ABSTRACT ALGEBRA AND SECONDARY SCHOOL MATHEMATICS: IDENTIFYING AND CLASSIFYING MATHEMATICAL CONNECTIONS

By

Ashley Luan Suominen

(Under the Direction of AnnaMarie Conner)

ABSTRACT

Many stakeholders concur that secondary teacher preparation programs should include study of abstract algebraic structures, and most certification programs require an abstract algebra course for prospective mathematics teachers. However, research has shown that undergraduate students struggle to understand fundamental concepts and, upon completion of the course, are unable to articulate connections between abstract algebra and secondary school mathematics. This three-part study involved a textbook analysis, the creation of a comprehensive connections list, and a series of expert interviews. In the textbook analysis, I examined nine introductory abstract algebra textbooks to elaborate the mathematical connections that authors explicitly mentioned between concepts found in abstract algebra and secondary school mathematics. I identified any potential connections made in the text, categorizing them according to five types: alternative or equivalent representations, comparison through common features, generalization, hierarchical relationship, and real-world application. I then interviewed 13 mathematicians and mathematics educators involved in abstract algebra teaching and research to understand how they describe connections between abstract algebra and secondary mathematics. Participants' descriptions of connections reflected their experiences with the secondary curriculum and differed according to their individual conceptualizations of abstract algebra. That is, participants prioritized different sets of connections based on their views of abstract algebra. Identifying and characterizing the connections between abstract algebra concepts and secondary school mathematics concepts offers abstract algebra professors, in particular, additional knowledge that can be used to enhance undergraduate students' understandings of abstract algebra in addition to providing the vocabulary to discuss these connections.

Index Words: Abstract Algebra, Mathematical Connections, Mathematicians, Mathematics Educators, Secondary School Mathematics, Textbooks,

ABSTRACT ALGEBRA AND SECONDARY SCHOOL MATHEMATICS: IDENTIFYING AND CLASSIFYING MATHEMATICAL CONNECTIONS

By

Ashley Luan Suominen

B.S., Wheaton College, 2005

M.A.T., Wheaton College, 2007

M.S., Marquette University, 2011

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2015

ABSTRACT ALGEBRA AND SECONDARY SCHOOL MATHEMATICS: IDENTIFYING AND CLASSIFYING MATHEMATICAL CONNECTIONS

By

Ashley Luan Suominen

Major Professor:	AnnaMarie Conner
Committee:	Jeremy Kilpatrick
	Daniel Krashen
	Patricia Wilson

Electronic Version Approved:

Julie Coffield

Interim Dean of the Graduate School

The University of Georgia

May 2015

DEDICATION

All I am I owe to my mother. I attribute all my success in life to the moral, intellectual, and physical education I received from her. –George Washington

ACKNOWLEDGEMENTS

The completion of this project would not have been possible without the encouragement, kindness, and support of numerous people I will now take the time to thank.

First, I want to thank my mom who has always selflessly sacrificed her life in order to support the fulfillment of my dreams. Thank you for encouraging me to never give up on them despite hardships I was facing. Thank you for financially helping me further my education. Thank you for the countless hours spent on the phone, keeping me sane during times of high emotion and stress. Thank you for making me laugh when I wanted to cry. Thank you for being the greatest friend I could ever ask for or deserve. There are a multitude of reasons I am tremendously grateful for you, but ultimately I am who I am because of your unwavering and endless love. Thank you mom!

Second, I want to thank my dear grandparents. Even though my grandfather is not with us anymore, the impact he has made on my life with never be forgotten. Thank you for reinforcing the importance of education. Thank you for purchasing mathematics video and computer games for me to play at a young age. My love for mathematics first began while playing the fraction pogo stick game on your computer. Thank you for investing your time and money in the numerous school projects I had assigned to me. I will never forget the circus car made with painted noodles and the bridge made with toothpick size wooden sticks that we built together to earn me outstanding student of my district. It was these experiences that contributed to my love for education. I would not be where I am today without them, so thank you for initiating and nurturing my passion for mathematical learning.

Next, I want to thank my beloved husband. Even though we met after I started my Ph.D. program, you quickly made the decision to endure my crazy schedule and limited free time to pursue a relationship with me. I'm not entirely sure what you were thinking, but thank you. Thank you for being willing to sacrifice your career in order for me to pursue mine. Thank you for granting me time to research and write by taking on the bulk of the housework without too many complaints. Thank you for your continual understanding when I had to work nights and weekends, often taking away from the time we could be spending together. And mostly, thank you for your words of encouragement and repeated prayers to the Lord for strength and endurance; they enabled me to run the race with perseverance until completion.

I am extremely thankful to my major professor and doctoral committee whom have read countless drafts of my dissertation. Thank you for all the ways in which you have guided me throughout this journey to complete my PhD.

Thank you also to all the mathematicians and mathematics educators who participated in my study. I learned so much from your vast experience and knowledge of abstract algebra. I want to express my deepest gratitude to you for taking the time to meet with me amidst your busy schedules.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: RATIONALE FOR MATHEMATICAL CONNECTION STUDY

Educational committees and professional organizations concur that secondary teacher preparation programs should include the study of abstract algebraic structures (Leitzel, 1991; National Council of Teachers of Mathematics [NCTM], 2000; Conference Board of the Mathematical Sciences [CBMS], 2001, 2012). Moreover, NCTM (2001) described the subject matter taught in abstract algebra as an "essential component of contemporary mathematics" (p. 1). Consequently, most certification programs currently require a course in abstract algebra for all prospective secondary mathematics teachers. Despite its overall importance to mathematical learning, however, undergraduate and graduate students encountering the subject for the first time often struggle and fail to comprehend many of the fundamental concepts of this course (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron & Dubinsky, 1995). In fact, Leron and Dubinsky (1995) declared that both professors and undergraduate students view the teaching of abstract algebra as a disaster. If abstract algebra is such a valuable course, why has it proven to be so difficult to learn and teach in a meaningful way?

Mathematical Connections

As a mathematics educator, I assume that mathematical connections between concepts and unifying themes can help students understand mathematics. NCTM (2000) affirmed the importance of mathematical connections: "Thinking mathematically involves looking for connections, and making connections builds mathematical understanding. Without connections, students must learn and remember too many isolated concepts and skills" (p. 274). Thus, the connection standard asserts that mathematics educators should provide students with opportunities to “recognize and use connections among mathematical ideas, understand how mathematical ideas interconnect and build on one another to produce a coherent whole, and recognize and apply mathematics in contexts outside of mathematics” (p. 64). NCTM noted that these connections should be made explicit to students so that they may be aware of the connections to increase their mathematical understandings. Similarly, Carpenter and Lehrer (1999) affirmed that students’ understanding of a concept is limited when mathematical connections are not made, because each mathematical concept is developed in isolation. Unfortunately, however, Zazkis and Leikin (2010) revealed that beginning undergraduate students experience great difficulty when starting their undergraduate mathematics courses because of their inabilities to build connections between these courses and the secondary school mathematics curriculum:

> Without connections students have to rely on their memory only and to remember many isolated concepts and procedures. To connect mathematical ideas means linking new ideas to related ideas considered previously and solving challenging mathematical tasks by thinking how familiar concepts and procedures may help in the new situations. (p. 275)

These students’ mathematical understandings are hindered because mathematical ideas are learned in isolation without recognizing the coherent nature of mathematics.

Furthermore, the first CBMS (2001) report on the mathematical education of teachers acknowledged, “Unfortunately, too many prospective high school teachers fail to understand connections between [abstract algebra and number theory] and the topics of school algebra” (p. 40). Cofer (in press) confirmed this assertion when she considered how prospective secondary mathematics teachers communicate the conceptual connections between abstract algebra and their teaching practices. She discovered that prospective teachers were unable to link the two despite having just finished an abstract algebra course. Bukova-Güzel, Ugurel, Özgür, and Kula

(2010) conducted a similar study in Turkey in which prospective secondary mathematics teachers could not communicate any connections between their undergraduate courses and the secondary school mathematics curriculum. Further, Cook (2012) asserted that the difficulties students have in learning abstract algebra are due to the lack of established connections between abstract algebra and school algebra. He affirmed that prospective teachers “do not build upon their elementary understandings of algebra, leaving them unable to communicate traces of any deep and unifying ideas that govern the subject” (p. xvi). As the research literature shows, the content connections between abstract algebra and school mathematics perceived by researchers, professional organizations, and educational committees are often not realized by prospective secondary mathematics teachers. Therefore, one characteristic of the teaching and learning of abstract algebra that might help students use abstract algebra in their teaching would be explicitly identifying the mathematical connections between that course and school mathematics.

The central theoretical lens for this study was the notion of a concept image (Tall & Vinner, 1981; Vinner, 1983). Tall and Vinner (1981) defined *concept image* as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). A concept image can be thought of as a cognitive network of mathematical connections between ideas. A concept definition is then the “verbal definition that accurately explains the concept in a non-circular way” (Vinner, 1983, p. 293). A person’s concept definition would then be the verbal description of his or her personal reconstruction of the concept image. Often, however, a person’s concept definition does not align with the formally accepted mathematical definition. Therefore, it is vital for students to recognize the mathematical connections between mathematical ideas to construct accurate concept definitions.

