STUDENTS' CONSTRUCTION OF ALGEBRAIC SYMBOL SYSTEMS

by

ERIK TILLEMA

(Under the Direction of LESLIE P. STEFFE)

ABSTRACT

The purpose of this study was to understand how eighth grade students used their symbolizing activity in interaction with a teacher-researcher to begin constructing an algebraic symbol system. Symbolizing activity—notation, diagrams, and natural language that the *students generated* in the process of solving quantitative problem situations—was taken as the basis for studying the construction of an algebraic symbol system. So, rather than take an algebraic symbol system as a given (e.g., start with problems that involved conventional algebraic notation like $3x^2 = 75$), the goal of the study was to understand students' generation of notation, diagrams, and natural language for their operations, schemes, and concepts. A central reason for this approach was to base definitions of algebraic symbol systems in students' generation of notation, diagrams, and natural language.

As a teacher-researcher, I taught three eighth graders at a rural middle school in Georgia in a constructivist teaching experiment from October 2005 to May 2006. All teaching episodes were videotaped with two cameras—one to capture student work and one to capture student teacher interaction. During the year, I posed quantitative problem situations that, from my perspective, involved multiplicative combinations, binomial reasoning, quadratic equations, and linear and quadratic functions. The analysis presented in my dissertation pertains to the problems

I posed that involved multiplicative combinations and led to the students finding the sum of the first so many whole numbers (e.g., 1 + 2 + … + 15). In retrospective analysis of the videotapes, I constructed second-order models that accounted for changes that students made in their mathematical way of operating and accounted for how they used their notation, diagrams, and natural language to symbolize their activity.

Students' multiplicative structures were significant resources in how they solved problems involving multiplicative combinations, how they produced the sum of the first so many whole numbers, and whether they engaged in recursive multiplicative reasoning. The data suggest that using one's notation to externalize the functioning of one's scheme and operating on this notation with operations that are internal or external to the scheme is one important aspect of constructing an algebraic symbol system.

INDEX WORDS: Algebra, Algebraic Reasoning, Notation, Symbol Systems, Algebraic Symbol Systems, Multiplicative Combinations, Combinatorial Reasoning, Two-Dimensional Multiplicative Reasoning, Operations, Scheme Theory, Quantitative Reasoning, Radical Constructivism, Teaching Experiment, Units-Coordination, Mental Imagery.

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ERIK TILLEMA

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by

ERIK TILLEMA

Major Professor: Leslie P. Steffe

Committee: Paola Sztajn
George Stanic

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
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DEDICATION

I dedicate this work to my wife, Amy, who has patiently listened to me develop my ideas during my doctoral work.

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CHAPTER 1: INTRODUCTION

Development of the Study

In order to understand this dissertation study, it is useful to set it in the context of the broader research study of which it was a part and to give a brief trajectory of the development of my ideas as a researcher. The study was part of a three-year constructivist teaching experiment with eight middle grades students who ranged in school mathematics achievement. I was involved as a teacher-researcher and witness-researcher during all three years of the experiment. During the beginning of the students' sixth grade year, the research team[1] conducted selection interviews to choose the eight students, who we taught in pairs during their sixth, seventh, and eighth grade years.

One question that framed the broader research study was: What might be taken as indicators of algebraic reasoning in middle grades students? This research question marks the broader study as basic research in the sense that its purpose was to define algebraic reasoning based on ways of reasoning that middle grades students produced in quantitative problem contexts. That is, the goal was not to take algebraic reasoning or algebra as a given. Rather, the goal was to identify characteristics of middle grades students' reasoning that I (and my colleagues) could use to help define what constituted algebraic reasoning for these middle grades students. This goal opens the possibility to contribute to efforts to redefine school algebra in order to reconceive of what might constitute K-12 curricula. The primary focus of the particular

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[1] Dr. Leslie P. Steffe oversaw and participated in all aspects of the study. The rest of the research team consisted the following graduate students—me, Amy Hackenberg, Zelha Tunc-Pekkan, Hyung Sook Lee, Kyle Schultz, Erin Horst—who contributed during different portions of the three-year study.

approach I have described is to root what is taken as algebra in models of middle grade students' reasoning.

To achieve this goal, it is essential to "see" students' reasoning, which entails building a consensual domain of communication with the students. Building this consensual domain of communication is what sparked my interest in symbol systems because human communication is predicated on the use of symbols. Prior to and during the broader research study, I began reading accounts of the development and use of natural language (people's primary symbol system) and its relationship to non-observable thought processes (e.g., Brouwer, 1929/1998; Dewey, 1933/1989; Piaget, 1951; von Glasersfeld, 1995; Vygotsky, 1930/1978). This investigation helped me develop the theoretical ideas that I have used as a basis for my dissertation study.

My dissertation study is focused on defining what it means to construct an algebraic symbol system, a significantly more specialized symbol system than natural language. In investigating this area of research, I made the decision to present students with quantitative problem situations as a starting place for their reasoning as opposed to presenting them with tasks that involved conventional algebraic notation (e.g., $3x = 4$, $y = 3x + 5$). I will address this decision in the second chapter, which provides a theoretical model of students' symbolizing activity, and in the concluding chapter that suggests conclusions and implications of my study. In these chapters, I will argue that algebraic symbol systems should be characterized more broadly than students' use of conventional algebraic notation. This broader definition has its genesis in my interaction with the students.

Having set the context of the larger research study and commented on the development of my ideas, I now want to provide an overview of my dissertation study. The data for my study comes from my work with three eighth grade students. During their eighth grade year, I investigated the students' reasoning in four related mathematical areas. From my point of view, the problem situations that the students investigated involved multiplicative combinations, binomials, quadratic equations of the form $ax^2 = b$, and linear and quadratic functions.[2] In the case studies, I analyze the students' ways of operating in the first mathematical area. The goal of this analysis is to provide a model of the students' non-observable mental activity along with how the students produced and used notation for their activity.

Problem Statement

There is an effort in the United States to assure that all students have the opportunity to learn algebra (National Council of Teachers of Mathematics [NCTM], 2000; Moses & Cobb, 2001). This effort stems from views that algebra is a powerful web of knowledge and skills (Kaput, 1998) and that passing an algebra course is essential for full political and economic participation in society (Moses & Cobb). In response, schools are offering more algebra courses, emphasizing taking algebra courses in earlier grades, and requiring students to pass an algebra course as a mandatory high school graduation requirement (Silver, 1997). Despite the increased offering of and access to algebra courses, reports suggest that many students find these courses incomprehensible and fail to pass them (RAND Mathematics Study Panel, 2002; The National Commission on Mathematics and Science Teaching, 2000).

These reports suggest that simply increasing students' access to algebra courses is not sufficient to improve student learning of algebra. As a response to this problem, Kaput (1998), Silver (1997), and Fouche (1997) have each advocated for a broad shift in the way in which school algebra is conceived. This shift is from algebra as a one-year course to algebra as a set of ideas that students can develop as they progress through their K-12 education. The impetus for

[2] I do not claim that the students saw the situations in this way.

this shift is to help students develop reasoning throughout their schooling in ways that open the possibility for more students to succeed in algebra courses.

For such a shift to be successful, researchers and teachers have to understand what might constitute algebraic reasoning for a variety of different levels of student in varying age ranges. For example, a third grade student might use strategic additive reasoning to solve the following problem.

> *The Money Problem*: Jim has $17. His Grandma gave him some more money. Now he has $35. How much money did his Grandma give him?

In using strategic additive reasoning, he might consider $35 as composed of two parts, $17 and an unknown amount, and reason that the unknown amount can be found by adding $20 to $17 to make $37 where the unknown amount would have to be $18 because $37 is $2 more than $35. In this case, a teacher or research might judge that the student had reasoned algebraically in an additive context. So, conceiving of algebra in this way opens up the potential for a number of areas of research to contribute to redefining what might constitute school algebra.

To contribute to this redefinition of algebra, researchers conducting research on algebra learning have approached algebraic activity as generalizations of whole number arithmetic (e.g Falkner, Levi, & Carpenter, 1999; Herscovics, & Kieran, 1980; Kieran & Cahlouh, 1993), modeling scientific phenomena (e.g., Izsak, 1999, 2000; Janvier, 1996; Nemirovsky, 1996), problem solving (e.g., Bednarz & Janvier, 1996), and coordinating representations (Kaput, 1989; Schoenfeld, Smith, & Arcavi, 1993). Other researchers have taken functional approaches to algebra (Chazan, 2000; Heid, 1996; Kieran, Boileau, & Garancon, 1996; Schwarz, 1999), and have investigated how students operate on or with unknowns (Carraher, Schliemann, & Brizuela, 2001; Filloy & Rojano, 1989; Herscoviks & Lincheski, 1994).

In many of these research endeavors, one aspect of researchers' analyses involves how students produce and use notation. However, this aspect is often not the central focus of analysis, which has led to recent calls for making it a more central feature of research (e.g., Sfard, 1995; Yackel, 2000). In response to these calls, researchers have taken recent interest in the role that notating one's activity plays in the learning process (Brizuela, 2004; Cobb, Yackel, & McClain, 2000; Kirshner, 2004; MacGregor & Stacey, 1997). For instance, in recent books, researchers have focused on the role of social negotiation in the notating process (Cobb, Yackel, & McClain, 2000) and the wide variety of notations that students use (Brizuela, 2004). Less attention has been given to how notation functions in relation to the mental processes that students use as they reason mathematically. Because less attention has been paid to this issue, researchers and teachers lack knowledge about how to help students' grow notation out of their reasoning.

That this issue is a contributing factor to students' lack of success in learning to reason algebraically is visible in the research literature, even when it has not been the primary focus of analysis. For instance, numerous researchers have provided images of how students use conventional mathematical notation (e.g., Booth, 1988; Clement, 1982; Filloy & Rojano, 1989; Hart, 1981; Herscoviks & Lincheski, 1994; Kieran, 1989; Knuth, Stephens, McNeil & Alibali, 2006; Rosnick & Clements, 1980). These images often suggest a wide gap between how a proficient user of mathematical notation might conceive of the meaning for this notation and how a student conceives of similar notation. Although this research leaves unaddressed the issue of how students *produce* notation, it does suggest that students often do not use notation in ways that are meaningful to them.

These images are supported by research that has examined students' impressions of learning to reason mathematically (e.g., Schoenfeld, 1989; Weinstein, 2004). A common impression of learning to reason algebraically is that it is about manipulating conventional algebraic notation. With this impression, the bulk of what is to be learned are the correct rules and procedures for manipulating a conventional system of notation. One consequence of such an impression is that students commit to memory how to manipulate notation in proscribed ways, but this manipulation of notation often lacks any connection to their reasoning (Mason, 1996; Chazan, 2000).

For example, it is common for students to use cross multiplication to solve a host of problems with no consideration of whether such a problem involves reasoning with proportions. In such a situation, students may or may not be able to produce proportional reasoning and cross multiplication may or may not be an appropriate way of solving the problem. In any of these cases, if the student has memorized a proscribed rule, his reasoning is not reflected in the notation he uses to solve the problem. According to Mason (1996), the literature on student misconceptions is riddled with similar examples of students' use of notation in ways that are divorced from their reasoning.

If students are left with the impression that algebraic reasoning is about manipulating conventional notation in proscribed ways, they often lose a sense of themselves as active agents in the learning process. So, learning mathematics can become something that seems external to their way of reasoning. When students take mathematics as external to their way of reasoning, they may not conceive of notation as something that they can produce and use to stand in for this reasoning. Such a view of notation can suppress the power that mathematical symbols might hold if students saw themselves as integrally involved in creating this notation.

One research approach that focuses on students' generative mathematical activity as a basis for inquiry into research in mathematics education is an ontogenetic approach (e.g., Steffe, 1992, 1994, 2003; Thompson, 1994). In an ontogenetic approach, researchers build models of how students use their current mathematical ways of operating to construct new, more powerful ways of operating (Steffe & Thompson, 2000). These models have provided potent explanatory constructs for students' mathematical activity through the construction of models of students' mathematics. Although this research has been developed in a number of areas, two areas where it has been less developed are in students' algebraic reasoning and more specifically on how students produce and use notation in their reasoning.

Research Questions

The purpose of this study is to understand how students' transform their mental imagery and operations into notation[3] and how this notation transforms their mental imagery and operations. The following research questions guide this study:

1) What aspects of multiplicative and quantitative ways of operating are symbolized and what changes do students make in their multiplicative and quantitative ways of operating in interaction with a teacher-researcher?
2) What mental imagery and operations do students produce in the context of solving quantitative problem situations?
3) How does students' notation function in the process of constructing an algebraic symbol system?
4) What conventions do students produce in the context of their notating activity and what role does social interaction play in this process?

[3] I have written the problem statement using the term production and use of notation to make it more reader friendly. The term symbolizing activity is something that I consider to be broader. I will define the term "symbolizing activity" in Chapter 2.

Rationale

In the problem statement, I suggested three reasons for conducting this study. First, students struggle to learn algebra so efforts are needed to understand what it might mean for middle grades students to learn to reason algebraically. Second, teachers and researchers lack basic knowledge about how students produce and use notation and what role this activity plays in students' learning. This lack of basic knowledge has a variety of consequences. One consequence is that students' reasoning is frequently divorced from the notation they produce. A second consequence is that students often turn to memorizing proscribed rules for manipulating notation. A third consequence, which motivates the third reason for conducting this study, is that mathematics often seems external to students' ways of reasoning when notation does not grow out of their reasoning. So, a third reason for conducting this study is to provide images of how students might come to view themselves as active agents in the process of producing and using notation for their reasoning. In this rationale, I give a fourth, fifth, and sixth reason for conducting this study.

Reason 4: Improving the Teaching of Algebra

There have been recent calls for improving the teaching of algebra (The National Commission on Mathematics and Science Teaching, 2000) along with research reports suggesting that teachers in the United States are frequently unprepared or under prepared to teach algebra (Post, Harel, Behr, & Lesh 1991; Stigler et al., 1999). One source for improving the teaching of algebra is through a better understanding of how students learn to reason algebraically. Understanding better how students learn to reason algebraically can contribute to research based curricula suggestions that are aimed at improving the teaching of algebra. Further, this research will be available in various forms for other researchers and teachers to critique and discuss. Such critiques and discussion can lead to improving teacher education programs through a better understanding of what it means for middle grades students to reason algebraically. Such critiques and discussions with middle grades teachers can also lead to refining and improving algebra teaching (both the authors and other middle grades teachers).

On a more personal note, I will shortly be a teacher educator. One source for improving mathematics teaching is through teacher education programs. In order to have teacher education programs that help teachers build their mathematical ways of operating in ways that might be useful for their teaching, it is important to have teacher educators who have worked both extensively and intensively with students. This work with students is important because it provides a common ground for teachers and teacher educators to have discussions about teaching. Therefore, one primary goal of this study is for me, a future teacher educator, to learn more about students' ways of reasoning that are algebraic and about ways that I can act that might bring forth these ways of reasoning. This research will provide both an experiential basis for such actions as well as an analytic basis for understanding these actions. So, it will give me a strong grounding for thinking about how to structure teacher education courses that are aimed at preparing teachers for teaching algebra.

Reason 5: Providing Models of Social Interaction Compatible with Radical Constructivism

Radical constructivism has been criticized for not accounting for social interaction and ethics in its theory of knowing (e.g., Cobb, 1996; Lerman, 1996, 2000; Lewin, 2000). This study seeks to respond to von Glasersfeld's (2000) call for models of social interaction that are compatible with radical constructivism. It will do so through investigating the social processes involved in students creating conventions as they produce and use notation. Von Glasersfeld's assertion (1985, 1995) that all knowledge has a subjective component precludes the possibility of notation taking on an observer-independent meaning. Therefore, my conception of notation, as being conventional is that two (or more) people operate as if they attribute compatible meanings to a particular kind of notation. Therefore, the processes of social negotiation among participants in the study that leads to this agreement serves as a foundation in my thinking about how notation becomes conventional.

Two types of interaction may lead to establishing conventions in notation. First, students may interact with each other and such interactions may form a basis for the students to establish conventions in the notation they produce. Second, the teacher and students will interact and through this interaction students may establish conventions in their notation. While the intent of this research is not to provide an extensive model of social interaction, it will certainly begin to establish a model of social interaction that is compatible with radical constructivist theory. In particular, as I mentioned in the Development of the Study Section of this introductory chapter, I am particularly interested in formulating consensual domains of communication with students. Part of formulating these consensual domains of communication is establishing conventions in systems of notation. So, this research is a beginning for establishing how social interaction contributes to establishing conventions in systems of notation.

Reason 6: Understanding the Relationship between Language and Mathematics

The final reason for conducting the study is philosophical and grows out of a personal interest. I am interested in better understanding the relationship between language and mathematics. One way to better understand this relationship is by understanding the relationship between how students produce and use notation, a language for their mathematical reasoning, and students' mathematical reasoning. Debates in the philosophy of mathematics often center on the relationship of language to mathematics (e.g., Bernays, 1930/1998; Brouwer, 1929/1998;

Mancosu, 1998; Weyl, 1921/1988). However, these debates are not based in empirical data, rather they are usually based in personal experience or historical accounts. So, for me, this study is a way to contribute to my own academic interests and possibly to contribute to both philosophical and historical understanding of the role of language in mathematics.

CHAPTER 2: THEORETICAL MODEL OF STUDENTS' MATHEMATICAL LEARNING AND SYMBOLIZING ACTIVITY

Radical Constructivism: A Guiding Theory

The guiding theory for this study is radical constructivism, a theory of rational knowing that Ernst von Glasersfeld developed (1975, 1984, 1995) and is based in Piaget's (1970a, 1970b) genetic epistemology. Both Piaget's and von Glasersfeld's theories spring from the observation of numerous philosophers that knowledge cannot be a copy of an observer independent reality. That is to say, if knowledge were a copy of an observer independent reality, then a person would have to have access to what is being copied, but the only access to this reality that a person has is through his own ways of perceiving and conceiving. To avoid this circularity, von Glasersfeld suggests a reformulation of the notion of reality, which he terms a person's experiential reality—the reality that a person builds up through his ways of perceiving and conceiving. This orientation to knowledge motivates the two basic principles of radical constructivist thought:

- knowledge is not passively received but built up by the cognizing subject;
- the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality. (von Glasersfeld, 1995, p. 18)

Knowledge, then, is considered to be viable in a given context, rather than something that reflects an observer independent reality. In the domain of concepts, I use the term viability to mean conceptual coherence or non-contradictory fit among the largest possible conceptual network of structures.

Given that a person cannot transcend his ways of perceiving and conceiving, this suggests he is in constant interaction with his environment[4] and a person's thoughts and ideas are constantly in interaction with each other (Steffe, 1996). The first type of interaction would include a person's interaction with sticks, animals, and other people. Interactions of this type that a person might construe as mathematical might include ordering sticks in a particular spatial pattern, interacting with a teacher who presents a mathematical task, or interacting with another student in solving a mathematical task.

The second type of interaction, interaction among constructs within a person, accounts for the interaction among non-observable mental processes that are involved in the constructive process. A person may or may not be aware of these non-observable mental processes and the interaction among them. An example of this type of interaction would be a person taking two ideas that for them had previously been unrelated and putting them together in a novel way to produce a new idea. Both types of interaction are necessary in the construction of mathematics, but individually each is insufficient on its own to explain the construction of mathematics. However, taken together they constitute the two domains of the interaction that enable a person to construct a mathematical reality.

A Model for Mathematical Learning

Having given a brief overview of radical constructivist theory, I present a theoretical model for mathematical learning. Von Glasersfeld (1987, 1995, 2001) and Steffe (1991, 1996) initially developed this model, which was based in Piaget's work. However, numerous researchers in mathematics education have made substantial contributions to the model (e.g., Cobb, 1996a; Olive & Steffe, 2002; Thompson, 1996; 2000). Although my understanding of this

[4] A person's environment is from an observer's perspective everything that is not that person.

model has been significantly refined because of my interaction with the students and subsequent use of the model in my data analysis, in a certain sense, it was one of the theoretical tools with which I began the study. I will present the model by defining terms and providing examples for these definitions.

Schemes

Development of Scheme Theory. According to von Glasersfeld (1995, p. 64), Piaget's notion of a scheme grows out of the reflexes he observed in newborn children. For instance, a child's rooting reflex is the reflex of turning one's head to root for the breast when the cheek of the child is rubbed. So, a reflex or fixed action pattern is something that is triggered by a stimulus and produces a particular activity. Piaget considered these reflexes to be hard wired into our genetic material. Given this position, he realized that these reflexes could only be explained as a result of natural selection, which suggested to him that it was not the activity that was important, but rather the results of the activity (i.e., babies with rooting reflexes were more likely to be well fed and survive). Once he made this observation along with the observation that these reflexes were not as fixed as people suggested, the notion of a scheme that contained three parts—an assimilatory mechanism, an activity, and a result—was born.

Mental Imagery and Operations. Piaget (1970a) did much to develop the notion of a scheme and applied it to the various levels of development that he identified from basic sensory motor schemes all the way to the construction of schemes in logical mathematical contexts. He eventually characterized a scheme as that which is repeatable or generalizable about a mental operation or a network of mental operations. A mental operation is an *interiorized action* that requires mental material on which to operate. So, for instance, a student might imagine partitioning a square that is 13 units by 13 units into four parts (a 10 unit by 10 unit square, two

10 by 3 unit rectangles, and a 3 unit by 3 unit square). In this case, the operation is partitioning and the mental material on which this operation is carried out is the student's concept of a square. I call this mental material *mental imagery*. Mental imagery may have a visual component to it, but I use the term more broadly to encompass images that a person makes in the context of their mental and physical experience, which need not be visual in nature.

A person creates this mental material by re-presenting a previous part of experience. So, the word *re-presentation* simply means to regenerate some previous part of experience. The ability to re-present mental material on which to operate is not necessarily immediate. Rather, a person may need to operate on material that is perceptually available prior to producing mental imagery on which he can operate. For example, a person may be able to operate on a square that he produces in a computer microworld prior to being able to re-present this square and operate on in it exclusively mentally. When a person is constrained in this way, I consider his activity to be *enactive* in the sense that his activity cannot be carried out exclusively on mental material.

I consider the construction of mental imagery and operations to be cyclically linked together in the following sense: Mental imagery on which a person operates in a particular situation is the result of an abstraction from a previous cycle of operations. In the square example, a person's mental image of a square is the result of operations that produced the abstraction for his concept "square". So, operations produce images. At the same time, the images that a person produces constrain the operations that a person calls forth in a given situation (cf. Thompson, 1996). For instance, depending on a person's image of a square, partitioning it into 4 parts may or may not be a conceivable operation to be performed on the image.

For the sake of clarity, it should be noted that neither mental operations nor mental imagery are observable phenomena. Rather, they are a researcher's construct for helping to make a model of non-observable mental processes to which a researcher has no actual access. Focusing on mental operations and mental imagery shifts a researcher's attention away from making characterizations of externally observable behaviors and focuses it on making accounts of the mental processes that might explain these observable behaviors (cf. Schoenfeld, Smith, & Arcavi, 1993). So, behavioral indicators are used to make inferences about a student's mental operations and mental imagery. Making a model of these non-observable mental processes is something that can be built up only over time and through multiple observations of another person's activity. One particularly useful way of making inferences about the mental operations and the kinds of mental imagery that a student produces is by observing how he operates on material that is perceptually available both to the researcher and the student. So, for instance, having a student operate on a square that he produces provides a good occasion to make inferences about the student's mental operations and mental imagery.

Assimilation. Von Glasersfeld's (1995) reformulation of Piaget's notion of made explicit the three parts of a scheme—the assimilatory mechanism, an activity and a result. The assimilatory part of a scheme, from the perspective of an acting agent, consists of recognizing the experiential situation as similar to one that has been previously experienced. Olive & Steffe (2002) point out that the experiential situation should not be considered to exist somewhere in the mind. Rather, the records of operations used in past experience are activated in the process of assimilation. Once the records of these operations are activated, they produce a recognition template through which a person creates an experiential situation. The experiential situation that the person creates may be considered similar to one he has previously experienced. The recognition template, then, may trigger an activity that has been associated with the recognition template. From an observer's perspective, the situation may contain elements that are quite different from situations that have triggered the activity in previous situations, but a person may not take these differences into account when he begins acting.

Activity and Results. The activity of a scheme is a composition of operations performed on some mental material. For instance, the activity of a scheme might be to partition a rectangular array into four pieces, disembed one of those pieces, and iterate it three times. I consider this activity to be goal directed in the following sense—when a person creates an experiential situation through assimilation, the associated activity that is triggered calls forth an expected result. The expected result is the expectation of what the activity of the scheme will produce. When a person actually produces a result through his activity, he assimilates the results he has produced to his expected result.

Acts of Learning

Perturbations. If the results a person produces are different from his expected result, this may create a perturbation, which can lead to reviewing the initial experiential situation. If the new result is disappointing or unpleasant, the person may exclude the experiential situation from ones that trigger the activity of the scheme. On the other hand, if the result is surprising or pleasing, a person may take into account previously unaccounted for aspects of the experiential situation, which may lead to a new recognition template and the construction of a new scheme. In either case, an act of learning, an accommodation in a person's schemes, has taken place.

Two Types of Accommodations.[5] A functional metamorphic accommodation is a reorganization of schemes in the context of their use. Such an accommodation may take place when a student is presented with a novel problem for which they have many of the conceptual structures available to them prior to operating in this context, might occur over an extended period of interaction, or as a result of monitoring one's mathematical activity. Such an accommodation may lead to the construction of a new scheme. The new scheme may lead to a perturbation at the level of reflection where a person may review old schemes in relation to the new scheme, and these old schemes may be reorganized to integrate the new scheme. In this reorganization, a person may construct other new operations, schemes, or concepts. I consider this type of accommodation to mark a significant shift in a students' way of operating.

For example, a student who has been unable to solve problems involving reversible multiplicative reasoning might work on problems that involve partitioning, mentally breaking apart quantities, and iterating, mentally repeating quantities. At the end of this work, a teacher might present a picture of a rectangular bar and ask the student to solve the following problem:

> *The Candy Bar Problem:* My candy bar (the given rectangular bar) is fifteen inches long. It is five times the length of your candy bar. Can you make a picture of your candy bar using my candy bar?

If the student were able to solve the problem, this would mark a significant shift in their way of operating because it would mark the beginning of the students' ability to engage in reversible multiplicative reasoning. Such a solution might transform both the students whole number multiplying and dividing schemes along with his fraction schemes.

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[5] I focus on the two types of accommodations that will appear in the case studies. For a more thorough characterization of kinds of accommodations see Steffe 1991.

A functional accommodation (non-metamorphic) is a change in a person's scheme, while the scheme is in use. The difference between this type of accommodation and a functional *metamorphic* accommodation is that a functional accommodation does not change the students' ways of operating in as dramatic a way as a functional metamorphic accommodation. So, a functional accommodation might include learning to use already constructed operations or schemes in novel situations. For example, a student might solve the following problem.

> *The Cake Problem:* Suppose that you cut a cake into eight pieces. You take two of the pieces. How much of the cake did you get?

A student who learns that his multiplying schemes are relevant in this context might come to see the amount he has produced as both two eighths and one fourth. Two eighths because he has two one eighth size pieces each of which when taken eight times makes the whole and one fourth because he can take two eighths four times to make the whole cake.

Self-Regulation and Monitoring. One source of accommodations[6] is through acts of self-regulation, and the subsequent monitoring that self-regulation can engender. For example, a student might begin counting 5 on from 7.[7] In this process, the student might lose track of how many times he has counted beyond 7. When he losses track of the number of times he has counted, he might re-initialize his counting. Re-initializing counting would be an act of *self-regulation* because a student had not reached some expected goal (in this case, the expected goal might be a definite ending point to his counting). When he re-initializes his scheme, the student might *monitor* his counting acts in order to meet his expected goal (i.e., nine is one, ten is two, etc.). Following Steffe (1991a), I have use the word self-regulation to mean an activity that

[6] It should be noted that I consider one of the most frequent occasions in which a person makes an accommodation to be while he or she is interacting with other people.
[7] Note this example is adapted from Steffe (1991a) and is presented in a simplified form.

occurs at the level of a scheme whereas monitoring occurs at the level of the activity of the scheme. In this instance, self-regulation was the act of re-initializing the student's counting scheme and monitoring occurred when the student kept track of his counting acts. Such activity can occasion accommodations in a student's scheme.

These two concepts—self-regulation and monitoring—are of particular interest for my dissertation study because students frequently used notation in ways that led to acts of self-regulation and subsequently they used their notation to monitor their activity. For instance, a student might produce notation for a situation, realize from his use of notation that he needed to produce his activity again, and then subsequently used his notation to monitor this activity. The students also frequently used notation as a way to monitor their activity where the act of self-regulation was anticipatory. That is to say, the student began using notation as a way to monitor their activity, when they felt uncertain about what the results of their activity would produce. So, the act of self-regulation preceded any activity in the situation and they considered notation to be a viable way of monitoring their activity.

Reflective and Reflected Abstraction. Any time a student makes an accommodations this indicates a *reflective* abstraction—the projection of something borrowed from the preceding level to a higher level—has taken place (Piaget, 2001/1977; von Glasersfeld, 1995). However, as my distinction between a functional and functional metamorphic accommodation suggest there is a wide range of variation in how much a person might reorganize his ways of operating after a reflective abstraction occurs. In the example of the functional accommodation I gave above, the student's activity might create a new recognition template. Such a recognition template might trigger his multiplying schemes when he was naming fractional quantities. In this case, what would be projected to a higher plane would be the new recognition template. Such a projection might be considered more minimal because of the amount of reorganizing which might occur as a result of this recognition template being projected.

On the other hand, in the example of the functional metamorphic accommodation I gave above, the student might be considered to have constructed a new operation that combines iterating and partitioning into one psychological structure. Steffe (2003) has termed this operation the splitting operation. In this case, the operation would be projected to a higher plane. When the new operation was integrated with other schemes, it could lead to further accommodations of these schemes. In this case, the contents of what was projected to the higher plane might create more significant reorganizations at this higher plane.

A *reflected* abstraction is the process through which a person becomes consciously aware of the reorganization that a reflective abstraction has created (Piaget, 1977/2001; von Glasersfeld, 1995). So, it is a retroactive thematization of what has been projected to a higher plane, which requires a person to some extent to become consciously aware of his activity. This model suggests that conceptualized knowledge of a situation lags behind knowing how to act in the situation. Therefore, if a person has conceptualized knowledge of their operating in the situation, then it is possible to infer that a reflected abstraction occurred.

Reflective and reflected abstractions are sometimes used as a research tools to identify static states or levels that a person achieves (e.g., Goodson-Espy, 1998; Hart, 1981). In the case studies, one of my goals is to place reflective and reflected abstraction in the context of accommodations. The purpose of this goal is to place these research tools in the dynamic context of learning rather than to use them to identify static states. This goal is in line with recent research that has suggested ways to make reflective and reflected abstraction a part of the dynamic processes of learning (Simon, Tzur, Heinz, & Kinzel, 2004).

A Model of Mathematical Symbol Systems

Having defined some of the basic tools that I use to understand mathematical learning, I now present a model of the construction of a mathematical symbol system. This model is presented in three parts. In the first part, I examine some of the characteristics that I take as necessary for a person to construct a symbols system. In this part, I use examples from children's development of spoken language. In the second part, I examine some aspects that are unique to constructing a mathematical symbol system and I suggest some of the activity that is a part of constructing a mathematical symbol system. I call this activity *symbolizing activity.* In the third part, I present my view of communication and look at some of the social processes that I consider to be involved in the symbolizing process.

The Development of Spoken Language[8]

From Experience to Concepts. When a person has abstracted his experience into a concept, I mean that a person takes some aspect from a set of experiences that he considers to be similar and constitutes this aspect as part of a generalized phenomenon. For example, a young child might form a dog concept through his experiences of petting the dog, chasing the dog, and feeding the dog. In order to abstract the particular experiences into a generalized phenomenon, several things would need to occur. First, the child would have to isolate the dog from other sensory signals and treat it as a unitary whole. Von Glasersfeld (1995) has called the operation that isolates sensory signals and treats them as a unitary whole the unitizing operation. Second, the child would have to consider each encounter with the dog to be an encounter with the same

[8] I consider that a similar thing occurs in the case of written language except that it is not a sound image but a graphic image formed from our visual experience with written words. Based on the observation that children learn to write later than they learn to speak, I infer that producing a written language involves more complexity than a spoken language.

dog. Piaget (1954) has called this object permanence and it serves as an important basis for the construction of concepts.

From Auditory Experience to Sound Images. As part of the process of abstracting a concept, a person may associate a sound image (e.g., an abstraction formed from hearing a word) with his concept. Both the sound image and concept are abstracted from experience. The sound image is abstracted from a particular kind of experience, auditory experience. The concept may be abstracted from physical or mental experience. The two need to be differentiated and coordinated in order for a word to function symbolically.

An example from children's early use of language may be useful in illustrating my point here. When a child first hears a word like “bottle”, it may be in the context of drinking milk from his bottle. From the perspective of the child, the experience of seeing the bottle, drinking the milk, and hearing the word may all be part of one experience where the sensory signals have not been differentiated from one another. In order for the child to use the word, the child needs to be able to isolate the bottle in his visual experience, differentiate this visual experience from the auditory experience, and coordinate the two kinds of sensory experience.

The differentiation and coordination leads to an association between a sound image and a concept, which I consider essential for a word to serve as a symbol. In fact, for a word to be a symbol, I consider that the sound image has to call forth the concept and conversely the concept must call forth the sound image. So, a two-sided psychological phenomenon is formed between a particular sound image and abstractions that a person has made from the functioning of his schemes.

Von Glasersfeld (1995) has differentiated between this two-sided psychological phenomenon and what he has termed *signaling*. An example that suggests how a word can be used to signal might help to make this distinction. Such an example might be when a person says, “sit”, to a dog. In this situation, the dog is certainly able to isolate the sound waves that produce the word “sit”. However, the action of sitting is simply an associated behavior not something that is a conceptual affair. That is to say, the dog probably does not have a concept for sitting. So, the sound image does not serve a function outside of the particular context in which it is heard. This situation means that the sound image is in one to one correspondence with a particular behavior that is enacted at the time of having isolated the sound waves.

The notion of signaling may account for the large number of student misconceptions reported in the mathematics education literature that are related to how students use conventional notation. That is, a student may make associations of what to do with particular notation, but this activity may not be rooted in the mathematical concept that is intended by a proficient user of the notation. So, for instance a student may learn to multiply “tops” and “bottoms” of fractions to produce the correct result for a fraction multiplication problem, but the student may not have produced any associated fraction multiplication concept. That the student multiplied suggests that he has some concept of multiplication, but this does not mean that he has a concept of fraction multiplication.

From Recognition to Re-presentation. Von Glasersfeld (1991) has suggested that the ability to recognize precedes the ability to re-present. The example of the child and the bottle is useful in bringing out this distinction. At first, a child may only be able to use a word in the context of having the bottle present in his visual experience. That is, the child is able to *recognize* the bottle when the bottle is in his visual field, which may occasion his use of the word bottle. Eventually, a child becomes able to re-present his bottle experiences in the absence of the visual material of the bottle (Piaget, 1951). That is, a person is able to regenerate his past bottle experiences, which means he is able to produce mental imagery of some type for these experiences without needing the perceptual material in his visual field.

The ability to produce a re-presentation is the third central feature of a symbol system that I consider. As part of the process of abstracting a concept, a person also makes an associational link between a re-presentation and his concept. This assertion is similar to the one I made about sound images. Here, the association means that a person upon hearing a word, or upon assimilating an experiential situation to his concept is able to use this activity to re-present some previous part of experience that has been associated with the sound image and concept. Conversely, a re-presentation can also be used to call forth either the sound image, or the person's concept. In Figure 2.1, I have adapted a graphic from von Glasersfled (1995, p. 131), to show these two-way associational links.

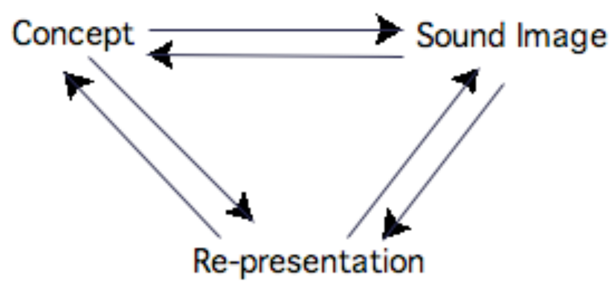

Figure 2.1: A graphic for the two way associational links that are established

An example might be useful to help illustrate Figure 2.1. For instance, a person might hear the word "apple", which would call forth the person's object concept for an apple. In this case, the object concept would simply be abstracted from his apple experiences. When the word functions symbolically, a person may use the occasion of hearing the word to re-present some particular apple experience. He might re-present his weekly experience of picking out Fuji apples from the store or his yearly childhood experience of picking apples in the fall. Whatever the particulars of the re-presentation, I consider the ability to re-present previous parts of experience to be possible if a word is to function symbolically.

A Relationship between the Particular and the General. As I stated above, I consider the abstraction of a concept to suggest that a person has constituted a certain aspect of a sequence of experiences as being similar. By associating a word with some aspect of these experiences, the word points to the aspect that a person considers common to all of the experiences without being fixed to any particular of the experiences. This observation suggests that a word that is a symbol has a variable quality to it because it symbolizes a concept, which is abstracted from more than one particular experience. However, in re-presenting a concept, the re-presentation must be of particular properties of this concept. That is, in re-presenting a dog, the dog must be of a particular color, size, and shape. So, it is not possible to re-present a dog that is both small and large even though the word dog can be used to re-present either a small dog or a large dog. So, words serve as a kind of placeholder.

In fact, once a person becomes a proficient user of words, the quality of being a placeholder gives words even more power. It gives them more power because the person can use them to call forth a re-presentation when necessary, but the word can also point to an unimplemented re-presentation. For example, a friend might begin telling a story about his drive to Alaska. He may begin describing a hike he took in the Rocky Mountains, but then quickly switch the focus of the story to his stay in Anchorage, Alaska. If the listener is a proficient user of symbols he might not re-present any experiences he has associated with the Rocky Mountains. Rather, the words would simply point to his abstracted re-presentation and concept. The listener, however, might re-present his experiences of Anchorage that seem to be the focal point of the story. If the speaker were to return to describing the Rocky Mountains, the listener might then produce a re-presentation of his experiences of the Rocky Mountains. So, a person who is proficient at using symbols is able to use them to point to an unimplemented re-presentation, re-presenting relevant parts of past experience as is necessary to make the words interpretable.

Creating a Mathematical Symbol System

The Difficulty of Mathematical Symbol Systems. Thus far in my discussion, I have focused on spoken language, giving some basic examples. I want to comment briefly on some of the particular difficulties of the construction of a mathematical symbol system. An example may serve to illustrate my point. In all of the previous examples, the words I have chosen have been rooted in a person's perceptual experience. That is, a dog, a bottle, an apple, and Anchorage are all words that can call up some kind of mental image that has been abstracted from perceptual experiences.[9] On the other hand, consider the words "two flowers". Here, the word "two" is not something that could be abstracted from perceptual experience because if it had been it would have to be a property of a person's perception of the flowers. For it to be a property of a person's perception, it would have to be a property of each individual flower, but this clearly cannot be the case.[10] This situation suggests that mathematical symbol systems are particularly difficult to construct and there construction involves quite a bit of creativity.

Abstracting Mathematical Schemes into Concepts. The difficulty as well as creativity arises in part because mathematical concepts are produced through mental experience not some kind of sensory motor or perceptual experience (Sfard, 2000). Keeping with my definition of a concept, to have abstracted a mathematical concept means that a person has taken some mental

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[9] This statement is not to claim that the words only call up mental images of perceptual experiences. For example, consider a person who owns an apple farm. The word apple may also call forth his experiences of classifying apples into different types.

[10] Note this differs from saying, "red flowers", where "red" could be considered a property of each flower and so I would consider it to be a property of a person's perception of the flowers.

experiences as constituting a generalized phenomenon. So, I consider a person to have abstracted a *mathematical concept* when he abstracts the structure and functioning of his mathematical schemes into a program of operations (von Glasersfeld, 1982). Here, I consider a program of operations to be a kind of “recipe” for how a person would create something without actually having to carry out the activity that would produce a particular result.

Once a person has constructed a mathematical concept, he may simply assimilate an experiential situation to his concept without having to implement the activity that produces a particular result. In this regard, a person’s scheme becomes significantly compressed so that the activity and the result in a certain sense are contained in the assimilatory part of the scheme. Assimilation still plays a crucial role because a person still needs to assimilate an experiential situation to a concept prior to taking it as a given in the situation.

Hackenberg (2005) provides an example of the difference between a student who had constructed a fraction as a concept and a student who had not. In her example, the student who had constructed a fraction as a concept was able to take both her making of a fraction from a unit quantity (i.e., given a unit making three fourths of the unit) and her making of the unit quantity from the fraction (i.e., taking four thirds of three fourths to make the unit quantity) as given in experiential situations. So, the student was able to state reciprocal relationships between two quantities without actually having to carry out any activity to produce these reciprocal relationships. For example if quantity A was three fourths of quantity B, then this implied to her that quantity B must be four thirds of quantity A. This differed from a second student in her study who could produce this reciprocal relationship as a result of his activity in an experiential situation, but he could not take it as given in the experiential situation. So, if quantity A was three fourths of quantity B, he could make quantity B by making four thirds of quantity A, but the reciprocal relationship was not immediate to him.

The Role of Externalization. This characterization of a mathematical concept suggests that once a person has abstracted a mathematical concept, he can read this concept into experiential situations. In the example above, the student who abstracted a fraction as a concept was simply able to state a fractional relationship between two quantities as if this relationship was a part of the situation. On the other hand, for the student who had not abstracted a fraction as a concept, he could operate in the situation but could not treat this activity as if it were apart of the situation itself. When a person is able to read his mathematical activity into an experiential situation, I consider him to have *externalized* his activity.[11] In this case, one projects his abstracted activity into the situation. Such a projection of a person's activity is what I take as constituting the person's mathematical reality. This process of *externalization* is an important part of the symbolizing process because it enables a person not to carry out activity that he may have initially produced to abstract his concept.

Symbolizing Activity[12]. Thus far, I have defined a number of terms that have aided in my characterizing what constitutes a symbol system. I now want to introduce an important term that is related to the activity that I consider a part of actually constructing a mathematical symbol system. I call this activity *symbolizing activity*. I define symbolizing activity to be creating perceptually available material (i.e., notation, diagrams, language, drawing a square, etc.) that stands in for aspects of a person's schemes and concepts (cf. Cobb, 2000). By perceptually available material, I simply mean the creation of something that is available to a person's senses

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[11] I consider this process to be similar to the one that leads to object permanence where a person considers an object to be independent of the operations he used to produce the object.
[12] The term symbolizing activity comes from the study where I was trying to characterize the activity of creating notation, diagrams, etc.

(in this case, both the students' and my own). So, when a student speaks, the act of speaking creates sound waves that are perceptually available. Similarly, when a student draws a diagram or writes something, he is creating material that is perceptually available. One of the nice features of written notation is that it creates a record of a student's activity, which opens the possibility that he can use this record to re-present parts of his activity. In my study, I will analyze both the written records and the verbal language that the students produced.

I have conceived of symbolizing activity both as a *tool in action* and as a *tool in reflection*. As a tool in action, a person may use some form of notation to help him engage in acts of self-regulation and monitoring to coordinate his activity in a particular situation. As a tool in reflection, he may use the notation as a tool to externalize his activity or re-present part of this activity without producing the activity in full. I take both of these activities as an essential part of the activity that is involved in constructing a symbol system.

It should be noted that the students' symbolizing activity serves a two-fold purpose when I do my analysis for the case studies. First, I use it to infer the students' unobservable mental imagery and mental operations. In this way, I use it to create a model of the students' schemes and concepts. Second, I also use it to make a model of how the symbolizing activity is functioning for the students in the situation. This second kind of model is intended to model the functioning of the students' symbolizing activity as they reason about the situations.

Communication, Intersubjectivity, and Convention in the Symbolizing Process

In the first two parts of this section, I have examined the development of language as a symbol system, suggesting some of its crucial characteristics, and then looked more specifically at mathematical symbol systems. In this final part of the chapter, I want to address how I consider communication between two (or more) people to occur. In this discussion, I will define intersubjective agreement and suggest how I have conceived of establishing conventions. All of these processes are a part of social interaction, which I consider to be an essential part of the construction of a symbol system.

From the very beginning of linguistic experience, other people are involved in learning language. For instance, a parent may point to a particular object and utter a word with the goal of focusing his child's attention on this particular object. Such an event opens the possibility for the child to make a coordination between hearing the word and the experience of seeing the object. Once a child has made this coordination and is able to create a re-presentation of the object, then as I have suggested earlier the word functions as a symbol. Communication between two people is then able to proceed as long as the people who are communicating are able to establish compatible meanings for words. That is to say, as long as each person is able to re-present meanings for words in a way that allows each to make an interpretation of what the other is saying. When two people are able to establish compatibility of meanings, I consider them to have established intersubjective agreement (von Glasersfeld, 1985).

The notion of intersubjective agreement is founded on each person in a conversation being able to establish a model of the other person's conceptions. For example, if person A and person B are talking to one another, person A must establish a model of the conceptions of person B and person B must establish a model of the conceptions of person A (Bauersfeld, 1980). To establish this model, person A must assimilate person B's words to his concepts and re-presentations and person B must engage in a similar activity. Von Glasersfeld (1985) suggests that establishing these types of models is the highest form of reality.

The extent to which each person's model of the other person is tested depends on the context. For example, in most everyday conversations, a person may do very little to check in what way his meanings are compatible with the meanings of the people with whom he communicates. In this situation, a person may simply tacitly assume that there is intersubjective agreement and may not examine this assumption unless there is some indication that this assumption is unwarranted. At the level of teaching and learning, a teacher may explicitly try to establish intersubjective agreement with his students as part of his classroom discourse. For example, a teacher may ask questions of a student to investigate or further investigate a student's activity. At the level of research, a researcher may try to establish scientific models of other people's meanings, which may undergo rigorous critique and refinement as part of the process of establishing intersubjective agreement among researchers.

It is in the context of intersubjective agreement that I think about the establishment of conventions. I define the establishment of *conventions* as two (or more) people engaged in activity that is aimed at establishing a compatible meaning for a particular symbol. For example, one person might make attempts to clarify a meaning that he attributes to a particular symbol to the other person. This activity would be activity that is aimed towards resuming the assumption that each person attributes a compatible meaning to a particular symbol. This notion of convention makes no attempt to suggest in what ways two people's meanings might be compatible. Rather, it focuses simply on the activity of attempting to establish compatibility so that communication might proceed.

CHAPTER 3: CONCEPTUAL FRAMEWORK FOR QUANTITATIVE REASONING AS A BASIS FOR CONSTRUCTING ALGEBRAIC SYMBOL SYSTEMS

I present this chapter in three parts. The first part provides a definition of quantitative reasoning, and some broad characteristics that I conjecture will be a part of the students' algebraic reasoning. I also provide three reasons to use quantitative reasoning as a basis for algebraic reasoning. I support these reasons with research literature and provide a comparison and contrast of my particular approach with other approaches to algebraic reasoning. In the second part, I specifically examine two bodies of research that are related to how students produce symbol systems. In analyzing these two bodies of research, I situate the work I am doing within this research. Finally, in the third part, I provide a rationale for the particular problems that I used to investigate the students' symbolizing activity. The goal of these problems was to expand the students' multiplicative reasoning. I provide several levels of reasoning for why I choose the particular problems that I did and base this reasoning partially in Piaget's research.

Quantitative Reasoning as a Basis for Algebraic Symbol Systems

What is Quantitative Reasoning?

Following Thompson (1994a), I consider a person to have established a quantity when he introduces a property to his object concept, a unit of measure for this property, and a way of assigning a numerical value to the property. I take reasoning to be the purposeful use of schemes and concepts in a context. So, I take quantitative reasoning to be a person's purposeful use of his schemes and concepts to create and relate quantities (Thompson, 1988, 1993). An example of a situation where a person produces a quantity and an example in which a person relates two quantities may help to illustrate these definitions.

To illustrate the components of my definition of a quantity, I provide the following hypothetical situation. A child might introduce a quantity into a situation in the following way. He might introduce length as a *property* of the fence in his backyard. Here, the child might introduce length from the activity of visually scanning the fence. Upon initial construction of such a property, a child's conception of length may not entail using a unit of measure or assigning a numerical value to the property. A child might introduce these two things by visually scanning from the first fence post to the second fence post, from the second fence post to the third fence post, etc., and enumerating the number of times he makes this visual scan. This hypothetical activity would include what Steffe (1991b) has termed a person's unitizing operation where the child applies his unitizing operation to his activity of visually scanning between fence posts to create units.

To illustrate quantitative reasoning, I present a second hypothetical situation. A student might be presented with the following problem situation: Bob and Joan have identical cakes. Bob cuts his cake into sevenths and eats six of his pieces. Joan cuts her cake into eighths and eats seven pieces. How much cake did each person have left over and who ate more cake? To solve this problem, a student might consider the amount that each person ate in relation to the whole cake in order to find the amount that each person had left over. The student might then compare the amount that each person had left over to find who ate more cake. Creating, comparing, and relating quantities would be an essential part of the solution I have described above and so this activity would involve quantitative reasoning.

What is algebraic reasoning?

I consider algebraic reasoning to be only a part of quantitative reasoning. So, I take quantitative reasoning to be a broader phenomenon than algebraic reasoning. However, as I suggested in the introduction, a major purpose of this dissertation study is to investigate what might constitute algebraic reasoning for eighth grade students. So, rather than begin with a fixed definition of algebraic reasoning, I am allowing this definition to grow out of the study. Nonetheless, in this section, I will provide three broad characteristics of reasoning that I conjecture will be important features of students' algebraic reasoning. As I discuss each of these broad characteristics, I connect them to the goal of studying students' construction of an algebraic symbol system.

The first broad characteristic that I conjecture will be part of students' algebraic reasoning is they may begin to abstract their schemes into concepts, enabling them to operate with the structure of their schemes. Abstracting a scheme into a concept means that a student would no longer need to enact the activity of his schemes. Rather, he would be able to use verbal language or written notation to symbolize the activity of his scheme. This criterion is not sufficient to differentiate algebraic reasoning from quantitative reasoning because students abstract schemes into concepts, in situations that most people would not consider algebraic. For example, Steffe, von Glasersfeld, Richards, and Cobb (1983) take as an indicator of the construction of a number *concept* a student taking the number word "seven" as symbolizing counting from one to seven without actually having to produce the activity of counting. This example suggests that simply abstracting the activity of a scheme into a concept might not constitute algebraic reasoning. So, I consider the second part of the statement operating with the structure of one's schemes to be a crucial aspect of algebraic reasoning. When a student has abstracted his schemes into concepts and can operate with the structure of his schemes, this opens the possibility for a student to begin relating his mathematical schemes and concepts. For example, a student might begin to compare and contrast different types of problems in order to find similarities and differences in these problems. In doing so, a student might identify problems as similar to one another by comparing the activities of his scheme across different problem situations. A student engaged in such activity would be operating with the structure of his schemes.

The second broad characteristic that I conjecture will be a part of the students' algebraic reasoning will be operating recursively. The notion of operating recursively means that a person will externalize the results of his scheme and operate on these results with operations external or internal to the scheme. Operating recursively enables a person to sustain activity in a problem situation because it enables him to produce a particular result and then take this result as input for further operating. As problem situations become more complex, I infer that the ability to operate on the results of one's activity will be central to solving the problems. For example, a student might produce a rate graph for a runner who is running at 5 miles per hour. Next, he might produce a second rate graph for a person who starts running at 6 miles per hour and 3 miles behind the first person. Then, he might operate on these two rate graphs in order to find when the second runner overtakes the first runner. Here, the student would be operating on the results of his activity (the two rate graphs) in order to continue reasoning about the situation.

The third broad characteristic that I conjecture will part of students' algebraic reasoning will be the use of verbal and written notation as a way of symbolizing the activity of the students' schemes and concepts. Many characterizations of algebra and algebraic reasoning involve the notion of a symbol system (e.g., Bell, 1996; Charbonneau, 1996; Mason, 1987;

Radford, 1996). Although I agree with Mason's (1996) warning against a rush to symbols, I nonetheless expect that one feature of the students' algebraic reasoning will be the production and use of symbols in creative ways that open the possibility for them to sustain their reasoning in a quantitative problem situation. However, when I make this statement, I want to make clear that I do not necessarily mean the use of conventional algebraic notation. Rather, I expect students to begin using notation in creative ways that helps enable them to reason further about problem situations.

Why Use Quantitative Reasoning as a Basis for Algebraic Reasoning and Algebraic Symbol Systems?

Steffe (1991b) has suggested the unitizing operation along with segmenting operations, partitioning operations, disembedding operations, and iterating operations as basic mental operations that are central in the construction of quantities and quantitative reasoning. I regard the development of these operations as important not just in the construction of schemes and concepts that produce extensive (e.g., height, length, money, etc.) and intensive quantities (e.g., temperature, rate, etc.), but also important in the construction of schemes and concepts that produce "purely mathematical quantities" (e.g., negative numbers, functions, polynomials, complex numbers, etc.). I make this point because of the frequent separation between extensive and intensive quantities, and "purely mathematical quantities". From my perspective, quantitative operations, schemes, and concepts are used to produce "purely mathematical quantities" and so such a focal point can help to lend coherence to students' mathematical educations as they progress in school. That is, focusing on developing quantitative operations as a basis for constructing extensive, intensive, and "purely mathematical quantities" makes for a sense of relatedness among the three.[13]

This position is in contrast to more formal approaches that treat "purely mathematical quantities" exclusively as entities that are to be defined formally (e.g., Cucco, 1993). However, it is in agreement with researchers who have characterized the construction of algebraic objects as a transition from process to object (e.g., Kieran, 1992; Moscovich, Schoenfeld, & Arcavi, 1993; Sfard & Linchevski, 1994; Sfard, 1995). That is, these researchers have found that students often regard their mathematical activity first as a process, which can eventually lead to the construction of a mathematical object. Such a view means that these researchers take students' activity as a central source for the construction of what I have called "purely mathematical quantities" as opposed to simply creating these objects through definition. For example, a student may view an expression like "3x + 2" as a process (three times an unknown number plus two) prior to the student being able to square this expression or take one half of this expression. The explanatory construct that is used for a process to become and object is reification (which is not dissimilar from the notion of a reflective abstraction). One difference between the process object approach and my approach is that the process object approach does not take as a central focus of analysis students' *quantitative* operations, schemes, and concepts. So, the researchers do not always explicitly regard notation like "3x + 2" as symbolizing a quantity (i.e., a length or area).

I consider a continued focus on students' quantitative operations, schemes, and concepts as central not only to developing students' algebraic reasoning but also in developing algebraic symbol systems. I consider it central to the development of algebraic symbol systems for three reasons: it opens the possibility for students' quantitative reasoning to be reflected in notation

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[13] For an example of an approach to complex numbers that could be used to focus on quantitative operations see http://www.geocities.com/eriktill/Situation3.

they produce, it opens the possibility to build students' mental imagery for problems situations, and it relates algebraic symbol systems to students' experiential realities.

Algebraic reasoning often entails creating relationships among known and unknown quantities (Bednarz & Janvier, 1996; Carraher, Schlieman, Brizuela, & Earnest, 2006; Dossey, 1998; Hackenberg, 2005; Smith III & Thompson, in press). Frequently, when students use notation for unknown quantities, researchers have found that the notation does not stand in for a quantitative property of an object concept (e.g., Booth, 1988; Clement, 1982; Kieran, 1989) or a relationship among two or more quantities (Knuth, Stephens, McNeil & Alibali, 2006; Kieran, 1981; Lee & Wheeler, 1989). These findings suggest that students frequently do not relate the notation they produce to their quantitative reasoning or have not developed their quantitative reasoning in ways that would support generating notation as a result of this reasoning. So, the first reason that I choose quantitative problem situations as a basis for investigating the construction of an algebraic symbol systems is that it opens the possibility for students to produce and use notation that stands in for both known and unknown quantities and their reasoning about these quantities.

