Asset allocation and optimal selling rule with regime switching and partial observation

by

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(Under the direction of Qing Zhang)

Abstract

Regime Switching model was receiving increasing attention as researchers searching good models to capture prices of financial assets. Using a regime switching model we study asset allocation problem with one risk free asset and one risky asset. We characterize the value function in terms of solutions of a partial differential equation. We use Viscosity solution and Markov chain approximation for its numerical solution. The second part is concerned with stock selling rule. We use our regime switching model to find the optimal timing to sell under a logarithmic utility function.

A key component in this thesis is that we consider the models with partial information. We resort to Wonham filter to recover necessary information required for optimal control of the problems under consideration.

Index words: Asset allocation, regime switching, optimal selling time

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Chapter 1

Introduction

When a person wants to invest her/his wealth, she/he has numerous choices. All these choices can be categorized into two types: risky investments or risk-free investments. Risky investments, such as stocks, bring the possibility for bigger profits as well as the risk of possible loss. Risk-free investments, such as saving accounts, secure predictable amount of profit but at a relatively low return rate. When making investment discussions, the problem of balancing wealth between risk-free and risky investment, such as bonds and stocks, constantly comes up. In this thesis, the first problem we will work on is to find the optimal proportion of these two different assets. The term " asset allocation" refers to the process of spreading wealth across different types of financial asset classes. The modern financial market offers the opportunity of move money from one class to another. The decision of optimal investment portfolio is not done only at the beginning of the investment. Instead, it can be done in the duration of a long time investment, which makes a dynamic asset allocation strategy possible.

Among the two types of investment, the price of a risk-free asset is easier to model. It can be assumed to satisfy the ordinary differential equation for a continuous compounding bank account. For the modeling of the price for a risky asset, such as a stock, we choose a regime switching method.

Regime Switching model was gaining more attention as researchers try to find a good model for the prices of financial products. Traditionally, a geometric Brownian motion (GBM) has been widely used in finance to capture the movement of stock prices because of its tractability. However, practitioners and researchers also noticed its imperfections. One of its weakness is to keep the market parameters, such as return rate and volatility, constant, which is only valid for a very short period of time. Therefore, we modify the geometric Brownian motion with a regime-switching model, so that it is capable to characterize market behavior in a longer time horizon. In the mean time, it still allows feasible mathematical analysis on the related pricing model.

Regime-switching model assumes that the market has finite many modes. Under different market mode, there are different sets of market parameters such as the rate of returns, volatility or the risk-free interest rates. The market movement from one mode to another mode over time is often assumed to follow a Markov process.

For example, the market is often characterized as bull or bear market which shows the intuition behind the regime-switching model, see [11],[2].

Only recently have researchers started to recognize the importance of regime-switching model in asset allocation. To name a few, we refer to [1],[3]. When the underlying Markov process is observable, in [24], a closed form portfolio selection can be developed using a mean-variance technique. For a hidden Markov model of a special structure, an optimal trading strategy has been presented in [19]. In [15], nearly-optimal asset allocation strategies was developed to maximize the expected returns.

In a regime switching model, the market mode can be assumed to be observable or unobservable. When it is unobservable, certain technique has to be used to convert the problem into an observable one. In many cases, a filter equation can be applied, see [20], [21].

In this thesis, we consider a model where the underlying Markov process can only be partially observed through stock prices. Our model is different from [19], because in this paper the relationship between the instantaneous rates of return and the Markov chain is not decided by a state matrix. Therefore, a different method of filtering has to be applied.

Due to the specific modeling of the stock price, we use the filter developed by Wonham [23], which is referred to as Wonham filter and is given by the solution of a system of stochastic differential equations. In their paper [21], similar method has been used and a classical probabilistic solution is proved for the logarithmic utility and power utility function.

The associated partial differential equation that we use to characterize the value function in this thesis has no available analytical solution for a general utility function. For its numerical solution, we turn to the notion of viscosity solutions. The theory of viscosity solutions applies to certain type of partial differential equations. It was first introduced by Crandall and Lion [10]. We take advantage of the fact that the viscosity solution of partial differential equations can be non-differentiable and merely continuous. We prove that the value function in this asset allocation problem is indeed a viscosity solution for the underlying partial differential equation.

Finite difference method is often used to compute a viscosity solution. However, in our problem, due to the specific degenerate feature of the stochastic differential equation, finite difference method is not applicable. So we turned to alternative ways to calculate the value function, the solution to the key partial differential equation. We use the Markov chain approximation method, which was developed by Kushner [6], [7]. This is a very intuitive solution to the classical HJB equation. It relates closely to the probabilistic interpretation of the stochastic differential equations.

If the underlying Markov chain that governs the market mode has a large state space, the generator matrix of the Markov chain can be very large and the resulting differential equation can be very difficult to solve. A two-time-scale approach will be used to reduce computational complexity, see [17]. We used the two-time-scale Wonham filter in [17], developed the limit problem to reduce the dimension of the computation scheme and proved that the value function of the original problem converges to that of the limit problem.

The second problem in this thesis concerns how to choose the optimal selling time for a risky investment. In practice, if one wants to profit from speculation in the marketplace, after buying a stock, the timing for selling it becomes crucial. Setting up a selling rule is a practical strategy. This selling rule can be a target price range like in [16], or an optimal selling time as in [18]. In [16], a policy based on a target price and a stop-loss price is obtained by solving a set of two-point boundary value differential equations. In [18] , a strategy is constructed for "bubble stocks" so that the investor can decide when to sell a stock that has a rapid growth rate and then a rapid rate of decline by computing the probability of the positive growth rate and sell the stock when this probability becomes too low.

Using a regime switching model to describe the stock price, we compute an optimal selling rule through variational inequality sufficient condition and nonlinear filtering. The results are similar with that of [18], but more general in a sense that it is not just for "bubble stocks".

Chapter 2

Asset allocation: Problem Setup

When considering investment, the problem of balancing wealth between risk free investment such as bonds, and risky investment such as stocks, constantly comes up. In this thesis, we consider a continuous-time market setting with one risk free investment and one risky investment. Their prices are denoted by $P_1(t)$ and $P_2(t)$, respectively. The risk free investment $P_1(t)$ pays a constant interest rate of $r > 0$. The risky investment $P_2(t)$ is modeled according to a revised Black-Scholes model. Black-Scholes model has been widely used in finance to capture the movement of stock prices. It assumes that the stock price follows a geometric Brownian motion. That is,

$$dP_2(t) = \mu P_2(t)\,dt + \sigma P_2(t)\,dW_t,$$

where W_t is a standard Brownian motion, μ is a constant return rate of the stock and σ is its constant volatility. However, the assumption that the stock prices maintain constant rate of return and volatility is generally not true in the market place. So, in order to better capture the movement of the market and still maintain the tractability of Black-Scholes model, we add a built-in regime-switching feature to $P_2(t)$. Namely, $P_1(t)$ and $P_2(t)$ satisfies

$$\begin{aligned} dP_1(t) &= P_1(t) r dt, \\ \frac{dP_2}{P_2} &= \mu(\alpha(t))dt + \sigma dW(t), \end{aligned}$$

where $\alpha(t) \in \mathcal{M} = \{1, 2, \cdots, m\}$ is a continuous-time Markov process that governs the market mode, and $\mu(i)$ is the return rate of the stock when the market mode is i. Note that this is a model where the rate of return follows the market trend, but the volatility is assumed to be constant. For notation simplicity, we use $\mu(t)$ and μ_i interchangeably.

The wealth function of the investor is denoted $\xi(t)$, and the portion that is applied for the risky investment is $u(t)$. At each time t, $u(t)\xi(t)$ is put into risky investment and $(1-u(t))\xi(t)$ is put in risk-free investment. We will assume self-financing, which means $\xi(t)$ equals to the sum of the values of the above investments and no external funds are transferred to it or from it. There is no cash inflow or outflow and no short sell. Therefore we must have

$$\frac{d\xi(t)}{\xi(t)} = (1 - u(t))rdt + u(t)(\mu(\alpha(t))dt + \sigma dW(t)). \tag{2.1}$$

Suppose the initial time is s, and the initial wealth is $\xi(s) = y$. We assume the investor did not consume any amount of the investment until a fixed future time T. The investor's objective is to dynamically adjust $u(t)$ over time to maximize the expectation of a utility function $\Phi(\xi(T))$.

This type of asset allocation with regime switching have been studied in finance literature, see [19]. However, up until now, the Markov process $\alpha(t)$ has been generally considered to be observable, which is not the case in the marketplace. So our goal is to consider that the information of $\alpha(t)$ is not directly available and develop a method for this type of asset allocation problems.

In control theory, $u(t)$ is considered a feedback control. Denote the filtration generated by $P_2(t)$ as $\mathcal{F}_t$. The control u is *admissible* if u is progressively measurable with respect to $\{\mathcal{F}_t\}$ and $u(t) \in [0, 1]$ for all $t \in [0, T]$. Denote the set of admissible control by $\mathcal{A}$.

Definition A process $\{X_t\}_{t\geq 0}$ is said to be progressively measurable with respect to the filtration $\{\mathcal{F}\}_{t\geq 0}$ if, for all $t \geq 0$, the mapping $(s, w) \in [0, t] \times Q \to X(s, w)$ is $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$-measurable.

The difficulty involved in this problem lies both in the underlying Markov chain with only partial observation through the stock price $P_2(t)$ and the resulting complexity of the associated stochastic differential equations especially when the dimension of $\alpha(t)$ is large.

To be more specific, the price of the stock, we can observe it from the marketplace. But the market mode, $\alpha(t)$, we can not observe. No one can know for sure if today is a bullish market or a bearish market. $\alpha(t)$ is basically a "hidden Markov chain." What we can do is to observe $P_2(t)$ and make valid estimation of $\alpha(t)$, which is why we say we have "partial" observation of $\alpha(t)$. $P_2(t)$ is considered a function of $\alpha(t)$, and in order to estimate the state of $\alpha(t)$ based on $P_2(t)$ we need an non-linear filter algorithm, which is why we turn to Wonham filter. This is because the specific structure of $P_2(t)$:

$$dP_2(t) = P_2(t)(\mu(\alpha(t))dt + \sigma dW(t)),$$

fits perfectly into the Wonham filter theorem conditions. The results allows us to compute the conditional probability of $\alpha(t)$ given the past observation $\mathcal{F}_t = \{P_2(r), s < r < t\}$.

The search for the optimal control results in a special type of partial differential equations, for which an analytic solution is often not available. We turns to viscosity solutions and Markov chain approximations for its numerical solutions.

Chapter 3

Wonham filter

Since $\alpha(t)$ is not directly observable, we can not approach the asset allocation problem directly. However, note that $P_2(t)$ is observable in the marketplace. Since $P_2(t)$ is a function of $\alpha(t)$, we can treat $P_2(t)$ as a 'signal' that may lead to useful information of $\alpha(t)$. Wonham filter can offer a very good result for this type of nonlinear filtering. We will be using the following result about the Wonham filter, which can be found in [23], [17], and [13].

Let $\alpha(t)$ be a continuous-time Markov chain having finite state space $\mathcal{M} = \{1, ..., m\}$, and generator $Q = (q_{ij}) \in \mathcal{R}^{m\times m}$. Consider a function $y(t)$ of the Markov chain that is observable with additive Gaussian noise. Let $y(t)$ denote the observation measurement given by

$$dy(t) = f(\alpha(t))dt + \sigma dW(t), y(0) = 0, \tag{3.1}$$

where σ is a positive constant and $W(t)$ is a standard Brownian motion. Let $p_i(t)$ denote the conditional probability of $\alpha(t) = i$ given the observations up to time t, i.e.,

$$p_i(t) = P(\alpha(t) = i | y(s) : s \le t);$$

for $i = 1, ..., m$. Let $p(t) = (p_1(t), ..., p_m(t)) \in \mathcal{R}^{1\times m}$. Then the *Wonham filter* is given by

$$dp(t) = p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m} f(i)p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t), \tag{3.2}$$

$p(0) = p_0$, being the initial probability, where

$$A(t) = \text{diag}(f(1), ..., f(m)) - \sum_{i=1}^{m} f(i)p_i(t)I.$$

Here, I is the identity matrix of dimension $m \times m$.

Recall that

$$dP_2/P_2 = \mu(\alpha(t))dt + \sigma dW(t),$$

so

$$d\log(P_2) = [\mu(\alpha(t)) - \frac{\sigma^2}{2}]dt + \sigma dW(t),$$

Define $y(t) = \log(P_2(t))$. Then

$$dy(t) = [\mu(\alpha(t)) - \frac{\sigma^2}{2}]dt + \sigma dW(t),$$

Since $\log(P_2)$ is observable, so is $y(t)$. Using the Wonham filter result, we can set up a Wonham filter for $\alpha(t)$ by the following

$$dp(t) = p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t). \tag{3.3}$$

where $p(0) = p_0$ being the initial probability, and

$$\begin{aligned} A(t) &= \mathrm{diag}(\mu(1) - \frac{\sigma^2}{2}, ..., \mu(m) - \frac{\sigma^2}{2}) - \sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)I. \\ &= \mathrm{diag}(\mu(1) - \frac{\sigma^2}{2}, ..., \mu(m) - \frac{\sigma^2}{2}) - \sum_{i=1}^{m}\mu(i)p_i(t)I + \frac{\sigma^2}{2}\sum_{i=1}^{m}p_i(t)I. \\ &= \mathrm{diag}(\mu(1), ..., \mu(m)) - \textstyle\sum_{i=1}^{m}\mu(i)p_i(t)I. \end{aligned}$$

Denote $\tilde{\alpha}(t) = \sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)$, we then have

$$\begin{aligned} dp(t) &= p(t)Qdt - \frac{1}{\sigma^2}\tilde{\alpha}(t)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t) \\ &= p(t)Qdt + \frac{p(t)A(t)}{\sigma}\left(\frac{dy(t) - \tilde{\alpha}(t)dt}{\sigma}\right). \end{aligned} \tag{3.4}$$

Let

$$d\hat{v} = \frac{d\log(P_2) - \tilde{\alpha}dt}{\sigma}, \quad \hat{v}(0) = 0.$$

Note that is called *an innovation process.*

We next show that $E\hat{v}(t)=0$, $E(\hat{v}(t))^2=t$.

$$\begin{aligned} d\hat{v} &= \frac{dy(t)-\tilde{\alpha}dt}{\sigma} \\ &= \frac{1}{\sigma}\left(\left[\mu(\alpha(t))-\frac{\sigma^2}{2}\right]dt+\sigma dW(t)-\tilde{\alpha}dt\right) \\ &= \frac{1}{\sigma}\left(\left[\mu(\alpha(t))-\frac{\sigma^2}{2}\right]dt+\sigma dW(t)-\sum_{i=1}^{m}\left[\mu(i)-\frac{\sigma^2}{2}\right]p_i(t)dt\right) \\ &= \frac{1}{\sigma}\left(\mu(\alpha(t))dt+\sigma dW(t)-\sum_{i=1}^{m}\mu(i)p_i(t)dt\right) \end{aligned}$$

Note that $\sum_{i=1}^{m}\mu(i)p_i(t)=E[\mu(\alpha(t))|y(s):s\le t]$. Hence,

$$E\hat{v}(t)=E\int_0^t d\hat{v}=E\frac{1}{\sigma}\int_0^t\left(\mu(\alpha(x))dx+\sigma dW(x)-\sum_{i=1}^{m}\mu(i)p_i(x)\right)dx=0,$$

$$\begin{aligned} E(\hat{v}(t))^2 &= E(\textstyle\int d\hat{v})^2 \\ &= E\left(\int\frac{1}{\sigma}\left(\mu(\alpha(t))dt+\sigma dW(t)-\sum_{i=1}^{m}\mu(i)p_i(t)dt\right)\right)^2 \\ &= \frac{1}{\sigma^2}E\left(\int\sigma dW(t)\right)^2=t. \end{aligned}$$

To prove $\hat{v}$ is also a standard Brownian motion, we define

$$\mathcal{F}_t^y=\sigma\{y(r):s\le r\le t\}.$$

It follows that

$$E[\hat{v}(t)|\mathcal{F}_s^y]-\hat{v}(s)=E\left[\int_s^t\left(\mu(\alpha(x))dx+\sigma dW(x)-E[\mu(\alpha(x))|\mathcal{F}_x^y]\right)dx|\mathcal{F}_s^y\right]=0.$$

i.e. $\hat{v}(t)$ is an $\mathcal{F}_t^y$ martingale.

From the definition of $d\hat{v}$, we have

$$\sigma d\hat{v}=dy(t)-\widetilde{\alpha}dt.$$

Hence,

$$dy(t)=\widetilde{\alpha}dt+\sigma d\hat{v}.$$

Or

$$d\log(P_2(t)) = \widetilde{\alpha}dt + \sigma d\hat{v}.$$

Therefore, we may rewrite the stock price equation of P_2 by the following

$$\frac{dP_2}{P_2} = (\widetilde{\alpha}(t) + \frac{\sigma^2}{2})dt + \sigma d\hat{v}.$$

Notice that both $\tilde{\alpha}(t)$ and $d\hat{v}$ are observable.

Because

$$\widetilde{\alpha}(t) = \sum_{i=1}^{m}[\mu(i) - \sigma^2/2]p_i(t) = \sum_{i=1}^{m}\mu(i)p_i(t) - \sigma^2/2,$$

we can simplify our notation and obtain

$$\frac{dP_2}{P_2} = \widehat{\alpha}(t)dt + \sigma d\hat{v},$$

by letting

$$\widehat{\alpha}(t) = \sum_{i=1}^{m}\mu(i)p_i(t).$$

Now the dynamic of the wealth function $\xi(t)$ can be reformulated by

$$\begin{aligned} \frac{d\xi(t)}{\xi(t)} &= [1 - u(t)]rdt + u(t)(\widehat{\alpha}dt + \sigma d\hat{v}) \\ &= [(1-u)r + u\widehat{\alpha}]dt + u\sigma d\hat{v}. \end{aligned} \tag{3.5}$$

We want to find the optimal control u which will maximize

$$J(s, y, u(\cdot)) = E_{sy}(\Phi(\xi(T))),$$

where T is a specific future time. The optimal expected performance v is given by

$$v(s, y) = \sup_{u(\cdot)} J(s, y, u(\cdot)),$$

where $\xi(s) = y$ is the initial data. $v(s, y)$ is called a value function.

Note that in the dynamic of $\xi(t)$, both $\hat{\alpha}(t)$ and $\hat{v}(t)$ are driven by $p(t)$. The state process should include both $\xi(t)$ and $p(t)$. And the value function should have variables s, y, and p, where p is the initial probability vector $p(s)$.

Denote $Z(t) = \log \xi(t)$ and $z = \log y$, then we have

$$\begin{aligned}
dZ(t) &= [(1-u(t))r + u(t)\hat{\alpha}(t) - (1/2)(u(t)\sigma)^2]dt + u(t)\sigma d\hat{v}, \\
dp(t) &= p(t)Qdt + \frac{p(t)A(t)}{\sigma} d\hat{v}(t), \\
Z(s) &= z, \\
p(s) &= p.
\end{aligned}$$

Then

$$J(s, z, p, u(\cdot)) = E_{sz}(\Phi(\exp(Z(T)))),$$

$$v(s, z, p) = \sup_{u(\cdot)\in\mathcal{A}} J(s, z, p, u(\cdot)),$$

and $v(T, z, p) = \Phi(e^z)$.

Let $Y(t) = (Z(t) \vdots p(t))'$, where A' denote the transpose of the matrix (or vector) A. Then

$$dY(t) = \begin{pmatrix} (1-u)r + u\hat{\alpha} - (1/2)(u\sigma)^2 \\ Q'p(t)' \end{pmatrix} dt + \begin{pmatrix} u\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix} d\hat{v}.$$

Let

$$f(t, Y, u) = \begin{pmatrix} (1-u)r + u\hat{\alpha} - (1/2)(u\sigma)^2 \\ Q'p(t)' \end{pmatrix}$$

and

$$\Sigma(t, Y, u) = \begin{pmatrix} u\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix}.$$

Then

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t).$$

Define

$$H(t, Y, P, G) = fP + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')G\} \tag{3.6}$$

where P is an $1 \times (m+1)$ vector and G is an $(m+1) \times (m+1)$ matrix. Here, fP should be understood as the inner product of two vectors.

By Dynkin's formula,

$$E^{s,z,p}[v(T, Y(T))] = v(s, z, p) + E^{s,z,p} \int_s^T \{\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2})\} dt.$$

And

$$\sup_u E^{s,z,p}[v(T, Y(T))] = v(s, z, p) + \sup_u E^{s,z,p} \int_s^T \{\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2})\} dt.$$

Since

$$\sup_u E^{s,z,p}[v(T, Y(T))] = v(s, z, p),$$

we then have

$$\sup_u E^{s,z,p} \int_s^T \{\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2})\} dt = 0 \quad \text{for all } s \text{ and } T.$$

So

$$\sup_u \{\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2})\} = 0$$

Therefore, the value function $v(s, Y)$ should satisfy the following Partial Differential Equation

$$\frac{\partial v}{\partial s} + \sup_u H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) = 0 \tag{3.7}$$

with the boundary condition $v(T, z, p) = \Phi(\mathrm{e}^z)$, where $z = \log y$, y is the initial wealth, and p is the initial probability vector. Conventionally, this is called a Hamilton-Jacobi-Bellman (HJB) equation.

Chapter 4

Viscosity Solutions

An analytical solution to equation (3.7) is difficult to obtain (if not impossible). It is not even clear if the equation (3.7) has a classical solution. In this thesis, We use viscosity solution to characterize the dynamics of the system.