Importance of School Algebra and Abstract Algebra

> Abstract algebra is a generalization of school algebra in which the variables can represent various mathematical objects, including numbers, vectors, matrices, functions, transformations, and permutations, and in which the expressions and equations are formed through operations that make sense for the particular objects: addition and multiplication for matrices, composition for functions, and so on. (Findell, 2001, p. 9)

Algebra has been a focal point of mathematics education research and curriculum reform movements over the past several decades (Kaput, 1998; NCTM, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSS], 2010). Given the fact that algebra is a chief entry point to higher levels of mathematics, the subject has acted as a gatekeeper for many by preventing students from future academic success and employment opportunities (Kaput, 1998; NGA & CCSS, 2010; National Research Council [NRC], 1998). As a result, many educators have argued that algebra should be integrated into the school mathematics curriculum at an earlier age, playing a more prominent role in all grade levels (e.g., Carpenter, Franke, & Levi, 2003; Kaput, 1998; Kieran, 1992). In fact, the NCTM (2000) asserted that if algebra is viewed as a strand of mathematical thinking throughout the Grades K–12 curriculum, then "[elementary school] teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school" (p. 37). These recommendations recognize the importance of the development of algebraic thinking in children and young adults.

The importance of abstract algebra lies in the opportunities it allows prospective secondary mathematics teachers to develop accurate concept images and concept definitions (Tall & Vinner, 1981) that help them explain and unite concepts found in school mathematics, which can include concepts found in school algebra as well as school geometry. For instance, Gallian (1990) explained the importance of abstract algebra because it exposes undergraduate students to the “terminology and methodology of algebra” (p. xi). Exploring terminology such as *identity*, *inverse*, *commutativity*, and *equivalence* is often common practice in an abstract algebra course, enabling students, including prospective secondary mathematics teachers, the opportunities to develop deeper conceptual understandings of these concepts. Findell (2001) asserted:

> When the population of students in an abstract algebra course includes future teachers (which may be almost always), these big ideas, such as inverse and identity, are particularly important because they can help teachers connect advanced mathematics with high school mathematics in ways that can strengthen and deepen their understandings of the mathematics they will teach. (p. 13)

Using precise mathematical language is vital to prospective secondary mathematics teachers as they attempt to communicate and teach their future students school mathematics. Furthermore, Papick, Beem, Reys, and Reys (1999) suggested that abstract algebra enables students to experience “a rigorous examination of arithmetic properties in various algebraic structures [that] deepens the understanding of traditional arithmetic and accentuates the importance of axiomatic mathematics” (p. 306). The results of such a rigorous examination ideally provide prospective secondary mathematics teachers opportunities to develop more solidified conceptual understandings of school arithmetic and algebraic structures and properties found in the curriculum. The benefits of an abstract algebra course described by previous research parallel the Common Core Standards for Mathematical Practice by encouraging students to “attend to precision,” “look for and make use of structure,” and “reason abstractly” (NGA & CCSS, 2010).

Prospective secondary mathematics teachers do not, however, access the full benefits of learning abstract algebra when mathematical connections are not recognized between abstract algebra and secondary school mathematics. The failure to recognize such connections undermines the purpose of that course and thus represents a significant problem in collegiate mathematics education. For this reason, something must be done to help abstract algebra students recognize and construct mathematical connections to other mathematical ideas. Additional research must be conducted to identify and describe these connections so that students can be made explicitly aware of them. In the remainder of this chapter, I explain the research questions and how my research study began to address these concerns.

Research Questions

Although several research studies have focused on the teaching and learning of abstract algebra, little research has been done on the mathematical connections between abstract algebra and school mathematics. Given that the importance of the undergraduate course in abstract algebra for teacher preparation programs is often explained by the course's ability to theoretically rationalize as well as unite school mathematics concepts, providing the research to explicitly outline these mathematical connections is important. Therefore, I formulated the following research questions in an effort to make these connections between school mathematics and tertiary abstract algebra more explicit:

1) What mathematical connections are explicitly stated between abstract algebra and secondary school mathematics in abstract algebra textbooks, and how are these connections discussed?
2) Which mathematical connections between abstract algebra and secondary school mathematics do mathematicians and mathematics educators identify, and how do they describe them?

Purpose of This Study

The purpose of investigating the mathematical connections between abstract algebra and school mathematics was complex. First, in this study, I accepted the rationale that abstract algebra should be a requirement of teacher preparation programs and thus it should aim to align abstract algebra concepts to the secondary school mathematics curriculum. The results from this research further illuminated the content connections between abstract algebra and secondary school mathematics. The more thorough examination of such ideas can cultivate more accurate concept images and concept definitions within prospective mathematics teachers. Second, practicing secondary mathematics teachers may also find the detailed mathematical connections beneficial in providing a theoretical explanation for certain topics of school mathematics that would ultimately enhance their students' learning. Professional development programs for secondary mathematics teachers, in particular, can be organized around these results by explicitly discussing these connections. Third, identifying and characterizing such connections can serve as a teaching supplement for abstract algebra professors who may not have considered the secondary school mathematics curriculum in their teaching. This knowledge can then be used to enhance undergraduate students' understandings of abstract algebra by providing the vocabulary to discuss these connections. The results of this study further the existing research on the teaching and learning of abstract algebra by suggesting new ways to think about teaching and learning the course.

The following chapter reviews the relevant literature on the teaching and learning of abstract algebra. I then discuss the different definitions of *mathematical connections* found in literature. Lastly, I elaborate on the limited amount of existing research directly addressing mathematical connections identified by undergraduates, primarily prospective secondary mathematics teachers, in tertiary mathematics.

CHAPTER 2: LITERATURE REVIEW

The purpose of this chapter is to summarize and critique literature relevant to the research questions. In the first section, I discuss research concerning the teaching and learning of abstract algebra. Next, I explain several different perspectives of mathematical connections and discuss how these perspectives informed my definition of *mathematical connections*. In this section I also summarize the types of mathematical connections identified in previous research. Finally, I elaborate on past research directly addressing mathematical connections in tertiary mathematics.

Teaching and Learning of Abstract Algebra

Research on the teaching and learning of abstract algebra has indicated the conceptual understanding of undergraduate students in abstract algebra is less than satisfactory (Dubinsky et al., 1994; Hazzan & Leron, 1996). Leron and Dubinsky (1995) asserted that the teaching of abstract algebra is considered to be a failure by both professors and undergraduate students. As a result, many undergraduate and graduate students, including prospective teachers, struggle to grasp even the most fundamental concepts of this course (Dubinsky et al., 1994; Hazzan & Leron, 1996; Leron & Dubinsky, 1995). For many of these students, taking abstract algebra is the first time they experience a higher level of mathematical abstraction and formal proof. It is often the first tertiary course in which teachers expect students to "go beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al., 1994, p. 268). Nevertheless, it is widely acknowledged that abstract algebra is an essential part of undergraduate mathematical learning despite its high level of difficulty at the collegiate level (Gallian, 1990; Hazzan, 1999; Selden & Selden, 1987).

Even with the importance of learning abstract algebra, the known difficulties of the subject, and an increasing amount of research on the teaching and learning of tertiary mathematics, few studies concentrate solely on the teaching and learning of abstract algebra. In this section, I summarize past research on the teaching and learning of abstract algebra, which can be classified into three categories: teaching methods (Freedman, 1983; Pedersen, 1972; Thrash & Walls, 1991), student learning (Asiala, Brown, Kleiman, & Mathews, 1998; Brown, DeVries, Dubinsky, & Thomas, 1997; Leron, Hazzan, & Zazkis, 1995), and proof writing (Hart, 1994; Selden & Selden, 1987; Weber 2001). Mostly, the literature review provides a synopsis of research in the first two categories; namely, teaching methods and student learning. A thorough investigation of that research clearly shows there is a limited amount of current research in the teaching and learning of abstract algebra. Furthermore, even less research exists about the mathematical connections between abstract algebra and school mathematics.

Teaching Methods. In response to an early awareness of the difficulties students were having in learning abstract algebra, several researchers introduced alternative teaching methods. One of the earliest papers on teaching abstract algebra was Pedersen's (1972) presentation of a learning activity that involved 10 paper equilateral triangles, which students folded in various ways to evolve the noncyclic 6-group. This hands-on, discovery-learning activity encouraged students to develop mental concept images and accurate concept definitions of the noncyclic 6-group in a unique way. Similarly, Huetinck (1996) introduced the SNAP learning activity in which students initially rotated and translated an equilateral triangle on an overhead transparency sheet to explore all possible orientations to introduce basic group theory ideas. Students then used a nine-peg 3×3 square array board with three rubber bands to discover patterns through various reorientations of the rubber bands. Larsen (2004, 2009) utilized both permutations and symmetries of an equilateral triangle and the SNAP learning activity to understand how students reinvent the concepts of group and group isomorphism.