Explicitly analyzing students' quantitative reasoning enables me as a researcher to keep as a central part of my research focus how students constitute quantitative knowns and unknowns and how students' constitution of quantities enables or constrains the notation they produce for these quantities. This approach contrasts with some other approaches to understanding algebraic symbol systems where the primary focus has been on how students use conventional algebraic notation (e.g., Herscoviks & Linchevski, 1994; Filloy & Rojano, 1989). This type of research has enabled the researchers to investigate how students use conventional algebraic notation, but it does not relate the students' use of notation to the students' quantitative reasoning. Without this connection, it makes it difficult to infer how students' reasoning is related to the notation that they use and leaves unaddressed how students produce notation for their reasoning. Further, as Schoenfeld, Smith, and Arcavi (1993) argue it can place too much emphasis on observable behaviors, which may or may not be good indicators of robust reasoning.

A second reason for using quantitative reasoning as a basis for the construction of algebraic symbol systems is that it affords opportunities for students' to build imagery. Geometric and quantitative images have played an important role in the historical development of what are recognized today as algebraic identities (Eves, 1953) and more specifically have played a role in the transition from geometric symbol systems to what is recognized as algebraic symbol systems (Charbonneau, 1996; Radford, 1996). Further, building students' imagery is seen as a source of creative mathematical and scientific thought (DiSessa, 1987; Goldin, 1987a; Hadamard, 1949; Thompson, 1996) and provides a basis for mathematical and linguistic symbol systems (Dougherty, 2004; Dougherty & Zillox, 2003; Kaput, 1987; Kieren, 1994; Piaget, 1951; Thompson, 1995). Quantitative problem situations afford the opportunity for students to base their mathematical activity in imagery that is related to the quantities in the situation.

An example may be useful to suggest in what ways focusing on quantitative reasoning can help build students' imagery. So, consider the following problem:

> *The Hallway Problem.* Suppose that the length of a hallway is twice the width of the hallway. The area of the hallway is 72 square feet. Can you find the length and width?

A student may solve this by finding all of the factors of 72 and finding one factor that is twice the other factor. This type of solution may be based purely in calculation. That is, a student may simply have relied on knowing that area can be evaluated through multiplication. In making a solution that is based in calculation, the student may not have constituted the area as a quantity that is related to the length and width of the hallway. Such a solution, then, would not open the possibility for the student to notate his solution in a way that relates the three quantities in the situation. Having students symbolize these quantities using pictures, diagrams, etc. (e.g., drawing a rectangle) opens the possibility for them to create a quantitative relationship in the situation. Establishing these quantitative relationships may open the possibility for the student to symbolize the situation using notation such as $2w = l$ and $A = wl$, so that $A = 2w^2 = 72$, which is unlikely to result from an approach that is rooted exclusively in calculation. Using a quantitative situation does not guarantee that a student would produce any particular notation, but it does open the possibility for the students' notation to reflect quantitative relationships that he creates in the situation.

In the previous paragraph, one facet of my discussion is compatible with researchers who have approached algebraic reasoning as generalized arithmetic (e.g., Carpenter, Franke & Levi, 2000; Carpenter & Levi, 2003; Peck & Jenks, 1988). These researchers have suggested that working on problem situations with students that focus on relating numbers, not on calculating with numbers, can serve as an important basis for algebraic reasoning in the early elementary grades. These researchers use statements of problems such as 6 + 6 = 7 + □ as opposed to problems such as 6 + 6 = to help students view mathematical notation as being about relating numeric statements as opposed to simply calculating a result. This research has not focused on using quantitative images as a basis for building up these notational statements, but nonetheless supports the notion that efforts need to be made to move away from having mathematical notation simply used as a calculational tool.

A third reason for using quantitative reasoning as a basis for the construction of an algebraic symbol system is that it can help students' constitute their mathematical reality as a part of their experiential reality. That is, students can come to see mathematical reasoning as a part of their daily lives and experiences. Further, quantitative problem situations can provide students motivation for solving problems and can help them to sustain interest in mathematics (Chazan, 2000; Confrey, 1998). I make these assertions because quantitative problem situations are derived from experiential situations. For example, consider the following problem:

> *The Sandwich Problem:* Suppose you have five kinds of meat and four kinds of bread. How many sandwiches could you make?

The solution to such a problem might include a student developing an image of making a sandwich by choosing a kind of meat and kind of bread. This type of imagery is derived from the experience of making a sandwich, which is a common experience for many people. Focusing on quantitative problem situations opens the possibility for students' construction of an algebraic symbol systems that is rooted in semantic meanings rather than exclusively syntactic meanings (Kaput, 1998).

Other Researchers' Approaches to Notation and Symbol Systems

In the previous section, I argued for three reasons for using quantitative reasoning as a basis for the construction of algebraic symbols systems. In this section, I will investigate two types of research that relate to students' construction of symbol systems. The first type of research investigates students' conceptions and uses of conventional notation and misconceptions about this notation. The second type of research investigates students' coordination of representations and their production and use of symbols.

Researchers have investigated students' conceptions and uses of the equality symbol (e.g., Behr, Erlwanger, Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth, Stephens, McNeil, &Alibali, 2006; Sanenz-Ludlow & Walgamuth, 1998), students operating with and on conventionally notated unknowns (e.g., Carraher, Schliemann & Brizuela, 2001; Filloy & Rojano, 1989; Hart, 1980; Herscovics & Linchevski, 1994; MacGregor & Stacey, 1997), and students' misconceptions when using conventional notation (e.g., Brown & Burton, 1978; Erlwanger, 1973; Krishner & Awtry, 2004; van Lehn, 1982). In a certain regard, this type of research is tangential to my study because my focus is not on investigating how students use (or misuse) conventional notation, but rather how they produce and use notation, how this functions in their reasoning, and how notation becomes conventional in the process of interaction. Nonetheless, it is one of the larger bodies of research that specifically focuses on students' use of conventional notation. So, it is pertinent to address the body of research here.

This research has helped to expose students' conceptions for notation of which a proficient user of notation may not be explicitly aware. Gaining this awareness can be useful in interacting with students over how they might produce and use notation. Moreover, it has provided insight into how students use particular symbol. For instance, the research on the equality symbol has suggested that students frequently treat the symbol as a sign to do something, as opposed to a symbol that relates two quantities. In response, researchers have modified problem situations in order to help engender the meaning of the equality symbol as a symbol that relates two quantities.

As part of conducting this research, the researchers have carried on debates about students' ability or lack of ability to use conventional notation. From my perspective, these debates are often rooted in treating a particular notation as having a non-context dependent meaning. For example, the debate about students' use of an unknown has centered on whether or not students can use a letter like "*x*" when reasoning about equations. In the debates, what is often disregarded is the context in which an unknown is used and how the use of notation relates to a person's current conceptions. For instance, I consider it to be different to use notation for an unknown in an additive context versus a multiplicative context. Moreover, even in a multiplicative context, a student can have a wide variety of conceptions of an unknown. For example, if a person has developed a conception of multiplication as primarily repeated groups, then a symbol like "$2x$" might be regarded as two groups of an unknown quantity, x. On the other hand, if a person has developed a conception of multiplication of two as a multiplicative operation, then a symbol like "$2x$" might be regarded as a dilation (or stretch) of an unknown quantity by two. So, for me, more precision about the meaning that a student attributes to a particular kind of notation would contribute to this debate.

One way to establish more precision is to examine how students' produce and use notation, which is a second way that researchers have investigated the construction of symbol systems. These researchers have investigated this issue by investigating students' coordination of multiple representations (e.g., Janvier, 1987; Kaput, 1987, 1989; Lesh, Post, & Behr, 1987a, 1987b; Moschovich et al., 1993; Schoenfeld, et al., 1993; Thompson, 1989) or more recently have investigated the symbols that students produce (Brizuela, 2004; Izsak, 2000; Nemirovsky & Monk, 2000). Researchers investigating students' coordination of multiple representations have conducted research on how students coordinate graphs, tables, and symbolic equations (Moschovich et al., 1993; Schoenfeld, et al., 1993) or translate one representation into a second (Janvier, 1987; Kaput, 1987, 1989; Lesh, Post, & Behr, 1987a, 1987b). According to Thompson (1994b), initially, the term representation was ill-defined among these researchers, which sometimes led to treating representations as static symbols for a particular concept. As this area of research developed some of the researchers focused less on the representations themselves and more on representable activity (e.g., Kaput, 1993; Moschovich, et al., 1993; Thompson 1994b). The investigation of representing as an activity is compatible with my study and use of the words symbolizing activity. Further, it is in line with researchers more recent interests in examining how students produce notation (Brizuela, 2004; Izsak, 2000; Lehrer, Schauble, Carpenter, & Penner, 2000; Nemirovsky & Monk, 2000). In contrast to many of these studies, I am specifically focusing on quantitative reasoning as a basis for algebraic symbol systems.

Students' Multiplicative Reasoning as a Context for Exploring Symbolizing Activity

In the previous parts of this chapter, I have used literature to compare and contrast my approach to other approaches to algebraic symbol systems and algebraic reasoning. In this final part of the chapter, I will present some research on students' multiplicative reasoning that provides a more local motivation for the particular problems I analyze in the case studies. In the case studies, I analyze the students' work on problems that from my perspective involved multiplicative combinations. Although the overarching purpose of my study was to understand students' symbolizing activity, it was in this particular context that I explored this issue.

I conceive of students' multiplicative reasoning in the context of students' units coordinating activity where a units coordination involves an insertion or injection of one unit into a second unit (Steffe, 1992). Using this conception, the most basic multiplicative concept for say 6 x 5 would involve mentally inserting or injecting five units into each of six units to produce the result 30. Steffe (1992) has suggested that students who can coordinate three levels of units would take the result 30 as itself a unit which would entail seeing 30 as consisting of a unit of 6 units each containing 5 units.

In my study, all three students were coordinating three levels of units prior to operating. So, I wanted to investigate in what ways I could expand their multiplicative reasoning. One of Piaget's (1958) distinctions between formal and concrete operations served as a basis for thinking about how I might expand the students' multiplicative reasoning (see also, Behr, Harel, Post, & Lesh, 1994). An example of this distinction based in how children experiment scientifically will help to illustrate my choice of problems. A child operating in the concrete operations stage might, when asked to determine which combinations of four clear liquids when combined produce a colored solution, experiment with the actual liquids to find which combinations of liquid produced colored solutions. In doing so, a child might not actually systematically keep track of the outcomes that were tested. On the other hand, a child operating in the formal operations stage might first imagine all possible combinations of two, three, and four liquids and only afterwards test what the results of these outcomes would be.

So, children in the concrete operations stage frequently operate sequentially and experiment with physical objects to produce possible outcomes. On the other hand, children in the formal operations stage produce all possible combinations in thought prior to producing any particular outcome. I took this as indication that problems aimed at engendering students' combinatorial reasoning might be an area in which the students might expand their multiplicative reasoning. So, I presented problems like the following:

> *The Dinner Menu Problem:* There are six kinds of dinners and five kinds of dessert on a menu at a restaurant. How many possible meals could you make?

In solving problems like this problem, I anticipated the students would create a new type of unit, a dinner-dessert unit. This type of unit is different from a unit of units of units structure in that the units that the student counts produce a new kind of unit that contains a pairing of a dinner and a dessert. I have called this new type of unit a pair because it contains an element from each of the composite units in the situation. Further, to solve this problem multiplicatively suggests that a student has considered all possible dinner dessert combinations as outcomes. So, this type of problem opens the possibility of investigating and extending the students multiplicative reasoning in the way that I was interested in investigating.

Also, I introduced these contexts as a way to investigate how the students' activity might be seen as related to a whole number variable concept and how their activity might be used to develop a concept of the sum of for example the first 15 whole numbers. The reader will get plenty of analysis of these concepts as they read the case studies. These two concepts were the local goals I had for these types of problems. I did, however, introduce this activity in the context of the larger teaching experiment for two other reasons.

The first reason for introducing these problems was because as I suggested it created a new type of unit a pair. This type of unit might be considered two-dimensional because, for example, the unit can be classified both as a unit that contains a particular dinner and a unit that contains a particular dessert. This type of unit can then be a basis for two-dimensional discrete units that are created from two one-dimensional composite units. So, this type of reasoning served as a basis for my work with binomials and quadratic equations where the students' reasoned both in discrete and continuous quantitative problem situations. These problem situations involved creating two-dimensional symbols—arrays and areas—relating the linear dimensions to the two-dimensional units, and partitioning these arrays and areas. The second reason for exploring how students produced this type of unit was that I was also subsequently interested in working on problems that involved functions. Conceiving of the Cartesian coordinate system as sets of ordered pairs requires being able to produce units of the type I suggested above. So, these types of problems also served as a basis for the future problems that I planned to present to the students.

CHAPTER 4: METHODS AND METHODOLOGY

Constructivist Teaching Experiments

In this section, I present the teaching experiment as a methodology for conducting scientific research on mathematics learning. I begin by presenting the two roles—a teacher-researcher and a witness-researcher— that researchers play when involved in a teaching experiment. Then, I examine the three kinds of conjectures—on the spot, ongoing, and broad—that guide the experiment. Finally, I present the retrospective analysis that leads to producing models of students' mathematics learning.

Two Kinds of Researchers

The Teacher-Researcher. Researchers conducting a constructivist teaching experiment work in teams of at least two members and use teaching as a method for investigating learning (e.g., Cobb & Steffe, 1983; Confrey & Lachance, 2000; Steffe & Thompson 2000; Thompson, 1979). One of the members of the research team acts as a teacher-researcher and the other members as witness-researchers. The teacher-researcher works with pairs of or individual students on mathematics activities to form an understanding of the students' mathematical ways of operating. In order to form this understanding, the teacher-researcher assumes that the mathematical reality of the students differ from his own. So, a primary goal of a constructivist teaching experiment is for the teacher-researcher to de-center from his way of operating in order to make a model of his students' mathematical activity.

To achieve this goal, the teacher-researcher works cyclically between intuitive and analytic interactions with the students. A teacher-researcher begins by operating intuitively and responsively to harmonize with a student's mathematical activity. Then, the teacher-researcher may begin to work more analytically and attempt to engage the student in mathematical interactions that aim at provoking perturbations with a goal of engendering accommodations in the student. In the course of working more analytically, a teacher-researcher may present tasks that create an unexpected response from the student, which may call for the teacher-researcher to work again in an intuitive way with the student.

In the previous paragraph, I have described the activity of the teacher-researcher as he interacts with students. The analytic work of the teacher-researcher, however, extends outside of the interactions with students. The teacher-researcher also engages in conceptual analysis of his knowledge. This type of analysis is aimed to develop the teacher-researcher's ideas about what might be necessary to construct a particular concept. So, he might analyze what he conjectures will be necessary to construct a concept of a linear function and transformations of this function. He, also, engages in analysis of the students' activity that is based on watching video of the previous teaching episodes to begin making a model of the students' mental activity. This type of analysis helps him formulate tasks to present to students in future episodes. Such work has been termed creating a hypothetical learning trajectory, a possible pathway that the teacher-researcher and the students may take (Simon, 1995; Steffe, 2004).

This pathway may be established at several different levels. For instance, a teacher-researcher formulates tasks that he plans to present over the course of several episodes and he also formulates long-term goals (i.e., year long) where he builds up sequences of tasks that he expects may be related. The hypothetical learning trajectory is constantly being shaped by the actual learning trajectory, the actual path that he and the students take. So, the teacher-researcher may need to significantly modify both his short-term and long-term goals based on his understanding of the students' ways of operating. In this way, the students and the teacher-researcher co-construct the actual learning trajectory.

The Witness-Researcher. The witness-researcher's primary function is to provide alternative perspectives for the teacher-researcher during all phases of the teaching experiment. The witness-researcher is present during all teaching episodes and helps in planning the activities that will take place during the episodes. During the teaching episodes, the witness-researcher suggests possible tasks to present to students when he sees a pathway that the teacher-researcher may not. However, the teacher-researcher is the person primarily in charge of the interaction with the students and so is free to decline the suggestions of the witness-researcher (Steffe & Thompson, 2000). In helping to plan future activities, the witness-researcher and teacher-researcher actively discuss and debate their current models of the students' mathematical activity. These discussions are aimed at refining the models that each of the researchers has of the students' mathematical activity.

Conjectures

I take a conjecture to be a dynamic supposition that is based on a teacher-researcher's current model of a student's way of operating (Confrey & LaChance, 2000). Three types of conjectures guide a teaching experiment. The first kind of conjecture occurs in the process of interacting with students. This type of conjecture is an on the spot conjecture, which guides the types of problems that the teacher-researcher poses to a student and are based in the creative contributions of a student. That is, based on a student's activity during a teaching episode, a teacher-researcher may make a conjecture about activity that he thinks a student can productively engage in and pose a problem based on this conjecture. The second kind of conjecture occurs outside of the interaction with the student. This type of conjecture is an ongoing conjecture that is formed during the analytic work that occurs outside the interaction with the student. For example, a teacher-researcher might conjecture that a student can solve a problem that involves a transformation of a function once the student is able to solve a problem that does not involve a transformation of a similar function. So, he might plan activities that are aimed at testing if the student can produce a transformation of a function. He would, then, test out his conjecture during a future teaching episode by presenting these problems to the student. The third conjectures are broad conjectures that guide the overall study. I will return to these conjectures later on to present two broad conjectures that guided this particular study.

Data Analysis

The first type of analysis is ongoing analysis. I have already described this kind of analysis, which occurs by watching videos of the teaching episodes and debating and planning future episodes. The second type of analysis that occurs after the data has been collected is retrospective analysis. The purpose of the retrospective analysis is to make models of the students' ways of operating mathematically. Such a model, a second order model, is an attempt to explain the way that a student operated in mathematical situations, to abstract the important qualities of these ways of operating, and to account for changes that a student made in his ways of operating. This model is one created as an analytic tool that the researcher uses to create a record of the path on which he and the students took and accommodations that the students made during the interactions. This work may highlight certain paths as fruitful and certain paths as more limited in provoking the construction of more robust mathematical ways of operating. Such models can inform the design of future studies and curricula.

The process for constructing a second order model is an iterative process (Lesh & Lehrer, 2000). The researcher watches video of all of the teaching episodes, creates preliminary models of the students' ways of operating, and then re-examines the video data to test for consistencies and inconsistencies in the preliminary models. Repetition of this process may eventually lead the researcher to think the model accounts for the activities of the student. During this process, the researchers on the research team continue to share their models of the students' mathematical activity to continue to get feedback from one another and to aid in further revision of the models.

Viability and Generalizability

In building second order models, a teacher-researcher attempts to create a viable—a consistent and coherent—explanatory model of a student's mathematical ways of operating. In this sense, the impetus is on the teacher-researcher to be continuously testing his current model of a student against their experience of the student in mathematical interaction. This process of testing is a process through which the teacher-researcher attempts to bring forth student's independent mathematical contributions, which may not be consistent with their current model of the student. That is, the teacher-researcher may find out in interaction with the student that the student's own creative mathematical acts may not be the actions the researcher anticipated given the model he had built. In this event, the teacher-researcher should be compelled to modify the model he has made of the student in order to broaden his model to incorporate the student's new or unexpected way of operating.

Witness-researchers and the broader research community are also essential in the creation of viable models. I have discussed how the witness-researcher provides a second perspective. Once a researcher has created a model of the students' mathematical activity, he should then present his model to the broader research community where outside researchers can examine the model and engage the model maker in further discussion and debate. This discussion and debate serves for further refinement of the researcher's model, which opens the possibility to make the model more viable.

One goal in making these models is that the researcher's model should help to explain more than simply a single student in a given context. Therefore, one goal of the researcher in model making is to generate theoretical constructs that help them to understand a broad range of phenomena, which they may use to help them understand future interactions with students. A researcher may test out the applicability of these theoretical constructs in future teaching situations with other students. The purpose of these tests is to see if the models contain theoretical constructs that can be applied to situations other than the one from which the original model was constructed. Through future interaction with students aimed at testing theoretical constructs, the researcher may create models containing constructs, which are able to explain a broader range of phenomenon. This activity is aimed at generalizability of a researcher's explanatory models.

My Teaching Experiment

The Broader Study

As I stated in chapter 1, the data that I analyze in the case studies was part of a three year teaching experiment whose broad purpose was to understand middle school students' reasoning. The teaching experiment began in October of 2003 and finished in May of 2006 at a rural middle school in northern Georgia. The initial selection interviews were conducted with 20 students in September and October of 2003. The students were selected from a range of different levels of mathematics classes and exhibited a range of school mathematics achievement.

In these selection interviews, we[14] presented students with specific tasks that had been identified in previous teaching experiments as important in the construction of whole number knowledge and fraction knowledge (See Appendix A). In particular, we wanted to identify students that had constructed different levels of number sequences. From these selection interviews, we selected eight students. We conjectured two of the students had constructed the initial number sequence, two had constructed the tacitly nested number sequence, two had constructed the explicitly nested number sequence, and two had conjectured the generalized number sequence (for more detail on these number sequences see Steffe, von Glasersfeld, Richards, & Cobb (1983)). We, then, taught these students during their sixth, seventh, and eighth grade years. Half of the students were boys and half were girls. Six of the students were Caucasian, one was African American, and one was Vietnamese. Two students participated in the study for most of the first two years and then did not participate for the third year.

Over the three-year period, I was a teacher-researcher, teaching two of the four pairs of students each semester and a witness-researcher during the other teaching sessions. The two pairs that I taught differed from year to year. At the end of the experiment, I had taught all of the students for at least one year. The teaching episodes lasted for thirty to forty-five minutes and were conducted during the students' regular school hours. We met with the students for two weeks, then took a week off, and repeated this cycle throughout the year. During the three-year period, we met with each pair of students approximately 90 times. We videotaped each of the teaching episodes using two cameras one to capture student work and the other to capture the student-student and student-teacher interaction (Figure 4.1).

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[14] When I use the word "we", I refer to the research team that was headed by Dr. Leslie P. Steffe. During the first year Amy Hackenberg and I were the sole graduate students involved. During the second year, Zelha Tunc Pekkan and Hyung Sook Lee joined Amy and me. During the third year, Erin Horst, Kyle Schultz, and I were the graduate students involved.

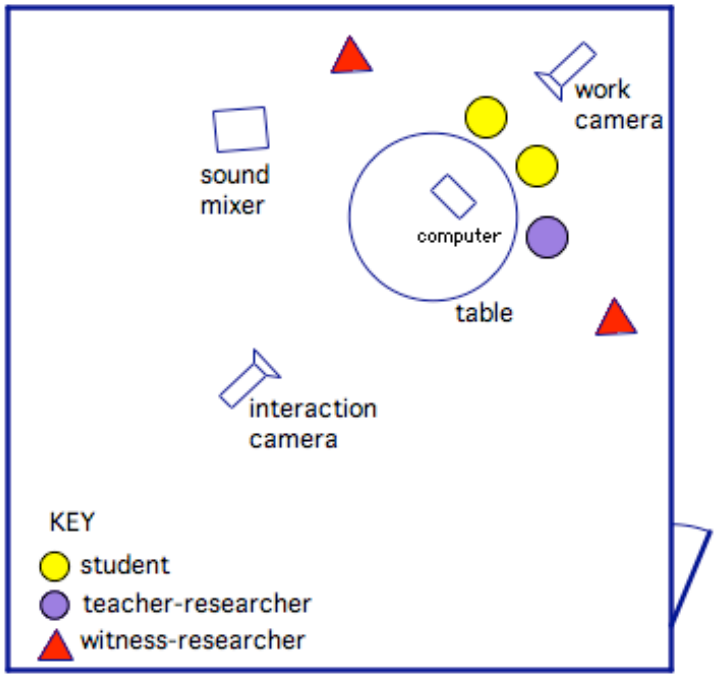

Figure 4.1: Room Setup

During most of the sessions, we worked in JavaBars or the Geometer's Sketchpad (GSP) although in some sessions I had the students work with paper and pencil. The intent of using the computer microworlds was to enable the students to engage in productive mathematical activity in an environment that supported students' quantitative reasoning. For example, JavaBars opens the possibility for students to create and enact a variety of different operations (i.e., partitioning, disembedding, iterating, etc.) on the rectangles and squares. In JavaBars, it is possible to create a square, horizontally partition it into fourteen parts, vertically partition it into fourteen parts, and then break the fourteen by fourteen square into two four by ten rectangles, a four by four square, and a ten by ten square (Figure 4.2). In GSP, it is possible to create segments, rectangles, and squares and operate on them using translations, rotations, reflections, and dilations.

Figure 4.2: A screen shot of JavaBars

The Data for the Case Studies

The variety of tasks that I posed over the three-year period to different students was large especially given that the tasks were tailored to each individual student. So, I will not give an overview of all of these tasks. Rather, in Appendix B, I give a summary of the types of problems that I posed to the *three* students who I worked with during the final year of the experiment. These three students are the students that I report on in the case studies. Each of them had constructed the generalized number sequence by the time they entered their eighth grade year. Two of these students (one a Caucasian female and the other a Caucasian male) were in an eighth grade algebra course and one (a Vietnamese male) was in an eighth grade pre-algebra course. I taught two of the students in a pair (two boys, one Caucasian and the other Vietnamese) and one by herself (Caucasian).

During the year, I taught the pair of boys a total of 27 times and the girl 26 times. One witness-researcher was at every teaching episode and at most episodes there were three witness-researchers. At the end of each week, I mixed the videotapes using a digital mixer into a single picture in picture video that could be watched on a computer. While mixing the videotapes, I watched many of the episodes, recording the problems I had posed, and writing possible new problems that I could present. Then, the research team met for between half an hour and an hour prior to the actual teaching episodes. During these meetings, the research team discussed previous episodes as a group, helped me refine the problems that I was going to present, and discussed current models and conjectures about the students' reasoning. During the actual teaching episodes, the witness-researchers intervened with suggestions that were usually made to me and then I incorporated these suggestions into the problems if I thought that it was appropriate.

At the end of the year, I watched all of the teaching episodes, looking for significant events and themes in the data. I took notes on the entire corpus of data, transcribing particular parts of the data for subsequent analysis. I, then, re-watched all of the video a second time and refined the notes that I had taken paying particular attention to the areas in the data that I thought were significant. At this point, I began writing the case studies, realizing that I might not be able to incorporate all of the data into the case studies. So, I focused on the data from the first four episodes and I began writing the case study of the boys. When I finished a first draft of this case study, I began writing the second case study of the girl. During the writing of this case study, I began to refine my understanding of the activity of the boys because it gave a point of comparison for the first case study. When I had finished a draft of both case studies (early December of 2006), I gave them to Dr. Steffe who gave me feedback and his perspective on each.

We continued to have discussions about the case studies and I continued to refine my models of the student's operating. This process continued through three more drafts of the case studies. During this time, I also began to re-watch other parts of the data and write about these other parts of the data. The purpose of this activity was to see in what ways my models of the students reasoning could serve as a basis for a broader interpretation of the students' activity throughout the year. These parts of writing are not part of the case studies, but I do consider them to have been a useful part of the analytic process. Also, as a part of this process, I used previous models of students' reasoning in order to help me interpret the phenomena that I was observing. This analytic process is consistent with methods that use an iterative video analysis (e.g., Cobb & Whiteneck, 1996; Lesh & Lehrer, 2000).

Broad Conjectures that Guided this Study

Two broad conjectures guided this study. The first conjecture is that the level to which a student has abstracted his schemes into concepts will have consequences on the ways in which a student symbolizes a quantitative situation. When a student has abstracted a mathematical concept, I conjecture he may more productively produce written notation for his activity. On the other hand, if a student has yet to abstract a concept from his schemes, it is more likely that the notation that he produces will be a record of his operating. In this case, he may not be able to use his notation as a tool in his reasoning. A second broad conjecture is that students will have significant difficulty having written notation symbolize quantitative relationships. This conjecture is simply based in the research literature, which suggests the difficulty that students' have in symbolizing their mathematical activity. One of the goals of this study is to identify how students can learn to use notation in meaningful ways to them.

CHAPTER 5: CARLOS AND MICHAEL

Introduction

In this case study, I examine the first four episodes of the teaching experiment I conducted with Michael and Carlos. The goal of the four teaching episodes was to explore if, and how, the students would coordinate their additive and multiplicative reasoning to produce a formula for the sum of the first *n* whole numbers. During the first two episodes, I presented the boys with problems similar to the following:

> Task 5.1, *The Outfits Problem*: Suppose you have four shirts and three pairs of pants. How many outfits could you make?

I used these problems as a way to explore Carlos's and Michael's multiplicative reasoning. During the third and fourth episodes, I posed problem situations such as:

> Task 5.2, *The Handshake Problem*: Suppose that there are four people in this room and each person wants to shake every other person's hand. How many different handshakes would there be?

The goal of this sequence of tasks was to see if the boys would use their activity in the Outfits Problem as a basis to reason multiplicatively about the Handshake Problem. That is, would they reason that each of four people could shake three other people's hands and then eliminate half of these handshakes because each handshake had been counted twice? Further, would they see their elimination of half the handshakes as producing the sum of the first three whole numbers (Figure 5.1a & Figure 5.1b)? In Figure 5.1a, the first column of letters symbolizes the four people. Each row after the vertical line symbolizes the three people with whom that person could shake hands. In Figure 5.1b, the red symbolizes the elimination of duplicate handshakes. This type of reasoning produces the sum as the result of multiplicative operations so it can be evaluated using these multiplicative operations (e.g., the sum of the first three whole numbers is half of four times three).

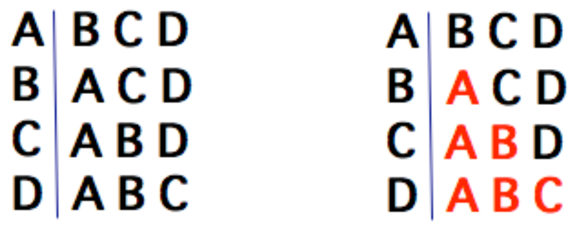

Figure 5.1a (left) Figure 5.1b (right): A diagram for a solution to the Handshake Problem[15]

Rather than produce a multiplicative solution for problems like the Handshake Problem, the boys each produced an additive solution. Each reasoned that the first person could shake three people's hands (everyone's hand but their own), the second person could shake two people's hands (everyone's hand but their own and the first person's hand), etc. So, *evaluating* the sum using multiplicative reasoning became a separate but associated problem to solving the quantitative problem situations.

Given that this was my initial experience with the boys, I presented them several tasks during the fourth episode where the starting place of reasoning was *only* evaluating the sum, rather than producing the sum in a quantitative situation and then evaluating it. These problem contexts were similar to the following:

> Task 5.3, *Sum Problem:* I wonder if you can find a quick way to evaluate the sum of the first 25 whole numbers?

In these contexts, the boys did find a way to relate their additive and multiplicative reasoning. For a full overview of the tasks Carlos and Michael worked on see Figure 5.2

	Carlos	**Michael**
Teaching Episode 1, 10/17/05	The Outfits Problem, Task 5.1; Three Card Combination Problem,	The Outfits Problem, Task 5.1; The Extension of the Outfits

[15] This figure is similar to one that Deborah, the third student in my study, produced to reason about this situation.

	Task 5.5	Problem, Task 5.8
Teaching Episode 2, 10/19/05	The Card Problem, Task 5.4; The Coin Problem, no task number or protocol number	Coin-Die Problem, Protocol 5.15
Teaching Episode 3, 11/02/05	The Flag Problem, Task 5.6	The Flag Problem, Task 5.6
Teaching Episode 4, 11/09/05	The Handshake Problem, Task 5.7; The Sum Problem, Task 5.3	The Handshake Problem, Task 5.7; The Sum Problem, Task, 5.3

Figure 5.2: Overview of the tasks I analyze that Carlos and Michael worked on

I present this overview as a way to understand the structure of this case study, which I have organized into six sections. I begin with three sections—one that examines Carlos's activity in problems like the Outfits Problem, one that examines his activity in problems like the Handshake Problem, and one that examines his activity in evaluating sums of consecutive whole numbers. I, then have three similar sections for Michael. This organization is based on my summary of how the boys reasoned during the first four episodes.

During the episodes, each boy saw these three types of problems as separate in the sense that each type of problem consumed their attention while they were engaged in producing their solutions. So, their predominant experience was of these three types of problems as distinct even though they saw them as associated with one another. During the analysis, I will provide an account that illuminates why I contend the boys' experience was this way, as well as provide indication of where they used their activity in one of these three situations to inform their activity in one of the other situations. Throughout this discussion, I build my working model of the role the students' symbolizing activity played in their solution of these problem situations.

Carlos's Activity to Solve the Outfits Problem and Similar Problems[16]

Overview

This section of the case study analyzes problems I presented to Carlos in the first two teaching episodes and is presented in three parts. In the first part, I analyze Carlos's multiplicative reasoning in the context of solving problems like the Outfits Problem and I analyze several features of his symbolizing activity while solving these problems. In the second parts, I analyze the scheme that Carlos used to solve the problems that I analyzed in the first part and I analyze important concepts that are related to this scheme. In the third part, I analyze Carlos's activity when solving two problems that opened the possibility for Carlos to reason recursively.

Carlos's Solution to Two Problems

Carlos's Initial Notation and Diagrams for the Outfits Problem. I presented the Outfits Problem, Task 5.1, to Carlos at the beginning of the first day of the teaching experiment.

> Task 5.1, *The Outfits Problem*: Suppose you have four shirts and three pairs of pants. How many outfits could you make?

> Protocol 5.1: Carlos's initial solution of the Outfits Problem on 10/17/05[17]
> C: [C sits for approximately 3 seconds after I pose the problem.] Twelve.
> T: How did you know that?
> C: 'Cause you multiply four times three and that's twelve.
> T: Could you draw me a picture of why it's four times three?
> C: [C draws a diagram of a shirt and writes "x 4" and a pair of pants and writes "x 3" (Figure 5.3)]. There would be one pair [C draws a line from the shirt to the pair of pants] so that would be one [C makes one tally mark below his picture] and then since there are four of them that would be three pairs [C adds two tally marks to the first one], 'cause you put the pants with the shirts and then you have another shirt left over and so you can

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[16] Carlos worked by himself on the first day of the teaching experiment.
[17] In the protocols, M stands for Michael, C for Carlos, D for Deborah, T for the teacher-researcher (the author), and W for a witness-researcher. Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue or interaction.

use that one with a different pair [C draws a fourth tally mark]. Okay, and then you can use it with one of the other shirts. So…okay hang on I messed up. [C writes next to the shirt he has drawn "B, R, G, W" and next to the pair of pants "B, W, R."] Let me see. There would be the blue shirt and the blue pants and then there would be the blue shirt and the white pants and then the blue shirt and the red pants [C writes "BB," "WB," and "BR" in a column][18]. And then you would do that for each one. And then…[quietly] and then it could be [C finishes writing out four columns]. So you count all these up and you get...there's three here, three here, three here, and three here. You add these two up you get six. You add these two up you get six. You add these two and you get twelve.

Figure 5.3: Carlos's diagram of a representative shirt and pair of pant

Initially, Carlos said "twelve" and "four times three," which provided little indication of why he solved the problem using multiplication. So, I asked him to provide an explanation that might give me some insight into how he would operate in the situation. In Carlos's first attempt to formulate an explanation to the problem, he drew a shirt and pair of pants. I take this diagram as indication that Carlos could take one shirt as a symbolizing *all* four shirts and similarly one pair of pants as symbolizing *all* three pairs of pants (Figure 5.3). This provides indication that Carlos's unit of one was iterable because he could take one unit, in this case either a shirt or pair of pants, as a symbol for all of the units in his number concepts without actually having to iterate

[18] Note what I have written is not a typographical error. Carlos wrote down WB rather than BW, but then wrote BR.

the unit of one to produce these units.[19] . Here, Carlos established a quantitative meaning for his concept of four as the *amount* of shirts in the situation where each shirt was identical to all of the other shirts (a similar statement could be made about the pairs of pants).

Then, Carlos drew a line from the shirt to the pair of pants, and subsequently, drew a tally mark below his diagram of the shirt and pants, concluding, “that would be one”. I infer that the line he drew between the shirt and pair of pants symbolized “putting” a pair of pants with a shirt (Figure 5.4a) and that the tally mark below the diagram symbolized the results of this interiorized action, an outfit (Figure 5.4b). I call the operation that Carlos used to “put” the shirt with the pair of pants his units coordinating operation; he took a unit from his concept of four and coordinated it with a unit from his concept of three. Because Carlos drew a line to notate making the units coordination and a tally mark to notate the results of this units coordination, I infer he had applied his unitizing operation to his units coordinating activity to produce a *unit* that contained one unit from his concept of four and one unit from his concept of three. I call the resulting units *pairs* because they are a *unity* that contains two units (i.e., for Carlos, an outfit was a unit that contained a shirt and a pair of pants).[20]

Figure 5.4a (Left) & Figure 5.4b (right): A diagram of Carlos’s mental imagery and operations.

After drawing the tally mark, Carlos continued his operating by “putting” all three shirts with three of the four pairs of pants (Figure 5.5a) and added two new tally marks as symbols for the results of this interiorized action, two new outfits (Figure 5.5b) [“…since there are four of them (shirts) that would be three pairs (outfits) cause you put the pants with the shirts”]. Here,

[19] Carlos came into the sixth grade with iterable units of one (Hackenberg, 2005).
[20] I will also refer to them as outfits in this situation.

Carlos appeared to be making a one to one correspondence between the first shirt and the first pair of pants, the second shirt and the second pair of pants, and the third shirt and the third pair of pants. Further, because Carlos only counted two new outfits after he produced the three outfits depicted by the solid lines in Figure 5.5b, I infer he was keeping track of the outfits he had produced.

Figure 5.5a (Left) & Figure 5.5b (Right): A diagram of Carlos's mental imagery and operations

Carlos then took the "left over" shirt and I infer that he made a new outfit ["…and so you can use that one (shirt) with a different pair (of pants)"] (See Figure 5.6).

Figure 5.6: A diagram of Carlos's mental imagery and operations

Carlos attempted to continue his activity after he created a fourth outfit ["Okay, and then you can use it (the pair of pants) with one of the other shirts."]. But he seemed to experience a perturbation ["So…okay hang on I messed up."]. I infer this perturbation arose from two aspects of his operating: his ability to keep track of which outfits he produced, and a novel aspect that he had introduced when he created the fourth outfit. The novel aspect was that he had produced two outfits using the same pair of pants and two different shirts. I infer this created a perturbation for Carlos because he wanted to be able to keep track of which outfits he had produced. To do so, he had to find a way to differentiate between the two outfits that he had created using the same pair of pants.

When he re-initialized his activity, Carlos introduced a way to help him keep track of which outfits he had produced—he ordered the shirts. He symbolized this ordering with “B, R, W, G”, and similarly he ordered the pairs of pants, symbolizing this ordering with “B, W, R”. To establish this ordering, he selected a property, color, of his shirt concept that he could use to differentiate each of the shirts from one another (a similar statement could be made about his pants concept). So the first unit of his concept of four was a different color from the second unit of his concept of four, etc. This *ordering* was a secondary meaning for his number concepts other than the *amount* meaning he had first established. He, then, began his units coordinating activity symbolizing the outfits he imagined producing with his notation “BB, WB, BR” (Figure 5.7). So he used both the amount and ordering meaning for his concept of three to make the first three outfits—the amount meaning to establish the total number of outfits he could make with the first shirt, and the ordering meaning to establish the order in which he produced the outfits.

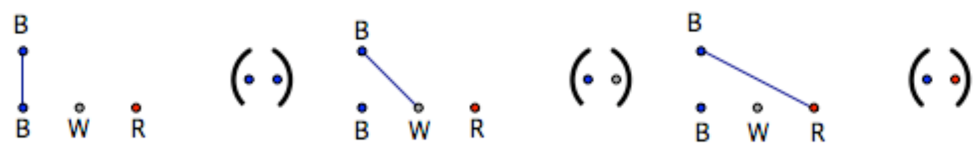

Figure 5.7: A diagram of Carlos’s mental imagery and operations

Carlos finished making the outfits in sequence imagining putting each particular pair of pants with each particular shirt, symbolizing these outfits with his notation. Carlos produced these outfits using a lexicographic ordering. A lexicographic ordering is an ordering that is similar to how words are organized in the dictionary. For instance, the two-letter word “*aa*” would appear before the two-letter word “*ab*” because “*a*” is before “*b*” in the alphabet. In the data excerpt, Carlos produced the first outfit with the first shirt and first pair of pants, the second outfit with the first shirt and the second pair of pants, the third outfit with the first shirt and the third pair of pants, the fourth outfit with the second shirt and first pair of pants etc., which established a lexicographic ordering.

Prior to writing out the final nine outfits, Carlos said, “and then you would do that for each one (shirt)”. Carlos’s language seemed to indicate an important feature of his way of operating. Carlos had united the three outfits and could anticipate using this sequence of operations for each of the remaining shirts, which would produce the result he expected—12 outfits. So, Carlos had identified a property of his shirt concept (color) to order the four shirts. Nonetheless, Carlos’s statement suggested he could take the blue shirt as *representative* of how each of the shirts would function in the situation. So, for Carlos, producing three outfits sequentially could symbolize engaging in these operations three more times because he considered the blue shirt to be identical to all of the other shirts (even though he did sequentially produce each of the outfits).

Carlos Produces a Tree Diagram. To investigate further Carlos’s operating in the situation, I present his activity to produce a tree diagram.

Protocol 5.2: Carlos produces a tree diagram for the Outfits Problem on 10/17/05

T: So could you make a chart now where there are pants on one side and shirts on the other side to show…[T intends for C to produce an array]
W: Like a square thing.
T: Yeah, like a square.
C: Yeah, I think so [C sits for 23 seconds in concentration. In a vertical column, he writes out the four letters, “B, R, W, G”, as symbols for the shirts. He writes the letters “B, W, R” vertically next to the letter “B”, as a symbol for the three outfits he can make with the first shirt. He repeats this process for the three remaining letters that are symbols for the shirts. He draws lines to connect for example the letter “B” with the letters “B, W, R” and draws a square around each of these sections of his tree diagram (See Figure 5.8). C has sat for a total of 1 minute in deep concentration while producing the diagram.]
W: That’s kind of like a tree with branches, isn’t it?
C: Yeah. [C, T, W all laugh]. So, these are kind of like the leaves.

Figure 5.8: Carlos's tree diagram

When Carlos made the diagram, he no longer appeared to sequentially produce twelve outfits. I make this inference based on the way in which he produced the diagram. He first wrote the letters "B", "R", "G", and "W", and then, next to each of these letters wrote "B", "W", and "R". He made the connecting lines at the end of making his diagram (almost as an afterthought), which suggested that he was no longer sequentially put each pair of pants with a particular shirt. Rather, I infer that each time he placed the three letters, "B", "W", "R", next to the letters that symbolized each shirt, he engaged in a units coordination between a particular shirt and a *representative* pair of pants. I use the term representative pair of pants in parallel to the meaning of an iterable unit except that a representative pair of pants symbolizes both an amount and ordering meaning for a person's number concept. In this case, the representative pair of pants symbolized three pants for which Carlos had established an ordering. So, putting a particular shirt with a representative pair of pants produced a representative outfit that symbolized the first three outfits Carlos had originally made (Figure 5.9). He engaged in this way of operating four times (one for each particular shirt) so he produced four representative outfits each of which symbolized three outfits. This way of operating provides indication that Carlos re-established that each of the pairs of pants served an identical role in the problem situation regardless of his having created an ordering for them.

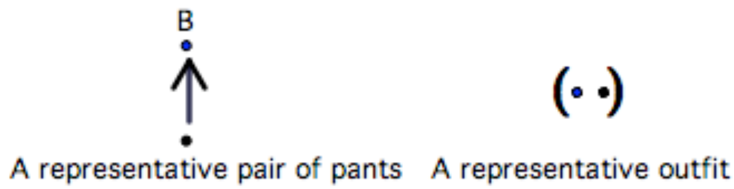

Figure 5.9: A diagram of Carlos's imagery and operations

It is important to point out the way in which Carlos's operating differed from his original way of operating. In this solution, he created a representative unit for the pants and put it with a particular shirt, which enabled him to create four representative outfits each of which symbolized three outfits. So, he iterated this units coordinating operation to make four representative outfits, each of which symbolized three outfits. In his previous solution, he had made a units coordination between each particular pair of pants and each particular shirt. Although based on his language, it appeared that Carlos could take the number of shirts as symbolizing the number of times he would produce three outfits sequentially. So, in the first solution, it appeared he took the blue shirt as representative of all the other shirts and it symbolized the number of times he could make three outfits. In this solution, he appeared to take a pair of pants as representative of the number of outfits he would produce with each particular shirt.

Carlos Produces an Array for the Outfits Problem. To investigate if Carlos could coordinate these two ways of operating, I present his activity to create an array for the Outfits Problem.

Protocol 5.3: Carlos produces an array for the Outfits Problem on 10/17/05

T: Can you make a rectangle with the shirts across one side and the pants on the other?
C: So, you want me to put the shirts on one side [C runs his finger vertically.]
T: Yeah, the shirts on one side [T runs his finger vertically mimicking C's actions.] and the pants on the other.
C: Oh, okay. I get what you're saying. [C writes the four letters vertically as symbols for the shirts and three letters horizontally as symbols for the pants.] So, just like that?
T: Yeah. Why don't you draw a dot for each pair of pants and each shirt?

C: [C draws horizontal lines extending out from each letter on the vertical axis (Figure 5.10a). He draws a vertical line from the horizontal axis and pauses momentarily when the two lines meet (Figure 5.10b). He then extends the line all the way up and makes dots at each intersection (Figure 5.10c). He draws in two more vertical lines extending from where each pair of pants is symbolized (Figure 5.10d). He draws in the four horizontal lines and fills in the dots (Figure 5.10e). C has sat for 1 minute and 10 seconds in deep concentration while making the array.]

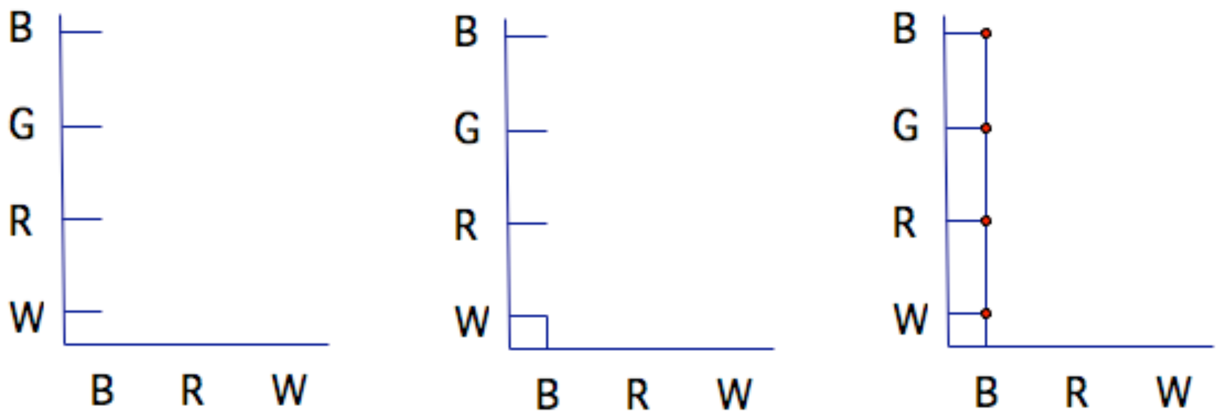

Figure 5.10a (left), 5.10b (center), and 5.10c (right): Replica of how Carlos made his array

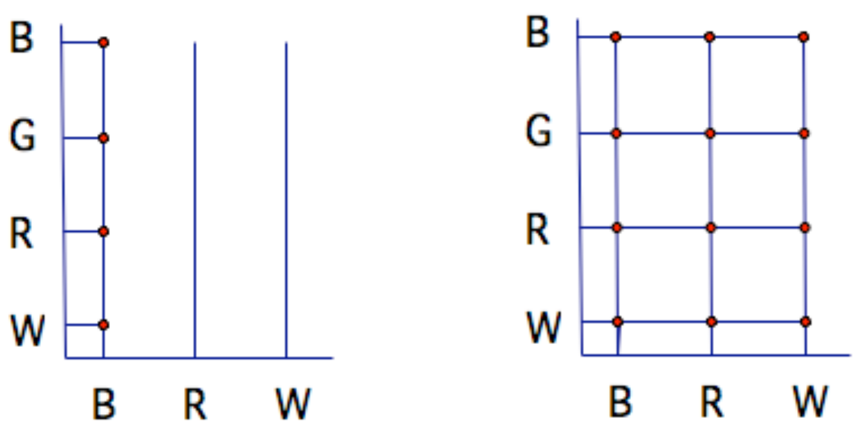

Figure 5.10d (left), and 5.10e (right): Replica of how Carlos made his array

Here, Carlos drew in horizontal lines from the vertical axis (Figure 5.10a) and drew a vertical line from the horizontal axis, pausing momentarily where the vertical and horizontal lines intersected (Figure 5.10b). Based on his pausing at this intersection, I infer that Carlos engaged in a units coordination. I infer that the units coordination was between a *representative* shirt and a particular pair of pants, which symbolized the 4 outfits that he could make with this particular pair of pants. I make this inference based on his extending the vertical line upward and filling in the four dots to "show" the outfits without further indication that he was imagining putting each shirt with the pair of pants (Figure 5.10c). Here, Carlos, treated the shirts as if they functioned identically even though he had ordered them. I make this inference based on his seeming to produce only one units coordination between the first pair of pants and a representative shirt and his taking this activity as indicating all of the outfits he could make with this pair of pants.

I infer that Carlos, then, used the pairs of pants in the situation as indication of the number of times he could produce 4 outfits without engaging in any further units coordinating activity to produce these outfits. I make this inference based on his drawing in the two vertical lines in his diagram, which I infer were symbols for producing 4 outfits two more times (Figure 5.10d). Here, Carlos engaged in a units coordination between a particular pair of pants and a representative shirt, which symbolized four outfits (as opposed to a particular shirt and a representative pair of pants). More importantly, he engaged in one units coordinating operation and then could take this as symbolizing all of the other units coordinating activity in which he could engage.

Carlos seemed to find the array that I suggested he make to be a novel symbol for his activity and one that he liked quite a bit. I make this assertion based on his solution of the next problem, the Coin-Die Problem. For this problem, Carlos and I played a game where he rolled a die and I flipped a coin. In the game, he recorded each new coin-die combination on his piece of paper. After recording three such combinations (i.e., "6T"), I asked Carlos how many possible coin-die combinations we could get. Rather than answer immediately, as he had in the Outfits Problem, Carlos sat in concentration for 30 seconds and then smiled broadly, and said, "twelve".

I infer that in solving this problem Carlos was operating on mental imagery that he had interiorized during his solution of the Outfits Problem. I make this inference because in making his explanation Carlos produced an array in a similar manner to the one he produced in the Outfits Problem. Carlos seemed very pleased with this new way of symbolizing his activity, smiling and saying, when I asked him to explain, “its like you showed me in the last problem”.

Carlos Produces Part of an Array for the Two-Deck Card Problem. The Outfits Problem involved small numbers and so was conducive for observing Carlos engage in the activity I described above. However, I wanted to test how Carlos would operate in situations that involved larger numbers. So, at the beginning of the second day of the teaching experiment I had Carlos and Michael play a card game. In the game, each of the boys had a deck of fifty-two cards. The game consisted of each of them choosing a card at random out of the deck, seeing who had the higher card, and recording the resulting pair of cards. The game was to give them an experience in which they might form pairs of cards. I then posed the following problem to Carlos and Michael.

> Task 5.4, *The Two Deck Card Problem:* If you were going to write down all the possible pairs of cards you could get, how many would you write down?

Carlos initially solved this problem by multiplying 52 times 52 using his standard computational algorithm for whole number multiplication. Michael seemed unsure of how to answer the question. So, I asked Carlos to explain to Michael how he had gotten his answer.

> Protocol 5.4: Carlos’s Solution of the Card Problem on 10/19/05
>
> T: Carlos could you explain why you multiplied?
> M: You finally found something I’m not good at.
> C: ‘Cause there are two pairs of decks that have fifty-two. So there are fifty-two possibilities of getting each, or kind of like this…[C begins to draw an array (Figure 5.11).] The same way you showed me before. So, its gonna be “K”, “A” [C is referring to the king thru ace in his deck], and then “A” thru “K” and then you would go all the way

up [C moves his pencil vertically] and keep on dotting all thirteen, no all fifty-two…[M interrupts C who does not appear to be done with his explanation. C begins to move his pencil horizontally.]
M: I don't get that. Why are you all talking about dots?

Figure 5.11: Carlos's Array

Carlos appeared to assimilate this situation with two representative units, one that symbolized the 52 cards in the first deck and one that symbolized the 52 cards in the second deck. I make this inference based on his saying, "its going to be 'K', 'A' and then 'A' thru 'K'". I infer that Carlos made one units coordinations simultaneously with saying, "you would go all the way up and keep on dotting all thirteen, no all fifty-two". Here, I infer he imagined taking a particular card from the first deck and putting it with a representative card from the second deck. This operation symbolized 52 two-card combinations that he could make with the particular card from the first deck and all of the cards from the second deck. When he swept his pencil horizontally, I infer that he switched his focus of attention to the horizontal dimension and was about to explain that he could do this for each of the 52 cards in the first deck. However, Michael interrupted his explanation in the situation.

This explanation provides corroboration that Carlos could operate as I suggested he had when he produced his array in the Outfits Problem. Moreover, his explanation suggested that he had conceptualized of his way of operating so that he could take his explanation as pointing to units coordinating activity that if carried out would produce pairs. So, his language and use of the array evoked operations that remained primarily unimplemented, but which Carlos could take as indication of why the situation involved multiplication.

An Analysis of Carlos's Symbolizing Activity. In Carlos's initial solution to the Outfits Problem, he used his notation and diagram as a tool for implementing operations that he could not immediately coordinate exclusively mentally. In this situation, Carlos began by using his notation to externalize the assimilatory part of his scheme by drawing a pair of pants and then a shirt. He, then, attempted to solve the problem again by operating on mental material and recording the results of this operating using tally marks. However, in the middle of his operating, he experienced a perturbation because he could no longer keep track of which outfits he had produced. So, he used his notation to help him order the shirts and also to order the pants. This notation helped him establish an ordering for the shirts and pants, which in turn helped him to organize the way in which he produced outfits. He, then, used the notation to monitor the activity of his scheme until he could operate once again on exclusively interiorized imagery. Here, the primary function of his notation was to help him monitor and coordinate his way of operating. So, it functioned for him as a tool in action in the sense that it helped him carry out the activity of his scheme.

When I suggested that he produce a new form of notation, he used this as an occasion to focus his attention on treating the pants in the situation as if they functioned identically. This new form of notation then shifted his focus of attention, which resulted in his producing a *representative* outfit that symbolized all of the outfits he could make with a particular shirt and all of the pairs of pants. Producing the array in his final solution of the Outfits Problem helped him enact a novel coordination between the two ways that he had initially used to solve the problem. So, the notation appeared to help him externalize his operating, which occasioned several accommodations.

The array that Carlos produced in the Card Problem he used in a different manner from how he had used the notation in the Outfits Problem. Carlos used his array simply to evoke operations that remained primarily unimplemented. So, the array for Carlos was primarily an illustrative tool, which he commented on several occasions helped him "see all the pairs". Carlos then was able to use the notation and diagrams to move from assimilating the situation almost directly to an explanation of the results he anticipated without implementing much of the activity he engaged in to produce these results. So, it functioned primarily as a tool to communicate and reflect on the results of his operating across these situations.

Further, Carlos attributed to me having "shown him" a new way of reasoning about the situations. I infer that Carlos's comment suggested that he attributed to me the kind of reasoning that he attributed to himself. For Carlos, this provided confirmation that his reasoning was viable in the situation. For this reason, I consider that Carlos and I had established a notational convention in the context of interacting. On the other hand, the arrays that Carlos produced were not meaningful to Michael because it symbolized the results of Carlos's operating in the situations and included only linguistic reference to the activity that needed to be carried out. At this point in the teaching experiment, Michael was not able to assimilate these words to an activity that he attributed to himself. So, the diagram served the function of being a conventional way for Carlos and I to communicate, but Michael did not attribute a mathematical meaning to Carlos's diagram. So, Michael did not establish the diagram as a conventional way to communicate mathematical meaning even though he was aware that Carlos and I took it to communicate a particular meaning.

Carlos's Lexicographic Units Coordinating Scheme and Related Concepts

Carlos's Scheme. Carlos's solutions to the Outfits Problem suggested that he had already constructed a scheme for solving such problems prior to entering the teaching experiment. I call Carlos's scheme his lexicographic units coordinating scheme because one of its primary features was the lexicographic ordering he established to monitor the way in which he made his units coordinations. Across these situations, Carlos made functional accommodations in his way of operating. I will highlight these functional accommodations as I discuss his scheme.

