\mathcal{T}_{short}$ such that the second coordinate $L$ of every element $([a],L) \in \mathcal{S}_{short} \cup \mathcal{T}_{short}$ obeys $L \leq 2\delta$.

(ii) A bijection $\sigma\colon \mathcal{S} \setminus \mathcal{S}_{short} \to \mathcal{T} \setminus \mathcal{T}_{short}$ such that, for each $([a],L) \in \mathcal{S} \setminus \mathcal{S}_{short}$ (where $a \in \mathbb{R}$, $L \in [0,\infty]$) we have $\sigma([a],L) = ([a'],L')$ where for all $\epsilon > 0$ the representative $a'$ of the coset $[a'] \in \mathbb{R}/\Gamma$ can be chosen such that both $|a' - a| \leq \delta + \epsilon$ and either $L = L' = \infty$ or $|(a' + L') - (a + L)| \leq \delta + \epsilon$.

**Example 4.2.5.** *Suppose*

$$\mathcal{S} = \{[2,\infty),[4,5),[2,4)\} \quad \textit{and} \quad \mathcal{T} = \{[3,\infty),[3,6)\}.$$

*There exists a* 2*-matching. In fact, take* $\mathcal{S}_{short} = \{[4,5),[2,4)\}$ *and* $\mathcal{T}_{short} = \{[3,6)\}$*. Moreover, we can set up a bijection* $\sigma'$ *as*

$$\sigma'([2,\infty)) = [3,\infty).$$

*However, there exists a* 1*-matching (which is better in the sense of distance defined below). In fact, take* $\mathcal{S}_{short} = \{[2,4)\}$ *and* $\mathcal{T}_{short} = \{\emptyset\}$*. Moreover, we can set up a bijection* $\sigma$ *as*

$$\sigma([2,\infty)) = [3,\infty) \quad \textit{and} \quad \sigma([4,5)) = [3,6).$$

*Note that in order to get a $\delta$-matching with finite $\delta$, there must be equal numbers of infinite-length bars from $\mathcal{S}$ and $\mathcal{T}$.*

*Definition* 4.2.6. If $\mathcal{S}$ and $\mathcal{T}$ are two multisets of elements of $(\mathbb{R}/\Gamma) \times (0, \infty]$ then the *bottleneck distance* between $\mathcal{S}$ and $\mathcal{T}$ is

$$d_B(\mathcal{S}, \mathcal{T}) = \inf\{\delta \geq 0 \,|\, \text{There exists a } \delta\text{-matching between } \mathcal{S} \text{ and } \mathcal{T}\}.$$

**Example 4.2.7.** *Suppose $\mathcal{S}$ and $\mathcal{T}$ are barcodes given in Example 4.2.5. $d_B(\mathcal{S}, \mathcal{T}) = 1$.*

Our construction associates to a Floer-type complex a concise barcode *for every* $k \in \mathbb{Z}$, so the appropriate notion of distance for this entire collection of data is:

*Definition* 4.2.8. Let $\mathcal{S} = \{\mathcal{S}_k\}_{k \in \mathbb{Z}}$ and $\mathcal{T} = \{\mathcal{T}_k\}_{k \in \mathbb{Z}}$ be two families of multisets of elements of $(\mathbb{R}/\Gamma) \times (0, \infty]$. The bottleneck distance between $\mathcal{S}$ and $\mathcal{T}$ is then

$$d_B(\mathcal{S}, \mathcal{T}) = \sup_{k \in \mathbb{Z}} d_B(\mathcal{S}_k, \mathcal{T}_k).$$

## 4.3 Lipschitz comparison

Example 4.2.2 easily implies

$$d_Q(CF_*(H_-, J_-), CF_*(H_+, J_+)) \leq \frac{1}{2} d_H(\phi^1_{H_-}, \phi^1_{H_+}). \tag{4.2}$$

Moreover, we know from Stability Theorem (Theorem 8.16 and Theorem 8.17 in [UZ15]),

$$d_B(\mathcal{B}(CF_*(H_-, J_-)), \mathcal{B}(CF_*(H_+, J_+))) \leq 2 d_Q(CF_*(H_-, J_-), CF_*(H_+, J_+)) \tag{4.3}$$

$$d_Q(CF_*(H_-, J_-), CF_*(H_+, J_+)) \leq d_B(\mathcal{B}(CF_*(H_-, J_-)), \mathcal{B}(CF_*(H_+, J_+))). \tag{4.4}$$

So we have

$$d_B \leq 2 d_Q \leq d_H. \tag{4.5}$$

Actually, for most part of this paper, we will only use a (much) weaker version of stability theorem, which only involves the generalized boundary depth (because our construction of invariant (1.11) is only based on the generalized boundary depth).

**Theorem 4.3.1.** *[Corollary 8.8 in [UZ15]] Suppose that $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ are $(\delta, \delta)$-quasiequivalent. Then for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have $|\beta_k((\partial_C)_{i+1}) - \beta_k((\partial_D)_{i+1})| \leq 2\delta$.*

*Remark* 4.3.2. In order to use Theorem 4.3.1, note any $(\delta_+, \delta_-)$-quasiequivalent is automatically $(\delta_+ + \delta_-, \delta_+ + \delta_-)$-quasiequivalent because $\delta_+, \delta_- \geq 0$. Therefore, Theorem 4.3.1 says

$$|\beta_k((\partial_C)_{i+1}) - \beta_k((\partial_D)_{i+1})| \leq 4d_Q(C_*, D_*).$$

*Remark* 4.3.3. (4.5) and Theorem 4.3.1 reflect a general principle of the well-known facts of Lipschitz continuity of spectral invariant and boundary depth. See [Oh05] and [Ush13].

# Chapter 5

# Build up numerical measurement

## 5.1 Existence of $p$-th root map

**Lemma 5.1.1.** *For any $a \in \mathbb{N}$, we have the following commuting diagram*

$$\begin{tikzcd} CF_k(H^{(p)}, J_{t+\frac{a}{p^2}})_\alpha \arrow[r, "R_{p^2}"] \arrow[d, "C_a"'] & CF_k(H^{(p)}, J_{t+\frac{a+1}{p^2}})_\alpha \arrow[d, "(R_{p^2})_*(C_a)"'] \\ CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "R_{p^2}"'] & CF_k(H^{(p)}, J_{t+\frac{1}{p^2}})_\alpha \end{tikzcd} \tag{5.1}$$

*where $C_a$ is some continuation map.*

*Proof.* Suppose $x(t + a/p^2)$ is a (capped) periodic orbit as a generator of $CF_k(H^{(p)}, J_{t+a/p^2})_\alpha$ and suppose there exists a Floer connecting orbit $u(s, t)$ satisfying parametrized partial differential equation (2.2) connecting $x(t+a/p^2)$ and $y(t)$ for some $y(t)$ as a generator of $CF_k(H^{(p)}, J_t)$, followed by a rotation on the parameter $t$ to $t + 1/p^2$. In other words, we have

$$x(t + a/p^2) \xrightarrow{u(s,t)} y(t) \xrightarrow{R_{p^2}} y(t + 1/p^2)$$

Then by rotating $u(s, t)$ to be $u(s, t + 1/p^2)$, we will also get a Floer connecting orbit, namely $u(s, t + 1/p^2)$, satisfying the following parametrized partial differential equation (used to construct

$(R_p)_*(C_\alpha))$,

$$\frac{\partial u}{\partial s} + (J_{t+1/p})_s(u(s,t))\left(\frac{\partial u}{\partial t} - X_{H_s}(t+1/p^2, u(s,t))\right) = 0, \tag{5.2}$$

connecting $x(t+(a+1)/p^2)$ and $y(t+1/p^2)$. In other words,

$$x(t+a/p^2) \xrightarrow{R_{p^2}} x(t+(a+1)/p^2) \xrightarrow{u(s,t+1/p^2)} y(t+1/p^2)$$

By a symmetric argument, there is a one-to-one correspondence between these two approaches. Therefore, this diagram commutes by definition of Floer continuation map. $\square$

*Proof of Proposition 1.3.1.* Because $R_{p^2}$ is well-defined, consider composition $C_1 \circ R_{p^2}$ (which is a map on $CF_k(H^{(p)}, J_t)_\alpha$ itself). We observe that

$$\begin{aligned}(C_1 \circ R_{p^2}) \circ (C_1 \circ R_{p^2}) &= C_1 \circ (R_{p^2} \circ C_1) \circ R_{p^2} \\ &= C_1 \circ ((R_{p^2})_*(C_1) \circ R_{p^2}) \circ R_{p^2} \\ &= (C_1 \circ (R_{p^2})_*(C_1)) \circ R_{p^2}^2\end{aligned}$$

where the second equality comes from Lemma 5.1.1. By setting

$$C_2 = C_1 \circ (R_{p^2})_*(C_1)$$

we have $(C_1 \circ R_{p^2})^2 = C_2 \circ R_{p^2}^2$. Applying Lemma 5.1.1 inductively, we can get

$$(C_1 \circ R_{p^2})^p = C_p \circ R_{p^2}^p = C_p \circ R_p$$

where $C_p$ is recursively defined by

$$C_p = C_1 \circ (R_{p^2})_*(C_{p-1}).$$

Therefore, in order to get the conclusion, set $C = C_p$ (that is determined by $C_1$) and set $C' = C_1$ so $S = C_1 \circ R_{p^2}$. $\square$

## 5.2 Well-definiteness of self-mapping cone

We will prove the following general lemma,

**Lemma 5.2.1.** *For two filtration preserving chain maps $\Phi$ and $\Psi$ on Floer-type complex $(C_*, \partial_*, \ell)$, if $\Phi$ and $\Psi$ are filtered homotopic to each other, then the associated self-mapping cones $Cone_C(\Phi)_*$ and $Cone_C(\Psi)_*$ are filtered isomorphic to each other.*

*Proof.* Suppose the chain homotopy between $\Phi$ and $\Psi$ is $K$, which preserves the filtration. Construct a map $F : Cone_C(\Phi)_* \to Cone_C(\Psi)_*$ by

$$F = \begin{pmatrix} \mathbb{I} & -K \\ 0 & \mathbb{I} \end{pmatrix}.$$

First $F$ preserves filtration. In fact, for any $(x, y) \in Cone_C(\Phi)_*$, we have

$$\begin{aligned} \ell_{co}(F(x,y)) = \ell_{co}((x - Ky, y)) &= \max\{\ell_C(x - Ky), \ell_C(y)\} \\ &\leq \max\{\ell_C(x), \ell_C(Ky), \ell_C(y)\} \\ &= \max\{\ell_C(x), \ell_C(y)\} \\ &= \ell_{co}((x,y)). \end{aligned}$$

Second, $F$ is a chain map because

$$\begin{aligned} \begin{pmatrix} \mathbb{I} & -K \\ 0 & \mathbb{I} \end{pmatrix} \cdot \begin{pmatrix} \partial_C & -\Phi \\ 0 & -\partial_C \end{pmatrix} &= \begin{pmatrix} \partial_C & -\Phi + K\partial_C \\ 0 & -\partial_C \end{pmatrix} \\ &= \begin{pmatrix} \partial_C & -\Psi - \partial_C K \\ 0 & -\partial_C \end{pmatrix} = \begin{pmatrix} \partial_C & -\Psi \\ 0 & -\partial_C \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I} & -K \\ 0 & \mathbb{I} \end{pmatrix}. \end{aligned}$$

Third, $F$ is an isomorphism because we have its inverse

$$F^{-1} = \begin{pmatrix} \mathbb{I} & K \\ 0 & \mathbb{I} \end{pmatrix}$$

which is easily seen to preserve the filtration too. $\square$

*Proof of Proposition 1.3.4.* By standard result of Floer theory explained Section 2.2, different choices of continuation maps $C_1$ and $C_2$ will result in different but filtered homotopic chain maps. Therefore, the first conclusion comes from a direct application of Lemma 5.2.1 to $\Phi = C_1 \circ R_p - \xi_p \cdot \mathbb{I}$ and $\Psi = C_2 \circ R_p - \xi_p \cdot \mathbb{I}$.

For the second conclusion, note that for the following diagram

$$\begin{tikzcd} CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "R_p"] \arrow[d, "\partial_k"'] & CF_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \arrow[r, "C"] \arrow[d, "(R_p)_*(\partial_k)"] & CF_k(H^{(p)}, J_t)_\alpha \arrow[d, "\partial_k"] \\ CF_{k-1}(H^{(p)}, J_t)_\alpha \arrow[r, "R_p"] & CF_{k-1}(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \arrow[r, "C"] & CF_{k-1}(H^{(p)}, J_t)_\alpha \end{tikzcd}$$

the associated gluing diagram (see Lemma 3.10 in [Sal97]) is

$$\begin{tikzcd} x^\alpha(t) \arrow[r] \arrow[d] & x^\alpha(t+1/p) \arrow[r] & x^\beta(t) \arrow[d] \\ y^\alpha(t) \arrow[r] & y^\alpha(t+1/p) \arrow[r] & y^\beta(t) \end{tikzcd}$$

which confirms that 1-dimensional moduli space $\mathcal{M}(x^\alpha(t), y^\beta(t))$ has its boundary

$$\bigcup_{y^\alpha(t)} \mathcal{M}(x^\alpha(t), y^\alpha(t), H^{(p)}) \times \mathcal{M}(y^\alpha(t), y^\beta(t), \mathcal{J})$$
$$\cup \bigcup_{x^\beta(t)} \mathcal{M}(x^\alpha(t), x^\beta(t), \mathcal{J}) \times \mathcal{M}(x^\beta(t), y^\beta(t), H^{(p)})$$

where $\mathcal{J}$ is a composition homotopy of homotopies arising from $J_t$ to $J_{t+\frac{1}{p}}$ (by rotation $R_p$) and then from $J_{t+\frac{1}{p}}$ back to $J_t$ (by continuation $C$). Therefore, by compactness of $\mathcal{M}(x^\alpha(t), y^\beta(t))$, $T = C \circ R_p$ commutes with $\partial_k$. Similarly, $S = C' \circ R_{p^2}$ commutes with $\partial_k$ for any $k \in \mathbb{Z}$. Then by definition of $\partial_{co}$, we can check

$$\begin{aligned}
\begin{pmatrix} \partial & -(T-\xi_p\cdot\mathbb{I}) \\ 0 & -\partial \end{pmatrix} \cdot \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} &= \begin{pmatrix} \partial T & -(T-\xi_p\cdot\mathbb{I})\circ T \\ 0 & -\partial T \end{pmatrix} \\
&= \begin{pmatrix} T\partial & -T\circ(T-\xi_p\cdot\mathbb{I}) \\ 0 & -T\partial \end{pmatrix} \\
&= \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \cdot \begin{pmatrix} \partial & -(T-\xi_p\cdot\mathbb{I}) \\ 0 & -\partial \end{pmatrix}.
\end{aligned}$$

Similarly, since $S$ commutes with $T$,

$$\begin{aligned}
\begin{pmatrix} \partial & -(T-\xi_p\cdot\mathbb{I}) \\ 0 & -\partial \end{pmatrix} \cdot \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} &= \begin{pmatrix} \partial S & -(T-\xi_p\cdot\mathbb{I})\circ S \\ 0 & -\partial S \end{pmatrix} \\
&= \begin{pmatrix} S\partial & -S\circ(T-\xi_p\cdot\mathbb{I}) \\ 0 & -S\partial \end{pmatrix} \\
&= \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \cdot \begin{pmatrix} \partial & -(T-\xi_p\cdot\mathbb{I}) \\ 0 & -\partial \end{pmatrix}.
\end{aligned}$$

□

## 5.3 Non-divisibility

*Proof of Proposition 1.3.7.* The first conclusion directly comes from the Theorem 1.3.5 (which will be proved in the next section) and definition of $\mathfrak{o}_X(\phi)$ in Definition 1.3.6. For the second conclusion, suppose $s_{max}$ is the largest multiple of $p$ smaller than $m_k$, the multiplicity of degree-$k$ concise barcode of $Cone_{CF(H^{(p)},J)_\alpha}(T_p - \xi_p\cdot\mathbb{I})$. Then $p \nmid m_k$ implies $\beta_{s_{max}p+1}(C) \neq 0$ and $\beta_{(s_{max}+1)p}(C) = 0$. Therefore,

$$\mathfrak{o}_X(\phi)_k \geq \beta_{s_{max}p+1}(\phi) - \beta_{(s_{max}+1)p}(\phi) = \beta_{s_{max}p+1}(\phi) \geq \beta_{m_k}(\phi).$$

By its definition again, $\mathfrak{o}_X(\phi) \geq \mathfrak{o}_X(\phi)_k \geq \beta_{m_k}(\phi)$. □

# Chapter 6

# Chain complex with a group action

Recall our set-up. For any given Hamiltonian $H$, there is a Floer chain complex $(CF_*(H^{(p)}, J_t), \partial_{H,J})$. Its self-mapping cone of linear map $T - \xi_p \cdot \mathbb{I}$, $(Cone_*(T - \xi_p \cdot \mathbb{I}), \partial_{co})$, is in general a filtered chain complex over Novikov field $\Lambda^{\mathcal{K},\Gamma}$ where $T$ is a strict lower filtration perturbation of rotation $R_p$, *i.e.*, $T = C \circ R_p = R_p + P$ where for any $x$ from domain, $\ell(P(x)) < \ell(x)$. Moreover, there exists a $\Lambda^{\mathcal{K},\Gamma}$-linear chain map $\mathcal{D}_T$ on $(Cone_*(T - \xi_p \cdot \mathbb{I}), \partial_{co})$, defined as an action of $T$ on each component. If in the $p$-th power situation defined in the introduction, there exists a $\Lambda^{\mathcal{K},\Gamma}$-linear chain map $\mathcal{D}_S$ on $(Cone_*(T - \xi_p \cdot \mathbb{I}), \partial_{co})$ such that $\mathcal{D}_S^p = \mathcal{D}_T$ where $S$ is a strictly lower filtration perturbation of rotation $R_{p^2}$. In this section, we will prove the important Theorem 1.13,

## 6.1 Reduce to a group action

Recall our definitions,

$$\mathcal{D}_T = \mathcal{D}_{R_p} + C_T \quad \text{and} \quad \mathcal{D}_S = \mathcal{D}_{R_{p^2}} + C_S \tag{6.1}$$

and by Proposition 1.3.4,

$$\mathcal{D}_T \partial_{co} = \partial_{co} \mathcal{D}_T \quad \text{and} \quad \mathcal{D}_S \partial_{co} = \partial_{co} \mathcal{D}_S \tag{6.2}$$

because $T\partial = \partial T$ and $S\partial = \partial S$. In particular, we know $\mathcal{D}_T$ and $\mathcal{D}_S$ exactly preserves filtration. Moreover, due to Floer continuation map, there exists a constant $\hbar > 0$ such that for any $x \in$

$Cone_*(T - \xi_p \cdot \mathbb{I})$,

$$\ell_{co}(C_T x) \leq \ell(x) - \hbar \;\text{ and }\; \ell_{co}(C_S x) \leq \ell(x) - \hbar. \tag{6.3}$$

Because of this, we have the following lemma.