In his dissertation, Larsen (2004) detailed the development of the instructional design theory of Realistic Mathematics Education (RME) and guided reinvention, which allowed him to instruct students using discovery-based learning activities while also being able to analyze how those students thought about abstract algebra topics. These discovery-based learning activities endeavored to build upon students' intuition and past mathematical knowledge to develop mathematical meaning of abstract concepts. For instance, Larsen (2004) described the learning activities in his dissertation, "In each teaching experiment, I started the process by having the students identify the properties common to all of the situations they had considered" (p. 133). Larsen and Lockwood (2013) then modified this work to explore students' understanding of parity as it relates to quotient groups. Cook (2012) paralleled this approach in the teaching and learning of rings, fields, and integral domains. He found that through these discovery-based learning activities, the participants were able to develop mathematical connections between abstract algebra concepts. For instance, Cook noticed that both of the participants began to recognize a connection between the structural features of $\mathbb{Z}_5$ and the modulus because of their similar features. He also noted,

> Starting with $\mathbb{Z}[x]$, both pairs of students were fully aware that the solution to the equation $x + a = a + b$ would be largely the same as the previous solution. Specifically, in this case, the students were able to make a connection between the integers and polynomials over the integers. (p. 124)

Through discovery-based learning activities, the participants in these studies were able to identify similar features and structures to develop mathematical connections between concepts.

Freedman (1983) offered another approach to teaching abstract algebra in detailing a unique lecture-based method that gradually required students to take an active part in the learning process through teaching. In this three-stage teaching method, students initially learned through traditional lecture. Then in the second stage, students were required to complete a project as well as do a little of the teaching. Finally, students were solely responsible to design objectives and prepare all lectures. Through this active participation in the teaching process, Freedman claimed the students in the study were able to gain a strong understanding of the topics by being able to clearly explain them to others in the class. Still other researchers (Brown, 1990; Czerwinski, 1994; Leganza, 1995) suggested the use of writing assignments in abstract algebra to enhance student involvement in learning. For instance, one of these assignments asked students to write about connections between a certain abstract algebra concept and real-world applications in order to make the abstract more concrete.

Another technique found in the literature regarding the teaching of abstract algebra is the use of computers. Gallian (1976) first suggested using Fortran to investigate finite groups. More recently, Leron and Dubinsky (1994), in response to the findings of Dubinsky et al. (1994), introduced key ideas related to group theory through the use of the programming language ISETL. That program allowed students to construct an algebraic environment or structure with a set of axioms and properties in order to work through specific examples and problems. Hodgson (1995b) reported successful learning results of students using ISETL to learn abstract algebra. Similarly, Asiala, Dubinsky, Mathews, Morics, & Oktaç (1997) implemented the ACE teaching cycle (Activities, Class discussion, and Exercises), where students rotated in groups between classroom activities and computer activities using the language ISETL. Asiala, Dubinsky, et al. affirmed that through this teaching method students demonstrated deep understandings of cosets, normality, and subgroups by their performance on the two given tests and a final exam as well as participants' responses during two sets of interviews.

Student Learning. In studying student understanding and the development of abstract algebra concepts, researchers (Brown et al., 1997; Dubinsky et al., 1994) have used the APOS (Action-Process-Object-Schema) theoretical framework. APOS theory consisted of four major components: constructing mental actions, constructing processes, constructing objects, and organizing them in schemas. A mental *action* is a repeatable transformation of objects requiring step-by-step instructions to perform an operation either explicitly or from memory. When the mental construction has been reflected upon and interiorized, the action has turned into a *process*. From a process, an individual can construct an *object* by recognizing the transformations acting on the constructed process. Finally, a *schema* of a specific concept is developed through the process of organizing the collection of actions, processes, objects, and other schemas related to this concept in such a way that other problem situations involving that concept can be confronted.

Dubinsky et al. (1994) utilized the APOS framework to understand high school teachers' development of group theory during a six-week workshop. The results of a written assessment at the end of the workshop and a follow-up interview with ten selected participants showed that participants developed understandings of group and subgroup in a parallel manner and depended on past mathematical knowledge such as set and function. Dubinsky et al. wrote, "There are a number of specific instances in which what is understood relative to one concept was used in constructing new understandings of another" (p. 23). In a follow-up study, Brown, DeVries, Dubinsky, and Thomas (1997) examined how abstract algebra undergraduate students understood binary operations, groups, and subgroups while using the ACE teaching cycle with classroom activities and computer activities using the language ISETL. Based on the participants' performance on three written exams taken throughout the course and their responses during two sets of interviews, Brown et al. found that students had difficulty fully understanding groups and subgroups without a deep understanding of binary operations.

In summary, much of the research affirms students' difficulties in learning fundamental concepts in group theory (Asiala, Dubinsky, et al., 1997; Larsen, 2004, 2009; Leron, Hazzan, & Zazkis, 1995), with little concentration on students' learning of other algebraic structures like rings or fields (Cook, 2012). Despite the overarching focus on discovery-based instructional approaches in the teaching of abstract algebra, none of the studies directly addressed the mathematics content connections between abstract algebra and school mathematics. Much of this research endeavored to build on students' intuitions about mathematics without discussing one of the primary sources of that mathematical knowledge; namely, school mathematics. Furthermore, even though abstract algebra is considered to be "a generalization of school algebra" (Findell, 2001), the explicit generalizations from school algebra to abstract algebra have not been thoroughly researched thus far.

Mathematical Connections

Prior to discussing the existing research about tertiary mathematical connections, it is important to define what is meant by *mathematical connections*. The Oxford English Dictionary (2014) defines a connection as "a relationship in which a person, thing, or idea is linked or associated with something else." A mathematical connection could then be defined as a relationship between a mathematical idea linked or associated to another mathematical idea. Businskas (2008) and Singletary (2012) provided similar definitions for a *mathematical connection* as "a true relationship between two mathematical ideas, A and B" (Businskas, 2008, p. 18) and "a relationship between a mathematical entity and another mathematical or nonmathematical entity" (Singletary, 2012, p. 10). However, both researchers emphasized in their work that the literature defines this relationship in different ways. For instance, some researchers viewed these connections as a characteristic of mathematics. Singletary (2012) called this perspective *mathematical connections as a part of a connected discipline*. Other researchers viewed mathematical connections as an artifact of student learning. Still another group of mathematics education researchers viewed mathematical connections as an active process of doing mathematics. In the following subsections, I examine all three perspectives in light of existing literature to provide a more robust understanding of the definition of *mathematical connections*.

Mathematical Connections: Characteristic of Mathematics. National education reports and existing literature give various descriptions of how mathematical connections are characteristic to mathematics. The NCTM characterized mathematics as "a web of closely connected ideas" (NCTM, 2000, p. 200) and "a unified discipline, a woven fabric rather than a patchwork of discrete topics" (NCTM, 1995, p. vii). Coxford (1995) described mathematical connections as the unifying themes or mathematical processes that relate different ideas in mathematics, which can be used across mathematical topics to "to draw attention to the connected nature of mathematics" (Businskas, 2008, p. 8). For instance, unifying themes such as function and variables and mathematical processes such as proof and problem solving are widely found in mathematics. Unlike Coxford's understanding of broad mathematical connections across the discipline, other researchers considered the more fine-grained concept-by-concept mathematical connections (Businskas, 2008; Skemp, 1987; Zazkis, 2000). Still another subset of literature defined mathematical connections as equivalent representations across mathematical ideas (Businskas, 2008; Chappell & Strutchens, 2001; Hodgson, 1995a). For example, researchers recognized the equivalence of various methods of solving systems of linear equations: algebraic manipulation, matrix systems, and graphing. Despite some slight differences in the literature about the type of mathematical connections, all of these researchers maintained the belief that mathematical connections exist and should be acknowledged throughout the discipline.

Mathematical Connections: Artifact of Learning. Businskas (2008) described this type of mathematical connection as "a process that occurs in the mind of the learner(s) and the connection is something that exists in the mind of the learner" (pp. 12–13). The constructivist notions surrounding this definition of mathematical connections often are linked to the work of Piaget, in which the connections are established by the learner as an attempt to organize or interiorize mathematical ideas into a schema.

The Piagetian notion of abstraction is widely found in the literature when considering the learning of abstract algebra. Hazzan (1999) explained:

> Students' tendency to rely on systems of numbers, when asked to solve problems about other groups, can be explained by one of the basic ideas of constructivism. That is, that new knowledge is constructed based on existing knowledge. Thus, unknown (hence abstract) objects and structures are constructed based on existing mental structures. (p. 76)

As described by Hazzan, abstraction is the process whereby the student constructs connections between concrete mathematical ideas and abstract or unknown mathematical ideas in order to enable his or her complete understanding. Noss and Hoyles (1996) concurred with this notion: "[Mathematical] meaning can be maintained by involvement in the process of acting and abstracting, building new connections whilst consolidating old ones" (p. 49). Similarly, Hiebert and Carpenter (1992) wrote, "A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections" (p. 67). In light of this literature, mathematical connections are a necessary component of learning. Mathematical connections must be established between preexisting schema or networks in order for unknown mathematical ideas to be fully understood by the learner. Ultimately, Hazzan (1999) and Hiebert and Carpenter (1992) concluded that the teacher plays a pivotal role in helping the students to construct these connections among mathematical ideas.