Carlos's first solution of the Outfits Problem suggested that he assimilated the situation with his iterable unit of one to produce his concept of four shirts and his concept of three pairs of pants. He, then, tried to make the twelve outfits that he expected he would make. In doing so, he lost track of which outfits he produced so he made a functional accommodation in his number concepts. He enacted an ordering for the units in his concept of three and an ordering for the units in his concept of four. He, then, used this ordering as a way to keep track of his units coordinating activity as he produced the twelve outfits in sequence. His language suggested that once he had made the first three outfits he treated the blue shirt as representative of each of the other shirts.

When Carlos produced the tree diagram to symbolize his solution of the Outfits Problem, he produced a second functional accommodation. In this situation, he appeared to make a units coordination between a particular shirt and a representative pair of pants to produce a representative outfit, which symbolized three outfits. He engaged four units coordinations one between each particular shirt and a representative pair of pants. When Carlos produced an array to symbolize his solution of the Outfits Problem, he combined his two ways of operating. He imagined taking a particular pair of pants and making a units coordination between it and a representative shirt, which symbolized the four outfits he could make with that particular pair of pants, and then, he could take these four pairs of pants as symbolizing the number of outfits he could produce with each pair of pants. Corroboration that Carlos incorporated these functional accommodations into his operating came during his explanation of the two-deck card problem where he appeared to operate similarly to how I described his operating in the Outfits Problem when he produced the array.

Carlos's Ordering, Whole Number Variable, and Pair Concept. Carlos provided indication that he had constructed an ordering concept.[21] I make this assertion based on Carlos's introduction of ordering the shirts and pairs of pants into his solution of the Outfit Problem. Here, Carlos independently initiated this activity as a result of a perturbation he experienced when he was trying to keep track of the number of outfits he produced in the problem situation. Moreover, Carlos introduced an ordering concept when he produced his solution of the Card Problem. In this problem, Carlos gave indication that he could assimilate a situation with what I have called a representative unit. A representative unit is a unit that symbolizes both an *amount* and *ordering* meaning for a composite unit. So, for example, a student can take the ace of spades to symbolize both the number of cards in a deck of cards and the way in which these cards will be ordered. When a student is able to operate in this way, it enables him to produce in what I will call a composite unit that is a unit of *ordered* units.

Furthermore, Carlos could make a units coordination between a particular unit and a representative unit. Being able to do so meant that Carlos took the representative unit as symbolizing *any but no particula*r of the units within one of the composite units. For example,

[21] I had ideas about these concepts prior to the teaching experiment. However, I had no operative definition. I consider the operative definition I provide here to be a critical result of the study because I learned it from Carlos, Michael, and Deborah.

Carlos could take a representative pair of pants and put it with a particular shirt where the representative pair of pants symbolized any but no particular of the pairs of pants. I will take this way of operating to be the most basic whole number variable concept. Carlos's way of operating included more than this way of operating because once he made a units coordination between a *particular* unit and a *representative* unit, he could, then, imagine doing this with *any but no particular* of the units in the second composite unit. He could operate in this way by imagining the unit that had been a *particular* unit as being a representative unit. So, for example, he could imagine putting a representative pair of pants with a particular shirt, which produced three outfits, and then, imagine that this particular shirt was a representative shirt (i.e., any but no particular of the shirts).

Both the notion of *ordering* and the notion of *any but no particular* seem to be central facets in the construction of a unit that I will call a person's pair concept. Carlos could enact a coordination that could lead to the abstraction of a pair concept, but he did not appear to have abstracted this concept. I consider a pair concept to contain two slots. I infer that a person abstracts a pair concept that contains two slots from putting each particular unit (e.g., each particular pair of pants) from a first composite unit with each particular unit (e.g., each particular shirt) from a second composite unit. So, the first slot in a pair concept contains records of having inserted, for example, each particular pair of pants. Furthermore, the first slot has records of having ordered the pairs of pants and selecting a representative pair of pants to symbolize any but no particular of these pairs of pants. I also make the same inferences about the second slot of a pair concept.[22] Here, Carlos could enact this concept by putting a particular unit in the first slot and a representative unit in the second slot, and then, imagine that the first slot was any but no

[22] This definition is a result of Michael's and Deborah's way of operating.

particular unit, but he had not abstracted this concept so that a representative unit could be inserted into each slot.

Contraindication of a Recursive Scheme of Operations[23]

The Three-Card Combination Problem. During the experiment, I posed two problems to Carlos that opened the possibility for him to recursively use his lexicographic scheme. The first problem was similar to the Card Problem. However, rather than involving two decks of cards, the problem involved three players where each player had one of the suits from a deck of cards (i.e., 13 cards each). Prior to having three players involved, Carlos and I began by playing a game similar to the one he and Michael had played in the Card Problem. In this game, he had 13 cards and I had 13 cards and I posed a similar problem, where Carlos solved the problem multiplying 13 times 13 and producing an array. The witness-researcher, then, joined the game, and we made two experiential three-card combinations. I asked Carlos the following problem.

> Task 5.5, *Three Card Combination Problem*: How many possible three-card combinations could we make?

> Protocol 5.5: Carlos's solution of the Three Card Combination Problem 10/17/05
>
> C: [sits for approximately 8 seconds thinking] Can I do it on the piece of paper?
> T: Mm-hmm [yes].
> [C writes x 13 below where he has already written 169. He computes this multiplication to find that it is two thousand one hundred ninety seven. He rewrites the answer above and writes as if he is going to divide two thousand one hundred ninety seven by three. He stops after he has written $3\overline{)2197}$, turns over his pencil, looks at the paper in concentration, and thinks for approximately 20 seconds.]
> W: Why are you dividing by three?

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[23] I did not work on problems in an organized manner that might have opened the possibility for Carlos to construct a recursive scheme of operations. Based on my interactions with Deborah, I have a better sense of a trajectory for how a person might construct a recursive scheme of operations. I do not consider either of these problems to have been particularly well sequenced for opening the possibility that Carlos might construct such a scheme. So, the data I present, here, is simply to suggest that Carlos did not come into the teaching experiment with a recursive scheme of operations.

C: Huh. Oh, cause like there is going to be like a pair of three [doubtfully]. No.
T: Cause there's gonna be…
C: Like a pair of three of triplets.
T: There are gonna be triples.
C: Yeah. [Almost immediately.] No wait that would be the answer [pointing to two thousand one hundred and ninety seven].
T: That would be the answer? How come?
C: 'Cause before I thought you were asking if there were…See if you could make like three eights in each [C means that he was thinking about the number of three of a kind].
T: Mm-hmm [yes].
C: So, I was going to divide it by three to see like how many times we would be able to get like three of a kind out there.
T: Oh, oh I see.
C: But that's not what you were asking.

Here, I infer that Carlos made a conjecture about what calculations might solve the problem. He knew that multiplication was relevant in the situation and so multiplied. However, the fact that he began to divide suggests that he had simply made a conjecture about calculations that might produce the correct results. His explanation for the division was that he thought the question was the number of three of a kinds (e.g., three eights) not the total number of three card combinations he could make. So, his division may have been related to an intuition that he had over counted. However, when the witness-researcher cast doubt on his division, I infer that he reviewed his activity in the other situations, realizing that he had not used division in these situations. So, he simply abandoned his idea to divide, given that there was social pressure to do so. I infer that in this situation Carlos used his notation to find an answer although it was not clear that he could not coordinate this answer with a way of operating.

The Coin Problem. The Three Card Combination Problem involved large numbers and so Carlos did not produce a way of operating that provided insight into what activity he might engage in to reason about situations that involved three composite units instead of two. However, I did present him with a problem that involved smaller numbers. This problem involved Michael, Carlos, and I each flipping a coin and recording the results. I asked them how many possible outcomes there could be and Carlos wrote out the following.

3 Heads 0 Tails
2 Heads 1 Tail
1 Head 2 Tails
0 Heads 3 Tails

Figure 5.12: Replica of Carlos's solution

In his solution, I infer that Carlos used the units coordinating activity he had engaged in during the Outfits and Card Problem in a novel way. He coordinated the *amount* of possible heads with the *amount* of possible tails, but he did not, as he had in his solutions with only two composite units, produce an *ordering* for the units in his concept of two. In fact, it is inaccurate to suggest that Carlos experienced the Coin Problem as a situation that involved three composite units. Rather, he seemed to only experience the situation as one that involved two composite units—one that indicated to him the number of heads that had appeared on the three coins and one that indicated the number of tails that appeared on the three coins. He, then, produced the possible outcomes by making a units coordination between the number of tails and the number of heads that could occur in a given turn. So, Carlos's solution was not multiplicative in the same way that his solution was to the Outfits Problem.

Both this problem and the Three Card Combination Problem provide contraindication that Carlos had come into the teaching experiment with a recursive scheme of operations. By a recursive scheme of operations, I mean that a person can take the results of his units coordinating activity as input for making a further units coordination. So, in this problem situation, it would mean that a person produced the four possible outcomes with two coins and used these *pairs* (the units that contained for instance a tails and a tails) as input for making a further units coordination. Such activity would produce a unit that contained a unit from each of the composite units in the situation (i.e., an outcome of tails, tails, and tails).

Carlos's Activity to Solve the Flag and Handshake Problem

Overview

This section of the case study is broken into four parts. In the first and second part, I analyze Carlos's activity in solving the Flag and Handshake Problem, which I presented to him during the third and the beginning of the fourth teaching episodes. I conclude the second part with a comparison of his activity in these two problem situations. In the third part, I argue that Carlos re-interiorized his lexicographic units coordinating scheme as well as began to abstract his scheme into a conceptual structure. In the final part, I examine two aspects of Carlos's symbolizing activity that I do not explicitly address during my previous analysis.

The Flag Problem

Carlos's Initial Activity on the Flag Problem. At the beginning of the third episode, I posed a problem that I intended to be similar to the problems Michael and Carlos had worked on in the first two episodes. The problem, which they worked on during the entire third episode, I call the Flag Problem.

> Task 5.6, *The Flag Problem*: You are the President of a new country. You need to design a flag that has two stripes. You have 15 colors to choose from. How many possible flags could you make?

After making a few flags in the The Geometer's Sketchpad (GSP) (Figure 5.13), I posed the problem to the boys.

Figure 5.13: A replica of one of the flags the boys produced

Protocol 5.6: Carlos's solution to the Flag Problem on 11/02/06

C: [C notates his solution to the problem as "15 x 15" and uses his multiplication algorithm to find the results should be 275 (an error in carrying out the algorithm). He begins to write out each color.]
T: Why don't you number each color?
C: [C completes writing out the list of colors and the associated list of numbers for each color. He writes out the first column of notation by writing "1,1", "1,2", etc. (Figure 5.14). C writes out the second column the same as the first. When C gets to the third column, he writes, "3", thirteen times, and then fills in the appropriate number next to each "3". He outlines each column with a rectangle. C writes the fourth and fifth column of notation in a similar manner to the third column.]….
W: [Six minutes have passed.] You guys are going to need more paper.
C: I already have the answer. That is [C points to where he has written "275"] how many pairs you could make. [M tells C not to give away the answer and continues working on his solution to the problem. C resumes making his list by writing the numbers from "6" up to "15" and then writes in the number "6" next to each of these numbers. He repeats this process for the seventh, eighth, and ninth column and about eight minutes have passed.]....
M: It just decreases by one every time.
T: [To M] Yeah, it just decreases by one every time.
C: [C starts the tenth column writing "10" four times. He counts the number of flags he has symbolized in the ninth column, finds he has symbolized seven flags, and writes "10" twice more. He then writes "10"; "11", etc. next to each "10" he has written to complete the tenth column. He repeats this process for the remaining columns.]….
M: [M finishes writing a similar list of notation.] See there is like fifteen and it decreases by one every time so it's fifteen plus fourteen plus thirteen plus twelve until you get to one.
T: Go ahead and write that down and see if you can figure out how many you got that way. [C finishes his list of notation. 10 minutes have passed since he began. C sits as if satisfied that he has solved the problem.] Carlos, I wonder if you could figure out how many pairs you got here?
C: Two hundred and seventy five. [C points to his multiplicative notation.]
T: Why don't you check that by figuring out what this [T points to Figure 5.14] is?
C: Okay. [C writes out the sum 15 + 14 +…+1 vertically.]
T: [To M & C] I wonder if you could figure out a quick way to add those up.

C: Fifteen times fifteen.

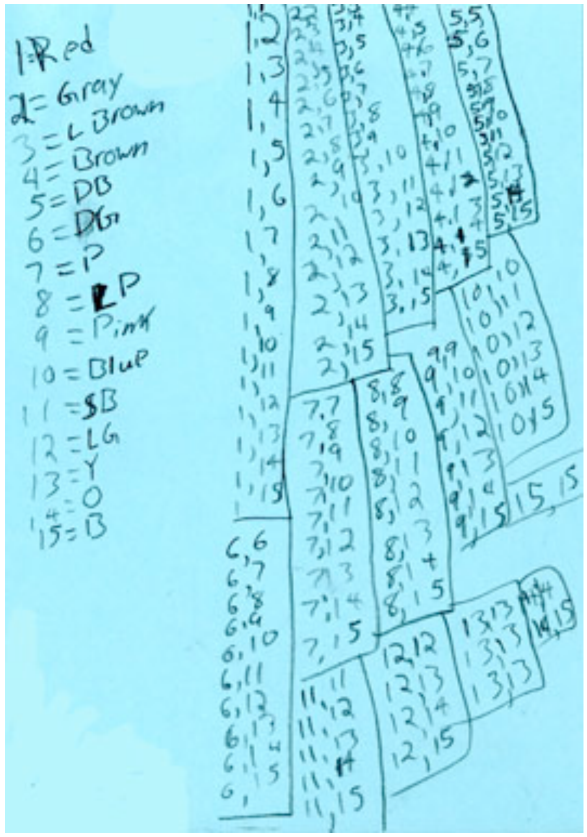

Figure 5.14: Carlos's notation for the flag problem

Based on Carlos producing the notation "15 x 15", I infer that Carlos anticipated the situation as one that he could solve using his lexicographic units coordinating scheme. So, his multiplicative notation symbolized the results he anticipated he would produce were he to actually carry out the activity of his scheme. In contrast to the Card Problem, where Carlos was content to use his multiplicative notation and the beginning of an array to symbolize his solution to the problem, he did begin to produce notation (Figure 5.14) for each of the flags. After producing the first five columns of this notation, he still remained convinced that he had already solved the problem by multiplying "15 x 15" ["I already have the answer"]. So, he seemed to take Figure 5.14 simply as a way to symbolize each of the flags in his anticipated results.

This might be explained simply by his lack of awareness that each column did not symbolize 15 flags. However, later in Carlos's solution, he provided indication that he was aware that each column contained one less symbol for a flag than the previous column did. I make this inference based on his counting the seven flags he produced in the ninth column and taking this as indication that he should only create six symbols for flags in the tenth column. Further, when I encouraged him to use Figure 5.14 to find the solution to the problem, he wrote out the sum 15 + 14 + … + 1, which suggests that he was aware that each column symbolized one less flag than the previous column.

The fact that this did not create a perturbation for Carlos suggests that Carlos had not differentiated his activity in this situation from his activity in previous situations that involved his lexicographic units coordinating scheme. Further, it suggests that a comparison of the additive and multiplicative notation was not enough to create a perturbation for Carlos. Rather, Carlos seemed to take the multiplicative notation to symbolize his anticipation of what the results would be in the situation and the additive notation to symbolize a way of quantifying the results he actually produced. Since he seemed unaware that the activity he had engaged in differed from the activity he had anticipated engaging in, I infer he took these two kinds of notation to symbolize the same amount.

This lack of differentiation suggests that simply producing one less flag in each column did not indicate to Carlos that he was operating differently in the situation. So, it is worth analyzing how Carlos operated in this situation and why he did not seem to experience this way of operating as different from his previous way of operating. To begin this analysis, I want to suggest that Carlos assimilated the Flag Problem in a different way than he had assimilated both the Outfits and Card Problem. In the Outfits Problem, Carlos assimilated the situation using two composite units that he differentiated between—one referred to the pants and one referred to the shirts. Similarly, Carlos assimilated the Card Problem with two different composite units—one symbolized his deck of cards and the other Michael's deck of cards.

On the other hand, Carlos's solution of the Flag Problem provides indication that he assimilated the situation using only one representative unit that symbolized one composite unit (fifteen colors with which to make flags). I make this inference based in part on how he notated the situation: he did not use an array, which he had used in most other problems up to this point. Using an array provides indication that a person has assimilated a situation using two composite units because to symbolize one's activity with an array requires drawing a diagram that has two axes—one for each composite unit. In part, he appeared to assimilate the situation differently because of his experience of making flags where he choose the colors to make the flags in GSP from only one list of colors.

While making flags in GSP, Carlos had observed that he could make flags that were "red-red", etc. and had specifically pointed out this feature of the problem to Michael. I take Carlos's ability to make flags of this type as indication that Carlos could disembed the color red from the list of 15 colors, treat it as independent from this list of 15 colors, and imagine putting it with the color red that remained a part of the list of 15 colors. Based on Carlos making this observation, I infer that when he began his activity to make flags that he disembedded a singleton unit (i.e., the color red) from the composite unit (i.e., the 15 colors). He, then, appeared to imagine sequentially putting the color red with each of the 15 colors on the list. So, he made a units coordination with the singleton unit and each particular unit in the composite unit. I make this inference based on the way he produced his notation for the situation, writing out "1,1", "1,2", etc.

When he reached the second column of flags, I infer he disembedded the second color, grey, from the list of 15 colors to begin making the second column of flags. Here, however, he did not use all 15 colors, which, in a certain regard, is consistent with how he operated in situations like the Outfits and Card Problem. That is, in the Outfits Problem, he no longer used a particular shirt to make any further outfits once he had put it with all of the pants. In this situation, once he had used the color red to make flags he did not use it to make any future flags. However, in this situation, he was operating with only one composite unit. So, he produced only fourteen flags (as opposed to fifteen) with the second singleton unit that he disembedded because he had eliminated red from the composite unit. This way of operating would have been similar to his operating in the Outfits Problem if he had eliminated the first pair of pants once he had produced all the outfits with the first shirt. He did not operate this way in the Outfits Problem because he differentiated shirts from pants prior to operating (i.e., he assimilated the situation with two composite units).

When Carlos got to the third column, he changed how he was writing out his notation. He wrote out the number "3" thirteen times, and then, filled in the appropriate number next to each "3" he had written. This way of notating suggests that Carlos was now imagining putting a particular color (the third color) with a representative color that symbolized the remaining colors. I infer this produced a representative flag that symbolized 13 flags. This explanation seems consistent with how I have suggested that Carlos operated in problems like the Card Problem where he seemed to put a particular card from the first deck of cards (e.g., the Ace) with a representative card from the second deck of cards (even though he could assimilate the situation with two representative units). Here, he seemed to disembed a particular color and then made a units coordination with a representative color that symbolized the number of flags he could make with that particular color.

Carlos's way of operating in the Flag Problem was a functional accommodation in that he assimilated the situation using only one composite unit, and then used this unit to create a singleton unit. He, then, used the singleton unit and a representative unit, which symbolized the remaining colors, to produce the flags in each column of his notation. I infer that this functional accommodation was a result of his using one composite unit to produce a singleton unit and a composite unit to produce the flags (a particular color and some portion of the list of 15 colors). I infer Carlos remained unaware of his functional accommodation based on his insistence that his multiplicative notation would produce the correct result. His lack of awareness provides indication that he did not differentiate between how he used the singleton unit (e.g., red) and how he used the remaining colors to create flags. Such a differentiation, I infer, could lead to an awareness of the possibility of creating a red-orange flag and an orange-red flag. Carlos gave no indication that he was aware that two such flags might be created.[24]

The fact that Carlos did not take the multiplicative and additive notation to symbolize two different amounts bothered me for a significant portion of the time I was conducting retrospective analysis. It bothered me because, as I have argued in the first section of this case study, Carlos could use his multiplicative notation to symbolize 15 pairs 15 times. So, I thought that when Carlos did not produce 15 pairs starting with his second column of notation that his additive notation *should* symbolize something different to him. Particularly, given that he

[24] I will suggest in Deborah's case study that simply having an awareness of these two different flags is insufficient for creating an *ordered* pair (e.g., considering as different a two of diamonds and the three of diamonds and the three of diamonds and the two of diamonds based on the order in which the cards were selected). However, I consider it a necessary condition for creating an ordered pair.

seemed to be aware that he had not produced units of equal numerosity in this situation and he seemed to take his multiplicative notation as symbolizing producing units of equal numerosity in other situations.

For him, I infer the two forms of notation symbolized the same thing because he had not differentiated his anticipated way of operating from the way he actually operated. To make an account of why he did not differentiate these two ways of operating, it is worth analyzing Carlos's way of operating in a bit more depth. Because Carlos was insistent that his notation 15 times 15 would result in the correct number of flags, I infer that after operating Carlos had reconstituted the singleton units he had disembedded from the unit of 15 as a second composite unit of 15. He produced this second composite unit as a *result* of his operating. So, he was aware of both the composite unit he had started with (the 15 colors) and the composite unit he had created by disembedding singleton units 15 times.

This inference along with my explanation of Carlos's functioning in situations like the Outfits Problem provides indication of why he did not differentiate these two types of notation. Remember, that I argued that Carlos could take a particular unit and put it with a representative unit, which produced a representative pair that symbolized some number of composite pairs (e.g., putting the blue shirt with a representative pair of pants produced a representative outfit that symbolized three outfits). Only after he had engaged in these operations could he treat the particular unit as if it was a representative unit, which enabled him to conclude the number of times he would produce this number of composite pairs (e.g., the blue shirt became a representative shirt which symbolized the number of times he could produce three outfits).

In this situation, I infer this way of operating would mean that Carlos would be aware of the number of flags he had symbolized in a particular column, and then subsequently the number of columns he had produced, but he would not be aware of both simultaneously. So, he wrote out the number of flags he had symbolized in each particular column with his notation, “15 + 14 +… + 1”, but was unaware that this symbolized 15 addends. He did seem to be aware that he had produced a second composite unit of 15, but he could not take this second unit of 15 as symbolizing the number of columns of notation he produced, while he was focused on the number in each column. This analysis suggests that Carlos did not produce the sum as a quantitative entity—his notation did not symbolize a simultaneous awareness of the number of addends and the amount in each addend where each addend was one less than the previous addend. I will analyze this situation more when I present Carlos’s activity to evaluate the sum in the third section of this case study. Here, I want to analyze Carlos’s subsequent activity to investigate if he became aware that his anticipated and actual way of operating differed.

Carlos Has an Insight. At this point during the episode, I asked Carlos to check his answer of 275 by evaluating his additive notation (and I asked Michael to figure out how many flags he had produced by evaluating his additive notation). Each of the boys added the results and provided the following responses.

Protocol 5.7: Carlos’s insight into the Flag Problem on 11/02/05

M: I got one hundred and ten possibilities.
C: I got one hundred and thirty.
T: Uh-oh. We got to figure out some way to add those numbers up and actually check.
[M & C both start adding numbers to check their answer again.]
W: [To T] Ask them if they notice a certain pattern…like fifteen plus one.
T: [To M & C] What is fifteen plus one?
M & C: Sixteen.
T: What is fourteen plus two?
M & C: Sixteen.
T: What is thirteen plus three?
M & C: Sixteen.
C: Oh. [C laughs.]
T: What is twelve plus four?
M & C: Sixteen.

T: I wonder if you could pair the numbers in that way.
W: Would that help you do it?
C: That's what I did with mine. I got that one with the five [C points to 15 and 5 in his notation.] and then once I…
M: [Interrupting C.] Wait a minute. I just had that. Every time you add fifteen you subtract that many. [M is referring to a pattern he has noticed that you can add 15 each time and then subtract an appropriate amount. M begins to actually carry out this activity. C looks at M's notation and watches what M is doing for about 30 seconds.]
T: Check yours out too Carlos.
C: Okay.
[About two minutes pass.]
W: What's Carlos doing over here? Are you doing the same way (as Michael)?
C: No, I am not subtracting. I just figured out something. That once you got that one [C points to 15], then you got that one and that one [C points to fourteen and one], then you make another fifteen, that one and that one [C points to thirteen and two] and make another fifteen. You keep on going all the way down until you get to seven and eight which (means) you have eight fifteens so you should have just timesed fifteen times eight and you would have got the answer [C crosses out where he has written "15 x 15".]

Carlos's activity is quite interesting. When he and Michael got different answers, he appeared to believe that there was a need to check which of their answers was correct. At this point, he gave no indication that he had related his answer of 130 to his answer of 275. Rather, the impetus for his check seemed to be that the numerosity of Michael's answer differed from his. When I intervened in the situation, suggesting the boys use strategic additions, Carlos assimilated my suggestion to his way of operating ["That's what I did with mine."]. He seemed to take my suggestion as indication that he no longer needed to check his answer. I make this inference because he stopped his activity and instead watched what Michael was doing. It was not until I explicitly asked him to check his answer again that Carlos did so.

After working for several minutes, Carlos seemed to have a moment of insight, "I just figured something out"[25], and then made the explanation that concluded with his remark, "so

[25] The comment, "I just figured something out", seemed to pertain to several aspects of the situation. Here, I will analyze how it pertained to Carlos's lexicographic units coordinating

you should have just timesed fifteen times eight". It is noteworthy that Carlos did not take my suggestion to make strategic additions to produce 16s. Rather, he had made strategic additions to produce eight 15s. I infer he introduced producing 15s rather than 16s because he had formed the goal of making a comparison between his anticipated way of operating that he had symbolized with 15 times 15 and his actual way of operating that he had symbolized with 15 + 14 + … + 1. That is, it appeared my and the witness-researcher's suggestion, along with Carlos's first calculation of 130 which was not 275 as he expected, helped Carlos to establish the goal of making a comparison between the results he had anticipated 15 times 15 and the results he had actually produced 15 + 14 +…+ 1.

In making this comparison, he found that his additive notation symbolized only eight 15s and I infer he was aware that his multiplicative notation symbolized fifteen 15s. After Carlos found that he, "should have timesed fifteen times eight", he immediately crossed out his notation, "15 x 15". This action was the first time he gave indication that he considered his anticipated way of operating, producing 15 flags fifteen times, to be different from his actual way of operating, producing 15 flags, then 14 flags, then 13 flags, etc. When he had added the numbers the first time, he produced a numeric result, 130, that was different than 275, but he gave no indication that he could relate this calculation back to his multiplicative notation or to the quantitative situation.

Because Carlos related these two ways of symbolizing the results, it opens the question of how he interpreted this relationship. I can think of two possible ways in which he might interpret this relationship. One possible explanation of Carlos's comparison is that he was exclusively comparing the two multiplication problems and he knew that 8 times 15 was not 15 times 15. If

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scheme. In the third section of the case study, I will analyze how it pertained to his finding sums of whole numbers.

this explanation were to hold, then, I would infer that Carlos was operating simply with the numbers in the situation and these numbers no longer symbolized quantities that Carlos related back to the problem situation. In this case, I might expect that Carlos would experience a perturbation, but he might not be able to resolve this perturbation because he would not be able to relate the different amounts back to the activity he produced in the Flag Problem.

A second possible explanation of Carlos's comparison is that he was comparing the results of his actual activity to the results of his anticipated activity where these two results continued to symbolize quantities in the problem situation. If this explanation were to hold, then I would infer that Carlos might try to account for why his second column symbolized only 14 flags as opposed to 15 flags. Up to this point in his operating, he did not seem to account for this because he remained unaware that the sum he had produced contained 15 addends. However, when he found that he could transform this sum into eight 15s, it opened the possibility that he would review his activity in the situation in order to account for how his activity differed. Note, I do not infer that it would necessarily indicate that he would become aware of the number of addends he symbolized with his additive notation, simply that he had not produced fifteen 15s.

The first possible explanation seems implausible because Carlos agreed later on in the episode with Michael's statement that, once "you have 1-2…you can't do 2-1 again because it's already used so that means you gotta do one more less every time". When Carlos initially produced his list of flags, he appeared to be unaware that he had eliminated a flag that he could have counted. His agreement with Michael, especially given that Carlos had disagreed frequently with Michael during the first two episodes, suggests that when Carlos produced the eight 15s he *did* review the quantitative situation to attempt to account for why he had produced less flags than he had anticipated.

Reviewing his activity in the quantitative situation opens the possibility that he would experience a perturbation at the level of reflection. I make this statement because it opens the possibility that this review would lead to a comparison and contrast of his activity in this situation and his activity in other situations like the Outfits Problem. For most of this episode, he seemed to consider this problem as similar to problems like the Outfits Problem. In order to look for further indication of whether this led to a perturbation at the level of reflection, I turn to an analysis of Carlos's activity in the Handshake Problem.

The Handshake Problem

Carlos's Solution of the Handshake Problem. I posed the Handshake Problem to both boys at the beginning of the fourth episode.

> Task 5.7, *The Handshake Problem*: Suppose there are ten people in this room. Each person would like to shake every other person's hand. How many total handshakes would there be?
>
> Protocol 5.8: Carlos's solution of the Handshake Problem on 11/9/05
>
> M: So it's one A. [M writes "1A" on his paper.] He can't shake his own hand can he? [M shakes his left and right hand.]
> T: A can't shake his own hand.
> C: Okay. [C writes "1A" on his paper. Below where he has written "1A", he writes "1 2". He pauses concentrating for 20 seconds and writes over where he has written "1A" with "1B" and where he has written "1 2" with "1C". He finishes his column of notation by writing "D, E, …, J" and then writes a "1" next to each letter. Off camera, C begins his second column with "2A, 2C, 2D, 2E"[26].]
> M: But didn't they already shake hands? Yes. [M is referring to where he has written the notation "2A"].
> C: [C pauses and subvocally says, "Oh". He erases the letters "A,C,D,E" in his second column of notation and writes "C,D,E,F". C, then, finishes his second and third column.]

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[26] By off camera, I mean this was not captured on the camera that was capturing the students' work. I could still see Carlos's actions on the camera that was capturing the student-teacher interaction. I am inferring what he wrote based on correlating his body movements on the student -teacher camera with examining the paper I collected from him. On his paper, it appears Carlos wrote what I am claiming he wrote and then erased it. It should be noted, however, that I am making this inference based on inspecting something that was erased so there are only partial markings of what was originally there.

M: Same problem as last time, minus one and keep going and going and going.
C: Minus two actually.
M: Uhnun (no).
C: [Emphatically] Unhun (yes), cause they can't shake their own hand. [M shakes his head, no.] [To T] Right? So, one couldn't shake A and two couldn't shake…[27]
T: You got to figure out what each of these means in terms of the problem. [T points to the letters and numbers in C's notation.]
C: [C writes "2 = B", "3 = C",…, "J = 10" in a column (Figure 5.15a). Three minutes pass during which C writes down the remaining columns of notation (Figure5.15b).]

B = 2	1B 2C 3D 4E 5F 6G 7H 8I 9J
C = 3	1C 2D 3E 4F 5G 6H 7I 8J
D = 4	1D 2E 3F 4G 5H 6I 7J
E = 5	1E 2F 3G 4H 5I 6J
F = 6	1F 2G 3H 4I 5J
G = 7	1G 2H 3I 4J
H = 8	1H 2I 3J
I = 9	1I 3J
J = 10	1J

Figure 5.15a (left) & 5.15b (right): Replica of Carlos's Notation

In his solution to the Handshake Problem, Carlos seemed to use the symbol "1A" because he overheard Michael's verbalization. He next wrote "1 2" and sat in concentration for 20 seconds. At this point, he wrote over where he had written "1A" with the symbol "1B". Here, I infer "1A" and "1 2" symbolized two handshakes. Further, I infer that he paused for two reasons. First, I infer he was trying to figure out both Michael's (and his own) use of the letter "A", rather than the number "1". Second, I infer he was trying to figure out how to use this notation so that he did not record a person shaking his own hand [T: "A can't shake his own hand. C: "Okay".]. Because Carlos selected to use the notation that contained both numbers and letters, I infer that he assimilated this situation with two representative units as opposed to one and each representative unit symbolized a composite unit. One symbolized the "handshakers" and the

[27] The previous four lines of transcript are a discussion of Carlos's perceived differences and Michael's perceived similarities between the Flag and Handshake Problem.

other the “handshakees”. This way of assimilating the situation suggests that Carlos incorporated the results of his operating in the Flag Problem into the assimilatory part of his scheme. That is, he had produced a second composite unit as a result of his activity in the Flag Problem and here he assimilated the Handshake Problem with two composite units as opposed to one.

Furthermore, Carlos seemed to make a correspondence between the first letter and the first number in the situation where “1” symbolized the same person as the letter “A”. I make this inference based on Carlos’s writing over where he had written “1A” with “1B”, which seemed to indicate that he rejected the handshake he had symbolized with “1A” because it symbolized a person shaking his own hand. Carlos proceeded to write the letters “C”, “D”, etc. and then filled in the “1” next to each letter. This way of notating suggests that Carlos made a units coordination between the first handshaker and a representative handshakee and this symbolized all 9 handshakes in which the first person could engage.

He, then, began his second column of notation with “2A, 2C, 2D, 2E”. The beginning of his second column of notation suggests that he continued to use his notation to help him eliminate a person shaking his own hand. Furthermore, it suggests that establishing an identity between particular units in each composite unit (e.g., handshaker “1” and handshakee “A”, and handshaker “2” and handshakee “B”) was not enough for Carlos to experience a perturbation when he symbolized the handshake “2A” because he did not recognize it as the same as the handshake he symbolized as “1B”. However, Michael’s comment, “But didn’t they already shake hands? Yes.”, seemed to help Carlos have a second moment of insight. I make this inference based on his halting his activity, subvocally uttering the word “oh”, and then erasing the first four letters he had written in the second column. Here, Carlos seemed to become aware that even though he had differentiated between “handshakers” and “handshakees” with his notation that some of the handshakes that he symbolized would count the same thing (i.e., when person 1 was the "handshaker" and person B the "handshakee" this would be the same as when person 2 was the "handshaker" and person A was the "handshakee").

This use of notation suggests that Carlos produced two different handshakes as a result of his operating, and then, considered the two handshakes to be identical. Operating in this way was different from how he operated in the Flag Problem because in the Flag Problem he appeared to be unaware that he might produce a flag that was red-orange and a flag that was orange-red (until long after he was done operating). Here, he produced both a handshake between person A and person 2 and person B and person 1. His assimilating the problem with two composite units appeared to enable him to consider both. So, producing two composite units in a situation like the Handshake Problem seemed to open the possibility for him to reason about making an ordering of the elements in the resulting handshake. I make this inference because it opened the possibility for Carlos to consider a particular *pair* that he produced as different or the same, depending on whether he considered the two results to symbolize different or similar things. Here, Carlos simply considered the two to be the same and eliminated one.

Carlos's solution of the Handshake Problem suggested that he had gained an awareness of several important feature of his lexicographic scheme. First, he appeared to have a heightened awareness of using two composite units when he used his scheme. I make this inference because he anticipated operating with two composite units prior to producing the results in the Handshake Problem. This anticipation is in contrast to how I have suggested he operated in the Flag Problem where he appeared to simply disembed a color from the list of 15 colors, as he needed it to make flags (e.g., when he got to the second column, he disembedded the color grey from the list of colors in order to use it and 14 colors to make the second column of flags). This way of operating made his scheme anticipatory in a new sense. Furthermore, he appeared to gain some awareness that *he* introduced a property to differentiate two composite units and could introduce this into a situation even if there was not a qualitative differentiation of such a property in the statement of the problem (e.g., shirts and pants).

These aspects of Carlos's solution suggest that his activity in the Flag Problem led to a perturbation at the level of reflection. Carlos's solution to the Handshake Problem, then, appeared to be part of the process of Carlos re-interiorizing his scheme to include the ways I have suggested above as new elements of this scheme. This process of re-interiorization suggests that Carlos engaged in a reflective abstraction where the elements I have suggested above were projected to a "higher plane" of operating. However, these new elements are quite substantive in that they have to do with issues of awareness and anticipation in Carlos's scheme. So, I infer that Carlos was also in the process of abstracting the structure and functioning of his scheme into a conceptual structure. This claim suggests that Carlos was also in the process of engaging in a reflected abstraction. I want to elaborate on several aspects of this process in the next part of this case study in order to suggest the role that externalization played in the re-interiorization of his scheme. Also, I want to elaborate on what I mean when I say that Carlos was in the process of abstracting a conceptual structure.

Carlos Re-interiorizes His Scheme and Abstracts a Conceptual Network of Related Tasks

The Role of Externalization in Carlos's Re-interiorization. In both the Flag Problem and the Handshake Problem, Carlos's activity provides indication that externalization was a crucial factor in his subsequent re-interiorization of his scheme. So, I want to examine in what way externalization played a role in these situations. In the Flag Problem, Carlos did not consider his actual way of operating, producing 15 flags, 14 flags, etc., as different from his anticipated way of operating, producing 15 flags 15 times, until he operated on the results of his lexicographic units coordinating with operations that were external to the scheme (i.e., adding 14 and 1 was not part of his lexicographic units coordinating scheme).

However, it was not just operating on the results of his scheme with operations external to it that seemed to help him reflect on his operating in the situation. I make this assertion because his first additive solution, where he found 130 as a result, did not seem to create a perturbation that he related to the functioning of his lexicographic units coordinating scheme. So, what seemed to occasion the perturbation was operating on the results of his scheme with operations that were external to it, but which he could relate to the operations that were involved in the scheme. By operating on the results of his scheme, Carlos treated these results as material for further operating. So, he operated recursively in the situation. This further operating seemed to occasion his reflection because it allowed him to produce a relationship between his anticipated way of operating and his actual way of operating.

In the Handshake Problem, a somewhat different type of externalization helped Carlos operate to engage in further reflection. Throughout Carlos's producing the first two columns of notation, Michael's utterances and actions seemed critical in Carlos modifying his activity in the situation. So, Carlos was able to assimilate Michael's comments to his own way of operating and use them as suggestions in this way of operating. Michael's statements were his own externalization of his way of operating not specifically directed to Carlos, but they seemed to serve the function of helping Carlos externalize his way of operating. I make this inference because Carlos modified his way of operating based on these utterances.

This activity may simply seem like an act of Carlos "copying" what Michael was doing, but I do not interpret it this way. I do not interpret it this way because of numerous occasions in the data where students would explain their reasoning to one another (or I would make a suggestion) and it did not occasion any change in the other student's activity.[28] So, here, I infer that Carlos took Michael's utterances as if they were externalizations of his own way of operating, which he could then use to modify his operating in the situation. For instance, Carlos's externalization of the first handshake using the notation "1A", after Michael said aloud "one A", seemed to enable him to assimilate the situation using two representative units that symbolized composite units. This differentiation subsequently allowed him to make the appropriate eliminations of the handshakes, which in turn enabled him to compare his activity in this situation with his activity in the Flag Problem.

Carlos Constructs a Conceptual Network. The fact that Carlos was beginning to make comparative statements about his way of operating in different situations ["minus two actually" "Unhun (yes), cause they can't shake their own hand"] provides indication that Carlos was abstracting the structure and functioning of his lexicographic units coordinating scheme into a conceptual network of related tasks. More specifically, I infer that Carlos's comments referred to his awareness that he was making two new eliminations each time he began a new column of notation—the person who had just been a handshaker and the person who was now a handshaker and could not shake his own hand.

This awareness allowed him to differentiate between three types of situations—those like the Outfits Problem, those like the Flag Problem, and those like the Handshake Problem. In the first type of situation, he made a units coordination between *all* of the elements of one composite unit and each of the elements of the other composite unit. In the second type of situation, he

[28] One such example is in the Flag Problem where I suggested making strategic additions to Carlos and Michael. In Protocol 5.6, each of them assimilated my suggestion to the way they had already operated in the situation. Carlos did eventually modify his way of operating based on my suggestion, but Michael did not. This lack of modification was a fairly frequent occurrence.

eliminated duplicate results like "1,2" and "2,1" by not using certain units from one of the two composite units when he was engaged in his units coordinating activity. In the third type of situations, he eliminated results like "1B" and "2A" and he also eliminated results like "1A". He made these eliminations by eliminating one *more* unit from one of the two composite units when he was engaged in his units coordinating activity.

This differentiation meant that he could begin to anticipate that situations might be of a particular type prior to his operating in the situation. This ability to characterize and anticipate situations, I infer, suggests that Carlos was beginning to build a network of related situations. The relatedness of the situations developed from his awareness of how he would operate were he to carry out his operations in the situation. As I suggested, Carlos still had to begin operating in the Handshake Problem in order to make the coordinations he did. So, he could not take the situation as itself symbolizing one of these three ways of operating prior to his activity (which would indicate a fully anticipatory scheme). Nonetheless, his scheme contained anticipatory aspects that suggested that he was in the process of abstracting a conceptual network of related tasks. In this case, Carlos's abstraction of a conceptual network suggests that he was making a conscious conceptualization of his operating—a reflected abstraction.

Highlighting Several Further Aspects of Carlos's Symbolizing Activity

Carlos's use of notation in the Flag and Handshake Problem provide a good example of how his symbolizing activity was instrumental in monitoring and coordinating his activity, which led to his reflecting on his activity. Here, I will not re-hash this aspect of Carlos's symbolizing activity. Suffice it to say that his symbolizing activity served both as a tool in action and as a tool in reflection in each of these problems. In this part, I do, however want to examine two aspects of Carlos's symbolizing activity that I have not already considered. First, I want to investigate how Carlos's use of notation in the Handshake Problem related to changes I can infer in his mental imagery. Second, I want to comment on Carlos's attempt to establish a convention for his system of notation in the Handshake Problem.

Re-presentation, Mental Imagery, and Notation. I do want to examine in a bit more detail my assessment of how Carlos's notation related to the mental imagery he produced in the Handshake Problem. I infer that it was Michael's utterance, "one A", that led Carlos to use the combination of a letter and a number as symbols for people. In attributing meaning to this notation, I have argued above that Carlos made a correspondence between the letters and numbers (i.e., 1 = A) and that he also used the notation to differentiate between "handshakers" and "handshakees". Subsequently, he eliminated "1A" from the list because it symbolized a person shaking his own hand. At this point in the problem situation, I infer Carlos's mental imagery was of people where he was figuring out what "1A" was to symbolize in terms of the problem situation.

On the other hand, when Carlos produced his second column of notation, he seemed to use his notation to take eliminating the self-handshake as a given. I make this inference based on his producing, "2A, 2C, 2D, 2E" to symbolize the first four handshakes without pausing to consider that he needed to eliminate the handshake where the second person was both the "handshaker" and "handshakee". Because he did not halt his activity in making the second column, I infer that he was no longer re-presenting an image of a person shaking his own hand but rather could simply take his elimination of the notation "2B" to symbolize his concept of a self-handshake. This inference suggests that Carlos used his notation to symbolize his re-presentation of a person shaking his own hand.

Michael's comment, "But didn't they already shake hands? Yes.", seemed to create a moment of insight for Carlos. I infer that Carlos used his own notation to re-present the meaning of what he had written in the second column where I infer his re-presentation was once again of people in the situation. The mental image he seemed to produce was of the equivalence of a handshake between two people whose role as "handshaker" and "handshakee" had interchanged. I make this inference based on his pausing at Michael's comment, and then modifying his notation in the second column to "2C, 2D, 2E, 2F" for the four handshakes he had already recorded. At this point, he finished his second and third column of notation where he produced his third column of notation without any hesitation. So, when he recorded the third column of notation, he seemed to take both the elimination of self-handshakes and the elimination of handshakes where the role of "handshaker" and "handshakee" had been interchanged as a given. I infer he was once again using his notation to symbolize his re-presentation of these two types of handshakes without actually producing this re-presentation. This statement seems to hold for all of the rest of the columns of notation he produced because he made them quickly without giving any further indication that he was re-presenting a mental image of people shaking each others hand.

Notational Convention. After Carlos produced the third column of notation, he and Michael had a disagreement about the similarity and differences between this problem and the Flag Problem. In part to settle this disagreement, and at my request to be clear about how he was using his notation, Carlos produced the notation in Figure 5.15a "(2 = B….9 = J"). I infer that Carlos produced this notation because he had formed the social goal of communicating his way of operating to Michael and I. So, I infer that Carlos was trying to make more explicit the meanings of his system of notation. Therefore, I consider Carlos's activity as directed towards establishing conventional meanings for the notation he used. Here, by conventional meanings, I mean that he produced this notation in part to make public meanings that he had already privately established in his system of notation.

Carlos Evaluates Sums of Whole Numbers

Overview

This section of the case study is presented in two parts. In the first part, I examine Carlos's activity in finding the sum of the first 15, 9, and 99 whole numbers. The sum of the first 15 and 9 whole numbers came out of Carlos's work on the Flag and Handshake Problem (which were presented during the third and first half of the fourth teaching episodes). However, towards the end of the fourth teaching episode I presented problems that were just about finding a quick way to evaluate sums of whole numbers. In the second part, I highlight two features of Carlos's symbolizing activity that I do not specifically highlight in my analysis during the first part.

Carlos's Activity in Finding Sums of Whole Numbers

The Sum of the First 15 Whole Numbers. I begin by presenting Carlos's activity in the context of solving the sum of the first 15 whole numbers.

> Protocol 5.9: Revisiting Carlos's solution to the sum of the first 15 whole numbers on 11/02/05
>
> C: [C writes in a vertical column the sum from 15 down to 1 (Figure 5.16a). He begins to evaluate the sum by making strategic additions (e.g., adding 15 and 5) (Figure 5.16b) and crosses elements of the sum from his vertical list as he uses them in his activity. During this process, he makes a mistake, which he does not notice. He finishes adding the final five elements of the sum in his head (10 through 14) to find that the total will be 130.]

15
14
13
12
11
10
9
8
7
6
5
4
3
2
+1

15
+5
20
9
+1
30
2
+8
40
3
+7
60
6
+4
70

Figure 5.16a (left) & 5.16b (right): Replicas of Carlos's notation

Carlos's activity in the situation suggested that he assimilated the situation using a strategic additive reasoning scheme that he had previously constructed. The activity of this scheme consisted of his using notation to symbolize a sequence of addition problems that had yet to be evaluated and subsequently strategically uniting addends into units that he selected to be convenient. In doing so, he used his notation to posit the sequence of addition problems as the material on which he intended to operate without actually operating on it (Figure 5.16a). Then, he united 5 and 15, etc. because he knew that together they made 20, an "easy" number with which to continue the process of evaluating the sum. As he engaged in these operations, he produced new notation to symbolize each of the uniting operations in which he engaged (Figure 5.16b). So, his notation helped him monitor his uniting operations, which enabled him to rearrange the elements of the sum in the process of evaluating it.

From an operational perspective, I infer this way of operating indicates that the result of using one's uniting operation contains the units used to produce these results. For instance, if 20 is the result of additively combining 3, 4, 6, and 7, then when a person has produced 20, he needs to maintain an awareness of the units that comprise it so that any arrangement of them would produce the same result. Since Carlos independently contributed this way of operating in the situation, I infer that he already had made such an abstraction.[29] At this point, I wanted to find out in what way Carlos might modify his strategy so that he might coordinate it with a multiplicative way of evaluating the sum. So, I made the following intervention.

> Continuation of Protocol 5.9: Revisiting Carlos's solution to the sum of the first 15 whole numbers on 11/02/05
>
> M & C: [M & C have different answers and so T suggests that each of them check their answers. W intervenes, suggesting to T that he point out the following pattern to M & C.]
> T: What is fifteen plus one?
> M & C: Sixteen.
> T: What is fourteen plus two?
> M & C: Sixteen.
> T: What is thirteen plus three?
> M & C: Sixteen. [C laughs]
> T: So, I wonder if you could pair the numbers that way?
> C: That's what I did with mine! I got that one (fifteen) with the five. [C does not immediately check his answer so T requests again that he check his answer. He begins by using his original notation for the sum crossing out "15" and then "14" and "1". He writes the addition problem "15 + 15" next to where he originally notated the sum. He continues to produce notation in this manner (Figure 5.17) until he is momentarily interrupted by one of M's comments. When he returns to working, he runs his fingers over the notation stopping at "1" and "14", "2" and "13", until he gets to "7" and "8" and writes 7 above his initial notation for the sum. C counts the number of units of 15 in his second column of notation.]
> W: What is Carlos doing over here? Are you doing the same way (as Michael)?
> C: No. I'm not subtracting. I just figured out something that once you got that one, then you got that one and that one [C crosses out "14" and "1"], then you make another fifteen that one and that one [C cross out "13" and "2"], and then make another fifteen and you keep on going all the way down until you get to seven and eight. You have eight fifteens so you should have just timesed fifteens times eight then you would have got the answer.

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[29] In the language of mathematics, this way of operating suggests Carlos operated as if addition was a binary operation and that (a + b + c + d) = (a + c) + (b + d). However, this description does not fit Carlos's experience of the situation because I do not think he would formalize his way of operating in this manner.

```
  15
 +15
 ---
  30
 +15
 ---
  45
 +15
 ---
  60
 +15
 ---
  75
 +15
 ---
  90
```

Figure 5.17: Replica of Carlos's new notation

It is interesting that Carlos concluded, "That's what I did with mine.", when I suggested the sequence of addition problems to him. Based on his comment, I infer that he engaged in uniting operations when I made my suggestion, then re-presented his initial activity, and assimilated the new activity to his strategic additive reasoning scheme. In fact, his lack of action, after my suggestion, provides indication that he considered this way of operating similar enough (if not identical) to his initial way of operating. So, he assumed he did not need to operate further.

When he did begin operating again, Carlos established the goal of *producing* 15s, which was evident from the new notation he used to monitor the addition problems he was solving (Figure 5.17).[30] So, Carlos appeared to restructure the elements of the sum by uniting two elements to make a 15, and then, uniting this 15 with another 15 he had previously produced. It is interesting that Carlos continued to use additive notation, which I take as indication that simply producing units of equal numerosity was not sufficient to call forth his multiplying schemes. This observation is consistent with why Carlos would not have differentiated this new

[30] I infer that Carlos choose 15 rather than 16 because he produced this sum in the context of solving the Flag Problem, where 15 was prominent in the quantitative situation.

activity from his old activity; he did not initially see it as a way to transform his strategic additive reasoning into multiplicative reasoning. Rather, he simply seemed to take it to be a different way of structuring his strategic additive activity. Nonetheless, Carlos made a modification in his strategic additive reasoning scheme where the activity of the scheme had been modified so he produced units of equal numerosity in the process of solving the problem.

After he was interrupted by Michael's comment, Carlos was unaware of the total number of 15s he had produced. This provides indication that he had not been *enumerating* these units as he produced them. However, rather than producing all of the 15s again, he formed the goal of finding how many total 15s he would produce in all, and compared this with the number that he had already produced. I make this inference based on his running his fingers over the numbers in his sum starting with 14 and 1, 13, and 2, etc., and then writing "7" at the top of the sum depicted in Figure 5.17. Subsequently, he counted the number of 15s that were recorded in his second form of notation to compare the total number of 15s he had produced to the number he expected to produce (Figure 5.17). Here, his notation was integral to his monitoring the 15s he had produced initially, and then, subsequently in enumerating the total number of 15s he anticipated producing.

His explanation, "I just figured out something…you should have just timesed fifteen times eight", provides indication that it was not until he was interrupted by Michael that he had the insight to relate his activity to his multiplying scheme.[31] I infer that this interruption led to Carlos modifying the goal of his activity to include both *producing* and *enumerating* the total number of 15s, which called forth his multiplying scheme. In this situation, Carlos used his

[31] He had already suggested that the sum could be evaluated by 15 times 15, but he suggested this relationship because he had not differentiated the activity of producing flags from the activity he anticipated he would use to produce flags.

notation to externalize the results of his strategic additive reasoning scheme. He, then, used these results as input for his multiplying scheme where the activity of his multiplying scheme was to enumerate the total number of composite units he produced in order to find a multiplication problem that he could use to evaluate the sum.

This coordination led to a second modification in his way of operating, namely one that led to a novel coordination among his strategic additive reasoning scheme and his multiplying scheme. I infer it was not until this point in his activity that Carlos differentiated this new way of operating from his initial way of operating. He differentiated it because he now seemed to view the activity of his strategic additive reasoning scheme in the context of his multiplying scheme, which I infer shifted how he viewed the results of his strategic additive reasoning scheme from producing a quick way to add the sum to a multiplicative way to evaluate the sum. It is worth noting that Carlos ran through most of the operations involved in his strategic additive reasoning scheme prior to making the coordination with his multiplying schemes. So, the coordination he made between these two schemes was sequential in nature in that he produced most of the results of his strategic additive reasoning scheme prior to implementing the operations of his multiplying scheme. This sequential way of operating provides corroboration of my analysis in the Flag Problem; Carlos could reason in sequence first about the amounts in the sequence of addition problems, and then, enumerate the number of units he had produced, but he did not appear to be able to do the two simultaneously.

Carlos finished his solution ahead of Michael and so I asked him if he could use a new piece of paper and show what he had done so that he could explain it to Michael and me (Figure 5.18).

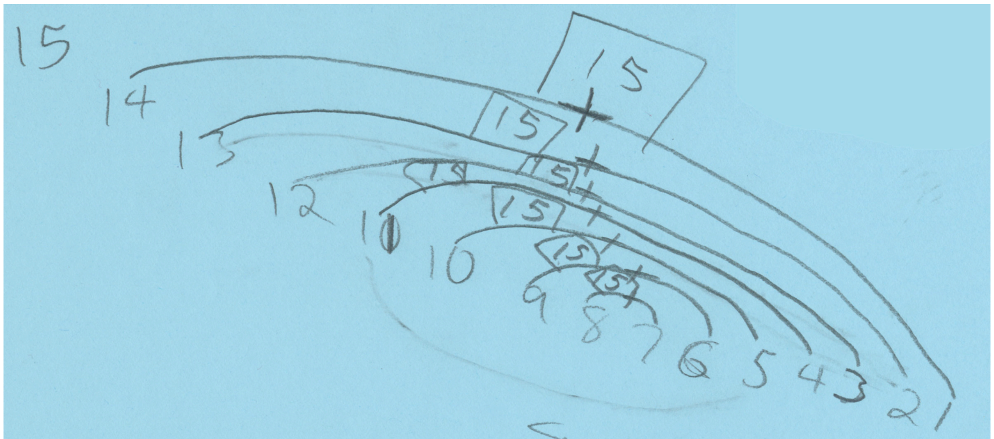

Figure 5.18: Carlos's notation[32]

Carlos's ability to produce notation for his activity suggested that he could re-present his way of operating and use notation to produce a record of how he had operated in the situation where his intent was to symbolize his activity. So, it appeared that he was able to conceptualize his way of operating, which I infer required producing an interiorized image of this way of operating that contained both the operations he produced in the situation and the material on which he operated. Then, he used his notation to externalize his interiorized image of his way of operating. So, in this instance, Carlos used his notation in a slightly different manner than he had previously. Namely, he used it as a tool to symbolize and reflect on his operating in the situation, rather than as a tool to help him monitor his way of operating in the situation.

It is noteworthy that Carlos used additive notation when he symbolized his solution of the problem. He had no difficulty remembering the multiplication he had suggested at the end of Protocol 5.9, but he found a multiplication problem only after I requested he do so. So, the he appeared to be aware of his restructuring of the sum into units of equal numerosity, but it was less clear that he was aware of the coordination he made between his strategic additive reasoning scheme and his multiplying scheme. I analyze Carlos's activity during the fourth teaching

[32] It should be noted that Carlos used additive notation here but later formed a multiplication problem and evaluated it.

episode, which occurred one week later, to investigate if the two modifications—producing units of equal numerosity with his strategic additive reasoning scheme and coordinating his strategic additive reasoning scheme and his multiplying scheme—were permanent, and so could be considered functional accommodations.

Carlos finds the sum of the first 9 and 99 whole numbers. At the beginning of the next episode, I posed the Handshake Problem to Carlos, which led to the sum of the first 9 whole numbers. He provided the following explanation for his solution of finding a quick way to evaluate the sum.

Protocol 5.10: Carlos's explanation for the sum of the first 9 whole numbers on 11/09/05

> C: I figured that just like the last time when I was adding them all and they all equaled fifteens, I just added them all up to ten to make it a lot easier. Then I figured that there would be a five and so once I added these [C points to the 10s he has notated] I got forty and then I just added the last one to get forty-five.