The theory of viscosity solutions applies to partial differential equations of the form $F(x, u, Du, D^2u) = 0$ where $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ and $S(N)$ is the set of symmetric $N \times N$ matrices. The notion of viscosity solutions was first introduced by Crandall and Lion for solving first-order Hamilton-Jacobi equations. The user's guide by Crandall , Ishii and Lion [10] offers a complete treatment of this topic. Readers are referred to [22] for applications to deterministic and stochastic control theory. Viscosity solution is useful for characterizing numerical solutions of partial differential equations of the form $F(x, u, Du, D^2u) = 0$ where Du is the gradient vector of u, D^2u is its Hessian matrix. The condition on F is that it has to be *proper* defined as follows.

Definition Function F is *proper* if it satisfies

$$F(x, r, p, X) \leq F(x, s, p, Y) \quad \text{whenever } r \leq s \text{ and } Y \leq X.$$

Definition Let Ω be an open subset of $\mathbb{R}^N$, F be proper and $u : \Omega \to \mathbb{R}$.

(a) u is a viscosity subsolution of $F(x, u, Du, D^2u) = 0$ in Ω if it is upper semicontinuous and for each $\phi \in C^2(\Omega)$ and local maximum point x_0 of $u - \phi$ we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(b) u is a viscosity supersolution of $F(x, u, Du, D^2u) = 0$ in Ω if it is lower semicontinuous and for each $\phi \in C^2(\Omega)$ and local minimum point x_0 of $u - \phi$ we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(c) u is a viscosity solution of $F(x, u, Du, D^2u) = 0$ in Ω if it is both viscosity subsolution and supersolution (hence continuous) of $F(x, u, Du, D^2u) = 0$.

Clearly, a classical solution $u \in C^2(\Omega)$ of $F(x, u, Du, D^2u) = 0$ is also a viscosity solution. However, a viscosity solution does not necessarily have to be differentiable. The reason that viscosity solutions are attractive is that it provides uniqueness theorems. It also allows a function that is merely continuous to be the solution of the PDE.

Recall that the value function $v(s, z, p)$ satisfies

$$\frac{\partial v}{\partial s} + \sup_u H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) = 0. \tag{4.1}$$

If we define

$$F(Y, v, Dv, D^2v) = \sup_u H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}),$$

then

$$v_t + F(Y, v, Dv, D^2v) = 0$$

is the equation under consideration. It is a classical parabolic equation. Note that $Dv = \frac{\partial v}{\partial Y}$ and $D^2v = \frac{\partial^2 v}{\partial Y^2}$ in this case.

Next, We first prove that the value function $v(s, z, p)$ is a viscosity solution of equation (3.7). Then we prove that the viscosity solution of equation (3.7) is unique. Therefore, a numerical viscosity solution is indeed a numerical solution for the PDE (3.7).

According to the above definition, in order to prove that the value function $v(s, z, p)$ is a viscosity solution to (3.7), we have to prove it is continuous, and

$$\frac{\partial \phi}{\partial s} + \sup_u \left\{ f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\} \right\} \leq 0,$$

for any $\phi \in C^2([0,T] \times \mathcal{R} \times [0,1])$ such that $v - \phi$ has a local minimum at (s,z,p); and

$$\frac{\partial \phi}{\partial s} + \sup_u \left\{ f \frac{\partial \phi}{\partial Y} + \frac{1}{2} \mathrm{tr}\{(\Sigma\Sigma') \frac{\partial^2 \phi}{\partial Y^2}\} \right\} \geq 0,$$

for any $\phi \in C^2([0,T] \times \mathcal{R} \times [0,1])$ such that $v - \phi$ has a local maximum at (s,z,p).

Before proving the continuity, we need the following condition for the utility function Φ.

$$|\Phi(y)| \leq K(1 + |\log y|^{k_1} + y^{k_2} + y^{-k_3}),$$

for $y \in (0,\infty)$, for some nonnegative constants K, $k_i, i = 1,2,3$.

Moreover, for any $y_1, y_2 \in (0,\infty)$, and some $K_1 > 0$, Either

$$|\Phi(y_1) - \Phi(y_2)| \leq K_1 |\log y_1 - \log y_2|$$

or, for some $\gamma < 1$ and $\gamma \neq 0$,

$$|\Phi(y_1) - \Phi(y_2)| \leq K_1 |y_1^\gamma - y_2^\gamma|.$$

These conditions are often imposed in the literature. With these conditions, we are ready to prove

Lemma 4.0.1. *$v(s,z,p)$ is continuous with respect to $s, z,$ and p.*

Proof. (1). $v(s,\cdot,p)$ is continuous with respect to z.

Fix (s,p). For given z_1, z_2 and u define

$$R(t) = \int_s^t [(1-u(x))r + u(x)\hat{\alpha}(x) - \frac{1}{2}(u(x)\sigma)^2]dx + \int_s^t u(x)\sigma d\hat{v}(x).$$

Let $Z_i(t), i = 1,2$, be defined as $Z_i(t) = z_i + R(t)$ then $Z_1(t) - Z_2(t) = z_1 - z_2$ and

$$\begin{aligned} |v(s,z_1,p) - v(s,z_2,p)| &\leq \sup_u |E\Phi(e^{z_1}e^{R(T)}) - E\Phi(e^{z_2}e^{R(T)})| \\ &\leq \sup_u |E\{[\Phi(e^{z_1}) - \Phi(e^{z_2})]e^{R(T)}\}| \\ &\leq K_1|z_1 - z_2|E(e^{R(T)}), \end{aligned}$$

Or

$$
\begin{aligned}
|v(s,z_1,p)-v(s,z_2,p)| &\le \sup_u |E\Phi(e^{z_1}e^{R(T)})-E\Phi(e^{z_2}e^{R(T)})| \\
&\le \sup_u |E\{[\Phi(e^{z_1})-\Phi(e^{z_2})]e^{R(T)}\}| \\
&\le EK_1|(e^{z_1R(T)})^\gamma-(e^{z_2R(T)})^\gamma| \\
&= K_1|e^{\gamma z_1}-e^{\gamma z_2}|E(e^{\gamma R(T)}).
\end{aligned}
$$

It suffices to show the boundedness of $E(e^{\gamma R(T)})$.

By Ito's differential rule, we have

$$
de^{\gamma R(t)} = e^{\gamma R(T)}[\gamma dR(t) + \frac{\gamma^2}{2}(u(t)\sigma)^2 dt].
$$

Taking expectation on both sides yields

$$
Ee^{\gamma R(t)} \le 1 + C\int_s^t Ee^{\gamma R(x)}dx,
$$

for some constant $C>0$. Then the Gronwall inequality implies

$$
Ee^{\gamma R(t)} \le e^{C(t-s)}, s\le t\le T.
$$

(2). $v(\cdot,z,p)$ is continuous with respect to s.

Fix (z,p). For a given $s,\Delta s>0$, and u define

$$
\begin{aligned}
\hat{Z}(t) &= Z(t-\Delta s), \\
\tilde{u}(t) &= u(t-\Delta s), \\
\tilde{p}(t) &= p(t-\Delta s), \\
\tilde{\alpha}(t) &= \textstyle\sum_{i=1}^m \mu(i)p(t-\Delta s).
\end{aligned}
$$

Write $Z(\cdot)$ in terms of $\tilde{u}(t)$:

$$
\begin{aligned}
Z(t) \quad = z+ &\int_{s+\Delta s}^{t+\Delta s}(1-\tilde{u}(x))r+\tilde{u}(x)\tilde{\alpha}(x)-\frac{1}{2}(\tilde{u}(x)\sigma)^2dx \\
+ &\int_{s+\Delta s}^{t+\Delta s}\tilde{u}(x)\sigma d\hat{v}(x).
\end{aligned}
$$

Let

$$\hat{Z}(t) = z + \int_{s+\Delta s}^{t} (1-\tilde{u}(x))r + \tilde{u}(x)\tilde{\alpha}(x) - \frac{1}{2}(\tilde{u}(x)\sigma)^2 dx + \int_{s+\Delta s}^{t} \tilde{u}(x)\sigma d\hat{v}(x).$$

Then

$$J(s+\Delta s, z, \tilde{p}, \tilde{u}) = E\Phi(e^{\hat{Z}(t)}).$$

Moreover,

$$Z(T) - \hat{Z}(t) = z + \int_{t}^{t+\Delta s} (1-\tilde{u}(x))r + \tilde{u}(x)\tilde{\alpha}(x) - \frac{1}{2}(\tilde{u}(x)\sigma)^2 dx + \int_{t}^{t+\Delta s} \tilde{u}(x)\sigma d\hat{v}(x),$$

$$\begin{aligned} |J(s,z,p,u) - J(s+\Delta s, z, \tilde{p}, \tilde{u})| &= |E\Phi(e^{Z(T)}) - E\Phi(e^{\hat{Z}(t)})| \\ \text{either} &\leq K_1 E|Z(T) - \hat{Z}(T) \leq K_1\sqrt{\Delta s} \\ \text{or} &\leq K_1|E(e^{\gamma Z(T)} - e^{\gamma \hat{Z}(T)})|. \end{aligned}$$

Note that $E(e^{\gamma Z(T)}) \leq K$.

By Cauchy-Schwarz inequality, we have

$$|E(e^{\gamma Z(T)} - e^{\gamma \hat{Z}(T)})|^2 \leq E(e^{\gamma(Z(T)-\hat{Z}(T))} - 1)^2 \leq K\Delta s.$$

The last inequality follows by Ito's rule.

(3). $v(s,z,\cdot)$ is continuous with respect to p.

Fix (s,z). For given p_1, p_2 and u, define

$$R_i(t) = \int_s^t [(1-u(x))r + u(x)\hat{\alpha}_i(x) - \frac{1}{2}u^2(x)\sigma^2]dx + \int_s^t u(x)\sigma d\hat{v}(x),$$

where $\hat{\alpha}_i(x) = \sum_{k=1}^m \mu(k)p_k^i(x), i = 1,2$.

Since $Z_i(t) = z + R_i(t), i = 1,2$, then $Z_1(t) - Z_2(t) = R_1(t) - R_2(t)$.

Either

$$\begin{aligned}|J(s,z,p_1,u)-J(s,z,p_2,u)| &= |E\Phi(e^{Z_1(T)})-E\Phi(e^{Z_2(T)})| \\ &= |E\Phi(e^{z+R_1(T)})-E\Phi(e^{z+R_2(T)})| \\ &\leq K_1E|Z_1(T)-Z_2(T)| = K_1E|R_1(T)-R_2(T)|,\end{aligned}$$

or

$$\begin{aligned}|J(s,z,p_1,u)-J(s,z,p_2,u)| &= |E\Phi(e^{Z_1(T)})-E\Phi(e^{Z_2(T)})| \\ &= |E\Phi(e^{z+R_1(T)})-E\Phi(e^{z+R_2(T)})| \\ &\leq K_1E|(e^{z+R_1(T)})^\gamma-(e^{z+R_2(T)})^\gamma| \\ &= K_1e^{\gamma z}E|e^{\gamma R_1(T)}-e^{\gamma R_2(T)}|.\end{aligned}$$

$$\begin{aligned}E|R_1(T)-R_2(T)| &= E\int_s^T u(x)\,[\hat{\alpha}_1(x)-\hat{\alpha}_2(x)]\,dx \\ &= E\int_s^T u(x)\sum_{k=1}^m \mu_k\,[p_k^1(x)-p_k^2(x)]\,dx \\ &\leq \|p_1-p_2\|.\end{aligned}$$

□

Now we are ready to prove the following theorem:

Theorem 4.0.2. *The value function $v(s,z,p)$ is the viscosity solution of HJB equation (3.7).*

Proof. To prove that $v(s,z,p)$ is a supersolution of the HJB equation, we recall the notation $Y(t)=(Z(t)\vdots p(t))'$. Therefore, we can write $Y_s=(z\vdots p)'$. Here A' stands for the transpose of a matrix A.

Suppose $v-\phi$ has a local minimum at (s,Y_s) in the neighborhood $N(s,Y_s)$. And let $(\theta,Y_\theta)\in N(s,Y_s)$,

$$v(\theta,Y_\theta)-\phi(\theta,Y_\theta)\geq v(s,Y_s)-\phi(s,Y_s).$$

We have

$$v(\theta,Y_\theta)\geq \phi(\theta,Y_\theta)+v(s,Y_s)-\phi(s,Y_s).$$

Define

$$\psi(t, Y_t) = \phi(t, Y_t) + v(s, Y_s) - \phi(s, Y_s).$$

It follows that

$$v(\theta, Y_\theta) \geq \psi(\theta, Y_\theta).$$

By Dynkin's formula, we have

$$\begin{aligned} E^{s,Y_s}\psi(\theta, Y_\theta) - \psi(s, Y_s) &= E^{s,Y_s}\int_s^\theta (\frac{\partial \psi}{\partial t} + f\frac{\partial \psi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \psi}{\partial Y^2}\})dt \\ &= E^{s,Y_s}\int_s^\theta (\frac{\partial \phi}{\partial t} + f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\})dt. \end{aligned}$$

Notice that $\psi(s, Y_s) = v(s, Y_s)$, and recall $v(\theta, Y_\theta) \geq \psi(\theta, Y_\theta)$. It follows that

$$\begin{aligned} E^{s,Y_s}v(\theta, Y_\theta) - v(s, Y_s) &\geq E^{s,Y_s}\psi(\theta, Y_\theta) - \psi(s, Y_s) \\ &= E^{s,Y_s}\int_s^\theta (\frac{\partial \phi}{\partial t} + f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\})dt. \end{aligned}$$

By optimality principle, $E^{s,Y_s}v(\theta, Y_\theta) \leq v(s, Y_s)$, or $E^{s,Y_s}v(\theta, Y_\theta) - v(s, Y_s) \leq 0$, we have hence

$$E^{s,Y_s}\int_s^\theta (\frac{\partial \phi}{\partial t} + f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\})dt \leq 0.$$

Take $u(t) = u$, a constant function on $[s, \theta]$. The above inequality hold for all value of u. Hence,

$$E^{s,Y_s}\int_s^\theta (\frac{\partial \phi}{\partial t} + \sup_u\{f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\}\})dt \leq 0.$$

Multiplying $\dfrac{1}{\theta - s}$ on both sides, and letting $\theta \to s$, we have

$$\frac{\partial \phi}{\partial t} + \sup_u\left\{f\frac{\partial \phi}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 \phi}{\partial Y^2}\}\right\} \leq 0.$$

This proves that v is a supersolution for the above PDE.

To justify that v is also a subsolution for the equation

$$\frac{\partial v}{\partial s} + \sup_u\left\{f\frac{\partial v}{\partial Y} + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 v}{\partial Y^2}\}\right\} = 0$$

we use dynamic programming principle. See page 176 in [22].

Suppose $v - \phi$ has a local maximum at (s, Y_s) in the neighborhood $N(s, Y_s)$. And let $(\theta, Y_\theta) \in N(s, Y_s)$,

$$v(\theta, Y_\theta) - \phi(\theta, Y_\theta) \leq v(s, Y_s) - \phi(s, Y_s).$$

We must have

$$v(\theta, Y_\theta) \leq \phi(\theta, Y_\theta) + v(s, Y_s) - \phi(s, Y_s).$$

Define a function

$$\psi(t, Y_t) = \phi(t, Y_t) + v(s, Y_s) - \phi(s, Y_s).$$

It follows that

$$v(\theta, Y_\theta) \leq \psi(\theta, Y_\theta).$$

By Dynkin's formula,

$$\begin{aligned} E^{s,Y_s}\psi(\theta, Y_\theta) - \psi(s, Y_s) &= E^{s,Y_s} \int_s^\theta \left(\frac{\partial \psi}{\partial t} + f \frac{\partial \psi}{\partial Y} + \frac{1}{2} \text{tr}\{(\Sigma\Sigma') \frac{\partial^2 \psi}{\partial Y^2}\} \right) dt \\ &= E^{s,Y_s} \int_s^\theta \left(\frac{\partial \phi}{\partial t} + f \frac{\partial \phi}{\partial Y} + \frac{1}{2} \text{tr}\{(\Sigma\Sigma') \frac{\partial^2 \phi}{\partial Y^2}\} \right) dt. \end{aligned}$$

Notice $\psi(s, Y_s) = v(s, Y_s)$, and recall $v(\theta, Y_\theta) \leq \psi(\theta, Y_\theta)$. Then

$$\begin{aligned} E^{s,Y_s} v(\theta, Y_\theta) - v(s, Y_s) &\leq E^{s,Y_s}\psi(\theta, Y_\theta) - \psi(s, Y_s) \\ &= E^{s,Y_s} \int_s^\theta \left(\frac{\partial \phi}{\partial t} + f \frac{\partial \phi}{\partial Y} + \frac{1}{2} \text{tr}\{(\Sigma\Sigma') \frac{\partial^2 \phi}{\partial Y^2}\} \right) dt. \end{aligned} \quad (4.2)$$

By dynamic programming principle, for every $\delta > 0$ there exists admissible control $u^\delta(\cdot)$ such that

$$v(s, Y_s) - \delta \leq E^{sY_s}\{v(\theta, Y_\theta)\},$$

where $Y_\theta = Y_\theta^{u^\delta(\cdot)}$ is under $u^\delta(\cdot)$. Combined with the inequality (4.2) we see that

$$-\delta \leq E^{s,Y_s} v(\theta, Y_\theta) - v(s, Y_s) \leq E^{s,Y_s} \int_s^\theta \left(\frac{\partial \phi}{\partial t} + f \frac{\partial \phi}{\partial Y} + \frac{1}{2} \text{tr}\{(\Sigma\Sigma') \frac{\partial^2 \phi}{\partial Y^2}\} \right) dt.$$

Let $\delta \to 0$. Then,

$$E^{s,Y_s} \int_s^\theta \left(\frac{\partial \phi}{\partial t} + f \frac{\partial \phi}{\partial Y} + \frac{1}{2} \text{tr}\{(\Sigma\Sigma') \frac{\partial^2 \phi}{\partial Y^2}\} \right) dt \geq 0.$$

This is true for all constant u. Consequently,

$$E^{s,Y_s}\int_s^\theta \left(\frac{\partial\phi}{\partial t}+\sup_u\left\{f\frac{\partial\phi}{\partial Y}+\frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2\phi}{\partial Y^2}\}\right\}\right)dt \geq 0.$$

Multiplying $\dfrac{1}{\theta - s}$ on both sides, and letting $\theta \to s$, we have

$$\frac{\partial\phi}{\partial t}+\sup_u\left\{f\frac{\partial\phi}{\partial Y}+\frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2\phi}{\partial Y^2}\}\right\} \geq 0.$$

This proves that v is a subsolution for the above PDE. □

Through this theorem we have obtained the existence for the solution of the HJB equation. The following concepts is needed in the proof of uniqueness of viscosity solution v:

Definition Let $f(s,z):[0,T]\times R^{m+1}$. Define the parabolic superjet by

$$\mathcal{P}^{2,+}f(s,z)=\{(a,q,X)\in \mathbf{R}\times\mathbf{R}^{m+1}\times\mathcal{S}(m+1): f(t,x)\leq f(s,z)+a(t-s)+q(x-z)$$
$$+\tfrac{1}{2}(x-z)X(x-z)+o(|t-s|+|x-z|^2) \text{ as } (t,x)\to(s,z)\},$$

and its closure is

$$\bar{\mathcal{P}}^{2,+}f(s,z)=\{(a,q,X)=\lim_{n\to\infty}(a_n,q_n,X_n)$$
$$\text{with } (a_n,q_n,X_n)\in\mathcal{P}^{2,+}f(s_n,z_n)$$
$$\text{and } \lim_{n\to\infty}(s_n,z_n)=(s,z)\}.$$

Similarly, we define the parabolic subjet $\mathcal{P}^{2,-}f(s,z) = -\mathcal{P}^{2,+}(-f)(s,z)$ and its closure $\bar{\mathcal{P}}^{2,-}f(s,z)=-\bar{\mathcal{P}}^{2,+}(-f)(s,z)$.

We have the following result. Its proof can be found in Fleming and Soner [22].

Lemma 4.0.3. *The $\mathcal{P}^{2,+}f(s,z)$ (resp. $\mathcal{P}^{2,-}f(s,z)$) consist of the set of*

$$\left(\frac{\partial\phi(s,z)}{\partial s},\frac{\partial\phi(s,z)}{\partial z},\frac{\partial^2\phi(s,z)}{\partial z^2}\right),$$

where $\phi\in\mathcal{C}^{1,2}([0,T]\times R^{m+1})$ and $f-\phi$ has a global maximum (resp. minimum) at (s,z).

We will use the following equivalent definition of viscosity solution.

Definition A continuous function $v(s,Y)$ with at most polynomial growth is a viscosity solution of

$$\frac{\partial v}{\partial s} + H(s,Y,\frac{\partial v}{\partial Y},\frac{\partial^2 v}{\partial Y^2}) = 0,$$

if

1. for all $(s,Y) \in [0,T] \times R^{m+1}$, and $(a,q,X) \in \mathcal{P}^{2,+}v(s,Y)$

$$a + \sup_u H(s,Y,q,X) \leq 0 \text{ in this case } v \text{ is a viscosity subsolution;}$$

and

2.for all $(s,Y) \in [0,T] \times R^{m+1}$, and $(a,q,X) \in \mathcal{P}^{2,+}v(s,Y)$

$$a + \sup_u H(s,Y,q,X) \geq 0 \text{ in this case } v \text{ is a viscosity supersolution.}$$

To prove the uniqueness of the viscosity solution for this PDE , we need to prove the comparison principle (Theorem 4.0.5). Before we do that we will state an important result needed in our proof.