Mathematical Connections: A Mathematical Activity. Still another perspective about mathematical connections is found in the literature—establishing connections as a mathematical process or activity. In some sense this final perspective can be characterized as the blending of the initial two perspectives. To be more specific, this perspective acknowledges that connections exist across mathematics and that the learner should be involved in the activity of establishing or identifying these connections. Boaler (2002) asserted:

> It seems to me that the act of observing relationships and drawing connections, whether between different functional representations or mathematical areas, is a key aspect of mathematical work, in itself, and should not only be thought of as a route to other knowledge. (p. 11)

The activity of making connections across mathematics, similar to those described by Coxford (1995), was seen as a significant aspect of doing mathematics (Boaler, 2002). In fact, NCTM (2006) characterized several mathematical activities that helped secondary students recognize the coherent nature of mathematics, which include: multiple representations, problem solving, proof, and real-world applications and mathematical modeling. These mathematical activities again align with Coxford's (1995) view.

Blending the Perspectives. The three perspectives highlighted in the literature—a characteristic of mathematics, an artifact of learning, and a mathematical activity—provide unique ways to consider mathematical connections. As Businskas (2008) and Singletary (2012) suggested, holding to only one of these perspectives seems unnecessary. Each provides different facets to help establish a more robust understanding of the definition. From my perspective, mathematical connections exist across the discipline and can be considered as both broad unifying themes and concept-by-concept links. In order to best learn new mathematical ideas, students should engage in mathematical activities that enable them to construct connections between existing knowledge and new unknown ideas. To address my first research question, I focused primarily on the mathematical connections as a characteristic of mathematics and more specifically the concept-by-concept mathematical connections. This type of connection was discussed with the mathematicians and mathematics educators during their interviews. I also considered, however, the other perspectives of mathematical connections when analyzing how mathematicians and mathematics educators talk about mathematical connections.

As a result of accepting a flexible definition of mathematical connections, I again go back to the definitions given by Businskas (2008) and Singletary (2012) as shown on page 15. Each of these definitions highlights the relationship that exists between mathematical ideas or entities, regardless of the perspective one has on mathematical connections. This relationship can take several forms. Previously established categories of mathematical concept-to-concept connections are displayed in Table 1 (Businskas, 2008; Singletary 2012). These categories aided me when analyzing the textbook and interview data.

Table 1

Categories of Mathematical Connections From Research

Category	Description
Alternative representation	One concept is represented in different ways such as symbolic (algebraic), graphic (geometric), pictorial (diagram), manipulative (physical object), verbal description (spoken), or written description.
Comparison through common features	Two concepts share some features in common, which allows a comparison through the concepts being similar, exactly the same, or not the same.
Equivalent representation	One concept is represented in different ways but within the same form (i.e., one concept could be represented in different ways symbolically).
Generalization	One concept is an example of specific instance of another concept.
Hierarchical relationship	One concept is a component of or included in another concept. Since one concept is included or contained in the other concept, a hierarchical relationship exists between two concepts.
Logical implication	One concept logically dependences on another concept. Often an *if-then* relationship exists between the two concepts.
Procedural	One concept can be used to find another concept. The first concept could be a type of procedure or connecting method used when working with the other concept.
Real-world application	One concept is an example of another concept in the real-world (i.e., a concept refers to another concept outside the current mathematical context).

Mathematical Connections to Tertiary Mathematics

In this final section, I examine literature that addresses mathematical connections established in tertiary mathematics. The research can be classified into two categories: the lack of identified mathematical connections by undergraduate students and the consequences of not establishing mathematical connections to learning. Much of the research concentrates on prospective secondary mathematics teachers, but other literature focuses on all tertiary mathematics students.

Previously Identified Mathematical Connections. Some of the earliest work that specifically identified mathematical connections between abstract algebra and secondary school mathematics was done by Usiskin (1974, 1975). In a first article, he examined properties of addition and multiplication of real numbers as they related to teaching and compared them to the structural properties of groups and fields. He declared that "all properties of the reals follow from the complete field properties" (Usiskin, 1974, p. 279). He then described the isomorphic relationship between linear and exponential functions through the map $x \to a^x$, which forms an isomorphic group under composition. Usiskin's second article highlighted similar mathematical connections; namely, connections between algebraic structures and known number systems and operators as well as the isomorphic connection between linear and exponential functions as seen in the properties of exponents. In this article, he also discussed connections between the group structure and solving simple linear equations, the multiplicative group of invertible 2×2 matrices and solving systems of linear equations, and additive and multiplicative groups with familiar number systems and groups of geometric transformations. Lastly, Usiskin detailed the connections between groups and trigonometric functions. For instance, the 2×2 matrix $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ forms a group under composition and this group describes the properties of rotation about the origin. In another article, Usiskin (2001) described the connections mentioned in these two articles as generalizations taught in abstract algebra.

The Conference Board of the Mathematical Sciences (CBMS), established by the American Mathematical Society (AMS) in partnership with the Mathematical Association of America, published two reports, the first published in 2001 with an updated version published in 2012, entitled *The Mathematical Education of Teachers (MET)* and *The Mathematical Education of Teachers II (MET II)* in which recommendations are made regarding the mathematical knowledge necessary for teaching at all grade levels: elementary, middle, and high school. In the chapters addressing secondary mathematics teachers, CBMS suggested ways in which undergraduate courses can better connect to secondary school mathematics given that prospective secondary teacher are not currently recognizing those connections. In the following paragraphs, I outline these suggestions as they pertain to the mathematical connections between abstract algebra and secondary school mathematics.

In the first MET report, CBMS (2001) focused primarily on the mathematical connection of generalization when describing the connections between abstract algebra and secondary school mathematics. They characterized abstract algebra as the examination of “mathematical structures that are the foundation for number systems and algebraic operations” (p. 40). Here we see that algebraic structures found in abstract algebra are generalizations of familiar number systems and operators. Another illustration of the emphasis placed on generalization connections in the MET report was given through a specific example: One task that can be incorporated into an abstract algebra course is “to show explicitly how the number and algebra operations of secondary school can be explained by more general principles” (p. 40). A second example was also given, in which abstract algebra students solve a linear or quadratic equation and for each step write out the necessary field property. In this example, the algebraic structures are again generalizations of operators but also the generalization of familiar solving procedures. In addition, the first MET report highlighted mathematical connections between abstract algebra and school geometry. Often these connections are overlooked by mathematicians and policymakers in favor of the connections between abstract algebra and school algebra. These mathematical connections again emphasized generalization connections by describing isometry and symmetry groups as generalizations of the geometry of transformations of regular polygons.

The second MET report provided additional specific examples of mathematical connections between abstract algebra and secondary school mathematics. CBMS (2012) recommended the study of ring and field structures as the underlying structures of operations with polynomials and rational functions. Another connection described in this report was between complex numbers and real number polynomials: "It would be quite useful for prospective teachers to see how $\mathbb{C}$ can be "built" as a quotient of $\mathbb{R}[x]$ and, more generally, how splitting fields for polynomials can be gotten in this way." CBMS also suggested a focus on mathematical concepts *inverse* and *identity.* For instance, they mentioned the similar idea behind the abstract algebra concept *inverse* and the secondary school mathematics concepts *additive* and *multiplicative inverse*, *inverse matrix*, and *inverse function*. The abstract algebra concept *isomorphism* was identified as important to draw connections through comparison of common features between "the real numbers and the multiplicative group of the positive real numbers given by the exponential and logarithm functions" (CBMS, 2012, p. 59).

Unknown Mathematical Connections. Bukova-Güzel et al. (2010) conducted a qualitative study in Turkey that considered prospective secondary mathematics teachers' perspectives of their undergraduate mathematics courses in connection to secondary teaching. Thirty-six students in their last year of their teacher preparation program from four different universities in Turkey were asked by their instructors or via email four open-ended questions regarding their undergraduate content courses in Calculus, Analytic Geometry, Linear Algebra, Abstract Mathematics, Topology, and Differential Equations. The purpose of the study was threefold: (a) identify student teachers' opinions on whether their mathematics courses fully prepared them to teach, (b) determine whether these courses were necessary to their understanding and abilities to teach, and (c) ascertain the participants' perceptions of the quality of these courses. Bukova-Güzel et al. found that 83% of the prospective secondary mathematics teachers did not see connections between the undergraduate content courses they had taken and the secondary school mathematics curriculum. However, 25% of the participants did believe that first-year undergraduate courses, such as Calculus and Analytic Geometry, were coherent and related to the secondary curriculum. One participant's response stated:

> Since the courses are based on memorizing theorems and passing exams, it is really hard for us to apply even useful knowledge. At least on my own behalf, I was better at secondary school mathematics topics when I graduated from secondary school. (p. 2236)

Furthermore, 42% of the participants recommended designing new undergraduate courses that were directly related to secondary school mathematics. In fact, 56% of the prospective secondary mathematics teachers felt insufficiently prepared mathematically to teach. We can see from these results that the participants did not feel that their undergraduate content courses made connections to the secondary school mathematics curriculum.

Cofer (in press) confirmed these results when she considered how prospective secondary mathematics teachers communicated the mathematical connections between abstract algebra and their teaching practices. Five upper division tertiary students, all of whom had recently completed coursework in abstract algebra, were involved in an interview study that asked them to explain abstract algebra concepts connected to the school mathematics curriculum, such as division by zero and even numbers. Cofer discovered that the students were unable to relate abstract algebra concepts to the school mathematics curriculum despite having just finished the abstract algebra course.