Carlos's explanation, which included reference to the previous episode, establishes that the functional accommodation he made to the activity of his strategic additive reasoning scheme was permanent and that he assimilated this situation to his strategic additive reasoning scheme. However, all of his verbal references in this problem situation were to *adding* and not to multiplying so in this situation he did not seem to independently initiate a coordination with his multiplying scheme.

The witness-researcher noticed this feature of Carlos's solution and made an on the spot conjecture that Carlos might produce a more multiplicative way of operating if the number of elements in the sum was large. So, he posed the problem of finding the sum of the first 99 whole numbers.

Protocol 5.11: Carlos's activity to find the sum of the first 99 whole numbers on 11/09/05

C: [C begins producing the ovals in Figure 5.19. T intervenes to see if C might figure out a quicker way to produce and enumerate the 100s.]
T: How many of those (one hundreds) do you think you would get Carlos?
C: I thought I was going to have forty-nine but I am not so sure now. [C continues to produce the notation and appears to be engaged in this process so T encourages him to continue. Once C has finished his first row of ovals he puts the numbers from one to eight above each of the first eight ovals. He points his pencil at the 9 in the ninth oval and stops writing the numbers above each oval. He resumes his activity of producing the units of 100 in the second row of ovals. C multiplies 50 times 100 and finds the sum would be 5000.[33]]

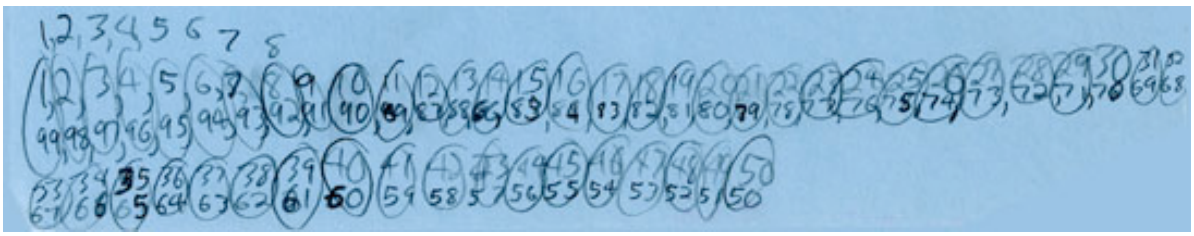

Figure 5.19: Carlos's notation for the sum of the first 99 whole numbers

In this situation, Carlos's response—he thought there would be "forty-nine" 100s—suggests that initially he did form the goal of finding a multiplication problem for evaluating the sum. This provides indication that he had formed the goal of coordinating his strategic additive reasoning scheme with his multiplying scheme. However, his lack of confidence suggests that he could not find a way to transform the sum into a multiplicative solution exclusively mentally. So, he began to monitor his production of the 100s using his notation.

When he completed the first row of notation, Carlos wrote the numbers "1" through "8" over the first eight ovals. I infer Carlos was beginning to *enumerate* the total number of 100s he had produced in the first row of ovals. When he reached the ninth oval, he appeared to establish a correspondence between the number of 100s he produced and the first element that each 100 contained. I make this inference because he pointed to where he had written "9" in the ninth 100

[33] Carlos made an extra unit of 100 by adding 50 and 50. In his explanation, he modified his answer to exclude this unit of 100 and concluded the sum would be 4950.

and then halted his activity.[34] Because Carlos did not finish producing all of the 100s prior to beginning to enumerate them, I infer that prior to operating Carlos had formed the goal not only to produce the 100s but also to enumerate them. Because Carlos formed both of these goals in the situation and maintained an awareness of each while he was operating, I infer that Carlos anticipated that the activity of his strategic additive reasoning scheme would produce results that he could enumerate using his multiplying scheme. So, I infer that the modification that led to his coordinating his strategic additive reasoning scheme with his multiplying scheme was permanent.

Further, Carlos enacted a new correspondence in making his solution to this problem. The new correspondence allowed him to attribute two meanings to the first unit in each 100. The first meaning was as an amount or a unit that contained so many singleton units (e.g., 32 is a unit of 32 units). The second meaning was 32 as the number of units of 100 that Carlos had already produced. That he was able to enact this dual meaning allowed him to embed his goal of enumerating the number of 100s in his activity of producing these 100s. So, in his notation for the 100 that contained 32 and 68, "32" simultaneously symbolized an amount and the number of 100s he had already produced.

In his solutions of the sum of the first 15 and 9 whole numbers, Carlos had produced most or all of the units and then subsequently enumerated these units so that he produced the units prior to achieving the goal of enumerating these units. By embedding his activity of enumerating the units in his activity of producing the units, he could achieve both goals simultaneously as opposed to the more sequential way he had achieved them in his solution of

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[34] Also, in a later explanation, Carlos referred to "the pattern" where "the first number (in each 100) was the same" as the number of 100s he had made.

evaluating the sum of the first 15 whole numbers. So, he enacted a coordination that embedded his multiplicative activity in the activity of his strategic additive reasoning scheme.

That he was aware of this correspondence was confirmed by his reference to this pattern in his final explanation of his solution to the problem. However, once Carlos made the correspondence between the first unit in each 100 and the number of 100s he had produced, he still remained unaware of what multiplication problem his activity would yield. I make this inference because although he stopped enumerating the 100s he continued producing these units by beginning the second row of ovals in Figure 5.19. So, he was not able to anticipate that the last 100 would contain 49, which would provide indication that there were 49 total 100s that he was going to produce.

To account for this way of operating, I appeal to the following explanation. I infer Carlos had not produced the sum as a *quantitative entity*. To do so, I infer entails a student constructing the sum as having a definite beginning and a definite ending. To establish a definite beginning and ending point, I infer the sum must be produced so that each element of the sum contains both an amount and position meaning and each element stands in relation to the previous element through a one more than relationship. In my analysis of the Flag Problem, I have suggested that Carlos was able to consider the amount that each addend symbolized or the total number of addends, but he was able to do so in sequence and not simultaneously. To operate on the sum as a quantitative entity appears to minimally require a simultaneous coordination between these two meanings. In this situation, Carlos was able to use his notation to embed the activity of enumerating the units in the activity of producing the units, but this embedding did not result in his curtailing his activity. It is possible that such an embedding might lead to producing the sum as a quantitative entity, but I was unaware of this possibility at the time I was presenting the problems to Carlos. So, I did not present any further problems similar to finding the sum of the first 99 whole numbers.

Carlos's Constructive Activity. Across these situations, Carlos demonstrated that he had constructed a stable and repeatable way of operating that allowed him to make a coordination between his strategic additive reasoning scheme and his multiplying scheme. The coordination among these schemes involved a functional accommodation to the activity of Carlos's strategic additive reasoning scheme, which opened up the possibility for him to make a coordination between this scheme and his multiplying scheme, a second functional accommodation. The first functional accommodation did not involve the construction of any new operations. That is, based on Carlos's initial activity in solving the sum of the first 15 whole numbers, I could attribute a uniting operation to him. However, he learned to use this operation in a novel way by strategically uniting elements of the sum to produce units of equal numerosity.

In order to enumerate these units, he had to externalize the units—treat the units as if they "existed"—after he produced them. To help him achieve this goal, Carlos used his notation to record each time he produced a new unit of 15, 10, or 100. This allowed him to treat the units as if they "existed" in the following sense; he did not have to re-produce the uniting operations that he had used to produce these units initially. Instead, he could take his notation to symbolize these operations without having to engage in them. This way of operating suggests that he could use his notation to re-present his uniting operations if it was necessary to do so, but he could also take these operations as a given in his further operating. His ability to use his notation in this way helped him to establish as well as achieve both the goal of producing and enumerating the composite units. This dual goal drove his activity, helping him establish a coordination between his strategic additive reasoning scheme and his multiplying scheme.

The coordination Carlos made between his schemes seems to have been in transition. That is, when he used his schemes in sequence (as when he solved the first 15 whole numbers), he appeared simply to establish an associational link between the two schemes. On the other hand, when he solved the sum of the first 99 whole numbers, he embedded the activity of one scheme in the other scheme, which led him to achieve both the goal of producing and enumerating the units simultaneously albeit he had to produce all of these units prior to actually knowing the total number of them. This type of embedding activity seems central in the symbolizing process because it allows a person to compress his activity so that he need not run through all of the operations involved in each of his schemes.

Carlos's Reflective and Reflected Abstraction. Carlos's activity suggests that he had engaged in both a reflective and a reflected abstraction[35] in the process of making a coordination among his schemes. In constructing the coordination between his schemes, Carlos had brought together a strategic use of his uniting operation along with his ability to externalize the composite units he produced so that he could enumerate these units. It was this coordination that was projected to a higher level of operating. I judge it as a higher because it allowed Carlos to evaluate a sum using his multiplying schemes, which opened the possibility for Carlos to coordinate his additive and multiplicative reasoning. There is no evidence from the data to support that the correspondence that Carlos made in his final solution was projected to the new level because I did not have any further occasion to observe Carlos's coordination between his schemes.

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[35] The differentiation between reflective and reflected abstractions was not always made in English translations of Piaget's work. For some discussion of the differences between the two see von Glasersfeld (1995) or Campbell's introduction to Piaget (2001).

In Carlos's case, he appeared to have also established a reflected abstraction. I make this assertion based on his referring to his activity of evaluating the sum of the first 15 whole numbers when he was producing the sum of the first 9 whole numbers. So, Carlos had used his activity in evaluating the first 15 whole numbers as a way to conceptualize what activity he would engage in when evaluating the sum of the first 9 whole numbers. This provides indication that he had abstracted his way of operating as useful in *any* situation that involved evaluating the sum of the first so many whole numbers. Here, I infer Carlos's concept of the sum was of a sequence of addition problems that began with one and each addend increased by one. However, I do not infer that the addends were related by the position of each. I infer that this allowed him to consciously establish the goal of producing and enumerating the composite units and to provide a linguistic explanation of his activity in the situation. Once again, however, I cannot claim that he was consciously aware of the correspondence he produced in his final solution of the sum of the first 99 whole number; he did notice the pattern in the context of working on the sum of the first 99 whole numbers but this coordination was enacted in his activity. So, although he did provide a linguistic explanation for it, I have no evidence one way or the other as to if he would take this coordination as a way of operating in the future.

Highlighting Several Further Aspects of Carlos's Symbolizing Activity.

Carlos Relates Additive and Multiplicative Notation. In solving the sum of the first 15 whole numbers, Carlos appeared to learn something about how his additive and multiplicative notation could be related. I make this inference based on his figuring out that he could transform his additive notation into something that he could compare to his multiplicative notation. So, he not only learned something about the quantitative situation through this transformation, but he also seemed to learn something about how he could relate these two types of notation. In this section of the case study, I have examined the underlying way of operating that enabled him to make this relation, but here, I simply want to suggest that he learned something about the notation itself and how he was using it. The additive notation (i.e., 15 + 14 + … + 1) seemed to contain a new awareness that each unit (e.g., 14) in this string of addition problems symbolized being one less than the previous unit in the addition problems. The multiplicative notation seemed to contain a heightened awareness that it meant producing units of equal numerosity.

I want to suggest that this type of learning is in itself important in the construction of an algebraic symbol system because then symbols can be taken themselves as a context in which to reason. So, for instance, a student might try to relate a sum to a multiplication problem simply in a context where he or she is given notation that symbolizes a sum (as Carlos did in finding the sum of the first 99 whole numbers). I take this statement as somewhat problematic because I have not specified what mental operations might constitute someone's meaning for the word "sum" or "multiplication problem" and too often particular kinds of notation are taken as having some meaning independent of an agent. So, in making this point, I am simply suggesting that learning of this type might enable a student to use conventional notation[36] as a meaningful context in which to reason. I will suggest in the sixth section of this case study that even when conventional notation is used as a context for reasoning students often produce very different ways of operating.

The Role of Social Interaction in the Symbolizing Process. As a final note to this section of the case study, I want to analyze the role that social interaction played in Carlos's symbolizing

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[36] Here, I mean conventional notation as notation that is accepted by the mathematical community as symbolizing a particular concept where people operate as if this concept is the same for everyone. I make this observation because finding the sum of the first 99 whole numbers (along with several other similar problems) did not come from a quantitative problem situation (like the Flag or Handshake Problem), but simply from asking Carlos to evaluate the sum from 1 up to 99.

activity. In Carlos's solution, interaction with me seemed to play an integral role in Carlos's finding a way to transform the sum into a multiplication problem. For Carlos, my suggestion of addition problems, which he initially assimilated to his own way of operating, appeared to help him restructure his operations in a novel way in order to find the sum. This suggestion appeared to be a central part of highlighting a novel structuring of operations where I could already attribute to Carlos a strategic use of his uniting operations. So, I could use my linguistic exchange with Carlos to point to a way of operating that he could attribute to himself. In later interactions, I tried to intervene to get Carlos to curtail part of his activity for finding the sum of the first 99 whole numbers but my interventions were not sufficient to help curtail or alter Carlos's activity in the situation. This is not to say that I could not have intervened in a way that might have been more profitable for Carlos, but rather to suggest that my interventions as they were did not yield any further modifications. In fact, Carlos independently contributed structuring the situations so that they were multiplicative which occurred based on his own self-regulation.

Contrasting these two interventions is interesting from the perspective of the role a teacher-researcher can play in the symbolizing process. When Carlos was able to assimilate my verbal suggestions to his own way of operating, he was able to modify his activity and the symbols he produced then reflected this modification (e.g., Figure 5.18). However, when Carlos was unable to assimilate my verbal suggestions to his activity, he simply continued to operate in the situation without making modification to his way of operating. So, the symbols he produced remained similar to the one's he had begun producing (Figure 5.19).

Michael's Activity to Solve the Outfits Problem and Similar Problems[37]

Overview

In this section of the case study, I examine Michael's activity during the first two days of the teaching experiment during which I posed him the Outfits Problem and several similar problems. This section is divided into four different parts. The first part examines Michael's activity in the context of solving the Outfits Problems, which occasioned a functional metamorphic accommodation that led to Michael's construction of a lexicographic units coordinating scheme. The second part summarizes Michael's activity in the context of solving card problems where he assimilated these situations to his one-to-one correspondence scheme. This section is intended to suggest that the way Michael operated in his solution of the Outfits Problem was indeed very novel for him. The third part, then, examines a functional accommodation that Michael made in his lexicographic units coordinating scheme during his solution of a problem I call the Coin-Die Problem. Finally, I conclude with a summary of Michael's symbolizing activity during these two episodes.

Michael's Lexicographic Units Coordinating Scheme

Michael's Initial Notation and Diagrams for the Outfits Problem. On the first day of the teaching experiment, I posed the Outfits Problem, Task 5.1, to Michael and Deborah (when I posed the problem this time I posed it so there were three shirts and four pairs of pants). Michael responded by saying that you could make three outfits. I asked him to show me how he got his answer and so he began to work out on a piece of paper his answer.

Protocol 5.12: Michael's Solution to the Outfits Problem on 10/17/06

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[37] Michael worked with Deborah on the first day of the teaching experiment and with Carlos after that. I switched Michael to work with Carlos because Deborah seemed more facile with these situations than Michael.

M: Okay, you said four pairs of pants. [Begins to draw a picture of one pair of pants and one shirt and writes 4 next to the pants and 3 next to the shirts.] You could take one and make one outfit. Take another one and take another one and then you have one pants (left over). Wait do we do every possibility?
T: Every possibility you could possibly get.
M: [M draws three tally marks followed by four tally marks, making two rows of tally marks.] You could do that, that, that [M connects each tally mark in the first row with the first tally mark in the second row.] You could do a lot of possibilities of this one. [M draws lines between the rows of tally marks (Figure 5.20) that end up in a jumble.] There is three for every pants so basically twelve. [In the meantime, M's partner D has produced the notation, "A1, A2, A3,…,D1, D2, D3" where "A" signifies the first pair of pants and "1" signifies the first shirt, etc.]
T: How did you get that?
M: There is three shirts for every one pants that is a possibility. [M sweeps his pencil from the first tally mark in the second row to each of the tally marks in the first row, and repeats this process for each of the tally marks in the second row.]

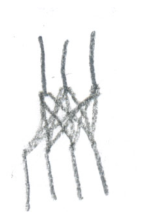

Figure 5.20: Michael's Tally Marks

In Michael's initial solution, "three", he created symbols for his concept of the three shirts and four pairs of pants by drawing a shirt and a pair of pants. This drawing (not shown, but similar to Carlos's drawing, Figure 5.3) suggests that he could take one shirt as a symbol for all of the shirts and similarly one pair of pants as a symbol for all of the pairs of pants. So, like Carlos, Michael seemed to assimilate the situation using his iterable unit of one where his number concepts symbolized the *amount* of shirts and the *amount* of pants.[38] In formulating a solution to the problem, Michael seemed to produce the mental material for each shirt and each pair of pants. In doing so, he imagined taking the first shirt and putting it with the first pair of

[38] Michael had constructed the generalized number sequence by the sixth grade a more powerful number sequence then the explicitly nested number sequence that Carlos had constructed by the sixth grade (Hackenberg, 2005). So, he too had constructed an iterable unit of one but his composite units were also iterable.

pants [“You could take one (shirt) and make one outfit.”]. He repeated this process for the second shirt and the second pair of pants and the third shirt and third pair of pants [“Take another one (shirt) and take another one (shirt) and then you have one pants (left over)”].

By doing so, he put the shirts in one-to-one correspondence with the pants, which led to the result he had first stated, “three”. I infer that Michael’s focus in making this one-to-one correspondence was on the mental action of putting a shirt with a pair of pants because his verbalization of his solution primarily focused on “taking a shirt”. So, in this situation, his concept of an outfit simply seemed to entail the action of putting a shirt with a pair of pants and did not entail taking the new entity, an outfit, as a unity that contained a shirt and pair of pants. As with Carlos, the mental action Michael used to put a shirt with a pair of pants I call his units coordinating operation. In Figure 5.21 below, I have illustrated only the units coordinating operations in which I hypothesize Michael engaged (contrast with Figure 5.4a & Figure 5.4b for Carlos) because as I suggested Michael’s outfit concept seemed to be created by the mental action of putting the shirt with the pair of pants.

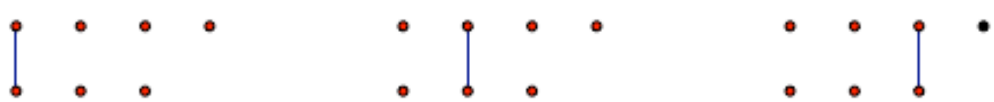

Figure 5.21: A diagram of Michael’s mental imagery and operations.

Michael’s verbalization of his solution seemed to provide occasion for him to enact the units coordinations that led to the result three and also brought to his awareness an element of the problem, which seemed to create a perturbation. I make the inference that he experienced a perturbation based on his saying, “Wait, do we do every possibility?” The perturbation seemed to arise from having a left over pair of pants that he thought he ought to be able to put with a shirt to create an outfit. So, he considered this pair of pants to be no different from any of the other pairs of pants. When he was done making outfits, he seemed to review his activity, realizing that in order to use the last pair of pants he would have to make an outfit with one of the shirts he had already used. This realization introduced the notion of possibility to Michael, which had been absent from his first solution. In his first solution, he had simply considered a pair of pants or a shirt as being "used" once he had made one outfit with it.

Michael re-initialized his activity in the situation, creating symbols in the form of tally marks for each pair of pants and each shirt. Here, he used his tally marks to help him enact an ordering for the shirts and similarly for the pairs of pants. I infer that he had established the goal of ordering between the shirts and pairs of pants because he anticipated he would need to put a shirt he had already used to make an outfit with the left over pair of pants. He then symbolized putting each of the shirts with the first pair of pants, by connecting the three tally marks in the first row with the first tally mark in the second row and saying, "you could do that, that, that" (See Figure 5.20). I infer based on Michael's language and action on his notation that he was making a units coordination between the first pair of pants and each of the shirts. Like Carlos, Michael made these units coordinations using a lexicographic ordering (i.e., taking the symbol for the first pair of pants and drawing a connecting line between it and the symbol for the first shirt, taking the symbol for the first pair of pants and drawing a connecting line between it and the symbol for the second shirt, etc.). So, he too produced an ordering and amount meaning for his number concepts in beginning to solve the problem.

In contrast to Carlos, after Michael finished his activity with the first pair of pants, he did not seem to anticipate that he would carry out this activity with each of the other pairs of pants. Rather, he actually carried out each of the units coordinations, which he symbolized by drawing a total of 12 connecting lines between the two rows of tally marks. I conjecture he operated in this way because once he had ordered the pairs of pants and ordered the shirts he treated them as if they were different from one another. I infer he continued to treat them as different from one another because this way of operating was novel to him.

However, once he had finished the activity with the tally marks Michael seemed to be able to treat the pants as if they functioned identically. I make this inference based on his subsequent explanation, "there is three (shirts) for every pants so basically twelve", which provides indication that after he was done creating outfits he was able to see that he had treated each of the pants in the same way. That is, he had made a units coordination between each pair of pants and all three shirts. He seemed to provide this explanation in part because of his experience of a distinct beginning and ending of the actions he performed on the tally marks. That is, he drew a line between the leftmost tally mark of the top row and leftmost tally mark in the bottom row, then drew a line from the middle tally mark of the top row to the leftmost tally mark of the bottom row, and finally drew a line from the right most tally mark in the top row to the leftmost tally mark in the bottom row. He, then, repeated this sequence of actions for all four tally marks in the bottom row. This explanation suggests that Michael could re-present his action on the tally marks to make a multiplicative explanation of his solution. However, unlike Carlos, the situation itself was not at the outset one that involved multiplication. Rather, he could interpret the situation as one involving multiplication once he had externalized his units coordinating activity by operating on the tally marks.

Michael Produces a New Form of Notation for the Outfits Problem. Both in verbalizing his solution and operating on the tally marks, Michael's outfit concept seemed to be centered on the mental action of putting a shirt with a pair of pants. So, his outfit concept seemed to be embedded in this mental action and it was unclear that he considered the outfits as distinct from the mental action that produced them. To help Michael produce a record of the outfits as distinct from his units coordinating activity, the witness-researcher asked Michael to use notation similar to the notation that Deborah had used in solving the problem.

Protocol 5.13: Michael uses new notation to produce a record of the outfits on 10/17/05

M: [M draws out four pairs of pants, labeling them “A” thru “D”, and three shirts, labeling them “1” thru “3”. M draws connecting lines between the pants labeled “A” and the shirt labeled “1”, and writes “A1” below. He repeats these actions until he gets to “B3” and then he ceases to draw the connecting lines (Figure 5.22) and simply finishes the list using notation “C1”, etc.]

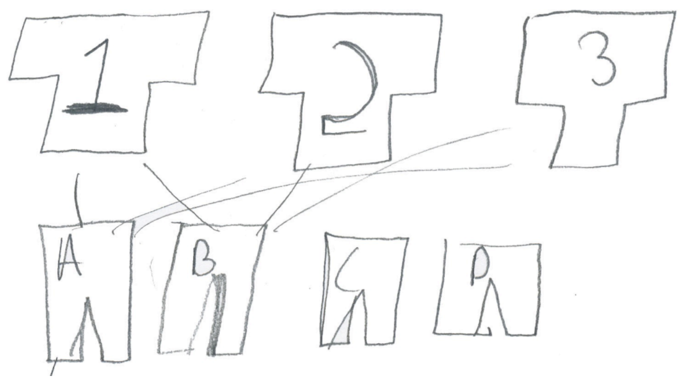

Figure 5.22: Michael’s Diagram of Pants and Shirts

In this solution to the problem, Michael drew lines between the pairs of pants and the shirts and then subsequently notated the outfits as, “A1, A2,…,D3”. This way of operating suggested that he differentiated between the units coordinating activity and the results of this mental action. So, in this solution, his outfit concept seemed to be differentiated from the mental action that produced the outfit. This differentiation suggested he had applied his unitizing operation to the results of his units coordinating activity to produce a *pair*. The distinction I am pointing to in this solution of the problem versus his verbalization and use of tally marks is one that I consider to be primarily about externalizing one’s way of operating. That is, Michael had all of the operations available to him to treat the outfits as a unity. By using his notation in the way he did, he seemed to make this distinction, externalizing the results of his operating from the operations that produced these results.

An Extension of the Outfits Problem. Because Deborah was waiting for Michael to finish his solution to the Outfits Problem, I posed the following extension of the problem.

> Task 5.8, *Extension of The Outfits Problem*: Suppose now that you have two pairs of shoes that you could put with each outfit. How many possible outfits could you make?

Michael heard me posing the problem to Deborah and so he wanted to solve the problem as well.

> Protocol 5.14: Michael's Solution of the Extension of the Outfits Problem on 10/17/05
>
> T: [D responds immediately with the answer 24.] Could you make a code to show that somehow?
> M: Wait, wait, I know what she is doing but I'm doing this. [M draws two pairs of shoes and in one puts the symbol "#" and in the other puts the symbol "?". He begins to write out each new unit sequentially giving him a list "A1#, A1?..." (Figure 5.23).] It's going to be very long.
> W: I like Michael's code.
> T: Mmhmm (yes).
> M: [About one minute passes and M as just written "C1?"] It's like doubling every time or like [M points to his list.]
> T: What's doubling?
> M: The amount of chances. Well, like each A1 is doubled again.
> T: Yeah.
> M: I get eighteen [M seems to have lost track of the number of outfits he originally produced.]
> T: Did you get the D's?
> M: Oops. I forgot there were four. [M finishes his list with the "D's" and finds there are 24.]

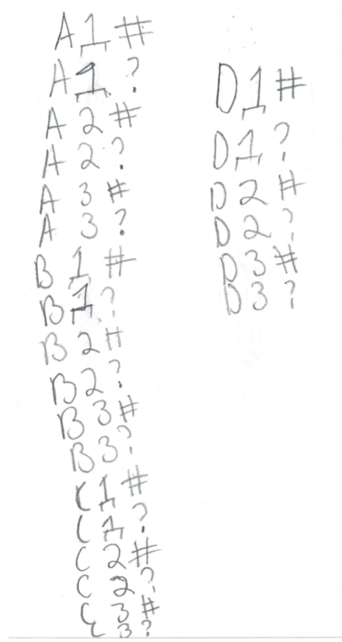

Figure 5.23: Michael's Notation for the Extension of the Outfits Problem

Interestingly, Michael was able to use his notation for the twelve outfits to make a continuation of his units coordinating activity. I infer that without using his notation to externalize the 12 outfits that he made just prior to the extension of this problem he would not have been able to solve the problem as he did. However, having created notation for the 12 outfits, he could take these as material with which he could make a continuation of his units coordinating activity. So, his notation seemed to play an integral role in his making a continuation of his activity. It should be noted that I presented this problem in two parts to Michael. So, Michael did not independently structure his activity recursively even though with the help of his notation he could operate on the results of his previous units coordinating activity.

Michael's comment, "It's like doubling every time…Well, like each A1 is doubled again", is interesting. It is interesting because it suggests Michael was aware that each individual outfit was doubling when he imagined putting it with the two pairs of shoes. However, this awareness was not enough for him to curtail his units coordinating activity. So, he did not seem to take his observation as indication that the total number outfits would double. This way of operating is consistent with my observation earlier that, while Michael was operating in these situations, he did not immediately experience them as situations that could be solved using multiplication even though he could re-present his activity and regard it as multiplicative. I infer that he did not experience them as situations involving multiplication because in order to operate in the situations Michael had to maintain a differentiation among the outfits (that was created by his ordering), even if, after the situation, he could regard them as identical.

That he referred to the units that were doubling in the situation by "A1" provides indication that he maintained an awareness of the outfits as containing both a shirt and a pair of pants. His new notation, "A1#" provides indication that he established new unities that also contained a pair of shoes. These new unities I call triples because each was a unity that contained a unit from each of three composite units and Michael appeared to maintain an awareness of each of the units in the triple. Because Michael seemed to be constrained to operating in a sequential manner in this situation, I infer he was able to create and keep track of these unities because of his use of notation in the situation. That is, while operating Michael seemed to treat each of the outfits as different, so he may have been able to create a few of these unities exclusively mentally, but it is doubtful he would have been able to keep track of them had he not used his notation.

Michael's Construction of a New Scheme. The Outfits Problem seemed to occasion a functional metamorphic accommodation that led to Michael's construction of a lexicographic units coordinating scheme. So, in contrast to Carlos, Michael did not seem to have a scheme already constructed for solving problems like the Outfits Problem. However, Michael did seem to have interiorized many of the operations that he used in making his solution to the Outfits Problem. I make this inference based on two aspects of his operating. First, he seemed to be able to carry out the operations involved in his scheme mentally, while he also enacted this activity and symbolized it using his notation. Second, he was able to re-present his activity and structure it multiplicatively. Both of these are indicators that many of the operations that Michael used were interiorized even though the coordination among these operations did not seem to be interiorized. Because this way of operating seemed novel to Michael, he did not always assimilate the subsequent experiential situations I posed to him using this scheme. Nonetheless, it was this problem situation that seemed to occasion his construction of a new scheme. So, I will provide a description of his scheme here in order to be able to compare his initial way of operating to his subsequent ways of operating.

Michael assimilated the Outfits Problem using his iterable unit of one to produce his concept of three and his concept of four. His introduction of possibility, after he made a one-to-one correspondence between the shirts and pants, was a central aspect of what was novel about Michael's way of operating. I will comment more on this introduction of possibility as I continue my exploration of Michael's activity. Once he introduced the notion of possibility into the situation, Michael's activity consisted of ordering each of the units in his concept of three and ordering each of the units in his concept of four. He, then, organized his sequential units coordination activity using a lexicographic ordering. When Michael carried out the units coordinating activity, he seemed to maintain the differentiation between each of the units in his concept of three and each of the units in his concept of four that his ordering had created. So, he carried out his units coordinating activity sequentially and did not give indication that he could curtail it.

I infer that Michael operated in this way because he had yet to interiorize a representative unit that symbolized a composite unit of ordered units. However, Michael's activity provides indication that he had interiorized two concepts that I consider would be central in constructing a representative unit. His unit of one was iterable. So, he could take one shirt as symbolizing three shirts. I make this assertion based on his initially drawing one shirt to symbolize the three shirts and one pair of pants to symbolize the four pairs of pants. Furthermore, Michael's use of the tally marks to order the units in his concept of three and order the units in his concept of four provides indication that he had an interiorized concept of ordering. It appeared that these two concepts had yet to be conjoined into the concept I have called a representative unit—a unit that symbolizes a composite unit of ordered units (e.g., a first shirt that symbolizes three ordered shirts).

The results of his scheme, during his solution with the tally marks, seemed undifferentiated from the action he used to produce these results. However, when he used his notation, "A1,…,D3", he appeared to establish the outfits as independent of the action that produced them, which I have taken as indication that he applied his unitizing operation to them, which created a pair—a unity that contained two units. In his solution to the extension of the Outfits Problem, the activity of Michael's scheme was similar but the results differed in that he produced triples, unites that contained three units, one from each of the composite units.

Michael's Reflective Abstraction. Given that I am suggesting that this situation occasioned the construction of a new scheme, I infer that a reflective abstraction was involved. Based on Michael's initial solution to the problem, using his one-to-one correspondence scheme, it was clear that Michael could engage in units coordinations to produce outfits. For Michael, the novel part of this situation appeared to be engaging in units coordinating activity in the context of ordering each of the units in his concept of three and four, which enabled him to order the pairs of pants and shirts. This ordering is what he used to monitor his activity. So, I infer what was projected to a higher plane was the units coordinating activity along with a new way to monitor this activity which relied on ordering the pairs of pants and ordering the shirts.

Michael Uses His One-to-One Pairing Scheme[39]

Summary of Michael's Activity with The Card Problems. I now want to briefly summarize Michael's activity in solving several problems that followed the Outfits Problem. I posed these problems to him at the end of the first teaching episode and into the beginning of the second episode. I make this summary for two reasons. First, I use it to suggest that Michael's way of operating in the Outfits Problem was indeed very novel for him. Second, I use it to investigate what might be required for introducing the notion of possibility into situations where a student is making pairs. After the Outfits Problem, all of the tasks I presented to Michael involved playing cards (like Task 5.4). I began all of the card problems with each student having an equal number of cards (13 cards, 6 cards, and 52 cards). I, then, had them play a game where they each drew a card randomly from the set of cards they had. They recorded the two-card combinations that they got and then repeated this process to make several experiential two-card combinations.

Analysis of Michael's Activity in the Card Problems. Michael assimilated these situations using his one-to-one pairing scheme. He operated similarly to how he had operated in the Outfits Problem when he found the results of this problem would be three. That is, he imagined putting the first card in one deck with the first card in the other deck until he had exhausted all of the cards. So, he figured out, for example, that there would be 52 possible two-card combinations when he and Carlos each had a deck of 52 cards. In these situations, Michael did not introduce

[39] I refer to it as his one-to-one pairing scheme as opposed to his one-to-one correspondence scheme because he provided indication that he was taking the results of units coordinating activity as a pair.

the notion of possibility into the situation. So, he did not consider that each card in one deck could go with any of the cards in the other deck. Therefore, once he had used a card to make a two-card combination he imagined this card to be no longer available for making any further two-card combinations. This way of operating suggests that the game where Michael replaced a card back into the deck was in itself insufficient for Michael to introduce the notion of possibility into these situations.

In each of these situations, Michael did experience a perturbation based on making a comparison between his answer and Carlos's or Deborah's answer who each found a multiplicative solution to the problem. In some of these situations, he did assimilate the other student's suggested way of operating to produce a way of operating that involved the notion of possibility, but other times he seemed only to get frustrated with the situations exclaiming things like, "she (Deborah) already told me the answer!", and "I don't get that!" (see Protocol 5.4).These exclamations suggest that introducing possibility into these situations was indeed a very novel way of operating for Michael. In order to understand more about what was novel in these situations for Michael, I turn to examining a second problem where he generatively introduced the notion of possibility.

Michael Makes a Functional Accommodation to His Lexicographic Units Coordinating Scheme

Michael Re-Introduces Possibility into the Situations. The card problems, I realized, were not helping Michael generatively produce the novel way of operating he introduced during the Outfits Problem. So, in the middle of the second teaching episode, I changed the problems to one's involving smaller numbers that were not of equal numerosity. One such situation was the Coin-Die Problem. To begin this problem, I simply had Michael flip a coin and Carlos roll a die. Without even posing a problem, Michael produced the following way of operating.

Protocol 5.15: Michael's Solution to the Coin-Die Problem on 10/19/05

M: [M flips a coin and C rolls a die. M & C each record "6T" on their paper after T asks them to record the result.] Wait a minute! [M writes "H, T" and "1,2,3,4,5,6".] Like I said the other day. [M draws in lines connecting "H" with the numbers "1,2,3,4".] So, like that times that [M points to his notation] and that times that. Times both of those. [M sweeps his pencil again over his notation as if he is making more connecting lines.] So, like thirty-six.
T: Thirty-six?
M: Wait a minute.
C: I got twelve.
M: Or that.
T: Why would it be that?
M: Cause it is six and six. [M draws in all of the connecting lines.]

Michael's exclamation in this situation, "Wait a second!", provides indication that he had an insight into the situation that he did not seem to produce when he was working on the card problems. In the card problems, Michael seemed to produce his solution based on operating on the actual material that he experienced in the situation, for example, two decks of 52 cards. So, the process of creating pairs was confined to how he could create these pairs with the actual material. Given this way of operating, once a card was used it was no longer available to be used with another card because it was already being used to make a two-card combination.

In this situation, on the other hand, Michael's insight was that he could interpret the situation as one that involved mental experimentation. The experiment, then, consisted of him putting an outcome on the coin with an outcome on the die to create a new kind of unit, a pair. So, what appeared to be novel about these situations was the notion of carrying out a mental experiment where any of the possible pairs could be attained as opposed to carrying out an experiment with material objects.[40]

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[40] This differentiation is one of the hallmarks of Piaget's (1958) differentiation between concrete and formal operations where the later involves mental experimentation to produce all of the possibilities and the former involves experimentation with perceptually available material.

I infer that this situation created a moment of insight for Michael because he took the coin as having two outcomes and the die as having six outcomes. So, the number of outcomes differed and he could not make a one-to-one pairing of these outcomes. This feature seemed to help him recall his way of operating from the earlier episode [“Like I said the other day”]. His recollection of how he operated along with his not using his one-to-one pairing scheme suggests that he was beginning to differentiate his new way of operating from his one-to-one pairing scheme. Nonetheless, Michael did not immediately produce the correct answer even after he had ordered the units in his concept of six and ordered the units in his concept of two and made four units coordinations. He symbolized this ordering with his notation “1, 2, 3, 4, 5, 6” and “T, H” and he symbolized his units coordinating activity by drawing the connecting lines between the letter “H” and the numbers “1,2,3,4” on his paper.

Once I threw doubt on his answer [“Thirty six?”] and Carlos suggested that he got twelve, Michael solved the problem in what appeared to be a novel way. He said there would be “six and six”, and subsequently drew all twelve lines on his notation. Given the short period of time that Michael took to respond, I infer he could not have sequentially imagined putting the heads on the coin with each of the six outcomes on the die. Rather, I infer he must have imagined putting the heads on the coin with a representative outcome on the die and took this to symbolize six pairs. I infer he engaged in a similar activity with the tails on the coin and a representative outcome on the die. This way of operating suggests that Michael did not have to engage in the sequential activity he had engaged in during his solution of the Outfits Problem. Rather, he appeared to produce a representative unit for the outcome on the die, and then, he iterated his units coordinating activity twice where each time he made a units coordination it symbolized six pairs. To confirm that this interpretation is viable, I present Michael's activity to produce a tree diagram for the same problem.

Michael Produces a Tree Diagram. After Michael produced several other forms of notation for the problem situation, I asked him if he could make a tree diagram.

Protocol 5.16: Michael Produces a Tree Diagram for the Coin-Die Problem on 10/19/06

M: [One minute and fifty-two seconds pass while M draws Figure 5.24.]
W: Okay, Michael, will you explain that? That's really, really, neat. That's really great Michael.
M: [Shrugs his shoulders.] Well, they're sort of like branches. Like H is the bottom, T is the top. [Looks at his diagram and quickly runs his pencil from the T on his diagram up each branch of the tree.]
W: How's that like Carlos's that he wrote over here? Could you guys kind of compare those and see if they are alike? [M looks at C's tree diagram for about 5 seconds]
M: Well T and H are the bases and one, two, three, four, five, six branch off [M opens his arms as if to show that the tails from the coin is with all six numbers from the die.]

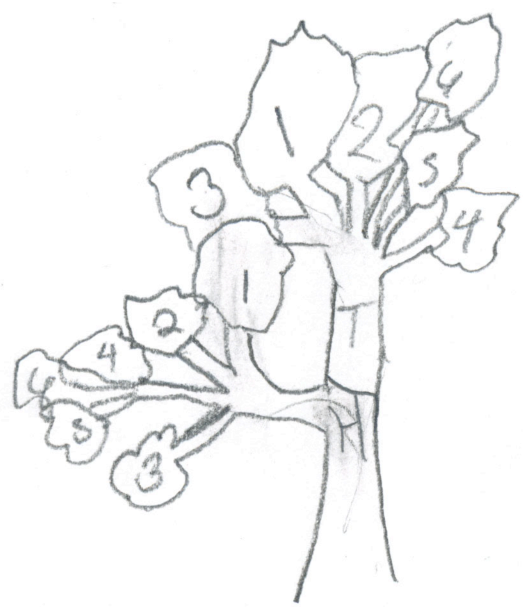

Figure 5.24: Michael's Tree Diagram

At the beginning of his explanation, Michael ran his pencil quickly up each branch of his diagram. I take this activity as indication that he was producing each of the pairs in sequence. Based on Michael's gesture with his arms, I infer that he subsequently produced the results a second time where this time he simply imagined putting the tails on the coin with a representative outcome on the die. This provides corroboration of how I suggested he had operated earlier in the situation. I infer that Michael could operate in this way because he considered each of the outcomes on the die to be no different from any of the other outcomes.

Michael's First Functional Accommodation. This event seems to be confirmation of the interpretation I made of Michael's way of operating above and provides indication that Michael did indeed make a functional accommodation. The functional accommodation was in his number concepts and I conjecture that it would allow Michael to assimilate these situations with his multiplying schemes. The functional accommodation in his number concepts was that he appeared to be able to create a representative unit that symbolized a unit of ordered units without actually having to carry out this ordering first. In this situation, Michael did not appear to produce a representative unit for both the coin and the die, but he did appear to produce a representative unit without actually having to carry out his sequential ordering and units coordinating activity.

I infer that this functional accommodation transformed the activity of Michael's scheme so that it was now multiplicative. It was multiplicative because Michael could imagine putting a representative unit from the die with a unit from the coin and he could iterate this operation to produce a composite number of pairs some number of times. I infer Michael could operate in this way because of the functional accommodation he had made to his number concepts. Given that Michael had not come into the teaching experiment having already constructed a representative unit it is worth analyzing what appeared to lead to his construction of a representative unit. I infer that he abstracted a representative unit from specifying a property of an object concept (e.g., the numbers on the die) and then iterating his unit of one while maintaining an awareness of the property that he had specified. I infer that this activity was crucial for Michael in abstracting a representative unit and is why he actually enacted the ordering of the units in his solution of the Outfits Problem and the Coin-Die Problem.

It should be noted that in the Outfits Problem, the activity of Michael's scheme was itself not multiplicative. I make this statement because in the Outfits Problem he seemed constrained to putting each shirt with each pair of pants in sequence where he actually needed to make and maintain the differentiation that his ordering created throughout his activity. When he was done with this activity, he could re-present the activity and structure it multiplicatively but this is different from the activity of the scheme itself being multiplicative. For the activity of his scheme to be multiplicative, he had to produce a representative unit that he could use in his units coordinating activity.

Highlighting Several Further Aspects of Michael's Symbolizing Activity

Michael's Symbolizing Activity in the Outfits Problem. In the Outfits Problem, Michael created symbols in the form of a picture of a pair of pants and a shirt. Then, Michael operated on mental material to produce the three outfits he anticipated making. After he was done producing these outfits, he seemed to experience a perturbation, which led to him creating new symbols for the situation, the tally marks. He used these symbols to help him order the pairs of pants and the shirts. Here, he did not seem to anticipate exactly how he would operate, but rather began operating with the tally marks in a way that allowed him to keep track of which outfits he had produced. To this point, Michael used his symbolizing activity as a tool in action, in the sense that he used it to help him coordinate and monitor his way of operating. When Michael re-presented his activity with the tally marks to produce a multiplicative explanation of the situation, I infer that the tally marks that he had used as a tool in action became a tool through which Michael was able to reflect on the situation. I make this inference based on my inference that Michael re-presented his activity on the tally marks to make his subsequent explanation, “three shirts for every pants”.

Michael’s Symbolizing Activity in the Extension of the Outfits Problem. At the witness-researcher’s request, Michael created new symbols, Figure 5.22, and the notation “A1…D3”, for the Outfits Problem. I infer his use of symbols helped Michael to differentiate between the mental action that produced the outfits and the outfits themselves. Here, the primary purpose of Michael’s notation seemed to be to externalize the results of his activity so he could treat these results as a pair, a unity that contained two units. This type of externalizing activity allowed Michael to reflect on the results of his activity, and then subsequently to make a continuation of that activity in the extension of the Outfits Problem. In the extension of the Outfits Problem, the symbols, “A1#, …, D3?”, that Michael produced once again became a tool in action because he used the notation monitor and coordinate his new way of operating. During this solution, Michael was able to use his notation to reflect in action, when he used his natural language to observe the pattern that “it is like doubling each time”. This symbolizing activity allowed him to observe a pattern in the activity of his scheme, but did not enable him to curtail his activity.

Michael’s Symbolizing Activity in the Coin-Die Problem. In the Coin-Die Problem, Michael appeared to make a functional accommodation in his way of operating. Subsequently, Michael created a symbol, the tree diagram, where he seemed to consider the tree diagram to contain both his old and his new way of operating. That is, he could use the tree diagram to imagine sequentially putting the tails on the coin with each of the six outcomes on the die (as was suggested by his running his pencil out each branch of the tree) and he could take it as a way to see making one units coordination between the outcome of tails on the coin and a representative outcome on the die (as was suggested by the surrounding arm gesture he made in the situation). Here, the symbols seemed to help him externalize and reflect on the functional accommodation he made in his way of operating.

Michael's Activity to Solve the Flag and Handshake Problems

Overview

In this section of the case study, I examine Michael's activity during the third and beginning of the fourth episodes. In the first part of this section, I will examine Michael's activity in the Flag Problem, which he worked on during the third episode. I will focus on confirmation that Michael's first functional accommodation was permanent and indication of further functional accommodations Michael made. In the second part of this section, I analyze concepts related to Michael's interiorization of various aspects of his scheme. In the third section, I analyze Michael's activity in the Handshake Problem, which he worked on during the beginning of the fourth episode. I will focus on how Michael's functional accommodations during the Flag Problem were incorporated into his operating in the Handshake Problem. In the final part, I will highlight the role Michael's self-regulation and monitoring played in his solution of these tasks in order to argue that Michael had abstracted a conceptual network of related tasks.

The Flag Problem

Confirmation of Michael's Functional Accommodation in the Coin-Die Problem. In the third teaching episode, Michael worked on the Flag Problem, Task 5.6, for the entire episode. After making a few flags in GSP (See Figure 5.13), I asked the boys how many possible flags they could make.

Protocol 5.17: Michael's Initial Solution to the Flag Problem on 11/02/05

M: We keep doing possibilities. [M counts the number of colors in the GSP menu.] Fifteen times two. Thirty.

T: Fifteen times two?
M: Thirty flags.

In solving the problem, I infer Michael made a units coordination between a representative unit that symbolized the 15 colors with the top stripe and a representative unit that symbolized the 15 colors with the bottom stripe where each units coordination symbolized producing 15 "flags" (note the flags were not two stripe flags so thirty referred to what I have just described.) This situation was the first time Michael assimilated the problems I posed to him using his multiplying schemes. This way of assimilating the situation provides confirmation that he made a functional accommodation in the Coin-Die Problem. This functional accommodation modified the activity of his lexicographic units coordinating scheme, which enabled him to recognize the activity of his scheme as multiplicative (as opposed to structuring the results multiplicatively after he had operated). Further, it suggests that Michael expected that he would need two composite units to engage in his units coordinating activity. Since there was only one list of 15 colors that Michael had used to create the experiential flags in GSP, I infer he used a second composite unit, the number of stripes, which he had identified.

Michael's Second Functional Accommodation. Michael's use of the stripes and the colors was not the solution I expected. So, I offered a suggestion to elicit Michael's further activity.

Protocol: 5.18 Michael's second solution of the Flag Problem on 11/02/05

T: Let's see, how many could you have if you had black down here? [T points to a flag that has the bottom stripe filled black.]
M: That takes off half.[41] Well no. What am I thinking? Is it how many flags there are or?
T: Nope you have to imagine that you could have a whole bunch more (flags that you could fill with two colors). And, the question is how many could you make with black as the bottom color? [T points to two flags that have black as the bottom color.]
M: Fourteen.

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[41] Michael's comment of "taking off half" provides further confirmation of my interpretation above because if the bottom color is fixed then no possible "flags" could be made with it.

T: Fourteen. How did you get that?
M: Because you take black off the possibilities and…
C: [Interrupting M] It should be fifteen.
M: Well, yeah fifteen because…
C: [Interrupting M] Black and black.
M: Yeah, black and black.
T: I wonder how many possible one's you could make total if you let this one [T points to the bottom stripe of the flag] be any possible color and this one [T points to the top stripe] be any possible color.
M: I'll do what I do. Let's see. [M begins to write out a list of the colors on a piece of paper.]
T: Why don't you number each color as well?
M: [M finishes writing out the list of colors and associated numbers. He sits in concentration for 20 seconds. He writes "1 red-14 colors", "2 orange-14 colors"[42], erases "14" in the second statement and writes "13" in its place. Then, he writes "3 yellow-12 colors". Off camera he writes fourteen "1s" in a column. Concentrating, he stares at his notation for 20 seconds at the end of which he puts his finger in the air and smiles. Next to where he has written "1", he fills in the numbers "2,3,…,15" to make a column that contains the notation "1-2,…, 1-15". He returns to the top of the column and writes "1-1" (Figure 5.25).][43]

1-1
1-2
1-3
1-4
1-5
1-6
1-7
1-8
1-9
1-10
1-11
1-12
1-13
1-14
1-15

Figure 5.25: Replica of Michael's first column of notation

When I asked him how many flags he could make with black as the bottom color, I infer that Michael entered a state of perturbation ["What am I thinking? Is it how many flags there are or?"] Because he eventually responded to my question by saying, "fourteen", I infer that he

[42] Note that the numbers "1" and "2" were the numbers that Michael had associated with the color red and the color orange as a result of my request to number each color.
[43] By off camera, I mean off the camera that was capturing student work. I make these inferences based on the student-teacher interaction camera and the work I collected from him.

assimilated the situation with only one representative unit, which symbolized the list of 15 colors. He appeared to partition this unit into two parts [“because you take black off the possibilities”] (Figure 5.26a), and make a units coordination between the color black and a representative unit that symbolized the remaining 14 colors (Figure 5.26b), which produced a representative flag that symbolized 14 flags (Figure 5.26c). So, Michael seemed to imagine only one composite unit (the 15 colors) and breaking this quantity into two parts (black and the remaining colors) in order to engage in his units coordinating activity. In Figures 5.26a, 5.26b, and 5.26c, I have used the outer brackets to suggest that Michael’s operating was taking place within one composite unit. I make the inference that he began operating with one composite unit, the 15 colors, because he left out the black-black flag, the only flag that by necessity required beginning with two different quantities prior to operating. To make this flag, he would have had to imagine a representative unit and the color black (a singleton unit) as independent from this list of colors.

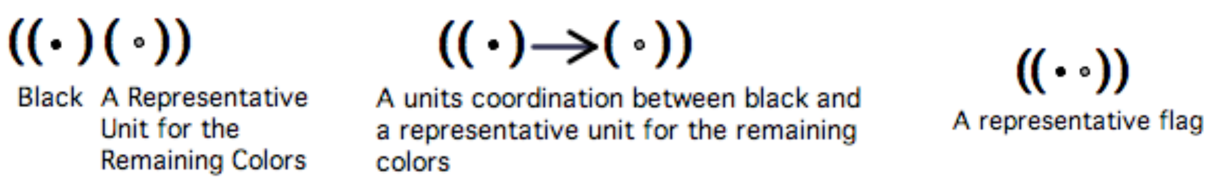

Figure 5.26a (right), 5.26b (middle), & 5.26c (left): A diagram of Michael’s mental imagery and operations

Although he did agree with Carlos that it was possible to make black-black flags, he wrote the words “1 red – 14 colors”, “2 orange – 13 colors”, “3 yellow – 12 colors”. So, it appears that this exchange with Carlos was not sufficient for Michael to change his way of operating. Rather, he appeared to continue operating in a manner consistent with how I have described above. So, I infer that this notation symbolized the number of flags he could make with the colors red, orange, and yellow. Because he originally wrote, “2 orange – 14 colors”, and then erased the “14” to replace it with “13”, I infer that Michael was not differentiating between the bottom and top stripe of the flag. So, once a color like red had been used from the list of colors, Michael considered it to be no longer available because then he would be counting flags he had already counted.

This way of operating seems sensible given that I have suggested that Michael was not producing two composite units. By producing two composite units, it is possible to imagine taking red from one list of colors (the first composite unit) and putting it with orange from the second list of colors (the second composite unit) and then imagining taking orange form the first list of colors and putting it with red from the second list of colors. This way of operating would create a sense of difference between these two flags. Without producing two composite units, flags that were, for example, red orange and orange red would be the same because the lists from which each color originated was the same. So, there would be no identifiable operative characteristic that would differentiate the flags. I use the word operative because both boys would have differentiated the flags had they seen them.

The way I have hypothesized that Michael was operating also helps to explain why he produced a new form of notation, 14 “1s”, and sat in concentration for 20 seconds. It helps explain it, because in his way of operating, he had introduced making a units coordination between *two* parts, red and representative unit, of the same composite unit, the 15 colors. I infer that this created a second perturbation for Michael because, as I suggested, he initially expected the situation to be one involving two composite units, the number of colors and the number of stripes. Here, he was breaking one composite unit into two parts, which seemed to create a feeling of dissonance for him.

I infer that this dissonance led to his producing new notation, which he thought might help him better monitor his activity. After producing 14 “1”s, he sat staring intently at this notation for 20 seconds, smiled raising his finger as if he had resolved his perturbation, wrote the numbers “2,3,…,15” next to the 14 “1s”, and returned to the top of the column of notation to write “1-1”. I infer that Michael’s intent when he wrote the 14 “1s” was to use his notation to symbolize the 14 flags that he anticipated he could make with red, but which he had not actually produced by sequentially making a units coordination between red and each of the fourteen other colors. So, his notation provided indication that he intended to drop down to a lower level of operating, sequentially putting red with each of the other colors, in order to monitor his activity. Dropping down to a lower level of operating was an act of self-regulation. I infer that this use of notation was Michael’s way of externalizing his operating in such a way that he could monitor this operating.

His notation seemed to help him become aware that he was using red independently of the list of 15 colors. So, rather than partitioning the list of colors into two parts, Michael seemed to disembed red from the list of colors, which meant that he produced one composite unit and a singleton unit with which he could make flags. I infer he sequentially put each color on the list (including red itself) with red (Figure 5.27), which produced 15 flags (Figure 5.28). I make this inference based on his filling in the numbers “2,3,…,15” and then returning to the top of the list and writing “1-1” which symbolized the red-red flag.

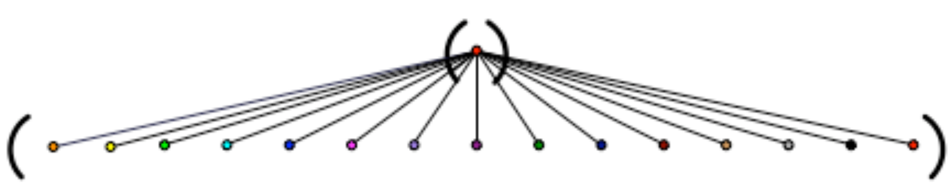

Figure 5.27: A diagram of Michael’s mental imagery and operations

Figure 5.28: A diagram of Michael's mental imagery and operations

The act of dropping to a lower level of operating in order to monitor his activity enabled him to make a functional accommodation to his lexicographic units coordinating scheme where he produced a singleton unit and a composite unit with which to make flags. I infer he remained unaware of having to produce two composite units in both the Outfits and Coin-Die Problem because, from his perspective, the two composite units were given in the statement of the problem. Here, Michael actually had to produce a singleton unit from the composite unit that he had used to assimilate the situation.

Michael's Third Functional Accommodation. In order to make this functional accommodation, I have suggested that Michael dropped to a lower level of operating in order to monitor the activity of his scheme. To investigate whether Michael subsequently returned to a higher level of operating and to investigate other consequences this functional accommodation had on Michael's operating, I present more of Michael's activity in solving the Flag Problem.

Protocol 5.19: Michael's further activity on the Flag Problem on 11/02/05

M: [M begins his second column with "2-2" and continues writing out his notation. He begins his fifth column of notation when C says he already has the answer. M tells him not to give it away. M looks at C's paper and sees that C is producing similar notation.] I am doing the same thing he is. Then, after the fifth I'm getting tired so just go ahead and…[M gestures as if it is going down one each time. He continues to write out the fifth column of notation]. I know a faster way, but I'll go ahead and do it like this. [M continues writing out his notation until he finishes his eighth column.] I know a faster way.
T: You are doing a great job.
M: [M begins the ninth column of notation.] It just decreases by one every time.
T: Yeah, it just decreases by one every time.
M: [M finishes all 15 columns of notation (Figure 5.29).] See it's like, there is fifteen and it decreases by one every time. So, it's like fifteen plus fourteen plus thirteen plus twelve plus until you get to one.

T: Why don't you write that out and see if you can figure out how many you got that way?
M: There is like one [M writes "1" next to where he has written "15".], two [M writes "2" next to where he has written "14".], three [M writes "3" next to where he has written "13". He finishes writing out the sum in this way producing Figure 5.30.]