**Lemma 6.1.1.** *$\mathcal{D}_T$ and $\mathcal{D}_S$ (if it exists) are invertible.*

*Proof.* First, because $\mathcal{D}_{R_p}^p = \mathbb{I}$ and $\mathcal{D}_{R_{p^2}}^{p^2} = \mathbb{I}$,

$$\mathcal{D}_T^p = \mathbb{I} - Q_T \;\text{ and }\; \mathcal{D}_S^{p^2} = \mathbb{I} - Q_S \tag{6.4}$$

where $Q_T$ is a combination of $C_T$ and $\mathcal{D}_{R_p}$, so strictly lowers filtration (by at least $\hbar$) and $Q_S$ is a combination of $C_S$ and $\mathcal{D}_{R_{p^2}}$, so also strictly lowers filtration (by at least $\hbar$). Then $\mathcal{D}_T \circ (\mathcal{D}_T)^p = (\mathcal{D}_T)^{p+1} = (\mathcal{D}_T)^p \circ \mathcal{D}_T$, so by (6.4),

$$\mathcal{D}_T \circ (\mathbb{I} - Q_T) = (\mathbb{I} - Q_T) \circ \mathcal{D}_T \quad \Rightarrow \quad \mathcal{D}_T Q_T = Q_T \mathcal{D}_T. \tag{6.5}$$

Similarly, $\mathcal{D}_S \circ (\mathcal{D}_S)^{p^2} = (\mathcal{D}_S)^{p^2+1} = (\mathcal{D}_S)^{p^2} \circ \mathcal{D}_S$, so by (6.4),

$$\mathcal{D}_S \circ (\mathbb{I} - Q_S) = (\mathbb{I} - Q_S) \circ \mathcal{D}_S \quad \Rightarrow \quad \mathcal{D}_S Q_S = Q_S \mathcal{D}_S. \tag{6.6}$$

For $\mathcal{D}_T$, on the one hand, we can define

$$B_T = (\mathbb{I} - Q_T)^{-1} = \mathbb{I} + Q_T + Q_T^2 + \dots.$$

It is a well-defined operator over $\Lambda^{\mathcal{K},\Gamma}$ because by (6.3) $\ell(Q_T^k(x))$ diverges to $-\infty$ (as $k \to \infty$) for any $x$. Moreover, by (6.5), $B_T \mathcal{D}_T = \mathcal{D}_T B_T$. On the other hand, for operator $B'_T = (\mathcal{D}_T)^{p-1} B_T$, we know

$$\mathcal{D}_T B'_T = (\mathcal{D}_T)^p B_T = (\mathbb{I} - Q_T)(\mathbb{I} - Q_T)^{-1} = \mathbb{I} = (\mathcal{D}_T)^{p-1} B_T \mathcal{D}_T = B'_T \mathcal{D}_T.$$

Therefore, $B'_T$ is the required inverse of $\mathcal{D}_T$. Similarly for $\mathcal{D}_S$, we can define

$$B'_S = (\mathcal{D}_S)^{p^2-1} B_S \;\text{ where }\; B_S = (\mathbb{I} - Q_S)^{-1} = \mathbb{I} + Q_S + Q_S^2 + \dots.$$

and then $\mathcal{D}_S B'_S = B'_S \mathcal{D}_S$, so $B'_S$ is the desired inverse of $\mathcal{D}_S$. $\square$

*Remark* 6.1.2. Note that $T$ and $S$ also satisfy perturbed group relations in the form of (6.4), therefore, the same argument in Lemma 6.1.1 implies both $T$ and $S$ are invertible too.

Note that by (6.4), $\mathcal{D}_T$ and $\mathcal{D}_S$ are *not* (but almost like in terms of strictly lower filtration perturbations) do not generate finite group actions (where $\mathcal{D}_{R_p}$ and $\mathcal{D}_{R_{p^2}}$ are group actions with order $p$ and $p^2$ respectively). However, from the following lemma, these can always be reduced to be group actions.

**Lemma 6.1.3.** *There exists $T'$ and $S'$ such that $\mathcal{D}_{T'}$ and $\mathcal{D}_{S'}$ are group actions, that is,*

$$\mathcal{D}_{T'}^p = \mathbb{I} \ \ \text{and} \ \ \mathcal{D}_{S'}^p = \mathcal{D}_{T'} \ (\text{so } \mathcal{D}_S^{p^2} = \mathbb{I}).$$

*Moreover, $T'$ and $S'$ are strictly lower filtration perturbations of $T$ and $S$ respectively.*

*Proof.* Denote $T^p = \mathbb{I} - P_T$ where $\ell(P_T x) \leq \ell(x) - \hbar$ for any $x$. We want to find $T'$ such that $(T')^p = (T^p + P_T) = \mathbb{I}$. Since $T$ is invertible by Remark 6.1.2, define

$$P_T^{(1)} = T^{-p} P_T \ \ \text{and} \ \ T' = T(\mathbb{I} + P_T^{(1)})^{\frac{1}{p}}$$

where $(\mathbb{I} + P_T^{(1)})^{\frac{1}{p}}$ is defined using the binomial expansion, that is,

$$(\mathbb{I} + P_T^{(1)})^{\frac{1}{p}} = \mathbb{I} + \binom{\frac{1}{p}}{1} P_T^{(1)} + \binom{\frac{1}{p}}{2} (P_T^{(1)})^2 + ... := \mathbb{I} + P_T^{(2)}$$

where $P_T^{(2)} = \binom{\frac{1}{p}}{1} P_T^{(1)} + \binom{\frac{1}{p}}{2} (P_T^{(1)})^2 + ...$ and it defines an operator over $\Lambda^{\mathcal{K},\Gamma}$. Hence, denote $P_T^{(3)} = T P_T^{(2)}$,

$$T' = T + P_T^{(3)}$$

which is the required (group action) $T'$ such that it is a strictly lower filtration perturbation of $T$. Moreover, $\mathcal{D}_{T'}^p = \mathcal{D}_{(T')^p} = \mathcal{D}_{\mathbb{I}} = \mathbb{I}$.

Now suppose $T = S^p$. We want to find $S'$ such that $(S')^p = T' = T + P_T^{(3)}$. Again, by Remark 6.1.2, define

$$P_S^{(1)} = S^{-p} P_T^{(3)} \quad \text{and} \quad S' = S(\mathbb{I} + P_S^{(1)})^{\frac{1}{p}}$$

where $(\mathbb{I} + P_S^{(1)})^{\frac{1}{p}}$ is defined using the binomial expansion, that is

$$(\mathbb{I} + P_S^{(1)})^{\frac{1}{p}} = \mathbb{I} + \binom{\frac{1}{p}}{1} P_S^{(1)} + \binom{\frac{1}{p}}{2} (P_S^{(1)})^2 + \ldots := \mathbb{I} + P_S^{(2)}$$

where $P_S^{(2)} = \binom{\frac{1}{p}}{1} P_S^{(1)} + \binom{\frac{1}{p}}{2} (P_S^{(1)})^2 + \ldots$. Denote $P_S^{(3)} = S P_S^{(2)}$,

$$S' = S + P_S^{(3)}$$

which is the required (group action) $S'$ such that it is a strictly lower filtration perturbation of $S$. Moreover, $\mathcal{D}_{S'}^p = \mathcal{D}_{(S')^p} = \mathcal{D}_{T'}$. □

To simplify the notation of proofs below, denote $[\cdot, \cdot]$ as commutator (of two matrices or operators). So $A$ and $B$ commutes if and only if $[A, B] = 0$. An important observation from definition of $T$ is $[T, P_T] = [P_T, T] = 0$. Then we have

**Corollary 6.1.4.** *For $\mathcal{D}_{T'}$ and $\mathcal{D}_{S'}$ constructed from Lemma 6.1.3, we have $[\mathcal{D}_{T'}, \partial_{co}] = 0$ and $[\mathcal{D}_{S'}, \partial_{co}] = 0$.*

*Proof.* First, we claim $[T', \partial] = 0$ and $[S', \partial] = 0$. In fact, starting from $[T, \partial] = 0$ and the definitions of $P_T^{(1)}$, $P_T^{(2)}$ and $P_T^{(3)}$, we have

$$\begin{aligned} [T, \partial] = 0 \Rightarrow [P_T, \partial] = 0 \Rightarrow [P_T^{(1)}, \partial] = 0 \Rightarrow [P_T^{(2)}, \partial] = 0 \\ \Rightarrow [P_T^{(3)}, \partial] = 0 \Rightarrow [T', \partial] = 0. \end{aligned}$$

Similarly, for $S'$, from $[S, \partial] = 0$, $[P_T^{(3)}, \partial] = 0$ and definitions of $P_S^{(1)}$, $P_S^{(2)}$ and $P_S^{(3)}$, we have

$$\begin{aligned} ([S, \partial] = 0], [P_T^{(3)}, \partial] = 0]) \Rightarrow [P_S^{(1)}, \partial] = 0 \Rightarrow [P_S^{(2)}, \partial] = 0 \\ \Rightarrow [P_S^{(3)}, \partial] = 0 \Rightarrow [S', \partial] = 0. \end{aligned}$$

Second, we claim $[T', T] = 0$ and $[S', T] = 0$. In fact, starting from $[P_T, T] = 0$, we have

$$[P_T, T] = 0 \Rightarrow [P_T^{(1)}, T] = 0 \Rightarrow [P_T^{(2)}, T] \Rightarrow [P_T^{(3)}, T] = 0 \Rightarrow [T', T] = 0.$$

Similar for $S'$, starting from $[S, T] = 0$ and $[P_T^{(3)}, T] = 0$, we have

$$\begin{aligned}([S, T] = 0, [P_T^{(3)}, T] = 0) \Rightarrow [P_S^{(1)}, T] = 0 \Rightarrow [P_S^{(2)}, T]& \\ \Rightarrow [P_S^{(3)}, T] = 0 \Rightarrow [S', T] = 0.&\end{aligned}$$

Third, we conclude $[\mathcal{D}_{T'}, \partial_{co}] = 0$ and $[\mathcal{D}_{S'}, \partial_{co}] = 0$. In fact,

$$\begin{aligned}\begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix} \cdot \begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} &= \begin{pmatrix} T'\partial & -T'(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial T' \end{pmatrix} \\ &= \begin{pmatrix} \partial T' & -(T - \xi_p \cdot \mathbb{I})T' \\ 0 & -\partial T' \end{pmatrix} \\ &= \begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} \cdot \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}\end{aligned}$$

where the second equality comes from the first part of two claims above. Similarly,

$$\begin{aligned}\begin{pmatrix} S' & 0 \\ 0 & S' \end{pmatrix} \cdot \begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} &= \begin{pmatrix} S'\partial & -S'(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial S' \end{pmatrix} \\ &= \begin{pmatrix} \partial S' & -(T - \xi_p \cdot \mathbb{I})S' \\ 0 & -\partial S' \end{pmatrix} \\ &= \begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} \cdot \begin{pmatrix} S' & 0 \\ 0 & S' \end{pmatrix}\end{aligned}$$

where the second equality comes from the second part of two claims above. □

*Remark* 6.1.5. Since $\mathcal{D}_T$ is a strictly lower filtration perturbation of $\mathcal{D}_{R^p}$, so is $\mathcal{D}_{T'}$ by its construction. Similarly, $\mathcal{D}_{S'}$ is a strictly lower filtration perturbation of $\mathcal{D}_{R_{p^2}}$. For orthogonality, strictly lower filtration perturbation behaves well as proved by the following easy lemma which will be used often later.

**Lemma 6.1.6.** *Strictly lower filtration perturbation preserves orthogonality. Specifically, given a set of orthogonal elements over $\Lambda^{\mathcal{K},\Gamma}$, say $\{v_1, ..., v_n\}$ and any strictly lower filtration for each $v_i$, that is $v_i + w_i$ where $\ell(w_i) < \ell(v_i)$, for $1 \leq i \leq n$,*

$$\{v_1 + w_1, ..., v_n + w_n\} \text{ are orthogonal over } \Lambda^{\mathcal{K},\Gamma}.$$

*Proof.* For any $\lambda_1, ..., \lambda_n \in \Lambda^{\mathcal{K},\Gamma}$,

$$\ell(\lambda_1(v_1 + w_1) + ...\lambda_n(v_n + w_n)) = \ell(\lambda_1 v_1 + ...\lambda_n v_n) = \max_{1\leq i\leq n}\{\ell(\lambda_i v_i)\} = \max_{1\leq i\leq n}\{\ell(\lambda_i(v_i + w_i))\}.$$

So $\{v_1 + w_1, ..., v_n + w_n\}$ are also orthogonal. □

## 6.2 Preparation

### 6.2.1 Orthogonal invariant complement

**Proposition 6.2.1.** *Suppose $V$ is acted by a group action $T$ such that $T^p = \mathbb{I}$ and it exactly preserves filtration. For any $T$-invariant subspace $V_1$, there exists an orthogonal complementary $T$-invariant $W$ in the sense that $V_1$ is orthogonal to $W$ and $V = V_1 \oplus W$.*

Recall if vector space $V$ is a representation of a group $G$ satisfying condition that field of scalars for $V$ has its [1]

$$char(\mathcal{K}) \nmid |G| \tag{6.7}$$

then given any $G$-invariant subspace $V_1 \leq V$, there exists a $G$-invariant complementary subspace $W \leq V$. Actually we can construct $W$ explicitly.

*Construction* 6.2.2. Taking any complement of $V_1$ (in the sense of vector space, no orthogonality is involved and not necessarily $G$-invariant), say $U$, consider the projection map $\pi_{V_1} : V \to V_1$ with

[1] See Theorem 4.1 and it equivalent conclusion like Lemma 2.2.11 in [Kow13]). Non-division between characteristic of field of scalars and order of group is the only hypothesis. When we apply to Novikov field $\Lambda^{\mathcal{K},\Gamma}$ and group $G$ with order $p$ here, $char(\mathcal{K}) \neq p$ implies $char(\Lambda^{\mathcal{K},\Gamma}) \neq p$. Therefore it satisfies this hypothesis.

respect to the decomposition $V = V_1 \oplus U$, and define

$$W = \ker\left(\frac{1}{|G|}\sum_{g\in G} g\cdot\pi_{V_1}\cdot g^{-1}\right). \tag{6.8}$$

It is easy to check that $W$ is $G$-invariant for any $g \in G$ and $\dim W = \dim U$. There is a useful observation that (6.8) implies each $x \in W$ satisfies

$$\frac{1}{|G|}\sum_{g\in G} g(\mathbb{I}-\pi_U)g^{-1}(x) = 0.$$

Therefore, we have

$$x = \frac{1}{|G|}\sum_{g\in G} g\cdot\pi_U\cdot g^{-1}(x) \quad \text{for all } x \in W. \tag{6.9}$$

In order to get Proposition 6.2.1, we need to show $W$ is orthogonal to $V_1$. We will use the following lemma from [UZ15],

**Lemma 6.2.3.** *Let $(V, \ell)$ be a filtered vector space over $\Lambda^{\mathcal{K},\Gamma}$ and let $V_1, U, W \leq V$ be such that $U$ is an orthogonal complement to $V_1$ and $\dim(U) = \dim(W)$. Consider the projection $\pi_U : V \to U$ associated to the direct sum decomposition $V = U \oplus V_1$. Then $W$ is an orthogonal complement of $V_1$ if and only if $\ell(\pi_U x) = \ell(x)$ for all $x \in W$.*

*Proof of Proposition 6.2.1.* Let $U$ be the orthogonal complement of $V_1$ that is used in the Construction 6.2.2. By Lemma 6.2.3, we will need to show for any $x \in W$, $\ell(x) = \ell(\pi_U x)$. Decompose $x = u + v$ where $u \in U$ and $v \in V_1$. By (6.9), we have

$$\begin{aligned} x &= \frac{1}{p}\left(\pi_U x + \sum_{i=1}^{p-1} T^i \pi_U T^{p-i} x\right) \\ &= \frac{1}{p}\left(u + \sum_{i=1}^{p-1} T^i \pi_U T^{p-i} u\right). \end{aligned}$$

The key step for the second line is that since $V_1$ is invariant, $\pi_U T^{p-i}(u+v) = \pi_U T^{p-i}u + \pi_U T^{p-i}v = \pi_U T^{p-i}u$ because $\pi_U T^{p-i}v = 0$. So we can write $v$ in terms of $u$, that is

$$v = -\frac{p-1}{p}u + \frac{1}{p}\sum_{i=1}^{p-1} T^i \pi_U T^{p-i}u.$$

Because $T$ exactly preserves filtration and $U$ is orthogonal to $V_1$, it follows that $\ell(v) \leq \ell(u)$. Therefore,

$$\ell(x) = \max\{\ell(u), \ell(v)\} = \ell(u) = \ell(\pi_U x).$$

Therefore, we get the conclusion. $\square$

**Example 6.2.4.** *For group action (guaranteed by Lemma 6.1.3) $\mathcal{D}_{T'} : Cone_k(T - \xi \cdot \mathbb{I}) \to Cone_k(T - \xi \cdot \mathbb{I})$, since $\ker(\partial_{co})$ is a $\mathcal{D}_{T'}$-invariant subspace of $Cone_k(T - \xi \cdot \mathbb{I})$ by Corollary 6.1.4, there exists an orthogonal complement of $\ker(\partial_{co})$ in $Cone_k(T - \xi \cdot \mathbb{I})$, denoted as $W$ which is also $\mathcal{D}_{T'}$-invariant.*

### 6.2.2 Restriction to 0-level

In this subsection, we will work on the vector space over universal Novikov field, that is, for $\Lambda^{\mathcal{K},\Gamma}$, $\Gamma = \mathbb{R}$. The advantage is that we can always rescale preferred element $x$ in the vector space such that $\ell(x) = 0$. Moreover, since we have seen that our obstruction (see (1.11)) will be constructed only from generalized boundary depth (which does not involve the specific value of end points), By Proposition 6.8 in [UZ15], it will be invariant under the coefficient extension.

From the idea of [Ush08], any orthogonalizable $\Lambda^{\mathcal{K},\mathbb{R}}$-space $(V, \ell)$ can be identified with $((\Lambda^{\mathcal{K},\mathbb{R}})^n, -\vec{\nu})$ (for some $n = \dim_{\Lambda^{\mathcal{K},\mathbb{R}}} V \in \mathbb{N}$) in Example 3.3.4 under an orthonormal basis, where $\ell(v) = -\vec{\nu}(\lambda_1, ..., \lambda_n)$ if $v$ is identified with a vector $(\lambda_1, ..., \lambda_n)$ under this basis. Therefore, for such $V$, we can associated a $\mathcal{K}$-module

$$[V] = V_{\leq 0}/V_{<0}$$

where

$$V_{\leq 0} := \{v \in V \mid \ell(v) \leq 0\} \text{ and } V_{<0} := \{v \in V \mid \ell(v) < 0\}.$$

In particular, denote $\Lambda^{\mathcal{K},\mathbb{R}_{\geq 0}} = \{\lambda \in \Lambda^{\mathcal{K},\mathbb{R}} \,|\, \nu(\lambda) \geq 0\}$ and $\Lambda^{\mathcal{K},\mathbb{R}_{>0}} = \{\lambda \in \Lambda^{\mathcal{K},\mathbb{R}} \,|\, \nu(\lambda) > 0\}$. Note that $\mathcal{K} \simeq \Lambda^{\mathcal{K},\mathbb{R}_{\geq 0}}/\Lambda^{\mathcal{K},\mathbb{R}_{>0}}$. There is a quotient projection $\pi : V_{\leq 0} \to [V]$ by, roughly speaking, taking the 0-level filtration term, that is, for $\lambda \in \Lambda^{\mathcal{K},\mathbb{R}}$,

$$\lambda = \sum_g a_g T^g \xrightarrow{\pi} a_0 \tag{6.10}$$

Then we can show

**Lemma 6.2.5.** *A $\Lambda^{\mathcal{K},\mathbb{R}}$-orthonormal set $\{e_1, ..., e_n\}$ reduces to a $\mathcal{K}$-linearly independent set $\{[e_1], ..., [e_n]\}$ under projection $\pi$ defined in (6.10). Conversely, for a set $\{e_1, ..., e_n\}$ over $\Lambda^{\mathcal{K},\mathbb{R}}$, if its reduction $\{[e_1], ..., [e_n]\}$ are $\mathcal{K}$-linearly independent, then $\{e_1, ..., e_n\}$ are $\Lambda^{\mathcal{K},\mathbb{R}}$-orthogonal. Therefore, in particular, $[V] = (\mathcal{K})^{\dim_{\Lambda^{\mathcal{K},\mathbb{R}}} V}$.*

*Proof.* Suppose $\{[e_1], ..., [e_n]\}$ are not $\mathcal{K}$-linearly independent. There exists $\eta_1, ..., \eta_n \in \mathcal{K}$, not all zero, such that

$$\eta_1[e_1] + ... + \eta_n[e_n] = 0.$$

Then

$$\eta_1 e_1 + ... + \eta_n e_n = \eta_1[e_1] + ... + \eta_n[e_n] + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\}.$$

So

$$\ell(\eta_1 e_1 + ... + \eta_n e_n) < 0 = \max_{1 \leq i \leq n} \{\ell(e_i) - \nu(\eta_i)\}.$$

Therefore, $\{e_1, ..., e_m\}$ are not $\Lambda^{\mathcal{K},\mathbb{R}}$-orthogonal.