Consequences of Not Establishing Mathematical Connections. Cook (2012) hypothesized in his dissertation that the difficulty students experience in learning abstract algebra is due to the lack of established connections between undergraduate mathematics and school mathematics. He wrote that prospective teachers "do not build upon their elementary understandings of algebra, leaving them unable to communicate traces of any deep and unifying ideas that govern the subject" (p. xvi). Similarly, Cuoco (2001) noted in an article about secondary teacher preparation programs, "Most teachers see very little connection between the mathematics they study as undergraduates and the mathematics they teach. This is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra" (p. 169). To put it simply, students are not recognizing or establishing the mathematical connections between abstract algebra and high school mathematics (Usiskin, 1988). Ultimately, this inability to make these connections hinders students' learning of advanced mathematics as well as hurts future teachers' ability to teach mathematics. Zazkis and Leikin (2010) affirmed:

> Without [mathematical] connections students have to rely on their memory only and to remember many isolated concepts and procedures. To connect mathematical ideas means linking new ideas to related ideas considered previously and solving challenging mathematical tasks by thinking how familiar concepts and procedures may help in the new situations. (p. 275)

There are few studies about mathematical connections in tertiary mathematics, and there is a great need for further research in this area. Previous researchers (or studies) have focused on the perceived mathematical connections of prospective secondary mathematics teachers. The perspectives of textbooks and professors have also not been considered in the literature. In the present study, I explored these gaps in the literature regarding mathematical connections in abstract algebra and secondary school mathematics.

CHAPTER 3: METHODOLOGY

In this chapter, I explain the specific methodology of this research. I first describe the rationale for the research design and the role of the researcher. Second, I explain the definition of *abstract algebra* used in this study. This definition provided the necessary boundaries to determine which abstract algebra concepts should and should not be included in this research. Next, I elaborate on the three-part data collection: textbook analysis, compilation of mathematical connection list, and participant interviews. Last, I discuss the methods of data analysis employed in this study.

Research Design

Qualitative research lent itself well to the present research study since it enabled the stories of the participants to be told. In describing qualitative research, Patton (2002) emphasizes the importance qualitative research places on understanding and capturing "the points of view of other people" (p. 21), which was at the heart of the present study. The purpose of this study was to understand the perspectives of mathematicians and mathematics educators about mathematical connections found in abstract algebra as expressed through interviews as well as the perspectives of written materials such as textbooks and previous literature. Thus, this study utilized two main sources of data: abstract algebra textbooks and interviews with mathematicians and mathematics educators involved in abstract algebra teaching and research.

Role of the Researcher

When conducting qualitative research, the role of the researcher must be considered given that the researcher's perspective influences the research process (Patton, 2002). My beliefs about and experiences with education and mathematics inevitability played a role in the research design of this study and the data collection and analysis. Therefore, I briefly describe my beliefs related to mathematics education and how those beliefs may have influenced my research.

As a student, I had several negative experiences in which mathematics was reduced to the memorization of procedural rules. These experiences challenged the ways I understood mathematics. To me, mathematics was the study of relationships between mathematical ideas about quantity, space, and structure across different mathematical domains. Mathematics was about recognizing existing patterns and formulating new conjectures based on these patterns. And ultimately, mathematics was a coherent, logical system in which every procedure originated from sound reason. The explanations behind the mathematics caused me to adore the subject. Many of my collegiate professors encouraged such beliefs about mathematics, so despite conflicting experiences with mathematics, I still adhere to my initial beliefs about mathematics as a complex, interrelated, beautiful system of mathematical ideas. Consequently, as a teacher, I explicitly discussed mathematical connections across various concepts and domains to help my students develop more accurate concept images and meaningful understandings of mathematics. It is my belief that students have an easier time conceptually understanding mathematics when new mathematical ideas are actively built upon previous experiences and prior knowledge.

It was important, in conducting this study, to recognize that I identify myself as both a mathematician and a mathematics educator. Having spent a significant portion of my graduate work in pure mathematics (particularly abstract algebra), I found it natural to pursue research in this area as it relates to education. When discussing my research passion with my peers, most complained about the difficulty level and uselessness of the course. Previous research about the teaching and learning of abstract algebra further illuminated these concerns. The combination of my beliefs about mathematics, my passion for abstract algebra, and my experiences with students and peers motivated this study on the mathematical connections between abstract algebra and school mathematics. Furthermore, my desire to conduct this study was rooted in the belief that teachers must deeply and conceptually know what they teach, and a course in abstract algebra can provide some of that necessary background knowledge about school mathematics to future teachers.

As a researcher, I recognize that I cannot separate myself my personal beliefs about mathematics and past experiences from my research. Therefore, I acknowledge that they likely influenced the ways I thought about, conceptualized, and continue to think about my research. It is possible that those beliefs acted as a lens for the mathematical connections I identified and the ways in which I categorized and described them. For these reasons, I reexamined the collected data several times, I relied on my analytic framework throughout my data collection and analysis, and I frequently met with my major professor to have a person with an additional perspective review and question my research findings.

Definition of Abstract Algebra

Renze and Weisstein (n.d.) define abstract algebra as "the set of advanced topics of algebra that deal with abstract algebraic structures rather than the usual number systems." In the undergraduate abstract algebra course, these algebraic structures include groups, rings, and fields. This study focused only on concepts associated with these algebraic structures and ignored any additional topics. Establishing such boundaries were necessary for this research because introductory abstract algebra courses vary in content, often including topics from linear algebra, number theory, and discrete mathematics. For the purpose of this study, I instead wanted to concentrate on what I considered the essential course concepts.

Data Collection

In this section, I describe the three parts of data collection for this study. First, I analyzed abstract algebra textbooks using the mathematical connections categories described in detail in Chapter 2 and summarized in Table 1. Second, I identified mathematical connections that were not explicitly stated in the textbooks. Third, I conducted interviews with mathematicians and mathematics educators specializing in abstract algebra. These parts of data collection aligned with the study's research questions (see p. 6).

Part One: Textbook Analysis. The first part of data collection in this study was to conduct an analysis of nine abstract algebra textbooks. The textbooks were chosen as the chief source of data about the abstract algebra curriculum since I believe that textbooks often drive the tertiary mathematics curriculum. Robitaille and Travers (1992) stated:

> Teachers of mathematics in all countries rely very heavily on textbooks in their day-to-day teaching, and this is perhaps more characteristic of the teaching of mathematics than of any other subject in the curriculum. Teachers decide what to teach, how to teach it, and what sorts of exercises to assign to their students largely on the basis of what is contained in the textbook authorized for their course. (p. 706)

I believe that although Robitaille and Travers were referring to Grades K–12 mathematics teachers, their observation is also true of tertiary mathematics professors. Also, I believe that mathematical connections between abstract algebra and school mathematics are more likely to be discussed in the abstract algebra class if they are found in textbooks. However, little research has been conducted on the analysis of abstract algebra textbooks. Capaldi (2012) analyzed abstract algebra textbooks to investigate the potential reader audience; level of detailed explanations, examples, and proofs; and content covered in the book. Using reader-oriented theory, Capaldi found discrepancies between the intended and the actual reader in regards to language maturity and level of details. Textbook examples were the only area in which the language maturity and level of details aligned between the intended reader and the actual reader. Similar research has been conducted on linear algebra textbooks concentrating again on the presentation of a certain concept (Cook & Stewart, 2014; Harel, 1987). No other research, at this point, exists studying textbooks related to tertiary algebra.

I examined nine abstract algebra textbooks for this research, based on a few criteria. First, I narrowed my focus to introductory undergraduate abstract algebra textbooks. While graduate textbooks may explicitly state mathematical connections between abstract algebra and secondary school mathematics, this study concentrated solely on undergraduate learning of abstract algebra. Second, all textbooks were published within the past 20 years from the start of the research study because they would be more likely to be used in undergraduate classrooms today. Next, I compiled a list of recently published abstract algebra textbooks by: (1) examining syllabi available online for introductory abstract algebra courses at more than 20 colleges and universities around the United States, (2) contacting five different textbook publishers about widely readily used abstract algebra textbooks, and (3) conducting online searches of textbook provider websites for textbooks that explicitly emphasis connections to school mathematics. The textbooks used in the study are listed in Table 2.

Because the focus of this study differed from that of previous research on mathematics textbooks, I utilized previous mathematical connections research described in the literature review as my analytic framework to identity and categorize any explicitly stated connections found in the abstract algebra textbooks (described in detail in Chapter 2 and summarized in Table 1). I defined connections to be *explicitly stated* when the authors without a doubt intended concepts to be linked in some way to secondary school mathematics. After analyzing the textbook data, I modified the initial analytic connection framework to include only the five mathematical connections categories found in the textbooks and better describe the types of connections based on the data (described in detail in Chapter 4 and seen in Table 3).