1-1	2-2	3-3	4-4	5-5	6-6	7-7	8-8	9-9	10-10	11-11	12-12	13-13	14-14	15-15
1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	
1-3	2-4	3-5	4-6	5-7	6-8	7-9	8-10	9-11	10-12	11-13	12-14	13-15		
1-4	2-5	3-6	4-7	5-8	6-9	7-10	8-11	9-12	10-13	11-14	12-15			
1-5	2-6	3-7	4-8	5-9	6-10	7-11	8-12	9-13	10-14	11-15				
1-6	2-7	3-8	4-9	5-10	6-11	7-12	8-13	9-14	10-15					
1-7	2-8	3-9	4-10	5-11	6-12	7-13	8-14	9-15						
1-8	2-9	3-10	4-11	5-12	6-13	7-14	8-15							
1-9	2-10	3-11	4-12	5-13	6-14	7-15								
1-10	2-11	3-12	4-13	5-14	6-15									
1-11	2-12	3-13	4-14	5-15										
1-12	2-13	3-14	4-15											
1-13	2-14	3-15												
1-14	2-15													
1-15														

Figure 5.29: Replica of Michael's notation[44]

15	1
14	2
13	3
12	4
11	5
10	6
9	7
8	8
7	9
6	10
5	11
4	12
3	13
2	14
+1	15

Figure 5.30: Replica of Michael's notation for the number of flags

Michael began his second column of notation with "2-2" and not "2-1". I infer this elimination was a result of his operating not based on the consideration that the flags symbolized by "1-2" and "2-1" could be counted as different. So, I have to account for how his operating

[44] Michael's notation spanned two pages and the third, fourth, and fifth column were actually two columns because he ran out of room at the bottom of the paper. So, it was not as nicely spatially organized as I have presented it here.

would lead to eliminating the flag symbolized by “2-1”, especially now that Michael seemed to be producing a composite unit and a singleton unit. I infer that when Michael disembedded red from the list of colors that he did not imagine this operation as producing a second composite unit of 15 colors. Rather, it simply provided him with the unit red with which to make new flags. So, when he reached the second color orange, he disembedded this from the composite unit that symbolized the 15 colors, but he had already eliminated from this list the color red because he had just used it. This elimination meant he had already imagined eliminating red prior to using the color orange to make new flags.

This way of operating suggests that when Michael began his second column of notation, he was still operating with only one composite unit. So, he disembedded a particular color each time he began a new column of notation and eliminated the unit that he disembedded when he was done making the column of notation that symbolized the flags he could make with this color (i.e., red was eliminated after the first column of notation was complete, orange was eliminated once the second column of notation was complete.) This way of operating was both similar to and different from how Michael had operated in the Outfits the Coin-Die Problem. It was similar to it, in that, once a pair of pants had been used it was eliminated from being used to make more outfits. It was different from how he had operated because he was using one composite unit to generate the singleton units and the composite unit that he used to make flags. So, operating in this way would have been like eliminating both the first pair of pants and the first shirt in the Outfits Problem after the first pair of pants had been used to make all the outfits.

At the beginning of writing his fifth and ninth column, Michael’s comments, “I know a faster way” and “It just decreases by one every time”, suggests that Michael anticipated what the results of his activity would be. At my encouragement, he did finish Figure 5.29. When Michael was done making Figure 5.29, he made the comment, “See it’s like, there is fifteen and it decreases by one every time. So, it’s like fifteen plus fourteen plus thirteen plus twelve plus until you get to one.” This comment suggests that he was aware of the number of addends in the situation as well as the amount of each addend. Figure 5.30 provides confirmation of this interpretation, where he numbered each of the addends to illustrate to me what he meant by his comment. So, Michael appeared to be aware of the number of columns and the number in each column in his notation as soon as he completed his activity.

This interpretation suggests that somewhere in the process of disembedding individual colors from the initial list of 15 colors he reconstituted these actions as producing a second composite unit of 15, which enabled him to establish the number of columns of notation he would produce. Establishing both the number of columns and the number in each column suggests that Michael made a units coordination between two representative units—one that symbolized the first composite unit he had used the one that symbolized the second composite unit he had produced from his disembedding activity.

This was a functional accommodation in his lexicographic units coordinating scheme and differed from how Carlos had operated both in situations that involved multiplication and in the Flag Problem. I have argued that Carlos made a units coordination between a representative unit and a particular unit to produce some number of composite pairs. He, then, subsequently was able to imagine the particular unit to be a representative unit, but only after he had established the composite number of pairs. Here, I am arguing that Michael made a units coordination between two representative units. Because this situation is more complicated than a situation where the boys operated multiplicatively, I want to present what such a units coordination might produce in a multiplicative situation, and then, come back to my analysis of this situation.

In the Outfits Problem, for a student to make a units coordination between two representative units, a student would imagine putting a representative shirt with a representative pair of pants and this would simultaneously symbolize the number of outfits he could produce with, for example, each shirt and the number of times he could produce this number of outfits. So, both representative units would symbolize any but no particular of the units in each composite unit. Carlos appeared to be able to make a units coordination with a particular unit and a representative unit, and then, he could imagine that the particular unit was a representative unit, but he did not appear to make a units coordination between two representative units.

This situation differed because each column contained one less than the previous column. I infer by the time he produced his notation for the fifth column he had abstracted this one less than relationship. So, I infer that at the beginning of his fifth column of notation he made a units coordination between a representative unit that symbolized the number of flags he would produce in that column and a representative unit that symbolized the column number. The first representative unit was representative because it contained his abstraction that each column would contain one less than the previous column and so symbolized any but no particular of the number of flags in each column. The second unit was representative because it symbolized any but no particular of the columns where each particular column had been produced by a particular singleton unit that Michael disembedded. Given this interpretation, I infer that by the fifth column Michael was simply using his notation to illustrate the results of his units coordination between the two representative units. The results of this units coordination was Michael's production of the sum as a quantitative entity. I make this assertion because each addend was related to the previous addend through an amount and ordering meaning, 15 symbolized the first addend, 14 was one less than 15 and was the second addend, etc, and Michael was aware of the total number of addends. I will focus on Michael's construction and evaluation of the sum as a quantitative entity in section six.

Michael's Scheme and Four Related Concepts

In the context of the Outfits Problem, I suggested that Michael made a functional metamorphic accommodation that led to the construction of his lexicographic units coordinating scheme. In the Outfits Problem, the scheme was functioning at a significantly lower level than the operations that constituted the scheme. In fact, I would consider the scheme to be pre-multiplicative because the activity of Michael's scheme could not be considered multiplicative even though he could re-present this activity and structure it multiplicatively. Given that I suggested that this problem situation was one in which a functional metamorphic accommodation occurred, it is sensible that a number of other functional accommodations would be associated with the functional metamorphic accommodation as Michael integrated the new scheme with his other operations, schemes, and concepts.

Therefore, it is not a surprise that over the course of the next three episodes, Michael made numerous functional accommodations that led to several subsequent interiorizations of the scheme. So, it is actually inappropriate to talk about Michael interiorizing his scheme in one fell swoop, but rather he interiorized various aspects of the scheme from operating in more problem situations where he made subsequent functional accommodations to his scheme. Along with each functional accommodation, a reflective abstraction occurred—a projection of operations from a lower to higher plane of operating. Since I have talked about the functional accommodations, in the earlier parts of this case study, here, I will present the new concepts that Michael seemed to construct and the operations that were projected to a higher plane of operating as I review each functional accommodation.

In the Coin-Die Problem, Michael introduced a new way of operating. He imagined putting a representative unit from the die with a particular unit from the coin and took this to symbolize producing six pairs. In this situation, the tree diagram that I requested that Michael make appeared to be an externalization of this way of operating where he seemed to be able to attribute both his initial way of operating (sequentially putting each outcome on the die with an outcome on the coin) and his new way of operating (putting a representative outcome on the die with an outcome on the coin) to the diagram. Here, what was projected to a higher plane was his construction of a representative unit that symbolized a unit of ordered units. This meant that when he assimilated situations he could assimilate them to a representative unit that symbolized both an amount and ordering meaning.

Two new concepts, an ordering concept and a whole number variable concept, appeared to be abstracted with this construction of a representative unit. Michael abstracted his enactment of an ordering, which he engaged in during the Outfits and Coin-Die Problem into a concept. This abstraction meant that he no longer had to carry out ordering the units. When he enacted an ordering, he appeared constrained to actually iterating his unit of one while focusing on the position that each iteration created in relation to the previous iteration. An example would be when he created the tally marks where it appeared he had to actually carry out the ordering by iterating a tally mark three times. Because Michael independently contributed this ordering, I infer he already had an ordering concept, but it did not seem to be integrated with his amount concept of number. So, his construction of a representative unit conjoined his amount and ordering concepts.

Michael's construction of a representative unit enabled him to operate with what I have termed a whole number variable concept. That is, Michael could make a units coordination with a particular unit from one of the composite units and a representative unit from the other composite unit. So, in the Coin-Die Problem, he put the tails on the coin with a unit that symbolized *any but no particular* of the units on the die. As I suggested in Carlos's case study, I consider this to be a minimal criteria in the construction of a whole number variable concept. So, I consider Michael to have constructed such a concept.

The second and third functional accommodation that Michael made suggested that he produced two composite units as a result of his operating in the Flag Problem. Moreover, he appeared to make a units coordination between the two representative units—one from each of the two composite units. These operations along with his awareness of the one less than relationship produced the sum as a quantitative entity, a new concept that Michael appeared to construct as a result of his operating in the Flag Problem. Here, Michael produced a second representative unit that symbolized a second composite unit as a result of the disembedding operations he performed on the first composite unit. I have argued that Michael began by making a units coordination with a particular unit and a representative unit in the Flag Problem. However, by the time he produced his fifth column of notation, he appeared to be have established this second unit as a representative unit. Moreover, I have argued that Michael was making a units coordination between two representative units, which is what gave him a simultaneous awareness of the both the number of columns he would produce and the number in each column.

Michael's operating suggests that he had produced a new type of unit, which I called in Carlos's case study a pair concept. This unit contains two slots and the two slots can be filled with two representative units each of which symbolize a composite unit of ordered units. I infer that Michael abstracted this unit from his activity of putting each particular unit (e.g., each particular pair of pants) from the first composite unit with each particular unit (e.g., each particular shirt) from a second composite unit. So, the first slot in his pair concept contained records of having inserted, for example, each particular pair of pants. Furthermore, the first slot had records of having ordered the pairs of pants and selecting a representative pair of pants to symbolize any but no particular of these pairs of pants. I make the same inferences about the second slot. Because Michael initially assimilated the Flag Problem with only one composite unit and produced the second composite unit from his disembeding a singleton unit, Michael's pair concept was constructed in a situation that involved a one less than relationship (as opposed to a multiplicative relationship). I now turn to an analysis of Michael's solution to the Handshake Problem to investigate how his construction of these new concepts were integrated into his operating.

The Handshake Problem

Michael's Solution of the Handshake Problem. I posed the Handshake Problem, Task 5.7, to Michael and Carlos during the fourth episode. After I presented the problem, Michael began to speak aloud as he produced notation.

Protocol 5.20: Michael's solution of the Handshake Problem 11/09/05

M: [Writes "10 people" on his paper and begins a column of notation that starts "1A, 2"] A can't shake his own hand, can he? [M gestures to make a handshake with his left and right hand.]
T: No, A can't shake his own hand.
M: [M erases the "A" and replaces it with a "B". M erases the "2" and writes "1C". He continues his notation writing "1D, 1E, 1F, 1G, 1H, 1I", counts the notation in his column, finds there are eight handshakes, and writes "1J". He begins a second column by writing "2A, 2" and looks up at his first column of notation.] No, wait. "B" is person "2". [M proceeds to write subscripts after each of the letters in his first column of notation. M then erases the "A" and writes "2A_1".] But didn't they already shake hands? Yes. [M erases "A_1", replacing it with "B_2".] Nooooh. [M erases "B_2", replacing it with "C_3".] I could have just wrote two three, two four, two five (i.e., "2-3", "2-4", "2-5"). Oh well.

[M finishes symbolizing all handshakes with the number "2". M smiles and looks at T.] Same problem as last time. Minus one, and keep going, and going, and going.[45]
C: Minus two actually.
M: Uhnun (no). [M continues until he has finishes writing notation for all the handshakes (Figure 5.31).]

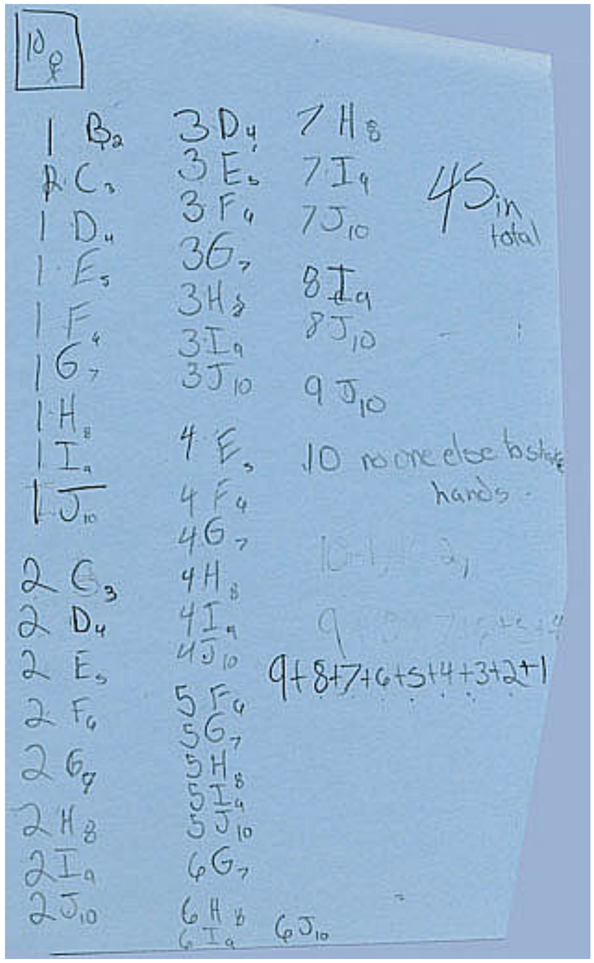

Figure 5.31: Michael's Notation for the Handshake Problem

Based on Michael's notation "1" and "A", I infer that Michael assimilated the situation using two representative units, each of which symbolized a composite unit of ordered units. The first representative unit, notated with "1", symbolized the 10 handshakers. The second representative unit, notated with "A", symbolized the 10 handshakees. This differentiation

[45] This is a reference to the Flag Problem.

suggests that Michael, prior to engaging in activity, anticipated he would need two composite units of ordered units. I infer he anticipated this because he assimilated a handshake to his concept of a pair, which contained records of two representative units. This way of operating suggests that Michael incorporated the results of his operating in the Flag Problem, his pair concept, into the assimilatory part of his scheme.

It, however, suggests that his pair concept contained something that I did not highlight in the Flag Problem. Namely, it appeared that Michael expected the two slots in his pair concept to be filled with two qualitatively differentiated units (the handshakers and the handshakees). In the Flag Problem, Michael assimilated the situation with only one composite unit and so this differentiation appeared to be made through an operational distinction—the second composite unit was created as a result of disembeding it from the first unit. I infer that the Flag Problem was useful in engendering this type of activity because there was only one list of colors and I infer constructing a concept of flag with two stripes (either ordered or unordered) is a more abstract concept than a handshake. By more abstract, I mean that I infer Michael's handshake concept had been abstracted from his actual experience of shaking hands where he easily identified a difference between himself (a handshaker) and the person's hand he shook (a handshakee). So, in this situation, Michael did not seem to create the second composite unit from disembedding it from the first, but rather through distinguishing between the two based on a qualitative differentiation.

His notation, "1A", I infer symbolized a units coordination that Michael made between the first unit in the first composite unit and the first unit in the second composite unit. He began to continue his activity by writing "2" below where he had written "1A" but halted his activity asking, "'A' can't shake his own hand, can he?" This question along with his gesture of shaking his own hand suggests that Michael engaged in an act of self-regulation that led to his monitoring the activity of producing handshakes. Here, his notation seemed to help him monitor his activity. I make this assertion because I infer that when he wrote “2” on his piece of paper without writing a letter or number after it that he was trying to figure out how to symbolize the next handshake. In order to figure this out, I infer he reviewed the meaning of the number “1” and the letter “A” that he had produced to symbolize the first handshake. In reviewing this meaning, he established an identity between these two symbols because they each symbolized the same person so the notation “1A” indicated a person shaking his own hand. He, then, rejected this type of handshake confirming with me that it was not a legitimate handshake to count.

He erased the letter “A” and replaced it with “B”, which symbolized a handshake between the first handshaker and the second handshakee (i.e., a units coordination between the first unit in the first composite unit and the second unit in the second composite unit). At this point, he erased the “2” and appeared to sequentially engage in units coordinating operations to produce the first nine handshakes, which he notated by writing, “1C, 1D,…,1J”.

Michael began the second column of notation with “2A, 2” and then examined his first column of notation. I infer that his notation once again helped him engage in an act of self-regulation that led to his monitoring the handshakes that he was producing. Here, I infer that Michael halted his activity because he was aware that B was the second letter of the alphabet and he maintained an awareness of eliminating self-handshakes. When he examined the first column of notation, I infer he re-presented the activity of creating an identity between person A and person 1. After re-presenting this activity, he seemed to establish an identity between person B and person 2 [“No, wait. B is person 2.”]. He, then, used notation to once again engage in an act of monitoring. The act of monitoring was to produce new notation, which identified the people that were to be considered the same in each of the two composite units. He did this by writing subscripts after each letter in the first column. He, then, wrote “$2A_1$” and after doing so said, “But didn’t they already shake hands? Yes.”, which suggests Michael was aware that “$1B_2$” and “$2A_1$” symbolized the same handshake.

When Michael made this identity and said, “I could have just wrote two three, two four, two five (i.e., “2-3”, “2-4”, “2-5”), I infer that Michael abandoned using two qualitatively differentiated composite units (handshakers and handshakees). Instead he appeared to create one composite unit, the 10 people, which he could use to produce both the handshakers and the handshakees. I infer he did so by disembeding a particular unit, which symbolized the second person, and a representative unit, which symbolized the remaining handshakees and made a units coordination between the two currently with saying, “two three, two four, two five”. This units coordination symbolized all the handshakes that the second person could make (although he notated this units coordination by saying three specific handshakes). This way of operating enabled Michael to eliminate self-handshakes and handshakes that he had already counted. So, a handshaker shaking a handshakees hand was the same as when these two people exchanged roles.

Note that this interpretation is different from my interpretation of his elimination of flags in the Flag Problem. In that problem, he did not consider the difference between the flags symbolized as “1-2” and flags symbolized by “2-1”; they were simply the same flag because he had assimilated the situation with only one composite unit (and produced two as a result of his operating). Because he assimilated this situation with two qualitatively differentiated composite units and then transitioned to using only one composite unit as material to produce the second, he considered both types of handshakes and rejected one of the two because the results, a handshake, were the same regardless of who was the handshaker and who was the handshakee. This way of operating opens up the possibility of considering the results of one's units coordinating scheme as an ordered pair. Here, Michael simply considered them to be the same and so eliminated the extra handshake, but the two handshakes were differentiated because he produced each one separately. I had no further occasion on which to observe Michael so I cannot comment on the permanence of this way of operating though Michael notating as he did suggests that this way of operating would be a permanent modification and hence a functional accommodation.

Once Michael got to the end of his second column of notation he said, "Same problem as last time. Minus one, and keep going, and going, and going." This statement seemed to refer to his awareness that he was going to produce the sum where each column of notation would contain one less than the previous column. So, I infer that when Michael finished his second column of notation he re-presented his activity and produced a units coordination between two representative units that he had created from one composite unit, the people. The first representative unit symbolized the number of handshakees for each person, which was one less each time. The second was the number of handshakers, which indicated to him the number of columns he would produce.

Michael Abstracts a Conceptual Network of Related Tasks

One of the interesting features of Michael's way of operating was the self-regulation that he engaged in as he operated. The acts of self-regulation meant that Michael halted the activity of his scheme and used his notation as a tool to monitor that activity. So, his notation was a central way in which he monitored the activity of his scheme but his notation does not account for the acts of self-regulation. This type of self-regulation suggests that Michael had to be able to be an observer of his own operating while he was in the midst of operating. This type of self-regulation seemed to arise because Michael operated at differing levels of interiorization. So, sometimes he would operate at a level of interiorization where he experienced a perturbation. To resolve this perturbation, Michael would return to a lower level of interiorization where he could monitor his activity and resolve the perturbation. So, Michael often experienced perturbations as he operated, suggesting he had certain expectations of how he would experience the activity of his scheme not just an expectation of what the results of his scheme would be.

This way of operating was in contrast to Carlos (at least in the Flag and Handshake Problem) whose acts of self-regulation appeared to be occasioned by Michael or me, rather than by something that was internal to the scheme itself. So, Carlos's acts of monitoring occurred from a source that was external to the scheme. This suggests that each boy could monitor the activity of his scheme, but there was a sense in which Michael was above the activity of his scheme. Being above the activity enabled him to engage in acts of self-regulation as he acted, which led to his monitoring. On the other hand, Carlos was more in the action of the scheme, which did not enable him to necessarily engage in acts of self-regulation even though he was able to interpret my suggestions or Michael's comments, which in the Flag Problem led to him reviewing the activity of his scheme and in the Handshake Problem led to his monitoring his activity while he acted.

Michael's ability to engage in acts of self-regulation that led to his monitoring the activity of his scheme ,while he was in the midst of acting, meant that Michael was able to characterize his activity as similar or different from other activity in which he engaged. Further, the functional accommodations that I have suggested that Michael made during his activity all deal with issues of anticipation and awareness of how he was operating. Indication that Michael was beginning to make comparison among the situation was given by Michael's comment that the Handshake Problem was the "same problem as last time" where last time referred to the Flag Problem. The ability to classify situations as of a certain type suggests an awareness of the structure and functioning of one's scheme and so suggests that Michael was beginning to abstract these types of tasks into a network of related situations.

Michael Evaluates Sums of Whole Numbers

Overview

This section of the case study is broken into three parts. The first part examines more of Michael's activity in the Flag Problem where he came up with a strategy for evaluating the sum. In this analysis, I will suggest that some of the elements of treating the sum as a quantitative entity were part of his solution and I will suggest a reason that he did not actually produce a multiplicative solution for evaluating the sum. In the second part of this section, I will examine Michael's activity when evaluating the sum of the first 25 and 99 whole numbers and several functional accommodations that Michael made to this activity. I posed these problems to Michael towards the end of the fourth teaching episode. In this part, I will argue that Michael abstracted a concept from his activity of solving these problems. In the final part of this section, I will analyze the role that social interaction had in the symbolizing process.

Revisiting Michael's Eliminating Activity in the Flag Problem

Once Michael had finished his activity that produced the flags in the Flag Problem, he notated the results as an addition problem (See Figure 5.30).[46] Recall that Michael had notated the number of addends along with the amount of each addend and had found the pattern that each

[46] This figure is in the section where I analyze Michael's activity in the Flag Problem.

addend was one less than the previous addend. Michael, then, began to evaluate the sum, introducing the following strategy.

Protocol 5.21: Michael evaluates the sum for the Flag Problem on 11/02/05

M: [M has just written the sum and he begins to figure it out.] It is twenty-nine [M writes "29"]. We need a calculator in here.
T: What are you doing?
M: Just subtracting by one.
T: Okay and your adding them all up. [M nods.] I wonder if you could figure out a quick way to add those guys up.
M: Fifteen plus fifteen minus one.
T: Mmhmm, yeah, okay.
C: Fifteen times fifteen.
M: [M ignores C's comment and continues with his own strategy.]That's twenty-nine and that is forty-four [M writes "44" below where he has written "29".] What? Subtract, subtract. [M looks at T and then erases where he has written "44". He writes below where he has written "29", "13" and adds 29 and 13 using his algorithm for addition, recording "42". M pauses and looks at his notation for 10 seconds.] So, it's like, never mind. So, add twelve and that's forty-four [M writes "44" below where he has written "42".]. No wait that is not forty-four. [M writes "54" over where he has written "44". M continues to evaluate the sum.]

When I intervened to ask Michael if he could find a quick way to find the sum, I infer he articulated the strategy he had already used to find 29 ["Fifteen plus fifteen minus one"]. He, then, proceeded to add fifteen to 29 concluding "that is forty-four". However, he appeared to experience a perturbation about what he should subtract from the resulting amount. I infer that this perturbation was related to coordinating his observation that, "It just decreases by one every time" (See Protocol 5.19), coupled with his choice to add 15 to 29. His observation that it just decreased by one each time was a recursive way of producing each successive addend where the amount of flags in a particular column could be found by reducing the amount of flags in the previous column by one. I infer that this way of operating was related to his awareness of reducing the number of colors that he could use to make flags by one each time he began a new column. This way of operating, however, did not seem to take into account the cumulative eliminations that he had made.

On the other hand, his choice to add 15 was related to the total number of colors with which he had begun. I infer what he was not considering was the total number of eliminations he had produced when he reached, for instance, the third column where he had reduced the number of colors by 2 in relation to 15. So, he was not sure how much to subtract because he had added 15 but 15 was not 1 more than 13. Michael appeared to try and figure this out by erasing where he had notated 44 and simply adding 13 to 29 to find out what the result would be. He, then, halted his activity where I infer he was trying to coordinate his two ways of operating. To make such a coordination, I infer would have involved relating 13 to 15 and not 13 to 14 which would have indicated to him that the amount he had produced this time was 2 less than the amount he had previously produced. He appeared unable to resolve this perturbation, himself, or at least judged that this way of operating was not as simple as just finding the sum using the standard addition algorithm so he began to add the numbers in sequence abandoning his strategy ["So, wait, never mind"]. I intervened to see if he might figure out a way to use his strategy.

Protocol 5.22: Michael's activity after my intervention on 11/02/05

T: Can I ask you a question? This one was fifteen times two minus one [T points to 29]. Is that what you were thinking? [M nods his head to indicate agreement.] So, then this one you thought would be fifteen times three minus…
M: No, I thought it was…oh, it would have been. You add fifteen to that and subtract two. You add fifteen again and subtract three. That is just confusing though. I'll just keep doing like this. It is easier [M is referring to adding the numbers in sequence. M finishes adding them up.] One hundred and ten possibilities.
C: [C finishes adding up his amount.] One hundred and thirty.
T: [T tells the boys that they have to find a way to add up those numbers so they can check. He suggests a strategy of adding 15 and 1, 14 and 2, etc. to the boys. M & C both begin to check their answers. M assimilates T's suggestion to his way of operating where he adds fifteen and subtracts an appropriate amount (See Protocol 5.7 for a full transcript of this interaction).]

M: [M begins to evaluate the sum adding 15 and subtracting 1, and then adding 15 and subtracting 2, etc. producing Figure 5.32. About 8 minutes pass during which M uses his strategy to find the result is 120, which agrees with the amount Carlos got the second time. He provides the following explanation of his strategy.] …. And then every time you do it like the first step, second step, so it is just basically going fifteen minus that [M points to a place where he has written "1" on his piece of paper], fifteen minus that [M points to where he has written "2" on his piece of paper], fifteen minus that [M points to where he has written "3" on his paper] going on and on and on.

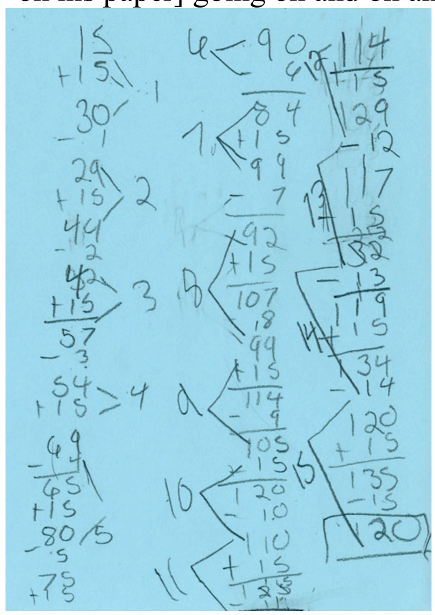

Figure 5.32: Michael's notation

Michael agreed with my statement that the first step of his strategy was to multiply 15 times 2 and subtract 1. However, when I suggested for the third step that he had multiplied 15 times 3 and subtracted some amount. At first, he disagreed but then seemed to interpret my statements in the context of how he had operated. I infer that he disagreed initially because he was not keeping track of the cumulative number of 15s he had added so that the successive steps were in essence independent of one another. That is, when he added 15 the first time and subtracted 1 and then added 15 a second time he did not see this as a third 15 but rather a new step in his process of adding 15s and then subtracting some amount. So, my suggestion seemed to help him resolve the perturbation of how much to subtract each time he added a new 15. He, then, produced this way of operating notating each step as he went. Here, he took each step to include adding a 15 and then subtracting an appropriate amount from the result in order to get the amount he should have added. Michael did not seem to take this activity as producing fifteen 15s that contained a sum that indicated the total number of flags and a sum that indicated the total number of eliminations. Rather, he seemed to take adding 15 and subtracting the appropriate amount as an individual step, which was independent of each of the other steps. This way of operating meant that he was not considering all of the eliminations as a quantitative entity but rather as a succession of actions that he performed on each successive 15.

However, Michael was aware that the addition problem contained 15 addends along with the numerosity of each addend. His awareness of the total number of addends and the amount of each addend seems to me a minimal condition for taking the sequence of addition problems as a quantitative entity because it opens the possibility of assigning a position to each of the addends as well as the possibility of having a sense that there is a beginning and ending for the addends. To see if Michael could use this awareness to evaluate the sum, I turn to his activity of solving the sum of the first 25 and 99 whole numbers. I posed these problems to Michael simply as problems where the goal was a to find a quick way to evaluate the sum.

Michael's Activity to Find the Sum of the First 25 and 99 Whole Numbers

The Sum of the First 25 Whole Numbers. I challenged Michael to find the sum of the first 25 whole numbers after he had finished solving the Handshake Problem. In his solution of the Handshake Problem, Michael sequentially combined the addends (i.e., 9 + 8 = 17, 17 + 7 = 24,, etc.). He remembered his way of operating in the Flag Problem, but concluded that this way of operating was too difficult to use again. Michael produced the following activity in finding the sum of the first 25 whole numbers.

Protocol 5.23: Michael's activity in finding the sum of the first 25 whole numbers on 11/09/05

M: [M attempts several times to use his way of operating from the Flag Problem without actually engaging in the process sequentially. He produces several forms of notation as he tries to do this. He gives up his attempts and re-writes the sequence of numbers from 1 up to 25. He adds the numbers from 25 down to 10 in sequence, apologizing that he cannot find a quicker way. When he gets to the numbers from 1 to 9, he runs his thumb and index finger over them.] That's ten [M points to where he has written 9 and 1], twenty [M points to where he has written 8 and 2], thirty [M points to where he has written 7 and 3], forty [M points to where he has written 6 and 4], forty-five.
T: I wonder if you could use that (strategy) on the big string (the numbers from 1 through 25)?
M: Yeah, that's…[M nods his head and smiles. He looks at his notation for 3 seconds.] So, it's basically twenty-five times twenty-four.
T: Show me how you got that.
M: [M writes out the sequence of numbers from 1 to 25, again.] No it's twenty-six times twenty-four. No…twenty-six [M draws a line between 25 and 1]…twenty six [M draws a line between 24 and 2]…twenty six [M draws a line between 23 and 3]. So, like half of them. [M continues drawing lines (Figure 5.33) until he gets to 20 and 6. He stops drawing lines and runs his thumb and index finger over this sequence stopping at 7 and 19, 8 and 18, etc. M draws a final line connecting 12 and 14 and counts the numbers from 12 down to 1. He writes "26 x 12" and evaluates it.]

Figure 5.33: Replica of Michael's Notation

Because Michael apologized while he was adding the numbers in sequence, I infer that he had entered a state of perturbation where he was searching for a quicker way to evaluate the sum. When he got to the numbers between 1 and 9, he intuitively began to make units of 10. It is hard to infer what created this way of operating other than Michael's unresolved perturbation and the possibility that he considered 10 to be a unit that was easy to use in additive contexts. So, like Carlos, Michael used his uniting operation to unite 9 and 1, 8 and 2, etc.[47] Once

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[47] This way of operating suggests Michael had made a similar abstraction to Carlos from his previous additive reasoning. Namely, he could unite elements of a sum in any order and the results would produce the same amount.

Michael engaged in these operations for the numbers between 1 and 9, he was able to assimilate my statement about the "big string" of numbers to his concept of the sequence of numbers from 1 to 25 and almost immediately found a multiplication problem that he thought he could use to evaluate the sum. Here, I infer that he re-presented his experience of producing and enumerating the units of 10, which called forth his multiplying schemes, which helped him to establish a conjecture about the new problem.

He made two conjectures about multiplication problems he thought might solve the problem based on his anticipation of how he would operate. I infer that he could not fully anticipate the results that his new way of operating would produce. So, he continued to produce and operate on his notation as a way of monitoring the operations in which he engaged. From an operational perspective, Michael operated in a way that was quite similar to the way Carlos had operated. Michael united the elements 1 and 25, 2 and 24, etc. and subsequently enumerated the number of 26s he had produced. I make these inferences based on him drawing lines on and running his fingers over his notation, and then counting the numbers from 12 down to 1 to enumerate them. So, he sequentially used his strategic additive reasoning scheme[48] and his multiplying scheme[49]. This provided indication that he like Carlos used the notation to externalize the 26s that he produced with his uniting operations. Then, he took these as given in the situation so that he could simply count the notation that he took to symbolize each of the 26s he had produced.

Michael's conjectures about multiplication problems and his subsequent reference to one-half suggested that he had formed the goal of producing and enumerating the 26s

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[48] The activity of this scheme was to unite elements of the sum to produce units of equal numerosity.

[49] The activity of this scheme was to enumerate composite units in order to find a multiplication problem he could evaluate using his algorithm for multiplication.

simultaneously. So, producing the units of equal numerosity called forth his multiplying schemes. However, he did not attain this goal during his initial solution. So, I infer that he had not fully resolved the perturbation he experienced in the situation even though he had found a quick way to solve the problem. When he finished evaluating the multiplication problem "26 x 12", he became confused about which of the addends he had not used to make a 26 (i.e., 13). So, I intervened to ask about his solution.

> Continuation of Protocol 5.23: Michael's activity in finding the sum of the first 25 whole numbers on 11/09/05
>
> T: How many 26s could you get?
> M: Twelve.
> T: Twelve. Show me where they are?
> M: [M counts the numbers from 1 up to 12.] Paired with like switched over with that (the numbers from 14 to 25). [M gestures with his hand as if he is folding a segment in half.]
> T: Yeah so that's a really good image they are switched over like that. Which one doesn't have a pair?
> M: Thirteen.

When I intervened, I infer that Michael re-presented his previous way of operating, reviewing the results of this way of operating (twelve 26s). Upon re-initializing his activity in the situation, he took the twelve 26s as a given in the situation. That is, he no longer needed to either produce or enumerate the number of units in the situation because he already knew what the results of the coordination between his strategic additive reasoning scheme and his multiplying scheme would yield. So, he was freed from having to engage in either activity, which I infer allowed him to search for a way that would produce these results using only one operation.

To do so, he introduced a new mental image for the sum, the sequence of numbers from 1 to 25 as a segment that had a midpoint, and a new operation that he performed on the segment, folding it onto itself. I infer that he generated this new image by re-presenting his action of running his fingers over his notation during his first solution. This action during his first solution gave him the experience of the sequence of numbers as a continuous quantity. Michael's image suggested that he had posited the sum as a quantitative entity, which allowed him to transform the operations on the elements of the sum (i.e., uniting 1 and 25, etc.) and enumerating these results, to an operation, folding the segment, that was performed on the sum itself. I infer Michael embedded the aspects of the sum he had already abstracted, the amount and ordering meanings for each addend, into this new mental image. I infer that he introduced the new operation, folding the segment, because his initial perturbation of finding a way to simultaneously produce and enumerate all of the units had yet to be resolved. The folding operation he introduced appeared to be suggested to him by the lines he drew between elements that were equidistant from the center of the sequence of numbers he had written on his paper.

His operation on the segment provides indication that he was making a part-to-part comparison between the number of numbers in the first part of the segment and the number of numbers in the second part of the segment, where his goal was to have the same number of numbers in each part of the segment, so that each would have a number with which it could be united. The midpoint of the segment then corresponded with the number in his number sequence that Michael referred to as "being alone". So, the folding operation on the segment symbolized both his initial production and enumeration of the 26s. This substitution of one operation for many operations allowed Michael to produce and enumerate all of the 26s simultaneously using only one operation. So, Michael's operation allowed him to treat a number like 12 as both an amount (i.e., a unit containing 12 units) and as a number that referred to the number of units of 26 he would produce.

Michael made two functional accommodations in his operating during this situation. The first was his strategic use of his uniting operation on the sequence of elements and his subsequent enumeration of the number of 26s. This accommodation involved a novel coordination among his strategic additive reasoning scheme and his multiplying scheme. Michael's second functional accommodation reconstructed this coordination in such a way that Michael enacted a new scheme. I consider this the enactment of a new scheme because while it contained the goals of the old schemes it involved the production of a new mathematical structure, the sum as a quantitative entity, and a new operation that symbolized the operations involved in each of the old schemes. This operation was quite different from the operations in either of the old schemes but the activities of each of the old schemes were embedded in this new operation. To find out if these accommodations were permanent, I present Michael's solution of the sum of the first 99 whole numbers.

Protocol 5.24: Michael's solution to the sum of the first 99 whole numbers 11/9/05

M: So, that would be like a hundred, hundred, hundred, times half [M creates a folding over gesture with his arms.] that would be let's see ahh…[M closes his eyes to concentrate.] There is one odd number in the middle. What is it? Ninety-eight so it would be…forty fives in the middle. No it isn't. Yes it is. No it isn't. It is forty-nine. [Emphatically] So, 50 is the odd man out. [M writes "49 50 49" on his paper and sits in concentration for 30 seconds. Then, he multiplies forty-nine by one-hundred and adds fifty.]
T: Tell me what you did. You did that so quickly, you didn't even…
M: Okay, since there is ninety-nine, you find, you like go one more lower, that's ninety-eight. Forty-nine plus forty-nine is ninety-eight. So, there has to be one in the middle fifty. It (fifty) is the one that like doesn't make it even. So, then I timesed forty-nine times one hundred and got four thousand nine hundred and added fifty.

I infer that Michael engaged in three representative uniting operations ["that would be like a hundred, hundred, hundred"] on his image of the elements of the sum and these operations elicited Michael's re-presentation of the sum as a segment and his operation of folding the segment. Here, he verbalized his operation on the segment as "times half", suggesting his concept of one half was called forth by his folding operation, and he began to search for the "one odd number in the middle". I infer he was aware that he could not take ½ of 99 because it was odd, so, instead he began searching for ½ of 98. Here, Michael used his concept of ½ as a multiplicative operation on his concept of 98. Michael once again had made a substitution where this time he substituted his concept of ½ as a multiplicative operation for the folding action he had performed on the segment. He wrote "49 50 49," which I infer symbolized the 49 numbers in the first part of the segment, the middle number, and the 49 numbers in the second part of the segment. He used this notation just prior to sitting in concentration for 30 seconds, so I infer that he used it as a tool to help him confirm that there were indeed 49 numbers in both parts of the segment. He then took 49 to symbolize the number of 100s he could make and so multiplied 49 times 100 and added 50.

In this situation, Michael had operated primarily on interiorized imagery where he used his notation only minimally to monitor his activity. When I asked him to explain how he had operated, it provided him an opportunity to reflect on his solution to the problem. His explanation provides indication that he had significantly curtailed his activity at this point in his operating. That is, his activity contained only reference to finding ½ of 98 with the goal of establishing the total number of 100s and the element of the sum that was in the middle. So, Michael used natural language to symbolize the events that had unfolded over a portion of 20 minutes. I infer that he would have been able to produce these operations were I to ask him more about his way of operating. Therefore, he could use natural language to point to the operations in which he engaged without needing to implement these operations in full.

Michael's Constructive Activity. Based on Michael's way of operating, I infer Michael first made a novel coordination between two schemes, and then, out of this novel coordination he constructed a new scheme. By the end of his activity, I infer he had transformed his scheme into a new *concept*. Initially, the activity of Michael's new scheme involved a novel coordination between his strategic additive reasoning scheme and his multiplying scheme. In engaging in this activity, Michael externalized each of the 26s that he produced with his uniting operation and could take his notation as symbolizing these 26s without having to re-produce them. So, he could operate as if these units existed and enumerate them by counting the numbers from 1 to 12. This way of operating suggested that there was a sense of reversibility to the operations that produced each of the composite units because he could take his notation both as symbolizing the results of his uniting operations (i.e., a unit of 26) and at the same time take it as symbolizing the elements he had used to produce the units (i.e., a unit of 1 and 25).

The result of this coordination between his schemes was Michael's awareness of twelve 26s. Michael externalized these results so he operated as if *all* of the 26s existed, allowing him to operate as if the units were given in the situation. By externalizing all of the 26s, he could search for an operation that would produce all of these units at once. This act of externalization was the precursor to Michael's construction of a new scheme. I make the assertion that he constructed a new scheme because his new way of operating contained the same goals as the coordination between the two old schemes but Michael's act of externalization had produced new givens and new operations that were not contained in either of the old schemes.

His awareness of the results of twelve 26s opened a search for an activity that would produce all of these units simultaneously. This led him to introduce a new image and operation in order to engage in the activity of his scheme. The new image altered how he assimilated these situations—he assimilated the situations using the symbol of a segment for his number sequence. So, this provides corroboration that he had constructed the sum as a quantitative entity. The new operation changed the activity of his scheme, which consisted of simply folding the segment. This folding contained both the goal of producing and enumerating the units of 26 and allowed him to do so simultaneously. The results were transformed in the sense that they were linked via one operation performed on the sum.

Up to this point in the episode, Michael used his notation primarily as a tool to externalize and monitor his way of operating in the situation. However, his final linguistic explanation appeared to be a tool in his construction of a new concept—his concept of "evaluating the sum of the first so many whole numbers". I make this inference because his language appeared to point to the operations and imagery that he had used. So, he could use his language to point to the operations and imagery he produced, but he did not actually have to implement in full the way of operating that he had produced. Rather, at this point, he appeared to simply view his explanation of taking one half of 98 to find 49 and then multiplying 49 by 100 and adding 50 as sufficient explanation for his way of operating.

One final point I want to make about Michael's symbolizing activity pertains to the image he produced. The image of the sequence of numbers as a segment allowed him to imagine inserting his number sequence from 1 up to a number of no particular specification into this segment. So, the symbol he used for his number sequence had a general quality to it, in that it was not tied to a particular instance of the sum. I infer that this type of symbol is the quantitative basis for using a more conventional algebraic symbol like the letter "*n*" to symbolize the relationship between the sum of any but no particular consecutive whole numbers that begins with 1 and a multiplication problem for evaluating this sum.

Michael's Reflective and Reflected Abstraction. Michael produced what might be considered two reflective abstractions, corresponding to his two functional accommodations. First, the initial segment of Michael's number sequence and his uniting operations were combined in a novel way to produce a new sequential way of operating in the context of the sum.

The new way of operating I infer re-organized Michael's notion of finding the sum so that it could be associated with a multiplication problem. I am making the judgment that this new way of operating was at a higher level because Michael was able to formulate a new connection between the first 25 whole numbers and his multiplying schemes.

The second reflective abstraction that Michael made pertained to his becoming an observer of his own way of operating. Here, his process of notating to externalize his first way of operating enabled him to re-interiorize his first way of operating, transforming this way of operating with a new mental image and operation to achieve the goal of simultaneously producing and enumerating all of the units in a given sum. This re-interiorization took the operations from his initial way of operating as given in the sense that Michael no longer needed to produce them in order to productively operate to find a multiplication problem that allowed him to evaluate the sum. Taking these operations as given meant that he considered them to be contained in his new way of operating, and so once again I could infer that Michael had reached a higher level of operating. At this level of operating, Michael needed only one operation to produce what he had initially produced with numerous instantiations of his uniting operation and enumeration of the results of his uniting operation. The input for this operation also changed because he now treated the sum as a quantitative entity. So, in re-structuring the situation, he operated on a newly constructed mathematical structure, a second indication that he was operating at a higher level than his previous way of operating.

The reflected abstraction—retroactive thematization—occurred in the context of him providing his final linguistic solution for the problem. Here, he appeared to simply use natural language to notate his solution of the problem. So, what he had abstracted from his activity in order to find a multiplication problem for the sum appeared to be three actions: finding the unit he was going to produce (e.g., 100), finding the number of these units by taking one half of the total numbers in the initial segment of his number sequence, and finding the unit that was the "odd man out" (e.g., 50). With these three pieces of information, Michael appeared to be aware that he could find a multiplication problem that would help him to evaluate the sum. So, I infer that he had abstracted a general way of operating that allowed him to transform the sum into a multiplication problem and that he was aware of this way of operating.

Social Interaction and the Symbolizing Process

In this final part of the third section, I want to examine the role that social interaction played in Michael's solution to the problem. For Michael, my initial suggestion in the Flag Problem (i.e., adding 1 and 15, 2 and 14, etc.) did not alter his way of operating. Therefore, I infer he did not assimilate my suggestion to his own activity in the situation. So, he experienced my suggestion as an interruption rather than as a useful suggestion in the situation. In order to use my suggestion, Michael would have had to stop his own way of operating, and assimilate my suggestion to operations that he was not using in the situation. Based on his subsequent activity, I cannot claim that he could not have modified his activity at that moment had I insisted that he use my suggestion. However, neither he nor I had co-constructed teaching episodes as events where I insisted that students use my suggestions even though I did frequently intervene as a way to test or extend their ways of operating. I infer that part of Michael's enjoyment about coming to the teaching episodes was that he had the opportunity to solve mathematics problems, which sometimes meant that he disregarded my interventions because this took away "the fun".

In the context of evaluating of the first 25 whole numbers, Michael did use my suggestion to use his way of operating on the "bigger string" once he had produced this way of operating on the numbers from 1 to 9. I infer he used my suggestion because he could assimilate it to a way of operating that he had just produced. So, he considered my suggestion as highlighting a potentially useful way of operating that he attributed to himself. His experience of this intervention was much different than his experience of the first intervention because he had produced the way of operating and I had simply highlighted that it might be useful.

Subsequently, my intervention, when Michael became confused about the middle number in the sum of the first 25 whole numbers, provided the occasion for Michael to reflect on his previous way of operating and produce a new way of operating. Michael's new way of operating included a new mental image and operation that I cannot attribute to some quality of the interaction that we had but rather to Michael's productive activity in the situation. So, while my intervention provided the occasion for reflection, I cannot attribute a causal relationship between my intervention and Michael's subsequent activity. So, my interaction with Michael in this problem situation opened the possibility for scheme construction and reflection on this construction, but I cannot use it as a causal explanatory construct for the operations Michael produced and the reflection he engaged in.

CHAPTER 6: DEBORAH

Introduction

In this case study, I examine the first five teaching episodes of the teaching experiment I conducted with Deborah. When I began the teaching episodes, my goal was to explore if, and how Deborah would coordinate her additive and multiplicative reasoning to produce a formula for the sum of the first *n* whole numbers. The first problem I posed to Deborah was the One Deck Card Problem.

> Task 6.1, *The One Deck Card Problem*: How many two-card hands could you make with a deck of fifty-two-cards?

Both Deborah and her partner Michael seemed stumped by this problem.[50] Deborah said she understood the problem, but was not sure how to proceed to solve the problem. So, I posed the Outfits Problem and its extension.

> Task 6.2, *The Outfits Problem*: Suppose you have three shirts and four pairs of pants. How many outfits could you make?

> Task 6.3, *Extension of the Outfits Problem*: Suppose now that you have two pairs of shoes that you could put with each outfit. How many possible outfits could you make?

Deborah used her multiplicative reasoning to solve each of these problems rapidly and then returned to the One Deck Card Problem, while Michael continued to work on his solution of the Outfits Problem. Deborah figured out that she could solve the One Deck Card Problem by adding the first 51 whole numbers. So, Deborah's solution of the Outfits Problem seemed to help her

[50] Remember that Deborah and Michael worked together for the first teaching episode. After that episode, Deborah worked by herself, and Michael and Carlos worked together.

conceptualize a solution to the One Deck Card Problem. However, Deborah did not produce this sum using her multiplicative reasoning.

I continued to investigate Deborah's multiplicative reasoning in problems like the Outfits Problem and its extension during the first, fourth, and fifth teaching episodes. I specifically focused on problem situations that involved three composite units (like the Extension of the Outfits Problem, which includes shirts, pants, and shoes) during the fourth and fifth teaching episodes. I devoted the second and third teaching episodes to problems like the One Deck Card Problem that could lead to sums of whole numbers, with the goal of eliciting Deborah's multiplicative reasoning, rather than just simply her additive reasoning. Figure 6.1 is an overview of the tasks that Deborah worked on that I analyze.

	Deborah
Teaching Episode 1, 10/17/05	The One Deck Card Problem, Task 6.1; The Outfits Problem and its Extension, Tasks 6.2 & 6.3; The Two Suits Card Problem, Task 6.4
Teaching Episode 2, 10/19/05	The Two Deck Card Problem, Task 6.5; The Handshake Problem, Task 6.6
Teaching Episode 3, 10/24/05	The Committee Problem, Task 6.7; Work on producing the formula for the sum of the first *n* whole numbers.
Teaching Episode 4, 11/02/05	The School Supplies Problem, Task 6.8; The Picnic Problem, Task 6.9
Teaching Episode 5, 11/07/05	The Candy Problem, Task 6.10

Figure 6.1: Overview of the five teaching episodes

I present this overview as a way to understand the structure of this case study, which I have organized into three sections: in the first, I analyze Deborah's multiplicative reasoning in problems like the Outfits Problem; in the second, I analyze Deborah's activity towards producing a formula for the sum of the first *n* whole numbers; and in the third, I analyze Deborah's solutions to problems where she began to transform her lexicographic units coordinating scheme into a recursive scheme of operations. I will use the first section as a way to establish how Deborah's multiplicative reasoning served as a foundation for her reasoning in the second and third sections. Throughout this discussion, I build my working model of the role Deborah's symbolizing activity played in her solution of these problem situations.

Deborah's Activity to Solve the Outfits Problem and Similar Problems

Overview

In this section of the case study, I analyze Deborah's activity during three of the early episodes in the teaching experiment. I begin by analyzing her activity in the context of solving the Outfits Problem and a Card Problem that I presented to her on the first day of the teaching experiment. These problems I use to illustrate the operations Deborah produced when solving problems involving multiplicative combinations. After analyzing these two problems, I examine Deborah's activity in the context of working on three problem situations that involved recursive multiplicative combinations, the School Supplies, the Picnic, and the Candy Problem. I posed these problems during the fourth and fifth episodes of the teaching experiment. For these three problems, I analyze how the operations Deborah produced in solving the first two problems supported her way of operating in these novel contexts.

Deborah's Solution of Three Problems

Deborah's Notation and Diagrams for The Outfits Problem. On the first day of the teaching experiment, I presented Deborah with the Outfits Problem, Task 6.2.

Protocol 6.1: Deborah's solution of the Outfits Problem on 10/17/05

D: Twelve.
T: Can you show how you are thinking about that?
D: [D takes a piece of paper.] Four pairs of pants?
T: Yeah, four pairs of pants and three shirts.

D: [D draws four pairs of pants and three shirts (Figure 6.2).] A, B, C, D, one, two, three [D says the letters and numbers as she labels each shirt and pair of pants in her diagram. D writes “A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3” (Figure 6.3). D counts by ones the notation she has produced.] Twelve.
T: Okay, and what did you do?
D: I just multiplied four by three and that’s twelve. But to show it, I just, I just gave them each a letter or a number and just like A one thru three [D draws a line to show grouping “A1, A2, A3”], B one thru three [D draws a line to show grouping “B1, B2, B3”], C one thru three [D draws a line to show grouping “C1, C2, C3”], and D one thru three [D draws a line to show grouping “D1, D2, D3”].

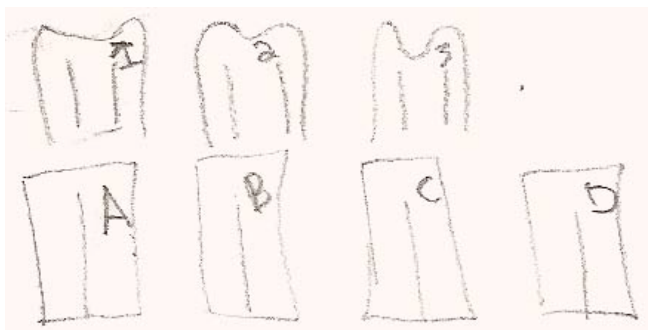

Figure 6.2: Deborah diagram for the Outfits Problem

Figure 6.3: Deborah’s notation for the Outfits Problem

Deborah seemed to take this situation simply as one that involved multiplication. In fact in her verbal explanation, she explicitly differentiated between how she got the answer and how she showed that she got the answer ["I just multiplied four by three and that's twelve. But to show it,…]. Here, I will analyze Deborah's activity "to show" her solution to me. However, when I analyze her lexicographic units coordinating scheme and some of the underlying concepts, I will come back to an analysis of why she assimilated this and other situations directly to her concept of multiplication.

I infer from Figure 6.2 that Deborah operated in a way, which differed from how Carlos and Michael operated in solving the Outfits Problem. Recall that Michael and Carlos each produced a picture of a shirt and pair of pants (like Figure 5.2) and then began to engage in units coordinating activity. When each of them had engaged in some units coordinating activity (i.e., putting a shirt with a pair of pants), they halted this activity and ordered the shirts and pairs of pants so that they could keep track of which outfits they had produced. So, both Carlos and Michael ordered the shirts and pants only after they had begun to produce outfits.

On the other hand, Deborah ordered the shirts, "1", "2", "3", and ordered the pairs of pants, "A", "B", "C", "D", prior to producing any outfits with her units coordinating activity. So, Deborah anticipated that to show her solution to me she would need to order each of the shirts and each of the pairs of pants. Like Carlos and Michael, she, then, sequentially imagined putting each shirt with each pair of pants. As with Carlos and Michael, I call the mental action of putting a shirt with a pair of pants her units coordinating operation. I infer she applied her unitizing operation to the results of her units coordinating activity based on her establishing a separate location to notate the outfits (Figure 6.3). This meant that she was producing *pairs*, a unit that contained two units (i.e., a shirt and a pair of pants). When she engaged in this way of operating, she produced the pairs according to a lexicographic ordering (i.e., the first outfit she produced was with the first pairs of pants and first shirt, the second outfit she produced was with the first pair of pants and the second shirt, etc.).

During her subsequent explanation [“like A one thru three, B one thru three, C one thru three, D one thru three.”], she produced activity that was multiplicative. While making this explanation, she appeared to imagine putting each pair of pants with a representative shirt (Figure 6.4a) to produce a representative outfit (Figure 6.4b), which symbolized three outfits. So, she iterated her units coordinating activity four times to produce four representative outfits where each representative outfit symbolized three outfits. Deborah’s way of operating suggested that she treated the shirts as if each functioned identically even though she had differentiated them (i.e., a first shirt, a second shirt, etc.). So, she no longer sequentially put each shirt with each pair of pants, suggesting that she had begun to curtail her activity in the situation. Note that the representative outfits that she produced were produced using a *particular* pair of pants and a *representative* shirt.

Figure 6.4a (left) & 6.4b (right): Diagram of Deborah’s mental imagery and operations

The Extension of the Outfits Problem. Deborah seemed quite facile with the Outfits Problem and with providing an explanation for it. So, I posed her the extension of the Outfits Problem, Task 6.3.

Protocol 6.2: Deborah’s solution of the Extension to the Outfits Problem on 10/17/05

T: Are you ready to think about another problem related to this one? [D nods.] So you have your twelve outfits right?

D: Yeah.
T: Now say you had two pairs of shoes you could put with each of the outfits that you have, how many outfits would you get?
D: Twenty-four. [M asks a question. T responds and then returns to talking with D.]
T: Could you make a code to show that somehow?
D: Like give one pair of shoes (the label) "X" and one "Y" and then you put, like you could have an "A1X" and you could have an "A1Y" for each one so then you would just multiply it by two. [After making this statement, D uses her old notation to make a code (Figure 6.5).]