Conversely, suppose $\{e_1, ..., e_n\}$ are not $\Lambda^{\mathcal{K},\mathbb{R}}$-orthogonal. There exist $\lambda_1, ...\lambda_n \in \Lambda^{\mathcal{K},\mathbb{R}}$, not all zero, such that

$$\ell(\lambda_1 e_1 + ... + \lambda_n e_n) < \max_{1 \leq i \leq n} \{\ell(e_i) - \nu(\lambda_i)\} = 0 - \min_{1 \leq i \leq n} \{\nu(\lambda_i)\}.$$

If we rescale $\lambda_i$ on both sides such that $\min_{1\leq i\leq n}\{\nu(\lambda_i)\} = 0$, then still we have the inequality. However, reducing to the 0-filtration level, which is the highest filtration level, we have

$$[\lambda_1][e_1] + ...[\lambda_n][e_n] = 0 \text{ where } [\lambda_i] \text{ in } \mathcal{K}.$$

Because of our rescaling, not all $[\lambda_i]$ are zero, which means $\{[e_1], ..., [e_n]\}$ are not $\mathcal{K}$-linearly independent.

In particular, if $\{e_1, ..., e_n\}$ is an $\Lambda^{\mathcal{K},\mathbb{R}}$-orthonormal basis of $V$, then $\dim_{\mathcal{K}}[V] \geq n$. Meanwhile, $[V]$ is a submodule of $[\Lambda^{\mathcal{K},\mathbb{R}}]^n = \mathcal{K}^n$. So $\dim_{\mathcal{K}}[V] = n = \dim_{\Lambda^{\mathcal{K},\mathbb{R}}} V$. $\square$

Not only can we reduce spaces, but also we can reduce maps. For a $\Lambda^{\mathcal{K},\mathbb{R}}$-linear map $A$ on $(V, \ell)$ which is exact filtration preserving (for instance, $\mathcal{D}_{T'}$ or $\mathcal{D}_{S'}$), under an orthonormal basis, $A \in M_{n\times n}(\Lambda^{\mathcal{K},\mathbb{R}_{\geq 0}})$. Note that then $A(V_{<0}) \leq A_{<0}$ which implies we have a well-defined reduced map of $A$, denoted as $[A]$,

$$[A] : [V] \to [V].$$

**Example 6.2.6.** *Suppose under an orthonormal basis,*

$$A = \begin{pmatrix} 1+T^2 & T^6 & T^2+T^4 \\ T^4 & 2 & T^6-T^{10} \\ 2 & T^2 & 5+T^2 \end{pmatrix}.$$

*Then*

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

**Example 6.2.7.** *Suppose $A$ is exact filtration preserving and $\lambda \in \Lambda^{\mathcal{K},\mathbb{R}}$,*

$$A^n = \lambda \cdot \mathbb{I} \xrightarrow{\textit{reduces to}} [A]^n = [\lambda] \cdot \mathbb{I}.$$

*In particular, for any $x \in V$, $Ax = \lambda x$ reduces to $[A][x] = [\lambda][x]$.*

### 6.2.3 Irreducible condition

**Lemma 6.2.8.** *$\mathcal{K}$ satisfies "irreducible condition" if and only if $\Lambda^{\mathcal{K},\Gamma}$ satisfies "irreducible condition".*

*Proof.* One direction "$\Leftarrow$" is trivial because $\mathcal{K} \hookrightarrow \Lambda^{\mathcal{K},\Gamma}$. We will just prove the other one "$\Rightarrow$". Suppose not. Then there exists some $x \in \Lambda^{\mathcal{K},\Gamma}$ and number $q$ such that

$$x^p = \xi_p^q \tag{6.11}$$

but $p$ does not divide $q$. The general form of $x$ is $x = a_m t^{\lambda_m} + a_{m+1} t^{\lambda_{m+1}} + \dots$ where $a_m \neq 0$ and $\lambda_m < \lambda_{m+1} < \dots$ (which diverges to infinity). If $\lambda_m \neq 0$, then

$$x^p = a_m^p t^{p\lambda_m} + \dots$$

such that the lowest degree $p\lambda_m$ is either strictly positive or strictly negative, which by equation (6.11) above, forces $a_m^p = 0$. So $a_m = 0$. Contradiction. Now we are left to the case that $\lambda_m = 0$, so we may rewrite $x$ as $x = a_m + a_{m+1} t^{\lambda_1} + \dots$ where $a_m \in \mathcal{K}$. Therefore, (6.11) implies $a_m^p = \xi_p^q$ which contradicts the hypothesis that $\mathcal{K}$ satisfies "irreducible condition". $\square$

Here comes a perturbed version of the lemma above, which will be used later.

**Lemma 6.2.9.** *Suppose $\mathcal{K}$ satisfies "irreducible condition". For any $g(x) \in \Lambda^{\mathcal{K},\Gamma_{>0}}[x]$ with deg$(g(x)) <$ $p$,*

$$x^p = 1 + g(x) \text{ is solvable in } \Lambda^{\mathcal{K},\Gamma};$$

*while for any $q$ not divisible by $p$,*

$$x^p = \xi_p^q + g(x) \text{ is not solvable in } \Lambda^{\mathcal{K},\Gamma}.$$

*Proof.* The second conclusion is easier to prove. Suppose it is solvable with some solution $x$. Then $\ell(x)$ is necessarily 0. So reduced to $[V]$,

$$[x]^p = \xi_p^q$$

for some $q$ not divisible by $p$ in $\mathcal{K}$. It contradicts to the condition that $\mathcal{K}$ satisfies "irreducible condition".

For the first conclusion, we will run an inductive process to solve $x$ piece by piece. First note that since $\ell(x)$ is necessarily 0, we can write

$$x = a_0 t^{g_0} + a_1 t^{g_1} + a_2 t^{g_2} + \dots.$$

where $g_0 = 0$ and $g_i > 0$ for any $i \geq 1$. Moreover, we arrange the exponents to be in a strictly increasing order, that is, $g_i < g_{i+1}$, which, by definition, diverge to infinity. Denote $G_x = \{g_i \mid g_i \text{ is an exponent of } x\}$. Then denote

$$S = \left\{ \sum_{g \in G_x} n_g g \,\middle|\, n_g \in \mathbb{N} \cup \{0\} \right\}.$$

Note that for any given number $\lambda \in \mathbb{R}_{\geq 0}$, there are only finitely many linear combinations from $S$ whose values are no greater than $\lambda$ (so $S$ is discrete). Therefore, it makes sense to define, for any $n \in \mathbb{N} \cup \{0\}$, $s_n$ representing the $n$-th smallest combination from $S$. For example, $s_0 = 0 = g_0$ and $s_1 = g_1$. It's not clear in general what the expression of $s_n$ is, but for sure, $s_n \leq g_n$ because there might be combination of smaller terms still not exceeding $g_n$.

Now denote $V_{<s_k} = \{v \in V \mid \ell(v) < -s_k\}$ and projection $\pi_{s_k} : V_{\leq 0} \to V_{\leq 0}/V_{<s_k}$. Note when $k = 0$, $\pi_{s_k} = \pi$ defined in (6.10). For initial step, apply $\pi_{s_0}$ to $x^p = 1 + h(x)$ and we get

$$(\pi_{s_0} x)^p = 1 + \pi_{s_0}(h(x)),$$

that is, $a_0^p = 1$. Indeed, $\pi_{s_0}(h(x)) = 0$ because by definition of $h(x)$, each of its coefficients has valuation strictly bigger than 0 which implies the lowest valuation of $h(x)$ is strictly bigger than $s_0$. Since $\mathcal{K}$ contains all the $p$-th root of unity, we can solve $a_0$. Suppose for $s_n$, we can solved $a_0, \dots, a_{m(n)}$ for some $m(n) \in \mathbb{N}$. For $s_{n+1}$, there are two cases. Either, there is no new coefficient $a_i$ appearing to be solved. Then we are done with this step and then move to $s_{n+2}$. Or new $a_{m(n)+1}$ appears. However, new $a_{m(n)+1}$ always appears in accompany with its formal power $t^{g_{m(n)+1}}$ and the lowest exponent it contributes is $pa_0^{p-1}a_{m(n)+1}t^{g_{m(n)+1}}$. Moreover, as this term is not in the previous inductive step, we know $(s_n <)\, g_{m(n)+1} = s_{n+1}$ by definition of $s_i$. Therefore,

$$(\pi_{s_{n+1}}x)^p = F(a_0, ..., a_{m(n)}, t) + pa_0^{p-1}a_{m(n)+1}t^{g_{m(n)+1}},$$

where $F(a_0, ..., a_{m(n)}, t)$ is some combination of $a_0, ..., a_{m(n)}$ and powers of $t$ with exponents no greater than $s_{n+1}$.

On the other hand, from the similar reason as above, $\pi_{s_{n+1}}(h(x))$ will not contain any $a_{n(m)+1}$ (or any $i \geq n(m)+1$) because the lowest exponent it contributes from its formal power is already bigger than $s_{n+1}$. Therefore,

$$1 + \pi_{s_{n+1}}(h(x)) = 1 + G(a_0, ..., a_{m(n)}, t),$$

where $G(a_0, ..., a_{m(n)}, t)$ is some combination of $a_0, ..., a_{m(n)}$ and powers of $t$ with exponents no greater than $s_{n+1}$. From the inductive step of $s_n$, we have already known $a_0, ..., a_{m(n)}$ and the following equation

$$F(a_0, ..., a_{m(n)}, t) + pa_0^{p-1}a_{m(n)+1}t^{g_{m(n)+1}} = G(a_0, ..., a_{m(n)}, t)$$

is linear on $a_{m(n)+1}$, therefore, $a_{m(n)+1}$ can be solved. $\square$

From now on, we will always assume $\mathcal{K}$ satisfies "irreducible condition" and $\Gamma = \mathbb{R}$. For brevity, we introduce the following notation.

*Definition* 6.2.10. For any $x \in V$ and an operator $A$ such that $A^p = \lambda \cdot \mathbb{I}$ for some scalar $\lambda \in \Lambda^{\mathcal{K},\Gamma}$, we denote

$$V_x = \text{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, Ax, ..., A^{p-1}x \right\rangle$$

and call it *the cyclic span (of $A$) by $x$*.

Note that by Lemma 6.2.8 and Lemma 4.15 in [PS14], if $\lambda = \xi_p^q$ for $1 \leq q \leq p-1$, we know $\{x, Ax, ..., A^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-linearly independent, so $\dim_{\Lambda^{\mathcal{K},\Gamma}} V_x = p$. However, we can get a stronger result as follows.

**Corollary 6.2.11.** *Let $V$ be a $\Lambda^{\mathcal{K},\Gamma}$-vector space associated with an $\Lambda^{\mathcal{K},\Gamma}$-linear exact filtration preserving operator $A$ such that*

$$A^p = \xi_p^q \cdot \mathbb{I}$$

*for $1 \leq q \leq p-1$. Then for any $x \in V$, $\{x, Ax, ..., A^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal.*

*Proof.* Rescale $x$ such that $\ell(x) = 0$ if necessary. The cyclic span $V_x$ of $A$ is an $A$-invariant subspace. Then, $[V_x]$ is an $[A]$-invariant subspace where

$$[V_x] = \mathrm{span}_{\mathcal{K}} \left\langle [x], [A][x], ..., [A]^{p-1}[x] \right\rangle$$

is a cyclic span of $[x]$ since $[A]^p = \xi_p^q \cdot \mathbb{I}$. Because $\mathcal{K}$ satisfies "irreducible condition", we know $p \,|\, \dim_{\mathcal{K}}[V_x]$. But $\dim_{\mathcal{K}}[V_x] \leq p$. The rigidity $\dim_{\mathcal{K}}[V_x] = p$ implies $\{[x], [A][x], ..., [A]^{p-1}[x]\}$ are $\mathcal{K}$-linearly independent. By Lemma 6.2.5, $\{x, Ax, ..., A^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal. □

**Corollary 6.2.12.** *Under condition of Corollary 6.2.11, if $y \in V$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $V_x$, then cyclic span $V_y$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $V_x$.*

*Proof.* Rescale $y$ and $x$ such that $\ell(x) = \ell(y) = 0$ if necessary. By Lemma 6.2.5, condition $y$ being $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $V_x$ implies $[y]$ being $\mathcal{K}$-linearly independent from cyclic $\mathcal{K}$-span $[V_x]$. We claim

$$\left\{[y], [A][y], ..., [A]^{p-1}[y], [x], [A][x], ..., [A]^{p-1}[x]\right\}$$

are $\mathcal{K}$-linearly independent. In fact, $\mathcal{K}$-span of all these reduced $2p$ elements, denoted as $V_{x,y}$, is an $[A]$-invariant subspace. By "irreducible condition", its dimension is either $p$ or $2p$. If it is $p$, then, by Corollary 6.2.12,

$$\dim_{\mathcal{K}}([V_y] \cap [V_x]) = \dim_{\mathcal{K}}[V_y] + \dim_{\mathcal{K}}[V_x] - \dim_{\mathcal{K}} V_{x,y} = p + p - p = p.$$

Then since $p$ is prime, $[V_y] = [V_x]$, which is a contradiction. Then by Lemma 6.2.5, the original $2p$ elements $\{y, ..., A^{p-1}y, x, ..., A^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal. So, in particular, $V_y$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $V_x$. □

*Remark* 6.2.13. Note that in the situation of both Corollary 6.2.11 and Corollary 6.2.12, if $A$ satisfies $A^p = \mathbb{I}$ (that is $q = 0$), then we can't directly conclude that $\{x, Ax, ..., A^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal (even $\Lambda^{\mathcal{K},\Gamma}$-linearly independent) because "irreducible condition" does not apply here. To get the expected result on the multiplicity of $p$, more structure of self-mapping cone will be used later.

### 6.2.4 Filtration optimal pair

The following lemma is key to construct singular value decomposition later. [2]

**Lemma 6.2.14.** *Let $(V_1, \ell_1)$ and $(V_2, \ell_2)$ be $\Lambda^{\mathcal{K},\Gamma}$-vector space and let $A : V_1 \to V_2$ be any nonzero $\Lambda^{\mathcal{K},\Gamma}$-linear map. Then there exists some $y_* \in V_1 \setminus \{0\}$ such that, for all $y \in V_1 \setminus \{0\}$,*

$$\ell_2(Ay_*) - \ell_1(y_*) \geq \ell_2(Ay) - \ell_1(y). \tag{6.12}$$

*Proof.* Because $\ker A$ is a subspace of $V_1$, by Corollary 2.17 and 2.18 in [UZ15], there exists an orthogonal basis of $V_1$, say $(v_1, \ldots, v_r, v_{r+1}, \ldots, v_n)$ such that $(v_{r+1}, ..., v_n)$ is an orthogonal basis for $\ker A$ where $r = rank(A)$. Meanwhile, let $(w_1, ..., w_m)$ be an orthogonal basis for $V_2$. Represent $A$ by an $m \times n$ matrix $\{A_{ij}\}$ over $\Lambda^{\mathcal{K},\Gamma}$ with respect to these two bases, so that $Av_j = \sum_{1 \leq i \leq m} A_{ij} w_i$ for each $j \in \{1, ..., n\}$. For any $y = \sum_{1 \leq j \leq n} \lambda_j v_j$ in $V$, we have

$$\begin{aligned} \ell_2(Ay) = \ell_2\left(\sum_{1\leq i\leq m}\left(\sum_{1\leq j\leq n} A_{ij}\lambda_j\right) w_i\right) &= \max_{1\leq i\leq m}\left(\ell_2(w_i) - \nu\left(\sum_{1\leq j\leq n} A_{ij}\lambda_j\right)\right) \\ &= \ell_2(w_{i(y)}) - \nu\left(\sum_{1\leq j\leq n} A_{i(y)j}\lambda_j\right) \end{aligned}$$

where $i(y) \in \{1, ..., m\}$ is the index attaining the maximum in the middle. By the definition of the valuation $\nu$,

$$-\nu\left(\textstyle\sum_{1\leq j\leq n} A_{i(y)j}\lambda_j\right) \leq \max_{1\leq j\leq n}\left(-\nu(A_{i(y)j}) - \nu(\lambda_j)\right) = -\nu(A_{i(y)j(y)}) - \nu(\lambda_{j(y)})$$

[2] This lemma is exactly the same as Lemma 3.5 in the early version of [UZ15]. For submitted version, this lemma has been deleted for brevity. For reader's convenience, we add/repeat it here.

where, again, $j(y) \in \{1, ..., n\}$ is the index attaining the maximum in the middle. Also due to the orthogonality of $(v_1, ..., v_n)$,

$$\ell_1(y) = \max_{1 \le j \le n} (\ell_1(v_j) - \nu(\lambda_j)) \ge \ell_1(v_{j(y)}) - \nu(\lambda_{j(y)}),$$

so

$$\begin{aligned} \ell_2(Ay) - \ell_1(y) &\le \big(\ell_2(w_{i(y)}) - \nu(A_{i(y)j(y)}) - \nu(\lambda_{j(y)})\big) - \big(\ell_1(v_{j(y)}) - \nu(\lambda_{j(y)})\big) \\ &= \ell_2(w_{i(y)}) - \nu(A_{i(y)j(y)}) - \ell_1(v_{j(y)}). \end{aligned} \tag{6.13}$$

Now choose $(i_0, j_0)$ among $(i, j) \in \{1, ..., m\} \times \{1, ..., n\}$ so that

$$\ell_2(w_{i_0}) - \nu(A_{i_0 j_0}) - \ell_1(v_{j_0}) \ge \ell_2(w_i) - \nu(A_{ij}) - \ell_1(v_j) \tag{6.14}$$

for all $i$ and $j$. Then due to the orthogonality of $(w_1, ..., w_m)$, $\ell_2(Av_{j_0}) = \max_{1 \le i \le n}(\ell_2(w_i) - \nu(A_{ij_0}))$. So using (6.14), we have

$$\ell_2(Av_{j_0}) - \ell_1(v_{j_0}) = \max_{1 \le i \le m} (\ell_2(w_i) - \nu(A_{ij_0}) - \ell_1(v_{j_0})) = \ell_2(w_{i_0}) - \nu(A_{i_0 j_0}) - \ell_1(v_{j_0})$$

Given any $y \in C$, (6.14) holds for $i = i(y), j = j(y)$, so using (6.13) we get $\ell_2(Av_{j_0}) - \ell_1(v_{j_0}) \ge \ell_2(Ay) - \ell_1(y)$. Therefore $y_0 = v_{j_0}$ obeys the desired optimality property. $\square$

Roughly speaking The proof of the main theorem goes as follows. Step one: prove the conclusion in a special case that $C_T = C_S = 0$ in (6.1). Then it's easy to check

$$\mathcal{D}_{T'} = \mathcal{D}_T = \mathcal{D}_{R_p} \quad \text{and} \quad \mathcal{D}_{S'} = \mathcal{D}_S = \mathcal{D}_{R_{p^2}}.$$

We call this *unperturbed case*. Step two: we will prove the conclusion in the general case, that is, *perturbed case*. We emphasize here that Step one will give a clear picture of the general algebraic strategy of the proof. The difference in the proofs between unperturbed case and perturbed case is mild. The majority of the algebraic construction can be adopted directly from Step one to Step two.