Table 2

Included Abstract Algebra Textbooks

Author	Title	Publication year
Cuoco & Rotman	*Learning Modern Algebra: From Early Attempts to Prove Fermat's Last Theorem*	2013
Dummit & Foote	*Abstract Algebra (3rd ed.)*	2004
Fraleigh	*A First Course in Abstract Algebra (7th ed.)*	2003
Gallian	*Contemporary Abstract Algebra (8th ed.)*	2013
Hillman & Alexanderson	*Abstract Algebra: A First Undergraduate Course (5th ed.)*	1994
Hodge, Schlicker, & Sundstrom	*Abstract Algebra: An Inquiry-based Approach*	2014
Nicholson	*Introduction to Abstract Algebra (4th ed.)*	2012
Nicodemi, Sutherland, & Towsley	*An Introduction to Abstract Algebra with Notes to the Future Teachers*	2007
Shifrin	*Abstract algebra: A geometric approach*	1996

While reading each abstract algebra textbook in its entirety (explanation sections, homework problems, and any additional material), I documented when the book made an explicit connection to secondary school mathematics content. For instance, Nicholson (2012) used students' prior knowledge of the geometric series and convergence to prove that if a is nilpotent in R, then $1 - a$ and $1 + a$ are units. He then made the connection between this proof and students' knowledge of the geometric series when he wrote:

> In elementary algebra it is proved that, if $x \in \mathbb{R}$, and $|x| < 1$, the geometric series $1 + x + x^2 + \cdots$ converges for any real number x with $|x| < 1$ and equals $(1 - x)^{-1}$ in this case. … We recognize that $1 + a + a^2 + \cdots$ makes sense in any ring R when a is nilpotent, which then provides a formula for $(1 - a)^{-1}$. (p. 167)

Since the proof required the prior knowledge of geometric series and convergence, this connection was classified as a hierarchical relationship.

For each textbook, I created a list of every identified mathematical connection, noted where each connection was found in the text, and classified the type of connection based on the analytic connection framework. As previously mentioned, I modified the framework to more effectively categorize the data. Next, I constructed separate tables for each connection category detailing all the connections found in the textbook. I then created a list of the stated mathematical connections in preparation for participant interviews that included all of the adjusted categories except for the real-world application category. I purposely excluded that category because it was unique to a small number of textbooks and did not specifically address the study's research questions about school mathematics. In order to ensure accuracy in the classifications of connections, I reexamined all of the mathematical connections found in the abstract algebra textbooks a second time prior to moving on to the next part of the study. The list of mathematical connections from the textbook analysis can be seen in Appendix B.

Part Two: Compile Mathematical Connection List. After I formulated an initial list of mathematical connections from the abstract algebra textbooks, I used my own knowledge of abstract algebra and secondary school mathematics to add four additional mathematical connections to the list; namely, cyclic group with imaginary unit i, ideal with subset and number systems, kernel with nullspace of a matrix, and subgroup with subset. This comprehensive list of mathematical connections was used for interviews and can be seen in Appendix C.

Part Three: Participant Interviews. Last, I conducted interviews with mathematicians and mathematics educators. This qualitative research activity enabled me to explore the perspectives of the participants as they shared their experiences and knowledge regarding abstract algebra and secondary school mathematics. I used purposive case homogeneous sampling to choose participants who had expertise in abstract algebra (Patton, 2002; Roulston, 2010). To be more specific, mathematicians and mathematics educators were chosen if they had taught or were currently teaching an undergraduate abstract algebra course or were involved in abstract algebra research of some kind. These participants ranged from pure mathematicians that have only published in pure mathematics journals with little to no experience with the secondary school curriculum to mathematics educators that have only published in mathematics education journals with several years of secondary school teaching experience. In total, I interviewed 13 mathematicians and mathematics educators.

I initially contacted each participant through an email in which I introduced myself, outlined the purpose of the study, and discussed why I specifically chose him or her to participate in this research. I then asked when each participant would be available to set up an interview. Through this initial contact, I established that this study was about the mathematical connections between abstract algebra and secondary school mathematics that could be explored in an abstract algebra course. I also briefly mentioned how identifying these connections will be beneficial for all undergraduate students learning abstract algebra regardless of their major.

Each participant was involved in one interview that lasted between 40 and 70 minutes. A semistructured interview protocol was used, consisting of a predetermined list of open-ended questions that included knowledge questions regarding abstract algebra and secondary school mathematics and reflection questions about the formulated list of connections (see Appendix A). The types of questions used in the study were based on question types given in Patton (2002), Taylor and Bogdan (1984), and Zazkis and Hazzan (1999). Given the range of participants' experiences with the secondary school mathematics curriculum, I prepared in advance a list summarizing some of the curriculum topics found in secondary school mathematics based on NGA and CCSS (2010) in case a participant was not familiar with the curriculum. During the interview, participants were given the option of including their names or affiliations in the research results. Participants who chose to include their names are listed below. In addition to those who chose to identify themselves, one assistant professor from a small liberal arts college and one assistant professor from a university that specializes in graduate programs in education participated in the study.

- Daniel Anderson, University of Iowa, Mathematics Department
- George Andrews, The Pennsylvania State University, Department of Mathematics
- Tanya Cofer, Northeastern Illinois University, Mathematics Department
- Al Cuoco, Education Development Center, Director of the Center for Mathematics Education
- Joseph Gallian, University of Minnesota Duluth, Department of Mathematics and Statistics
- Timothy Fukawa-Connelly, Drexel University, School of Education
- Brian Katz, Augustana College, Mathematics Department
- Sean Larsen, Portland State University, Fariborz Maseeh Mathematics and Statistics Department
- Joseph Rotman, University of Illinois Urbana-Champaign, Department of Mathematics
- Zalman Usiskin, University of Chicago, Director of School Mathematics Project
- Rose Zbiek, The Pennsylvania State University, Department of Curriculum and Instruction

All of the interviews from this study were audiotaped and transcribed within a week of the interview. At the end of each interview after transcription, I modified the mathematical connections list to reflect the participant's responses. Every explicit mathematical connection mentioned during the interview by each participant was added to the connections list. These data served dual purposes: first, to identify mathematical connections between abstract algebra and secondary school mathematics, and second, to ensure the validity of identified conceptual connections through the practice of theoretical sampling (Charmaz, 2000).

Data Analysis

In this section, I elaborate on the several types of analysis that I employed. I used grounded theory when analyzing the textbook and interview data, so even though coding was initially based on the aforementioned mathematical connection categorizes, I adjusted those categories and established new categories to reflect additional types of mathematical connections emerging from the data.

To address the first research question, I analyzed each abstract algebra textbook individually in its entirely. I listed all identified mathematical connections and coded them thematically based on the aforementioned connection categorizes as well as modified or newly created connection categories emerging from the data (Charmaz, 2000; Patton, 2002; Taylor & Bogdan, 1984). In Chapter 4, I documented and elaborated themes found across multiple textbooks. I then created a comprehensive list of mathematical connections including all the connections from the three main categories in preparation for the interviews (Appendix B). The categories included: alternative or equivalent representations, comparison through common features, and hierarchical relationship. I selected these classifications were specifically because they were most predominant in the majority of the abstract algebra textbooks. I also excluded real-world application connections because these connections were made to areas outside the scope of this research; namely, connections were made to subjects other than secondary school mathematics. Chapter 4 discusses the expanded list of the identified connections of all types.

To answer the second research question, I analyzed each transcribed interview through an inductive and iterative coding process. Each time I examined the data, I concentrated on different aspects of mathematical connections: characteristic of mathematics, artifact of learning, a mathematical activity, and emerging mathematical connections. Since I modified the list of mathematical connections after each interview to include the previous interviewee's identified mathematical connections, my data analysis was continuous throughout the data collection process. During the first stage of data analysis, I concentrated solely on concept-by-concept connections so that I could modify the comprehensive list of mathematical connections. I employed theoretical sampling throughout the interviews to ensure validity in the list of mathematical connections (Charmaz, 2000). In particular, I added to the connection list every mathematical connection explicitly mentioned by the participants. However, if two sequential participants disagreed with a connection, I removed the connection from the list. The final comprehensive list of concept-by-concept connections after the completion of the interviews can be seen in Appendix D.

I conducted a second stage of analysis on the interview data upon the completion of the interviews. During this stage, I coded and analyzed the interviews thematically in light of the various definitions of mathematical connections described in detail in Chapter 2. I used the adjusted analytic connection framework from the textbook analysis to classify the concept-to-concept connections mentioned by the participants. In addition, the historical development of abstract algebra provided a lens for me to understand the additional types of connections the participants discussed. For instance, four participants mentioned the historical role of solving polynomial equations in the development of abstract algebra. Those four participants often concentrated on connections related to solving polynomial equations when describing the mathematical connections between abstract algebra and secondary school mathematics.

In analyzing the transcripts a third time, I was able to elaborate on other aspects of mathematical connections discussed by the participants. For instance, six participants concentrated on unifying themes such as proof or function during certain parts of the interview in addition to concept-by-concept connections. In addition, three participants alluded to mathematical connections as an artifact of learning or as a mathematical activity in mentioning the development of students' mental constructions of abstract algebra or the importance of doing proof, respectively. I detail the various connection descriptions given by the participants in the interviews in Chapter 5.