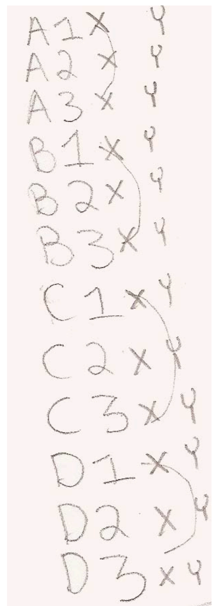

Figure 6.5: Deborah adds to her old notation

Based on Deborah's immediate response, "twenty-four", I infer that Deborah once again assimilated the situation to her concept of multiplication. When I asked her to produce a code, she appeared to take a representative *old* outfit, "A1", that she took as symbolizing all twelve outfits and imagined putting it with each pair of shoes ["…you could have an 'A1X' and you could have an 'A1Y'"]. Deborah took these two implementations of her units coordinating operation, which produced two *new* outfits, as indication that she could produce two new outfits with each of the old outfits ["…you could have an "A1X" and you could have an "A1Y" for each one so then you would just multiply it by two"]. Based on her operating in this way, I infer that she could treat each of the outfits as if they functioned identically. This observation is somewhat similar to her treating each of the shirts identically in the first situation. However, in this situation, the outfits referred to the number of times she would engage in the activity of putting the two shoes with a particular outfit. So, rather than iterate her units coordinating activity, she took twelve as the number of times she would produce two new outfits where each new outfit would contain a pair of pants, a shirt, and a pair of shoes.

I want to highlight two aspects of Deborah's way of operating. First, Deborah produced a unit that I have called a triple, a unit that contains three units. In this case, the new outfits contained a pair of pants, a shirt, and a pair of shoes. I infer Deborah was aware of all three of these units because she was able to create notation that symbolized each of them. It should be noted that Deborah took producing two triples as sufficient for producing all of the triples. She was able to operate in this way because she considered all of the pairs (i.e., the outfits) to function identically—which is the second aspect of Deborah's way of operating that I want to highlight. This way of operating provides indication that she could use a representative pair (i.e., an outfit that symbolized 12 outfits) to engage in further units coordinating activity. I would not call this way of operating recursive because I posed the problem to Deborah in two parts. Nonetheless, it did suggest the possibility that she might independently structure a situation recursively because it suggested that she could use a representative pair in making a continuation of her activity.

Deborah's Solution of The Two Suits Card Problem. In the Outfits Problem, Deborah treated the shirts identically, where she put a representative shirt with a particular pair of pants, which symbolized the *number of outfits*, three, she could produce with a particular pair of pants. In the extension of the Outfits Problem, Deborah treated the outfits identically, where she put each pair of shoes with a representative outfit, where the representative outfit symbolized the *number of times* she could produce two outfits. To investigate if Deborah could coordinate these two ways of operating in a problem situation that involved large numbers, I present Deborah's activity in the Two Suits Card Problem, Task 6.4.

> Task 6.4, *The Two Suits Card Problem*: Suppose you have all the hearts and Michael has all the clubs. How many different two-card combinations could you make?
>
> Protocol 6.3: Deborah's solution to the Two Suits Card Problem on 10/17/05
>
> D: [M sorts out the hearts and clubs from the deck of cards. W has M and D make a few experiential pairs and poses the problem. D writes "13 x 13" on her piece of paper and calculates.]
> T: Go ahead what did you do?
> D: I said thirteen times thirteen because there is a card [D pulls out a card from her deck] and there is that many different…[sweeping her finger across space as if imagining a row of possible cards that could go with the one card she has pulled from her deck]. He could pull out thirteen different cards to go with that.
> T: Okay.
> D: And then he could pull out thirteen to go with that one [D turns over a second card from her deck] and so forth and so forth.

Here, Deborah appeared to imagine taking the first card from her deck and putting it with a representative card from Michael's deck to produce a representative two-card combination that symbolized 13 of these two-card combinations ["He could pull out thirteen different cards to go with that."]. She, then, appeared to imagine putting the second card from her deck and putting it with a representative card from Michael's deck to produce a second representative two-card combination that symbolized 13 two-card combinations ["And then he could pull out thirteen to go with that one"]. In doing so, she treated Michael's cards as if they functioned identically because she could take a units coordination with a representative card as symbolizing all of the possible two-card combinations she could make.

After this second use of her units coordinating operation, she halted her activity and took this as indication of all of the possible pairs she could produce ["and so forth and so forth"]. Halting her activity provides indication that she operated as if all of the cards in her deck functioned identically. Because Deborah could operate as if the cards in Michael's deck were identical and the cards in her deck were identical, this opened the possibility for her to take a single implementation of her units coordinating operation as symbolizing all of the possible pairs she could produce. This way of operating was a coordination of the two ways of operating that she had produced and was a further curtailment of her activity.

Deborah's Lexicographic Units Coordinating Scheme and Related Concepts

Based on Deborah's way of operating during the initial episode, I infer she came into the teaching experiment having already constructed a lexicographical units coordinating scheme. Further, in all of the situations, it appeared that she could take her multiplicative calculation and notation to symbolize the activity of her scheme without actually having to produce this activity. I will analyze her scheme and the functional accommodations she made to this scheme. Then, I will analyze Deborah's associated whole number variable concept, ordering concept, and pair concept. I will conclude with a discussion of how she used her notation.

Deborah's Scheme. In the Outfits Problem, Deborah assimilated the situation with two composite units (one for the pairs of pants and one for the shirts). Then, prior to engaging in units coordinating activity, Deborah organized the situation by differentiating the units within each composite unit (e.g., pants "A, B, C, D" and shirt "1, 2, 3"). This way of organizing the situation suggests that Deborah's scheme was anticipatory in that she anticipated she would have to use all possible shirts and all possible pants to make all possible outfits. She, then, engaged in units coordinating activity where she made functional accommodations in this units coordinating activity across the three problems I have presented.

She solved the Outfits Problem by sequentially putting each of the shirts with each of the pairs of pants. In her subsequent explanation, she appeared to make a functional accommodation so that she put a representative shirt with each particular pair of pants and this activity symbolized producing three outfits. This activity was indication of why Deborah took these situations as multiplicative—she iterated her units coordinating activity. To make such an accommodation, Deborah had to treat the shirts as if each functioned identically even though she had differentiated between them. In the Extension of the Outfits Problem, Deborah made another functional accommodation. She imagined putting each shoe with a representative outfit, using the representative outfit to symbolize the number of times she would have to engage in this activity to produce all of the *new* outfits. Here, she had to treat all of the outfits as if they functioned identically even though she had differentiated between them.

Deborah combined the two ways of operating she had produced in the Two Suit Card Problem, which was a third functional accommodation. She imagined putting her first card with a representative card of Michael's and taking this as symbolizing the first thirteen two-card combinations and repeated this mental action one more time to symbolize the second thirteen two-card combinations she could make. So, she treated Michael's cards as if they functioned identically. She then halted her activity ["…and so forth and so forth"], providing indication that she treated her cards identically where she expected to engage in the mental action that produced thirteen pairs thirteen times. So, it appeared she could produce one units coordination to symbolize producing all the two-card combinations (although she did produce two when she provided her explanation of the Two Suits Card Problem).

Deborah's Whole Number Variable, Ordering, and Pair Concepts. In these situations, Deborah's activity provides indication that she had abstracted a whole number variable concept. I make this assertion based on her activity in the Outfits Problem. In this problem, she imagined taking a particular pair of pants and putting it with a representative shirt. By doing so, she took the representative shirt as symbolizing any but no particular of the shirts. When a student can operate in this way, I take this way of operating as the minimal requirement for having constructed a whole number variable concept. It is important to note what contained in this way of operating was Deborah's ordering concept. So, Deborah's number concepts contained both an ordering and a amount meaning. This enabled her to operate with a representative unit that symbolized any but no particular of the units within a composite unit.

Deborah's activity suggested that she had abstracted more than just these two concepts. In fact, it suggested that she had abstracted a concept that contained each of the two previous concepts. I call this concept her *pair* concept. Her pair concept consisted of two slots. I infer she had abstracted a pair concept that contained two slots from putting each particular unit (e.g., each particular pair of pants) from a first composite unit with each particular unit (e.g., each particular shirt) from a second composite unit. So, the first slot in her pair concept had records of having inserted, for example, each particular pair of pants. Furthermore, the first slot had records of having ordered the pairs of pants and selecting a representative pair of pants to symbolize any but no particular of these pairs of pants. I also make the same inferences about the second slot of her pair concept.[51] I make the inference that Deborah had abstracted a pair concept based, in part, on her ordering the shirts and pants in the Outfits Problem prior to engaging in units coordinating activity. This way of organizing the situation suggests that she anticipated having to put each particular pair of pants into the first slot of her pair concept and each particular shirt into the second slot of her pair concept to produce all possible outfits.

In these situations, I had no occasion to observe if Deborah considered these slots to be ordered (e.g., counting as different the two of diamonds and the three of diamonds and the three

[51] I consider a representative pair to be one the bases for a producing a two-dimensional concept of multiplication from two one-dimensional composite units.

of diamonds and the two of diamonds based on the order in which the two cards were selected). I make this statement to make it clear that at this point I am assuming that the two slots were *differentiated* because, for example, the qualitative property that was associated with each of the two composite units was different (e.g., pants were different from shirts). As I suggested in Carlos and Michael's case study, I consider differentiation based on a qualitative property to be a precursor to having abstracted a concept of *ordering*.

Deborah's activity in the Two Suit Card Problem provides indication of how her ordering and whole number variable concept functioned in the context of her using her pair concept. When she made her explanation of this problem, she simply took a card out of her deck and used it as the first card to be inserted into the first slot of her pair concept. In doing so, she did not specifically consider the two of hearts or any particular card to be the first card (her ordering concept). Rather, the first card in the first slot was simply one she chose from her deck and any card could serve. So, any but no particular of the units could go in the first slot and symbolize all of the other cards (her whole number variable concept). Similarly, Deborah's explanation, "[Michael] could pull out thirteen different cards to go with that", suggests that she imagined the first card in the second slot of her pair concept could be any of Michael's thirteen cards. So, any but no particular of Michael's cards could be inserted into the second slot to symbolize all of his cards.

Assimilating the situations to her concept of a pair appeared to enable Deborah to assimilate the situation as one that involved multiplication without actually engaging in any units coordinating activity. I make this assertion, in part, based on her verbal explanation for the Outfits Problem, where she explicitly differentiated between how she got the answer and how she showed that she got the answer ["I just multiplied four by three and that's twelve. But to show it,…]. I take this verbal differentiation as indication that she assimilated this situation with her concept of a pair, which had been abstracted from her previous multiplicative activity. So, I infer that when she assimilated, for example, the Outfits Problem she simply identified the number of possible shirts and the number of possible pants and she took this as indication that she should multiply without actually having to engage in any units coordinating activity. That is, the situation was itself multiplicative to Deborah and she could find a way to show it, but she did not consider it necessary.

This way of operating meant that the assimilatory part of Deborah's scheme was anticipatory in the sense that it seemed to contain the activity and the results of her scheme. So, she could use notation like "4 x 3" or "13 x 13" to evoke operations that remained unimplemented, but she was confident she could implement these operations were she asked to provide a justification for such a multiplication. Deborah's way of operating provides indication of why there is a prevalence of the use of notation of two empty slots, "___ ___", where one imagines inserting some number of possibilities into each of the slots. From my perspective, this kind of notation is intended to evoke a concept that is similar to Deborah's pair concept.

Deborah's Notation and Natural Language. As I suggested above, Deborah's multiplicative notation and language appeared to contain her way of operating in the sense that she could take this notation as symbolizing operations she knew she could produce. In part, I make this assertion based on the way she used her other diagrams and notation (e.g., Figure 6.2, 6.3, & 6.5), which seemed to be solely an externalization of her anticipated way of operating. That is, she did not seem to use these diagrams and notation as a tool to monitor her activity or as a tool to reflect on this activity, but rather she seemed to use it as a way to illustrate to me why she thought the answer should be the multiplication problem she stated or wrote on her piece of paper.

To illustrate this point, it is useful to compare her use of notation and diagrams in the Outfits Problem with that of Carlos and Michael. In the Outfits Problem, both Carlos and Michael used their notation as a way to externalize their activity so that they could monitor this activity. On the other hand, Deborah used her notation and diagrams to serve a social goal—to illustrate to me why she multiplied. This way of operating suggests that Deborah's use of notation in this situation was primarily as a tool to establish a means of communicating with me. This way of operating suggests that Deborah was aware of how she would operate prior to operating. So, she could use her notation exclusively to illustrate to me her anticipated way of operating. I consider this way of operating as a basis for communicating about mathematical activity with another person because this level of awareness opens the possibility of comparing and contrasting one's own activity to another person's activity.

Deborah Produces the Sum of the First n Whole Numbers

Overview

This section of the case study is presented in four parts and reports on data from the first, second, and third teaching episodes conducted with Deborah. For several reasons, it is a somewhat more complicated story in comparison to the boy's story about producing the sum. It is complicated by the fact that Deborah produced both an additive and multiplicative way to reason about the sum and related these ways to quantitative problem situations. So, two themes will be under investigation simultaneously: the first will be how Deborah developed the sum as a quantitative entity and the second will be functional accommodations Deborah made in her lexicographic units coordinating scheme during her activity in these situations. The first part will specifically investigate how Deborah initially produced an additive concept of the sum and functional accommodations she made to her lexicographic units coordinating scheme. The second part will investigate how Deborah produced the sum in the context of her multiplicative reasoning and further functional accommodations she made in her lexicographic units coordinating scheme. The third part will investigate Deborah's abstraction of her multiplicative concept of the sum. The fourth part will investigate the association Deborah made between her additive concept of the sum and her multiplicative concept of the sum.

Deborah's Additive Reasoning for the Sum of the First 51 Whole Numbers

Deborah's Solution of the One Deck Card Problem. The first problem I posed to Deborah and Michael on the first day of the teaching experiment was Task 6.1, the One Deck Card Problem.

> Task 6.1, *The One Deck Card Problem*: How many two-card hands could you make with a deck of fifty-two-cards?
>
> Protocol 6.4: Deborah's notation for solving the One Deck Card Problem on 10/17/05
>
> T: [T poses Task 6.1 to the students. Neither seems certain of what to do. T tries elaborating the statement of the problem and so does W.]
> D: Do you put the cards to the side once you are done?
> T: No. Let's start with a slightly different problem and then we will come back to that problem.
> D: I get it. I just don't know how to figure it out.
> T: [T poses the Outfits Problem and its extension to D & M. D solves the problems quickly and begins working on Task 6.1, while M is finishing his solution to the extension of the Outfits Problem.]
> D: [D writes Figure 6.6a, looking at it intently for 15 seconds. Then, she writes Figure 6.6b. She sits in concentration for 15 seconds and then erases the "51" she has written in Figure 6.6b. She writes Figure 6.6c and sits again in concentration for 10 seconds. She writes the numbers "51, 50,…,42" and counts on her fingers to thirteen. She counts the number of numbers she has written and continues her column of notation down to "39". She continues her notation writing out a total of four columns (Figure 6.6d). Without prompting from T, she gives the following explanation.] The way I thought about it was like say you go through the two of diamonds and you go through every number that can go along with the two of diamonds and that would be fifty-one combinations. [D points to the 51 in Figure 6.6d.] And then, when you went to the three of diamonds, there would

only be fifty because you have already done the two of diamonds and the three of diamonds.
T: Okay.
D: And then so forth, so you go down one every time. [A bell momentarily interrupts D's explanation and she and M have a brief conversation about the school schedule.] And like so four of diamonds [D points to 49 in Figure 6.6d], five of diamonds [D points to 48 in Figure 6.6d], six of diamonds, seven of diamonds, eight of diamonds, nine of diamonds, ten of diamonds, jack, queen, king, ace. [D points to the associated number in the first column of notation of Figure 6.6d concurrently with saying each card name.] And then you start with another (suit). So like if you add all those together that would be like your… [D smiles and pauses.]
T: Total?
D: total.

2 ◊

2 ◊ 51

2 ◊ & 3 ◊
" & 4 ◊
"

Figures 6.6a (left), 6.6b (middle), & 6.6c (right): Replica of Deborah's notation

51	38	25	12
50	37	24	11
49	36	23	10
48	35	22	9
47	34	21	8
46	33	20	7
45	32	19	6
44	31	18	5
43	30	17	4
42	29	16	3
41	28	15	2
40	27	14	1
39	26	13	

Figure 6.6d: Replica of Deborah's notation

Given that Deborah was not initially sure of how to solve the problem ["I get it. I just don't know how to figure it out."], I infer that she had entered a state of perturbation. Based on the facility with which Deborah produced pairs in the Outfits Problem, and its extension, I infer that her perturbation arose from having a goal of producing pairs, but not being sure of how to make these pairs. Solving the Outfits Problem seemed to help her conceptualize how she might produce pairs in the One Deck Card Problem. Specifically, producing a unit of ordered units (e.g., the first shirt, the second shirt, etc.) seemed to help provide her with insight into how to operate in the One Deck Card Problem. So, she appeared to make a functional accommodation in the assimilatory part of her lexicographic units coordinating scheme. The functional accommodation was that she could find a way to make pairs in a situation that she initially considered involving only one composite unit (i.e., the deck of fifty two cards) as opposed to two composite units.

I make the inference that initially Deborah only considered one composite unit in the situation because she appeared to imagine drawing the two of diamonds from the deck of cards and putting it with a representative card from the remaining 51 cards in the deck, which symbolized producing 51 two-card combinations. I make this inference based on her notation in Figures 6.6a and 6.6b. So, Deborah's functional accommodation was to assimilate the situation with only one unit of ordered units (i.e., the deck of 52 cards), to disembed the first unit (i.e., the two of diamonds), and make a units coordination between the disembedded unit and a representative unit from the remaining deck of cards. After writing Figure 6.6b, she stared at her notation for 15 seconds. During this time, I infer she was engaged in activity similar to her activity with the two of diamonds except with the three of diamonds and the four of diamonds. I take her erasing the "51" that she had written next to the two of diamonds as indication that she had become aware that neither the three of diamonds nor the four of diamonds would produce 51 two-card combinations. So, she appeared to eliminate two-card combinations that she considered she had already produced. Her explanation, "And then, when you went to the three of diamonds, there would only be fifty (two-card combinations) because you have already done the two of diamonds and the three of diamonds", provides corroboration that she made these eliminations.

When she wrote Figure 6.6c, I infer she was imagining making two-card combinations with the five of diamonds and her list symbolized the three two-card-combinations that she had to eliminate (where the quotation mark in the third row symbolized the two of diamonds and the five of diamonds). Here, she seemed to be using her notation to confirm a pattern that she had noticed—the number of eliminations would increase by one every time so the number of two-card combinations would decrease by one. I make this inference based on her concentrating for 10 seconds after writing Figure 6.6c, followed by her writing out the first column of notation in Figure 6.6d down to "42". When she wrote Figure 6.6d, she appeared to have stopped her units coordinating activity and was using her notation as a way to quantify the pattern she had noticed. So, her notation symbolized her units coordinating activity without her actually having to produce the activity to know what the results of this activity would be.

She halted making Figure 6.6d and counted on her fingers from one to thirteen. Here, I infer that she was figuring out the number of diamonds in a deck of cards and making a correspondence between the number of diamonds in the deck and the number of numbers she had written in the first column of her notation. So, I infer that when she began producing Figure 6.6d, she was simply quantifying that she could produce one less two-card combination with each subsequent card she used. Quantifying this pattern with Figure 6.6d seemed to open the question as to when she would complete quantifying the pattern. So, I infer that she stopped producing Figure 6.6d in order to make definite the number of numbers she had already produced and the number of numbers that she considered necessary to produce in order to have completed the first column of her notation. This was an act of self-regulation that led to her monitoring her activity of quantifying the number of pairs she produced.

Deborah's Use of Notation. Deborah's notation in Figures 6.6a, 6.6b, 6.6c, and 6.6d appeared to help her create a record of her activity that helped her monitor this activity. Moreover, she used this record to externalize her way of operating so that she could take particular parts of her previous operating as a given in the situation and then make a continuation of her operating. So, for instance, as I suggested above, I infer that Deborah ordered the cards in the deck, disembedded the two of diamonds from the deck of cards, and made a units coordination with this card and a representative card from remaining deck of cards to produce 51 two-card combinations. She recorded this activity in Figures 6.6a and 6.6b. She, then, took this notation as symbolizing these operations without having to produce this activity again. So, here, she used her notation as a placeholder for having operated in a way that had produced 51 two-card combinations. So, she could take this activity as a given in the situation and continue operating. Figures 6.6c and 6.6d operated in a similar manner. So, Deborah used her notation as a tool in action to help her monitor and externalize her way of operating.

Based on her subsequent explanation, I infer that Deborah could also use her notation to re-present her activity. For example, she pointed to the "51" in her sum as she said, "say you go through the two of diamonds and you go through every number that can go along with the two of diamonds and that would be fifty-one combinations." She made a similar explanation for each of the numbers in the first column of Figure 6.6d. Using her notation in this way was a tool in reflection in the sense that she used her notation to re-present her activity as she made an explanation to me. Re-presenting her activity in this way opened the possibility for her to reflect on how she had operated in the situation. Indication that she did reflect on her activity in the situation was suggested by her more compressed explanation as she pointed to each number in the first column of Figure 6.6d and associated a card with each particular number. Once she reached the end of the first column she simply stated that you would begin with a new suit of cards, rather than associating each number in her sum with a particular card.

Deborah's compressed explanation also provides indication that she was using her verbal explanation for the social goal of communicating her solution to me. Here, Deborah appeared to have abstracted her way of operating in this situation so that she could use her verbal explanation to establish a convention with me about how she had operated in the situation and what her notation symbolized. This way of operating was similar to how she had operated during the Outfits Problem, the extension of the Outfits Problem, and the Two Suits Card Problem. It suggests that Deborah was aware enough of her activity so that she could use her notation and verbal explanation for this social goal.

Deborah's Functional Accommodations. As I suggested above, operating in this situation suggested that Deborah had made a functional accommodation in the assimilatory part of her lexicographic units coordinating scheme. This functional accommodation enabled her to make pairs in a situation that she assimilated with only one unit of ordered units. Deborah's subsequent actions and explanation provide indication that she had made a second functional accommodation in the context of operating in this situation. The functional accommodation was that Deborah appeared to produce two units of ordered units as a *result* of operating in the situation even though the statement of the problem only referred to one composite unit.

To produce two units of ordered units, Deborah would have to constitute her disembedding activity that produced a single card each time as producing a second composite unit. I take as indication that she constituted her disembedding activity as a second composite unit because she curtailed her explanation of associating a particular card with each number in her sum. To curtail her activity in this way suggests that she expected that there would be 52 cards one that could be associated with each number in her sum. These 52 cards constituted the second composite unit, which as I stated above she produced as a result of her activity. The first composite unit had been used to produce the number of two card combinations, which she had symbolized with the numbers in her sum (i.e., 51, 50, etc.).

Note that this way of operating was different than how she had operated during the first three problems. It was different in that she produced a second composite unit as a result of her activity rather than as part of assimilating the situation. So, Deborah did not appear to take this situation as one that involved multiplication as she had with the previous situations. Because Deborah's solution was not multiplicative, I infer that when she made eliminations she simply took these eliminations as not counting two card combinations that she had already made. So, for example, when she eliminated the two of diamonds and the three of diamonds, I infer she made this elimination based on her already having made that particular two-card combination. Nonetheless, her explanation suggests she was aware of making the eliminations, which opened the possibility that she might consider the two of diamonds and the three of diamonds and the three of diamonds and the two of diamonds as the same or different. To do so would involve Deborah seeing the situation as one that involved two composite units, and further, to *order* the selection from these composite units (e.g., first a two of diamonds and second a three of diamonds versus first a three of diamonds and second a two of diamonds). Because Deborah assimilated the situation using only one composite unit, her activity in the situation did not appear to involve this type of ordering.

Deborah's Reflective and Reflected Abstraction. I infer that Deborah made both a reflective and reflected abstraction in relation to her first functional accommodation. I make this inference because Deborah was initially unable to operate in this situation. However, she found a way to operate and was able to *explain* her way of operating, which suggests that by the end of her activity she was aware of how she had operated. So, I infer that what was projected to a higher plane, her reflective abstraction, was the new assimilatory part of her scheme and the associated operations of disembedding a unit from one composite unit in order to make pairs.

The functional accommodation that led to this reflective abstraction was somewhat minor. I consider it a relatively minor shift in her way of operating because I could attribute all of the operations that she used in making her solution to her prior to having made the solution. However, these operations enabled Deborah to produce the sum as a quantitative entity. I make this inference because the notation she used to symbolize the sum contained both an amount and ordering meaning. That is, 51 referred to the *number* of pairs that could be made with the *first* card that she had considered (the two of diamonds). As I suggested in Michael's case study, this seems like a minimum requirement for producing the sum of the first so many whole numbers as a quantitative entity because each of the amounts in the sum is related to the previous amount based on the position in which it occurs. Further, it establishes the sum as having a definite beginning and ending point where the beginning point is the first addend and the ending point is the last. In this situation, Deborah used her additive notation to quantify what she considered an activity that had a definite beginning and ending and so established the sum as a quantitative entity. To evaluate the sum, Deborah made strategic additions (e.g., adding 51 and 49 to make

100, etc.) over the course of the rest of the episode and eventually found the number of pairs.[52] So, I infer that Deborah considered the operations that she had used to produce the sum to be additive in nature.

Her reflected abstraction, retroactive thematization, I infer occurred during the process of her making her explanation of her solution to me. Here, she appeared to be well aware of how she had operated in the situation and how her operating had produced the sum. So, while the functional accommodation was somewhat minor, the consequences of it were quite significant because it led to the construction of a new mathematical concept, the sum as a quantitative entity.

Deborah Produces the Sum with Multiplicative Reasoning

Deborah's Solution to the Handshake Problem. Given that Deborah had independently initiated finding a sum to solve the One Deck Card Problem, I wanted to revisit problems that could be solved by finding a sum, during the next episode. However, I wanted to investigate if Deborah might relate her additive and multiplicative reasoning. So, at the beginning of the second episode, I gave her a deck of cards and took a deck of cards for myself. We made a few experiential two-card combinations by each choosing at random a card from our respective decks. I then posed Task 6.5, the Two Deck Card Problem, to her.

> Task 6.5, *The Two Deck Card Problem:* If you were going to write down all the possible two-card combinations you could get, how many would you write down?

Deborah responded, "I did that problem yesterday!", and explained her solution of the One Deck Card Problem. I asked her how she would solve the problem if she considered as different the two of diamonds from my deck and three of diamonds from her deck, and the three of diamonds from my deck and the two of diamonds from her deck. She responded in that case the answer

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[52] She worked on making these additions, while she waited for Michael to solve other problems that she solved more quickly.

would be “fifty two times fifty two”. I asked her a few question about her solution to the problem, and then, I posed the Handshake Problem, Task 6.6, to Deborah.

> Task 6.6, *The Handshake Problem*: Suppose there are ten people in this room. Each person would like to shake every other person’s hand. How many total handshakes would there be?

> Protocol 6.5: Deborah’s solution to the Handshake Problem on 10/19/05

> D: [D writes out Figure 6.7a by first writing the column of letters shown before the vertical line. She then writes out the first four rows, which takes 26 seconds. After finishing the first four rows, she counts the first row of letters and runs her pencil vertically over the first column.] Ninety.
> T: Ninety?
> D: Yeah do you want me to write the rest of these out?
> T: Well, go ahead and tell me how you got ninety.
> D: These are the people [D points to the first column of Figure 6.7a.] There is person A, person B, person C, person D, person E, person F, person G, person H, person I, and person J.
> W: Can she do it using addition?
> T: Hold on, hold on, I want her to explain how she got ninety first.
> D: Oh wait, I’m wrong, I’m wrong. [D is examining her notation.] Because if you say person A has shaken hands with everybody [D points to her first row of notation in Figure 6.7a], then person B has shaken hands with everybody [D points to her second row of notation in Figure 6.7a], but he has already shaken hands with person A [D erases “A” from the second row in Figure 6.7a]. So, so person A has already shaken hands with everybody [D erases “A” from the third and fourth rows in Figure 6.7a.] So, there is no person A. And then, like they can’t shake their own hand so that’s how I got that top one (row). Shakes everybody’s hand except for itself. Cause these are the people. [D draws in the line in Figure 6.7a that separates the first column from the rest of the letters.] And ninety is not right because person A has already shooken person B’s hand and person B shaking person A’s hand is the same thing. Alright, so, then you do that and then C has already shook B’s hand [D erases the “B” in the third row.], and person D has already shook B and C’s hand [D erases “B” and “C” in the fourth row. See Figure 6.7b to see the end result of this erasing.], and then, like every time you are going to take away one. I’m just going to write them all out and then I am going to erase them. [D writes out nine letters in each row (Figure 6.7c).] So, E has already shaken all their hands. F has already shaken all their hands. G has already shaken all their hands. H has already shaken all their hands. I has already shaken everyone’s hand except for J. J has shaken everybody’s hand. [As D says each of the eliminations, she erases the appropriate letters in Figure 6.7c to make Figure 6.7d.]

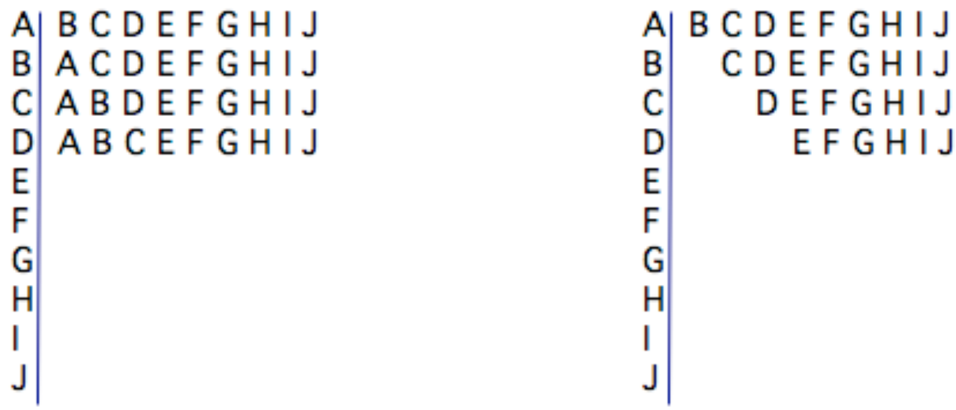

Figure 6.7a (left) & 6.7b (right): Replica of Deborah's array for the Handshake Problem

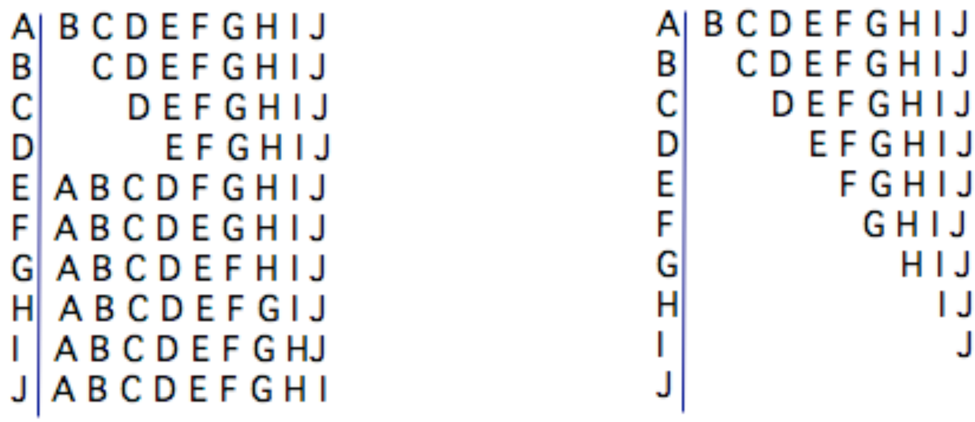

Figure 6.7c (left) & 6.7d (right): Replica of Deborah's array for the Handshake Problem

To solve this problem, I infer that Deborah assimilated the situation with one composite unit of order units, the ten handshakers. She symbolized these ten people in her first column of notation. She, then, appeared to make sequential units coordinations between the first person (the handshaker) and all of the remaining people (the handshakees), which symbolized the handshakes for the first person. I infer that to produce the handshakees she disembedded each of the individual units from the initial composite unit that she had used to assimilate the situation. She repeated this activity for each of the first four rows, then halted this activity and counted the number of handshakes that she had symbolized in the first row.

That she only counted the first row of notation and then multiplied suggests several things about her activity. First, it suggests that she was aware that all of the rows symbolized the same number of handshakes so she seemed to be aware that she had eliminated exactly one handshake from each row, the self-handshake. Second, it suggests that she established a second composite unit of ordered units, the handshakees, which she could use with each of the handshakers. She established this second composite unit through her disembedding activity. I infer that producing two composite units meant that she recognized the situation as multiplicative So, once she established a second composite unit of ordered units she assimilated her activity to her pair concept and simply imagined inserting the number of handshakers into the first slot and the number of handshakees into the second slot, which led to her calculating a total of 90 handshakes. Note that the letters in the rows seemed to function in two different capacities—to identify the handshakees (the person with whom person A shook hands) and to identify the handshakes themselves (the results of two people shaking hands).

When Deborah gave her response of 90 handshakes, I infer she had yet to identify that person A shaking person B's hand was the same as person B shaking person A's hand. When the witness-researcher intervened to ask if she could make an addition problem, she seemed to take this as an occasion to reflect on her activity, saying, "Oh wait, I'm wrong, I'm wrong". I infer that she had associated her additive activity from the One Deck Card Problem with making eliminations, and so this suggestion appeared to bring to her awareness that she could eliminate some of the handshakes.

So, the witness-researcher's suggestion appeared to occasion an act of self-regulation and Deborah used her notation to monitor the results that her units coordinating activity had produced. Note that Deborah did not produce the activity again. Rather, she simply took the results of her activity, the 90 handshakes, as material on which to operate further. I make this inference based on her erasing parts of her notation that she had already written and then writing out all 90 handshakes and eliminating from these handshakes. So, she used her notation to externalize her *pair concept* and took this concept as material on which to operate further.

The way in which Deborah began to eliminate handshakes was interesting. She appeared to eliminate person A both from shaking hands with person B ["then person B has shaken hands with everybody, but he has already shaken hands with person A."], and from shaking hands with everyone else [So, so person A has already shaken hands with everybody. So, there is no person A."]. So, she erased "A" from the second, third, and fourth rows. This way of operating suggests that she identified two locations in her notation where person A's handshakes and been symbolized—along the first row of her diagram and in the second column of her diagram (Figure 6.8).

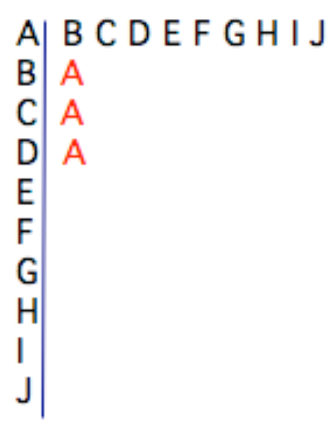

Figure 6.8: Deborah identifies two locations where she has symbolized person A's handshakes

She, then, interrupted her eliminating activity to clarify with me what her notation symbolized ["And then, like they can't shake their own hand so that's how I got that top one (row)…"]. When she returned to making eliminations, she no longer identified two locations in her notation where she had symbolized each person's handshakes. So, when she got to the second row, she eliminated person A because he had completed all of his handshakes. When she got to the third row she had to eliminate up to person B because person A and person B had completed their handshakes. This way of operating led to a recursive way of making her eliminations where she took the previous row as indication of a person having completed all of his handshakes. So, each time she moved down one row she eliminated one more person ["every time you are going to take away one"]. This way of operating meant that her eliminations were sequential and on the rows of the array, which contrasted from her first way of eliminating where she identified two locations, a row and column, where each handshake had been symbolized.

At this point, I asked her to quantify her results to compare them to her original answer.

Protocol 6.6: Deborah quantifies the results of her activity on 10/19/05.

T: Okay, so, how many have you got? Can you write an addition problem for that?
D: Nine plus eight plus seven plus six plus five plus four plus three plus two plus one. [D writes the sum vertically.] Ten [D crosses out "9" and "1" in her sum.], twenty [D crosses out "8" and "2" in her sum.], thirty [D crosses out "7" and "3" in her sum.], forty [D crosses out "6" and "4" in her sum.], five [D crosses out "5" in her sum and writes "45" at the bottom of the sum.]….
W: What would you have to do to the ninety to get forty-five?
D: Divide it by two.
W: Why would you divide it by two?
D: Cause that equals forty-five. [D points to where she has written the sum.]
T: How back in the situation can you think about it? Why would you have to divide the ninety by two?
D: [D sits in concentration for 25 seconds.] Cause…cause…I'm not sure. I think it has something to do with like this person shakes everybody's hand [D points to the first row of her diagram] and this person doesn't shake anybody's hand [D points to the last row of her diagram.] so they sort of like make up for each other. So, you like divide it by two because they are like opposites [D runs her eraser along the diagonal of Figure 6.7d.]….
T: Now the question is when you were counting all of them,
T & W: [together] how many did you erase?
D: Half of them.
T: How come you erased half of them?
D: Because they have already shook their hand.

At first, Deborah evaluated the sum using strategic additive reasoning to find that she had symbolized 45 handshakes, which she knew that she could find by dividing 90 by 2. This assessment appeared to be based purely on a numeric calculation. So, I tried to focus her attention on the quantitative situation in making her explanation ["How back in the situation can you think about it?"]. Deborah began making her explanation based on the individual rows (i.e., referring to person A and J as opposites) but then transitioned to operating on the array itself (i.e., sweeping her eraser across the diagonal of the array). This way of operating suggested that she was trying to coordinate the sequential eliminations she performed on the rows of the array with her observation that these sequential eliminations when taken together made half of the original array she had produced.

Deborah appeared to take this as indication that she had written each handshake in her array exactly twice. However, she seemed unsure of how to explain why she had written each handshake in her array exactly twice based on her multiplicative reasoning. I make this inference based on Deborah's doubtfulness ["I'm not sure."] of relating her division by two back to the problem situation. At the time, I was intuitively aware that Deborah could not relate her second kind of elimination to the problem situation, but I was unaware of what problems might enable Deborah to make such a relation. Now it is clear to me that work with problems that involved making ordered pairs may have opened this possibility. I make this statement because I infer that to relate her eliminations to the quantitative problem situation would involve creating an *ordered* pair.

Doing so might have enabled Deborah to temporarily consider two handshakes as different depending on who she considered to be the handshaker and who she considered to be the handshakee. Her way of operating did open the possibility of considering two different types of handshakes because she recorded each with her notation, but she seemed to consider the two to simply be the same thing. I take this way of operating as contra-indication that Deborah's pair concept contained records of ordering the two slots. Rather, the two slots in Deborah's pair concept seemed simply to be differentiated because she considered the two composite units to have different referents (e.g., handshakees and handshakers).

So, I infer that Deborah made her sequential eliminations based on the *results* of the activity of producing the handshakes, not based on a feature of this activity itself. That is, Deborah could use her notation to monitor the results of her activity and was able to consider that person A shaking person B's hand was the same as person B shaking person A's hand. So, she could relate her eliminations on the rows back to the quantitative problem situation.

However, it appeared to be the figurate aspects of her notation that helped her make the observation that exactly half of the handshakes had been erased by her sequential eliminations. By figurate aspects of her notation, I mean that she seemed to observe that her sequential eliminations on the rows of the array produced exactly half of the rectangle that she had used to symbolize the 90 handshakes she had first created. This way of operating produced the sum as a quantitative entity in a new way—by erasing half of the array that Deborah produced. So, I infer that Deborah recognized her new way of operating as producing the quantity she had produced with her additive reasoning in the One Deck Card Problem, but she produced it in this context with different operations (i.e., her units coordinating operation followed by her partitioning operation). In a certain regard, her way of operating was similar to how she made her eliminations in the One Deck Card Problem: she considered any person whose handshakes had already been completed to be eliminated from future handshaking activity. However, it differed in that she produced this activity in the context of operating on the results of her multiplicative reasoning, which was not the case in the One Deck Card Problem.

Deborah's Functional Accommodation. My interpretation of Deborah's solution of the Handshake Problem suggests that Deborah had taken the results of her operating in the One Deck Card Problem (i.e., producing a second composite unit) and incorporated these results into the assimilatory part of her scheme. I make this inference because Deborah produced all ten handshakers prior to engaging in her units coordinating activity. This way of operating differed from the One Deck Card Problem in that rather than envisioning drawing one card at a time and then making a units coordination. She produced all of the handshakers first and then began her units coordinating activity. She eventually produced two composite units—10 handshakers and 9 handshakees—and assimilated this activity to her pair *concept*. This way of operating was a functional accommodation because it enabled Deborah to assimilate problems that referred to only one quantity in the statement of the problem to her concept of a pair.

I want to point out that the multiplicative way of operating appeared to be related to the quantitative problem situation (i.e., 10 handshakers and 9 handshakees for each person), but the division by two which produced the eliminations appeared to be only associated with the quantitative problem situation in that it produced the sum as a quantitative entity. Deborah could relate this sum to the quantitative problem situation because she recognized the sum as something that she could produce with her sequential eliminations, but she did not seem to relate her division to creating ordered pairs. Deborah was able to substitute a single operation, partitioning the array, for the sequential operations, eliminating the individual handshakes because she recognized that each produced the sum and each eliminated the extra handshakes. Deborah's functional accommodation enabled her to produce the sum as a quantitative entity in a new way—with her partitioning operations. She recognized this way of operating as highly useful because it enabled her to evaluate the sum with a multiplication and division problem whereas she had previously only been able to evaluate it using strategic additions.

A Brief Analysis of the Questions I asked Deborah. At the time of conducting the study, I was unaware of what questions I might have asked Deborah to help her produce an ordered pair even though I had some awareness that she could not fully relate her partitioning to the quantitative problem situation. Moreover, it was also not clear that she established that the array contained two sums—one that symbolized the handshakes and one that symbolized the eliminations. So, I want to analyze the two questions I asked her and possible alternatives to these questions that may have opened the possibility for further functional accommodations.

The first question I asked Deborah was, "Okay, so, how many (handshakes) have you got? Can you write an addition problem for that?" This question focused only on the handshakes that were left after her eliminations and not on quantifying the eliminations she made in each row. I could have asked Deborah to write two sums one that symbolized the number of handshakes in each row and one that symbolized the number of eliminations she made in each row. This type of question seems like it would have been within Deborah's zone of potential construction given that she appeared to establish each row as containing nine handshakes and she established the total number of handshakes left over after she made eliminations. So, it seems plausible that she would have been able to assess how many total handshakes she had eliminated from each row.

The first type of question might have opened the possibility for investigating why the number of eliminations were the same as the number of handshakes. So, the second possible question might have been: When you were counting all of them (the handshakes), how many times did you record each handshake? With follow up questions to elicit activity to find where each of person A’s handshakes that she had counted were in her notation and where the duplicates of these handshakes might be found in her notation. I asked instead, “Now the question is when you were counting all of them, how many did you erase?” This question was focused on her eliminating activity but not on her eliminating activity in relation to the activity she had used to produce the ninety handshakes.

Were Deborah able to engage in this type of activity, she would have decomposed the ninety handshakes into two sums based on the handshakes that Deborah had counted and where in her notation she had eliminated these same handshakes. In Figure 6.9, I have colored the first row red to show the handshakes she counted for person A and the second column blue (excluding the letter “B”) these same handshakes that she had eliminated. For person B, she would have found only eight handshakes (the second row excluding the A) she counted and eight handshakes she had eliminated (the third column excluding the two C’s).

A	B	C	D	E	F	G	H	I	J
B	A	C	D	E	F	G	H	I	J
C	A	B	D	E	F	G	H	I	J
D	A	B	C	E	F	G	H	I	J
E	A	B	C	D	F	G	H	I	J
F	A	B	C	D	E	G	H	I	J
G	A	B	C	D	E	F	H	I	J
H	A	B	C	D	E	F	G	I	J
I	A	B	C	D	E	F	G	H	J
J	A	B	C	D	E	F	G	H	I

Figure 6.9: Replica of what Deborah’s full notation would have looked like

This type of structuring would have involved highlighting having eliminated each of the handshakes she counted or that each handshake had been counted in two different ways (the row when person A was a handshaker and the column when person A was a handshakee). I infer that this activity was within Deborah’s zone of potential construction because in her eliminating activity with person A she had identified the two locations where she had symbolized the handshakes of person A (see Figure 6.8).

I infer that these two activities could have opened the possibility for Deborah to become aware that the ninety handshakes could be decomposed into two sums where one sum was the number of eliminations and the other was the number of handshakes she had counted. Furthermore, that the two sums were produced from considering each of the people handshakers and handshakees. I highlight these two hypothetical trajectories both to suggest what Deborah did seem to produce on her own as well as future interventions that might be useful when working with a student like Deborah.

Deborah Abstracts Her Multiplicative Way of Producing the Sum into a Concept

The Committee Problem. I began the next episode posing the Committee Problem to Deborah to find out if she would regenerate her way of operating from the previous episode.

> Task 6.7, *The Committee Problem*: Suppose you want to select a two-person committee from a group of 25 people. How many possible committees could you have?
>
> Protocol 6.7: Deborah's solution and explanation of the Committee Problem on 10/24/05
>
> D: [D calculates 25 x 24 and uses her multiplication algorithm to calculate that it is 600 people. She, then, divides 600 by 2 and gets 300. T asks her to explain her multiplication and division.] I multiplied twenty-five, the number of people, times the number of different people that could go with each person and I got six hundred and I divided it by two.
> T: And why did you divide it by two again?
> D: Because half of them you would have to take away because they are repeats. [T asks D what the addition problem would be and D states the sum of the first 24 whole numbers.]

Deborah's response suggests that she had abstracted her multiplicative way of operating from the Handshake Problem into a concept and that she had associated this way of operating with problem situations that she recognized could be evaluated using a sum. I make this inference because she assimilated the situation to a way of computing the result. I infer that making these computations also contained an image of an array partitioned into two pieces where one of these two pieces was the sum. I make this inference because the array she produced in the Handshake Problem seemed to give rise to her multiplication and division computation. Although she did not reference the array in this situation, she had referenced it in several other problem situations. So, I infer that she had interiorized the image of this array and when she computed using her multiplication and division algorithm that her algorithms contained an image of an array partitioned into two pieces where the resulting amount was a numerical value (i.e., 300) and a quantitative entity, the sum (i.e., a 25 by 24 array partitioned into 2 parts which produced rows containing 24, 23, etc.).

Because Deborah seemed to have succinctly described her activity using natural language, I wanted to find out if she could use algebraic notation to symbolize this activity. So, I asked her the following problem.

Protocol 6.8: Deborah's algebraic notation on 10/24/05

T: Okay, so let's suppose there are n people. Could you make a formula for n people?
D: Okay. [D writes $n(n\text{-}1) = x$].
T: And then what would…
D: Then you would divide x by…no. Yeah, you would divide x by two.
T: Divide x by two?
D: Well you could just do this. [D writes "$n(n -1) \div 2 = x$".] And that is the number of committees. [At T's request, D writes out what each letter signifies (Figure 6.10a).]
T: And, what does n times $n -1$ divided by two stand for?
D: The way to figure out how many possible committees there could be before you take away repeats.
T: Why don't you write that down what the whole formula stands for?
D: This whole thing.
T: Not the x, just the n times $n\text{-}1$ divided by two.
D: [D writes only over the formula $n(n\text{-}1)$ (Figure 6.10b).]
T: Okay and what does the divided by two stand for?
D: Taking away the number of repeats. [D writes over the divided by two (Figure 6.10b).]

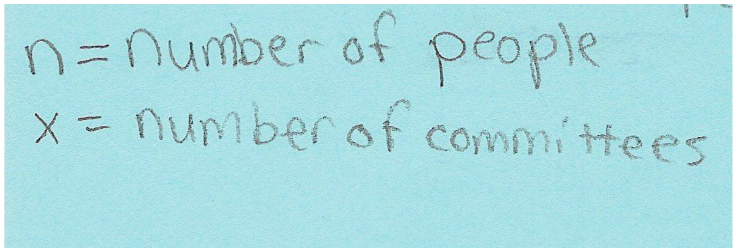

Figure 6.10a: Deborah's notation

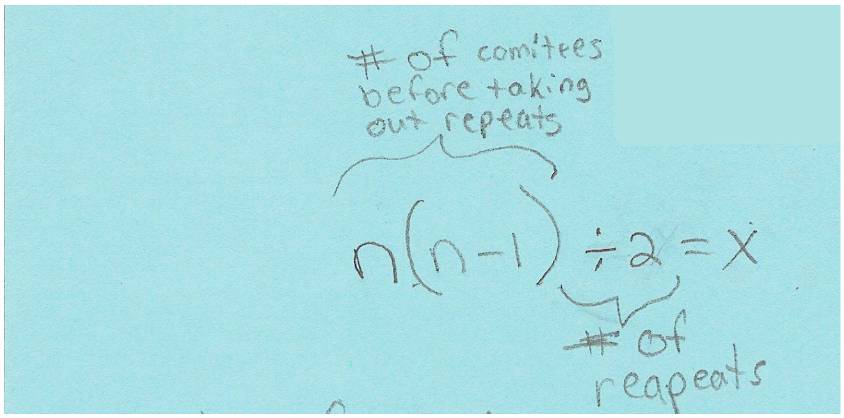

Figure 6.10b: Deborah's notation

This way of operating suggests that she had created an algorithm for finding solutions to problems like the Committee Problem. Deborah's algorithm, which she symbolized with her algebraic notation, symbolized a multiplicative concept of the sum she had abstracted from operating in the Handshake, Committee and other problems. The concept appeared to be the following program of operations: to produce a composite unit of ordered units, disembedd a second composite units of ordered units, imagine a units coordination between a representative unit from each, which I infer produced an interiorized image of an array. She, then, imagined partitioning this array into two parts, which produced the desired result. At this point, she appeared to be operating on a quantitative symbol, the array, and not with the committees, themselves, even though she could relate her notation back to referents in the quantitative situation.

Deborah's Notation. Deborah's use of notation was interesting from several perspectives. The first interesting feature of Deborah's algebraic notation was the way in which she produced the notation. She first produced the total number of committees with duplicates, which was what x initially symbolized. She then began by suggesting that she should divide x by two to make the total number of committees after the elimination of repeats. So, at first, it appeared that she was imagining x as a letter that symbolized evaluating the quantity $n(n-1)$. However, when she decided to write her equation as "$n(n-1)\div 2$", she appeared to be able to imagine operating on an unevaluated multiplicative quantity (i.e., $n(n-1)$). This way of notating suggested she could take her unevaluated multiplication problem as material on which to carry out further operations. I conjecture that Deborah was able to operate this way because she imagined her multiplicative notation as producing an array where the array could be operated on without actually having to evaluate the multiplication problem. That is, any array could be partitioned into two parts without having to specify what the result of multiplying the dimensions would be. This enabled her to use her notation to point to a way of operating without actually having to take into account evaluating this notation with each successive operation she symbolized with her notation.

A second and related interesting feature of Deborah's use of notation was that she took her algorithm as needing to be produced in two steps. The first step was the multiplication problem that produced the array and the second was the division problem that resulted in half of the array. This way of operating provides corroboration that the eliminating activity that she engaged in was not embedded in her pair concept, but was an associated operation that would produce the desired result (i.e., it did not result from producing ordered pairs). This assertion is further confirmed by my question to Deborah about what "$n(n - 1) \div 2$" symbolized. Even after clarifying with me what I was asking, Deborah responded by labeling each of the two parts of her notation, rather than the entire notation itself.

Deborah Relates Her Additive and Multiplicative Concept of the Sum

Deborah Uses Dots to Symbolize the Sum. I wanted to investigate if Deborah might use her additive concept of the sum—a quantity with a definite beginning and ending point where each position in the sum contained one more than the previous position—to produce the multiplication problem in her formula. So, I asked her the following problem.

Protocol 6.9: Deborah uses dots to symbolize the sum on 10/24/05

T: [D has just finished solving the Committee Problem with six people both additively and multiplicatively and knows the result will be fifteen.] I wonder if you could find out a way to represent the sum using dots?
D: What do you mean?
T: Like, how would you represent five using dots?
D: Like one, two, three, four, five. [D makes five dots.] One, two, three, four. [D makes a second row of four dots. D produces Figure 6.11a.]
T: I wonder if you could use the dots to show the multiplication formula. [D is confused by this request and T is unsure how to proceed. T asks W for help. W reviews with D what she has done to produce the multiplication formula.]
W: Could you show five times six using the dots?
D: You mean like the whole number…[D begins to make separate dots from where she has made the dots for the sum.]
W: Using the same dots you used before.
D: Oh. [D concentrates for 5 seconds and then draws a second triangle of dots above the bottom half of the dots, but only makes four rows of dots (Figure 6.11b).]
W: Where is five times six divided by two? Can you draw a line to show it?
D: [D looks at Figure 6.11b for 15 seconds.] Wait, that is not right. I needed a whole other row. [D draws another row with five dots in it.] Then, split it in half. [D draws a line between the two halves of her diagram (Figure 6.11c).]

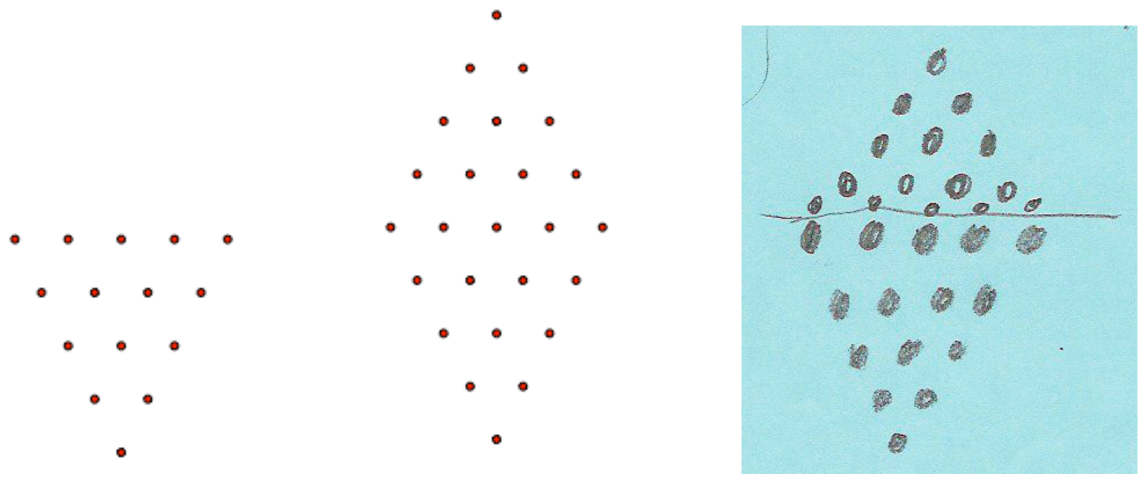

Figure 6.11a (left), 6.11b (center), & 6.11c (right): Deborah's diagram with dots

Deborah's initial diagram, Figure 6.11a, suggests that Deborah had indeed produced an interiorized image of her additive concept of the sum during the One Deck Card Problem and could externalize what this might "look like" using dots. Here, she was able to externalize this image with her diagram and to operate on it. She operated on it further by iterating her additive concept of the sum to make a second copy of it. Although she did not initially make the correct amount, the fact that she produced a copy of the sum suggested that she anticipated the multiplication problem would contain two copies of the sum. I infer that her anticipation was a result of knowing that 5 x 6 was 30 and knowing that the sum she symbolized contained 15 dots. Using the sum to produce the multiplication problem was a way for her to begin with her additive concept of the sum and use it to produce the multiplication problem. Doing so seemed to help her establish an identity between the multiplication problem and twice the sum.

So, Deborah appeared to become aware that the multiplication problem contained two copies of the sum. Unfortunately, I did not ask her to engage in operations to recompose the resulting thirty dots into five 6s and then use these five 6s to make two sums based on decomposing each 6 into part of each sum (i.e., the first 6 into a 1 and a 5, the second 6 into a 2 and a 4, etc.). So, it remained unclear if she took the relationship that she established as primarily established because each would contain the same number of dots (i.e., 2 times 15 could be symbolized by 30 dots as could 5 times 6) or if she engaged in some kind of re-composition of the 30 dots.

The difference I am pointing to here is similar to the one I was pointing to when I analyzed the questions I asked Deborah in the Handshake Problem. The difference is between two quantities being equivalent because operations performed to make each produce the same result versus an embedding of operations into the *activity* of producing these results. So, for instance, when a student has embedded one activity into the other, I would infer that she could take producing five 6s as producing two sums because producing the five 6s would contain the activity of decomposing each unit of six into two units (i.e the first 6 decomposed into a 5 and a 1, etc.).