Theorem 4.0.4. (Crandall, Lions and Ishii [10]) *For $i = 1,2$, let Ω_i be locally compact subsets of R^{m+1}, and $\Omega = \Omega_1 \times \Omega_2$, let v_i be upper semi-continuous in $[0,T] \times \Omega_i$, and $\bar{P}^{2,+}_{\Omega_i} v_i(t,Y)$ the parabolic superjet of $v_i(t,Y)$, and ϕ be twice continuous differentiable in a neighborhood of $[0,T] \times \Omega$. Set*

$$\omega(t,Y_1,Y_2) = v_1(t,Y_1) + v_2(t,Y_2)$$

for $(t,Y_1,Y_2) \in [0,T] \times \Omega$, and suppose $(\hat{t},\hat{Y_1},\hat{Y_2}) \in [0,T] \times \Omega$ is a local maximum of $\omega - \phi$ relative to $[0,T] \times \Omega$. Moreover let us assume that, there is an $r > 0$ such that for every $M > 0$ there exists a C such that for $i = 1,2$

$$a_i \leq C \text{ whenever } (a_i,q_i,X_i) \in \bar{P}^{2,+}_{\Omega_i} v_i(t,Y),$$
$$|Y_i - \hat{Y_i}| + |t - \hat{t}| \leq r \text{ and } |v_i(t,Y_i)| + |q_i| + \|X_i\| \leq M. \tag{4.3}$$

Then for each $\epsilon > 0$ there exists $X_i \in S(m+1)$ such that

1.

$$(a_i, D_{Y_i}\phi(\hat{t}, \hat{Y}), X_i) \in \bar{P}^{2,+}_{\Omega_i} v_i(\hat{t}, \hat{Y}) \text{ for } i = 1, 2,$$

2.

$$-\left(\frac{1}{\epsilon} + \|D^2\phi(\hat{Y})\|\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2(\phi(\hat{Y})) + \epsilon(D^2\phi(\hat{Y}))^2, \tag{4.4}$$

3.

$$a_1 + a_2 = \frac{\partial\phi(\hat{t}, \hat{Y_1}, \hat{Y_2})}{\partial t}. \tag{4.5}$$

Now $v(s, Y) = v(s, z, p) = \sup_{u(\cdot)\in\mathcal{A}} J(s, z, p, u(\cdot)) = \sup_{u(\cdot)\in\mathcal{A}} E_{sz}(\Phi(\exp(Z(T))))$. Since we assume Φ has at most linear growth, then v has at most linear growth.

We can apply the idea in Flemming and Soner [22] and prove the following comparison principle.

Theorem 4.0.5. *If $v_1(s, Y)$ and $v_2(s, Y)$ are continuous in (s, Y) and are respectively viscosity subsolution and supersolution of (3.7) with at most linear growth, namely, there exists K*

$$v_i(s, Y) \leq K(1 + |Y|) \text{ for all } (s, Y) \in [0, T] \times R^{m+1}, i = 1, 2,$$

then

$$v_1(s, Y) \leq v_2(s, Y) \text{ for all } (s, Y) \in [0, T] \times R^{m+1}. \tag{4.6}$$

Proof. First we observe that for $\rho > 0, \tilde{v_1} = v_1 - \rho/(T - t)$ is also a subsolution of (3.7) and satisfies

$$\begin{cases} (i) \dfrac{\partial\tilde{v_1}}{\partial t} + H(s, Y, \dfrac{\partial\tilde{v_1}}{\partial Y}, \dfrac{\partial^2\tilde{v_1}}{\partial Y^2}) \leq -\dfrac{\rho}{(T-t)^2}, \\ (ii) \lim\limits_{t\uparrow T} \tilde{v_1}(t, Y) = -\infty. \end{cases} \tag{4.7}$$

If we can prove that $\tilde{v_1} \leq v_2$ for any $\rho > 0$ then $v_1 \leq v_2$ will follows by letting $\rho \to 0$.

Hence it suffice to prove the comparison under the additional assumption

$$\frac{\partial v_1}{\partial t} + H(s, Y, \frac{\partial v_1}{\partial Y}, \frac{\partial^2 v_1}{\partial Y^2}) \leq -\frac{\rho}{T^2}.$$

We want to prove $v_1 \leq v_2$. Suppose this is not true. Then there exists

$$(t, Z) \in (0, T) \times \Omega \text{ and } v_1(t, Z) - v_2(t, Z) = \delta > 0. \tag{4.8}$$

We next prove that this leads to a contradiction.

For any $0 < \alpha < 1$ and $0 < \gamma < 1$, we define

$$\begin{aligned} \Theta_{\alpha\gamma}(s, Y_1, Y_2) &= \frac{1}{\alpha}|Y_1 - Y_2|^2 + \gamma \mathrm{e}^{T-s}(Y_1^2 + Y_2^2), \\ \Phi(s, Y_1, Y_2) &= v_1(s, Y_1) - v_2(s, Y_2) - \Theta_{\alpha\gamma}(s, Y_1, Y_2). \end{aligned}$$

Since $v_1(s, Y_1)$ and $v_2(s, Y_2)$ satisfy the linear growth condition, we have

$$\lim_{|Y_1|+|Y_2| \to \infty} \Phi(s, Y_1, Y_2) = -\infty.$$

Since $\Phi(s, Y_1, Y_2)$ is continuous in (s, Y_1, Y_2) it has a global maximum at a point (s^0, Y_1^0, Y_2^0). Notice that

$$\Phi(s^0, Y_1^0, Y_1^0) + \Phi(s^0, Y_2^0, Y_2^0) \leq 2\Phi(s^0, Y_1^0, Y_2^0).$$

By the definition of Φ we have

$$\begin{aligned} v_1(s^0, Y_1^0) - v_2(s^0, Y_1^0) - 2\gamma \mathrm{e}^{(T-s^0)}(|Y_1^0|^2) + v_1(s^0, Y_2^0) \\ -v_2(s^0, Y_2^0) - 2\gamma \mathrm{e}^{(T-s^0)}(|Y_2^0|^2) \leq 2v_1(s^0, Y_1^0) - 2v_2(s^0, Y_2^0) \\ -\frac{2}{\alpha}|Y_1^0 - Y_2^0|^2 - 2\gamma \mathrm{e}^{T-s^0}(|Y_1^0|^2 + |Y_2^0|^2). \end{aligned}$$

Then

$$\begin{aligned} -v_2(s^0, Y_1^0) - 2\gamma \mathrm{e}^{(T-s^0)}(|Y_1^0|^2) + v_1(s^0, Y_2^0) - 2\gamma \mathrm{e}^{(T-s^0)}(|Y_2^0|^2) \\ \leq v_1(s^0, Y_1^0) - v_2(s^0, Y_2^0) - \frac{2}{\alpha}|Y_1^0 - Y_2^0|^2 \\ -2\gamma \mathrm{e}^{T-s^0}(|Y_1^0|^2 + |Y_2^0|^2). \end{aligned}$$

Consequently, we have

$$\frac{2}{\alpha}|Y_1^0 - Y_2^0|^2 \leq v_1(s^0, Y_1^0) - v_1(s^0, Y_2^0) + v_2(s^0, Y_1^0) - v_2(s^0, Y_2^0). \tag{4.9}$$

By the linear growth condition, there exists K

$$v_i(s, Y) \leq K(1 + |Y|) \text{ for all } (s, Y) \in [0, T] \times R^{m+1}, i = 1, 2.$$

So

$$|Y_1^0 - Y_2^0|^2 \leq \alpha K(1 + |Y_1^0| + |Y_2^0|). \tag{4.10}$$

In addition, the choice of (s^0, Y_1^0, Y_2^0) implies $\Phi(s^0, 0, 0) \leq \Phi(s^0, Y_1^0, Y_2^0)$. Together with $\Phi(s^0, 0, 0) \leq K(1 + |Y_1^0| + |Y_2^0|)$ we then have

$$\begin{aligned} \gamma \mathrm{e}^{(T-s^0)}(|Y_1^0|^2 + |Y_2^0|^2) &\leq v_1(s^0, Y_1^0) - v_2(s^0, Y_2^0) - \frac{1}{\alpha}|Y_1^0 - Y_2^0|^2 - \Phi(s^0, 0, 0) \\ &\leq 3K(1 + |Y_1^0| + |Y_2^0|). \end{aligned}$$

It follows that

$$\frac{\gamma \mathrm{e}^{(T-s^0)}(|Y_1^0|^2 + |Y_2^0|^2)}{(1 + |Y_1^0| + |Y_2^0|)} \leq 3K.$$

Consequently, there exists C_γ such that

$$|Y_1^0| + |Y_2^0| \leq C_\gamma. \tag{4.11}$$

This inequality implies that the sets $\{Y_1^0\}$, and $\{Y_2^0\}$ are bounded by C_γ independent of α. We can extract convergent subsequences also denote (s^0, Y_1^0, Y_2^0). Moreover, from the inequality (4.10) we conclude that there exists t_0, Y_0 such that

$$\lim_{\alpha \to 0} Y_1^0 = Y_0 = \lim_{\alpha \to 0} Y_2^0 \text{ and } \lim_{\alpha \to 0} s^0 = t_0.$$

Also, use the above limit and equation (4.9) we have

$$\lim_{\alpha \to 0} \frac{2}{\alpha}|Y_1^0 - Y_2^0|^2 = 0. \tag{4.12}$$

Since Φ achieves its maximum at (s^0, Y_1^0, Y_2^0), Upon applying Theorem 4.0.4 we know that there exists numbers a_1, a_2 and $A_1, A_2 \in S(m+1)$ such that

$$(a_1, \frac{2}{\alpha}(Y_1^0 - Y_2^0) + 2\gamma e^{(T-s^0)} Y_1^0, A_1) \in \bar{P}^{2,+} v_1(s^0, Y_1^0), \text{ and}$$
$$(-a_2, -\frac{2}{\alpha}(Y_1^0 - Y_2^0) + 2\gamma e^{(T-s^0)} Y_2^0, -A_2) \in \bar{P}^{2,+} - v_2((s^0, Y_2^0).$$

By $\bar{P}^{2,+} - v_2((s^0, Y_2^0) = -\bar{P}^{2,-} v_2((s^0, Y_2^0)$, we can obtain

$$(a_2, \frac{2}{\alpha}(Y_1^0 - Y_2^0) - 2\gamma e^{(T-s^0)} Y_2^0, A_2) \in \bar{P}^{2,-} v_2((s^0, Y_2^0).$$

By the equivalent definition of viscosity solution we have

$$a_1 + H(s^0, Y_1^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) + 2\gamma e^{(T-s^0)} Y_1^0, A_1) \le -c,$$
$$a_2 + H(s^0, Y_2^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) - 2\gamma e^{(T-s^0)} Y_2^0, A_2) \ge 0,$$

where $c = \rho/T^2 > 0$. Combine the above inequality, then we can get

$$\begin{aligned} c \;\le a_2 - a_1 \quad &+ H(s^0, Y_2^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) - 2\gamma e^{(T-s^0)} Y_2^0, A_2) \\ &- H(s^0, Y_1^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) + 2\gamma e^{(T-s^0)} Y_1^0, A_1). \end{aligned}$$

In view of theorem 4.0.4, we have

$$a_1 - a_2 = \frac{\partial \Theta_{\alpha\gamma}(s, Y_1, Y_2)}{\partial s} = \gamma e^{(T-s^0)}((Y_1^0)^2 + (Y_2^0)^2).$$

So, $a_1 - a_2 \to 0$ when $\gamma \to 0$.

Now denote

$$L_2 = H(s^0, Y_2^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) - 2\gamma e^{(T-s^0)} Y_2^0, A_2)$$

and

$$L_1 = H(s^0, Y_1^0, \frac{2}{\alpha}(Y_1^0 - Y_2^0) + 2\gamma e^{(T-s^0)} Y_1^0, A_1)$$

then we obtain $c \le a_1 - a_2 + (L_2 - L_1)$. we need to approximate $L_2 - L_1$ to discover our contradiction.

To simplify our notation we denote

$$f_1 = f(s^0, Y_1^0, u), \;\; f_2 = f(s^0, Y_2^0, u)$$

$$C_1 = \Sigma(s^0, Y_1^0, u), \;\; C_2 = \Sigma(s^0, Y_2^0, u)$$

and

$$p_\alpha = \frac{2}{\alpha}(Y_1^0 - Y_2^0), \;\; q_{1\gamma} = 2\gamma e^{(T-s^0)} Y_1^0, \;\; q_{2\gamma} = 2\gamma e^{(T-s^0)} Y_2^0.$$

Then

$$\begin{aligned} L_2 - L_1 &= \sup_u \left(f_2(p_\alpha - q_{2\gamma}) + \frac{1}{2}\mathrm{tr}\{(C_2C_2')A_2\} \right) - \sup_u \left(f_1(p_\alpha - q_{1\gamma}) + \frac{1}{2}\mathrm{tr}\{(C_1C_1')A_1\} \right) \\ &\leq \sup_u f_2(p_\alpha - q_{2\gamma}) - f_1(p_\alpha - q_{1\gamma}) + \frac{1}{2}\mathrm{tr}\{(C_2C_2')A_2 - (C_1C_1')A_1\} \\ &\leq \sup_u (f_2 - f_1)p_\alpha - f_2 q_{2\gamma} + f_1 q_{1\gamma}) + \frac{1}{2}\mathrm{tr}\{(C_2C_2')A_2 - (C_1C_1')A_1\}. \end{aligned}$$

Notice that f_1, f_2 are bounded, $p_\alpha \to 0$ as $\alpha \to 0$. $q_{1\gamma}, q_{2\gamma} \to 0$ as $\gamma \to 0$.

$$\mathrm{tr}\{(C_2C_2')A_2 - (C_1C_1')A_1\} = \mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1' \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ 0 & -A_1 \end{bmatrix} \right).$$

In view of Crandall-Ishii's maximum principle (4.0.4), we have

$$\begin{bmatrix} A_2 & 0 \\ 0 & -A_1 \end{bmatrix} \leq D^2\Theta_{\alpha\gamma}(s^0, Y_1^0, Y_2^0) + \epsilon(D^2\Theta_{\alpha\gamma}(s^0, Y_1^0, Y_2^0))^2.$$

Now

$$D^2\Theta_{\alpha\gamma}(s^0, Y_1^0, Y_2^0) = \frac{2}{\alpha}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + 2\gamma e^{T-s^0} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

and

$$\begin{aligned}(D^2\Theta_{\alpha\gamma}(s^0,Y_1^0,Y_2^0))^2 &= \frac{8}{\alpha^2}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix} + \frac{8\gamma e^{T-s^0}}{\alpha}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix} + 4\gamma^2 e^2(t-s^0)\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} \\ &= \frac{8+8\gamma\alpha e^{T-s^0}}{\alpha^2}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix} + 4\gamma^2 e^2(t-s^0)\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix}.\end{aligned}$$

So

$$\begin{aligned}\mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & -A_1\end{bmatrix}\right) &\le \frac{2}{\alpha}\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix} \\ &+(2\gamma e^{T-s^0}+4\epsilon\gamma^2e^2(t-s^0))\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} \\ &+\epsilon\frac{8+8\gamma\alpha e^{T-s^0}}{\alpha^2}\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix}.\end{aligned}$$

Letting $\gamma \to 0$, we have

$$\begin{aligned}&\mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & -A_1\end{bmatrix}\right) \\ \le\ &\frac{2}{\alpha}\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix} + \epsilon\frac{8}{\alpha^2}\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix}.\end{aligned}$$

Taking $\epsilon = \frac{\alpha}{4}$, we have

$$\mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & -A_1\end{bmatrix}\right) \le \frac{4}{\alpha}\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1'\end{bmatrix}\begin{bmatrix} I & -I \\ -I & I\end{bmatrix},$$

$$\mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1' \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & -A_1 \end{bmatrix}\right) \le \frac{4}{\alpha}\mathrm{tr}(C_2C_2' - C_2C_1' - C_1C_2' + C_1C_1');$$

so

$$\mathrm{tr}\left(\begin{bmatrix} C_2C_2' & C_2C_1' \\ C_1c_2' & C_1C_1' \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & -A_1 \end{bmatrix}\right) \le \frac{4}{\alpha}\mathrm{tr}([C_2 - C_1][C_2' - C_1']) = \frac{4}{\alpha}\|C_2 - C_1\|^2$$

$$= \frac{4}{\alpha}\|\Sigma(s^0, Y_2^0, u) - \Sigma(s^0, Y_1^0, u)\|^2 \le C\frac{4}{\alpha}|Y_2^0 - Y_1^0|^2.$$

In the above inequality we assume Lipschitz continuity on $\Sigma(s, Y, u)$. By equation (4.12) we see that $\mathrm{tr}\{(C_2C_2')A_2 - (C_1C_1')A_1\} \to 0$ when $\gamma \to 0, \alpha \to 0$. This finishes the proof that $L_2 - L_1 \to 0$ as $\gamma \to 0, \alpha \to 0$. Now

$$c \le a_2 - a_1 + L_2 - L_1 \to 0 \text{ as } \gamma \to 0 \text{ and } \alpha \to 0.$$

This contradicts the fact that $c > 0$. We conclude that the assumption of $v_1(t, Z) > v_2(t, Z)$ for some time $t > 0$ is false.

This proves the uniqueness of the value function. □

Chapter 5

Markov Chain Approximation

In the previous chapter, we have proved that the value function v is the unique viscosity solution of the PDE

$$\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) = 0 \tag{5.1}$$

with the boundary condition $v(T, z, p) = \Phi(\mathrm{e}^z)$, where $z = \log y$, y is the initial wealth, and p is the initial probability vector.

A common choice to compute the value function is to use finite difference approximation. It is proved in [7] that finite difference approximation typically leads to Markov chain approximation. But finite difference method is not applicable in this problem. The obstacle is due to the fact that the matrix $a(u, p) = \Sigma\Sigma'$ is not diagonally dominant. To apply finite difference method, one needs

$$a_{ii}(u, p) - \sum_{j:j \neq i} |a_{ij}(u, p)| \geq 0.$$

We can see that this is not satisfied in our setup. For example, if $p(t)$ is 2-dimensional which means the markov chain $\alpha(t)$ has two states. Since

$$A(t)P(t)' = \begin{pmatrix} \mu_1 - \hat{\alpha} & 0 \\ 0 & \mu_2 - \hat{\alpha} \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} (\mu_1 - \hat{\alpha})p_1 \\ (\mu_2 - \hat{\alpha})p_2 \end{pmatrix},$$

it follows that

$$\Sigma(t, Y, u) = \begin{pmatrix} u\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix} = \begin{pmatrix} u\sigma \\ \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \\ \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2 \end{pmatrix}.$$

So

$$\Sigma\Sigma' = \begin{pmatrix} u\sigma \\ \\ \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \\ \\ \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2 \end{pmatrix} (u\sigma \; \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \; \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2),$$

$$\Sigma\Sigma' = \begin{pmatrix} u^2\sigma^2 & u(\mu_1 - \hat{\alpha})p_1 & u(\mu_2 - \hat{\alpha})p_2 \\ \\ u(\mu_1 - \hat{\alpha})p_1 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})^2 p_1^2 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2 \\ \\ u(\mu_2 - \hat{\alpha})p_2 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2 & \frac{1}{\sigma^2}(\mu_2 - \hat{\alpha})^2 p_2^2 \end{pmatrix}.$$

In order to have

$$a_{ii}(u,p) - \sum_{j:j\neq i} |a_{ij}(u,p)| \geq 0,$$

we must have

$$u\sigma^2 \geq |\mu_1 - \hat{\alpha}|p_1 + |\mu_2 - \hat{\alpha}|p_2$$

$$\frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})^2 p_1^2 \geq u|\mu_1 - \hat{\alpha}|p_1 + \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2$$

$$\frac{1}{\sigma^2}(\mu_2 - \hat{\alpha})^2 p_2^2 \geq u|\mu_2 - \hat{\alpha}|p_2 + \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2.$$

Suppose $\mu_1 > \mu_2$, hence $\mu_1 - \hat{\alpha} > 0, \mu_2 - \hat{\alpha} < 0$. We then hope to have

$$u\sigma^2 \geq (\mu_1 - \hat{\alpha})p_1 + (\hat{\alpha} - \mu_2)p_2,$$

$$(\mu_1 - \hat{\alpha})p_1 \geq u\sigma^2 + (\mu_2 - \hat{\alpha})p_2,$$

$$(\mu_2 - \hat{\alpha})p_2 \leq u\sigma^2 + (\mu_1 - \hat{\alpha})p_1.$$

The third condition holds but in order to satisfy the first two condition, we must have

$$u\sigma^2 = (\mu_1 - \hat{\alpha})p_1 + (\hat{\alpha} - \mu_2)p_2.$$

However, u is the control function that should be varying between 0 and 1.

We will apply Kushner's Markov chain approximation method to numerically solve this PDE. See [6].

The idea of the Markov chain approximation method is to discretize the control problem so as to find numerical solutions. In particular, the continuous-time state variables of the control problem are approximated by a discrete-time Markov chain so that the value function corresponding to the discrete-time Markov chain converges to the value function of the continuous-time control problem.

The key requirement in finding the proper Markov chain approximation is to satisfy the " local consistency conditions," which basically means that the approximating chain should have local properties that are consistent with that of the original chain. Recall that we denote $Y(t) = (Z(t) \vdots p(t))'$, and $Y(t)$ evolves according to the stochastic process

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t).$$

So the approximating chain $Y^h(t)$ should satisfy the following "local consistency conditions":

$$E^{h,u}_{z,p,n} \Delta Y^h_n = f(t, Y(t), u(t))\Delta t^h(Y, u) + o(\Delta t^h(Y, u))$$

$$covar^{h,u}_{z,p,n} \Delta Y^h_n = \Sigma(t, Y(t), u(t))\Sigma(t, Y(t), u(t))'\Delta t^h(Y, u) + o(\Delta t^h(Y, u))$$

If we can find approximating chain $Y^h(t)$ whose transition probability $P^h(Y, Z|u)$ and time step functions $\Delta t^h(Y, u)$ satisfy the "local consistency conditions" then we can use it to compute the value function $v^h(s, Y^h)$ for the approximating chain. For a detailed discussion of this method, see [6].