## 6.3 Unperturbed $p$-cyclic singular value decomposition

In this subsection, as mentioned above, we will work on the special case - unperturbed case. The idea is the following

$$\begin{tikzcd}
\boxed{\text{eigenspace decomposition of action } \mathcal{D}_{R_p}} \arrow[r, "(a)"] \arrow[d, "(c)"] & \boxed{\text{singular value decomposition of } \partial_{co} \text{ compatible with } \mathcal{D}_{R_p}} \arrow[d, "(b)"] \\
\boxed{\text{special care of eigenvalue } 1} & \boxed{\text{singular value decomposition of } \partial_{co} \text{ compatible with } \mathcal{D}_{R_{p^2}}}
\end{tikzcd}$$

First, by Example 6.2.4, there exists a $\mathcal{D}_{R_p}$-invariant orthogonal complement of $\ker(\partial_{co})$ in $Cone_{k+1}(T - \xi_p \cdot \mathbb{I})$, denoted as $W$. Let's start from $(a)$. Since $\mathcal{D}_{R_p}$ is diagonalizable, its eigenvalues are $\{1, \xi_p, ..., \xi_p^{p-1}\}$. Therefore, we have the following decompositions

$$\ker(\partial_{co}) = E_0 \oplus E_1 \oplus \ldots \oplus E_{p-1} \tag{6.15}$$

and

$$W = F_0 \oplus F_1 \oplus \ldots \oplus F_{p-1} \quad \text{and} \quad \mathrm{Im}(\partial_{co}) = G_0 \oplus G_1 \oplus \ldots \oplus G_{p-1} \tag{6.16}$$

where $E_i$, $F_i$ and $G_i$ are eigenspaces of $\mathcal{D}_{R_p}$ of eigenvalue $\xi_p^i$ in their corresponding (invariant) subspaces. Note that it is possible that some of them are trivial. Study the algebraic relation between different eigenspaces is crucial. From linear algebra, different $E_i$'s (so are $F_i$ and $G_i$) are $\Lambda^{\mathcal{K},\Gamma}$-linearly independent over $\Lambda^{\mathcal{K},\Gamma}$. Now we will show they are actually mutually orthogonal to each other. This is the following lemma.

**Lemma 6.3.1.** *$\{E_i\}_{i=1}^{p-1}$ are mutually orthogonal to each other. So are $\{F_i\}_{i=1}^{p-1}$ and $\{G_i\}_{i=1}^{p-1}$.*

*Proof.* Assume all the basis elements have filtration 0. $\mathcal{D}_{R_p}$ acting on $E_i$ implies $[(\mathcal{D}_{R_p})] = \mathcal{D}_{R_p}$ acting on $[E_i]$. Moreover, by Example 6.2.7, eigenspace $E_i$ (of eigenvalue $\xi_p^i$) reduces to $\mathcal{K}$-space $[E_i]$ which is a subspace of eigenspace (of eigenvalue $[\xi_p^i] = \xi_p^i$) of $[\ker(\partial_{co})]$. Because different eigenspaces $[E_i]$ are $\mathcal{K}$-linearly independent, by Lemma 6.2.5, $E_i$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $E_j$ for all $i \neq j$. A similar argument holds for $\{F_i\}_{i=0}^{p-1}$ and $\{G_i\}_{i=0}^{p-1}$. □

On the other hand, $W$ is isomorphic to $\mathrm{Im}(\partial_{co})$, so in particular,

$$\dim_{\Lambda^{\mathcal{K},\Gamma}}(W) = \dim_{\Lambda^{\mathcal{K},\Gamma}}(\mathrm{Im}(\partial_{co})). \tag{6.17}$$

$\partial_{co}$ and $\mathcal{D}_{R_p}$ commutes gives rise to an important observation that $\partial_{co}$ brings a $\mathcal{D}_{R_p}$-invariant subspace into a $\mathcal{D}_{R_p}$-invariant subspace. In particular, we have $\partial_{co}(F_i) \leq G_i$, so $\dim_{\Lambda^{\mathcal{K},\Gamma}}(F_i) \leq \dim_{\Lambda^{\mathcal{K},\Gamma}}(G_i)$. (6.17) implies the dimensions are actually equal for each $i$. So restrictions

$$\partial_{co}|_{F_i} : F_i \to G_i,$$

are isomorphisms between two (smaller) filtered $\Lambda^{\mathcal{K},\Gamma}$-vector spaces. Then

**Proposition 6.3.2.** *There exists a singular value decomposition of* $\partial_{co} = (\partial_{co})_{k+1} : Cone_{k+1}(T - \xi_p \cdot \mathbb{I}) \to \mathrm{Im}(\partial_{co})$ *(compatible with action* $\mathcal{D}_{R_p}$ *in the sense that they span eigenspaces of eigenvalues* $\xi_p^q$ *for* $0 \leq q \leq p-1$*).*

*Proof.* Choose an orthogonal basis for each $E_i$. Together all $0 \leq i \leq p-1$, they will form an orthogonal basis for $\ker(\partial_{co})$ by Lemma 6.3.1. By Theorem 3.5 in [UZ15], there exists a singular value decomposition of each $\partial_{co}|_{F_i}$. Again together all $0 \leq i \leq p-1$, they will form a singular value decomposition of $\partial_{co}|_W$ again by Lemma 6.3.1. □

Now for (b), we have $\mathcal{D}_S = \mathcal{D}_{R_{p^2}}$ and therefore $\mathcal{D}_S$ acts on each eigenspace of $\mathcal{D}_{R_p}$. By discussion above, we will work on $E_i$ and $\partial_{co}|_{F_i} : F_i \to G_i$ piece by piece for each $1 \leq i \leq p-1$.

On $E_i$, for any $x \in E_i$, by Lemma 6.2.11, $\{x, \mathcal{D}_S x, ..., \mathcal{D}_S^{p-1} x\}$ as a $\Lambda^{\mathcal{K},\Gamma}$-orthogonal set occupies a $p$-dimensional subspace of $E_i$. Choose (if $\dim(E_i) > p$) a $y \in E_i$ such that it is orthogonal to cyclic span $V_x$. By Lemma 6.2.12, $V_y$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $V_x$. So it occupies another $p$-dimensional subspace of $E_i$. Inductively going through this process, we find an orthogonal basis of $E_i$ in the form of $p$-tuple,

$$(x, \mathcal{D}_S x, ..., \mathcal{D}_S^{p-1} x, y, \mathcal{D}_S y, ..., \mathcal{D}_S^{p-1} y...). \tag{6.18}$$

On $\partial_{co}|_{F_i} : F_i \to G_i$, we will formulate a theoretic process (in accordance with the algorithmic process, Theorem 3.5 in [UZ15] of finding a singular value decomposition) which is the following lemma. The idea of generating singular value decomposition is by orthogonally cutting down $p$-dimensional subspaces. In general, we have

**Lemma 6.3.3.** *For a filtered isomorphism $A : (V_1, \ell_1) \to (V_2, \ell_2)$ with group action by $S$ such that $S^p = \xi_p^q \cdot \mathbb{I}$ for $1 \leq q \leq p-1$ (so $S$ exactly preserves filtration), there exists a pair of dimension $p$ subspaces in the cyclic span form of $(V_y, V_x)$ where $A(V_y) = V_x$ such that there exists an orthogonal complement pair $(W_1, W_2)$ in the sense that*

*(1) $A(W_1) = W_2$;*

*(2) $V_1 = V_y \oplus W_1$ and $W_1$ is orthogonal to $V_y$;*

*(3) $V_2 = V_x \oplus W_2$ and $W_2$ is orthogonal to $V_x$.*

*Proof.* By Lemma 6.2.14, there exists an optimal pair $(x^*, y^*)$ where $A(y^*) = x^*$ such that for all $y \in V_1$,

$$\ell_1(y^*) - \ell_2(x^*) \leq \ell_1(y) - \ell_2(Ay). \tag{6.19}$$

Since $S$ exactly preserve filtrations,

$$\ell_1(S^i y^*) - \ell_2(S^i x^*) \leq \ell_1(y) - \ell_2(Ay) \tag{6.20}$$

for all the $i = 0, ..., p-1$. By Lemma 6.2.12 we know elements from $\{x^*, ..., S^{p-1}x^*\}$ are orthogonal, so are elements from $\{y^*, ..., S^{p-1}y^*\}$. So consider the span $V_{x^*}$ and denote its orthogonal complement as $W_2$ and its preimage under $A$ as $W_1 = A^{-1}(W_2)$. Now we only need to show proposition (2) above. We will show this inductively (with finitely many steps). By Lemma 2.15 in [UZ15],

$$W_1 \perp \langle y^* \rangle \ \text{and} \ \langle Sy^* \rangle \perp W_1 \oplus \langle y^* \rangle \ \Rightarrow \ W_1 \perp \langle y^*, Sy^* \rangle .$$

Then

$$W_1 \perp \langle y^*, Sy^* \rangle \text{ and } \left\langle S^2 y^* \right\rangle \perp W_1 \oplus \langle y^*, Sy^* \rangle \;\Rightarrow\; W_1 \perp \left\langle y^*, Sy^*, S^2 y^* \right\rangle.$$

Inductively, we will get the conclusion.

Indeed, in order to prove $W_1 \perp \langle y^* \rangle$, which is equivalent to the statement that for any $v \in W_1$, $\ell_1(y^*) \leq \ell_1(y^* - v)$, we note since $W_2$ is orthogonal to $\langle x_* \rangle$, $\ell_2(A(y^* - v)) \geq \ell_2(x)$, so by optimal choice (6.19) we get the conclusion. Now let's prove $\langle Sy^* \rangle \perp W_1 \oplus \langle y^* \rangle$. Actually, the proof is similar. Again, this is equivalent to show for any $v \in W_1 \oplus \langle y^* \rangle$, $\ell_1(Sy^*) \leq \ell_1(Sy^* - v)$. Because $W_2 \oplus \langle x^* \rangle$ is orthogonal to $\langle Sx^* \rangle$, $\ell_2(A(Sy^* - v)) \geq \ell_2(Sx^*)$. Then by optimal choice (6.20), we get the conclusion. in general, the same argument works for the proof of $\left\langle S^{i+1} y^* \right\rangle \perp W_1 \oplus \left\langle y^*, ..., S^i y^* \right\rangle$. □

*Remark* 6.3.4. Note that the condition $S^p = \xi_p^q \cdot \mathbb{I}$ of Lemma 6.3.3 is only used to formulate a cyclic span $V_{x^*}$.

Inductively applying Lemma 6.3.3 to $\partial_{co} = \partial_{co}|_{F_i}$, $V_1 = F_i$ and $V_2 = G_i$, we get a singular value decomposition of $\partial_{co}|_{F_i}$ for each $1 \leq i \leq p-1$ in $p$-tuple. Together, we get a singular value decomposition of $\partial_{co}|_W$, that is

$$\begin{array}{cccccc} y_1 & \mathcal{D}_S y_1 & \dots & \mathcal{D}_S^{p-1} y_1 & y_2 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ x_1 & \mathcal{D}_S x_1 & \dots & \mathcal{D}_S^{p-1} x_1 & x_2 & \dots \end{array} \tag{6.21}$$

and this is the required $p$-tuple form. In summary, we get the following proposition.

**Proposition 6.3.5.** *Assume $\mathcal{D}_S = \mathcal{D}_{R_{p^2}}$. There exists a singular value decomposition in p-tuple form as (6.18) and (6.21) of*

$$\partial_{co}\big|_{\bigoplus_{1 \leq i \leq p-1} E_i \oplus F_i} : \bigoplus_{1 \leq i \leq p-1} E_i \oplus F_i \to \bigoplus_{1 \leq i \leq p-1} G_i$$

*which is compatible with action $\mathcal{D}_{R_p}$ in the sense that cyclic spans generate each eigenspace.*

Now, we deal with (c). By Remark 6.2.13, we will take a special consideration of eigenspace of eigenvalue 1. Denote $K_0 = E_0 \oplus F_0$ (eigenspace of eigenvalue 1 of $\mathcal{D}_{R_p}$ in $Cone_{k+1}(T - \xi_p \cdot \mathbb{I})$). We claim

**Proposition 6.3.6.** *The restriction map $\partial_{co}|_{K_0} : K_0 \to G_0$ only contributes (in p-tuple) 0-length bars.*

*Proof.* On the one hand, we know precisely the generators of $K_0$. Indeed, for each generating loop (for rotation), say $x(t)$,[3] denote $x(t)_{(i)} = x\left(t + \frac{i}{p^2}\right)$ where $0 \leq i \leq p^2 - 1$. Then

$$Cone_{k+1}(T - \xi_p \cdot \mathbb{I}) = \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \begin{array}{c} (0, x_i(t))..., (0, x_i(t)_{(p^2-1)}) \\ (y_j(t), 0)..., (y_j(t)_{(p^2-1)}, 0) \end{array} \right\rangle_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}},$$

where $x_i(t)$'s have indices equal to $k+1$ and $y_j(t)$'s have indices equal to $k+2$. Moreover, these (initial) generators are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal. On the one hand, from the definition of $\mathcal{D}_{R_{p^2}}$, we know $K_0$ is generated by the element in the following form. Denote

$$v_i = x_i(t) + x_i(t)_p + ... + x_i(t)_{p^2-p} \text{ and } w_j = y_j(t) + y_j(t)_p + ... + y_j(t)_{p^2-p}. \tag{6.22}$$

Then,

$$K_0 = \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \begin{array}{c} (0, v_i), (0, Sv_i), ..., (0, S^{p-1}v_i) \\ (w_j, 0), (Sw_j, 0), ..., (S^{p-1}w_j, 0) \end{array} \right\rangle_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}. \tag{6.23}$$

Note that generators are all eigenvector of eigenvalue 1 of $\mathcal{D}_{R_p}$ because $\mathcal{D}_S$ commutes with $\mathcal{D}_{R_p}$ and more importantly, they are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal. On the other hand, by definition of $\partial_{co}$, for any eigenvector of eigenvalue 1 of $\mathcal{D}_{R_p}$ in the form of $(0, z(t))$,

$$\partial_{co}((0, z(t))) = \begin{pmatrix} \partial & -(T - \xi_p \cdot \mathbb{I}) \\ 0 & -\partial \end{pmatrix} \cdot \begin{pmatrix} 0 \\ z(t) \end{pmatrix} = ((\xi_p - 1)z(t), -\partial z(t)).$$

[3] All these loops are normalized to be filtration 0 by adjusting cappings.

Applying this relation to $(0, S^h w_j)$ for $0 \leq h \leq p-1$, we get

$$\partial_{co}(0, S^h w_j) = \Big((\xi_p - 1)S^h w_j, -\partial S^h w_j\Big).$$

By Lemma 6.1.6, since $\partial$ strictly decreases the filtration, orthogonality of $\{(S^h w_j, 0)\}_{h=0}^{p-1}$ is equivalent to orthogonality of $\{\partial_{co}(0, S^h w_j)\}_{h=0}^{p-1} = \{(\xi_p - 1)S^h w_j, -\partial S^h w_j)\}_{h=0}^{p-1}$ which are also eigenvectors of eigenvalue 1 of $\mathcal{D}_{R_p}$. Therefore, instead of (6.23), we can write

$$K_0 = \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \begin{array}{c} (0, v_i), ..., (0, S^{p-1} v_i) \\ \partial_{co}(0, w_j), ..., \partial_{co}(0, S^{p-1} w_j) \end{array} \right\rangle_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}. \tag{6.24}$$

So,

$$\mathrm{Im}(\partial_{co}|_{K_0}) = \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \partial_{co}(0, v_i), ..., \partial_{co}(0, S^{p-1} v_i) \right\rangle_{1 \leq i \leq m}$$

because $(\partial_{co})^2 = 0$. Moreover $\{\partial_{co}(0, v_i), ..., \partial_{co}(0, \mathcal{D}_S^{p-1} v_i)\}$ are orthogonal by the same reason as above. In other words, for each degree $k+1$, we get $p$-tuple singular value decomposition of $\partial_{co}|_{K_0}$ with building block

$$\begin{array}{cccc} (0, v_i) & (0, Sv_i) & \dots & (0, S^{p-1} v_i) \\ \downarrow & \downarrow & \dots & \downarrow \\ \partial_{co}(0, v_i) & \partial_{co}(0, Sv_i) & \dots & \partial_{co}(0, S^{p-1} v_i) \end{array}$$

(together with those mapped to 0 which are also in $p$-tuple). Moreover, these $p$-tuples only contribute 0-length bar because it's easy to check $\ell_{co}((0, v_i)) = \ell_{co}(\partial_{co}(0, v_i))$ for any $v_i$ in (6.22). $\square$

Put all (a), (b) and (c) together, we confirm the unperturbed version of the main theorem.

## 6.4 Perturbed $p$-cyclic singular value decomposition

Now we are working with general $\mathcal{D}_T$ and $\mathcal{D}_S$. By Lemma 6.1.3, we can reduce to group actions $\mathcal{D}_{T'}$ and $\mathcal{D}_{S'}$. Once we have group action, by Lemma 6.2.1 and Lemma 6.1.4, there exists an orthogonal $\mathcal{D}_{T'}$-invariant decomposition of the domain, denoted as $\ker(\partial_{co}) \oplus W$ where $W$ is also $\mathcal{D}_{T'}$-invariant.

Next, we want to find the eigenvalue decomposition of $\mathcal{D}_{T'}$. Different from $\mathcal{D}_{R_p}$, the exact values of eigenvalues are not obvious here. However, we have

**Lemma 6.4.1.** *For any eigenvalue $\lambda$ of $\mathcal{D}_{T'}$,*

$$\lambda = \xi_p^i + \left\{ \begin{array}{c} \textit{strictly lower} \\ \textit{filtration terms} \end{array} \right\}.$$

*for some $i \in \{0, ..., p-1\}$.*

*Proof.* This is a direct application of Lemma 6.2.9 because characteristic polynomial of $\mathcal{D}_{T'}$ is in the form of $\lambda^p = 1 + g(\lambda)$ for some polynomial $g(x) \in \Lambda^{\mathcal{K},\mathbb{R}_{>0}}[x]$ with degree at most $p-1$. In fact, $\lambda^p$ is part of the summand of product of all the diagonal entries (so contains $p$-many $\lambda$) and 1 is part of the summand of product of (possibly perturbed) all the entries involving $-1$. Other summand of product contains at most $(p-1)$-many $\lambda$. $\square$

Now denote, for $0 \leq i \leq p-1$,

$$\Theta(i) = \left\{ \xi_p^i + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\} \right\} \tag{6.25}$$

where generic element in $\Theta(i)$ is denoted as $\theta(i)$. We remark here that different perturbations of strictly lower filtration terms will give rise to different eigenvalues. But for each $0 \leq i \leq p-1$,

$$\#\{\theta(i) \in \Theta(i)\} = \#\{\xi_p^i \text{ as eigenvalue of } \mathcal{D}_{R_p}\}.$$

where $\#\{\cdot\}$ means the algebraic multiplicity of the set $\{\cdot\}$. Moreover, perturbed operator $\mathcal{D}_{T'}$ is not necessarily diagonalizable. However, similar to (6.15) and (6.16), we have

**Lemma 6.4.2.** $\ker(\partial_{co})$*, $W$ and* $\mathrm{Im}(\partial_{co})$ *can be orthogonally decomposed as*

$$\ker(\partial_{co}) = \tilde{E}_0 \oplus \tilde{E}_1 \oplus \ldots \tilde{E}_{p-1} \tag{6.26}$$

*and*

$$W = \tilde{F}_0 \oplus \tilde{F}_1 \oplus \ldots \oplus \tilde{F}_{p-1} \quad \textit{and} \quad \mathrm{Im}(\partial_{co}) = \tilde{G}_0 \oplus \tilde{G}_1 \oplus \ldots \oplus \tilde{G}_{p-1} \tag{6.27}$$

*where*

$$\tilde{E}_i = \bigoplus_{\theta(i)\in\Theta(i)} E_{\theta(i)}, \quad \tilde{F}_i = \bigoplus_{\theta(i)\in\Theta(i)} F_{\theta(i)}, \quad \tilde{G}_i = \bigoplus_{\theta(i)\in\Theta(i)} G_{\theta(i)}$$