Given the ease with which she used the sum to establish the multiplication problem, I wanted to investigate if Deborah could produce algebraic notation that related her concept of the sum as produced through multiplication and division to her additive concept of the sum. In order to make this investigation, I presented Deborah with the problem of symbolizing the first n whole numbers with a triangle.

Protocol 6.10: Deborah uses dots to symbolize the sum of the first n whole numbers on 10/24/05

D: [At W's suggestion, D begins making her triangle with one dot. She makes two rows of dots (Figure 6.12).] Keep going?
T: Yeah.
D: To how many though? [D makes the third and fourth row of dots.] How many do I go to though? [D makes the fifth row.]
T: Okay, you can do this dot, dot, dot thing [T is referring to making "…" which he has showed D how to do in a previous problem.], and that is going to mean that it is going to go on to some number that we don't know. So, why don't you make a last row down here somewhere? [D draws a line in Figure 6.12.] How many dots are in this row?
D: I don't know.
T: You don't know. So, what should we call it?

D: n [D writes "n" in Figure 6.12.]
T: Yeah. Can you figure out a multiplication problem for that array you just made? [T points to the triangle.]
D: *n* times (*n* -1) divided by two.
….
T: What would *n* times *n* -1 be?
D: It would be this whole triangle and another one. [After some discussion of how to use notation, D produces Figure 6.13 and smiles.] [53]

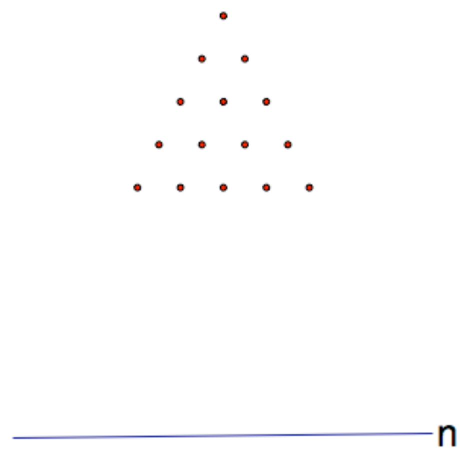

Figure 6.12: A replica of Deborah's diagram for *n* people

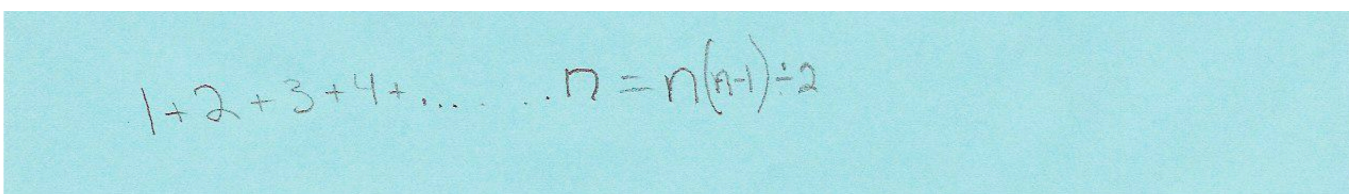

Figure 6.13: Deborah's algebraic equivalence

As in the problem with 6 people, Deborah appeared to take her multiplicative notation as symbolizing two copies of the sum and her division by two as creating the triangle she produced in Figure 6.12. She, then, produced Figure 6.13 where I infer that she set the two equal to one

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[53] Asking this question was a mistake in a certain sense because to this point Deborah had worked on finding sums that involved one less than the number of people in the statement of the problem. So, I should have asked her to symbolize the sum of $(n - 1)$ dots rather than *n*. When she produced Figure 6.13, she did not take this into account in her notation and so produced a formula that was incorrect. It should have been $(n + 1)n \div 2$. However, this was the first time she made this conflation. So, I infer it was not a conflation that was related to something that she did not understand about the problem situations. Rather, she based the right hand side of her algebraic notation on the pattern she had already established.

another because of the reversibility she had established between the two. That is, she could operate from the multiplicative formula to find the sum of the first *n* whole numbers (as she had in the Handshake Problem) and she could operate with the sum to produce the multiplication problem. So, the sum was no longer just the result of her multiplication and division. Rather, she took the multiplication problem as containing two sums.

Deborah's Notation. Deborah's comments, "Keep going? To how many though?", while producing the triangular figure for the sum suggest that producing the sum in this manner gave her a sense of indefiniteness in her experience. I infer that this indefiniteness arose from my request for her to produce a quantity that did not have a specified ending amount. Furthermore, when I asked her how many dots would be in the final row, she responded by saying "I don't know", but then was easily able to establish that she should label the final row *n*. Having Deborah engage in this activity, prior to notating the sum with algebraic notation, seemed quite useful because it gave her this experience of indefiniteness, which she then might attribute to the notation that she produced. So, the letter *n* in the left hand side of Figure 6.13 and the triangle could symbolize any but no particular sum because it was not clear when she would stop producing dots. Her ability to use notation in this way is strong corroboration that she had abstracted an additive concept of the sum from the One Deck Card Problem. The symbol, then, I assume symbolized the operations she had associated with the activity of additively producing the sum—a definite beginning and definite ending where each subsequent position had one more than the previous one. The right hand side symbolized her multiplicative concept of the sum and I analyzed her use of notation for this in the earlier section. So, she used her algebraic notation to symbolize the relationship she had developed between her additive and multiplicative reasoning across these quantitative situations.

Deborah's Recursive Scheme of Operations

Overview

This section of the case study investigates how Deborah began to produce a recursive lexicographic units coordinating scheme. I presented problems aimed at opening this possibility at the end of the third episode and then throughout the duration of the fourth and fifth episodes. Deborah's solution of the Extension of the Outfits Problem suggested to me that operating with three composite units might be in her zone of potential construction. However, I had presented the problem so that she only had to independently consider two composite units at a time. That is, in the initial Outfits Problem, she composed three shirts and four pants to make twelve outfits and then in the extension of the Outfits Problem she composed the twelve outfits with two shoes.[54] So, she did not independently produce three (or more) composite units where she needed to independently structure the situation by isolating two of them, composing them, and then composing the results of her first composition with a third composite unit.[55] So, in the first part of this section, I analyze Deborah's activity in three problem situations where Deborah had to independently structure the situation recursively. In my investigation of Deborah's activity in these three problems, I analyze functional accommodations Deborah made in her way of operating. In the second part of this section, I analyze Deborah's progress towards a recursive scheme of operations.

Before I begin analyzing the problem situations, I do want to give the reader an overview of several problems that I posed to Deborah that were unsuccessful. I posed problems that involved large numbers (e.g., Suppose you wanted to create a three letter password for a

[54] Mathematical notation to indicate this way of operating might be $3 \times 4 = 12$ and $12 \times 2 = 24$.
[55] Mathematical notation to indicate this way of operating might be $(3*4) * 2 = 24$ where a student has structured the problem in such a way that she chooses an order in which to compose the composite units.

computer. How many different possible passwords could you make?) or more complicated problems that involved ordering (e.g., Suppose you want to arrange five students in a row. How many different ways could you arrange them?). Deborah was able to make a conjecture that multiplication was involved in these problems, but they proved difficult for her. Often, she turned to numeric calculation that was sometimes correct and sometimes incorrect. In either case, the problems were too complicated for her to become aware of her way of operating. So, in these situations, Deborah used her multiplicative notation to calculate results but seemed unable to provide an explanation for these results that were rooted in generative mathematical activity.

Deborah's Solution of Three Problems

The School Supplies Problem. The witness-researcher noticed the difficulties I was having communicating with Deborah and intervened with the School Supplies Problem, Task 6.8.

Task 6.8, *The School Supplies Problem*: You have two pencils, three notebooks, and two book bags. How many pencil, notebook, book bag combinations could you have?

Protocol 6.11: Deborah's solution of the School Supplies Problem on 11/02/05

D: [D writes Figure 6.14.] You could have "1AY", "1A" [D pauses for approximately three seconds looking at her notation depicted in Figure 6.14.] "X", "1AZ". [D writes each as she says them.] You could have "1BX", "1BY", "1BZ". Then you could have "2AX", "2AY", "2AZ", "2BX", "2BY", "2BZ". [D counts the list by ones.] Twelve.
T: Twelve. Could you figure out a multiplication problem to figure that out?
D: Two times two times three.

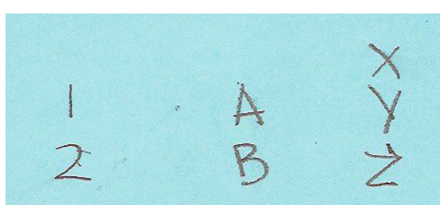

Figure 6.14: Deborah's notation

I infer from her use of notation that she assimilated the problem situation using her concept of two, two, and three, which contained a way to order the units within each of these composite units. She used her notation to make this ordering where "1" symbolized the first pencil, "2" the second pencil, "A" the first book bag, "B" the second book bag, "X" the first notebook, "Y" the second notebook, and "Z" the third notebook. Unlike the problems that involved only two composite units, Deborah did not prior to operating give indication that she was aware of a multiplication problem that would give her the total number of pencil, notebook, book bag combinations. Rather, she appeared to produce notation as a way to monitor the units coordinating activity she seemed to anticipate engaging in, but she seemed to need to engage in this activity to figure out what the results would be. Deborah's use of notation was interesting. It was interesting because she was able to anticipate that using her notation to order the pencils, book bags, and notebooks would be useful to her. So, using notation seemed to be something that Deborah had associated with helping her to monitor her activity in a situation where she could not immediately assimilate the situation to some kind of algorithmic calculation (i.e., carrying out her algorithm for multiplication).

She, then, imagined putting the first pencil with the first book bag, and then putting the results with the second notebook. She symbolized this combination with "1AY". I refer to the results of her coordinating activity a triple, a unit that contained one unit from each composite unit in the situation. When she began producing the second triple, she put the first pencil with the first book bag, symbolizing this activity with "1A" and paused. I infer she paused in order to figure out a way to use her notation to monitor which of the triples she had produced. For the first triple she produced, the spatial configuration of her notation appeared to play a role in her choice of which triple to make (the letters and number were all in a row). However, when she made the second triple she needed to find a way to better monitor her activity. Based on the way she produced the subsequent triples, I infer that she used a lexicographic ordering to help her monitor the way in which she produced triples. Here, Deborah did not curtail her activity while producing each of these triples so it appeared that she needed to use her ordering concept to monitor her activity.

This way of operating was similar to how Michael operated at the beginning of the first episode involving two composite units where he maintained the *differentiation* (not ordering because Michael had yet to abstract an ordering concept) between each of shirts and pants in order to operate when solving the Outfits Problem. Based on this way of operating, I considered Michael's activity to be pre-multiplicative. For the same reasons, I consider Deborah's activity in this situation to be pre-multiplicative. Nonetheless, this problem situation seemed to be novel for Deborah and appeared to occasion her construction of a recursive scheme of operations. The scheme that became a recursive scheme of operations was her lexicographic units coordinating scheme. I make this assertion because the activity she produced involved taking the results of her units coordinating activity (e.g., a pair symbolized by "1A") and using these results in further units coordinating activity to produce triples. Deborah had done this during the Extension of the Outfits Problem, but this was the first time that she independently structured a situation in this way.

I wanted to find out if Deborah could find a multiplication problem for the situation after producing her results. She was immediately aware that twelve could be produced from the multiplication problem 2 x 2 x 3. Here, I infer that Deborah's statement of this multiplication problem was primarily conjectural. By conjectural, I mean that she had maintained an awareness of all three composite units in the situation and could easily figure that the results of multiplying these three numbers gave her a number that was consistent with the number she had produced from her units coordinating activity. However, I was not convinced that it was coordinated with her activity in the sense that she did not seem to initially structure her activity in a way that was multiplicative.

To find out more about how Deborah structured the twelve triples and coordinated it with her multiplication problem, I present the following continuation of Protocol 6.12.

> Continuation of Protocol 6.11: Deborah's solution of the School Supplies Problem on 11/02/05
>
> T: Yeah, how did you get that multiplication problem?
> D: Because there is two pencils, two book bags, and three notebooks.
> T: Can you coordinate that with your list down here?
> D: What do you mean?
> T: Well, you got...[D begins to mark on her list of notation.] Yeah, go ahead. What were you going to do?
> D: [D draws circles around her notation, making four groups of three (Figure 6.15). "1AX", "1AY", "1AZ" is the first group, "1BX", "1BY", "1BZ" is the second group, etc. D pauses and changes her grouping strategy.] You have this group of numbers and this group of numbers so, that's two times…[D makes two circles around her notation. One circle contains all the combinations that begin with "1" and the other all the combinations that begin with "2".] that there is two groups of letters. There is this group and that group. [D groups all of the combinations in her list that have A as the middle letter and then all of the combinations in her list that have "B" as the middle letter.] And then there are three groups of these letters. [D points to X.]
> T: Mmhmm (Yeah.)
> D: So it's like that's a group [D circles all of the combinations with "X" in the third position.], that's a group [D circles all of the combinations with "Y" in the third position.], and that's a group. [D circles all of the combinations with "Z" in the third position.]

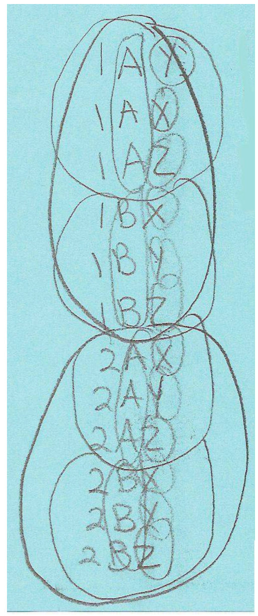

Figure 6.15: Deborah circles parts of her notation

Deborah began by circling four groups of three triples, where each group contained all triples that contained the same pencil book bag combination, which yielded four units of three triples. Deborah seemed aware that this accounted for multiplying four times three in the situation but I infer she was not satisfied with this result because it did not account for multiplying by two twice. I make this inference based on her pausing and seeming to change her activity. I infer that she reestablished the twelve triples and tried once again to account for the multiplication problem 2 x 2 x 3. When she produced two “groups of numbers” by circling all the triples that had “1” and all the triples that had “2”, it appeared that she was decomposing twelve into two units of six triples. She, then, made two more “groups”, where each “group” contained one of the two book bags (either A or B). Here, she appeared to re-establish the 12 triples and decomposed them into two new units of six triples. When she circled each of the letters, she once again appeared to establish the 12 triples and decompose it into three units of four triples. I make the inference that she re-established the 12 triples each time because each time she appeared to make her "groupings" according to all of the triples she had recorded with her notation. So, she did not appear to account for her multiplication problem 2 x 2 x 3.

At the time, I had an intuitive sense that Deborah was not wholly clear on why she had multiplied. So, I asked her if she could make a tree diagram, thinking that such a diagram might be useful in her structuring her activity as multiplicative.

Protocol 6.12: Deborah makes a tree diagram for the School Supplies Problem on 11/02/05

T: [D is unsure what a tree diagram is. So, T writes "1" and two branches coming off of it.] What letters could be paired with one?
D: A or B.
T: Yeah. So, I wonder if you could finish the diagram.
D: [D begins to produce Figure 6.16 by writing "1". She then draws two lines emanating from it and writes in "A" and "B". She pauses momentarily and draws three lines emanating from each of the letters "A" and "B" and writes "X", "Y", and "Z".] Like that? [D appears to consider herself finished with the diagram.]
T: Yeah. And then, what would you do with…
D: [D continues by writing a "2" and two lines emanating from it. She writes "A" and "B". She, then, writes three lines emanating from each of these letters and finishes her diagram.]

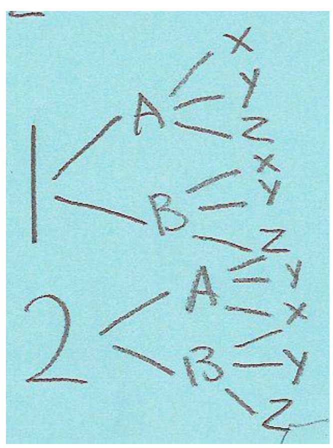

Figure 6.16: Deborah's tree diagram.

In making the tree diagram, Deborah appeared to imagine putting each of the book bags with the first pencil. I make this inference based on her producing the "1A" and "1B" branches of the tree diagram. She, then, appeared to imagine putting the first pencil book bag combination, symbolized by "1A", with a representative notebook. This activity symbolized producing three pencil book bag notebook combinations. Next, she appeared to imagine putting the second pencil book bag combination, symbolized by “1B”, with a representative notebook. This activity also symbolized producing three pencil book bag notebook combinations. I make the inference that she was imagining a representative notebook based on her producing three lines emanating from “1A” and “1B” prior to labeling the lines with the letters “X”, “Y”, and “Z”. Operating in this way was a functional accommodation in her recursive scheme of operations because she no longer made each triple in sequence. Rather, she could take putting a representative notebook with each pair as symbolizing producing three triples. This made the activity she engaged in multiplicative because she iterated her units coordinating operation twice to produce two representative triples each of which symbolized three triples.

In part, the tree diagram appeared to help Deborah produce a new way of operating because she became aware of engaging in two distinct “rounds” of units coordinating activity. During the first “round” of her units coordinating activity, she made two pairs symbolized by “1A” and “1B”. During the second round of her units coordinating activity, she produced six triples using the two pairs she had just produced. I make this inference about “rounds” based on her briefly pausing between the first and second “round” of her units coordinating activity while she examined the results she had produced after the first “round”. So, she appeared to become aware of using a pair as input for further units coordinating activity.

Once she had completed the first part of the diagram, Deborah appeared to produce the second half anticipating that it would be identical to the first half. In fact, in making the second half of the diagram, she used several of the letters “X”, “Y”, and “Z” in two ways (i.e she took the “Z” from “1BZ” and used it for “2AZ”, see Figure 6.16). This way of operating was interesting because it was not dissimilar from how Carlos operated in the context of two composite units. That is, Carlos appeared to imagine taking a particular element (e.g., the first card from the first deck of cards) from the first composite unit and making a units coordination with this unit and representative unit that symbolized all units in the second composite unit (e.g., all of the cards in the second deck of cards). He, then, could imagine doing this with each of the units in the first composite unit (e.g., each card in the first deck). Here, Deborah appeared to do a similar thing except with three composite units. That is, she took the first pencil from the first composite unit and made pairs with it. She used these two pairs as input for making triples. When she completed this activity, she could imagine engaging in this activity with each of the units in the first composite unit (i.e the pencil she had symbolized with "2"). So, she iterated the six triples to make twelve triples.

The Picnic Problem. Deborah said very little while producing the tree diagram in the School Supply Problem. So, much of my analysis is based on observing how she operated. In order to find out more about Deborah's ways of operating, I present her activity in the Picnic Problem, Task 6.9, which I presented her after the School Supplies Problem.

> Task 6.9, *The Picnic Problem*: You have four meats, five potato salads, and six desserts to choose from at a picnic. You can pick one of each to make a meal. How many possible meals could you make?
>
> Protocol 6.13: Deborah's solution to the Picnic Problem on 11/02/05
>
> D: [D writes "4 x 5 x 6" vertically and calculates 4 x 5 as 24[56] and takes 24 times 6.] One hundred and forty four.
> T: Could you make a diagram for that one?
> D: What do you mean?
> T: Like a tree diagram.
> D: Oh, I see. Meat, there is four meats right?
> T: Yeah.
> D: One, two, three, and four. [D writes "1", "2", "3", and "4". She draws five lines emanating from each (Figure 6.17).] Five. And, then, you could have what was the…the

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[56] I take this miscalculation as a non-important calculation error. So, throughout I will refer to her producing 20 combinations instead of 24.

potato salad. A, B, C, D, E, A, B, C, D, E, A, B, C, D, E, A, B, C, D, E [D writes each of these letters as she says them.] And then you would have six lines coming off of each of these. Do I have to do it?
T: Just make a couple.
D: [D draws six lines from the "1A" branch of the diagram and writes the letters "U", "V", "W", "X", "Y", and "Z".]

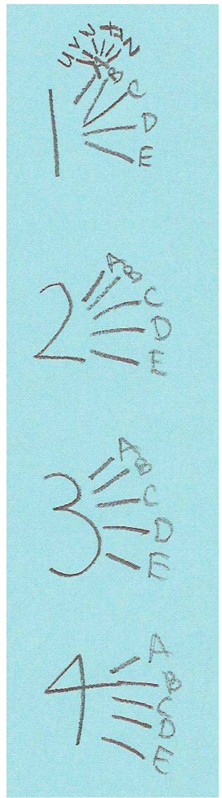

Figure 6.17: Deborah's tree diagram for the Picnic Problem

Deborah immediately assimilated this task to her multiplying schemes and figured the answer would be 4 x 5 x 6, performing the multiplication to find the results. This use of notation was interesting because she used it to symbolize activity that she anticipated. However, it appeared in part that this anticipation was "social" anticipation. That is, Deborah liked to know the results of her activity immediately and often was able to pattern what type of problems I presented her. So, she frequently calculated numerical results in situations based on patterning from the previous situation. To find out how she would operate, I asked her to produce a tree diagram.

Based on Deborah labeling the four meats “1”, “2”, “3”, “4”, I infer that Deborah anticipated using all four of the units from the first composite unit (the meats). This appeared to differ from her solution to the School Supplies Problem where she appeared to only anticipate using the first unit from the first composite unit. She, then, appeared to imagine putting a representative potato salad with a representative meat to produce a representative meat potato salad combination. This representative meat potato salad combination symbolized all twenty meat potato salad combinations she could produce. I make this inference based on her subsequently considering each of the meat potato salad combinations to function identically, which she did when she engaged in her subsequent units coordinating activity. This way of operating suggests that Deborah used her pair concept and inserted the four meats into the first slot and the five potato salads into the second slot and then took the results this produced as material on which to operate further.

In her subsequent units coordinating activity, she appeared to take a representative meat potato salad combination, which symbolized 20 meat potato salad combinations, and imagined putting it with a representative dessert, which symbolized six desserts [“And then you would have six lines coming off of each of these”]. She took her language as sufficient to symbolize what she would do for all of the rest of the meat potato salad combinations. So, it appeared that she treated all of the meat potato salad combinations identically. I also infer that she treated all of the desserts as if they were identical because she could simply refer to six lines coming off of the meat potato salad combination as indication of producing six triples without having to actually produce these triples. So, she produced the twenty pairs and then took one of these pairs as symbolizing all of the pairs and made a units coordination between this representative pair and a representative unit from the third quantity, which symbolized all of the triples that she could produce.

This way of operating was a second functional accommodation of her recursive scheme of operations because she produced *all* of the pairs first by making a units coordination between a representative meat and a representative potato salad, which symbolized 20 meat potato salad combinations. Here, she used her lexicographic units coordinating scheme similarly to how she had used it in the Two Suits Card Problem. To do so, she temporarily had to ignore the third composite unit and only focus on producing the results with the first two composite units. She, then, took the results of her activity, a representative meat potato salad combination, and used it as input for a second use of her lexicographic units coordinating scheme. The two composite units she used were a representative meat potato salad combination and a representative dessert. She appeared to take a single units coordination to symbolize all 120 triples she could produce, but did not actually need to produce. This way of operating suggests that the multiplicative notation, "4 x 5 x 6" symbolized her anticipation of engaging in these operations.

The Candy Problem. In both the School Supplies Problem and the Picnic Problem, the three composite units had different referents in the statement of the problem. To investigate how Deborah operated when the statement of the problem did not differentiate between three composite units, I present her activity in the Candy Problem, Task 6.10. To begin the Candy Problem, I had Deborah put a red, orange, yellow, and pink candy into each of three bags. She, then, drew a red candy out of the first bag, an orange candy out of the second bag, and an orange candy out of the third bag, recording this on her paper as "ROO". Once she had done this once, I presented Deborah with Task 6.10.

Task 6.10, *The Candy Problem*: How many possible color combinations could you get?

Protocol 6.14: Deborah's solution to the Candy Problem on 11/07/05

D: Sixteen.
T: How did you get that?
D: Four times four. I mean no twelve.
T: Twelve? How did you get twelve?
D: Four times three bags.
T: Why don't you go ahead and make another combination?
D: Red. [D pulls out a red candy from the first bag.] Orange. [D pulls out an orange candy from the second bag.]
T: Uh oh. Is it going to be the same one?
D: [D smiles.] Yellow. [D pulls out a yellow candy from the third bag. She writes "ROY" below where she has written "ROO"]
T: How many could you get if you had a red and an orange here [T points to where Deborah has recorded "RO" for the first two combinations that she pulled out of the bag.]
D: Two.
T: Two?
D: Wait no three, three, three. No. Yes, three.
T: What possible one's could you get?
D: Wait no four.
T: Four?
D: Yeah, you could get red orange orange, red orange yellow, red orange red, red orange pink. So, it would be four times four. It would be sixteen. Wait, it would be more than that. It would be like four times four times four.
T: Oh. Why is it that?
D: Cause there is four combinations in each bag and for each letter there is like four combinations and another four combinations. Cause there is three bags. [D calculates this multiplication problem at T's request.]
T: Could you represent all sixty-four using a diagram?
D: [D produces Figure 6.18.] And there would be four of those.

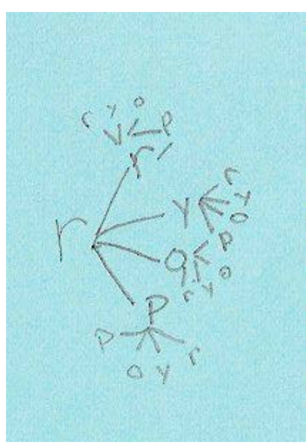

Figure 6.18: Deborah's tree diagram

Deborah's lack of certainty and her appearing to guess various multiplication problems suggested that she was unsure of what the results would be. So, making just one color combination did not seem to be enough to elicit Deborah's units coordinating activity that she had produced during the previous episode.[57] After making a second color combination, her response that she could have two combinations with red and orange suggested that she produced only one composite unit and that she eliminated red and orange from this composite unit and figured that there would be two colors left over to go with the red orange combination (yellow or pink). However, she appeared to review the two combinations she had already made and included the red orange orange combination in her count and so concluded that there would be three possible color combinations.

The red orange orange combination appeared to create a perturbation for Deborah because it suggested that she could have combinations where two of the three flavors were the same. So, I infer that Deborah produced a second composite unit of four. She, then, imagined putting each of the elements of this unit of four with the red orange combination ["Yeah you could get red orange orange, red orange yellow, red orange red, red orange pink"]. This activity seemed to elicit Deborah's units coordinating activity, which appeared to give her insight into further operating. I make this inference because she said, "so, it would be four times four".

This comment along with her further operating suggests that she imagined the first candy as always being red, the second candy could be any of four colors, and the third candy could be any of four colors, which yielded her response "four times four". So, to get this result Deborah

[57] One of the largest challenges for me as a teacher-researcher with Deborah was to engage and challenge her. She operated very quickly and was quite good at patterning both her activity and mine. So, it was often difficult for me to figure out when she was bored, when she was challenged, and how much activity I should expect her to produce prior to asking her to solve a problem.

imagined putting a particular unit, the red candy, with a representative unit from the first unit of four. This produced a representative pair, which symbolized four pairs. She, then, appeared to put the representative pair with a representative unit from the second unit of four, which produced a representative triple that symbolized sixteen triples.

In producing this solution, however, Deborah had introduced a singleton unit (the red candy). I infer that Deborah maintained an awareness of this third unit because she once again changed her answer to “four times four times four”. She gave the subsequent explanation, “for each letter there is like four combinations and another four combinations.” This explanation along with Figure 6.18 suggests that Deborah produced sixteen triples and then iterated these sixteen triples four times to produce the sixty-four triples. This way of operating was consistent with her way of operating in the School Supplies problem where I suggested she iterated six triples twice to produce twelve. I infer that she produced this way of operating again because she did not prior to operating produce three composite units. So, she could not have operated in the way she did in the Picnic Problem.

Producing three composite units in a situation that did not explicitly refer to the composite units in the statement of the problem was a modification in her way of operating. However, throughout the rest of the fifth episode Deborah only produced this way of operating with my interventions. So, the modification she made in this problem situation did not seem to modify the way she assimilated these situations, which means that I do not consider it a functional accommodation in her way of operating.

Assessing Deborah’s Progress Towards a Recursive Scheme

During these problem situations, Deborah had constructed a recursive scheme of operations where she could take the results of one application of her lexicographic units coordinating scheme as input for further use in a second application of her lexicographic units coordinating scheme. However, Deborah's activity in the Candy Problem suggests that she did not necessarily assimilate situations that did not contain reference to three composite units to one that involved three composite units.

This way of operating was a persistent constraint during the fifth teaching episode where Deborah assimilated problems similar to the Candy Problem with only one composite unit. She could produce three composite units if I made suggestions to her, but she did not independently do so. Moreover, it suggests that she was yet to abstract a triple concept—a unit that contains three blank slots and which could be filled with the number of possible choice for each slot—from her activity. So, her solution of the Candy Problem did not seem to mark a functional accommodation. One difficulty that occurred during the fifth episode was that Deborah would guess a multiplication problem without actually producing any units coordinating activity. So, I often questioned her result or made suggestions to help her produce some activity. This dynamic was not helpful for Deborah in patterning her own activity because Deborah did not necessarily generatively produce this activity.

Nonetheless, I would say Deborah's way of operating in the School Supplies, Picnic, and Candy Problem is sufficient indication that she had started making progress towards constructing a recursive scheme of operations. I make this inference because in each of these three situations she could take her units coordinating activity as input for further operating. One interesting aspect of these problems was that it appeared that in the School Supplies and Candy Problem Deborah produced a way of operating that was similar to how Carlos operated with two composite units. That is, she took a particular unit from one of the three composite units and made a units coordination between it and the second composite unit and then took the results of this way of operating and made a units coordination with the third composite unit. She, then, iterated the results a certain number of times, depending on how many units were in the first composite unit. This way of operating was similar to how Carlos operated with two composite units. It was only in the Picnic Problem that Deborah seemed to use her lexicographic units coordinating scheme twice in a way that was consistent with how she used it when she was operating in the context of only two composite units.

CHAPTER 7: SUMMARIES, CONCLUSIONS, AND DIRECTIONS FOR FUTURE RESEARCH

This final chapter is presented in three sections. The first section presents what I learned in relation to my first two research questions, which I present again here:

1) What aspects of multiplicative and quantitative ways of operating are symbolized and what changes do students make in their multiplicative and quantitative ways of operating in interaction with a teacher-researcher?
2) What mental imagery and operations do students produce in the context of solving quantitative problem situations?

In this section, I compare and contrast the students' reasoning. In doing so, I present the operations, schemes, and concepts I have analyzed. In the second section, I point to ways in which I consider the students' activity to be algebraic. As part of this section, I suggest five ways in which the students used their symbolizing activity to suggest the ways in which this activity might be considered part of constructing an algebraic symbol system. This part of the second section specifically seeks to address my third and fourth research questions, which I also present here:

3) How does students' symbolizing activity function in the process of constructing an algebraic symbol system?
4) What conventions do students produce in the context of their symbolizing activity and what role does social interaction play in this process?

In the final section, I begin to connect this research to a broader point of reference. I make this connection in two ways: by reconnecting with the research literature and by suggesting future directions for research.

The Students' Operations, Schemes, and Concepts

I present this section in three parts. The first part presents a summary of the concepts involved in constructing a representative unit and suggests why I consider the students' use of such a unit to be algebraic. The second part presents a summary of each of the three students operating across the problem contexts that I presented and highlights important features of their operating. In doing so, I provide an account of the construction of a pair concept and the construction of the sum as a quantitative entity. In the third part, I present five different levels of operating that I abstracted from the students' activity in these situations.

The Students' Construct Representative Units

In the case studies, I argued that all three students constructed what I termed a representative unit—a unit that conjoins ordering and amount meanings of number. Carlos and Deborah seemed to have constructed representative units prior to entering the teaching experiment and Michael constructed them in the course of his activity during the first two episodes. The students constructed representative units in the context of working on problems like the Outfits Problem.

> Task, 5.1*: The Outfits Problem:* You have three shirts and four pairs of pants. How many outfits can you make?

I judged that a student had constructed a representative unit when he could imagine taking a *representative* shirt and putting it with a *particular* pair of pants and have this mental action symbolize the three outfits he could make with this particular pair of pants and any but no particular shirt.

Carlos’s activity to solve the Outfits Problem is useful in highlighting the concepts I suggested were involved in constructing a representative unit. He solved this problem by assigning a color to each shirt and a color to each pair of pants. He, then, sequentially put each particular shirt with each particular pair of pants using a lexicographic ordering to keep track of which outfits he produced (I called this scheme his lexicographic units coordinating scheme). In this solution, he did not use a representative unit rather he imagined making each outfit in sequence. However, when he produced a tree diagram to symbolize his solution, he made a functional accommodation in his operating. He took the blue shirt and imagined putting it with a particular pair of pants and this mental action symbolized all of the outfits he could produce with the particular pair of pants and any but no particular shirt. Here, he treated the blue shirt as representative of how all the shirts would function.

I suggested two concepts were involved in constructing a representative unit—an iterable unit of one and an ordering concept. A student’s unit of one is iterable when he can assimilate a problem with a unit of one that symbolizes a composite unit (Steffe, 1992). For example, in the Outfits Problem, Carlos assimilated the situation by drawing one shirt and took this shirt to symbolize the three shirts. When a student can operate in this way, he does not have to actually iterate a unit of one to produce meaning for a composite unit. So, Carlos did not actually have to iterate his shirt concept three times to establish meaning for the words “three shirts”. When a students’ unit of one is iterable and he iterates the shirt in activity to produce a composite unit, Steffe has called the composite unit a unit of units. This concept is what gives the *amount* meaning to a representative unit.

The second concept involved in constructing a representative unit is an ordering concept. Based on the students’ activity, I took as indication that they had abstracted an ordering concept when they could identify a property of an object, such as color, to differentiate the units within their number concepts, but they no longer had to carry out this differentiation. So, in the Outfits Problem, Carlos carried out his differentiation of each shirts and each pair of pants, assigning a letter to each unit in his concept of three and each unit in his concept of four to symbolize the color that he imagined each shirt and pair of pants to be. In this situation, he actually carried out the differentiation in activity. However, in future problem situations, he was able to simply identify a property of an object concept like color without actually having to assign a particular color to each of the objects. So, he could take a property of his concept as establishing an ordering without having to carry out this ordering.

In addition, I suggested the differentiation could not be based on the qualitative property that a student introduced, but rather this qualitative property was used to establish a *position* for each of the units in the students' number concepts. So, in Carlos's solution of the Outfits Problem, the blue shirt he took to be the first shirt, the white shirt he took to be the second shirt, etc. Given that researchers ((e.g., Booth, 1988; Clement, 1982; Kieran, 1989) have found that students frequently use letters to symbolize *only* a qualitative property of an object (e.g., the letter "B" symbolizes the color blue), it is significant that all three students in my study used letters (and other notation) to symbolize the mental operations they used to produce amount and ordering meanings for their number concepts (i.e., quantitative properties).

I infer producing a representative unit involves some level of awareness that the unit is representative. I make this inference because a student *introduces* a qualitative property to differentiate the units, and only later, supposes that the qualitative property has no bearing on the situation. This way of operating seems distinctly algebraic because it involves some level of awareness of using a unit symbolically (i.e., the blue shirt symbolizes all of the other shirts). That is to say, one has to have some level of awareness that it is not the particular shirt or pair of pants that is of interest, but the operations that are used when operating with a particular shirt or pair of pants. This awareness was reflected in Carlos's statement in the Outfits Problem that "you could do that with each one (pair of pants)" after he sequentially put three shirts with the first pair of pants. Here, his statement referred to the operations of putting three shirts with each pair of pants. So, he took the first pair of pants as representative of the operations he would perform with the second, third, and fourth pair of pants. As I suggested above, all three students in my study were able to operate in this way, which, for the reasons I just suggested seems distinctly algebraic.

The Students' Pair Concepts, Sum Concepts, and Production of Triples

Deborah. Based on Deborah's activity in problems like the Outfits Problem, I abstracted a concept that I called Deborah's pair concept, which I suggested she had already constructed prior to entering the teaching experiment. She did make functional accommodations to this concept in the course of solving the problems I presented her. Deborah's solution of the Two Suit Card Problem, illustrates well the concept. In this situation, Deborah immediately

> Task 6.4*, Two Suits Card Problem,* Suppose you have all the hearts and Michael has all the clubs. How many different two-card combinations could you make?

assimilated the problem situation to one involving multiplication and calculated 13 times 13. To explain this calculation, she took a card from her deck and without looking at it asserted that she could put that with all thirteen of Michael's cards. Then, she took a second card and asserted she could do the same for this card. After this explanation, she halted her activity and said, "and so on and so forth" indicating she would continue to do this activity, but did not actually have to carry it out. In this solution, Deborah appeared to imagine putting a representative unit from her deck with a representative unit from Michael's deck two times and she took this activity to symbolize all the pairs she could make.

I argued that this way of operating had been abstracted from previous activity (not witnessed in this teaching experiment) of producing *particular* pairs with her lexicographic units coordinating scheme. The most basic form of the scheme that I observed was for a student to order the units within one composite unit (e.g., order the shirts) and order the units within a second composite (e.g., order the pants) and then sequentially put each unit from one composite unit with each unit from the other composite unit, keeping track of this activity with a lexicographic ordering. When a student operated in this way, I considered her to produce each particular pairs. In the Two Suits Card Problem, an example of a particular pair would be taking as a unit the two of hearts and the two of clubs. From this activity, Deborah seemed to have abstracted a unit that contained two slots, which I called her pair concept. The first slot in her pair concept contained a record of being any but no particular, for example, card from her hand and the second slot contained a record of being any but no particular card from Michael's hand.

This enabled her to take a representative unit that symbolized one composite unit and put it with a representative unit that symbolized a second composite unit, which symbolized all the pairs she could produce. For example, in the Two Suits Card Problem, she could take a representative card from her deck and imagine putting it with a representative card from Michael's deck and this symbolized all the two-card combinations she could make. One feature of Deborah's way of operating that I suggested provided indication that she had constructed this concept was that she did not identify any particular card from her deck or any particular card from Michael's deck when she operated. So, she appeared to operate with two representative cards—one for her deck of cards and one for Michael's deck of cards.

Furthermore, I argued that her pair concept was itself multiplicative because it had been abstracted from her multiplicative activity. Because the unit was multiplicative, she could simply identify the number of possible cards she had and the number of possible cards Michael had and multiply these together without actually having to produce any activity. This characteristic is why I called it a *concept*—Deborah had abstracted a program of operations that she could implement to explain why she multiplied, but she could also take this way of operating as a given in her activity without having to produce it.

Deborah made functional accommodations to her pair concept that I detailed in the case studies. Here, I focus on the two most important functional accommodations that she made to this concept. The first functional accommodation she made was in the context of solving the One Deck Card Problem. Deborah was initially unable to solve this problem.

> Task 6.1*, The One Deck Card Problem:* How many two-card hands could you make with a deck of fifty-two-cards?

She did solve the problem after she solved the Outfits Problem, which seemed to help her conceptualize the One Deck Card Problem. She assimilated the situation with one representative unit that symbolized the 52 cards. She imagined drawing a card from the deck, the two of diamonds, and putting it with a representative unit that symbolized the remaining cards. She took this activity to produce 51 two-card combinations. I described this activity as disembedding a singleton unit (the two of diamonds), and making a units coordination with it and a representative unit. She engaged in this activity for four cards in the deck (the two, three, four, and five of diamonds) where each time she re-established the deck of 52 cards. That is, she imagined drawing the three of diamonds and having 51 remaining cards with which to make pairs. She, however, eliminated the three of diamonds and the two of diamonds because she had made this two-card combination the first time. After four cards, she established that each successive card she used would make one fewer two-card combinations than the previous card because of the eliminations she made.

After she established this relationship, she stopped disembedding particular cards and produced a representative unit that symbolized all possible cards that she could disembed from the deck (i.e., each card could be disembedded from the deck and used in this way). I made this inference because she began notating her activity with the sum 51 + 50 + … + 1 and no longer appeared to imagine disembedding each particular card. She coordinated the first thirteen addends with a particular card in the deck (the two of diamonds with 51, etc.), but then she simply seemed to take this coordination as something she could do but did not actually have to carry out.

This functional accommodation was important for two reasons—Deborah produced the sum as a quantitative entity and she used one composite unit to make pairs. In the case studies, I argued that the result of her activity was additively producing the sum as a quantitative entity. I suggested that the minimal requirements for establishing the sum as a quantitative entity were to establish it has having a definite beginning and ending, to conceive of each addend as having a position and amount, and to establish that each addend was related to the previous addend through a one less than relationship (e.g., 50 was the second addend and was one less than 51). In this situation, Deborah appeared to establish the sum as a concept because she appeared to abstract a program of operations that she could use to produce the sum without having to actually produce these operations. This enabled her to notate the sum as, "51 + 50 + … + 1", after carrying out only some of the operations to produce the sum.

The second reason I argued this functional accommodation was significant was because Deborah used one composite unit to produce singleton units, which part way through her activity she constituted as a second composite unit with which to make pairs. This way of operating meant that Deborah could use two composite units that had the same referent (i.e., both referred to cards) to make pairs. She differentiated between the two composite units based on how she produced and used each—the first she took as given in the situation and the second she produced from her disembedding activity. Each unit within the second composite unit was a card that she imagined putting will all of the remaining cards. The first composite unit was a deck of cards that decreased by one every time because of the eliminations she made. This way of operating was an interiorization of her activity of making pairs because she could generate two composite units as a result of her activity and did not need to have the two composite units differentiated in the statement of the problem (e.g., a pair of pants and a shirt).

The second significant functional accommodation that Deborah made was when she solved the Handshake Problem. Deborah assimilated this problem with one representative unit

> Task 6.6, *The Handshake Problem*: Suppose there are ten people in this room. Each person would like to shake every other person's hand. How many total handshakes would there be?

that symbolized the ten people. She notated these ten people with the letters "A", "B", "C", etc. appearing to imagine that these letters symbolized the ten people each of whom was going to be a handshaker. Then she disembedded a second composite unit from the first, nine people who symbolized all of the people who could shake hands with person A, the handshakees (only nine people because she excluded self-handshakes). She disembedded a composite unit of nine four times and then recognized this as a situation where she could use her pair concept to solve the problem—ten possible people could be inserted into the first slot and nine possible people into the second slot, which produced ninety handshakes.

In this situation, Deborah's functional accommodation was to imagine that she was going to use all of the handshakers prior to operating and then to disembed a second composite unit that symbolized handshakees from the first composite unit. This differed from the One Deck Card Problem because she did not imagine taking a *particular* person and putting this particular person with the remaining people as she had imagined taking a particular card and putting it with the remaining cards. So, she anticipated that she would need two composite units prior to operating (even though the statement of the problem had only one composite unit). In addition, she appeared to introduce a qualitative differentiation between the handshakers and the handshakees. This functional accommodation was important because it opened the possibility that Deborah would produce the sum as a quantitative entity in the context of her multiplicative reasoning.

Subsequently, she operated on the results of her pair concept, which she notated with an array (See Figures 6.7a, 6.7b, 6.7c, 6.7d), to eliminate extra handshakes (she was not initially aware of producing extra handshakes, but became aware when the witness-researcher asked if she could produce an addition problem to solve the Handshake Problem). To eliminate the duplicate handshakes, she eliminated all of the extra handshakes in each *row* of her array. So, in person B's row, she eliminated the handshake with person A, in person C's row she eliminated the handshakes with person A and B, etc. She recognized the results of her eliminating activity as producing the sum as a quantitative entity. She also recognized that her eliminating activity produced half of the original array only after she was done making her eliminations. When I questioned her as to why her activity produced *half* of the array, she was not able to answer in terms of the quantitative problem situation. So, she related her eliminations on the rows back to the quantitative problem situation, but when she observed that all of the eliminations produced half of the array she was unable to provide an explanation that was based in the quantitative problem situation.

I took this as indication that she had yet to construct an *ordered* pair concept, which meant that the two slots in her pair concept were not ordered. An ordered pair seems to be central for the construction of permutations of two elements and combinations of two element subsets. Although Deborah did not construct an ordered pair, Deborah's operating provides good insight into how a student would construct an ordered pair concept. I conjecture a basis for constructing such a concept is to assimilate a situation with one composite unit with the anticipation of using this composite unit to produce a second composite unit. The important part of this way of operating is that the two composite units are considered to symbolize identical objects (e.g., people), and the person produces two composite units prior to operating (as opposed to a singleton unit and a composite unit). Deborah operated in both of these ways. A person then must be aware of *introducing* a qualitative differentiation between the two composite units (e.g. handshakers and handshakees) that is either considered as important to counting the results or not important to counting the results. In Deborah's case, she made a differentiation and then subsequently considered the differentiation not to be crucial when she counted the results (i.e., it was no different for person A to shake person B's hand than it was for person B to shake person A's hand).

I conjecture that a person has to make some record of how many possible ways she could count objects based on considering some kind of difference that she recognize as an ordering. Furthermore, a person has to abstract the process of making particular orderings into an ordering concept. When a person abstracts such a concept, I infer that she is aware that each handshake could be counted in two ways based on having considered each person both a handshaker and a handshakee. In this regard, the Handshake Problem is not intuitive from the perspective of building up an ordering concept because a person's experiential concept of a handshake probably does not contain a record of ordering the people involved (as counting the number of ways of seating two people in a row might). Deborah appeared to be aware that she was counting each handshake twice based on her division by two, but she did not use an ordering concept to account for these eliminations.

Nonetheless, she could assimilate the results of partitioning the array into two parts to her concept of the sum (which she had abstracted in the One Deck Card Problem) and she recognized this way of operating as useful because she could use these operations to produce and *evaluate* the sum with a multiplication and division problem. Deborah worked on several problems similar to the Handshake Problem during which she abstracted the multiplicative operations she used to produce the sum as a quantitative entity into a concept. This program of operations was to take a composite unit, disembedd a second composite unit from it that was one less than the original composite unit[58], insert these two composite units into the two slots of her pair concept, and then divide this amount by two.

In the process of abstracting this concept, Deborah made reference to the array she produced in the Handshake Problem several times. So, her interiorized image of the array seemed to be a crucial part of her producing the sum with multiplicative operations. She, eventually, symbolized these operations with algebraic notation as $n(n - 1) \div 2$ where I argued that she could do so because she had produced an array as a symbol in the Handshake Problem.

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[58] In the quantitative problem situations, the reason the second composite unit was one less was because of the elimination of self-handshakes.

That is, the array could symbolize an unevaluated multiplication problem because the dimensions did not need to be specified in order for Deborah to consider it multiplicative. So, Deborah could use her use algebraic notation to symbolize this unevaluated multiplication and operate on it by partitioning the array, which she symbolized with her division by two.

Deborah did not initially set this equation equal to an addition problem. I argued that she did not because she had only used her multiplication and division problem to produce and evaluate the sum as a result of her operating, but had not used her additive concept of the sum to produce the multiplication problem. So, I had her use dots to symbolize her additive concept of the sum of the first *n* whole numbers and she used these dots to produce the array she symbolized with her multiplication problem. In doing so, she iterated the sum twice to make the array. This activity seemed to help her establish reversibility between a particular instance of her pair concept and her additive concept of the sum, which enabled her to notate the equivalence of the sum of the first *n* whole numbers to her multiplication and division problem. Here, I call this a *particular* instance of her pair concept because she could use her pair concept in situations where the two composite units were not necessarily one less than each other as they were in the situations when she produced the sum multiplicatively. So, although her algebraic notation symbolized any but no particular array where one dimension was one less than the other dimension, the dimensions of this array stood in a particular relationship and she could use her pair concept more generally (e.g., when one composite unit was two more than the other composite unit etc.).

To this point, I have summarized aspects of Deborah's pair and sum concept along with the operations she used to produce these concepts. I now want to examine the progress she made towards a recursive multiplicative concept with a generalized slot concept. I asserted that this concept could be abstracted from a student structuring a situation with a recursive use of her lexicographic units coordinating scheme. Deborah began work towards such a concept when she produced triples, a unit that contained an element from three composite units (e.g., a three card combination that contained the two of hearts, the three of spades, and the ten of clubs). Deborah's initial activity in these contexts highlighted the significance of her activity to use one composite unit with one referent to produce two composite units. I make this assertion because Deborah did not produce three composite units in situations that I presented to her until I posed the School Supplies problem that involved three composite units with different referents. She solved this problem by producing each triple in sequence and using her notation and a lexicographic ordering to keep track of which triples she had produced.

> Task 6.8, *The School Supplies Problem*: You have two pencils, three notebooks, and two book bags. How many pencil, notebook, book bag combinations could you have?

Subsequently, she produced a tree diagram. When she produced a tree diagram, she appeared to use her pair concept to produce all possible notebook book bag combinations and put her pair concept with a particular unit from the first composite unit. That is, she took the first pencil and put it with all six of the notebook book bag combinations. She, then, appeared to take this activity to symbolize the activity she would produce with the second pencil as well. So, she took a particular unit, the first pencil, and imagined putting it with her pair concept, the six notebook book bag combinations, and then imagined that the particular unit could be any but no particular of the pencils. Below, I will argue that this is similar to how Carlos operated with two composite units.

Deborah made progress towards constructing a recursive multiplicative concept by operating in several subsequent situations where the three composite units had the same referent (three bags of four candies), but she did not abstract a triple concept, a unit that contains three slots. Rather, she could produce triples with a recursive use of her lexicographic units coordinating scheme. I was surprised at how difficult this activity was given the facility with which Deborah operated with two composite units. Partially, this difficulty occurred because of my structuring of the tasks.

Michael. Having given a detailed summary of Deborah's way of operating, I use her as a basis for comparison with Michael and Carlos. Michael came into the teaching experiment having not already constructed a representative unit even though he had interiorized the two concepts, an iterable unit of one and an ordering concept, which I suggested were crucial for abstracting a representative unit. Therefore, these problem situations were novel for him. He spent the first two episodes interiorizing the coordination between these two concepts, which enabled him to construct a representative unit. Prior to interiorizing a representative unit, Michael experienced situations like the Outfits Problem as multiplicative only after he had operated. He experienced the situations this way because he sequentially put, for example, each shirt with each pair of pants. I argued that Michael operated in this way because he had to enact a conjoining of his ordering and amount meanings for number. So, in contrast to Deborah, Michael did not experience these situations immediately as multiplicative in nature. However, by the end of the second episode, Michael appeared to have interiorized the coordination between amount and ordering meanings for his number concepts and he used this coordination to construct a representative unit and assimilate these situations to his multiplying schemes.

Once he had interiorized a representative unit, he appeared to relatively quickly abstract a pair concept. I took this as corroboration that these situations were novel and argued that Michael had interiorized many of the operations involved in constructing a pair concept prior to operating in these situations. I suggested that Michael constructed a pair concept during his solution of the Flag Problem.

> Task 5.6, *The Flag Problem*: You are the President of a new country. You need to design a flag that has two stripes. You have 15 colors to choose from. How many possible flags could you make?

Michael was initially unsure of how to operate in this situation because he had the expectation that he would use two composite units and he assimilated the situation with only one. When he resolved this perturbation, his solution to this problem was similar to Deborah's solution to the One Deck Card Problem. That is, he assimilated the situation using one composite unit that symbolized the 15 colors, disembedded a singleton unit that symbolized a particular color, and imagined putting all 15 colors with this particular color. He engaged in this activity for a few particular units and then appeared to establish a second composite unit that symbolized all of the singleton units that he was going to disembed. So, as he said several times, he did not need to continue with his activity to know what the results of this activity would produce.

The primary difference between Michael's and Deborah's solution was the level of awareness each of them had of his or her eliminating activity. Michael did not consider the possibility of counting a flag that was, for example, orange-red after he produced a flag that was red-orange even though in this situation the statement of the problem did not rule it out. When Deborah made her eliminations, she seemed more aware that she could have counted, but did not count, the three of diamonds and the two of diamonds once she had already counted 51 two-card combinations with the two of diamonds. Deborah appeared to rule out counting both two-card combinations because she interpreted the statement of the problem in this way.

Michael's operating in the Flag Problem was important for both of the reasons I suggested Deborah's operating was important in the One Deck Card Problem—he abstracted the sum as a quantitative entity and he used one composite unit to produce a second composite unit. He produced the sum as a quantitative entity using similar operations to Deborah. As Michael said when he was finished solving the Flag Problem, “There is fifteen and it is like fifteen plus fourteen…” Here, he meant there are fifteen addends and the addends proceed in a one less than relationship. His initial production of the sum as a quantitative entity, as with Deborah’s, did not include any *quick* way to evaluate the sum. Also, in solving the Flag Problem, he used one composite unit with one referent, the colors, to produce a second composite unit. As with Deobrah, he did not initially produce the second composite unit in full. Rather, he eventually produced it after he had disembedded a few particular units and became aware that he would engage in these operations for each particular color. This way of operating suggested he constituted this disembedding activity as producing a second composite unit that symbolized all of the colors.

I argued that Michael’s activity provided indication that he had abstracted a *pair concept* because he appeared to produce two composite units, the first symbolized the remaining number of colors and the second symbolized the total number of units he was going to disembed. Moreover, he had abstracted the one less than relationship and so appeared to imagine putting a representative disembedded unit with a representative unit that symbolized the remaining colors, which symbolized all of the flags he could make. So, he appeared to imagine inserting any but no particular of the units he disembedded from the fifteen colors into the first slot and any but no particular of the units that remained into the second slot. This way of operating enabled Michael to provide a verbal description of the sum prior to his finishing notating all of the flags (he notated the flags as 1-1, 1-2 etc. where each number symbolized a particular color).

In the Handshake problem, Michael's activity suggests that his activity in the Flag Problem highlighted the importance of establishing two composite units prior to operating. I make this inference because Michael appeared to assimilate the Handshake Problem with two composite units that he differentiated between, the handshakers and the handshakees. He initially symbolized the handshakers with numbers and the handshakees with letters. Michael began symbolizing the first person's handshakes as, "1B", "1C", etc., and finished all of the handshakes in which the first person could engage. When he began the second column of notation, he established an identity between the letters and numbers, for example, "2" symbolized the same person as "B". He made this identity based on his activity to eliminate the self-handshakes and already counted handshakes.

When he made this identity, Michael appeared to operate with only one composite unit that symbolized the ten people, instead of two. This way of operating meant that he *introduced* the differentiation of handshaker and handshakee based on how he used the person in his activity. Once Michael got done with the second column of notation, he appeared to assimilate his activity to his pair concept. So, he appeared to imagine inserting a representative unit that symbolized any but no particular of the people he identified as handshakers into the first slot of his concept and a representative unit that symbolized any but no particular of the people he identified as handshakees into the second slot (where the handshakees decreased by one each time).

Michael, similarly to Deborah, made his eliminations based on establishing a differentiation and then an identity between two composite units. However, I argued that Michael's pair concept had yet to contain the same kind of multiplicative records that Deborah's pair concept did. I made this assertion because he was in the process of constructing this concept when he worked in problems like the Outfits Problem. I infer that by the time he solved the Handshake Problem it was a lateral learning goal for Michael to produce a pair concept that contained a record of multiplication where he might operate in a manner similar to how Deborah operated in the Two Suits Card Problem.

Because Michael had yet to produce a pair concept that contained multiplicative records, there was no possibility for him to produce the eliminations he made in the context of his multiplicative reasoning. So, Michael neither established the sum with his multiplicative reasoning nor did he establish the eliminations in the context of this multiplicative reasoning. This way of operating meant he could not, in this context, find a way to use his multiplicative reasoning to evaluate the sum nor was it a possibility for him to establish permutations of two elements or combinations of two element subsets. As with Deborah, he did produce many of the operations that I infer a person uses to produce these concepts. However, Deborah operated at a higher level because she actually established her eliminating activity in the context of her multiplicative reasoning, which is essential for producing the sum using one's multiplicative reasoning, as well as producing permutations of two elements and combinations of two elements.

So, Michael had only constructed an additive concept of the sum, which meant that his initial construction of the sum as a quantitative entity did not include multiplicative operations that he could use to evaluate the sum. Michael did find a way to evaluate the sum, but he found this way outside of the context of a problem that involved making pairs. He evaluated the sum with operations that were external to the operations that he used to initially construct a concept of the sum. Michael's evaluation of the sum involved his insight of strategically pairing numbers to make units of equal numerosity (e.g., in evaluating the sum, $25 + 24 + \ldots + 1$, he introduced making units of 26). I argued that during this activity Michael introduced a new mental image of the sum as a segment, which he introduced by re-presenting his activity of running his fingers over the notation he produced to symbolize the sum. Michael's sum concept already contained position and amount meanings and he used this new mental image to imagine folding the sum onto itself to produce and enumerate the number of units of 26 he could make. So, he transformed his additive concept of the sum into a multiplicative way to evaluate the sum. In his solution of the sum of the first 99 whole numbers, he demonstrated that he had abstracted a concept of how to evaluate the sum and notated this concept with his natural language. His natural language was a description of the operations he would perform were he to actually evaluate the sum.