The value function is

$$v(s, Y_0) = \sup_u E[\Phi(\exp(Z(T)))|Y(s) = Y_0],$$

where $Y_0 = (z, p)$ is the initial condition. By the principle of dynamic programming,

$$v(s - \Delta t^h, Y_0) = \sup_u E[v(s, Y(s))|Y(s - \Delta t^h) = Y_0].$$

The approximation function v^h should have the same property

$$v^h(s - \Delta t^h, Y_0) = \sup_u E[v^h(s, Y(s))|Y(s - \Delta t^h) = Y_0].$$

The degenerate structure of the noise covariance matrix suggests that the part of the transitions of any approximating Markov chain which approximates the effects of the "noise" would move the chain in the directions $\pm\Sigma(s, Y, u)$. Let the state space S_h be such that

$$Y \pm h\Sigma(s, Y, u) \in S_h, \text{ for } Y \in S_h,$$

and

$$Y \pm e_i h \in S_h, \text{ for } Y \in S_h,$$

We use the following steps to choose a set of transition probability $P^h(Y, Z|u)$ and time step functions $\Delta t^h(Y, u)$ to satisfy the "local consistency conditions." First we consider the stochastic process

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

as having two different components, represented respectively by

$$dY(t) = \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

and

$$dY(t) = f(t, Y(t), u(t))dt.$$

We choose two different sets of transition probability and time step functions, so these two SDE's individual "local consistency conditions" can be satisfied. Then we combine them to obtain a choice that can satisfy the "local consistency conditions" for the original state $Y(t)$.

(1) One set of transition probabilities for a locally consistent chain for the component represented by

$$dY(t) = \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

is $P_1^h(Y, Y \pm h\Sigma(s, Y, u)|u) = 1/2$. With these transition probabilities, the covariance of the state transition can be written as

$$\sum_Z (Z - Y)(Z - Y)' P_1^h(Y, Z|u) = \Sigma\Sigma' h^2$$

Then, if we define the interpolation interval $\Delta t_1^h(Y, u) = h^2$, $P_1^h(Y, Y \pm h\Sigma(s, Y, u)|u)$ is locally consistent.

(2) One possibility for the transition probability for the approximation to

$$dY(t) = f(t, Y(t), u(t))dt$$

is

$$P_2^h(Y, Y \pm e_i h|u) = f_i^{\pm}(t, Y, u) \times \textit{normalization},$$

where the normalization is

$$\frac{1}{Q_2^h(Y, u)} = \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y, u)},$$

$f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$. Define

$$\Delta t_2^h(Y, u) = \frac{h}{\sum_{i=1}^{m+1} f_i(t, Y, u)}.$$

The local consistency can be shown by the calculations

$$\sum_Z (Z - Y) P_2^h(Y, Z|u) = f(t, Y, u) \times \Delta t_2^h(Y, u),$$

where $Z \in \{Y \pm e_i h, i = 1, \cdots, m\}$, and

$$\sum_Z (Z - Y)(Z - Y)' P_2^h(Y, Z|u) = o(\Delta t_2^h(Y, u)).$$

(3) Combine the above "partial" transition probabilities from the diffusion and drift component to get

$$P^h(Y, Y \pm h\Sigma(s, Y, u)|u) = \frac{1}{2Q^h(Y, u)},$$

$$P^h(Y, Y \pm e_i h|u) = f_i^{\pm}(t, Y, u) \frac{h}{Q^h(Y, u)}$$

where

$$Q^h(Y,u) = 1 + h\sum_{i=1}^{m+1} |f_i(Y,u)|,$$

and

$$\Delta t^h(Y,u) = \frac{h^2}{Q^h(Y,u)}.$$

To show that the local consistency is satisfied, we see that

$$\sum_{Z\in S_h} (Z-Y)P^h(Y,Z|u) = f(t,Y,u)\frac{h^2}{Q^h(Y,u)},$$

where $Z \in \{Y \pm e_i h, i = 1, Y \pm h\Sigma(s,Y,u), \cdots, m\}$, and

$$\sum_{Z\in S_h} (Z-Y)(Z-Y)'P^h(Y,Z|u) = \Sigma\Sigma'\frac{h^2}{Q^h(Y,u)} + o(\Delta t_2^h(Y,u)).$$

The numerical scheme for the value function is

$$v^h(s-\Delta t^h(Y,u),Y) = \sup_u [\sum_{Z\in S_h} P^h(Y,Z|u)v^h(s,Z)] \tag{5.2}$$

with

$$v^h(T,z,p) = \Phi(e^z), (z{:}p) \in S_h. \tag{5.3}$$

For calculation purposes, it will be better if we can find a constant interpolation intervals Δt^h. This can be done by defining

$$\overline{Q}^h = \sup_{u,p} Q^h(Y,u).$$

Then the following are locally consistent:

$$\Delta t^h = h^2/\overline{Q}^h$$

$$P^h(Y, Y \pm h\Sigma(s,Y,u)|u) = 1/2\overline{Q}^h,$$

$$P^h(Y, Y \pm e_i h|u) = f_i^{\pm}(t,Y,u)h/\overline{Q}^h,$$

$$P^h(Y,Y|u) = (\overline{Q}^h - Q^h(Y,u))/\overline{Q}^h.$$

Let

$$\mathcal{F}_h(\phi)(Y) = \sup_u [\sum_{Z \in S_h} P^h(Y, Z|u)\phi(Z)].$$

Then the scheme for computing the value function approximation can be rewritten as

$$\begin{aligned} v^h(s,Y) &= \mathcal{F}_h(v^h(s + \Delta t^h(Y,u), \cdot))(Y), Y \in S_h, \\ v^h(T,z,p) &= \Phi(e^z), (z \vdots p) \in S_h. \end{aligned}$$

In order to use the Barles-Souganidis method [4] to prove the desired convergence, we need to check the following condition:

$$\mathcal{F}_h(\phi_1) \leq \mathcal{F}_h(\phi_2) \text{ if } \phi_1 \leq \phi_2 (\text{monotonicity}).$$

For $0 < h < 1$, there exists a solution v^h to the computation scheme and a constant K such that $\|v^h\| \leq K$(stability).

For every "test function" $w \in C^{1,2}(\mathbf{R}^{m+1})$,

$$\lim_{(t,q) \underset{h \downarrow 0}{\to} (s,p)} h^{-1}[\mathcal{F}_h(w(t+h, \cdot))(q) - w(t,q)]$$

$$= \frac{\partial w}{\partial s} + H(s, Y, \frac{\partial w}{\partial Y}, \frac{\partial^2 w}{\partial Y^2}) \text{ (consistency)}.$$

We have the consistency because

$$\begin{aligned} & \lim_{(t,q) \underset{h \downarrow 0}{\to} (s,p)} h^{-1}[\mathcal{F}_h(w(t+h, \cdot))(q) - w(t,q)] \\ = & \lim_{(t,q) \underset{h \downarrow 0}{\to} (s,p)} \frac{\sup_u [\sum_Z P^h(q, Z|u) w(t+h, Z)] - w(t,q)}{h} \\ = & \lim_{(t,q) \underset{h \downarrow 0}{\to} (s,p)} \frac{\sup_u \left[\sum_Z P^h(q, Z|u)[w(t+h, Z) - w(t+h, q)]\right] + w(t+h, q) - w(t,q)}{h} \\ = & \frac{\partial w}{\partial s} + H(s, Y, \frac{\partial w}{\partial Y}, \frac{\partial^2 w}{\partial Y^2}). \end{aligned}$$

Since $P^h(Y, Z|u) \geq 0$, the monotonicity is immediate.

$$\begin{aligned}\|\mathcal{F}_h(\phi_1)(Y) - \mathcal{F}_h(\phi_2)(Y)\| &= \left\| \sup_u \left[\sum_{p\in S_h} P^h(Y,Z|u)[\phi_1(Z) - \phi_2(Z)] \right] \right\| \\ &\le \sup_u [\sum_{p\in\Sigma_0^h} P^w(p,q)] \, \|\phi_1 - \phi_2\| \\ &= \sup_u \|\phi_1 - \phi_2\| .\end{aligned}$$

Therefore $\mathcal{F}_h$ is a contraction mapping. The fixed point v^h of this contraction mapping is the solution of (5.2). This proves the stability.

Define

$$v^*(s,Y) = \limsup_{(t,Z)\underset{h\downarrow 0}{\to}(s,Y)} v^h(t,Z)$$

$$v_*(s,Y) = \liminf_{(t,Z)\underset{h\downarrow 0}{\to}(s,Y)} v^h(t,Z)$$

Lemma 5.0.6. *v^* is a viscosity subsolution of equation (5.1), and v_* is a viscosity supersolution.*

Proof. In order to prove that v^* is a viscosity subsolution, we suppose that ϕ is a test function such that $v^* - \phi$ has a strict local maximum at (s,Y). Then there is a sequence converging to zero denoted by h, such that $v^h - \phi$ has a local maximum at (t_h, Y_h) which converges to (s,Y) as $h \downarrow 0$.

$$v^h(t_h,Y_h) - \phi(t_h,Y_h) \ge v^h(t_h+h,Y_h) - \phi(t_h+h,Y_h),$$

$$\phi(t_h+h,Y_h) - \phi(t_h,Y_h) \ge v^h(t_h+h,Y_h) - v^h(t_h,Y_h).$$

By the monotonicity we proved above,

$$\mathcal{F}_h(\phi(t_h+h,\cdot))(Y_h) - \phi(t_h,Y_h) \ge \mathcal{F}_h(v^h(t_h+h,\cdot))(Y_h) - v^h(t_h,Y_h).$$

Since v^h is the solution of (5.2), the right side is 0. We divide by h and let $h \downarrow 0$. By the consistency, we have

$$\frac{\partial \phi}{\partial s} + H(s,Y,\frac{\partial \phi}{\partial Y},\frac{\partial^2 \phi}{\partial Y^2}) \ge 0$$

Therefore, v^* is a viscosity subsolution.

Similarly, suppose that $\phi \in C^{1,2}$ is a test function such that $v_* - \phi$ has a strict local minimum at (s, Y). Then there is a sequence converging to zero denoted by h, such that $v^h - \phi$ has a local minimum at (t_h, Y_h) which converges to (s, Y) as $h \downarrow 0$.

$$v^h(t_h, Y_h) - \phi(t_h, Y_h) \leq v^h(t_h + h, Y_h) - \phi(t_h + h, Y_h),$$

$$\phi(t_h + h, Y_h) - \phi(t_h, Y_h) \leq v^h(t_h + h, Y_h) - v^h(t_h, Y_h).$$

By the monotonicity we proved above,

$$\mathcal{F}_h(\phi(t_h + h, \cdot))(Y_h) - \phi(t_h, Y_h) \leq \mathcal{F}_h(v^h(t_h + h, \cdot))(Y_h) - v^h(t_h, Y_h).$$

Since v^h is the solution of (5.2), the right side is 0. We divide both sides by h and let $h \downarrow 0$. By the consistency, we have

$$\frac{\partial \phi}{\partial s} + H(s, Y, \frac{\partial \phi}{\partial Y}, \frac{\partial^2 \phi}{\partial Y^2}) \leq 0$$

Therefore, v_* is a viscosity supersolution. □

Theorem 5.0.7. *As $h \to 0$ the solution v^h of (5.2) converges locally uniformly to the unique continuous viscosity v of (5.1).*

Proof. By Lemma 5.0.6, v^* is a viscosity subsolution of equation (5.1). By comparison result for viscosity solutions, $v^* \leq v$. Similarly, $v_* \geq v$. Since $v_* \leq v^*$, we have proved

$$\lim_{(t,Z) \underset{h \downarrow 0}{\to} (s,Y)} v^h(t, Z) = v(s, Y).$$

□

Chapter 6

Separable Case

We have proved that v is the unique viscosity solution of the PDE

$$\frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) = 0 \tag{6.1}$$

with the boundary condition $v(T, z, p) = \Phi(\mathrm{e}^z)$, where $z = \ln y$, y is the initial wealth, and p is the initial probability vector. When the utility function is of the form $\Phi(x) = x^k$, we can simplify the numerical solution even further by variable separation.

Recall that

$$H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) = \sup_u \{ f \frac{\partial v}{\partial Y} + \frac{1}{2} \mathrm{tr}\{(\Sigma\Sigma') \frac{\partial^2 v}{\partial Y^2}\}\}.$$

Let

$$f(t, Y, u) = \begin{pmatrix} (1-u)r + u\hat{\alpha} - (1/2)(u\sigma)^2 \\ Q'p(t)' \end{pmatrix}.$$

If we denote $f(t, Y, u) = (f_u, f_p)'$, then

$$f \frac{\partial v}{\partial Y} = f_u \frac{\partial v}{\partial z} + f_p \frac{\partial v}{\partial p},$$

where $f_p \frac{\partial v}{\partial p}$ is the inner product of the two vectors.

Recall that

$$\Sigma(t, Y, u) = \begin{pmatrix} u\sigma \\ \frac{A(t)p(t)'}{\sigma} \end{pmatrix}.$$

We can denote $\Sigma(t, Y, u) = (c_u, c_p)'$, then

$$(\Sigma\Sigma') \frac{\partial^2 v}{\partial Y^2} = \begin{pmatrix} c_u^2 & c_u c_p' \\ c_p c_u & c_p c_p' \end{pmatrix} \begin{pmatrix} \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 v}{\partial z \partial p} \\ \frac{\partial^2 v}{\partial p \partial z} & \frac{\partial^2 v}{\partial p^2} \end{pmatrix}.$$

So,

$$\frac{1}{2}\text{tr}\{(\Sigma\Sigma')\frac{\partial^2 v}{\partial Y^2}\} = \frac{1}{2}\left(c_u^2\frac{\partial^2 v}{\partial z^2} + c_u c_p'\frac{\partial^2 v}{\partial p\partial z} + c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right) + \frac{1}{2}\text{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right).$$

The PDE becomes

$$\begin{aligned} 0 = & \frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) \\ = & \frac{\partial v}{\partial s} + \sup_u\left\{f_u\frac{\partial v}{\partial z} + f_p\frac{\partial v}{\partial p} + \frac{1}{2}\left(c_u^2\frac{\partial^2 v}{\partial z^2} + c_u c_p'\frac{\partial^2 v}{\partial p\partial z} + c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right) + \frac{1}{2}\text{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right)\right\} \\ = & \frac{\partial v}{\partial s} + \sup_u\left\{f_u\frac{\partial v}{\partial z} + \frac{1}{2}c_u^2\frac{\partial^2 v}{\partial z^2} + f_p\frac{\partial v}{\partial p} + c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right\} + \frac{1}{2}\text{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right). \end{aligned}$$

Suppose that the value function has the form

$$v(s, z, p) = y^k w(s, p) = e^{kz} w(s, p).$$

Then

$$\begin{aligned} \frac{\partial v}{\partial s} &= e^{kz}\frac{\partial w}{\partial s}, \\ \frac{\partial v}{\partial z} &= k e^{kz} w(s, p), \\ \frac{\partial v}{\partial p} &= e^{kz}\frac{\partial w}{\partial p}, \\ \frac{\partial^2 v}{\partial z^2} &= k^2 e^{kz} w(s, p), \\ \frac{\partial^2 v}{\partial z\partial p} &= \left(\frac{\partial^2 v}{\partial p\partial z}\right)' = k e^{kz}\frac{\partial w}{\partial p}, \\ \frac{\partial^2 v}{\partial p^2} &= e^{kz}\frac{\partial^2 w}{\partial p^2}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 = & \frac{\partial v}{\partial s} + H(s, Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}) \\ = & e^{kz}\frac{\partial w}{\partial s} + \sup_u\left\{f_u k e^{kz} w(s, p) + \frac{1}{2}c_u^2 k^2 e^{kz} w(s, p)\right. \\ & \left. + f_p e^{kz}\frac{\partial w}{\partial p} + c_u c_p k e^{kz}\frac{\partial w}{\partial p}\right\} + \frac{1}{2}\text{tr}(c_p c_p' e^{kz}\frac{\partial^2 w}{\partial p^2}) \end{aligned}$$

Therefore, if the value function has the form $v(s,z,p) = y^k w(s,p) = e^{kz}w(s,p)$, then

$$\begin{aligned}\frac{\partial w}{\partial s} &+ \sup_u \left\{ f_u k w(s,p) + \frac{1}{2}c_u^2 k^2 w(s,p) \right.\\ &+ \left. f_p \frac{\partial w}{\partial p} + c_u c_p k \frac{\partial w}{\partial p} \right\} + \frac{1}{2}\mathrm{tr}(c_p c_p' \frac{\partial^2 w}{\partial p^2}) = 0. \end{aligned} \tag{6.2}$$

This is a reduced PDE that only contains the variables s and p.

Theorem 6.0.8. *If $w(s,p)$ is the viscosity solution of the PDE (6.2), then $v(s,z,p) = e^{kz}w(s,p)$ is the viscosity solution of the HJB equation (3.7).*

Proof. Suppose $w(s,p)$ is the viscosity solution of the PDE (6.2), then

$$\begin{aligned}\frac{\partial \phi}{\partial s} &+ \sup_u \left\{ f_u k w(s,p) + \frac{1}{2}c_u^2 k^2 w(s,p) \right.\\ &+ \left. f_p \frac{\partial \phi}{\partial p} + c_u c_p k \frac{\partial \phi}{\partial p} \right\} + \tfrac{1}{2}\mathrm{tr}(c_p c_p' \tfrac{\partial^2 \phi}{\partial p^2}) \leq 0. \end{aligned}$$

for all $\phi \in C^2$ such that $w - \phi$ has a local minimum at (s,p). Then $w(s,p) - \phi(s,p) \leq w(t,q) - \phi(t,q)$.

Suppose $v(s,z,p) = e^{kz}w(s,p)$ and $\psi \in C^2$ such that $v - \psi$ has a local minimum at (s,z,p). That is,

$$e^{kz}w(s,p) - \psi(s,z,p) \leq e^{kx}w(t,q) - \psi(t,x,q) \tag{6.3}$$

for all (t,x,q)in a neighborhood $N(s,z,p)$.

(1) Let $t = s, q = p, x = z + \Delta z$ in (6.3) we have

$$e^{kz}w(s,p) - \psi(s,z,p) \leq e^{k(z+\Delta z)}w(s,p) - \psi(s, z+\Delta z, p),$$

or,

$$\psi(s, z+\Delta z, p) - \psi(s,z,p) \leq e^{k(z+\Delta z)}w(s,p) - e^{kz}w(s,p). \tag{6.4}$$

Therefore,

$$\frac{\psi(s, z+\Delta z, p) - \psi(s,z,p)}{\Delta z} \leq \frac{e^{k(z+\Delta z)} - e^{kz}}{\Delta z} w(s,p).$$

Letting $\Delta z \to 0$, we have

$$\frac{\partial \psi}{\partial z} \leq \frac{\partial v}{\partial z} \text{ at } (s,z,p). \tag{6.5}$$

Similarly, we have

$$\psi(s,z-\Delta z,p)-\psi(s,z,p)\le e^{k(z-\Delta z)}w(s,p)-e^{kz}w(s,p). \tag{6.6}$$

Add (6.4) and (6.6). We have

$$\psi(s,z+\Delta z,p)-2\psi(s,z,p)+\psi(s,z-\Delta z,p)\le e^{k(z+\Delta z)}w(s,p)+e^{k(z-\Delta z)}w(s,p)-2e^{kz}w(s,p).$$

Hence

$$\frac{\psi(s,z+\Delta z,p)-2\psi(s,z,p)+\psi(s,z-\Delta z,p)}{(\Delta z)^2}\le\frac{e^{k(z+\Delta z)}-2e^{kz}++e^{k(z-\Delta z)}}{(\Delta z)^2}w(s,p).$$

Letting $\Delta z\to 0$, we have

$$\frac{\partial^2\psi}{\partial z^2}\le\frac{\partial^2 v}{\partial z^2}\text{ at }(s,z,p). \tag{6.7}$$

(2) Let $x=z$ in (6.3). We also have

$$e^{kz}w(s,p)-\psi(s,z,p)\le e^{kz}w(t,q)-\psi(t,z,q).$$

Fix z and divide both sides by e^{kz}, we have

$$w(s,p)-\frac{\psi(s,z,p)}{e^{kz}}\le w(t,q)-\frac{\psi(t,z,q)}{e^{kz}},$$

for all (t,q) in the neighborhood $N(s,p)$. Because w is the viscosity solution of (6.2), we must have

$$\begin{aligned}\frac{1}{e^{kz}}\frac{\partial\psi}{\partial s}\quad&+\sup\nolimits_{u(\cdot)}\{f_u k w(s,z,p)+\tfrac{1}{2}c_u^2k^2w(s,z,p)\\&+f_p\frac{1}{e^{kz}}\frac{\partial\psi}{\partial p}+c_uc_pk\frac{1}{e^{kz}}\frac{\partial\psi}{\partial p}\}+\frac{1}{2}\frac{1}{e^{kz}}\mathrm{tr}(c_pc_p'\frac{\partial^2\psi}{\partial p^2})\le 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial\psi}{\partial s}\quad&+\sup\nolimits_{u(\cdot)}\{f_u k e^{kz}w(s,z,p)+\tfrac{1}{2}c_u^2k^2e^{kz}w(s,z,p)\\&+f_p\frac{\partial\psi}{\partial p}+c_uc_pk\frac{\partial\psi}{\partial p}\}+\frac{1}{2}\mathrm{tr}(c_pc_p'\frac{\partial^2\psi}{\partial p^2})\le 0.\end{aligned}$$

This is true for all value of z. So

$$\begin{aligned}\frac{\partial\psi}{\partial s}\quad&+\sup\nolimits_{u(\cdot)}\{f_u\frac{\partial v}{\partial z}+\frac{1}{2}c_u^2\frac{\partial^2 v}{\partial z^2}\\&+f_p\frac{\partial\psi}{\partial p}+c_uc_pk\frac{\partial\psi}{\partial p}\}+\frac{1}{2}\mathrm{tr}(c_pc_p'\frac{\partial^2\psi}{\partial p^2})\le 0.\end{aligned}$$

Consider (6.5) and (6.7). We have

$$\begin{aligned}\frac{\partial\psi}{\partial s} & +\sup_{u(\cdot)}\{f_u\frac{\partial\psi}{\partial z}+\frac{1}{2}c_u^2\frac{\partial^2\psi}{\partial z^2} \\ & +f_p\frac{\partial\psi}{\partial p}+c_uc_pk\frac{\partial\psi}{\partial p}\}+\frac{1}{2}\mathrm{tr}(c_pc_p'\frac{\partial^2\psi}{\partial p^2})\le 0.\end{aligned}$$

This proves that $v(s,z,p) = e^{kz}w(s,p)$ is a viscosity subsolution of (3.7). The proof for supersolution is similar. □

Recall that the numerical scheme for the value function is

$$v^h(s-\Delta t, Y) = \sup_u[\sum_Z P^h(Y,Z|u)v^h(s,Z)] \tag{6.8}$$

with

$$v^h(T,z,p) = \Phi(z), z \in S_h. \tag{6.9}$$

Now with $v(s,z,p) = e^{kz}w(s,p)$ we can simplify this scheme and have

$$w^h(s-\Delta t, p) = \sup_u[\sum_q P^h(p,q|u)\delta(q)w^h(s,p)], \tag{6.10}$$

with

$$w^h(T,p) = 1. \tag{6.11}$$

6.1 Numerical Example

In order to test the numerical scheme in this chapter, we compare the value function from the Markov chain approximation and from the Monte Carlo simulation. To take advantage of the separable case, we assume the utility function is $\Phi(x) = x^{1/2}$.