*are direct sums of generalized eigenspaces $E_{\theta(i)}$, $F_{\theta(i)}$ and $G_{\theta(i)}$ of $\mathcal{D}_{T'}$ of (perturbed) eigenvalue $\theta(i)$ respectively.*

*Proof.* The decompositions come from a general fact in linear algebra that an operator can be decomposed as a direct sum of generalized eigenspaces as long as the field of scalars contains the corresponding eigenvalues (and this is guaranteed by Lemma 6.2.9). The proof of orthogonality is exactly the same as Lemma 6.3.1 together with Lemma 6.1.6. □

By the same argument as above, we have isomorphisms $\partial_{co}|_{\tilde{F}_i} : \tilde{F}_i \simeq \tilde{G}_i$ as a direct sum of family of isomorphisms $\partial_{co}|_{F_{\theta(i)}} : F_{\theta(i)} \simeq G_{\theta(i)}$ for each $\theta(i) \in \Theta(i)$ for $0 \leq i \leq p-1$. Similar to Proposition 6.3.2,

**Proposition 6.4.3.** *There exists a singular value decomposition of $\partial_{co} : Cone_{k+1}(T - \xi_p \cdot \mathbb{I}) \to \mathrm{Im}(\partial_{co})$ (compatible with $\mathcal{D}_{T'}$ in the sense of that they span its generalized eigenspaces).*

Now assume $\mathcal{D}_{T'} = \mathcal{D}_{S'}^p$, then $\mathcal{D}_{S'}$ acts on each $\tilde{E}_i$, $\tilde{F}_i$ and $\tilde{G}_i$ because it acts on each $\tilde{E}_{\theta(i)}$, $\tilde{F}_{\theta(i)}$ and $\tilde{G}_{\theta(i)}$. Moreover, for instance on $\tilde{E}_i$, if $\Theta(i) = \{\theta(i)_1, ..., \theta(i)_m\}$, then for any $x \in \tilde{E}_i$ with expression $x = x_1 + ...x_m$ where $x_j \in E_{\theta(i)_j}$ for $1 \leq j \leq m$,

$$\begin{aligned}\mathcal{D}_{S'}^p x = \mathcal{D}_{T'} x = \theta(i)_1 x_1 + ... + \theta(i)_m x_m &= \xi_p^i (x_1 + ... + x_m) + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\} \\ &= \xi_p^i x + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\}.\end{aligned} \tag{6.28}$$

In other words, for $x \in \tilde{E}_i$, $\mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, ..., \mathcal{D}_{S'}^{p-1} x \right\rangle$ is not an invariant subspace any more (due to the perturbed terms). However, still we have (perturbed version of Corollary 6.2.11),

**Lemma 6.4.4.** *For $1 \le i \le p-1$, if $x \in \tilde{E}_i$ (or $\tilde{F}_i$, $\tilde{G}_i$) , then $\{x, \mathcal{D}_{S'}x, ..., \mathcal{D}_{S'}^{p-1}x\}$ are $\Lambda^{\mathcal{K},\Gamma}$-orthogonal.*

*Proof.* By Lemma 6.2.5, $\Lambda^{\mathcal{K},\Gamma}$-orthogonality of $\{x, \mathcal{D}_{S'}x, ..., \mathcal{D}_{S'}^{p-1}x\}$ is equivalent to $\mathcal{K}$-linearly independent of $\{[x], [\mathcal{D}_{S'}][x], ..., ([\mathcal{D}_{S'}])^{p-1}[x]\} = \{[x], \mathcal{D}_{R_{p^2}}[x], ..., \mathcal{D}_{R_{p^2}}^{p-1}[x]\}$. Meanwhile, cyclic span $V_{[x]}$ (over $\mathcal{K}$) is $\mathcal{D}_{R_{p^2}}$-invariant because (6.28) reduces to

$$\mathcal{D}_{R_{p^2}}^{p}[x] = \xi_p^i[x].$$

Hence, "irreducible condition" implies the desired $\mathcal{K}$-linear independence. $\square$

Also similar to Corollary 6.2.12, we have

**Lemma 6.4.5.** *For $1 \le i \le p-1$, if for a given $y \in \tilde{E}_i$ (or $\tilde{F}_i$ and $\tilde{G}_i$), it is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $\mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, ..., \mathcal{D}_{S'}^{p-1}x \right\rangle$, then*

$$\mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle y, ..., \mathcal{D}_{S'}^{p-1}y \right\rangle \text{ is } \Lambda^{\mathcal{K},\Gamma}\text{-orthogonal to } \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, ..., \mathcal{D}_{S'}^{p-1}x \right\rangle.$$

*Proof.* By Lemma 6.2.5, we only need to consider the set

$$\{[x], \mathcal{D}_{R_{p^2}}[x], ..., \mathcal{D}_{R_{p^2}}^{p-1}[x], [y], \mathcal{D}_{R_{p^2}}[y], ..., \mathcal{D}_{R_{p^2}}^{p-1}[y]\}.$$

These span a $\mathcal{D}_{R_{p^2}}$-invariant subspace. Then the argument of Corollary 6.2.12 (with "irreducible condition") implies the $\mathcal{K}$-linearly independence. $\square$

Then we have a perturbed version of Proposition 6.3.5,

**Proposition 6.4.6.** *Assume $\mathcal{D}_{S'}$ exists. There exists a singular value decomposition in p-tuple form as (6.18) and (6.21) of*

$$\partial_{co}|_{\bigoplus_{1\le i\le p-1}\tilde{E}_i\oplus\tilde{F}_i} : \bigoplus_{1\le i\le p-1} \tilde{E}_i \oplus \tilde{F}_i \to \bigoplus_{1\le i\le p-1} \tilde{G}_i.$$

*Proof.* For each $\tilde{E}_i$, for any $x \in \tilde{E}_i$, we get a $p$-dimensional subspace $\text{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, ..., \mathcal{D}_{S'}^{p-1} x \right\rangle$ by Lemma 6.4.4. Choose $y \in \tilde{E}_i$ such that $y$ is $\Lambda^{\mathcal{K},\Gamma}$-orthogonal to $\text{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x, ..., \mathcal{D}_{S'}^{p-1} x \right\rangle$. By Lemma 6.4.5, we get another $p$-dimensional subspace. Inductively apply this process until we run out of the dimension of $\tilde{E}_i$. Thus together for all $1 \leq i \leq p-1$, we get a basis of $\ker(\partial_{co})$ as in (6.18), that is,

$$\{x, \mathcal{D}_{S'}^2 x ..., \mathcal{D}_{S'}^{p-1} x, y, \mathcal{D}_{S'}^2 y ..., \mathcal{D}_{S'}^{p-1} y, ...\}. \tag{6.29}$$

Now we will deal with $\partial_{co}|_{\tilde{F}_i} : \tilde{F}_i \to \tilde{G}_i$. By Remark 6.3.4, we can apply Lemma 6.3.3 to $\partial_{co}|_{\tilde{F}_i}$, $V_1 = \tilde{F}_i$ and $V_2 = \tilde{G}_i$. The only difference is replacing cyclic span $V_{x^*}$ by (non-invariant) subspace $\text{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle x^*, ..., \mathcal{D}_{S'}^{p-1} x^* \right\rangle$. Thus we get a correspondence as in (6.21), that is,

$$\begin{array}{cccccc} y_1 & \mathcal{D}_{S'} y_1 & \dots & \mathcal{D}_{S'}^{p-1} y_1 & y_2 & ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ x_1 & \mathcal{D}_{S'} x_1 & \dots & \mathcal{D}_{S'}^{p-1} x_1 & x_2 & ... \end{array} \tag{6.30}$$

Together we get the desired singular value decomposition in $p$-tuple. $\square$

*Remark* 6.4.7. If applying Lemma 6.3.3 to $\partial_{co}|_{F_{\theta(i)}} : F_{\theta(i)} \to G_{\theta(i)}$, it is not quite clear the singular value decomposition for this restriction is orthogonal or not to another restriction $\partial_{co}|_{F_{\theta'(i)}}$ of another perturbed eigenvalue $\theta'(i)$ since the reduced $\mathcal{K}$-space of both $F_{\theta(i)}$ and $F_{\theta'(i)}$ are subspaces of eigenspace space of the same eigenvalue $\xi_p^i$, so Lemma 6.4.2 does not apply directly. Similar situation for $\tilde{E}_i$. In other words, each basis element of singular value decomposition for $\partial_{co}|_{\tilde{F}_i} : \tilde{F}_i \to \tilde{G}_i$ is, in general, a $\Lambda^{\mathcal{K},\Gamma}$-linear combination of elements from $F_{\theta(i)}$ and $G_{\theta(i)}$.

Last but not least, denote $\tilde{K}_0 = \tilde{E}_0 \oplus \tilde{F}_0$, we have a similar result as Proposition 6.3.6.

**Proposition 6.4.8.** *The restriction map $\partial_{co}|_{\tilde{K}_0} : \tilde{K}_0 \to \tilde{G}_0$ only contributes (in p-tuple) 0-length bars.*

*Proof.* Similar to proof of Proposition 6.3.6, it's easy to show

$$\tilde{K}_0 = \text{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \begin{array}{c} (0, \tilde{v}_i), (0, S'\tilde{v}_i), ..., (0, (S')^{p-1}\tilde{v}_i) \\ (\tilde{w}_j, 0), (S'\tilde{w}_j, 0), ..., ((S')^{p-1}\tilde{w}_j, 0) \end{array} \right\rangle_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}, \tag{6.31}$$

where $\tilde{v_i}$ and $\tilde{w_j}$ are strictly lower filtration perturbations of $v_i$ and $w_j$ as defined in (6.22). More important, these elements are pairwise orthogonal. For $\tilde{w_j}$, on the one hand,

$$T\tilde{w_j} = (T' - P_T^{(3)})\tilde{w_j} = \tilde{w_j} + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\}.$$

On the other hand, since $[S', T] = 0$,

$$\partial_{co}(0, (S')^h \tilde{w_j}) = \left( (\xi_p - 1)(S')^h \tilde{w_j} + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\}, -\partial(S')^h \tilde{w_j} \right).$$

By Lemma 6.1.6, orthogonality of $\{(S')^h \tilde{w_j}, 0)\}_{h=0}^{p-1}$ implies orthogonality of $\{\partial_{co}(0, (S')^h \tilde{w_j})\}_{h=0}^{p-1}$. Meanwhile, we can rewrite

$$\tilde{K}_0 = \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \left\langle \begin{array}{c} (0, \tilde{v_i}), (0, S'\tilde{v_i}), ..., (0, (S')^{p-1}\tilde{v_i}) \\ \partial_{co}(0, \tilde{w_j}), \partial_{co}(0, S'\tilde{w_j}), ..., \partial_{co}(0, (S')^{p-1}\tilde{w_j}) \end{array} \right\rangle_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

Therefore, we have the correspondence as in (6.3),

$$\begin{array}{cccc} (0, \tilde{v_i}) & (0, (S')\tilde{v_i}) & \dots & (0, (S')^{p-1}\tilde{v_i}) \\ \downarrow & \downarrow & \dots & \downarrow \\ \partial_{co}(0, \tilde{v_i}) & \partial_{co}(0, (S')\tilde{v_i}) & \dots & \partial_{co}(0, (S')^{p-1}\tilde{v_i}) \end{array}$$

It's easy to check $\ell_{co}((0, \tilde{v_i})) = \ell_{co}(\partial_{co}(0, \tilde{v_i}))$ for any perturbed $\tilde{v_i}$, so these only contribute 0-length bars. $\square$

### 6.4.1 Proof of main theorem

*Proof.* Proposition 6.4.6 and Proposition 6.4.8. $\square$

# Chapter 7

# Filtered homotopy category

In this section, we will prove all the Lipschitz continuity propositions that have been advertised in the introduction part. Recall these three propositions.

- **Proposition 1.4.1** For any Hamiltonian $H$ and $G$, we have

$$d_Q(Cone(H)_*, Cone(G)_*) \leq 3p \cdot ||H - G||_H.$$

- **Proposition 1.4.2** Denote $\beta_i(\phi_H)$ as the length of $i$-th bar in degree-$k$ verbose barcode of $Cone(H)_*$ and $\beta_i(\phi_G)$ as the length of $i$-th bar in degree-$k$ verbose barcode of $Cone(G)_*$. We have for every $i \in \mathbb{Z}$,

$$|\beta_i(\phi_H) - \beta_i(\phi_G)| \leq 4\, d_Q(Cone(H)_*, Cone(G)_*).$$

- **Proposition 1.4.3** For any closed symplectic manifold $(X, \omega)$ and $\phi, \psi \in Ham(X, \omega)$, we have, for divisibility sensitive invariants,

$$|\mathfrak{o}_X(\phi) - \mathfrak{o}_X(\psi)| \leq 24p \cdot d_H(\phi, \psi).$$

The others quickly follows from the first proposition. Among various methods, we will prove it by an idea from *triangulated category*.

## 7.1 Algebraic set-up

*Definition* 7.1.1. For a $\Lambda$-linear map $F : (V, \ell_1) \to (W, \ell_2)$ between two orthogonalizable $\Lambda$-space, $F$ is called a *$\delta$-morphism* if there exists a $\delta \geq 0$ such that for any $v \in V$,

$$\ell_2(Fv) \leq \ell_1(v) + \delta.$$

In particular a filtration preserving map is a 0-morphism.

Note that by definition, a $\delta$-morphism is automatically a $\eta$-morphism for any $\delta \leq \eta$.

*Definition* 7.1.2. Given two Floer-type complexes $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$, suppose chain map $\Phi : C_* \to D_*$ is a $\delta_+$-morphism and $\Psi : C_* \to D_*$ is a $\delta_-$-morphism. Then we call $\Phi$ and $\Psi$ are *$(\delta_+, \delta_-)$-homotopic* if there exists a degree-1 $(\delta_+, \delta_-)$-morphism $K : C_* \to D_{*+1}$ such that

$$\Phi - \Psi = K \circ \partial_C + \partial_D \circ K.$$

where then $K$ is called a *$(\delta_+, \delta_-)$-homotopy.*

**Example 7.1.3.** *Using Definition 7.1.2, definiton of $(\delta_+, \delta_-)$-quasiequivalence between $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ (defined in Definition 4.2.1) can be rephrased as*

- *there exist a $\delta_+$-morphism $\Phi : C_* \to D_*$ and a $\delta_-$-morphism $\Psi : D_* \to C_*$;*
- *$\Psi \circ \Phi$ and $\mathbb{I}_{C_*}$ are $(\delta_+, \delta_-)$-homotopic and $\Phi \circ \Psi$ and $\mathbb{I}_{D_*}$ are $(\delta_+, \delta_-)$-homotopic.*

Now consider the following algebraic set-up. For two Floer-type complexes $(C_*, \partial_C, \ell_C)$ (simply denoted as $C$) and $(D_*, \partial_D, \ell_D)$ (simply denoted as $D$), we will study the following diagram

$$\begin{tikzcd}
C \arrow[r, "S"] \arrow[d, "\phi"'] \arrow[dr, "P"] \arrow[dd, bend right=60, "K_C"'] & C \arrow[d, "\phi"] \arrow[dd, bend left=60, "K_C"] \\
D \arrow[r, "S'"] \arrow[d, "\psi"'] \arrow[dr, "Q"] \arrow[dd, bend right=60, "K_D"'] & D \arrow[d, "\psi"] \arrow[dd, bend left=60, "K_D"] \\
C \arrow[r, "S"] \arrow[d, "\phi"'] \arrow[dr, "P"] & C \arrow[d, "\phi"] \\
D \arrow[r, "S'"] & D
\end{tikzcd}$$

where

- $S$ and $S'$ 0-morphism chain maps;
- $\phi$ and $\psi$ are $(\delta_+, \delta_-)$-quasiequivalent with homotopies $K_C$ and $K_D$;
- $S' \circ \phi$ and $\phi \circ S$ are $(\delta_+, \delta_-)$-homotopic with homotopy $P$;
- $S \circ \psi$ and $\psi \circ S'$ are $(\delta_+, \delta_-)$-homotopic with homotopy $Q$.

Here we give a supporting example from our story to this algebraic set-up.

**Example 7.1.4.** *When we are considering the chain complexes with (almost) rotation actions constructed from different Hamiltonians $H$ and $G$, that is, the following diagram for some continuation maps $C_H$, $C_G$ and $C_{H,G}$,*

$$\begin{tikzcd} CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "R_p"] \arrow[d, "C_{H,G}"'] & CF_k(H^{(p)}, J_{t+\frac{1}{p}})_\alpha \arrow[r, "C_H"] \arrow[d, "(R_p)_*(C_{H,G})"] & CF_k(H^{(p)}, J_t)_\alpha \arrow[d, "C_{H,G}"] \\ CF_k(G^{(p)}, J_t)_\alpha \arrow[r, "R_p"] & CF_k(G^{(p)}, J_{t+\frac{1}{p}})_\alpha \arrow[r, "C_G"] & CF_k(G^{(p)}, J_t)_\alpha \end{tikzcd}$$

*Because the second (right) diagram above is not necessarily commutative (where continuation map commutes with boundary operator but not necessarily commutes with other continuation maps), the following collapsed diagram is then not necessarily a commutative diagram,*

$$\begin{tikzcd} CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "T^H"] \arrow[d, "C_{H,G}"'] & CF_k(H^{(p)}, J_t)_\alpha \arrow[d, "C_{H,G}"] \\ CF_k(G^{(p)}, J_t)_\alpha \arrow[r, "T^G"] & CF_k(G^{(p)}, J_t)_\alpha \end{tikzcd} \tag{7.1}$$

*where $T^H = C_H \circ R_p$ and $T^G = C_G \circ R_p$. Meanwhile, we can symmetrically form continuation map $C_{G,H} : CF_k(G^{(p)}, J_t)_\alpha \to CF_k(H^{(p)}, J_t)_\alpha$. Likewise, we have a (not necessarily commutative) diagram,*

$$\begin{tikzcd} CF_k(G^{(p)}, J_t)_\alpha \arrow[r, "T^G"] \arrow[d, "C_{G,H}"'] & CF_k(G^{(p)}, J_t)_\alpha \arrow[d, "C_{G,H}"] \\ CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "T^H"] & CF_k(H^{(p)}, J_t)_\alpha . \end{tikzcd} \tag{7.2}$$

*However, we can prove these two diagrams are not far from being commutative. In fact,*

**Lemma 7.1.5.** *The diagram (7.1) is commutative up to a filtration-shifted homotopy with filtration shifted up to $p||H-G||_H$. More precisely, $T^G \circ C_{H,G}$ and $C_{H,G} \circ T^H$ are $(p\int_0^1 \max_X(H-G)dt, p\int_0^1 -\min_X(H-G)dt)$-homotopic with homotopy $P$. The same conclusion holds for diagram (7.2) with a homotopy $Q$.*

*Proof.* By Floer gluing argument, we can glue the homotopy inducing $T^G$ and the homotopy between $(H^{(p)}, J)$ and $(G^{(p)}, J)$ so that it gives a new chain map between $CF_*(H^{(p)}, J)_\alpha$ and $CF_*(G^{(p)}, J)_\alpha$. Similarly, for $C_{H,G} \circ T^H$, we get another chain map. By the discussion in Section 2.3, two induced chain maps (from different homotopies) are homotopic to each other with certain filtration shifts. Moreover, the filtration shift is given by the standard energy estimations and the definitions of $H^{(p)}$ and $G^{(p)}$. $\square$

*Therefore, (7.1), (7.2), Lemma 7.1.5 and Example 4.2.2 together gives rise to the following diagram compatible with the algebraic set-up above.*

$$\begin{tikzcd}
CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "T^H"] \arrow[d, "C_{H,G}"'] \arrow[dr, "P"] \arrow[dd, bend right=60, "K_H"'] & CF_k(H^{(p)}, J_t)_\alpha \arrow[d, "C_{H,G}"] \arrow[dd, bend left=60, "K_H"] \\
CF_k(G^{(p)}, J_t)_\alpha \arrow[r, "T^G"] \arrow[d, "C_{G,H}"'] \arrow[dr, "Q"] \arrow[dd, bend right=60, "K_G"'] & CF_k(G^{(p)}, J_t)_\alpha \arrow[d, "C_{G,H}"] \arrow[dd, bend left=60, "K_G"] \\
CF_k(H^{(p)}, J_t)_\alpha \arrow[r, "T^H"] \arrow[d, "C_{H,G}"'] \arrow[dr, "P"] & CF_k(H^{(p)}, J_t)_\alpha \arrow[d, "C_{H,G}"] \\
CF_k(G^{(p)}, J_t)_\alpha \arrow[r, "T^G"] & CF_k(G^{(p)}, J_t)_\alpha
\end{tikzcd}$$

Now we remark that Proposition 1.4.1 can be easily proved if we can prove the following general algebraic proposition.