CHAPTER 4: MATHEMATICAL CONNECTIONS IN TEXTBOOKS

In this chapter, I describe the explicitly stated mathematical connections between abstract algebra and secondary school mathematics found in abstract algebra textbooks. I first discuss how the initial mathematical connection framework was adjusted to be more suitable for the textbook data. Next, I elaborate on the mathematical connections found in the abstract algebra textbooks by connection category. Categorizing written texts was not a simple task. The textbook authors, for instance, may have intended to make a generalization connection when in fact the written text is more of a comparison connection. In addition, there is a fine line between these types of connections. Some hierarchical connections could also be generalizations depending on how the authors presented the connections. In short, the findings in this chapter are those that were most explicitly stated in the abstract algebra textbooks.

Mathematical Connection Framework

After reading through the nine abstract algebra textbooks, I adjusted the initial mathematical connection framework described in detail in Chapter 2 to better fit the data. For instance, the initial alternative representation category included oral or written descriptions, but textbooks always have a written description of a concept and never have an oral one, so those characteristics were omitted. In addition, the initial generalization category was defined as: “One concept is an example of specific instance of another concept.” However, abstract algebra textbooks are filled with examples of specific concepts, but the textbooks also generalize school mathematics concepts. For example, several abstract algebra textbooks introduced isomorphism and then provided the well-known example of exponential laws to illustrate the concept of isomorphism. In spite of this example, isomorphism is not a generalization of exponential laws. As a result, this example was not included in the connection list, nor were any other examples of concepts. The algebraic structures, however, generalize the number systems and operators used in all of mathematics, so I altered the generalization category to reflect the latter situation. I also omitted several initial categories from the study because I found no mathematical connections of those types in the abstract algebra textbooks. The adjusted mathematical connections framework is given in Table 3.

Table 3

Categories of Mathematical Connections Found in Textbooks

Category	Description
Alternative representation	One concept is represented in different ways such as symbolic (algebraic), graphic (geometric), pictorial (diagram), or manipulative (physical object).
Comparison through common features	Two concepts share some features in common, which allows a comparison through the concepts being similar, exactly the same, or not the same.
Generalization	One concept is a generalization of another specific concept.
Hierarchical relationship	One concept is a component of or included in another concept. Since one concept is included or contained in the other concept, a hierarchical relationship exists between two concepts.
Real-world application	One concept is an example of another concept in the real-world (i.e., a concept refers to another concept outside the current mathematical context).

Mathematical Connections by Category

Next, I elaborate on the findings of mathematical connections explicitly stated in the nine abstract algebra textbooks by connection category.

Alternative representation. This connection category represents a concept in multiple ways such as symbolic (algebraic), graphic (geometric), pictorial (diagram), or manipulative (physical object). The alternative representation connections in the nine abstract algebra textbooks are given in Table 4 followed by a description of those connections.

Table 4

Mathematical Connections: Alternative Representation

Abstract Algebra Concept	Secondary School Mathematics Concept	No. of Textbooks
Group	Geometric transformations, Solving linear equations	7
Permutation group	Function, table, etc.	5

Seven of the nine abstract algebra textbooks introduced groups or specific types of groups using geometric transformations alongside the formal definition. For instance, Fraleigh (2003) and Nicodemi, Sutherland, and Towsley (2007) introduced the concept of a group by solving linear equations, drawing a Cayley (or operation) table, and discussing the symmetries of a triangle and square alongside the formal definition. These varied approaches allowed students to learn the concept of a group algebraically, geometrically, and pictorially. If the abstract algebra professor also used manipulatives in class to do the geometric transformations, as many of the textbooks suggested, then students would also have a physical object understanding. These diverse ways of thinking about a group, as a result, can enable students to develop more a robust understanding.

Five abstract algebra textbooks introduced several ways to write a permutation group, all of which relied on the students' previous knowledge. For instance, Fraleigh (2003) and Gallian (2013) presented students with the function form: $\sigma(1) = 3,\ \sigma(2) = 2,\ \sigma(3) = 4$, and $\sigma(4) = 1$ for a permutation σ on the set $X_4 = \{1, 2, 3, 4\}$. They also then suggested the array form: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$, which may be more familiar to students with a knowledge of matrix systems. Hillman and Alexanderson (1994) introduced both of these forms and also illustrated the permutation group in a table in which the input was 1, 2, 3, and 4 and the output was 3, 2, 4, 1, respectively.

Comparison through common features. This connection category allowed for two concepts to be compared as being similar, exactly the same, or not the same because the concepts share some common features. Table 5 summarizes the comparison connections I found in the abstract algebra textbooks. A detailed description of the most frequently made connections follows.

Table 5

Mathematical Connections: Comparison Through Common Features

Abstract Algebra Concept	Secondary School Mathematics Concept	No. of Textbooks
Algebraic structures & properties	Number systems, arithmetic operators	8
Congruence	Solving linear equations	1
Fundamental theorem of algebra	Polynomial roots	2
Homomorphism, kernel, image	Mathematical modeling	1
Polynomial ring	Polynomial operators and vocabulary	8
Quaternions	Complex numbers	2
Unit	Invertible matrices	2

Eight of the nine abstract algebra textbooks introduced algebraic structures such as group, ring, or field by comparing their defining properties to those of familiar number systems and arithmetic operators. Five of these textbooks explicitly stated and described these connections. For instance, Cuoco and Rotman (2013) wrote, “The main idea is to abstract common features of integers, rational numbers, complex numbers, and congruences, as we did when we introduced the definition of commutative ring” (p. 191); Hodge, Schlicker, and Sundstrom (2014) wrote, “Rings are algebraic objects that share the same basic structure as the integers” (p. 89); and Nicodemi et al. (2007) wrote, “The arithmetic of fields is similar to the arithmetic of the rational numbers” (p. 89). All of these textbooks also stated or explained that the set of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ each form a group under addition because these sets share the four common features with groups: closure under addition, associativity, zero as the identity element, and negative numbers as inverse elements.

Arithmetic operators such as addition and multiplication were also used in this comparison. For instance, the set of integers $\mathbb{Z}$ forms a group under addition but not under multiplication. The known arithmetic operators—addition, subtraction, multiplication, and division—were also compared to a field, given that it is a smallest algebraic structure in which all of these operators can be performed by nonzero set elements. Most textbooks also noted that known arithmetic operators have the same features as binary operators; namely, two set elements are combined to obtain one set element. Hillman and Alexanderson (1994) wrote, “Our notation for the operation has been the same as for multiplication in our familiar number system” (p. 74). This relationship is not surprising given that arithmetic operators are specific examples of binary operators. Focusing on the properties and characteristics of known number systems and arithmetic operators enabled the textbook authors to compare them to the newly introduced algebraic structures by drawing on shared and unshared features.

The polynomial ring was compared in eight of the nine abstract algebra textbooks to an abundance of polynomial information from secondary school mathematics. Much of the vocabulary used with polynomials in secondary algebra, such as *coefficient*, *degree*, and *polynomial equality*, was stated to be the same as the vocabulary used with polynomial rings. For instance, Nicodemi et al. (2007) wrote, “In high school algebra, the polynomials studied usually had coefficients that were either integers or rational numbers. We will *extend the scope* of that investigation to consider polynomials with coefficients in other commutative rings" (p. 111, emphasis added). In fact, five textbooks explicitly mentioned the different types of polynomial coefficients found in secondary algebra and abstract algebra. Fraleigh (2003) also pointed out, "We will be working with polynomials from a slightly different viewpoint than the approach in high school algebra or calculus" (p. 198) when he contrasted the vocabulary used for the symbol x as a variable with polynomials in secondary algebra and indeterminate with polynomial rings in abstract algebra. Similarly, all of the textbooks mentioned the similar types of operations performed on polynomials and polynomial rings. For example, Dummit and Foote (2004) wrote, "The operations of addition and multiplication which make $R[x]$ into a ring are the same operations familiar from elementary algebra: addition is componentwise" (p. 234). The eight textbooks then discussed polynomial long division using a comparison of similar features and vocabulary with either numerical long division or polynomial long division.

Five of the abstract algebra textbooks provided an explanation and proof of the fundamental theorem of algebra. Two of those textbooks compared secondary students' informal experiences with that theorem to the more formal presentation found in abstract algebra. For instance, Nicodemi et al. (2007) drew attention to the roots of a quadratic having two real or complex roots through factoring or graphing. Thus, secondary school mathematics textbooks often conclude that any nonzero polynomial has at most the same number of roots as the polynomial's degree. These two abstract algebra textbooks then highlighted the common features of polynomial roots found in secondary school mathematics to introduce the formality of the theorem and its proof.

The similar features of four-dimensional quaternions to two-dimensional complex numbers were compared in two of the nine abstract algebra textbooks as follows. Secondary school students are taught that complex numbers have the form $a + bi$ where *a* and *b* are real numbers with the imaginary unit i such that $i^2 = -1$. Similarly, abstract algebra students are taught that quaternions have the form $a + bi + cj + dk$ where *a*, *b*, *c*, and *d* are real numbers and $i^2 = j^2 = k^2 = ijk = 1$. The two abstract algebra textbooks noted that quaternions paralleled complex numbers in the way elementary operations are performed. For instance, a complex number $z = a + bi$ has a complex conjugate $\bar{z} = a - bi$ and a quaternion $q = a + bi + cj + dk$ has a conjugate $\bar{q} = a - bi - cj - dk$. One difference of quaternions is that quaternions are not commutative, so quaternion multiplication works differently than complex number multiplication. However, addition is quite similar for both number systems in that the real terms of the numbers are added together as well as the like i, j, k elements. Last, if two complex numbers are equal, then the real components and the imaginary components are equal; likewise, if two quaternions are equal (if $a + bi + cj + dk = r + si + tj + uk$), then the individual components are equal ($a = r, b = s, c = t$, and $d = u$).