Assume $T = 0.5$, a half year time frame. With initial investment of \$1000, the value function $v(s,x,p)$ for different initial time s compared with the data from Monte Carlo simulation is shown in Table 6.1 .

With initial investment of \$1000, the value function $v(s,x,p)$ for different initial probability p compared with the data from Monte Carlo simulation is shown in Table 6.2 .

Table 6.1: Different initial time

s	$v(s, \log(1000), 0.8)$	MC
0	33.5278	33.8943
0.1	33.2449	33.4353
0.2	32.8735	33.2977
0.3	32.4707	32.4048
0.4	32.0480	32.1506

Table 6.2: Different initial probability

p	$v(0.2, \log(1000), p)$	MC
0	32.5956	32.9894
0.1	32.5964	32.7831
0.2	32.5981	32.8393
0.3	32.6024	32.8257
0.4	32.6140	32.9907
0.5	32.6511	32.9886
0.6	32.6963	33.0124
0.7	32.7415	33.1261
0.8	32.7868	33.1319
0.9	32.8321	33.2506
1.0	32.8588	33.2377

Chapter 7

Two-Time-scale Approximation

When the underlying Markov chain has a large state space, it is difficult to obtain an optimal asset allocation. To deal with this problem, we use a method that is called two-time-scale approximation. The method is effective for Markov chains whose states can be divided into a number of weakly irreducible classes. The Markov chain fluctuates rapidly among different states within a weakly irreducible class, but jumps less frequently from one weakly irreducible class to another.

Let us first summarize the results of time-scale separation in Markov chains. Assume the generator of the Markov chain is of the form:

$$Q^{\varepsilon} = \frac{1}{\varepsilon}\tilde{Q} + \widehat{Q},$$

where both $\widetilde{Q}$ and $\widehat{Q}$ are generators. Let's assume

$$\widetilde{Q} = \operatorname{diag}\left(\widetilde{Q}^{1}, \cdots, \tilde{Q}^{l}\right).$$

For each $k = 1, \cdots, l, \widetilde{Q}^{k}$ is the weakly irreducible generator corresponding to the states in $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$, for $k = 1, \cdots, l$. The state space is, therefore, decomposed into $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l = \{s_{11}, \ldots, s_{1m_1}\} \cup \cdots \cup \{s_{l1}, \ldots, s_{lm_l}\}$,

Note that $\widetilde{Q}$ governs the rapidly changing part and $\widehat{Q}$ describes the slowly varying components. As $\varepsilon \to 0$, the underlying Markov chain jumps so fast within $\mathcal{M}_k$, that it is no longer useful to distinguish the states in $\mathcal{M}_k$. That is why we can lump all the states in each $\mathcal{M}_k$ into one state. This reduces the number of states in the state space dramatically and results in a aggregated process $\overline{\alpha}^{\varepsilon}(\cdot)$:

$$\overline{\alpha}^{\varepsilon}(t) = k, \text{ when } \alpha^{\varepsilon}(t) \in \mathcal{M}_k.$$

Definition A generator $Q(t)$ is said to be weakly irreducible if, for each fixed $t \geq 0$, the system of equations

$$\nu(t)Q(t) = 0,$$
$$\sum_{i=1}^{m} \nu_i(t) = 1$$

has a unique solution $\nu(t) = (\nu_1(t), \cdots, \nu_m(t))$ and $\nu(t) \geq 0$.

We are going to apply a couple of results from [15] and [17]. For convenience, the notation here is also mostly consistent with these two papers.

Assuming $\widetilde{Q}^k$ to be weakly irreducible, the following results have been shown in [5] section 7.5.

(a) $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, which is a continuous-time Markov chain generated by

$$\begin{aligned} \overline{Q} &= \nu \widetilde{Q} \widetilde{\mathbb{1}}, \\ \nu &= \operatorname{diag}(\nu^1, \cdots, \nu^l), \quad \widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \cdots, \mathbb{1}_{m_l}), \end{aligned}$$

where ν^k is the quasi-stationary distribution of $\widetilde{Q}^k$, $k = 1, \cdots, l$, $\mathbb{1}_l = (1, \cdots, l)' \in \mathbb{R}^l$ is an l-dimensional column vector with all components being equal to 1, $\operatorname{diag}(D^1, \cdots, D^r)$ is a block-diagonal matrix with appropriate dimensions.

(b) For any bounded deterministic $\beta(\cdot)$,

$$E\left(\int_0^T (I_{\{\alpha^\varepsilon(t)=s_{ij}\}} - \nu_j^k I_{\{\alpha^\varepsilon(t)=k\}})\beta(t)dt\right)^2 = O(\varepsilon),$$

where I_A is the indicator function of a set A.

(c) Let $\overline{P}(t) = \widetilde{\mathbb{1}}(\exp \widetilde{Q}t)\nu \in \mathbb{R}^{m\times m}$. Then

$$|\exp(Q^\varepsilon t) - \overline{P}(t)| = O(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}),$$

for some $\kappa > 0$.

To apply the two-time-scale Wonham Filters, we state the result from Section 2.2 in [17]. Let $\alpha^\varepsilon(t)$ be a continuous-time Markov chain having finite state space $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup$

$\mathcal{M}_l = \{s_{11}, ..., s_{1m_1}\} \cup \cdots \cup \{s_{l1}, ..., s_{lm_l}\}$, and generator Q^ε. Consider a function $y^\varepsilon(t)$ of the Markov chain that is observable with additive Gaussian noise. Let $y^\varepsilon(t)$ denote the observation measurement given by

$$dy^\varepsilon(t) = f(\alpha^\varepsilon(t))dt + \sigma dW(t), y^\varepsilon(0) = 0, \tag{7.1}$$

where σ is a positive constant and $W(t)$ is a standard Brownian motion. Let $p^\varepsilon_{ij}(t)$ denote the conditional probability of $\alpha^\varepsilon(t) = s_{ij}$ given the observations up to time t, i.e.,

$$p^\varepsilon_{ij}(t) = P(\alpha^\varepsilon(t) = s_{ij} | y^\varepsilon(s) : s \le t);$$

for $i = 1, ..., l$ and $j = 1, \cdots, m_i$. Let

$$p^\varepsilon(t) = (p^\varepsilon_{11}(t), \cdots, p^\varepsilon_{1m_1}(t), \cdots, p^\varepsilon_{l1}(t), \cdots, p^\varepsilon_{lm_l}(t)) \in \mathcal{R}^{1\times m}.$$

Let

$$\widehat{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} \sum_{j=1}^{m_i} f(s_{ij}) p^\varepsilon_{ij}(t).$$

Then the Wonham filter is given by

$$dp^\varepsilon(t) = p^\varepsilon(t)Q^\varepsilon dt - \frac{1}{\sigma^2}\widehat{\alpha}^\varepsilon(t)p^\varepsilon(t)A^\varepsilon(t)dt + \frac{1}{\sigma^2}p^\varepsilon(t)A^\varepsilon(t)dy^\varepsilon(t), \tag{7.2}$$

where

$$A^\varepsilon(t) = \mathrm{diag}(f(s_{11}), \cdots, f(s_{1m_1}), \cdots, f(s_{l1}), \cdots, f(lm_l)) - \widehat{\alpha}^\varepsilon(t)I.$$

and $p^\varepsilon(0) = p_0$ being the initial probability. Here, I is the identity matrix of dimension $m \times m$, $m = m_1 + m_2 + \cdots + m_l$.

Let's assume the Markov process that governs the market mode is represented by $\alpha^\varepsilon(t)$. Then $\alpha^\varepsilon(t)$ affects the stock price P^ε_2 which then affects the wealth function $\xi^\varepsilon(t)$. Recall that

$$\frac{dP^\varepsilon_2}{P^\varepsilon_2} = \mu(\alpha^\varepsilon(t))dt + \sigma dW(t).$$

So

$$d\log(P^\varepsilon_2) = [\mu(\alpha^\varepsilon(t)) - \frac{\sigma^2}{2}]dt + \sigma dW(t).$$

Since $\log(P_2^\varepsilon)$ is observable, we can set up a Wonham filter for $\alpha^\varepsilon(t)$ by the following

$$dp^\varepsilon(t) = p^\varepsilon(t)Qdt - \frac{1}{\sigma^2}\widetilde{\alpha}^\varepsilon(t)p^\varepsilon(t)A^\varepsilon(t)dt + \frac{1}{\sigma^2}p^\varepsilon(t)A^\varepsilon(t)d\log(P_2^\varepsilon), \tag{7.3}$$

where

$$\widetilde{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i}[\mu(s_{ij}) - \frac{\sigma^2}{2}]p_{ij}^\varepsilon(t),$$

$$A^\varepsilon(t) = \operatorname{diag}(\mu(s_{11}) - \frac{\sigma^2}{2}, \cdots, \mu(s_{1m_1}) - \frac{\sigma^2}{2}, \cdots, \mu(s_{l1}) - \frac{\sigma^2}{2}, \cdots, \mu(lm_l) - \frac{\sigma^2}{2}) - \widetilde{\alpha}^\varepsilon(t)I.$$

and $p^\varepsilon(0) = p_0$ being the initial probability.

Rewrite (7.3) as follows:

$$\begin{aligned} dp^\varepsilon(t) &= p^\varepsilon(t)Q^\varepsilon dt - \frac{1}{\sigma^2}\widetilde{\alpha}^\varepsilon(t)p^\varepsilon(t)A^\varepsilon(t)dt + \frac{1}{\sigma^2}p^\varepsilon(t)A^\varepsilon(t)d\log(P_2) \\ &= p^\varepsilon(t)Q^\varepsilon dt + \frac{p^\varepsilon(t)A^\varepsilon(t)}{\sigma}\left(\frac{d\log(P_2) - \tilde{\alpha}^\varepsilon(t)dt}{\sigma}\right). \end{aligned} \tag{7.4}$$

Let

$$d\hat{v}^\varepsilon = \frac{d\log(P_2^\varepsilon) - \tilde{\alpha}^\varepsilon dt}{\sigma}.$$

$\hat{v}^\varepsilon$ is an innovation process.

From the definition of $d\hat{v}^\varepsilon$, we have $\sigma d\hat{v}^\varepsilon = d\log(P_2^\varepsilon) - \tilde{\alpha}^\varepsilon dt$. Hence,

$$d\log(P_2^\varepsilon) = \tilde{\alpha}^\varepsilon dt + \sigma d\hat{v}^\varepsilon.$$

Therefore, we can replace the dynamic of P_2 by the following

$$\frac{dP_2}{P_2} = (\tilde{\alpha}^\varepsilon(t) + \frac{\sigma^2}{2})dt + \sigma d\hat{v}^\varepsilon.$$

Notice that both $\tilde{\alpha}^\varepsilon$ and $d\hat{v}^\varepsilon$ are observable.

Because

$$\widetilde{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i}[\mu(s_{ij}) - \frac{\sigma^2}{2}]p_{ij}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i}\mu(s_{ij})p_{ij}^\varepsilon(t) - \frac{\sigma^2}{2} = \widehat{\alpha}^\varepsilon(t) - \frac{\sigma^2}{2},$$

$dP_2^\varepsilon/P_2^\varepsilon = \widehat{\alpha}^\varepsilon(t)dt + \sigma d\hat{v}^\varepsilon$, where $\widehat{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i}\mu(s_{ij})p_{ij}^\varepsilon(t)$.

Therefore, $\xi^\varepsilon(t)$ has the following dynamics:

$$\frac{d\xi^\varepsilon(t)}{\xi^\varepsilon(t)} = (1-u(t))rdt + u(t)(\widehat{\alpha}^\varepsilon(t)dt + \sigma d\hat{v}^\varepsilon), \tag{7.5}$$

where $\xi^\varepsilon(s) = y$.

Denote $Z^\varepsilon(t) = \log \xi^\varepsilon(t)$ and $z = \log y$, then we have

$$\begin{aligned} dZ^\varepsilon(t) &= [(1-u(t))r + u(t)\widehat{\alpha}^\varepsilon(t) - \frac{1}{2}(u(t)\sigma)^2]dt + u(t)\sigma d\hat{v}, \\ dp^\varepsilon(t) &= p^\varepsilon(t)Q^\varepsilon dt + \frac{p^\varepsilon(t)A^\varepsilon(t)}{\sigma}d\hat{v}(t). \end{aligned}$$

We can write the reward and value functions in terms of the new state variables

$$J^\varepsilon(s,z,p,u(\cdot)) = E_{sz}(\Phi(\exp(Z^\varepsilon(T)))),$$

$$v^\varepsilon(s,z,p) = \sup_u J^\varepsilon(s,z,p,u(\cdot)),$$

and $v^\varepsilon(T,z,p) = \Phi(e^z)$. Let us refer to the problem of computing v^ε as problem P^ε, the original problem.

Let $Y^\varepsilon(t) = (Z^\varepsilon(t) \vdots p^\varepsilon(t))'$, where A' denote the transpose of the matrix (or vector) A. Then

$$dY^\varepsilon(t) = \begin{pmatrix} (1-u)r + u\hat{\alpha}^\varepsilon - (1/2)(u\sigma)^2 \\ Q^{\varepsilon\prime} p^\varepsilon(t)' \end{pmatrix} dt + \begin{pmatrix} u\sigma \\ \dfrac{A^\varepsilon(t)p^\varepsilon(t)'}{\sigma} \end{pmatrix} d\hat{v}^\varepsilon.$$

Let

$$f^\varepsilon(t,Y,u) = \begin{pmatrix} (1-u)r + u\hat{\alpha}^\varepsilon - (1/2)(u\sigma)^2 \\ Q^{\varepsilon\prime} p^\varepsilon(t)' \end{pmatrix}$$

and

$$\Sigma^\varepsilon(t,Y,u) = \begin{pmatrix} u\sigma \\ \dfrac{A^\varepsilon(t)p^\varepsilon(t)'}{\sigma} \end{pmatrix}.$$

Then

$$dY^\varepsilon(t) = f^\varepsilon(t, Y^\varepsilon(t), u(t))dt + \Sigma^\varepsilon(t, Y^\varepsilon(t), u(t))d\hat{v}^\varepsilon(t).$$

Define

$$H^\varepsilon(t,Y,P,G)=\sup_u\{f^\varepsilon P+\frac{1}{2}\mathrm{tr}\{(\Sigma^\varepsilon\Sigma^{\varepsilon\prime})G\}\} \tag{7.6}$$

where P is an $1\times(m+1)$ vector and G is an $(m+1)\times(m+1)$ matrix. Here, fP should be understood as the inner product of two vectors.

By Ito's formula, the value function $v^\varepsilon(s,Y)$ should satisfy the following HJB Equation

$$\frac{\partial v^\varepsilon}{\partial s}+H^\varepsilon(s,Y^\varepsilon,\frac{\partial v^\varepsilon}{\partial Y},\frac{\partial^2 v^\varepsilon}{\partial Y^2})=0 \tag{7.7}$$

with the boundary condition $v^\varepsilon(T,z,p)=\Phi(\mathrm{e}^z)$, where $z=\log y$, y is the initial wealth, and p is the initial probability vector. Using the same proof as in Chapter 3, we can see that $v^\varepsilon(s,z,p)$ is the unique viscosity solution of HJB equation (7.7).

It is shown in Section 3 of [17] that $p^\varepsilon(t)\to p^0(t)$ as $\varepsilon\to 0$, where

$$p^0(t)=(\nu^{\,1}\bar{p}_1(t),\cdots,\nu^{\,l}\bar{p}_l(t))=\bar{p}(t)\nu.$$

To determine $\bar{p}(t)$, we note the weak limit of $y^\varepsilon(\cdot)=\log P_2^\varepsilon(\cdot)$ is given by

$$dy(t)=[\bar{\mu}(\overline{\alpha}(t))-\frac{1}{2}\sigma^2]dt+\sigma dW(t),\ \ y(0)=0,$$

where

$$\bar{\mu}(i)=\sum_{j=1}^{m_i}\mu(s_{ij})\nu_j^i.$$

The corresponding conditional probability $\bar{p}(t)$ is decided by

$$\bar{p}(t)=\bar{p}(0)+\int_0^t\bar{p}(u)\overline{Q}du-\frac{1}{\sigma^2}\int_0^t\widetilde{\alpha}(u)\bar{p}(u)\overline{A}(u)du+\frac{1}{\sigma^2}\int_0^t\bar{p}(u)\overline{A}(u)dy(u),$$

with initial condition

$$\bar{p}(0)=p_0\widetilde{\mathbb{1}},$$

where

$$\widetilde{\alpha}(t)=\sum_{i=1}^{l}\bar{\mu}(i)\bar{p}_i(t)-\frac{1}{2}\sigma^2,$$

and

$$\overline{A}(t)=\mathrm{diag}(\bar{\mu}(1)-\frac{1}{2}\sigma^2,\cdots,\bar{\mu}(l)-\frac{1}{2}\sigma^2)-\widetilde{\alpha}(t)I.$$

Using this filter, the SDE can be reformulated as

$$\begin{aligned} d\overline{Z}(t) &= [(1-u(t))r + u(t)\widehat{\mu}(t) - (1/2)(u(t)\sigma)^2]dt + u(t)\sigma d\hat{w}, \\ d\bar{p}(t) &= \bar{p}(t)\overline{Q}dt + \frac{\bar{p}(t)\overline{A}}{\sigma}d\hat{w}, \end{aligned}$$

where

$$d\hat{w} = \frac{dy(t) - \widetilde{\alpha}(t)dt}{\sigma}$$

is an innovation process. And

$$\widehat{\mu}(t) = p^0(t)\mu = \bar{p}(t)\nu\mu.$$

Here $\mu = (\mu_1 \cdots \mu_m)'$ and the corresponding reward and value functions are

$$J^0(s, z, \bar{p}, u(\cdot)) = E_{sz}(\Phi(\exp(\overline{Z}(T)))),$$

$$\bar{v}^0(s, z, p) = \sup_{u(\cdot)} J^0(s, z, \bar{p}, u(\cdot)),$$

with $\bar{v}^0(T, z, \bar{p}) = \Phi(e^z)$. Let us refer to this problem as problem P^0, or the limit problem.

Let $\overline{Y}(t) = (\overline{Z}(t) \vdots \bar{p}(t))'$, where A' denote the transpose of the matrix (or vector) A. Then

$$d\overline{Y}(t) = \begin{pmatrix} (1-u)r + u\widehat{\mu} - (1/2)(u\sigma)^2 \\ \\ \overline{Q}'\bar{p}(t)' \end{pmatrix} dt + \begin{pmatrix} u\sigma \\ \\ \dfrac{\overline{A}(t)\bar{p}(t)'}{\sigma} \end{pmatrix} d\hat{w}.$$

Let

$$\overline{f}(t, Y, u) = \begin{pmatrix} (1-u)r + u\widehat{\mu}(t) - (1/2)(u\sigma)^2 \\ \\ \overline{Q}'\bar{p}(t)' \end{pmatrix}$$

and

$$\overline{\Sigma}(t, Y, u) = \begin{pmatrix} u\sigma \\ \\ \dfrac{\overline{A}(t)\bar{p}(t)'}{\sigma} \end{pmatrix}.$$

Then

$$d\overline{Y}(t) = \overline{f}(t, \overline{Y}(t), u(t))dt + \overline{\Sigma}(t, \overline{Y}(t), u(t))d\hat{w}(t).$$

Define

$$\overline{H}(t, \overline{Y}, P, G) = \sup_u \{\overline{f}P + \frac{1}{2}\mathrm{tr}\{(\overline{\Sigma\Sigma'})G\}\} \tag{7.8}$$

where P is an $1 \times (l+1)$ vector and G is an $(l+1) \times (l+1)$ matrix. Here, $\overline{f}P$ should be understood as the inner product of two vectors.