**Proposition 7.1.6.** *With the algebraic set-up above, there exist finite constants $\Delta_+$ and $\Delta_-$ such that self-mapping cones of complex $C$ and complex $D$ with respect to map $S$ and $S'$ respectively, $Cone_C(S)$ and $Cone_D(S')$, are $(\Delta_+, \Delta_-)$-quasiequivalent. Moreover, $\Delta_+ + \Delta_- \leq 6(\delta_+ + \delta_-)$.*

In other words, by Example 7.1.3, we are looking for $\Delta_+$-morphism and $\Delta_-$-morphism with $(\Delta_+, \Delta_-)$-homotopies. Assuming Proposition 7.1.6, we have

*Proof of Proposition 1.4.1.* Denote

$$\delta_+ = p\int_0^1 \max_X (H-G)dt \quad \text{and} \quad \delta_- = p\int_0^1 -\min_X (H-G)dt.$$

Let $S = T^H - \xi_p \cdot \mathbb{I}$ and $S' = T^G - \xi_p \cdot \mathbb{I}$. First, they are 0-morphisms because $T^H$ and $T^G$ preserves filtration. Moreover, they are chain maps because $T^H$ and $T^G$ commutes with boundary operator $\partial$. Second, $C_{G,H} \circ C_{H,G}$ are $(\delta_+, \delta_-)$-quasiequivalent. Third, $S' \circ C_{H,G}$ and $C_{H,G} \circ S$ are $(\delta_+, \delta_-)$-homotopic by Lemma 7.1.5.Therefore, all the conditions in the algebraic set-up are satisfied. By Proposition 7.1.6, there exists a $(\Delta_+, \Delta_-)$-quasiequivalence between $Cone(H)_*$ and $Cone(G)_*$. So we have

$$d_Q(Cone(H)_*, Cone(G)_*) \leq \frac{\Delta_+ + \Delta_-}{2} \leq 3(\delta_+ + \delta_-) = 3p \cdot ||H-G||_H.$$

Thus we get the conclusion. □

For the rest of this section, we will focus on the proof of Proposition 7.1.6.

## 7.2 Homotopy category

*Definition* 7.2.1. Considering the following category, denoted as $\mathcal{F}$: [1]

- Object in $\mathcal{F}$: Floer-type complex $(C_*, \partial_C, \ell_C)$;
- Morphism in $\mathcal{F}$: 0-morphism chain map.

Note that by this definition, any $\delta$-morphism $\phi$ with $\delta > 0$ is *not* a morphism in $\mathcal{F}$. On the other hand, fixing this constant $\delta \geq 0$, we can define a map on $\mathcal{F}$ itself, denoted as $\Sigma^\delta$ by

- $\Sigma^\delta((C_*, \partial_C, \ell_C)) = (C_*, \partial_C, \ell_C + \delta)$;
- $\Sigma^\delta(\phi) = \phi$.

[1]It is routine to check that $\mathcal{F}$ is an additive category.

It is easy to see this map is a functor. This functor does not change morphism and shift up the filtration of any filtered chain complex by $\delta$. In short notation, $\Sigma^\delta C := \Sigma^\delta((C_*, \partial_C, \ell_C))$ and

$$\ell_{\Sigma^\delta C} = \ell_C + \delta.$$

Therefore, for any $\delta$-morphism $\phi : C \to D$, we know

$$\Sigma^\delta C \xrightarrow{\phi} D$$

is a well-defined morphism in $\mathcal{F}$ because $\ell_D(\phi(x)) \leq \ell_C(x) + \delta = \ell_{\Sigma^\delta C}(x)$ for any $x \in C$.

**Example 7.2.2.** *Suppose $C$ and $D$ are $(\delta_+, \delta_-)$-quasiequivalent with chain maps $\phi$ and $\psi$. By discussion above, we have well-defined maps in $\mathcal{F}$,*

$$\Sigma^{\delta_+ + \delta_-} C \xrightarrow{\phi} \Sigma^{\delta_-} D \xrightarrow{\psi} C \quad \text{and} \quad \Sigma^{\delta_+ + \delta_-} D \xrightarrow{\phi} \Sigma^{\delta_+} C \xrightarrow{\psi} D.$$

*Meanwhile, there is an obvious map $i_{\delta_- + \delta_+} : \Sigma^{\delta_+ + \delta_-} C \to C$ by identity map on $C$ as a vector space. We emphasize that in category $\mathcal{F}$,*

$$\Sigma^{\delta_+ + \delta_-} C \neq C,$$

*because their filtrations are different. Therefore, $i_{\delta_+ + \delta_-}$ is not an identity map. Moreover, by definition of $(\delta_+, \delta_-)$-quasiequivalence, we know*

$$\psi \circ \phi \ \text{ is filtered homotopic to } \ i_{\delta_+ + \delta_-} \ \text{ in } \mathcal{F}$$

*and*

$$\phi \circ \psi \ \text{ is filtered homotopic to } \ i_{\delta_+ + \delta_-} \ \text{ in } \mathcal{F}.$$

Now consider the following "smaller" category defined as

*Definition* 7.2.3. Define (filtered) homotopy category of Floer-type complex $K(\mathcal{F})$ as

- Object in $K(\mathcal{F})$: the same as object in $\mathcal{F}$;

- Morphism in $K(\mathcal{F})$: filtered homotopy class (in $\mathcal{F}$).

**Example 7.2.4.** *The condition that $C$ and $D$ are $(\delta_+, \delta_-)$-quasiequivalent with chain maps $\phi$ and $\psi$ is equivalent to the condition $\psi \circ \phi = \phi \circ \psi = i_{\delta_+ + \delta_-}$ in $K(\mathcal{F})$.*

Using this language, we transfer the problem of constructing certain chain maps with (desired) bounded filtration shifting homotopies into an existence question of morphisms in $K(\mathcal{F})$ up to a finite filtration shift. Therefore, we can restate Proposition 7.1.6 in this new category as

**Proposition 7.2.5.** *Suppose we have the algebraic set-up above. There exist finite positive constants $\Delta_+$ and $\Delta_-$ such that there exist morphism $\Phi$ and $\Psi$ in $\mathcal{F}$ to form the following sequence of maps,*

$$\Sigma^{\Delta_+ + \Delta_-} Cone_D(S') \xrightarrow{\Psi} \Sigma^{\Delta_+} Cone_C(S) \xrightarrow{\Phi} Cone_D(S') \xrightarrow{\Psi} \Sigma^{-\Delta_-} Cone_C(S)$$

*and*

$$\Phi \circ \Psi = \Psi \circ \Phi = i_{\Delta_+ + \Delta_-} \text{ in } K(\mathcal{F}).$$

*Moreover, $\Delta_+ + \Delta_- = 6(\delta_+ + \delta_-)$.*

First, we note $K(\mathcal{F})$ is a triangulated category, therefore, for the following diagram in $K(\mathcal{F})$,

$$\begin{tikzcd}
C \arrow[r, "S"] \arrow[d, "\phi"] & C \arrow[r, "\iota"] \arrow[d, "\phi"] & Cone_C(S) \arrow[r, "\pi"] \arrow[d, "h"] & C[1] \arrow[d, "\phi[1]"] \\
D \arrow[r, "S'"] & D \arrow[r, "\iota"'] & Cone_D(S') \arrow[r, "\pi"'] & D[1]
\end{tikzcd} \tag{7.3}$$

where $\iota$ is inclusion and $\pi$ is projection. By 4. Corollary in [GM, p. 242], we know if the first $\phi$, the second $\phi$ and the fourth $\phi[1]$ are isomorphisms, then the middle morphism $h$ is also an isomorphism (existence of this map is given by axiom one (TR1) of triangulated category). The basic tool to prove this is the exactness of functor $\mathrm{Hom}(Cone_D(S'), -)$ and $\mathrm{Hom}(-, Cone_C(S))$ together with Five Lemma. Second, here our situation is $\phi$ and $\psi$ is not necessarily invertible (because $i_{\delta_+ + \delta_-}$ is not identity). But we still can prove the following technical proposition saying $h$ is "close to be invertible" in $K(\mathcal{F})$ modulo enough shift of filtration, which will directly imply proposition 7.1.6.

**Proposition 7.2.6.** *Apply exact functor* $\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), -)$ *to the diagram (7.3) shifted by* $\Sigma^{\delta_+}$ *on the first row. For* $i^3_{\delta_+ + \delta_-} \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), Cone_D(S'))$*, there exists a* $j$ *in* $\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} Cone_C(S))$ *such that* $h \circ j = i^3_{\delta_+ + \delta_-}$*.*

*Remark* 7.2.7. Note that filtration of $Cone_D(S')$ is just $\ell_D$ since $S'$ preserves filtration. If $f : C \to D$ is well-defined in $K(\mathcal{F})$ then $f$ is also well-defined from $\Sigma^\delta C \to D$ for any $\delta \geq 0$. Moreover, for any chain map $f$ and any $i_\delta$, we know

$$f \circ i_\delta = i_\delta \circ f. \tag{7.4}$$

*Proof.* We will closely follow the proof of Five Lemma. Starting from $i^3_{\delta_+ + \delta_-} = i_{3(\delta_+ + \delta_-)} \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), Cone_D(S'))$, due to the following diagram,

$$\begin{tikzcd}
\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} Cone_C(S)) \arrow[r, "\pi\circ"] \arrow[d, "h\circ"] & \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_C(S'), \Sigma^{\delta_+} C) \arrow[d, "\phi\circ"] \\
\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), Cone_D(S')) \arrow[r, "\pi\circ"] & \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), D)
\end{tikzcd}$$

and (7.4) with Example 7.2.4, we know

$$\pi \circ i^3_{\delta_+ + \delta_-} = i^3_{\delta_+ + \delta_-} \circ \pi = \phi(i^2_{\delta_+ + \delta_-} \circ \psi \circ \pi).$$

Therefore, there exists an element, say, $i^2_{\delta_+ + \delta_-} \circ \psi \circ \pi \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} C)$ with image under $\phi$ to be $\pi \circ i^3_{\delta_+ + \delta_-}$. Then composed with $S'$, we have

$$S' \circ \pi \circ i^3_{\delta_+ + \delta_-} = i_{\delta_+ + \delta_-}(i^2_{\delta_+ + \delta_-} \circ S \circ \pi) = 0$$

so

$$i^2_{\delta_+ + \delta_-} \circ S \circ \pi = 0. \tag{7.5}$$

Therefore, combining with (7.5) and relation $\psi \circ S' = S \circ \psi$,

$$S(i^2_{\delta_+ + \delta_-} \circ \psi \circ \pi) = i^2_{\delta_+ + \delta_-}(S \circ \psi \circ \pi) = i^2_{\delta_+ + \delta_-}(\psi \circ S' \circ \pi) = \psi(i^2_{\delta_+ + \delta_-} \circ S \circ \pi) = 0.$$

Since the top row is exact, there exists some $z \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} C)$ such that $\pi \circ z = i^2_{\delta_+ + \delta_-} \circ \psi \circ \pi$, so by (7.4) we can choose $z$ to be in the form

$$z = i_{\delta_+ + \delta_-} \circ z'$$

where $z' \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{2\delta_+ + \delta_-} C)$ (such that $\pi \circ z = \psi \circ \pi$). Next, considering the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} C) \arrow[r, "\iota\circ"] \arrow[d, "\phi\circ"] & \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} Cone_C(S)) \arrow[d, "h\circ"] \\
\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), D) \arrow[r, "\iota\circ"] & \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), Cone_D(S'))
\end{tikzcd}$$

and element

$$i^3_{\delta_+ + \delta_-} - h(z) = i_{\delta_+ + \delta_-}(i^2_{\delta_+ + \delta_-} - h(z')).$$

By commutativity of previous diagram, we know this element is $\ker(\pi\circ)$. In fact,

$$\pi \circ i^3_{\delta_+ + \delta_-} - \pi \circ h(z) = i^3_{\delta_+ + \delta_-} \pi - \phi(i^2_{\delta_+ + \delta_-} \circ \psi \circ \pi) = 0.$$

So by exactness of lower row, we know there exists some $u \in \mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), D)$ such that $\iota(u) = i_{\delta_+ + \delta_-}(i^2_{\delta_+ + \delta_-} - h(z')) = \phi(\psi(i^2_{\delta_+ + \delta_-} - h(z')))$. Then we can take a preimage (under $\phi$)

$$w = \psi(i^2_{\delta_+ + \delta_-} - h(z'))$$

in $\mathrm{Hom}(\Sigma^{3(\delta_+ + \delta_-)} Cone_D(S'), \Sigma^{\delta_+} C)$. Therefore

$$\iota(u) = \iota \circ \phi(w) = h \circ \iota(w) = i^3_{\delta_+ + \delta_-} - h(z)$$

so

$$i^3_{\delta_+ + \delta_-} = h(z + \iota(w)).$$

Hence we have shown that there exists a preimage of $i^3_{\delta_++\delta_-}$ under map $h$, that is

$$j := z + \iota(\psi(i^2_{\delta_++\delta_-} - h(z'))).$$

Therefore, we find a $j$ such that $h \circ j = i^3_{\delta_++\delta_-}$. $\square$

*Proof of Proposition 7.2.5.* Similarly to the proof of Proposition 7.2.6, we can apply contravariant functor $\mathrm{Hom}(-, \Sigma^{-2\delta_+-3\delta_-}Cone_C(S))$ on the first row of diagram (7.3) shifted by functor $\Sigma^{\delta_+}$ so we will get some $j' \in \mathrm{Hom}(Cone_D(S'), \Sigma^{-2\delta_+-3\delta_-}Cone_C(S))$ such that

$$j' \circ h = i^3_{\delta_+-\delta_-}. \tag{7.6}$$

Combined with Proposition 7.2.6, we have the following composition,

$$\Sigma^{3\delta_++3\delta_-}Cone_D(S') \xrightarrow{j} \Sigma^{\delta_+}Cone_C(S) \xrightarrow{h} Cone_D(S') \xrightarrow{j'} \Sigma^{-2\delta_+-3\delta_-}Cone_C(S). \tag{7.7}$$

Now $j$ and $j'$ are not necessarily the same since $i^3_{\delta_++\delta_-}$ is not an isomorphism. But taking advantage that $i^3_{\delta_++\delta_-}$ commutes with any morphism, we have

$$(j' \circ h \circ j) \circ h = j' \circ (h \circ j) \circ h = j' \circ i^3_{\delta_++\delta_-} \circ h = (j' \circ h) \circ i^3_{\delta_++\delta_-} = i^6_{\delta_++\delta_-}$$

and

$$h \circ (j' \circ h \circ j) = h \circ (j' \circ h) \circ j = h \circ i^3_{\delta_+\delta_-} \circ j = (h \circ j) \circ i^3_{\delta_++\delta_-} = i^6_{\delta_++\delta_-}.$$

Denote $g = j' \circ h \circ j$, then we can improve chain of maps (7.7) into

$$\Sigma^{6\delta_++6\delta_-}Cone_D(S') \xrightarrow{g} \Sigma^{\delta_+}Cone_C(S) \xrightarrow{h} Cone_D(S') \xrightarrow{g} \Sigma^{-5\delta_+-6\delta_-}Cone_C(S). \tag{7.8}$$

Hence, we will get the conclusion by setting $\Delta_+ = \delta_+$, $\Delta_- = 5\delta_+ + 6\delta_-$ and $\Phi = h$ and $\Psi = g$. $\square$

## 7.3 Proofs of Proposition 1.4.2 and Proposition 1.4.3

### 7.3.1 Proof of Proposition 1.4.2

*Proof of Proposition 1.4.2.* By Remark 4.3.2, $(\Delta_+, \Delta_-)$-quasiequivalence is in particular $(\Delta_+ + \Delta_-, \Delta_+ + \Delta_-)$-quasiequivalence. For any given $\epsilon > 0$, there exists a $(\Delta_+, \Delta_-)$-quasiequivalence between $Cone(H)_*$ and $Cone(G)_*$ with

$$\Delta_+ + \Delta_- \leq 2d_Q(Cone(H)_*, Cone(G)_*) + 2\epsilon$$

Moreover, by Proposition 4.3.1, we know

$$|\beta_i(\phi_H) - \beta_i(\phi_G)| \leq 2(\Delta_+ + \Delta_-) \leq 4d_Q(Cone(H)_*, Cone(G)_*) + 4\epsilon.$$

As $\epsilon$ is arbitrarily chosen, we get the conclusion. □

### 7.3.2 Proof of Proposition 1.4.3

*Proof of Proposition 1.4.3.* Clearly, we only need to prove the following conclusion

$$\mathfrak{o}_X(\phi)_k - \mathfrak{o}_X(\psi)_k \leq 24p \cdot d_H(\phi, \psi) \tag{7.9}$$

for each degree $k$. Indeed, for any $\epsilon > 0$, there exists some $k \in \mathbb{Z}$ such that $\mathfrak{o}_X(\phi) \leq \mathfrak{o}_X(\phi)_k + \epsilon$. Meanwhile, for this $k$, we have $\mathfrak{o}_X(\psi)_k \leq \mathfrak{o}_X(\psi)$. Therefore, (7.9) will implies

$$\mathfrak{o}_X(\phi) - \mathfrak{o}_X(\psi) \leq \mathfrak{o}_X(\phi)_k - \mathfrak{o}_X(\psi)_k + \epsilon \leq 24p \cdot d_H(\phi, \psi)$$

and by symmetry, we get the other direction.

Now suppose $\phi = \phi_H$ and $\psi = \psi_G$. Proposition 1.4.1 and Proposition 1.4.2 give

$$|\beta_i(\phi) - \beta_i(\psi)| \leq 12p \cdot ||H - G||_H.$$

If $o_X(\phi)_k$ is realized by some index $s_0 \in \mathbb{N}$, then

$$\begin{aligned} \mathfrak{o}_k(\phi) - \mathfrak{o}_k(\psi) &\leq (\beta_{s_0p+1}(\phi) - \beta_{(s_0+1)p}(\phi)) - (\beta_{s_0p+1}(\psi) - \beta_{(s_0+1)p}(\psi)) \\ &\leq (\beta_{s_0p+1}(\phi) - \beta_{s_0p+1}(\psi)) + (\beta_{(s_0+1)p}(\psi) - \beta_{(s_0+1)p}(\phi)) \leq 24p \cdot ||H - G||_H. \end{aligned}$$

So we get the conclusion by taking the infimum of generating functions for $\phi$ and $\psi$. $\square$

# Chapter 8

# Chaotic model

As explained in the introduction part, the key that we can conclude our main result lies in a chaotic model - egg-beater model, $\Sigma_g$ (for $g \geq 4$). In this section, we will do a concrete computation combining all the ingredients from previous chapters to draw the following conclusion, which is a special case of our main theorem.