Carlos. Carlos's activity suggested he had a concept that was a precursor to a pair concept. Based on comparing Michael's and Carlos's initial activity this statement is somewhat surprising. That is, Carlos immediately recognized the Outfits Problem as one that involved multiplication whereas Michael did not. I have argued that this difference was because Carlos was very familiar with pairing contexts and already had constructed a representative unit whereas these contexts seemed to be novel to Michael. In Carlos's case study, I asserted that when Carlos solved problems like the Outfits Problem, he took a particular unit (e.g., a particular pair of pants) and imagined putting it with a representative unit (e.g., a representative shirt), which he took to symbolize the composite number of pairs (e.g., three outfits) that he could produce with that particular unit. Then I suggested that Carlos could let the unit that had been a particular unit be a representative unit, and this activity symbolized the number of times he would engage in these operations. I took this way of operating as indication that Carlos had yet to interiorize a pair concept. But, he had an enactive pair concept and had abstracted a "pair-like" unit that had two slots, where one of the two slots had to remain fixed in his activity.

A concrete example might be useful to fill in this description. For example, in the Two Deck Card Problem, Carlos imagined taking a particular card from one of two decks of cards

> Task 5.4, *The Two Deck Card Problem:* If you were going to write down all the possible pairs of cards you could get, how many would you write down?

and putting this card with a representative unit that symbolized the second deck of cards. Carlos imagined that this produced a 1 by 52 column in an array. Carlos, then, fixed the second dimension (second deck of cards), which he now considered to be 52 units and imagined the unit that had been a particular card to be a representative card. This final mental action produced a 52 by 52 array. This description suggests that Carlos could reason about only one of the dimensions at a time and the other dimension had to be fixed with a particular number of units. I infer part of the reason that Carlos liked arrays so much was that it enabled him to produce a novel coordination where he first fixed one of the two composite units and then fixed the other composite unit.

Carlos had little difficulty assimilating the Flag Problem and made flags in a manner that was similar to how Deborah operated in the One Deck Card Problem and Michael operated in the Flag Problem. That is, he disembedded a singleton unit, which symbolized a particular color, and imagined putting it with all other possible colors. He notated this activity in columns with notation like “1,1”, “1,2”, etc. where the numbers symbolized a particular color. However, he remained convinced throughout most of the problem that the results he produced would be 15 times 15 even though he like Michael eliminated some of the flags and produced a sum that was 15 + 14 + … + 1. It was not until Carlos found a way to evaluate this sum using strategic additive reasoning to make eight 15s that he became aware that 15 times 15 would produce the incorrect result.

I argued that Carlos was aware that his disembedding activity had produced a second composite unit that had the same referent as the first, but that he did not construct the sum as a quantitative entity. I argued that he did not construct the sum as a quantitative entity because one of the slots of his pair concept remained fixed. Because of this constraint, Carlos would be aware of the number of flags he had symbolized in a particular column, and then subsequently the number of columns he had produced, but he would not be aware of both simultaneously. So, when he symbolized the sum with his additive notation, he was aware only of the number of flags in each particular column. Moreover, he did not seem to be aware that he had made eliminations of some of the flags until he compared the number of 15s he actually produced, 8, with the number of 15s he expected to produce, 15. This comparison appeared to create occasion for him to review his actual activity and compare it to his expected activity, which suggested to him that he had eliminated some of the flags. This way of operating was in contrast to Michael and Deborah who seemed to be aware earlier in their activity of having made eliminations so that a multiplicative answer like 15 times 15 would be incorrect.

In the Handshake Problem, after seeing Michael use both letters and numbers, Carlos used similar notation. I infer in figuring out how to use this notation to symbolize his answer Carlos produced two composite units one that symbolized the handshakers and one that symbolized the handshakees. He, then, appeared to imagine having a particular handshaker shake hands with a representative handshakee, which symbolized all the handshakes in which the first person could engage. When he engaged in this activity, he eliminated the self-handshake even though he had differentiated between the two composite units.

When he began the second column of notation, which symbolized the second person's handshakes, he eliminated the self-handshake again, but did not eliminate any other handshakes until he heard Michael ask the question "But didn't they already shake hands?" Michael's statement occasioned an act of reflection on the part of Carlos, and he established an identity between each of the units in the two composite units that he had initially differentiated between. Carlos, then, eliminated handshakes according to which he had already counted (e.g., person 1 shaking person B's hand was the same as person 2 shaking person A's hand). I argued in the case study that this identity was rooted in Carlos's productive mathematical reasoning. However, I conjecture that Carlos would have to abstract a pair concept prior to these eliminations opening the possibility of leading to a concept of an ordered pair that might enable his construction of two element permutations and combinations.

In this situation, Carlos once again did not appear to produce the sum as a quantitative entity and I infer it was for similar reasons as in the Flag Problem. As with Michael, I asked Carlos questions that involved simply evaluating the sum of the first 25 and 99 whole numbers. In these situations, Carlos did find a way to produce and enumerate units of equal numerosity (e.g., units of 26 and units of 100) using his strategic additive reasoning, but this activity did not seem to help Carlos construct the sum as a quantitative entity. Rather, he made a sequential coordination between producing the units of, for example, 100 by notating addition problems like 99 and 1, 98 and 2 etc. and then enumerating the number of 100s he produced. I conjectured that in the case studies that the difficulty he had with finding a quicker way to evaluate the sum was related to his lack of construction of the sum as a quantitative entity. So, he operated on the elements of the sum but did not operate on the sum itself.

Five Levels of Operating

Based on these summaries of the students' construction of various schemes and concepts and accommodations they made to these, I want to suggest my model of the various levels at which a student might use his lexicographic units coordinating scheme. In describing the levels of operating, I will refer back to the Outfits Problem in order to capture how a student operating at a particular level would solve this task. I observed all of the levels of operating in my study although none of the students were constrained to the first or second level. At each level, I will suggest some tasks that a student might work on and the constraints a teacher might meet when working on the tasks with students at a particular level. The tasks that I suggest will be appropriate for students working at the particular level, but would be appropriate for students working at the higher levels as well (even though these student might conceive of these tasks in a different way).

The most basic way in which the students operated in my study was to put the units within two composite units in one-to-one correspondence with each other. I do not consider this activity to be a level of using one's lexicographic units coordinating scheme, but nonetheless it is important to mention because Michael did begin by assimilating the situations in this way. I pointed out, in the case studies, that this way of operating was not focused on possibility, but rather on exhausting the "material" that was available in a situation. That is, to operate in this way a student imagines, for example, taking one card from one deck of cards and a second card from a second deck of cards and putting them together. Once cards are used in this way the student considers that the cards have been exhausted and cannot be used to make any more two-card combinations.

In contrast, the first level of the lexicographic units coordinating scheme that I observed was when a student ordered the units in both composite units and sequentially put *each* unit from one composite unit with *every* unit from a second composite unit, keeping track of his activity with a lexicographic ordering. In the Outfits Problem, all three students engaged in this activity and symbolized the activity by sequentially producing notation like “1A”, “1B”, “1C”, etc. where “1A” symbolized the first shirt paired with the first pair of pants, etc. For the first level of operating, it was common for the student to use notation like “1A”, “1B”, etc. However, using this notation did not mean that the student was constrained to this level of operating. I considered this way of operating to be pre-multiplicative because, when a student operated this way, the students considered each particular outfit to be different. This difference is based on differentiating each of the units within each of the composite unit (e.g., assigning a different color to each).

For a student constrained to the first level of operating, the following tasks might be appropriate:

1) You have three shirts and four pairs of pants. How many different outfits could you make?
2) In the context of rolling a die and flipping a coin to make some experiential pairs, how many possible outcomes could you get?
3) In the context of having a two students each flip a coin to make some experiential pairs, how many possible outcomes could you get?

These types of tasks would be appropriate because the total number of outcomes is small so that a student could easily notate all of the possible outcomes. Because a student constrained to the first level of operating would not consider these situations to be multiplicative, it is unlikely that the student would be able to keep track of all of the outcomes exclusively mentally. So, being able to easily notate all of the outcomes would be essential in the students’ solution of the problem. Furthermore in the first two tasks, there are not the same number of, for example, shirts and pants. Because the number of shirts and pants is not equal, a student may experience a perturbation that he might not if he initially puts the shirts and pants in one-to-one correspondence. That is, he may end up with a left over pair of pants and want to make an outfit out of this pair of pants. This type of perturbation may occasion a student to introduce creating more than one outfit with each particular shirt. The second problem has many of the same features as the first, but students in my study appeared to experience it as slightly more difficult because they did not necessarily have the everyday experience of pairing an outcome on the die with an outcome on a coin. The third problem might be used to test if the student was differentiating between outcomes like tails on his coin and heads on the other student's coin or if he simply considered these outcomes to be the same. Work on problems like the third task might bring to a student's awareness differentiating between outcomes based on making a distinction between two composite units.

For a student constrained to the first level of operating, I might explore some tasks like these. However, many of the tasks I presented to the students in my study would not be accessible to a student who was constrained to this level of operating because tasks that involved larger numbers would likely have too many outcomes for a student to successfully keep track of all of them. So focusing on developing the student's multiplicative reasoning in other domains might be a more fruitful avenue of work than to proceed with more tasks like the harder one's found in this study.

The second level of the lexicographic units coordinating scheme that I observed was when a student imagined putting a representative shirt with a particular pair of pants, which symbolized creating three outfits. He, then, repeated these operations for each pair of pants to produce three outfits four times. In this situation, I argued that a student iterated his units coordinating activity because he imagined putting the representative shirt with each particular pair of pants. When a student operated this way, he frequently used a tree diagram to symbolize this operating. The first three tasks that I presented above would be appropriate for a student operating at the second level of the scheme. However, tasks like the following would also be appropriate for students operating at the second level as well.

4) You have the two thru ace of hearts and your friend has the two thru ace of clubs. How many two-card combinations can you make?

5) There are ten people in this room. If each person wants to shake everyone else's hand, how many handshakes would there be?

In the fourth situation, the numbers are larger and so such a problem would probably not be feasible for a student operating at the first level. However, I infer that a student who is operating at the second level of the scheme would be able to solve this problem because he could imagine putting a representative unit that symbolizes his deck of cards with each particular card in his friend's deck of cards. Furthermore, it is likely that the student could solve the fifth task by assimilating the situation using one representative unit, disembedding a particular unit that symbolized a particular handshaker, and producing a representative unit for the handshakees. In this situation, a student would eliminate duplicate handshakes based on the results being the same (i.e. a handshake between two people are the same regardless of who is the handshaker). For a student operating at this level, the handshaker would always be a particular handshaker while the handshakee would be a representative handshakee. So the student would not be able to consider handshakes as ordered pairs, which I infer would involve considering that handshaker and handshakee as functioning identically in the context of solving the fifth problem. Furthermore, based on the activity of the students in my study, it appears that a student constrained to the second level of the scheme would not abstract the sum as a quantitative entity from a problem like the fifth task even if he produced a sequence of addition problems in the context of operating in this context.

The third level of the lexicographic units coordinating scheme and the one that I infer Carlos had abstracted his “pair like” concept from was his activity to put a representative shirt with a particular pair of pants and subsequently imagining that this particular pair of pants was a representative pair of pants. The difference between this level and the previous level is that the student does not actually engage in putting the representative shirt with each pair of pants. Rather, he first considers the shirt to be a representative shirt, and then, considers the pair of pants to be a representative pair of pants. In the case studies, I argued that doing so produced a composite number of pairs a composite number of times.

At this level, a person does not simultaneously consider both the shirt and the pair of pants to be representative units. I take this as indication that a person can enact operations that makes an array multiplicative as Carlos seemed able to do, but Carlos could not take as a given that an array was multiplicative. In addition, as Carlos demonstrated, he could eliminate extra, for example, handshakes by monitoring which handshakes he had already counted. Like a student operating at the second level, a student operating at this level would not be able to create an ordered pair because the student has yet to have abstracted a pair concept. However, a student operating at this level would be able to solve the following problems that probably would not be appropriate for a student constrained to the second level of operating.

6) You have a deck of cards and your friend has a deck of cards. How many two-card combinations could you make?

7) You have a deck of cards. How many two-card combinations can you make?

A student constrained to either of the first two levels of the scheme might not be able to solve either of these problems because of the large numbers involved. In the case of the seventh task, a student operating at the third level of the scheme might be able to notate the sequence of addition problems prior to actually producing the full results (i.e., the student would not have to imagine putting a representative card with each particular card in the deck). However, Carlos's activity suggests that the student would not necessarily construct the sum as a quantitative entity in this context. Constructing the sum as a quantitative entity appears to entail simultaneously coordinating two representative units, which entails constructing the fourth level of the scheme.

The fourth level of the lexicographic units coordinating scheme is when a student has abstracted a pair concept from his or her operating. As I suggested earlier, a pair concept is a unit that contains two slots into which the number of possibilities for each composite unit can be inserted. Both Deborah and Michael appeared to have abstracted this concept. Deborah had abstracted it in the context of her multiplicative reasoning (prior to entering the teaching experiment) and Michael appeared to have abstracted it in the context of solving a problem that involved addition, the Flag Problem. So, Michael's concept did not contain multiplicative records, as did Deborah's concept.

Based on my interaction with the students, I infer that the difference between the third and fourth level is quite significant and is not exclusively related to carrying out the activity of the scheme. I make this statement because Carols did not appear to construct a pair concept and Michael did even though Michael began operating at a lower level than Carlos. I accounted for this difference based on the fact that these situations were novel to Michael. Therefore, I infer that a significant reorganization occurs in one's multiplicative concepts when he or she interiorizes the fourth level of operating.[59]

Once at this fourth level of operating, I infer that a student has all of the operations that are necessary to construct permutations of two elements and therefore an ordered pair. In my summary of Deborah's and Michael's activity, I suggested what operations seemed to be

[59] For the previous levels, I make a similar inference, but all the students in my study eventually could easily operate at the first two levels because of their previously interiorized operations.

essential to this construction, even though neither of them actually constructed permutations of two elements. I infer making such a construction would lead to a concept that I have called an ordered pair where the two slots in a person's pair concept contain records of ordering the slots in that concept. The following tasks might be useful to help a student develop ordering in his pair concept.

8) A circle has five points on its circumference. Suppose that there is an ant that can crawl in a straight line between any two of these points. He wants to know how many different paths he can crawl taking into account the direction he crawls. How many possible paths could he crawl?

9) A teacher is going to seat people in a two-person row. She has six people she can choose from. How many possible rows could she make?

10) You are the President of a new country. You need to design a flag that has two stripes. You have 15 colors to choose from. How many possible flags could you make?

The eighth and ninth situation rely on an experience of considering different crawling in a particular direction or being seated on the left versus the right. These two types of problems may help to begin to develop an ordered pair in the context of the student's multiplicative reasoning, which I infer can lead to the abstraction of ordering in a student's pair concept. The tenth problem might be a good problem to present once a student has begun to work on constructing a concept of an ordered pair because a student might then consider several different scenarios for the task (the king wants to count as different red-orange or orange-red, the king does not want to count these two flags as different, the king does not want the stripes to be the same color, etc.). I did use this task for the boys and each was able to solve it even though neither had constructed the concept of an ordered pair. So, this problem is a problem that can be solved using the third level of the scheme, but the problem can be extended once a student has constructed an ordered pair concept.

A fifth level of operating is to recursively use one's lexicographic units coordinating scheme. The only student who independently operated in this way was Deborah. It was interesting that to recursively use her lexicographic scheme Deborah began by operating at each of the first three levels that I suggested a student might operate when she assimilated a situation with her lexicographic scheme. Recursively using one's lexicographic units coordinating scheme is significant because once a person abstracts a triple concept, I infer that she abstracts a generalized slot concept that is the basis for combinatorial reasoning. I was surprised at how difficult the problems I presented to Deborah were for her, given the facility with which she operated in situations that she assimilated using two composite units. She did begin to produce a recursive scheme of operations, but did not abstract a triple concept or a generalized slot concept.

For problems that a student assimilates using three composite units (or more), I would follow a similar sequence as I suggested above, but I would begin to incorporate permutations of three (or more) elements as a student became facile with problems like one through four. I make this assertion because permutations become more essential when presenting situations that could be solved with multiplicative combinations.

Algebraic Aspects of the Students' Activity

As I argued above, I consider the use of a representative unit as distinctly algebraic. In this section of the chapter, I will investigate three further aspects of the students' activity that I consider to be algebraic—abstracting concepts, abstracting a conceptual network of related tasks, and the students' symbolizing activity. In the section on the students' symbolizing activity, I consider five aspects of this activity that I consider to be algebraic in nature. Throughout my analysis, I will use examples from the summaries I provided above and re-visit some other examples from the case studies.

Abstracting Concepts

The students in my study all abstracted concepts in the context of interacting. I consider the abstraction of concepts to be one of the ways in which the students operated that seems algebraic in nature. I make this statement because when a student abstracts a concept the activity that they produced to construct the concept is no longer necessary. Rather, they can implement this activity or can simply use notation or verbal description to point to the concept without implementing operations to produce it. To do so takes a level of compression in one's operating that is not present when a student actually has to enact a scheme to produce a result. For this reason, I consider the students activity towards abstracting concepts to be algebraic in nature.

A Pair and "Pair-Like" Concept. Because of the novelty of these situations, Michael appeared not to have abstracted a pair concept at the beginning of the teaching experiment. On the other hand, both Carlos and Deborah appeared to have abstracted their operating in problems like the Outfits Problem into a multiplicative concept, which enabled them to assimilate their activity directly to calculating a result for the problem. I have argued that Deborah had a concept that was at a higher level of interiorization, but nonetheless neither of the students actually had to implement the operations of their scheme to assimilate these situations to their multiplicative concept. I want to suggest that this way of operating was algebraic in nature because it involves having a generalized concept of multiplication that has been associated with particular quantitative contexts. This type of generalized concept of multiplication enabled both Carlos and Deborah to implement the activity of their scheme in order to explain their solutions to me, but each also simply took the situations as multiplicative.

The difference in the level of interiorization in their concepts seemed to enable Deborah to take a two-dimensional array as a multiplicative object without having to produce operations to consider it multiplicative. I infer this enabled her ultimately to arrive at the formula for producing the sum with a multiplication and division problem $n(n - 1) \div 2$. I make this assertion because she appeared to produce an image of an array and this image she could symbolize with an unevaluated multiplication problem, which suggests she took the array to itself be multiplicative even when she could not evaluate a particular multiplication problem.

On the other hand, Carlos appeared to have to enact operations to produce *arrays* that he considered to be multiplicative (even though he could consider the problems to be multiplicative without enacting operations). The difference I am pointing to here is that Carlos's "pair like" concept did not enable him prior to operating to take an array to be multiplicative. I make this assertion based on the summary above of his "pair like" concept where one slot appeared to be fixed. Because one slot was fixed he could imagine taking a particular card and putting it with a representative unit, which made the first column of an array (e.g., a column that was 1 unit by 52 units). Subsequently, he could take the second slot as fixed (e.g., containing 52 units) and allow the first slot to be a representative unit, which created an array.

A Sum Concept and a Concept for Evaluating the Sum. Both Michael and Deborah abstracted the sum as a quantitative entity. Abstracting the sum as a quantitative entity enabled them to assimilate new situations with their concept of the sum, where each addend in the sum had both a position and amount meaning. This way of operating meant that each had constructed the sum as a quantitative entity in the context of pairing problems, but both demonstrated that they could subsequently take various forms of notation to symbolize this concept without having to be in a pairing context. In addition, in evaluating the sum, each of them produced a quantitative symbol that I infer is the basis for taking algebraic notation to symbolize any but no particular sum. Deborah actually notated an algebraic equivalence between the sum and a multiplication and division problem whereas Michael only verbalized a general formula for how to evaluate the sum.

In both cases, however, I argued that the students took their way of operating as applicable to any sum not just a particular sum. Michael could operate as he did because the mental image that he introduced of the sum as a segment and the operations he performed on this mental image did not appear to be constrained to a particular sum. In fact, at the end of his activity, Michael appeared to recognize a way of producing the results for sums involving an odd number of addends: find what unit he would produce (i.e., units of 100 or 26 in the case of the sum of the first 99 and 25 whole numbers) when he strategically added elements of the sum, find the number of these units by taking one half of one less than the total number of addends in the sum, and find the number that was in the middle (e.g., 50 or 13).

Similarly, Deborah could operate on an array without producing a particular result because taking half of an array did not depend on actually evaluating the number of elements in the array. As I suggested in the last section, this way of operating provides indication that Deborah could simply take an array as a multiplicative structure without having to produce operations that made it multiplicative. So, she could take identifying the relationship between the two dimensions of the array (i.e., there would be one less in the rows than in the columns) as sufficient for considering it to be multiplicative. In addition, Deborah demonstrated she could also symbolize any sum with additive notation when she symbolized the sum of the first n whole numbers with dots and algebraic notation.

Carlos found a general way of operating when evaluating the sum. In order to find this general way of operating, I infer he had to have some concept of the sum as a sequence of addition problems, but without constructing the sum as a quantitative entity (i.e., where addends had both position and amount meanings) he actually had to implement a number of sequential operations in order to evaluate the sum. This example is an interesting for considering aspects of algebraic reasoning. It is interesting because Carlos found a general way of operating, which in a certain regard is algebraic. However, without producing the sum as a quantitative entity, he was constrained to sequentially engaging in this activity and so he could not compress this activity. I consider a compression of activity to be distinctly algebraic. In particular, it contrasted with Michael's verbalization of how to evaluate the sum that compressed twenty minutes of activity into a short statement and Deborah's algebraic notation, which compressed several days of activity into a relationship between her additive way of producing the sum and her multiplicative way of producing the sum. Nonetheless, as I argue below, Carlos's use of notation in these situations did have certain characteristics that I would also consider as algebraic.

Abstracting a Conceptual Network of Related Tasks

One aspect of all of the students operating that I want to argue is algebraic was that they all abstracted a conceptual network of related tasks. I consider this algebraic because it suggests that a person is aware, to some extent, of his or her own operating and is comparing and contrasting these tasks to his or her expectation of the outcomes. Furthermore, it is relevant to algebraic reasoning because it provides students an opportunity to build up networks of conceptually related tasks. I infer that what is particularly important in this process is the sense of relatedness that a student can build up as a result of operating across situations. The level of relatedness that the students were able to create depended on their previously interiorized operations, schemes, and concepts, the functional accommodations each made as they operated, and the extent to which each embedded new operations into old ways of operating. When a student was unable to embed new operations in old ways of operating, then I considered the tasks to be associated, but not necessarily contained in one another. Below, I will analyze in what ways the students related their operating across the various quantitative problem situations that I presented them.

Carlos. I argued at the beginning of Carlos's and Michael's case study that the three sections of the case study corresponded to the boy's experiencing three distinct types of problems. This distinctness was because the operations that they produced in each of these kinds of tasks were not embedded in one another. For example, Carlos did not embed his additive solution of the Handshake and Flag Problem in a multiplicative solution in the sense that he did not produce the additive solution as contained in his multiplicative reasoning. Similarly, his evaluation of the sum was a separate but related task to the first two types of problems. Nonetheless, Carlos did begin to compare and contrast his activity in the Handshake, Flag, and Outfits Problem.

Carlos's differentiation of his activity in the Outfits Problem from his activity in the Flag Problem enabled him to differentiate his multiplicative way of operating from his additive way of operating. He did not differentiate these two activities until he operated on the results of his additive reasoning with operations that were external to his lexicographic units coordinating scheme. Namely, it came through his evaluation of the sum where he related the number of 15s he anticipated producing, 15, with the number of 15s he actually produced, 8. When he became aware of this difference, it appeared to occasion a review of his activity in the Flag Problem and he compared his actual activity to his anticipated activity. This enabled him to compare and contrast these two activities and he seemed to gain awareness that he had eliminated some of the flags.

In the Handshake Problem, Carlos's comment, "minus two actually", prior to his finishing all the handshakes suggested that he had become aware of his eliminating activity in the midst of this activity. This particular comment pertained to his elimination of self-handshakes and the person who had just been a handshaker. For example, in his third column of notation, which symbolized the handshakes for the third handshaker, he was aware of both eliminating the third person from shaking his own hand and the second person who had just been a handshaker. This enabled him to make a comparison of the activity of his scheme in differing situations based on the activity he produced not just based on the results of this activity. I consider it important that Carlos compared his activity not just the results because it suggests that he was aware of his activity in the process of acting. In addition, it suggested that he was beginning to make a comparison between activities that he initially experienced as similar. I consider this type of work to be central in the process of thematizing one's activity, which is a significant part of what I would consider to be algebraic activity.

Michael. I organized Michael's case study similarly to Carlos's, in three sections, because I argued he did not embed his operations from one context into his operations in the other contexts. In part, this embedding did not occur because these contexts were novel to Michael so by the time he constructed operations that would enable him to assimilate a problem like the Outfits Problem as a situation of multiplication, I presented him problems in which he produced a sum. So, the activity he produced in the early problems served as a way for him to conceptualize his activity in the later problems, but he did not for example embed his activity of producing the sum additively in his multiplicative reasoning.

Because Michael did not initially have a scheme constructed for the situations, the way in which he related these tasks was quite interesting. The relationship that he created between the tasks was primarily related to his working at varying levels of interiorization in solving the tasks. So, he recognized these tasks as related because his most basic way of operating, ordering the units within two composite units (or a singleton unit and a composite unit) and then sequentially making a units coordination between each particular unit in one composite unit and each particular unit in the other composite unit could be used in any of the situations (what I have called the first level of a lexicographic units coordinating scheme). Michael used this way of operating as a way to resolve perturbations he experienced when he was operating at a higher level of interiorization and could not resolve the situation at this higher level of interiorization.

So, for example, in the Flag Problem, when Michael experienced a perturbation when making flags with a particular unit and a representative unit, which symbolized making a composite number of flags, he dropped down to a lower level of operating to resolve this perturbation. Michael's ability to do so created a sense of relatedness among the tasks because he was aware that he could resolve these tasks by dropping down to his most basic way of operating. It also suggests that Michael was aware of how he was operating and engaged in acts of self-regulation that enabled him to function as I have suggested.

In addition, Michael also related the Handshake and Flag Problem because he recognized that each produced the sum as a quantitative entity. He abstracted this concept from his operating in the Flag Problem and recognized it in the Handshake Problem. So, he experienced the two problems as related to one anther based on the quantitative entity that each would produce. Here, Michael's comparison did seem to be of the results his activity would produce, but he could make this comparison prior to having to produce these results in full, which suggests that he identified the activity of his scheme as similar to the activity he had produced in the Flag Problem. In a subsequent problem of evaluating the sum 25 + 24 + …+ 1, he assimilated his activity of evaluating the sum of the first 25 whole numbers to his concept of the sum and introduced a new mental image in this situation that enabled him to evaluate the sum. This meant that evaluating the sum was associated with the Handshake and Flag Problem because he had produced his concept of the sum in these contexts even if he did not find a quick way to evaluate the sum that was related to the pairing situations.

Deborah. Of the three students, Deborah developed the highest level of relatedness among the tasks that I presented to her. She was the only student that produced the sum multiplicatively. Although she could not relate the partitioning of her array, which led to her division of 90 by 2 in the Handshake Problem, to the quantitative situation, she did relate it through the results that it produced, the sum as a quantitative entity. I have argued that she could not relate the partitioning of her array to the quantitative problem situation because she had yet to produce an ordered pair. However, she was able to relate the sum back to the quantitative problem situation because she could produce the sum in the context of her multiplicative reasoning by sequentially eliminating the extra handshakes.

I infer that Deborah experienced these situations as related through more than just an association of the kinds of activity she engaged in because the operations she produced were embedded in one another. For instance, she could assimilate situations that involved only one composite unit to her pair concept because she disembedded a second composite unit from the composite unit she used to assimilate the situations. For example, in the Handshake Problem, she assimilated the situation with one composite unit that symbolized the ten people and used this unit to produce a second composite unit that also symbolized the number of people that could shake each person's hand. This enabled her to operate multiplicatively, which was similar to how she operated when she assimilated a situation with two composite units (e.g., the shirts and the pants).

The one example where Deborah did not embed her operations was in the case of making her eliminations in the Handshake Problem. In this problem, she could operate on the 90 handshakes she symbolized with an array by sequentially eliminating handshakes. Doing so meant that she compared individual handshakes to make these eliminations. So, for example, she eliminated person A from shaking person C's hand, since the two had already shaken hands when person A was a handshaker. However, I argued this way of operating was not embedded in her pair concept because she did not make these eliminations based on an ordering of the two slots in this concept. Had she operated in this way then I would consider the eliminations to be embedded in her pair concept because then her eliminating activity would have been contained in her pair concept.

Symbolizing Activity

Having considered two facets of the students' activity that I considered algebraic, I want to suggest five ways in which the students used their symbolizing activity that seems in the province of constructing an algebraic symbol system. Each of these aspects, at varying levels, is related to the students using their notation to become consciously aware of how they operated. Moreover, it enabled them to externalize their operations, schemes, and concepts, hold these operations, schemes, and concepts at a distance, and operate on them further. I consider this activity to be an important part of constructing an algebraic symbol system. None of these activities by itself is sufficient to characterize an algebraic symbol system, but taken together they seem to be important features of such a system.

Self-Regulation and Monitoring. I have referred to some of the students' symbolizing activity as activity that is used as a tool in action. This type of activity is produced while a student is implementing the operations of his or her scheme (i.e., while acting) and is a way for students to use notation to engage in acts of self-regulation and monitoring. For example, in the Outfits Problem, when Carlos first used a picture of a shirt and a picture of a pair of pants, he was externalizing both his object concepts of a shirt and pair of pants and his number concepts. In this situation, he began his activity after producing this notation, but experienced a perturbation because he could not use the notation he had produced to keep track of the number of outfits he anticipated making. So, he halted his activity (an act of self-regulation) and produced new notation, the letters "B, R, W, G" and "B, W, R", that he could use to monitor which outfits he had produced. This act of self-regulation and subsequent monitoring implied that Carlos had to be able to be an observer of his own activity while he was engaged in this activity.

A second example of this way of using notation was Michael's activity in the Flag Problem where he initially produced notation by writing "1-red 14 flags", which symbolized the number of flags each could produce with the first color red. I argued that he imagined putting a particular unit, the color red, with a representative unit to symbolize 14 flags, he appeared to experience a feeling of disequilibrium from operating in this way, and so he halted his activity. When he began his activity again, he listed with his notation (i.e., "1-1", "1-2"etc.) each individual flag he could be make with the color red. Halting his activity was an act of self-regulation and listing the flags with his notation was a way for Michael to monitor the activity of producing flags. This way of operating suggests that Michael experienced a perturbation and used his notation as a means to resolve this perturbation where his notation helped him become more aware of the activity that he expected to produce.

Re-presentation and Reflection. A second way in which the students used their notation was to re-present their activity and reflect on this activity, which I called using notation as a tool in reflection. Deborah's use of notation in the One Deck Card Problem is a good example of this use of notation. In this problem, she began by using her notation to externalize her activity of imagining drawing a particular card from the deck and putting it with a representative card that symbolized the remaining cards in the deck. She engaged in this activity with four particular cards.

After producing this activity for four cards, I infer she re-presented some part of her activity to establish the structure of her way of operating: taking a particular card and putting it with a representative card to produce some number of two card combinations where each subsequent time she produced one less two card combination. I make this assertion because she stopped notating particular cards she imagined drawing from the deck (e.g., the two of diamonds), and the cards she would eliminate when using a particular card, and instead began notating the sum 51 + 50 +…+ 1. This new notation suggested that she used her old notation as a way to re-present her activity and reflect on what this activity would produce were she to carry it out in full. Here, her notation appeared critical in her operating because she could use the notation to symbolize her initial activity and take it as a given until she produced a sufficient amount of this activity and wanted to review this activity. This way of using notation seems distinctly algebraic because a student uses their notation to "step outside" of their activity and observe it prior to completing the activity in full.

Recursive Operating. The students also used their notation as a way to engage in recursive operating. To operate in this way, their notation had to symbolize both where something had come from and how it could be used further. I will give two examples of this type of activity from the data I presented in the case studies. The first example of this activity was in Carlos's evaluation of the sum of the first 99 whole numbers where he symbolized producing units of 100 by making an oval with the two units he imagined adding together (e.g., 99 and 1). Then he began counting the number of ovals he had produced and made an association between the first unit in each oval and the number of ovals he had counted. This way of operating enabled him to externalize the results of his strategic additive reasoning scheme and use the results, units of 100, as input for further operating by enumerating these units of 100. In order for this activity to be sensible, the place where he symbolized each unit of 100 with his notation had to symbolize both where the units of 100 came from (an additive combination of two other units) and how he could use these units of 100 in further activity. Furthermore, operating in this way, opened the possibility of embedding his activity of enumerating the units of 100 in the activity of producing these units of 100.

A second example of this type of recursive operating occurred during Deborah's solution of the Handshake Problem where she produced the entire array with the goal of operating on this array to produce the eliminations of extra handshakes. So, she symbolized her pair concept with an array and operated on this concept with operations that produced the eliminations. This type of operating was recursive in nature because she operated on the results of her pair concept with operations that were external to the one's she used to produce a symbol for this concept. For Deborah to operate in this way, the array had to symbolize both where the results came from and also how she could use these results in further operating. Moreover, this situation opened the opportunity for the operations that she used that were external to her pair concept to be re-interiorized in a way that was internal to her concept (although it did not in this particular case). Both these example suggest that the students used notation to operate recursively, which opened the possibility for them to embed operations, schemes, and concepts into one another.

Symbolizing Structural Relationships. Both Deborah and Michael symbolized a structural relationship between their additive and multiplicative reasoning. Deborah used conventional algebraic notation to symbolize this relationship and it was developed over the course of three episodes. I have argued that the notation she used to symbolize this relationship symbolized a program of operations that could be implemented in any but no particular case. This meant that the notation symbolized the *operations* themselves in that they did not pertain to a particular instance in which she had used these operations. So, she used her notation to symbolize whatever trace these operations left from operating across these situations. Michael's use of verbal notation to symbolize the relationship between his additive and multiplicative reasoning seemed to contain similar characteristics. That is, it contained the operations he had produced to evaluate the sum in the sense that this notation pointed to the trace these operations left from operating across these situations.

Social Interaction and Convention. I consider the activity of establishing conventions to be part of creating an algebraic symbol system because establishing conventions suggests that a person is engaged in the activity of making a model of another person's operations, schemes, and concepts (or their own in certain cases). This type of activity requires a certain level of awareness of one's own way operating. I will provide three examples of when the students established conventions in their systems of notation and the role that social interaction played in these examples. The first example is when I requested that Carlos produce an array in the Outfits

Problem. Once Carlos figured out what I meant by an array, he considered this to be a conventional way of communicating about his activity in problems like the Outfits Problem. For Carlos, it appeared to be a way for him to consider that he and I attributed the same meanings to a particular symbol. This suggested that he had attained some level of awareness of this operating and how this notation symbolized his operating. In the Two Deck Card Problem, Carlos used the array to try and explain his solution to Michael, which suggested he was using this notation to try and establish a convention with another student.

The second example is Deborah's use of notation in the Outfits Problem where the primary reason that she appeared to produce this notation was to illustrate her concept to me. So, the purpose of using her notation was purely to establish a conventional way of communicating with me. This way of operating suggested that Deborah had conceptualized her own way of operating in a manner so that she could use notation with the intent of communicating a particular way of reasoning to me. So, she intentionally used notation for the purpose of communication. This way of operating is a basis for establishing intersubjective agreement among teacher and student or student and student. I make this assertion because it requires a student to have abstracted her way of operating so that she no longer *has* to produce the activity involved in this way of operating, but can produce it for the purposes of communication.

A third example of establishing conventions was Michael's use of subscripts in the Handshake Problem. Remember that Michael, initially symbolized handshakes as "1B", "1C", etc. where "1B" symbolized a handshake between the first handshaker and the second handshakee. When he began notating the handshakes for the second handshaker, he realized he would have to eliminate some handshakes and established an identity between handshakers and handshakees by notating "$1B_2$", etc. to show that person B and person 2 were the same.

I consider this to be establishing a convention because he introduced the notation as a way to clarify to himself what his own notation symbolized. Here, he had to become an observer of his own activity and doing so enabled him to establish this convention in his notation. I consider this to be an act of creating a convention because he identified to himself that he actually did not need to symbolize people with letters and numbers, acknowledging that he could symbolize the situation using only numbers. So, he became aware that the notation that he used to symbolize a particular event, a handshake, could be done in more than one way and the way he choose depended on what he considered to be more convenient. This type of activity involves deciding what particular kind of notation might best communicate one's operating to self and others, which is a form of creating conventions. All three of these examples have to do with awareness of self in relation to others, which I infer is an important part of constructing an algebraic symbol system because a person often uses the symbol system as part of a tool to communicate their reasoning with others.

A Broader Perspective on the Students' Activity

I present this section in two parts. In the first part, I connect the study back to the broader research literature that I used to situate this study. In making this connection, I specifically examine in what ways my research connects to two recent works on students' symbolizing activity. In the second part, I connect the students' ways of operating to a broader scope of algebraic activity. In making this connection, I suggest conjectures that could serve to guide future research.

Re-Connecting with the Research Literature on Algebraic Reasoning and Symbolizing Activity

In the literature review, I discussed previous research on algebraic reasoning and symbolizing activity in order to situate and define this study. Specifically, I focused on quantitative reasoning as a basis for algebraic reasoning, which has support in other scholarly work (e.g., Carraher et al., 2006; Kaput, 1998; Smith & Thompson, in press; Thompson, 1988; Steffe, 1991b), and on imagistic conceptions as a basis for building symbol systems (e.g., Charbonneau, 1996; DiSessa, 1987; Goldin, 1987; Hadamard, 1949; Kaput, 1987; Piaget, 1951; Thompson, 1996). In addition, I suggested that researchers have shown recent interest in students' symbolizing activity (e.g. Brizuela, 2004; Cobb, Yackel, & McClain, 2000). I now want to suggest how my study contributes to these existing bodies of research.

In characterizing algebra, Kaput (1998) suggests that a central aspect of algebraic reasoning is generalized quantitative reasoning. This statement is useful as a general guide in conducting research, but provides little insight into specific areas of research such as *how* a student generalizes their quantitative reasoning and *what* is entailed in this process from the perspective of the student (cf. Thomas & Tall, 2001). My research provides an example of what this activity looks like. More specifically, it provides a model of operations that are involved in the construction of what I have termed a pair concept, a whole number variable concept, and a concept of the sum as a quantitative entity. Each of these concepts was a generalization of the students' quantitative reasoning (e.g., Michael's concept of the sum was usable outside of pairing contexts where he used it to construct a concept of evaluating the sum). So, this study contributes to the dialogue among researchers who are interested in understanding how students might come to generalize their quantitative reasoning and opens for debate in what ways this generalization might be considered algebraic.

Similarly, researchers have suggested and investigated students' imagistic conceptions as a basis in which mathematical symbol systems might be profitably rooted. This study contributes to this body of research in that it provided models of the students' mental imagery and the operations they produced along with the notation that they used to symbolize this mental imagery and operations. Furthermore, it contributes to efforts to hold together students' observable and non-observable mathematical activity in order to suggest how students' symbolizing activity can be seen as the externalization of their mental imagery and operations. Researchers have distinguished between external (notation, diagrams, etc.) and internal representations (operations and mental imagery) (e.g., Goldin, 1987), and they have often endeavored to study one or the other of these. This work contributes to understanding the relationship between the two by modeling them together.

More specifically two recent research endeavors—Brizuela's (2004) book and Cobb, Yackel, and McClain's (2000) edited book—have contributed to researchers understanding of students' symbolizing activity. Brizuela asserts that students' generation of notation is an integral part of student learning. Warren's (2007) recent review of this book suggests that much more research is needed to understand in what ways students' symbolizing activity is integral in the learning process. The research that I have presented here seeks to contribute to research like Brizuela's research that is concerned with students' generation of systems of notation, while addressing Warren's concern about the ways in which students' symbolizing activity contributes to their learning. Above, I have suggested five ways in which this activity not only contributed to the students learning, but also aspects of this activity that I considered to be algebraic.

Cobb, Yackel, and McClain's (2000) edited book provides multiple contributions that investigate the function of students' symbolizing activity from a variety of theoretical perspectives. This work specifically examines aspects of students' symbolizing activity as they are related to classroom tasks and activity. Thompson (2001) points out that this book does not address psychological aspects of the symbolizing process. So, one way in which my study contributes to recent work on students' symbolizing activity is that it examines some of these psychological aspects.

As a final note and prelude to the next part of this conclusion, I want to briefly re-visit my choice to use problems that I characterized as involving combinatorial reasoning. I made this choice based on Piaget's (1958) characterization of some of the differences between concrete and formal operations, and his identification of combinatorial reasoning as requiring formal operations. That is (very broadly), investigating areas that involve formal operations is critical for engendering and understanding students' construction of algebraic reasoning. However, using combinatorial situations as a basis for engendering students' algebraic reasoning has remained largely uninvestigated in mathematics education research. So, my study contributes to the research literature on algebraic reasoning by exploring this underdeveloped area. In the next section, I will provide some connections between combinatorial reasoning and algebraic reasoning, in order to suggest some future directions for research.

Connections to a Broader Scope of Algebraic Activity and Directions for Future Research

A Whole Number Variable and Function Concept. During the case studies, I suggested that making a pairing between a particular unit (e.g., a particular pair of pants) and a representative unit (e.g., a shirt that symbolized three shirts) was the most basic form of a whole number variable. I made this suggestion because a student made a pairing of two units and this pairing involved one of the units being any but no particular unit. I consider this way of operating to be central to constructing a function concept.

To generate a function, a person needs to produce *pairs*, a unity that contains two units that satisfy some specified relationship. So, a particular unit from what is considered to be the independent variable is paired with a particular unit that is considered to be the dependent variable through this specified relationship. Thus producing pairs seems to be a central aspect of enacting a function concept. In addition, I *conjecture* that a general function concept on which a person can operate (i.e., a concept that does not need to be enacted) involves imagining that any but no particular value for the independent variable (a representative unit) can be paired with any but no particular value for the dependent variable (a second representative unit). A person with a general function concept could make a particular function, then, by specifying a relationship that identifies how the pairs could be produced, but would not actually need to enact this paring.

An example of a general function concept may help to illustrate my point. A person might consider $f(x)$ to symbolize a pairing between a continuous independent and dependent variable and $g(x) = f(x) + 1$ to be a related function that is a translation of the first function. The person could enact a pairing given a specific relationship, but would not have to enact this relationship in order to relate the two functions. Rather, the notation "$f(x)$" would symbolize the pairing between any but no particular value of the independent variable with any but no particular of the dependent variable and would not depend on producing a particular function.

Note that in this example, the independent and dependent variable are continuous, which is more advanced than the situations I presented that involved only whole numbers. Nonetheless, it is useful for imagining what a general function concept might look like. In the case of the students in my study, it appears that Deborah's pair concept would be an analogue with whole numbers for the kind of pairing that might lead to a general function concept. The analysis I have just made is based on conjectures about what constitutes a function concept and general function concept, as well as its relationship to the unit I have called a pair concept. I consider these conjectures to establish an important future direction of research—namely, investigation of this question: what is the relationship between a pair concept and students' construction of functional reasoning?

Two-Dimensional Multiplication. I have argued that a pair concept enables a person to take as a given that a two-dimensional array is a multiplicative object. I make this assertion because this concept enables a person to assimilate situations involving two composite units that have been differentiated to a concept that she considers multiplicative. This assertion opens up the question as to what is necessary for a student to consider a two-dimensional object such as a rectangle as multiplicative, and what is necessary for a student to begin with an array or rectangle and produce the dimensions of this array or rectangle. I conjecture that this activity is closely related to the operations that produce a concept of taking a square root. I consider these issues to be areas for future research.

Series. I conjecture that the operations that Deborah and Michael produced to construct the sum as a quantitative entity are the basis for constructing series more generally. In Deborah's and Michael's case, the particular relationship was a one-more-than relationship, but I conjecture that it is a lateral learning goal to create a different relationship (e.g., a series with a three-more-than relationship that begins with seven as opposed to one). These are conjectures and seem important to test in future research.

Two Further Areas of Research. In addition to the three suggestions I have already made, I want to suggest two further research areas. First, I consider the models of the students' multiplicative reasoning to be initial models that may be tested and refined to produce superseding models of students' multiplicative reasoning in these kinds of contexts. The three students in my study gave me insight into these issues, but working with them opened as many questions as it answered. In particular, I consider this work to be a starting place for understanding how students produce combinatorial reasoning. In this area of reasoning, a number of questions remain open based on my models, such as: How do students produce permutations? How do students produce combinations? What is necessary to abstract a generalized slot concept for multiplication?

Finally, I want to suggest that more research needs to be conducted on how students produce and use notation. I consider my work to point to an orientation that was fruitful, but to have provided only a preliminary model of this activity. The underlying orientation was to investigate how the students' generated and used notation in quantitative contexts. Taking this orientation enabled me to observe many aspects of notational use that might have been hidden had I taken conventional notation as a starting place for interacting with the students. Nonetheless, the models I have presented only begin to suggest some of the ways in which students' use of notation might be considered algebraic and what is involved in the construction of an algebraic symbol system. Therefore, I consider this activity to be open to further research.

One of the most significant things that I learned from conducting the study was that modeling the students' use of notation was dependent on my making models of the students' operations, schemes, and concepts. I make this assertion because without the models of this mental activity it would have been difficult to establish what meaning the students attributed to the notation they produced. Creating a way to think about the meaning that a student likely attributed to the notation was essential in how I modeled the students' use of notation. So, in future research that investigates notation, I consider it to be critical to also make models of the students' operations, schemes, and concepts.

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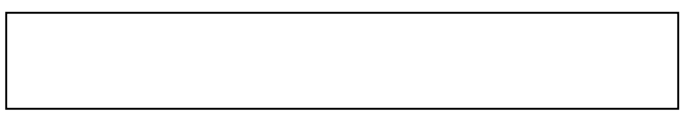
APPENDIX A: Selection Interview Guide

These questions were used with the 20 sixth grade students during selection interviews in September and October of 2003 (cf. Chapter 4). The goals of the selection interviews were to make an initial determination of each student's fractional and multiplicative reasoning, and to assess each student's openness to the interviewer and to this mathematical activity. Questions 1 through 6a were asked of almost all students. Question 6b was asked of some students, including all three students in my study. Questions 7 and 8 were asked of only a few students, including Deborah, but not including Michael or Carlos.

1. Here is a drawing of a candy bar.



Use a pencil to mark how you could share it fairly among five people, so that all people get an equal amount.

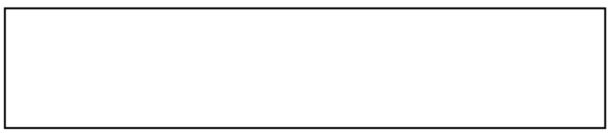
Are you sure that everyone would have a fair share? (If the student says yes, go to *. If not, let the student try again with another copy of the bar or make adjustments on the original bar until she or he is sure.)

*Do you think you got the pieces *exactly* right? (If the student says yes, go to **. If the student says no, let her or him adjust until she or he believes it's exactly right. If the student doesn't believe it's exactly right but it's close enough, go to **.)

Sara doesn't think you got the pieces exactly right (intention is to throw doubt on the situation). Can you think of a way to show Sara that you did? (If the student can think of a way, let her or him try it and go to *. If the student cannot think of a way, suggest as follows.) Could you use scissors to show Sara you got the pieces exactly right?

***Can you think of another way to show Sara? (Can keep asking about other ways, depending on time and intuitive impressions of the student.)

2. Here is another candy bar.



The candy bar is to be shared fairly among six people, so each person gets an equal share.) Make *one* mark to show your share.

Pretend that you can take your piece out of the bar. How could you use it to show that it's a fair share? (If the student moves finger along the bar or makes more marks, go to *.)

*(Give student another identical candy bar.) Use this candy bar to make your share. (If the student does not cut off her or his share, then go to this question.) What could you do to get your part separate from the candy bar? (If the student does not cut off his share with scissors, then gently suggest that.) Now how can you use that part to show it's a fair share?

3. a. Here is a candy bar and several separate pieces that were made from identical candy bars. (Give the student a 1/3-piece, 1/4-piece, 1/6-piece, and a 1/8-piece.) The candy bar is to be shared fairly among six people, so that each person gets an equal share. Which one of these pieces would be your piece? How do you know it's going to be fair?

(Use (b) only if student does not do well on (a).)

b. Here is a candy bar and a piece that was taken from an identical candy bar. (Give the student a 1/8-piece.) The candy bar is to be shared fairly among six people so that each person gets an equal share. Is this piece a fair share? (Probe the student's justification for her or his response.)

4. The drawing below shows my piece of string. Think of a piece of string that it is *seven* times longer than mine. (Pause.) Can you draw what you're thinking of? (If the student has difficulty with "long," restate the question using "big." If the student draws one continuous line without breaks or marks, go to *.)

*How do you know that your string is seven times longer than mine? (If the student doesn't know, give student a copy of the string and go to **.)

**Could you use this to show me that your string is seven times longer than mine?

5. a. The drawing below shows my piece of string. Think of *your* piece of string so that mine is five times longer than yours. (Pause.) Can you draw what you're thinking of? (If the student iterates the string five times, go to *.)

*Is yours five times longer than mine? (The student will hopefully say yes.) Ah, but I said I wanted *mine* to be five times longer than yours. (If the student then draws a piece of string shorter than the string, go to **.)

**Can you show for sure that mine is five times longer than yours?
(If the student has a lot of trouble with (a), go to (b).)

b. Think of piece of string so that mine is *twice* as long as yours (could also try "two times longer than yours"). Can you draw what you're thinking of? Can you show for sure that mine is twice as long as (or "two times longer than") yours? (If the student can do this problem, ask the same question with three times as long or four times as long.)

6. a. The Giant Soda at the convenience store is 24 ounces. That's eight times the amount of soda that Stephanie drank. What would you do to find out how much Stephanie drank? (Ask the student to do it, if she or he suggests something that seems feasible. Probe to get at operations, behind what the student says verbally.) Can you draw a picture of what you are thinking?

b. Camika has $21. That's one-seventh as much money as Rickard has. What would you do to find out how much money Rickard has? (Ask the student to do it, if she or he suggests something that seems feasible. Probe to get at operations, behind what the student says verbally.) Can you draw a picture of what you are thinking?

7. The submarine sandwich is to be shared fairly among four people, you and three friends, so that everyone gets an equal piece.

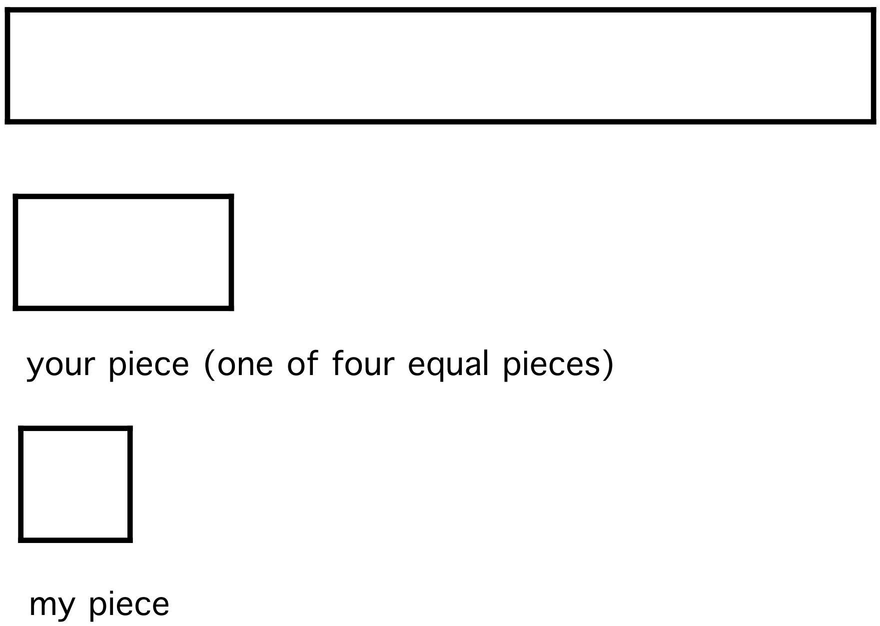
a. Make one mark to show your piece.

b. You decide to share your piece equally with me (because I'm hungry!) What part of the sandwich do I get?

c. What fraction is my piece of the whole sandwich?

(If the student has trouble with this task, try (d), which will be on a separate page for the student.)

d. The piece below shows your piece. The next piece below shows my piece.



My piece is what part of the whole sandwich?

8. At a small party, five friends split a submarine sandwich fairly. You take one of their pieces and share it fairly among four people. How much of the whole sandwich do three of these people get?

APPENDIX B: Sample Tasks from the Eighth Grade Year

I give an overview of the problems I presented that involved multiplicative combinations at the beginning of the case studies. So, in this appendix, I give examples of the three other types of problems I posed to the students.

Sample binomial reasoning problems that were symbolized with paper and pencil and in JavaBars:

1) (Discrete Case) Your friend owns a theater that has 13 rows and 13 columns of seats. He has two aisles in the theater. One runs horizontally the other vertically. The horizontal aisle is after the tenth row and the vertical aisle is after the tenth column. The aisles break the theater into four sections (like figure 4.2). Can you make a picture of the theater and find out the number of seats in each of the four sections and the total number of seats?

2) (Continuous Case) Suppose a farmer has a field that is 64 square kilometers. It has a cross fence in it that breaks the field into four subfields. One of the subfields is 9 square kilometers. Can you figure out the areas of the other subfields and the linear dimensions of each subfield?

3) (Continuous Case) Suppose a farmer has a field that is broken into four subfields. You know that the side of the entire field is $n + 3$ kilometers long and the area of the entire field is 169 square kilometers. Can you figure out areas of the subfields and the length of the side of the field?

4) Can you use a 1 km by 1 km field to make a field that is 12/5km by 12/5 km. Can you figure out the area of the whole field using the binomial pattern[60]?

5) Can you use the binomial pattern to figure out the area of a 20 by 20 field? How about a 30 by 30 field, etc?

6) Can you use the binomial pattern to figure out the area of a field that is $(a + b)$ on a side?

Sample quadratic equation problems that were symbolized with paper and pencil and in JavaBars:

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[60] I used the words binomial pattern once the students had established the visual pattern that is displayed in Figure 4.2.

1) A football field is seven kilometers on one side and seven plus seven plus seven kilometers on the other side. Can you make the field in JavaBars and find a way to symbolize what you made with paper and pencil? How many seven squared areas do you have? How much total area is there?

2) The length of a football field is three times the width. The area of the football field is 243 square yards. Can you make the field in JavaBars and figure out the width and length of the field?

Sample linear and quadratic function problems that were symbolized with paper and pencil and in GSP:

1) Can you make a variable segment of length x and use this segment to make a variable square whose area is x^2 (in GSP it is possible to make a square that varies in size in relation to the side of the square)? Can you make a graph in the Cartesian Coordinate System that graphs the length of the side versus the area of the square (in GSP it is possible to measure a variable length and area and further to plot a movable point in the Cartesian Coordinate System that traces the locus of points in the Cartesian Coordinate System)? Can you use your variable length x to make a variable rectangle that is $2x^2$ and a variable square that is $4x^2$? What do you think the graphs of these functions will look like in comparison to the last graph? Can you make graphs for each?

2) Suppose a runner is running at 5 miles per hour. Can you make a variable segment on the x-axis to symbolize the amount of time he has run? Can you use this segment to make the number of miles he has run on the y-axis? Can you make a rate graph of the runner's time versus the distance the runner has traveled? Can you make an equation for the line you made? Suppose the runner's friend starts running three miles south of him at a pace of 6 miles per hour. Can you make a rate graph for this runner? Can you make an algebraic expression for the second runner? Can you find where the two graphs intersect and interpret this in the quantitative problem situation? Can you use your two equations to solve for when the two runner's graphs will intersect?