By Ito's formula, for the reformulated SDE, the value function $\bar{v}^0(s, Y)$ should satisfy the following HJB Equation

$$\frac{\partial \bar{v}^0}{\partial s} + \overline{H}(s, \overline{Y}, \frac{\partial \bar{v}^0}{\partial \overline{Y}}, \frac{\partial^2 \bar{v}^0}{\partial \overline{Y}^2}) = 0, \tag{7.9}$$

with the boundary condition $\bar{v}^0(T, z, \bar{p}) = \Phi(\mathrm{e}^z)$, where $z = \log y$, y is the initial wealth, and $\bar{p} = p_0 \widetilde{1\!\!1}$ is the initial probability vector. Using the same proof as in Chapter 3, we can see that $\bar{v}^0(s, z, p)$ is the unique viscosity solution of HJB equation (7.9).

Define

$$\widetilde{p}^\varepsilon(t) = \overline{p}^\varepsilon(t)\nu,$$

$$\overline{p}^\varepsilon(t) = \overline{p}^\varepsilon(0) + \int_0^t \overline{p}^\varepsilon(u)\overline{Q}du - \frac{1}{\sigma^2}\int_0^t \check{\alpha}^\varepsilon(u)\overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)du + \frac{1}{\sigma^2}\int_0^t \overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)dy^\varepsilon(u),$$

with initial condition

$$\overline{p}^\varepsilon(0) = p_0 \tilde{1\!\!1},$$

where

$$\check{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} \bar{\mu}(i)\overline{p}_i^\varepsilon(t) - \frac{1}{2}\sigma^2,$$

and

$$\overline{A}^\varepsilon(t) = \mathrm{diag}(\bar{\mu}(1) - \frac{1}{2}\sigma^2, \cdots, \bar{\mu}(l) - \frac{1}{2}\sigma^2) - \check{\alpha}^\varepsilon(t)I.$$

The following theorem has been proved in [17]:

Theorem 7.0.1. *The following hold.*

(a) $\widetilde{p}^\varepsilon(t)$ is an approximation to $p^\varepsilon(t)$ for small ε. More precisely,

$$E|\widetilde{p}^\varepsilon(t) - p^\varepsilon(t)|^2 = O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right),$$

for some constant $\kappa > 0$.

(b) $\bar{p}^\varepsilon(t)$ converges weakly to $\bar{p}(\cdot)$ in $C([0,T];\mathbb{R}^m)$, where $C([0,T];\mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued continuous functions defined on $[0,T]$.

We notice that $\widetilde{p}^\varepsilon(t)$ has much lower dimension than $p^\varepsilon(t)$ and the approximation can be fairly accurate. So it is reasonable to consider computation of the value function using $\widetilde{p}^\varepsilon(t)$ instead of $p^\varepsilon(t)$.

Through similar argument, the original SDE

$$\begin{aligned} dZ^\varepsilon(t) &= [(1-u(t))r + u(t)\hat{\alpha}^\varepsilon(t) - (1/2)(u(t)\sigma)^2]dt + u(t)\sigma d\hat{v}, \\ dp^\varepsilon(t) &= p^\varepsilon(t)Q^\varepsilon dt + \frac{p^\varepsilon(t)A^\varepsilon(t)}{\sigma} d\hat{v}(t). \end{aligned}$$

can be reformulated into

$$\begin{aligned} d\overline{Z}^\varepsilon(t) &= [(1-u(t))r + u(t)\widehat{\mu}^\varepsilon(t) - (1/2)(u(t)\sigma)^2]dt + u(t)\sigma d\hat{w}^\varepsilon, \\ d\bar{p}^\varepsilon(t) &= \bar{p}^\varepsilon(t)\overline{Q}dt + \frac{\bar{p}^\varepsilon(t)\overline{A}^\varepsilon(t)}{\sigma} d\hat{w}^\varepsilon, \end{aligned}$$

where

$$d\hat{w}^\varepsilon = \frac{d\log p_2^\varepsilon(t) - \check{\alpha}^\varepsilon(t)dt}{\sigma}$$

is an innovation process. And

$$\widehat{\mu}^\varepsilon(t) = \tilde{p}^\varepsilon(t)\mu = \bar{p}^\varepsilon \nu\mu.$$

Here $\mu = (\mu_1 \cdots \mu_m)'$.

Recall that

$$J^\varepsilon(s,z,p,u(\cdot)) = E_{sz}(\Phi(\exp(Z^\varepsilon(T)))),$$

$$v^\varepsilon(s,z,p) = \sup_u J^\varepsilon(s,z,p,u(\cdot)),$$

and $v^\varepsilon(T,z,p) = \Phi(e^z)$.

Let $\overline{Y}^\varepsilon(t) = (\overline{Z}^\varepsilon(t)\vdots\bar{p}^\varepsilon(t))'$, where A' denote the transpose of the matrix (or vector) A. Then

$$d\overline{Y}^\varepsilon(t) = \begin{pmatrix} (1-u)r + u\widehat{\mu}^\varepsilon - (1/2)(u\sigma)^2 \\ \\ \overline{Q}'\bar{p}^\varepsilon(t)' \end{pmatrix} dt + \begin{pmatrix} u\sigma \\ \\ \dfrac{\overline{A}^\varepsilon(t)\bar{p}^\varepsilon(t)'}{\sigma} \end{pmatrix} d\hat{w}^\varepsilon.$$

Let

$$\overline{f}^{\varepsilon}(t, Y, u) = \begin{pmatrix} (1-u)r + u\widehat{\mu}^{\varepsilon} - (1/2)(u\sigma)^2 \\ \\ \overline{Q}'\overline{p}^{\varepsilon}(t)' \end{pmatrix}$$

and

$$\overline{\Sigma}^{\varepsilon}(t, Y, u) = \begin{pmatrix} u\sigma \\ \\ \dfrac{\overline{A}^{\varepsilon}(t)\overline{p}^{\varepsilon}(t)'}{\sigma} \end{pmatrix}.$$

Then

$$d\overline{Y}^{\varepsilon}(t) = \overline{f}^{\varepsilon}(t, \overline{Y}^{\varepsilon}(t), u(t))dt + \overline{\Sigma}^{\varepsilon}(t, \overline{Y}^{\varepsilon}(t), u(t))d\hat{w}^{\varepsilon}(t).$$

Define

$$\overline{H}^{\varepsilon}(t, \overline{Y}, P, G) = \sup_u \{\overline{f}^{\varepsilon} P + \frac{1}{2}\mathrm{tr}\{(\overline{\Sigma}^{\varepsilon}\overline{\Sigma}^{\varepsilon\prime})G\}\} \tag{7.10}$$

where P is an $1 \times (l+1)$ vector and G is an $(l+1) \times (l+1)$ matrix. Here, $\overline{f}P$ should be understood as the inner product of two vectors.

By Ito's formula, for the reformulated SDE, the value function $\bar{v}^{\varepsilon}(s, Y)$ should satisfy the following HJB Equation

$$\frac{\partial \bar{v}^{\varepsilon}}{\partial s} + \overline{H}^{\varepsilon}(s, \overline{Y}^{\varepsilon}, \frac{\partial \bar{v}^{\varepsilon}}{\partial \overline{Y}}, \frac{\partial^2 \bar{v}^{\varepsilon}}{\partial \overline{Y}^2}) = 0 \tag{7.11}$$

with the boundary condition $\bar{v}^{\varepsilon}(T, z, \bar{p}) = \Phi(\mathrm{e}^z)$, where $z = \log y$, y is the initial wealth, and $\bar{p} = p_0\widetilde{1\!\!1}$ is the initial probability vector. Using the same proof as in Chapter 3, we can see that $\bar{v}^{\varepsilon}(s, z, p)$ is the unique viscosity solution of HJB equation (7.11). $\bar{v}^{\varepsilon}(s, z, p)$ is an approximation for the original value function $v^{\varepsilon}(s, z, p)$. Suppose the optimal control for $v^{\varepsilon}(s, z, p)$ is u_*^{ε} and the optimal control for $\bar{v}^{\varepsilon}(s, z, p)$ is $\bar{u}_*^{\varepsilon}$ then

$$v^{\varepsilon}(s, z, p) = \sup_u E\left[\Phi(\exp(Z^{\varepsilon}(T)))\right] = E\left[\Phi(\exp(Z^{\varepsilon}_{u_*^{\varepsilon}}(T)))\right],$$

$$\bar{v}^{\varepsilon}(s, z, p) = \sup_u E\left[\Phi(\exp(\overline{Z}^{\varepsilon}(T)))\right] = E\left[\Phi(\exp(\overline{Z}^{\varepsilon}_{\bar{u}_*^{\varepsilon}}(T)))\right].$$

We have

$$
\begin{aligned}
& v^\varepsilon(s,z,p) - \bar{v}^\varepsilon(s,z,\bar{p}) \\
= \quad & \sup_u E\left[\Phi(\exp(Z^\varepsilon(T)))\right] - \sup_u E\left[\Phi(\exp(\overline{Z}^\varepsilon(T)))\right] \\
= \quad & E\left[\Phi(\exp(Z^\varepsilon_{u^\varepsilon_*}(T)))\right] - E\left[\Phi(\exp(\overline{Z}^\varepsilon_{\bar{u}^\varepsilon_*}(T)))\right] \\
\leq \quad & E\left[\Phi(\exp(Z^\varepsilon_{u^\varepsilon_*}(T)))\right] - E\left[\Phi(\exp(\overline{Z}^\varepsilon_{u^\varepsilon_*}(T)))\right] \\
= \quad & E\left[\Phi(\exp(Z^\varepsilon_{u^\varepsilon_*}(T))) - \Phi(\exp(\overline{Z}^\varepsilon_{u^\varepsilon_*}(T)))\right] \\
\leq \quad & K_1 E\left|Z^\varepsilon_{u^\varepsilon_*}(T) - \overline{Z}^\varepsilon_{u^\varepsilon_*}(T)\right|.
\end{aligned}
$$

For the last inequality, we need the utility function to satisfy

$$|\Phi(y_1) - \Phi(y_2)| \leq K_1|\log y_1 - \log y_2|.$$

Therefore,

$$
\begin{aligned}
& v^\varepsilon(s,z,p) - \bar{v}^\varepsilon(s,z,\bar{p}) \\
\leq \quad & K_1 E\left|\int_s^T \left((1-u^\varepsilon_*)r + u^\varepsilon_*\widehat{\alpha}^\varepsilon - \frac{1}{2}(u^\varepsilon_*\sigma)^2\right)dt - \int_s^T \left((1-u^\varepsilon_*)r + u^\varepsilon_*\widehat{\mu}^\varepsilon - \frac{1}{2}(u^\varepsilon_*\sigma)^2\right)dt\right| \\
= \quad & K_1 E\left|\int_s^T u^\varepsilon_*(\widehat{\alpha}^\varepsilon - \widehat{\mu}^\varepsilon)dt\right| \\
= \quad & K_1 E\left|\int_s^T u^\varepsilon_*(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t))\cdot\mu dt\right| \\
= \quad & K_1 E\int_s^T u^\varepsilon_*\left|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right|\cdot\mu dt \\
\leq \quad & K_1\int_s^T u^\varepsilon_* E\left|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right|\cdot\mu dt \\
= \quad & O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right), \quad \text{by theorem 7.0.1.}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \bar{v}^\varepsilon(s,z,p) - v^\varepsilon(s,z,\bar{p}) \\
= \ & \sup_u E\left[\Phi(\exp(\overline{Z}^\varepsilon(T)))\right] - \sup_u E\left[\Phi(\exp(Z^\varepsilon(T)))\right] \\
= \ & E\left[\Phi(\exp(\overline{Z}^\varepsilon_{\bar{u}^\varepsilon_*}(T)))\right] - E\left[\Phi(\exp(Z^\varepsilon_{u^\varepsilon_*}(T)))\right] \\
\leq \ & E\left[\Phi(\exp(\overline{Z}^\varepsilon_{\bar{u}^\varepsilon_*}(T)))\right] - E\left[\Phi(\exp(Z^\varepsilon_{\bar{u}^\varepsilon_*}(T)))\right] \\
= \ & E\left[\Phi(\exp(\overline{Z}^\varepsilon_{\bar{u}^\varepsilon_*}(T))) - \Phi(\exp(Z^\varepsilon_{\bar{u}^\varepsilon_*}(T)))\right] \\
\leq \ & K_1 E\left|\overline{Z}^\varepsilon_{\bar{u}^\varepsilon_*}(T) - Z^\varepsilon_{\bar{u}^\varepsilon_*}(T)\right| \\
\leq \ & K_1 E\left|\int_s^T \left((1-\bar{u}^\varepsilon_*)r + \bar{u}^\varepsilon_*\widehat{\alpha}^\varepsilon - \frac{1}{2}(\bar{u}^\varepsilon_*\sigma)^2\right)dt - \int_s^T \left((1-\bar{u}^\varepsilon_*)r + \bar{u}^\varepsilon_*\widehat{\mu}^\varepsilon - \frac{1}{2}(\bar{u}^\varepsilon_*\sigma)^2\right)dt\right| \\
= \ & K_1 E\left|\int_s^T \bar{u}^\varepsilon_*(\widehat{\alpha}^\varepsilon - \widehat{\mu}^\varepsilon)dt\right| \\
= \ & K_1 E\left|\int_s^T \bar{u}^\varepsilon_*(p^\varepsilon(t) - \widetilde{p}^\varepsilon(t))\cdot \mu dt\right| \\
= \ & K_1 E\int_s^T \bar{u}^\varepsilon_* \left|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right| \cdot \mu dt \\
\leq \ & K_1 \int_s^T \bar{u}^\varepsilon_* E\left|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\right| \cdot \mu dt \\
= \ & O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right), \quad \text{by theorem 7.0.1.}
\end{aligned}
$$

Combine the above inequalities, we conclude

$$
|v^\varepsilon(s,z,p) - \bar{v}^\varepsilon(s,z,\bar{p})| = O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right).
$$

We are going to prove that $\bar{v}^{\varepsilon} \to \bar{v}^0$ as $\varepsilon \to 0$. Suppose that the optimal control for $\bar{v}^{\varepsilon}$ is u^{ε} and the optimal control for $\bar{v}^0$ is u^0. Then

$$\begin{aligned}
& \bar{v}^{\varepsilon}(s,z,\bar{p}) - \bar{v}^0(s,z,\bar{p}) \\
= \;& \sup_u E\Phi(\exp(\overline{Z}^{\varepsilon}(T))) - \sup_u E\Phi(\exp(\overline{Z}(T))) \\
= \;& E\Phi(\exp(\overline{Z}^{\varepsilon}_{u^{\varepsilon}}(T))) - E\Phi(\exp(\overline{Z}_{u^0}(T))) \\
\leq \;& E\Phi(\exp(\overline{Z}^{\varepsilon}_{u^{\varepsilon}}(T))) - E\Phi(\exp(\overline{Z}_{u^{\varepsilon}}(T))) \\
= \;& E\left[\Phi(\exp(\overline{Z}^{\varepsilon}_{u^{\varepsilon}}(T))) - \Phi(\exp(\overline{Z}_{u^{\varepsilon}}(T)))\right] \\
\leq \;& K_1 E\left|\overline{Z}^{\varepsilon}_{u^{\varepsilon}}(T) - \overline{Z}_{u^{\varepsilon}}(T)\right|
\end{aligned}$$

$$\begin{aligned}
& \bar{v}^0(s,z,\bar{p}) - \bar{v}^{\varepsilon}(s,z,\bar{p}) \\
= \;& \sup_u E\Phi(\exp(\overline{Z}(T))) - \sup_u E\Phi(\exp(\overline{Z}^{\varepsilon}(T))) \\
= \;& E\Phi(\exp(\overline{Z}_{u^0}(T))) - E\Phi(\exp(\overline{Z}^{\varepsilon}_{u^{\varepsilon}}(T))) \\
\leq \;& E\Phi(\exp(\overline{Z}_{u^0}(T))) - E\Phi(\exp(\overline{Z}^{\varepsilon}_{u^0}(T))) \\
= \;& E\left[\Phi(\exp(\overline{Z}_{u^0}(T))) - \Phi(\exp(\overline{Z}^{\varepsilon}_{u^0}(T)))\right] \\
\leq \;& K_1 E\left|\overline{Z}_{u^0}(T) - \overline{Z}^{\varepsilon}_{u^0}(T)\right|.
\end{aligned}$$

It suffices to prove that $E|\overline{Z}_u^\varepsilon(T) - \overline{Z}_u(T)| \to 0$ as $\varepsilon \to 0$ for any $u(t)$.

$$\begin{aligned}
& E|\overline{Z}_u^\varepsilon(T) - \overline{Z}_u(T)| \\
= \; & E\left|\int_s^T \left((1-u)r + u\widehat{\mu}^\varepsilon - \tfrac{1}{2}(u\sigma)^2\right) dt - \int_s^T \left((1-u)r + u\widehat{\mu} - \tfrac{1}{2}(u\sigma)^2\right) dt\right| \\
= \; & E\left|\int_s^T u(\widehat{\mu}^\varepsilon - \widehat{\mu})dt\right| \\
\leq \; & E\int_s^T |u(\widehat{\mu}^\varepsilon - \widehat{\mu})|\, dt \\
\leq \; & \int_s^T E\,|u(\widehat{\mu}^\varepsilon - \widehat{\mu})|\, dt \\
\leq \; & \int_s^T E|u| \cdot |\bar{p}^\varepsilon(t) - \bar{p}(t)| \cdot |\nu| \cdot |\mu| dt.
\end{aligned}$$

Note that

$$|\widehat{\mu}^\varepsilon(t) - \widehat{\mu}(t)| = |\bar{p}^\varepsilon(t) - \bar{p}(t)| \cdot |\nu| \cdot |\mu|.$$

By Theorem 7.0.1, $\bar{p}^\varepsilon(t)$ converges weakly to $\bar{p}(\cdot)$ in $C([0,T];\mathbb{R}^l)$, as $\varepsilon \to 0$. This proves $E|\overline{Y}_u^\varepsilon(T) - \overline{Y}_u(T)| \to 0$ as $\varepsilon \to 0$, which means that $|\bar{v}^\varepsilon(s,z,\bar{p}) - \bar{v}^0(s,z,\bar{p})| \to 0$ as $\varepsilon \to 0$.

As we discussed before, the limit problem has lower dimension, therefore, its optimal control is much easier to obtain computationally. Once we know the optimal control, or nearly-optimal control, for the limit problem, how do we decide the optimal control for the original problem? We can construct a nearly-optimal control of the original asset allocation problem from the nearly-optimal control of the limit problem. Suppose $\bar{u}(t, \bar{Z}(t), \bar{p}(t))$ is the nearly-optimal control for the limit problem such that

$$|J^0(s,z,\bar{p},\bar{u}(\cdot)) - \bar{v}^0(s,z,\bar{p})| < \delta,$$

for any $\delta > 0$. Assume $\bar{u}(t, \bar{Z}(t), \bar{p}(t))$ is Lipschitz we can prove that a nearly-optimal control of the original problem can be constructed as the following

$$u(t, Z^\varepsilon(t), p^\varepsilon(t)) = \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)), \tag{7.12}$$

where

$$p^\varepsilon(t) = (p_{11}^\varepsilon, \cdots, p_{1m_1}^\varepsilon, \cdots, p_{l1}^\varepsilon, \cdots, p_{lml}^\varepsilon),$$

and

$$\bar{p}^\varepsilon(t) = (p^\varepsilon_{11} + \cdots + p^\varepsilon_{1m_1}, \cdots, p^\varepsilon_{l1} + \cdots + p^\varepsilon_{lml}).$$

Theorem 7.0.2. *With the limit problem, the original problem, and the utility functions stated as above, if the nearly-optimal control for the limit problem is Lipschitz, then the control defined in (7.12) is $\delta-$optimal for the original problem for sufficient small ε.*

Proof. Under the constructed control, the state $Z^\varepsilon(t)$ satisfies

$$\begin{aligned} dZ^\varepsilon(t) &= [(1 - \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)))r + \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\widehat{\alpha}^\varepsilon(t) - \frac{1}{2}(\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\sigma)^2]dt \\ &\quad + \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\sigma d\hat{v}, \\ dp^\varepsilon(t) &= p^\varepsilon(t)Q^\varepsilon dt + \frac{p^\varepsilon(t)A^\varepsilon(t)}{\sigma} d\hat{v}. \end{aligned}$$

For the limit problem, the corresponding system dynamics are given by

$$\begin{aligned} d\overline{Z}(t) &= [(1 - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t)))r + \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\mu}(t) - (1/2)(\bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\sigma)^2]dt \\ &\quad + \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\sigma d\hat{v}, \\ d\bar{p}(t) &= \bar{p}(t)\overline{Q}dt + \frac{\bar{p}(t)\overline{A}}{\sigma} d\hat{v}, \end{aligned}$$

where $\widehat{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} \mu(s_{ij})p^\varepsilon_{ij}(t) = p^\varepsilon(t)\mu$, and $\widehat{\mu}(t) = \bar{p}(t)\nu\mu$. Here μ is to denote the column vector $(\mu_1, \mu_2, \cdots, \mu_m)'$. Notice that

$$\begin{aligned} d(Z^\varepsilon(t) - \overline{Z}(t)) &= \Big[(\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)) - \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)))\, r \\ &\quad + \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\widehat{\alpha}^\varepsilon(t) - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\mu}(t) \\ &\quad + \frac{1}{2}(\bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\sigma)^2 - \frac{1}{2}(\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\sigma)^2\Big]\, dt \\ &\quad + (\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)) - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t)))\sigma d\hat{v}. \end{aligned}$$

Hence,

$$Z^\varepsilon(t) - \overline{Z}(t) = \int_s^t A^\varepsilon(t)dt + \int_s^t B^\varepsilon(t)d\hat{v},$$

where

$$A^\varepsilon(t) = (\bar{u}(t, Z_t^\varepsilon, \bar{p}_t) - \bar{u}(t, Z_t^\varepsilon, \bar{p}_t^\varepsilon))\, r + \bar{u}(t, Z_t^\varepsilon, \bar{p}_t^\varepsilon)\widehat{\alpha}_t^\varepsilon - \bar{u}(t, Z_t^\varepsilon, \bar{p}_t)\widehat{\mu}_t$$
$$+\frac{1}{2}(\bar{u}(t, Z_t^\varepsilon, \bar{p}_t)\sigma)^2 - \frac{1}{2}(\bar{u}(t, Z_t^\varepsilon, \bar{p}_t^\varepsilon)\sigma)^2.$$

If $\bar{u}(t, z, \bar{p})$ is Lipschitz, then

$$|\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)) - \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)| \le K|\bar{p}(t) - \bar{p}^\varepsilon(t)|,$$

$$\begin{aligned} & |\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\widehat{\alpha}^\varepsilon(t) - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\mu}(t)| \\ \le\ & |\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))\widehat{\alpha}^\varepsilon(t) - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\alpha}^\varepsilon(t)| \\ & +|\bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\alpha}^\varepsilon(t) - \bar{u}(t, Z^\varepsilon(t), \bar{p}(t))\widehat{\mu}(t)| \\ \le\ & K|\bar{p}^\varepsilon(t) - \bar{p}(t)| + |\widehat{\alpha}^\varepsilon(t) - \widehat{\mu}(t)| \\ =\ & K|\bar{p}^\varepsilon(t) - \bar{p}(t)| + |(p^\varepsilon(t) - \bar{p}(t)\nu)\mu|. \end{aligned}$$

Therefore,

$$\begin{aligned} & |(\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)))^2 - (\bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t)))^2| \\ =\ & |[\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)) - \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))][\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)) + \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))]| \\ \le\ & 2|\bar{u}(t, Z^\varepsilon(t), \bar{p}(t)) - \bar{u}(t, Z^\varepsilon(t), \bar{p}^\varepsilon(t))|. \end{aligned}$$

As proved in [17], $p^\varepsilon(t) \Rightarrow \bar{p}(t)\nu, \bar{p}^\varepsilon(t) = p^\varepsilon(t)\tilde{\mathbb{1}} \Rightarrow \bar{p}(t)$, as $\varepsilon \to 0$. Here "$\Rightarrow$" denote weakly convergence. In view of Skorohod representation, we may assume $p^\varepsilon(t) \to \bar{p}(t)\nu$ w.p.1, and $\bar{p}^\varepsilon(t) \to \bar{p}(t)$ w.p.1.