**Theorem 8.0.1.** *When $p \geq 3$, $power_p(\Sigma_g) = +\infty$. In particular, $aut(\Sigma_g) = +\infty$.*

## 8.1 Count multiplicity

This section is devoted to prove Proposition 1.5.1. First, we will state CZ-index formula for generator of $CF_*(\Sigma_g, \phi_\lambda)_\alpha$ to help us label the indices later in this section. Recall from construction of egg-beater model, for each fixed point $z$, it will naturally come up with another $2p-1$ intermediate points, denoted as $z_1, ..., z_{2p-1}$ and set $z = z_0$. Using coordinates, we denote

$$z_{2j} = (x_{2j}, y_{2j}) \quad \text{and} \quad z_{2j+1} = (x_{2j+1}, y_{2j+1}).$$

By (41) in [PS14], we know

$$(x_{2j+1}, y_{2j+1}) = (-y_{2j+2}, x_{2j}) \quad \text{and} \quad (x_{2p}, y_{2p}) = (x_0, y_0).$$

So we will expect our formula only involves $x$ and $y$ coordinate of *even* index intermediate points. Here it is,

**Theorem 8.1.1.** *[Theorem 3.5 in [AKKKPRRSSZ15]] In egg-beater model (1.12), the CZ-index of fixed point $z$ with intermediate points $z_1, \ldots, z_{2p-1}$ is given by*

$$\mu_{CZ}(z) = 1 + \frac{1}{2}\sum_{j=0}^{p-1}(\mathrm{sign}(x_{2j}) - \mathrm{sign}(y_{2j})). \tag{8.1}$$

Note that each summand $\mathrm{sign}(x_{2j}) - \mathrm{sign}(y_{2j})$ only takes value $-2, 0$ and $2$, so support of index is $[-p+1, p+1]$. Moreover, we know

**Corollary 8.1.2.** *For each degree $k \in [-p+1, p+1]$, there are $\binom{2p}{k+p-1}$-many fixed points $z$ coming in with their p-tuple (1.13). Moreover, action of each generator of degree $k+1$ is strictly bigger than any generator of degree $k$ with action gap proportion to $\lambda$ up to a constant.*

*Proof.* For the first conclusion, suppose, within all the possible choices for $\mathrm{sign}(x_{2j})$ and $-\mathrm{sign}(y_{2j})$, we have $a$-many 1 and $b$-many $-1$, then by (8.1), we know $a - b = 2(k-1)$. Meanwhile, $a + b = 2p$, therefore, we need to choose $a = k+p-1$-many 1's. The second conclusion comes from Proposition 5.1 in [PS14] with (8.1). □

Now consider self-mapping cone of egg-beater model, denoted as

$$Cone_{\Sigma_g}(H_\lambda)_* := Cone_{CF(\Sigma_g, \phi_\lambda)}(T - \xi_p \cdot \mathbb{I})_*.$$

First, by a standard result from Floer theory, see Theorem 2.4.2, $H_*(CF(\Sigma, \phi_\lambda)_\alpha) = 0$ because $\alpha$ is non-contractible, so

$$H_*(Cone_{\Sigma_g}(H_\lambda)) = 0. \tag{8.2}$$

Therefore, in terms of barcode, there are only finite length bars, possibly with zero length. Meanwhile, by rank-nullity theorem, it is easy to check

$$\dim(\ker(\partial_{co})_k) = \dim(\mathrm{Im}(\partial_{co})_{k+1}) = \dim(CF_k(\Sigma_g, \phi_\lambda)_\alpha) = p \cdot \binom{2p}{k+p-1}. \tag{8.3}$$

Therefore, total multiplicity of verbose barcode of $Cone_{\Sigma_g}(H_\lambda)_*$ is

$$\sum_k \dim\ker(\partial_{co})_k = \frac{1}{2}\sum_k \dim(Cone_{\Sigma_g}(H_\lambda)_k) = p \cdot 2^{2p}.$$

Therefore, to prove Proposition 1.5.1, we need to prove there are exactly $(p-1)\cdot 2^{2p}$-many zero length bars in total for $Cone_{\Sigma_g}(H_\lambda)_*$. The way to prove this is by running the algorithm, see Theorem 3.5 in [UZ15], to get a singular value decomposition so that we can compute barcode in this concrete example. Actually, we will show for each degree $k \in [-p+1, p+1]$, there are $(p-1)\cdot\binom{2p}{k+p-1}$-many zero length bars. We will first demonstrate this by an explicit computational example in next section.

*Remark* 8.1.3. Structure of mapping cone is the key to succeed computing barcode and count its multiplicity. If we work only on $CF_*(\Sigma_g, \phi_\lambda)_\alpha$, in general, we don't have enough information of Floer boundary operator of $CF_*(\Sigma_g, \phi_\lambda)_\alpha$ to compute the associated barcode.

## 8.2 Barcode of self-mapping cone of egg-beater model I

First we briefly recall the process of generating a singular value decomposition of a filtration preserving linear map $F : (V, \ell_1) \to (W, \ell_2)$ when specializing $\Gamma$ being trivial (here egg-beater model satisfies). For general process, see Theorem 3.5 in [UZ15]. Given an ordered orthogonal basis $(v_1, ..., v_n)$ for $V$ and $(w_1, ..., w_m)$ for $W$, we will run the Gaussian elimination by choosing pivot column where the optimal index pair $(i_0, j_0)$ lies in. Here optimal index pair $(i_0, j_0)$ means

$$\ell_1(v_{j_0}) - \ell_2(w_{i_0}) \le \ell_1(v_j) - \ell_2(w_i) \quad \text{for all } (i,j) \in \{1,\ldots,n\}\times\{1,\ldots,m\}. \tag{8.4}$$

Therefore, after Gaussian elimination, $v_j$ is modified to be $v'_j = v_j - cv_{j_0}$ for some constant $c \in \mathcal{K}$. Moreover, we know

$$\ell(v'_j) = \ell(v_j) \quad \text{and} \quad \ell(Fv'_j) \le \ell(Fv_j).$$

Since $F$ is filtration preserving,

$$\ell(v_j') - \ell(Fv_j') = 0 \quad \text{implies} \quad \ell(v_j) - \ell(Fv_j) = 0. \tag{8.5}$$

Each step will start from choosing a pivot column and end up with deleting this column from consideration of pivot column for next step. Here is an example relating with our model. [1]

**Example 8.2.1.** *Let $p = 3$. For $Cone_{\Sigma_g}(H_\lambda)_{-1} \xrightarrow{(\partial_{co})_{-1}} Cone_{\Sigma_g}(H_\lambda)_{-2}$, our initial (orthogonal) bases are*

$$\big((b_1,0),(\phi_\lambda(b_1),0),(\phi_\lambda^2(b_1),0),...,(b_6,0),(\phi_\lambda(b_6),0),(\phi_\lambda^2(b_6),0)\big)$$

*and*

$$\big((0,a),(0,\phi_\lambda(a)),(0,\phi_\lambda^2(a))\big)$$

*for $Cone_{\Sigma_g}(H_\lambda)_{-1}$ where $b_i$ (for $i = 1,...,6$) and $a$ are fixed points; and*

$$\big((a,0),(\phi_\lambda(a),0),(\phi_\lambda^2(a),0)\big)$$

*for $Cone_{\Sigma_g}(H_\lambda)_{-2}$. First, by action functional formula for each fixed point (see (40) in [PS14]), we can generically get, for $s \in \{0,1,2\}$,*

$$\ell_{co}((\phi_\lambda^s(b_i),0)) < \ell_{co}((\phi_\lambda^s(b_j),0)) \quad whenever \quad i < j.$$

*Therefore, under the boundary map $\partial_{co}$, we have,*

$$\partial_{co}(\phi_\lambda^s(b_i),0) = \begin{bmatrix} \partial & -C\circ R_3 + \xi_3\cdot\mathbb{I} \\ 0 & -\partial \end{bmatrix} \begin{bmatrix} \phi_\lambda^s(b_i) \\ 0 \end{bmatrix} = \begin{bmatrix} L(\phi_\lambda^t(a)) \\ 0 \end{bmatrix}$$

[1]We recommend reader to go through this example carefully before reading the proof of Proposition 1.5.1 right after it.

*where $L(\phi_\lambda^t(a))$ is a linear combination of generators $\phi_\lambda^t(a)$ for $t \in \{0,1,2\}$. Meanwhile,*

$$\partial_{co}((0, \phi_\lambda^s(a)) = \begin{bmatrix} -C(\phi_\lambda^{(s+1) \bmod 3}(a)) + \xi_3 \phi_\lambda^s(a) \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_\lambda^{(s+1) \bmod 3}(a) + \xi_3 \phi_\lambda^s(a) \\ 0 \end{bmatrix}$$

*because continuation map $C$ is always in the form*

$$C(x) = x + \left\{ \begin{matrix} \textit{strictly lower} \\ \textit{filtration terms} \end{matrix} \right\}$$

*and by the speciality of generators (with lowest grading), there is no strictly lower filtration terms. In other words, if $(\partial_{co})_{-1}$ is represented by a matrix, it will be a 3 by 21 matrix as*

$$\begin{bmatrix} \xi_3 & 0 & -1 & * \dots * \\ -1 & \xi_3 & 0 & * \dots * \\ 0 & -1 & \xi_3 & * \dots * \end{bmatrix}. \tag{8.6}$$

*More importantly, we note*

$$\ell_{co}((0, \phi_\lambda^s(a)) - \ell_{co}((\phi_\lambda^t(a), 0)) = \ell(a) - \ell(a) = 0$$

*while for any $*$ position in the matrix (8.6),*

$$\ell_{co}(((\phi_\lambda^s(b_i), 0)) - \ell_{co}(0, \phi_\lambda^t(a)) > 0 \tag{8.7}$$

*therefore, our first choice of pivot column should come from one of the first three columns. Let's take column one. After Gaussian elimination, we have*

$$\begin{bmatrix} \xi_3 & 0 & 0 & * \dots * \\ -1 & \xi_3 & -\xi_3^2 & * \dots * \\ 0 & -1 & \xi_3 & * \dots * \end{bmatrix}$$

*and* $((0,\phi_\lambda^2(a))$ *is changed to* $(0,\phi_\lambda^2(a)+\xi_3^2 a)$. *Again, by the same reason, our second pivot column can be taken as the second column, so Gaussian elimination will give*

$$\begin{bmatrix} \xi_3 & 0 & 0 & *\dots* \\ -1 & \xi_3 & 0 & *\dots* \\ 0 & -1 & 0 & *\dots* \end{bmatrix}$$

*and* $(0,\phi_\lambda^2(a)+\xi_3^2 a)$ *is changed to* $(0,\phi_\lambda^2(a)+\xi_3\phi_\lambda(a)+\xi_3^2 a)$. *Note that second factor is in the kernel of* $R_3-\xi_3\cdot\mathbb{I}$. *Thus we get two elements in the singular value decomposition,* $(0,a)\to(-\phi_\lambda(a)+\xi_3 a,0)$ *and* $(0,\phi_\lambda(a))\to(-\phi_\lambda^2(a)+\xi_3\phi_\lambda(a),0)$. *Both of them will give zero length bars. The choice of next pivot column will start from column corresponding to generator* $(\phi_\lambda^s(b_i),0)$. *But by (8.7) and (8.5), we know none of them will give zero length bars. Moreover, by (8.3), multiplicity of degree-*$(-2)$ *verbose barcode is* 3. *So multiplicity of degree-*$(-2)$ *concise barcode is* 1.

## 8.3 Barcode of self-mapping cone of egg-beater model II

For any $p$-tuple generator of $CF_*(\Sigma,\phi_\lambda)_\alpha$ denoted as $\{z,\phi(z),...,\phi^{p-1}(z)\}$, for the span $V=\mathrm{span}_{\mathcal{K}}\langle z,\phi(z),...,\phi^{p-1}(z)\rangle$, the operator $R_p-\xi_p\cdot\mathbb{I}$ on $V$ is represented by the matrix

$$Q_p=\begin{bmatrix} -\xi_p & 0 & \dots & \dots & 1 \\ 1 & -\xi_p & \dots & \dots & 0 \\ 0 & 1 & -\xi_p & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -\xi_p \end{bmatrix} \tag{8.8}$$

and it has rank $p-1$. Its kernel is

$$\ker(R_p-\xi_p\cdot\mathbb{I})=\mathrm{span}_{\mathcal{K}}\left\langle \xi_p^{p-1}z+\xi_p^{p-2}\phi(z)+...+\phi^{p-1}(z)\right\rangle.$$

*Proof of Proposition 1.5.1.* For any degree $k$, after choosing "the standard" orthogonal bases as the generating loop with in the $p$-tuple forms, boundary map $(\partial_{co})_{k+1}$ can be represented as the following matrix

$$\begin{bmatrix} * & * & \dots & * & * \\ \boxed{P} & * & \dots & * & \vdots \\ 0 & \boxed{P} & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & * & \vdots \\ 0 & 0 & \dots & \boxed{P} & * \end{bmatrix} \tag{8.9}$$

where boxed $P = -Q_p$ in (8.8). Starting from the most left boxed $P$, we know after choosing pivot columns as describe in the Example 8.2.1 above, we will have $(p-1)$-many elements in the singular value decomposition which contribute to $(p-1)$-many zero length bars. Moreover, after the first $p-1$ step of Gaussian elimination, we will change the basis element corresponding to the $p$-th column in (8.9) in the form

$$v = v_* + \left\{ \begin{array}{c} \text{strictly lower} \\ \text{filtration terms} \end{array} \right\}$$

where $v_* \in \ker(R_p - \xi_p \cdot \mathbb{I})$. Therefore $\ell(v) - \ell((T^{H_\lambda} - \xi_p \cdot \mathbb{I})(v)) > 0$. Meanwhile, due to the nice position of boxed $P$ and the order that we start from the most left boxed $P$ and consecutively move to the last boxed $P$, we will eventually get

$$\frac{p-1}{p} \dim(CF_k(\Sigma, \phi_\lambda)_\alpha) = (p-1) \cdot \binom{2p}{k+p-1}$$

many zero length bars. By (8.5), we know after we used up all the pairs (under boundary map) from original basis having zero difference on the filtration, others will always give positive length bar. So in total, the multiplicity of zero length bars is

$$\sum_{k=-p+1}^{p+1} (p-1) \cdot \binom{2p}{k+p-1} = (p-1) \cdot 2^{2p}.$$

So we have exactly $2^{2p}$-many non-zero positive length bars in total. □

*Proof of Theorem 8.0.1.* Proposition 1.5.1, Lemma 8.1.2 and Theorem 1.4.5. □

# Chapter 9

# Product structure

## 9.1 Product of barcode

Given two Floer-type complexes $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ over $\Lambda^{\mathcal{K},\Gamma}$, we can form its tensor product

$$((C \otimes D)_*, \partial_\otimes, \ell_\otimes)$$

where

$$\partial_\otimes(a \otimes b) = \partial_C a \otimes b + (-1)^{|a|} a \otimes \partial_D b \tag{9.1}$$

where $|\cdot|$ denotes degree of element and

$$\ell_\otimes(a \otimes b) = \ell_C(a) + \ell_D(b).$$

Singular value decomposition of $(C \otimes D)_*$ is built from singular value decompositions of $C_*$ and $D_*$. Specifically, by Theorem 3.4.3, each Floer-type complex can be decomposed as a direct sum of *elementary complex* as follows,

$$\ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle x \rangle \to 0 \to \ldots \quad \text{and} \quad \ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \rangle \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle \partial y \rangle \to 0 \to \ldots.$$

Therefore, we have the following four types elementary complexes for the tensor product structure,

- $\ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \otimes z \rangle \to 0 \to \ldots$;
- $\ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \otimes z \rangle \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle \partial y \otimes z \rangle \to \ldots$;
- $\ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \otimes z \rangle \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \otimes \partial z \rangle \to \ldots$;
- $\ldots \to 0 \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle y \otimes z \rangle \to \mathrm{span}_{\Lambda^{\mathcal{K},\Gamma}} \langle \partial y \otimes z \pm y \otimes \partial z \rangle \to \ldots$.

If $y$ and $z$ are elements from singular value decompositions of $C_*$ and $D_*$, then orthogonality of each of these building blocks are guaranteed by Corollary 8.2 in [Ush13]. Therefore, we have the following proposition describing the barcodes of tensor product.

**Proposition 9.1.1.** *Barcode* [1] *of* $((C \otimes D)_*, \partial_\otimes, \ell_\otimes)$ *is given by carrying on the following operations for the original barcodes from* $(C_*, \partial_C)$ *and* $(D, \partial_D)$*,*

- *For* $\partial_C y = \partial_D z = 0$*,*

$$((\ell_C(y) \bmod \Gamma, \infty),\ (\ell_D(z) \bmod \Gamma, \infty)) \to \\ ((\ell_C(y) + \ell_D(z)) \bmod \Gamma, \infty).$$

- *For* $\partial_C y = x$ *and* $\partial_D z = 0$*,*

$$((\ell_C(y) \bmod \Gamma, \ell_C(y) - \ell_C(x)),\ (\ell_D(z) \bmod \Gamma, \infty)) \to \\ ((\ell_C(y) + \ell_D(z)) \bmod \Gamma, \ell_C(y) - \ell_C(x)).$$

- *For* $\partial_C y = 0$ *and* $\partial_D z = w$*,*

$$((\ell_C(y) \bmod \Gamma, \infty),\ (\ell_D(z) \bmod \Gamma, \ell_D(z) - \ell_D(w))) \to \\ ((\ell_C(y) + \ell_D(z)) \bmod \Gamma, \ell_D(z) - \ell_D(w)).$$

[1] Here degree is not specified.

- *For $\partial_C y = x$ and $\partial_D z = w$,*

$$(\ell_C(x) \bmod \Gamma, \ell_C(y) - \ell_D(x)),\ (\ell_D(w) \bmod \Gamma, \ell_D(z) - \ell_D(w)) \to$$
$$((\max\{\ell_C(x) + \ell_D(z), \ell_C(y) + \ell_D(w)\}) \bmod \Gamma, \min\{\ell_C(y) - \ell_D(x), \ell_D(z) - \ell_D(w)\}).$$

*Proof.* All the items are easy to verify. Here, we give the proof of the last item. If $\partial_C y = x$ and $\partial_D z = w$, then under the boundary map,

$$\partial_\otimes(y \otimes z) = \partial_C y \otimes z + (-1)^{|y|} y \otimes \partial_D z = x \otimes z + (-1)^{|y|} y \otimes w$$

since $x \otimes z$ is orthogonal to $y \otimes w$, for filtration, if $\ell_\otimes(x \otimes z) \geq \ell_\otimes(y \otimes w)$ (which is equivalent to $\ell_D(z) - \ell_D(w) \geq \ell_C(y) - \ell_C(x)$), then

$$\ell_\otimes(y \otimes z) - \ell_\otimes(x \otimes z + (-1)^{|y|} y \otimes w) = \ell_C(y) + \ell_D(z) - (\ell_C(x) + \ell_D(z)) = \ell_C(y) - \ell_C(x)$$

and if $\ell_\otimes(x \otimes z) \geq \ell_\otimes(y \otimes w)$ (which is equivalent to $\ell(y) - \ell(x) \geq \ell(z) - \ell(w)$), then

$$\ell_\otimes(y \otimes z) - \ell_\otimes(x \otimes z + (-1)^{|y|} y \otimes w) = \ell_C(y) + \ell_D(z) - (\ell_C(y) + \ell_D(w)) = \ell_D(z) - \ell_D(w).$$

□

**Example 9.1.2.** *Suppose we are given two Floer-type complexes,*

$$(C_*, \partial_C, \ell_C) \text{ (with trivial homology)} \quad \text{and} \quad (CF_*(M, \mathbb{I}) = CF_*(M, \mathbb{I})_{pt}, \partial_{\mathbb{I}}, \ell_{\mathbb{I}})$$

*where $M$ is a symplectically aspherical manifold associated with a Hamiltonian diffeomorphism being identity map. First, in order to have a well-defined Hamiltonian Floer chain complex with identity map, we should always think it as the limit of $CF_*(M, \epsilon f)$ for some small $\epsilon > 0$ and a Morse function $f : M \to \mathbb{R}$ when $\epsilon \to 0$. Then by standard Floer theory,*

$$CF_*(M, \epsilon f) \simeq CM_*(M, \epsilon f)$$

*where $CM_*$ represents a Morse chain complex. For barcode of $CM_*(M, \epsilon f)$, there are two types. One is infinite length bars which correspond to the generators of (Morse) homology of $M$ with respect to Morse function $\epsilon f$. The other is finite length bars whose lengths are, by Proposition 3.4 in [Ush11], at most $\epsilon ||f||$ (the total variation of $f$ on $M$), so they are called "$\epsilon$-small intervals".*

*Now denote*

$$\mathcal{B}(C_*) = \{(a, L) \ \textit{where } L \textit{ is finite}\},$$

*and*

$$\mathcal{B}(CF_*(M, \epsilon f)) = \{\{\epsilon\textit{-small intervals}\}, \ (c, \infty)\}.$$

*For its tensor product $(C \otimes_{\mathcal{K}} CF(M, \epsilon f))_*$, by Proposition 9.1.1, only the second and fourth items are considered, that is,*

$$((a, L), (c, \infty)) \to (a + c, L)$$

*and*

$$((a, L), \epsilon\textit{-small interval}) \to \epsilon\textit{-small interval}$$

*because the output of the operator on a pair of bars as in Proposition 9.1.1 always takes the minimal of lengths of two bars. Meanwhile, when $\epsilon \to 0$, $CM_*(M, \epsilon f)$ has trivial boundary operator, so, for each degree $j$,*

$$\begin{aligned} \textit{Betti number } b_j(M) &= \dim H_j(M, \mathcal{K}) \\ &= \dim CM_j(M, \epsilon f) = \dim CF_j(M, \epsilon f) \\ &= \dim HF_j(M, \mathbb{I}) = \{\#\textit{ of infinite length bars of degree } j\}. \end{aligned}$$

*In particular, given $k \in \mathbb{Z}$, for any degree $i, j$ such that $i + j = k$ (where $0 \leq j \leq 2n = \dim(M)$)*

$$\mathcal{B}(C_i \otimes_{\mathcal{K}} CF_j(M, \epsilon f)) = \left\{ (a + *, L) \,\middle|\, \begin{array}{c} (a, L) \in \mathcal{B}(C_i) \\ L \textit{ repeats } b_j(M)\textit{-many times} \end{array} \right\}.$$

*So the multiplicity of $(i,j)$-piece of degree-$k$ barcode of product is*

$$\{\text{multiplicity of } B(C_i)\} \times b_j(M). \tag{9.2}$$

## 9.2 Reproof of Polterovich-Shelukhin's result and generalization

**Lemma 9.2.1.** *Considering the following two maps between two Floer-type complexes $C \xrightarrow{S} C$ and $D \xrightarrow{\mathbb{I}} D$, then we have a chain isomorphism for any degree $k \in \mathbb{Z}$,*

$$Cone_{C\otimes D}(S\otimes \mathbb{I})_k \simeq \bigoplus_{i+j=k} Cone_C(S)_i \otimes D_j$$

*where the left side is mapping cone of $C \otimes D \xrightarrow{S\otimes\mathbb{I}} C\otimes D$.*

*Proof.* We have the following diagram

$$\begin{tikzcd}
\bigoplus_{i+j=k} C_i \otimes D_j \arrow[r, "S\otimes\mathbb{I}"] \arrow[d, "\phi"] & \bigoplus_{i+j=k} C_i \otimes D_j \arrow[r, "\iota"] \arrow[d, "\phi"] & \bigoplus_{i+j=k} Cone_C(S)_i \otimes D_j \arrow[r, "\pi"] \arrow[d, "h"] & \bigoplus_{i+j=k-1} C_{i-1} \otimes D_j \arrow[d, "\phi"] \\
(C\otimes D)_k \arrow[r, "S\otimes\mathbb{I}"] & (C\otimes D)_k \arrow[r, "\iota"'] & Cone_k(S\otimes\mathbb{I}) \arrow[r, "\pi"'] & (C\otimes D)_{k+1}
\end{tikzcd}$$

The top row is a distinguished triangle because tensor (of vector spaces) preserves distinguished triangle here [2] and direct sum of distinguished triangles is also distinguished triangle. Moreover, $\phi$ is an isomorphism by definition, so there exists a chain map $h$ being an isomorphism. $\square$

We can get the following result, which is a special case of Proposition 1.5.4.

**Lemma 9.2.2.** *For self-mapping cone of $\Sigma_g \times M$ where surface $\Sigma_g$ has genus $g \geq 4$ and $M$ is a symplectically aspherical manifold under map $T^{H_\lambda} \times \mathbb{I} - \xi_p \cdot \mathbb{I}$, its concise barcode of degree-1 has multiplicities not divisible by $p$ when $p$ is sufficiently large.*

*Proof.* Denote

$$Cone_{\otimes}(H_\lambda)_* := (Cone_{CF_*(\Sigma_g\times M, \phi_\lambda\times\mathbb{I})_{\alpha\times\{pt\}}}(T^{H_\lambda}\times\mathbb{I} - \xi_p\cdot\mathbb{I})_*, \partial_{co}).$$

[2] Note that in general, tensoring is only right exact. But here our category only consists of vector spaces, as a special case of flat module, tensoring here is then exact.

Then by Lemma 9.2.1, for degree $k = 1$,

$$Cone_{\otimes}(H_\lambda)_1 = \bigoplus_{i+j=1} Cone_{\Sigma_g}(H_\lambda)_i \otimes CF_j(M, \mathbb{I}).$$

For index $i$, by proof of Proposition 1.5.1, we know $i \in \{-p+1, ..., p+1\}$ and for each $i$, degree-$i$ concise barcode of of $Cone_{\Sigma_g}(H_\lambda)_*$, denoted as $m_1$, has multiplicity $\binom{2p}{i+p-1}$. Moreover, since $M$ is aspherical, $j \in \{0, ..., 2n\}$. In other words,

$$j = 1 - i \in \{-p, ..., p\} \cap \{0, ..., 2n\}.$$

So there are three cases.

- Case 1. When $p > 2n (\geq 2)$. Then by (9.2), we know

$$m_1 = \sum_{j=0}^{2n} \binom{2p}{p-j} b_j(M) = \binom{2p}{p} b_0(M) + \binom{2p}{p-1} b_1(M) + ... + \binom{2p}{p-2n} b_{2n}(M).$$

  where $p - 2n > 0$. On the one hand, for $1 \leq j \leq 2n$, $p \mid \binom{2p}{p-j}$, therefore, since $b_0(M) = 1$,

$$m_1 \equiv \binom{2p}{p} \bmod p.$$

  On the other hand, by Babbage's theorem in [B19], we know

$$\binom{2p}{p} = 2\binom{2p-1}{p-1} \equiv 2 \bmod p.$$

  Therefore, $m_1$ is not divisible by $p$.

- Case 2. When $p = 2n$. By the similar argument as above and $b_{2n}(M) = 1$,

$$m_1 \equiv \binom{2p}{p} b_0(M) + \binom{2p}{0} b_{2n}(M) \equiv 3 \bmod p.$$

  Therefore, since $p$ is even, $m_1$ is not divisible by $p$.

- Case 3. When $p < 2n$. Then by (9.2), we know

$$m_1 = \sum_{j=0}^{p} \binom{2p}{p-j} b_j(M) = \binom{2p}{p} b_0(M) + \binom{2p}{p-1} b_1(M) + ... + \binom{2p}{0} b_p(M).$$

So by the similar argument,

$$m_1 \equiv \binom{2p}{p} b_0(M) + \binom{2p}{0} b_p(M) \equiv 2 + b_p(M) \bmod p.$$

If $p > 2 + b_p(M)$, then $m_1$ is not divisible by $p$.

$\square$

By Theorem 1.4.5, Proposition 1.5.2 (which will be proved in the next section) and Lemma 9.2.2, we actually reproof one of the main result in [PS14] (under a stronger condition - $p$ is sufficiently large).

**Theorem 9.2.3.** *(Theorem 1.3 in [PS14]) For $\Sigma_g \times M$ where $\Sigma_g$ is a surface with genus $g \geq 4$ and $M$ is a symplectically aspherical manifold,*

$$power_k(\Sigma_g \times M) = +\infty,$$

*when $k$ is sufficiently large.*

In order to generalize from a symplectically aspherical manifold to any symplectic manifold $(M, \omega)$, we count multiplicity of concise barcode in a more subtle way since for $CF_*(M, \mathbb{I})$ (which is isomorphic to regular homology of manifold), Novikov field will be involved because (again since boundary operator is trivial)

$$CF_*(M, \mathbb{I}) \simeq HF_*(M, \mathbb{I}) = H_*(M, \mathcal{K}) \otimes \Lambda^{\mathcal{K}, \Gamma}$$

where $\Gamma \leq \mathbb{R}$. Specifically, adding a homotopy class $S \in \pi_2(M)/(\ker(\omega) \cap \ker(c_1))$ to each critical point $p$ will result in

- (a) filtration is shifted by $\int_{S^2} S^*\omega$;

- (b) CZ-index is shifted by $-2Nc_1(S)$,

where $N$ is minimal Chern number. However, in terms of barcode, shift of filtration (a) will not change the length of bars since the end point is defined modulo $\Gamma$. The issue will come from shift of degree because different from the symplectically aspherical case, support of index of $HF_*(M, \mathbb{I})$ might be infinite by the shift of index (b) (when $N$ is nonzero). Quantum Betti number (see Definition 1.5.3) is helpful for our counting in the sense that, similar to (9.2), the number of positive length bars that a $(i, j)$-piece will contribute is

$$\{\text{multiplicity of } B(C_i)\} \times qb_j(M). \tag{9.3}$$

*Proof of Proposition 1.5.2 and Proposition 1.5.4.* (a) By Proposition 9.1.1, there will be only the first two types to be considered. Indeed, due to the minimality of the length component in the last two type, it will be neglected eventually because any finite length bar contributed from $CF_*(M, \mathbb{I})_{\{pt\}}$ has $\epsilon$-small length. Moreover, from the operation in Proposition 9.1.1, especially the second type, the length of finite length bar keeps the same. Therefore, the smallest length of finite length bar of $Cone_{\otimes}(H_\lambda)_*$ is the same as the smallest length of finite length bar of $Cone_{\Sigma_g}(H_\lambda)_*$ coming from egg-beater model. Therefore, we draw the conclusion of Proposition 1.5.2 by Lemma 8.1.2.

(b) Fix the degree of product being 1. By (9.3) and Proposition 1.5.1, multiplicity of degree-1 concise barcode of $Cone_{\otimes}(H_\lambda)_*$ is

$$\sum_{k=-p+1}^{p+1} \binom{2p}{k+p-1} \cdot qb_{1-k}(M). \tag{9.4}$$

We can split (9.4) into two parts. One is

$$\binom{2p}{0} \cdot qb_p(M) + \binom{2p}{p} \cdot qb_0(M) + \binom{2p}{2p} \cdot qb_{-p}(M) \tag{9.5}$$

and

$$\sum_{k \neq \{-p+1, 1, p+1\}} \binom{2p}{k+p-1} \cdot qb_{1-k}(M). \tag{9.6}$$

Note that each binomial number in (9.6) is divisible by $p$, therefore divisibility of (9.4) depends only on (9.5). First, it is never equal to zero because $qb_0(M) \neq 0$. Moreover, by Babbage's theorem in [B19], we know

$$\binom{2p}{p} = 2\binom{2p-1}{p-1} \equiv 2 \bmod p.$$

Therefore, modulo $p$, we get the first conclusion of Proposition 1.5.4. For its second conclusion, by definition,

$$qb_p(M) = \sum_{s \in \mathbb{Z}} b_{p+2Ns}(M) \leq \sum_{i \text{ is odd}} b_i(M)$$

because $p + 2Ns$ is always odd. Similarly to $qb_{-p}(M)$; and

$$qb_0(M) = \sum_{s \in \mathbb{Z}} b_{2Ns}(M) \leq \sum_{i \text{ is even}} b_i(M)$$

because $2Ns$ is always even. Therefore, together, modulo $p$, we get (9.5) is at most $2\sum_{0 \leq i \leq 2n} b_i(M)$. So when $p > \sum_{0 \leq i \leq 2n} b_i(M)$, non-divisibility always holds. □

*Remark* 9.2.4. Here we remark that the non-divisibility requirement for Proposition 1.5.4 can sometimes be improved considerably. Here we give two examples.

(a) If $c_1(TM) = 0$, then since $qb_k(M) = b_k(M)$ for any $k \in \mathbb{Z}$, modulo $p$, (9.5) is equal to

$$b_p(M) + 2.$$

Therefore, if $p \nmid b_p(M) + 2$, then $p \nmid m_1$.

(b) If $M = \mathbb{C}P^n$, so $c_1(T\mathbb{C}P^n) = n+1$ and $b_k(\mathbb{C}P^n)$ is nonzero only when $k$ is even and $k \in [0, 2n]$. Therefore, back to (9.5), we only need to consider the middle term and modulo $p$, we get 2. Therefore, the non-divisibility holds for any prime $p \geq 3$ in this case.

# Bibliography


[AD14] Audin. M, Damian. M, *Morse theory and Floer homology.* Translated from the 2010 French original by Reinie Erne. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014.

[AKKKPRRSSZ15] Alvarez-Gavela, D., Kaminker, V., Kislev, A., Kliakhandler, K., Pavlichenko, A., Rigolli, L., Rosen, D., Shabtai, O., Stevenson, B., Zhang, J., *Embedding of free groups into asymptotic cones of Hamiltonian diffeomorphisms.* arXiv: 1602.05842v1.

[B19] C. Babbage, *Demonstration of a theorem relating to prime numbers*, The Edinburgh philosophical journal 1 (1819), 46-49.

[BD13] D. Burghelea and T. Dey. *Topological persistence for circle-valued maps.* Discrete Comput. Geom. **50** (2013), no. 1, 69–98.

[BO10] Buhovsky, L., Ostrover, Y., *On the uniqueness of Hofer's geometry.* arxiv:1008.0661.

[Car09] Carlsson, G., *Topology and data*, Bull. Amer. Math. Soc. 46 (2009), no. 2, 255-308.

[CCGGO09] F. Chazal, D. Cohen-Steiner, M. Glisse, L. Guibas, and S. Oudot. *Proximity of persistence modules and their diagrams.* Proceedings of the 25th Annual Symposium on Computational Geometry, SCG '09, 237–246. ACM, 2009.

[CdSGO12] Chazal, F., de Silva, V., Glisse, M., Oudot, S., *Structure and stability of persistence modules.* arXiv:1207:3674.

[Cr12] W. Crawley-Boevey. *Decomposition of pointwise finite-dimensional persistence modules.* arXiv:1201.0819.

[CS10] Carlsson, G., Silva, V., *Zigzag Persistence.* Foundations of Computational Mathematics, 10(4):367-405, August 2010.

[CZ05] Carlsson, G., Zomorodian, A., *Computing persistent homology.* Discrete Comput. Geom. 33 (2005), no. 2, 249-274.

[DW05] Derksen, H., Weyman, J., *Quiver representations.* Not. Am. Math. Soc. 52(2), 200-206 (2005).

[EP08] Entov, M., Polterovich, L., *Symplectic quasi-states and semi-simplicity of quantum homology.* (English summary) Toric topology, 47-70, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.

[EP09] Entov, M., Polterovich L., *Rigid subsets of symplectic manifolds.* Compositio Math. 145 (2009), 773-826.

[Flo89] A. Floer. *Symplectic fixed points and holomorphic spheres.* Comm. Math. Phys. **120** (1989), no. 4, 575–611.

[FO99] K. Fukaya and K. Ono. *Arnold conjecture and Gromov–Witten invariants.* Topology **38** (1999), 933–1048.

[FOOO09] Fukaya, K., Oh, Y.-G., Ohta, H., and Ono, K., *Lagrangian Intersection Floer Theory: Anomaly and Obstruction.* 2 vols. AMS, Providence, 2009. MR2553465.

[Gin10] Ginzburg, V., *The Conley conjecture.* Ann. of Math. (2) 172 (2010), no. 2, 1127-1180.

[GG15] Ginzburg, V., Gurel, B., *Non-contractible periodic orbits in Hamiltonian dynamics on closed symplectic manifolds.* arXiv: 1503.07145.

[Ghr08] R. Ghrist. *Barcodes: The persistent topology of data.* Bull. Amer. Math. Soc. **45** (2008), 61–75.

[GM] Gelfand, S.I., Manni Y. I., *Methods of homological algebra*, Springer, Berlin; New York: ISBN/ISSN 3540547460 (Berlin) 0387547460 (New York).

[Gut15] Gutt, J., *The positive equivariant symplectic homology as an invariant for some contact manifolds.* arXiv:1503.01443.

[Hof90] Hofer, H., *The topology of symplectic maps, Proceedings of the Royal Society of Edinburgh.* 115 (1990), 25-38.

[HLS15] Humiliere, V., Le Roux, F., and Seyfaddini, S., *Towards a dynamical interpretation of hamiltonian spectral invariants.* arXiv:1502.03834, to appear in Geometry & Topology.

[HS95] H. Hofer and D. Salamon. *Floer homology and Novikov rings.* In *The Floer memorial volume*, 483–524, Progr. Math., **133**, Birkhäuser, Basel, 1995.

[Kaw14] Kawasaki, M., *Superheavy lagrangian immersion in 2-torus.* arXiv:1412.4495, 2014.

[MS65] A. Monna and T. Springer. *Sur la structure des espaces de Banach non-archimédiens.* Nederl. Akad. Wetensch. Proc. Ser. A **68**=Indag. Math. **27** (1965), 602–614.

[MS04] D. McDuff, D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004. MR 2045629 (2004m:53154)

[Oh05] Oh, Y.-G., *Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds.* The breath of symplectic and Poisson geometr. Progr. Math. 232, Birkhauser, Boston, pages 525-570, 2005.

[Ono05] Ono, K., *Floer-Novikov cohomology and symplectic fixed points.* J. Symplectic Geom., 3 (2005), 545563.

[Par13] J. Pardon. *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves.* arXiv:1309:2370.

[Pol01] Polterovich, L., *The geometry of the group of symplectic diffeomorphism*, Birkhauser, 2001.

[PS14] Polterovich, L., Shelukhin, E., *Autonomous Hamiltonian flows, Hofer's geometry and persistence modules.*, Selecta Math. (2015), published online, doi: 10.1007/s00029-015-0201-2.

[RS93] Robbin, J., Salamon, D., *The Maslov index for paths*, Topology 32, 1993, 827-844.

[Sal90] Salamon, D., *Morse theory, the Conley index and Floer homology*, Bull. L.M.S. 22 113-140, (1990)

[Sal97] Salamon.D., *Lectures on Floer homology.* In Symplectic geometry and topology (Park City, Utah, 1997). AMS, Providence, 1999. MR1702944.

[Sey13] Seyfaddini, S., *$C^0$-limits of Hamiltonian paths and the Oh-Schwarz spectral invariants*, Int. Math. Res. Not. IMRN 21 (2013), 4920-4960.

[Sch00] Schwarz, M., *On the action spectrum for closed symplectically aspherical manifolds.* Pacific J. Math., 193(2): 419-461, 2000.

[Ush08] Usher, M., *Spectral numbers in Floer theories.* Compositio Math. 144 (2008), 1581-1592.

[Ush09] Usher, M., *Many closed symplectic manifolds have infinite Hofer-Zehnder capacity.* Transactions of the American Mathematical Society 364 (2012), no. 11, 5913-5943.

[Ush10] Usher, M., *The sharp energy-capacity inequality.* Communications in Contemporary Mathematics, 12 (2010), no. 3, 457-473.

[Ush11] Usher, M., *Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds*, Israel Journal of Mathematics, 184 (2011), 1-57.

[Ush13] Usher, M., *Hofer metrics and boundary depth.* Annales Scientifiques de l'cole Normale Suprieure 46 (2013), no. 1, 57-128.

[UZ15] Usher, M., Zhang, J., *Persistent homology and Floer-Novikov theory.* arXiv: 1502.07928, to appear in Geometry & Topology.

[Wei11] Weinberger, S., *What is ... persistent homology?* Notices Amer. Math. Soc. 58 (2011), 36-39.

[Vit92] Viterbo, C., *Symplectic topology as the geometry of generating functions.* Math. Annalen, 292:685-710, 1992.