Therefore,

$$E\left[Z^\varepsilon(t) - \overline{Z}(t)\right] = \int_s^t A^\varepsilon(t)dt \to 0, \text{ as } \varepsilon \to 0.$$

It follows that

$$|E\Phi(e^{Z^\varepsilon(T)}) - E\Phi(e^{\overline{Z}(T)})| \to 0, \text{ as } \varepsilon \to 0.$$

Consequently, we have

$$\begin{aligned}
& |J^\varepsilon(s,z,p^\varepsilon,u(\cdot)) - v^\varepsilon(s,z,p^\varepsilon)| \\
\leq\ & |J^\varepsilon(s,z,p^\varepsilon,u(\cdot)) - J^0(s,z,\bar{p},\bar{u}(\cdot))| \\
& +|J^0(s,z,\bar{p},\bar{u}(\cdot)) - \bar{v}^0(s,z,\bar{p})| \\
& +|\bar{v}^0(s,z,\bar{p}) - v^\varepsilon(s,z,p^\varepsilon)| \to 0, \text{ with } \bar{p} = p^\varepsilon\tilde{\mathbb{1}}.
\end{aligned}$$

□

Chapter 8

Optimal selling rule

In practice, a key step in stock speculation is the timing for selling. Setting up a selling rule is necessary for profitability. A selling rule can be given using a target price range like in [16], or an optimal selling time as in [18]. In [16], a policy based on a target price and a stop-loss price is obtained by solving a set of two-point boundary value differential equations. In [18] , a strategy is constructed for "bubble stocks" so that the investor can decide when to sell a stock that has a rapid growth rate and then a rapid rate of decline by computing the probability of the positive growth rate and sell the stock when this probability becomes lower.

Using a regime switching model to describe the stock price, we are going to compute an optimal selling rule through variational inequality sufficient condition and nonlinear filtering similar to what has been done in [18]. Our goal, however, is to extend the result in [18] to incorporate more general case.

Since Wonham filter has proved to be quite efficient in turning a partially observable problem into a completely observable one, in the rest of this paper, we will work on the optimal selling problem with partial observation.

Let's first state the variational inequality sufficient conditions for an optimal stopping problem. Let z be an n-dimensional vector, $F(z)$ an n-dimensional vector valued function and $\Sigma(z)$ an $(n\times m)$-dimensional matrix valued function of z. Let $W(t)$ be an m-dimensional Brownian motion process. Suppose $F(z)$ and $\Sigma(z)$ are regular enough so that solutions of the stochastic differential equation and initial condition

$$dz(t) = F(z(t))dt + \Sigma(z(t))dW(t), \;\; z(0) = z, \tag{8.1}$$

exist and are unique. Let $\mathcal{F}_t$ denote the $\sigma-$fields

$$\mathcal{F}_t = \sigma\{z(r) : 0 \leq r \leq t\}$$

generated by the past of $z(t)$. Let $U(z)$ be a twice continuously differentiable utility function. Let $\mathcal{A}$ denote a class of $\mathcal{F}_t$ stopping times. For each stopping time $\tau \in \mathcal{A}$, consider the expected utility

$$E[U(z(\tau))]. \tag{8.2}$$

The optimal stopping problem is to find τ in $\mathcal{A}$ which achieves maximum of (8.2).

We use the following variational inequality sufficient conditions for optimality for this problem. This is a simpler version than the one given in [18].

Theorem 8.0.3. *Let R be a region in E^n. Assume for each z in R that the solution of (8.1) with initial condition z is contained in R. Let $V(z)$ be a scalar valued function defined on R. Let $V(z)$ be regular enough so that Itô's stochastic differential rule holds for $V(z(t))$. Define the differential operator $A[V](z)$ by*

$$A[V](z) = V_z(z)F(z) + \frac{1}{2}\text{tr}(\Sigma(z)\Sigma(z)'V_{zz}(z)).$$

Let $V(z)$ be a solution of the variational inequality

$$A[V](z) \leq 0, \; V(z) \geq U(z),$$

$$(V(z) - U(z))A[V](z) = 0. \tag{8.3}$$

and let the condition

$$E\left[\int_0^\tau \|V_z(z(t))\Sigma(z(t))\|^2 dt\right] < \infty \tag{8.4}$$

hold for each stopping time τ in $\mathcal{A}$. For $z(t)$ the solution of (8.1) with initial condition z, let

$$\tau(z) = \textit{ first time } z(t) \textit{ hits } \{q : V(q) = U(q)\}. \tag{8.5}$$

Let

$$\tau(z) \in \mathcal{A} \textit{ for each } z \in R. \tag{8.6}$$

Then

$$V(z) = E[U(z(\tau(z)))] = \max_{\tau \in \mathcal{A}} E[U(z(\tau))].$$

That is, $\tau(z)$ is an optimal stopping time in $\mathcal{A}$ and $V(z)$ is the value function for the optimal stopping problem.

To satisfy the boundary condition (8.4), we see that if

$$\|V_z(z(t))\Sigma(z(t))\|^2 \leq K, \tag{8.7}$$

then

$$E\left[\int_0^\tau \|V_z(z(t))\Sigma(z(t))\|^2 dt\right] < KE(\tau),$$

which mean the condition (8.4) is satisfied for stopping times with finite expectations as long as (8.7) holds.

Let $S(t)$ denote the price of a stock at time t. It satisfies

$$\begin{aligned} dS(t) &= \mu(\alpha(t))S(t)dt + \sigma S(t)dW(t), \\ S(0) &= S_0, \;\; t \geq 0, \end{aligned}$$

where $S_0 > 0$ is the initial price, $\mu(i)$ is the expected return rate, σ is a constant, representing the stock volatility, $\alpha(t)$ is the markov process with generator Q, and $W(t)$ is a standard Brownian motion. The processes$\alpha(t)$ and $W(t)$ are independent.

Let $U(S)$ be a utility function. Let

$$\mathcal{F}_t = \sigma\{S(r) : 0 \leq r \leq t\}$$

denote the $\sigma-$fields generated by the past of the process $S(\cdot)$ up to times t.

We consider the problem of finding a $\mathcal{F}_t$ stopping time τ which maximizes the expected utility

$$E[U(S(\tau))].$$

This is an optimal stopping problem with partial observation.

Let $X(t)$ be the log price, i.e, $S(t) = S_0 \exp(X(t))$, then

$$\begin{aligned} dX(t) &= \left[\mu(\alpha(t)) - \frac{\sigma^2}{2}\right] dt + \sigma dW(t) \\ X(0) &= 0, \ \ t \geq 0 \end{aligned}$$

Let $p_i(t)$ denote the conditional probability of $\alpha(t) = i$ given the observations of $X(t)$ up to time t, i.e.,

$$p_i(t) = P(\alpha(t) = i | X(s) : s \leq t);$$

for $i = 1, ..., m$. Let $p(t) = (p_1(t), ..., p_m(t)) \in \mathcal{R}^{1\times m}$.

Since the value of $X(t)$ is observable and it is a function of $\alpha(t)$, we can set up a Wonham filter for $\alpha(t)$ by the following

$$dp(t) = p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dX(t), \tag{8.8}$$

$p(0) = p$, being the initial probability, where

$$\begin{aligned} A(t) &= \text{diag}(\mu(1) - \frac{\sigma^2}{2}, ..., \mu(m) - \frac{\sigma^2}{2}) - \sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)I. \\ &= \text{diag}(\mu(1) - \frac{\sigma^2}{2}, ..., \mu(m) - \frac{\sigma^2}{2}) - \sum_{i=1}^{m}\mu(i)p_i(t)I + \frac{\sigma^2}{2}\sum_{i=1}^{m}p_i(t)I. \\ &= \text{diag}(\mu(1), ..., \mu(m)) - \textstyle\sum_{i=1}^{m}\mu(i)p_i(t)I. \end{aligned}$$

Denote $\tilde{\alpha}(t) = \sum_{i=1}^{m}[\mu(i) - \frac{\sigma^2}{2}]p_i(t)$, we then have

$$\begin{aligned} dp(t) &= p(t)Qdt - \frac{1}{\sigma^2}\tilde{\alpha}(t)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dX(t) \\ &= p(t)Qdt + \frac{p(t)A(t)}{\sigma}\left(\frac{dX(t) - \tilde{\alpha}(t)dt}{\sigma}\right). \end{aligned} \tag{8.9}$$

Let

$$d\hat{v} = \frac{dX(t) - \tilde{\alpha}dt}{\sigma}.$$

As proved before, $\hat{v}$ is an innovation process. Moreover, we can write both X and p in terms of $\hat{v}$.

$$\begin{aligned} dX(t) &= \tilde{\alpha}(t)dt + \sigma d\hat{v},\ X(0) = 0, \\ dp(t) &= p(t)Qdt + \frac{p(t)A(t)}{\sigma} d\hat{v},\ p(0) = p. \end{aligned} \tag{8.10}$$

If the number of market modes is 2, $\mathcal{M} = \{1, 2\}$. Assume μ_1 is the bull market rate of return and μ_2 is the bear market rate of return, $2\mu_1 > \sigma^2 > 2\mu_2$. Assume $\alpha(t)$ has a generator

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix},$$

then

$$A(t) = \begin{bmatrix} \mu_1 - \mu_1 p_1(t) - \mu_2 p_2(t) & 0 \\ 0 & \mu_2 - \mu_1 p_1(t) - \mu_2 p_2(t) \end{bmatrix}$$

$$A(t) = \begin{bmatrix} \mu_1 p_2(t) - \mu_2 p_2(t) & 0 \\ 0 & \mu_2 p_1(t) - \mu_1 p_1(t) \end{bmatrix},$$

or

$$A(t) = \begin{bmatrix} (\mu_1 - \mu_2) p_2(t) & 0 \\ 0 & -(\mu_1 - \mu_2) p_1(t) \end{bmatrix}.$$

Therefore,

$$p(t)Q = \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 p_1(t) + \lambda_2 p_2(t) & \lambda_1 p_1(t) - \lambda_2 p_2(t) \end{bmatrix},$$

$$p(t)A(t) = \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} (\mu_1 - \mu_2) p_2(t) & 0 \\ 0 & -(\mu_1 - \mu_2) p_1(t) \end{bmatrix}$$

$$\tilde{\alpha}(t) = \mu_1 p_1(t) + \mu_2 p_2(t) - \frac{\sigma^2}{2}$$

It follows that

$$\begin{aligned} dX(t) &= \left(\mu_1 p_1(t) + \mu_2 p_2(t) - \frac{\sigma^2}{2}\right) dt + \sigma d\hat{v}, \\ dp_1(t) &= (-\lambda_1 p_1(t) + \lambda_2 p_2(t))dt + \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)p_2(t))d\hat{v}, \\ dp_2(t) &= (\lambda_1 p_1(t) - \lambda_2 p_2(t))dt - \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)p_2(t))d\hat{v}. \end{aligned} \tag{8.11}$$

Since $p_1(t) + p_2(t) = 1$ we can replace p_2 by $1 - p_1$,

$$dX(t) = \left(\mu_1 p_1(t) + \mu_2 - \mu_2 p_1(t) - \frac{\sigma^2}{2}\right) dt + \sigma d\hat{v}, \tag{8.12}$$

$$dp_1(t) = (-\lambda_1 p_1(t) + \lambda_2 - \lambda_2 p_1(t))dt + \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)(1 - p_1(t))d\hat{v}. \tag{8.13}$$

In the notation of Theorem 8.0.3,

$$z = \begin{pmatrix} X \\ \\ p_1 \end{pmatrix}, \Sigma(z) = \begin{pmatrix} \sigma \\ \\ \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)(1 - p_1(t)) \end{pmatrix}.$$

$$\begin{aligned} A[V](X, p_1) = \quad & V_X \left(\mu_1 p_1(t) + \mu_2 - \mu_2 p_1(t) - \frac{\sigma^2}{2}\right) + V_p(-\lambda_1 p_1(t) + \lambda_2 - \lambda_2 p_1(t)) \\ & \frac{1}{2}\sigma^2 V_{XX} + (\mu_1 - \mu_2)p_1(t)(1 - p_1(t))V_{Xp} \\ & \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1 - p_1(t))^2 p_1(t)^2 V_{pp} \end{aligned}$$

$$V_z(z)\Sigma(z) = \sigma V_X(X, p_1) + \frac{\mu_1 - \mu_2}{\sigma}(1 - p_1(t))p_1(t)V_p.$$

The optimization problem is to choose the stopping time τ to maximize

$$E[U(S(\tau))], \quad \text{or } E[U(e^{X(\tau)})]$$

subject to

$$\begin{aligned} dS(t) &= S(t)\left(\mu_1 p_1(t) + \mu_2 - \mu_2 p_1(t)\right) dt + S(t)\sigma d\hat{v}, \\ dp_1(t) &= (-\lambda_1 p_1(t) + \lambda_2 - \lambda_2 p_1(t))dt + \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)(1 - p_1(t))d\hat{v}. \end{aligned} \tag{8.14}$$

Recall the notation of Theorem 8.0.3,

$$z = \begin{pmatrix} S \\ \\ p_1 \end{pmatrix}, \quad \Sigma(z) = \begin{pmatrix} S\sigma \\ \\ \frac{1}{\sigma}((\mu_1 - \mu_2)p_1(t)(1-p_1(t)) \end{pmatrix}.$$

We also have

$$\begin{aligned} A[V](S,p_1) = \quad & V_S S\left(\mu_1 p_1 + \mu_2 - \mu_2 p_1\right) + V_p(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1) \\ & + \frac{1}{2} S^2 \sigma^2 V_{SS} + (\mu_1 - \mu_2) p_1 (1-p_1) S V_{Sp} \\ & + \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1-p_1)^2 p_1^2 V_{pp} \end{aligned}$$

$$V_z(z)\Sigma(z) = S\sigma V_S(S,p_1) + \frac{\mu_1 - \mu_2}{\sigma}(1-p_1(t))p_1(t)V_p.$$

Consider the utility function

$$U(S) = \ln(S)$$

and class of admissible $\mathcal{F}_t$ stopping times τ given by

$$\mathcal{A} = \{\tau : E(\tau) < \infty\}.$$

First we need to know if there are conditions under which it is optimal to sell the stock at $\tau = 0$ or it is optimal not to sell the stock at all. Suppose the rate of return is either μ_1 or μ_2 and $\mu_1 > \mu_2$. Then (8.12) implies that

$$E[\ln(S(t))] - \ln(S) = \mu_1 p_1(t) + \mu_2 - \mu_2 p_1(t) - \frac{\sigma^2}{2} \tag{8.15}$$

If $\sigma^2/2 \geq \mu_1$, then $\mu_1 - \mu_2 \leq \sigma^2/2 - \mu_2$. The right hand side of the above inequality will be

$$(\mu_1 - \mu_2)p_1(t) + \mu_2 - \frac{\sigma^2}{2} \leq (\frac{\sigma^2}{2} - \mu_2)p_1(t) + \mu_2 - \frac{\sigma^2}{2} = (\frac{\sigma^2}{2} - \mu_2)(p_1(t) - 1) \leq 0.$$

So $E[\ln(S(t))] \leq \ln(S)$. In this case, it is optimal to sell the stock immediately.

If $\sigma^2/2 \leq \mu_2$, then the right hand side of the inequality (8.15) will be

$$(\mu_1 - \mu_2)p_1(t) + \mu_2 - \frac{\sigma^2}{2} \geq \mu_2 - \frac{\sigma^2}{2} \geq 0$$

No matter what $p_1(t)$ will be, a positive rate of return is guaranteed. So there is no reason to sell the stock.

Assume $\mu_1 > \sigma^2/2 > \mu_2$ and let us find the optimal selling time in class $\mathcal{A}$ for this condition. We can assume $V(S, p_1)$ takes the form

$$V(S, p_1) = \ln(S) + f(p_1),$$

and try to find an appropriate $f(x)$.

$$A[\ln(S) + f(p_1)](S, p_1) = \mu_1 p_1 + \mu_2 - \mu_2 p_1 + f'(p_1)(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1) - \frac{1}{2}\sigma^2 + \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1-p_1)^2 p_1^2 f''(p_1). \tag{8.16}$$

Denote the right hand side as $B[f](p_1)$, the variational inequality reduces to

$$B[f](p_1) < 0, \;\; f(p_1) \geq 0, \;\; \text{and } f(p_1)B[f](p_1) = 0. \tag{8.17}$$

Note that

$$\begin{aligned} V_z(z)\Sigma(z) &= S\sigma V_S(S, p_1) + \frac{\mu_1 - \mu_2}{\sigma}(1-p_1)p_1 V_p \\ &= \sigma + \frac{\mu_1 - \mu_2}{\sigma}(1-p_1)p_1 f'(p_1), \end{aligned} \tag{8.18}$$

which is bounded if $f'(p_1)$ is bounded.

For Itô's differential rule to hold for $\ln(S(t)) + f(p_1(t))$, $f(p_1)$ must be at least once continuously differentiable. This implies: If $q \in (0, 1)$, and if q is a boundary point of an interval on which $f(p_1) = 0$, then $f'(q) = 0$.

So we look for a continuously differentiable $f(p_1)$ of (8.17) for which $f'(p_1)$ is bounded and $f'(q) = 0$ at boundary points q of intervals on which $f(x) = 0$.

Conditions (8.17) imply that if $f(p_1) \neq 0$, then $B[f](p_1) = 0$. From (8.16) this equation is

$$\mu_1 p_1 + \mu_2 - \mu_2 p_1 + f'(p_1)(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1) - \frac{1}{2}\sigma^2 + \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1-p_1)^2 p_1^2 f''(p_1) = 0$$

Since it is only involved $f'(p_1)$ and $f''(p_1)$, we will set $r(p_1) = f'(p_1)$ and solve

$$\mu_1 p_1 + \mu_2 - \mu_2 p_1 + r(p_1)(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1) - \frac{1}{2}\sigma^2$$
$$+\frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1-p_1)^2 p_1^2 r'(p_1) = 0.$$

Simplify this equation, then we have

$$\frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1-p_1)^2 p_1^2 r'(p_1) + r(p_1)(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1) = \frac{1}{2}\sigma^2 - \mu_1 p_1 - \mu_2 + \mu_2 p_1.$$

Denote $h = \frac{2\sigma^2}{(\mu_1 - \mu_2)^2}, k = \lambda_1 + \lambda_2, c = \mu_1 - \mu_2$ to shorten notation, we then have

$$\frac{(1-p_1)^2 p_1^2}{h} r'(p_1) + r(p_1)(\lambda_2 - kp_1) = \frac{1}{2}\sigma^2 - cp_1 - \mu_2.$$

Divide both sides by $(1-p_1)^2 p_1^2/h$, we have a classical first order ODE

$$r'(p_1) + r(p_1)\frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} = \frac{\sigma^2 h - 2hcp_1 - 2h\mu_2}{2(1-p_1)^2 p_1^2}.$$

The general solution for this equation is

$$r(p_1) = e^{-\int \frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} dp_1} \left[\int \frac{\sigma^2 h - 2hcp_1 - 2h\mu_2}{2(1-p_1)^2 p_1^2} e^{\int \frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} dp_1} dp_1 + C\right].$$

In order to have the desired boundary condition, we have to compute on the integrations.

$$\frac{1}{(1-p)p} = \frac{1}{p} + \frac{1}{1-p},$$
$$\text{so, } \frac{1}{(1-p)^2 p^2} = \left(\frac{1}{p} + \frac{1}{1-p}\right)^2 = \frac{1}{p^2} + \frac{1}{(1-p)^2} + \frac{2}{p} + \frac{2}{1-p}.$$

Therefore, the integral

$$\int \frac{1}{(1-p_1)^2 p_1^2} dp_1 = -\frac{1}{p_1} + \frac{1}{1-p_1} + 2\ln p_1 - 2\ln(1-p_1) = \frac{2p_1 - 1}{p_1(1-p_1)} + 2\ln\frac{p_1}{1-p_1}.$$

On the other hand,

$$\frac{p}{(1-p)^2 p^2} = \frac{1}{1-p}\left(\frac{1}{(1-p)p}\right) = \frac{1}{(1-p)p} + \frac{1}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p} + \frac{1}{(1-p)^2}.$$

So its indefinite integral is

$$\int \frac{p_1}{(1-p_1)^2 p_1^2} dp_1 = \ln p_1 - \ln(1-p_1) + \frac{1}{1-p_1} = \ln \frac{p_1}{1-p_1} + \frac{1}{1-p_1}.$$

Therefore, combine the two integration above, we then have

$$\int \frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} dp_1 = h\lambda_2 \left(\frac{2p_1 - 1}{p_1(1-p_1)} + 2\ln \frac{p_1}{1-p_1} \right) - hk \left(\ln \frac{p_1}{1-p_1} + \frac{1}{1-p_1} \right)$$

$$= (2h\lambda_2 - hk) \ln \frac{p_1}{1-p_1} + \frac{h(2\lambda_2 - k)p_1 - h\lambda_2}{p_1(1-p_1)}.$$

Recall $k = \lambda_1 + \lambda_2$. So $2\lambda_2 - k = \lambda_2 - \lambda_1$. We can further simplify the above integration as follows

$$\begin{aligned}
\int \frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} dp_1 &= h(\lambda_2 - \lambda_1) \ln \frac{p_1}{1-p_1} + \frac{h(\lambda_2 - \lambda_1)p_1 - h\lambda_2}{p_1(1-p_1)} \\
&= h(\lambda_2 - \lambda_1) \ln \frac{p_1}{1-p_1} - \frac{h(\lambda_1 - \lambda_2)}{1-p_1} - \frac{h\lambda_2}{p_1(1-p_1)} \\
&= h(\lambda_2 - \lambda_1) \ln \frac{p_1}{1-p_1} - \frac{h(\lambda_1 - \lambda_2)}{1-p_1} - h\lambda_2 [\frac{1}{p_1(1-p_1)}] \\
&= h(\lambda_2 - \lambda_1) \ln \frac{p_1}{1-p_1} - \frac{h(\lambda_1 - \lambda_2)}{1-p_1} - h\lambda_2 [\frac{1}{p_1} + \frac{1}{1-p_1}] \\
&= h(\lambda_2 - \lambda_1) \ln \frac{p_1}{1-p_1} - \frac{h\lambda_1}{1-p_1} - \frac{h\lambda_2}{p_1}.
\end{aligned}$$

So the general solution is

$$r(p_1) = \left(\frac{p_1}{1-p_1} \right)^{h(\lambda_1 - \lambda_2)} e^{\frac{h\lambda_1}{1-p_1} + \frac{h\lambda_2}{p_1}} \left[\int \frac{\sigma^2 h - 2hcp_1 - 2h\mu_2}{2(1-p_1)^2 p_1^2} e^{\int \frac{h(\lambda_2 - kp_1)}{(1-p_1)^2 p_1^2} dp_1} dp_1 + C \right]$$

$$= \left(\frac{p_1}{1-p_1} \right)^{h(\lambda_1 - \lambda_2)} e^{\frac{h\lambda_1}{1-p_1} + \frac{h\lambda_2}{p_1}} \left[\int h \frac{\sigma^2 - 2cp_1 - 2\mu_2}{2(1-p_1)^2 p_1^2} \left(\frac{p_1}{1-p_1} \right)^{-h(\lambda_1 - \lambda_2)} e^{-\frac{h\lambda_1}{1-p_1} - \frac{h\lambda_2}{p_1}} dp_1 + C \right].$$

Since $h > 0$,

$$\lim_{p_1 \to 1} \left(\frac{p_1}{1-p_1} \right)^{h(\lambda_1 - \lambda_2)} e^{\frac{h\lambda_1}{1-p_1} + \frac{h\lambda_2}{p_1}} = +\infty.$$

In order for $r(p_1)$ to be bounded at $p_1 = 1$, we must have

$$\lim_{p_1 \to 1} \left[\int h \frac{\sigma^2 - 2cp_1 - 2\mu_2}{2(1-p_1)^2 p_1^2} \left(\frac{p_1}{1-p_1} \right)^{-h(\lambda_1 - \lambda_2)} e^{-\frac{h\lambda_1}{1-p_1} - \frac{h\lambda_2}{p_1}} dp_1 + C \right] = 0.$$

Define $g(p_1)$ by

$$g(p_1) = \int_{p_1}^{1} -h\frac{\sigma^2 - 2cx - 2\mu_2}{2(1-x)^2x^2}\left(\frac{x}{1-x}\right)^{-h(\lambda_1-\lambda_2)} e^{-\frac{h\lambda_1}{1-x}-\frac{h\lambda_2}{x}}dx.$$

Let

$$r(p_1) = \left(\frac{p_1}{1-p_1}\right)^{h(\lambda_1-\lambda_2)} e^{\frac{h\lambda_1}{1-p_1}+\frac{h\lambda_2}{p_1}} g(p_1).$$

Calculation using L'hospital rule shows that

$$\lim_{p_1\to 1} \frac{g(p_1)}{(1-p_1)^{h(\lambda_1-\lambda_2)}e^{-\frac{h\lambda_1}{1-p_1}-\frac{h\lambda_2}{p_1}}} = \frac{-\sigma^2+2c+2\mu_2}{2\lambda_1} = \frac{2\mu_1-\sigma^2}{2\lambda_1}.$$

So

$$\lim_{p_1\to 1} r(p_1) = \frac{2\mu_1-\sigma^2}{2\lambda_1}.$$

To ensure that $r(p_1)$ is bounded and continuous, we look at the function $g(p_1)$'s behavior. Let denote the integrand of $g(p_1)$ as $j(x)$. Then

$$j(x) = h\frac{2(\mu_1-\mu_2)x + 2\mu_2 - \sigma^2}{2(1-x)^2x^2}\left(\frac{x}{1-x}\right)^{-h(\lambda_1-\lambda_2)} e^{-\frac{h\lambda_1}{1-x}-\frac{h\lambda_2}{x}}.$$

It is given by positive quantities times the linear term

$$2(\mu_1-\mu_2)x + 2\mu_2 - \sigma^2. \tag{8.19}$$

Now (8.19) will be positive on

$$\frac{\sigma^2-2\mu_2}{2(\mu_1-\mu_2)} < x \leq 1.$$

Therefore,

$$g(p_1) > 0 \text{ if } \frac{\sigma^2-2\mu_2}{2(\mu_1-\mu_2)} < p_1 < 1.$$

Since $2\mu_1 > \sigma^2$, (8.19) is positive near $x = 1$. Since $\sigma^2 > 2\mu_2$, (8.19) is negative near $x = 0$.

Also notice that

$$\lim_{x\to 0} j(x) = 0,\ \lim_{x\to 1} j(x) = 0.$$

Denote $z = (\sigma^2 - 2\mu_2)/[2(\mu_1-\mu_2)]$, then $0 \leq z \leq 1$.

To decide if $g(p_1)$ has a root in $(0,1)$ we need to compare the value of $I = \int_z^1 j(x)dx$ and $J = \int_0^z -j(x)dx$. Let $u = 1-x$,

$$I = \int_0^{1-z} h\frac{2(\mu_1-\mu_2)(1-u)+2\mu_2-\sigma^2}{2u^2(1-u)^2}\left(\frac{1-u}{u}\right)^{-h(\lambda_1-\lambda_2)} e^{-\frac{h\lambda_1}{u}-\frac{h\lambda_2}{1-u}}du.$$

Note that

$$\lim_{p_1\to 0} g(p_1) < 0,$$

if $I < J$. A sufficient condition for this to happen is $z > \frac{1}{2}$, which means

$$\frac{\sigma^2-2\mu_2}{2(\mu_1-\mu_2)} > \frac{1}{2}.$$

Simplifying this, we have

$$\sigma^2 > \mu_1+\mu_2.$$

To see that $I < J$, we check

$$J = \int_0^z h\frac{\sigma^2-2\mu_2-2(\mu_1-\mu_2)u}{2u^2(1-u)^2}\left(\frac{1-u}{u}\right)^{h(\lambda_1-\lambda_2)} e^{-\frac{h\lambda_1}{u}-\frac{h\lambda_2}{1-u}}dx$$

$$I = \int_0^{1-z} h\frac{2\mu_1-\sigma^2-2(\mu_1-\mu_2)u}{2u^2(1-u)^2}\left(\frac{u}{1-u}\right)^{h(\lambda_1-\lambda_2)} e^{-\frac{h\lambda_1}{u}-\frac{h\lambda_2}{1-u}}du.$$

Both I and J has positive integrand. J has larger integrand and longer integration range if $z > 1/2$.

So under the condition $2\mu_1 > \sigma^2 > 2\mu_2$ and $\sigma^2 > \mu_1+\mu_2$, we have the following result:

Lemma 8.0.4. *$r(p_1)$ has a unique root p^* in (0,1) which satisfies*

$$0 < p^* < \frac{\sigma^2-2\mu_2}{2(\mu_1-\mu_2)},$$

and $r(p_1)$ is positive on $(p^*, 1]$.

Proof. On the interval (0,1) the function $r(p_1)$ is given by positive quantities times $g(p_1)$, its roots should be the same as those of $g(p_1)$. The integrand of $g(p_1)$ is given by positive quantities times the linear term

$$2(\mu_1-\mu_2)x+2\mu_2-\sigma^2. \tag{8.20}$$

Now (8.19) will be positive on

$$\frac{\sigma^2 - 2\mu_2}{2(\mu_1 - \mu_2)} < x \leq 1.$$

Therefore,

$$g(p_1) > 0 \text{ if } \frac{\sigma^2 - 2\mu_2}{2(\mu_1 - \mu_2)} < p_1 < 1.$$

Since $2\mu_1 > \sigma^2$, (8.19) is positive near $x = 1$. Since $\sigma^2 > 2\mu_2$, (8.19) is negative near $x = 0$. Also

$$\lim_{p_1 \to 0} g(p_1) < 0.$$

Thus $g(p_1)$ and $r(p_1)$ must have a root p^* in $(0, \frac{\sigma^2 - 2\mu_2}{2(\mu_1 - \mu_2)})$. Since $g(p_1)$ is monotone increasing on this interval, p^* is unique. This implies $r(p_1)$ is positive on $(p^*, 1]$. □

Lemma 8.0.5. *For q in (0,1), let*

$$T(q) = \textit{ first time } p_1(t) \textit{ hits } [0, q].$$

Then

$$E(T(q)) < \infty.$$

Proof. We can assume the initial probability $p > q$; otherwise $T(q) = 0$ and $E[T(q)] = 0$.

For $p_1(t)$ given by

$$dp_1 = (-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1)dt + \frac{(\mu_1 - \mu_2)p_1(1 - p_1)}{\sigma} d\hat{v},$$

based on the proof of Theorem 2 on page 149 of [9], we have

$$P[p_1(t) < 1] = 1. \tag{8.21}$$

A solution $K(p_1)$ of the differential equation

$$(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1)K'(p_1) + \frac{1}{2}r^2(1 - p_1)^2 p_1^2 K''(p_1) + 1 = 0 \tag{8.22}$$

on [0,1] satisfying $K(q) = 0$ and $K'(p_1)$ bounded on $[q, 1]$ is given by

$$K(p_1) = \int_q^{p_1} \left[\frac{2}{r^2} e^{\frac{2}{r^2}(\frac{\lambda_1}{1-z} + \frac{\lambda_2}{z})} \left(\frac{z}{1-z} \right)^{\frac{2(\lambda_1 - \lambda_2)}{r^2}} \int_z^1 \left(\frac{1-y}{y} \right)^{\frac{2(\lambda_1 - \lambda_2)}{r^2}} \frac{e^{\frac{2}{r^2}(-\frac{\lambda_1}{1-y} - \frac{\lambda_2}{y})}}{(1-y)^2 y^2} dy \right] dz.$$

A derivation of this solution is included in the appendix.

Apply L'Hospital's rule to $K'(p_1)$, then we have

$$\lim_{p_1 \to 1} K'(p_1) = \frac{1}{2\lambda_1}.$$

So $K'(p_1)$ is bounded. Also note $K(p_1) \geq 0$ for $p_1 \geq q$.

Let

$$T(q) = \inf\{t : \;\; p_1(t) = q\}, \tag{8.23}$$

and for a fixed time T greater than 0, define

$$\tau_T = \min(T, T(q)).$$

From (8.21) and (8.23), we see that $p_1(s)$ is contained in $[q, 1]$ for $s < \tau_T$. Itô's formula implies that

$$\begin{aligned} K(p_1(\tau_T)) - K(p_1) \;\; &= \;\; \frac{\mu_1 - \mu_2}{\sigma} \int_0^{\tau_T} (1 - p_1(s))p_1(s)K'(p_1(s))dW(s) \\ &+ \int_0^{\tau_T} [(-\lambda_1 p_1 + \lambda_2 - \lambda_2 p_1(s))K'(p_1(s)) \\ &+ \frac{1}{2}\left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (1 - p_1(s))^2 p_1(s)^2 K''(p_1(s))\big] \, ds. \end{aligned}$$

Let $r = \frac{\mu_1 - \mu_2}{\sigma}$, then $K(p_1)$ being the solution of (8.22) implies that

$$K(p_1(\tau_T)) - K(p_1) = -\tau_T + \frac{\mu_1 - \mu_2}{\sigma} \int_0^{\tau_T} (1 - p_1(s))p_1(s)K'(p_1(s))dW(s).$$

Since the integrand in the stochastic integral is bounded and τ_T is bounded, the expected value of the stochastic integral is zero. Therefore

$$E[\tau_T] = K(p_1) - E[K(p_1(\tau_T))].$$

Let $T \to \infty$, then we have $\tau_T \to T(q)$. Since $K(p_1(\tau_T))$ is positive, we have

$$E[\tau_T] \leq K(p_1).$$

□

Theorem 8.0.6. *For $f(p_1)$ defined by*

$$f(p_1) = \begin{cases} 0 & \text{if } 0 \leq p_1 \leq p^*, \\ \int_{p^*}^{p_1} r(x)dx & \text{if } p^* \leq p_1 \leq 1. \end{cases} \tag{8.24}$$

the function $V(S, p_1) = \ln(S) + f(p_1)$ satisfies the conditions of Theorem 8.0.3, and

$$T(p^*) = 1st \text{ time } p_1(t) \text{ hits } [0, p^*] \tag{8.25}$$

is an optimal stopping time in the class $\mathcal{A}$.

Proof. Since $r(p^*) = 0$, $f(p_1)$ is continuously differentiable and is twice continuously differentiable, except at p^*. Hence Itô's differential rule holds for $\ln(S) + f(p_1)$. Lemma 8.0.4 implies that $f(p_1) \geq 0$. Notice that

$$B[f](p_1) = \begin{cases} \mu_1 p_1 + \mu_2 - \mu_2 p_1 - \frac{1}{2}\sigma^2 & \text{if } 0 \leq p_1 \leq p^*, \\ 0 & \text{if } p^* \leq p_1 \leq 1. \end{cases}$$

and since

$$p^* < \frac{\sigma^2 - 2\mu_2}{2(\mu_1 - \mu_2)},$$

we have that

$$B[f](p_1) \leq 0.$$

Since $f(p_1) = 0$ if $0 \leq p_1 \leq p^*$ and $B[f](p_1) = 0$ if $p^* \leq p_1 \leq 1$, $f(p_1)B[f](p_1) = 0$. Thus all the conditions of (8.17) are satisfied, which are equivalent to the conditions (8.3) for the function $V(S, p_1) = \ln(S) + f(p_1)$.

Because $r(p_1)$ is continuous on $[0, 1)$ and has a finite limit at $x = 1$, it is bounded. Because the boundedness of $f'(p_1)$ and equation (8.18), condition (8.4) is satisfied for stopping times in $\mathcal{A}$. Condition (8.6) follows from Lemma 8.0.5. Therefore, the conditions of Theorem 8.0.3 are satisfied and $T(p^*)$ in (8.25) is an optimal stopping time in $\mathcal{A}$. □

Remark Letting $\lambda_2 = 0$, the results here identical to that of [18], which is fairly reasonable. The price of " bubble stock" can also be considered as being modulated by a Markov chain process, but one with an absorbing state.

Chapter 9

Appendix

Definition The Markov chain or the generator Q is weakly irreducible if the system of equations

$$\nu Q = 0, \quad \text{and} \quad \sum_{i=1}^{m} \nu_i = 1$$

has a unique nonnegative solution. The nonnegative solution (row-vector-valued function) $\nu = (\nu_1, \ldots, \nu_m)$ is termed a quasi-stationary distribution. In addition, if ν is strictly positive, then we say the generator Q is irreducible.

Lemma 9.0.7. (*Gronwall's inequality.*) *Given a bounded measurable function $c(t)$, if*

$$0 \le h(t) \le c(t) + K \int_0^t h(u)du,$$

then

$$h(t) \le c(t) + K \int_0^t c(u)e^{K(t-u)}du.$$

Proof of Theorem 8.0.3:

Proof. By Itô's differential rule

$$dV(z(t)) = \left[V_z(z(t))F(z(t)) + \frac{1}{2}\text{tr}(\Sigma(z(t))\Sigma(z(t))' V_{zz}(z(t))) \right] dt + V_z(z(t))\Sigma(z(t))dW(t),$$

or,

$$V(z(t)) - V(z) = \int_0^t A[V](z(s))ds + \int_0^t V_z(z(s))\Sigma(z(s))dW(s).$$

This holds for each t, it also holds for a finite stopping time τ.

Condition (8.4) implies that for each stopping time τ in $\mathcal{A}$

$$E\left[\int_0^\tau V_z(z(s))\Sigma(z(s))dW(s) \right] = 0.$$

Hence for each τ in $\mathcal{A}$,

$$E\left[V(z(\tau))\right] = V(z) + \int_0^\tau A[V](z(s))ds. \tag{9.1}$$

From (8.3),

$$A[V](z) \le 0, \; V(z) \ge U(z),$$

therefore for each τ in $\mathcal{A}$,

$$E\left[U(z(\tau))\right] \le E\left[V(z(\tau))\right] \le V(z).$$

For

$$\tau(z) = \text{ first time } z(t) \text{ hits } \{q : V(q) = U(q)\},$$

we have

$$A[V](z(s)) = 0 \quad \text{on } 0 \le s < \tau(z)$$

and

$$U(z(\tau(z))) = V(z(\tau(z))).$$

Therefore (9.1) implies

$$E\left[U(z(\tau(z)))\right] = V(z).$$

□

Remark To solve the ODE, we do the following simplification

$$(-\lambda_1 p + \lambda_2 - \lambda_2 p)K'(p) + \frac{1}{2}r^2(1-p)^2p^2K''(p) + 1 = 0 \tag{9.2}$$

Since the equation involve only $K'(p)$ and $K''(p)$, let $y(p) = K'(p)$, we then have

$$(-\lambda_1 p + \lambda_2 - \lambda_2 p)y(p) + \frac{1}{2}r^2(1-p)^2p^2y'(p) + 1 = 0,$$

or

$$\frac{1}{2}r^2(1-p)^2p^2y'(p) + (-\lambda_1 p + \lambda_2 - \lambda_2 p)y(p) = -1.$$

Divide both sides by $r^2(1-p)^2p^2/2$, then

$$y'(p) + \frac{2(-\lambda_1 p + \lambda_2 - \lambda_2 p)}{r^2(1-p)^2p^2}y(p) = \frac{-2}{r^2(1-p)^2p^2}.$$

A general solution for this is

$$y(p) = e^{-\int \frac{2(-\lambda_1 p+\lambda_2-\lambda_2 p)}{r^2(1-p)^2p^2}dp}\left[\int \frac{-2}{r^2(1-p)^2p^2}e^{\int \frac{2(-\lambda_1 p+\lambda_2-\lambda_2 p)}{r^2(1-p)^2p^2}dp} + C\right].$$

To have $K'(p)$ bounded in $[q,1]$, we set

$$y(p) = \frac{2}{r^2}e^{\frac{2}{r^2}(\frac{\lambda_1}{1-p}+\frac{\lambda_2}{p})}\left(\frac{p}{1-p}\right)^{\frac{2(\lambda_1-\lambda_2)}{r^2}}\int_z^1 \left(\frac{1-y}{y}\right)^{\frac{2(\lambda_1-\lambda_2)}{r^2}}\frac{e^{\frac{2}{r^2}(-\frac{\lambda_1}{1-y}-\frac{\lambda_2}{y})}}{(1-y)^2y^2}dy.$$

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