When first introducing the concept of a unit, two of the abstract algebra textbooks compared it to invertible matrices. A unit refers to an element *a* in a ring that also has an inverse element b^{-1} in the ring such that $ab^{-1} = 1$, where 1 is the multiplicative identity. In other words, a unit is an element that has a multiplicative inverse in the ring. Even though students are first exposed to multiplicative inverses in studying reciprocals, matrix systems are often the first time students have experienced elements without multiplicative inverses. Thus, Hillman and Alexanderson (1994) and Nicholson (2012) both related unit to invertible by stating that unit is no different than invertible, so invertible matrices are unit elements. Thus, the abstract concept of unit is the same as the concept of invertibility first taught in secondary school mathematics.

Generalization. This connection category designates when one concept is a generalization of another concept. The generalizations explicitly stated in the nine abstract algebra textbooks are seen in Table 6, followed by a detailed description of most of the mathematical connections.

Table 6

Mathematical Connections: Generalization

Abstract Algebra Concept	Secondary School Mathematics Concept	No. of Textbooks
Algebraic structures	Number systems	5
Binary operators	Arithmetic operators and number systems	4
Direct product	Cartesian plane and ordered pairs	2
Inverse	Negatives; Multiplicative reciprocal	5
Irreducibility	Factoring polynomials	5
Quotient Field	Fractions, operations with fractions	5
Sign rule in a ring	Product of two negative numbers is positive	5

One of the most commonly discussed connections in the abstract algebra textbooks was how various algebraic structures (i.e., group, ring, field) were generalizations of familiar number systems. Five of the nine textbooks explicitly mentioned that generalization. For instance, several textbooks noted that the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ were familiar number systems that form groups under addition, and the nonzero elements of $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ form groups under multiplication. Hillman and Alexanderson (1994) stated, “The most basic number systems are examples of groups, and we all learn to deal with these early on” (p. 40), and Nicodemi et al. (2007) called these number systems “prototypes” under the “umbrella” of an arbitrary algebraic structure. Four textbooks then introduced the more general binary operator $*$ for the familiar addition and multiplication operators. The definition of a group is then developed having that understanding of a binary operator. A similar approach was used to introduce rings, integral domains, and fields.

The field of quotients or fraction fields is introduced in five of the abstract algebra textbooks as the generalization of fractions and operations on fractions. Those textbooks reviewed concepts such as equivalent fractions, equating two fractions, and operating on the fractions (addition and multiplication) prior to introducing the more general understanding of a fraction. For example, Dummit and Foote (2004) noted, "In more precise terms, the fraction $\frac{a}{b}$ is the equivalence class of ordered pairs (a, b) of integers with $b \neq 0$ under the equivalence relation $(a, b) \sim (c, d)$ if and only if $ad = bc$" (pp. 260–261). With this definition in mind, the textbook authors then explored the properties, operations, and proven results of quotient fields.

Five of the abstract algebra textbooks connected the concept of irreducible polynomials in a polynomial ring to prime polynomials and factoring. Upon introducing irreducibility, Fraleigh (2003) wrote, "The concept is probably already familiar. We really are doing high school algebra in a more general setting" (p. 213). In secondary school mathematics a polynomial is defined to be prime if it is unable to be factored, whereas in abstract algebra a polynomial is irreducible if it has no factorization of polynomials of lower degree than original polynomial. Thus, the latter is the generalization of the former for all polynomial rings.

Five of the textbooks also generalized the notion that the product of two negative numbers is positive. These textbooks utilized ring properties to prove four sign rules for negatives. Hillman and Alexanderson (1994) wrote, "The following result [sign proof] is a generalization of one of the rules of signs of elementary algebra" (p. 217). Nicodemi et al. (2007) noted, "It is interesting to see how to deduce these facts from the abstract properties of rings rather than from the elementary cookie-counting arguments that we usually use to explain the arithmetic of the natural numbers" (p. 84). These authors in particular allude to the importance of this connection in teaching by providing a ring based rationale for the positive result of multiplying two negative numbers and contrasting it with the rationale generally accepted in school mathematics.

Hierarchical relationship. A hierarchical relationship connection is one in which a concept is a component of or contained in another concept. The abstract algebra textbooks in this study mentioned secondary school mathematics concepts that were included in abstract algebra concepts. Table 7 contains a summary of topics involving those connections found in abstract algebra textbooks.

Table 7

Mathematical Connections: Hierarchical Relationship

Abstract Algebra Concept	Secondary School Mathematics Concept	No. of Textbooks
Algebraic structures & properties	Solving linear equations	6
Compass/geometric constructions	Ruler, circles, intersection, and other geometric concepts	7
Cyclic group	Division algorithm	1
Extension field/splitting field	Solving for the roots of a polynomial	5
Isomorphism	Function	3
Nilpotent	Geometric series and convergence	1
Permutation group	Function and function composition	3
Polynomial ring	Power series	1
Symmetry group	Rotation, reflection, function composition	7
Zero divisors	Solving quadratic equations by factoring	4

1) Given

$$x + a = b \qquad xa = b \qquad x * a = b$$

2) Determine the inverse of a under the operation and apply the operation with its inverse on the right of both sides of equation.

$$(x + a) + -a = b + -a \qquad (xa) \cdot \frac{1}{a} = b \cdot \frac{1}{a} \qquad (x * a) * a^{-1} = b * a^{-1}$$

3) Use the associative law under the operation to regroup the left side of the equation.

$$x + (a + -a) = b + -a \qquad x\left(a \cdot \frac{1}{a}\right) = b \cdot \frac{1}{a} \qquad x * (a * a^{-1}) = b * a^{-1}$$

4) The result of regrouping a with its inverse is the identity under the operation [let e be the unknown identity under the operation].

$$x + 0 = b + -a \qquad x \cdot 1 = b \cdot \frac{1}{a} \qquad x * e = b * a^{-1}$$

5) Combining x with the identity under the operation results in x itself.

$$x = b + -a \qquad x = b \cdot \frac{1}{a} \qquad x = b * a^{-1}$$

Figure 1. The properties needed to solve simple linear equations form a group.

Six of the abstract algebra textbooks seemed to rely on students' previous understandings of solving linear equations to serve as a foundation for the new algebraic structures. For instance, simple linear equations of the forms $a + x = b$ and $ax = b$ form groups under addition and under multiplication, respectively. Four of the textbooks asserted that the knowledge of solving such equations was an integral component in the definition of a group; namely, the properties used to solve a simple linear equation define the group structure. Figure 1 illustrates this information with three different operators: addition, multiplication, and the more formal binary operator. Three of the nine textbooks introduced a field in a similar manner by walking through the properties needed to solve a linear equation of the form $ax + b = cx + d$. To solve this problem one must utilize additive and multiplicative inverses, additive and multiplicative identities, additive and multiplicative associativity, closure under addition and multiplication, and the distributive law. These properties along with commutativity are integral components of understanding the definition of a field. In both instances the properties used in solving linear equations are contained in the definitions of various algebraic structures.

Similarly, five of the textbooks introduced the concept of extension field by building upon students' knowledge of solving for the roots of polynomial functions. For instance, if the given field is the real numbers $\mathbb{R}$, then the simple polynomial function is $f(x) = x^2 + 1$. Clearly, this polynomial does not have a solution in the field and is thus irreducible, so the question then arises whether or not a larger field that contains $\mathbb{R}$ would provide a root for the polynomial. Gauss answered that question by introducing the complex number system $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ where $i^2 = -1$. Thus, to solve the polynomial $f(x) = x^2 + 1$, the domain needs to be extended to the complex number system.

Analogously, four of the textbooks made a connection between students' previous knowledge of solving quadratic equations by factoring and the abstract algebra concept of zero divisors. These textbooks illustrated that to solve a quadratic equation, say $x^2 + 2x - 15 = 0$ where $x \in \mathbb{R}$, the students learned to factor the equation, $(x + 5)(x - 3) = 0$, and conclude the only way that a product can equal zero is if one of the factors is zero, $x + 5 = 0$ or $x - 3 = 0$, so the only two possible solutions of the equation are $x = -5$ or $x = 3$. Students are then asked to utilize this previous knowledge to understand zero divisors. Several of these textbooks then problematized solving quadratics in a modular ring. Thus, the understanding of solving for the roots of a quadratic was included in the abstract algebra concept of zero divisors.

Seven of the nine abstract algebra textbooks included a chapter or section on compass or geometric constructions, which are geometric applications of field theory. The geometric constructions first experienced in high school geometry using a compass and straightedge were given an algebraic context. Thus, students should first have a basic understanding of geometric concepts such as angles, circles, distance, intersection, regular *n*-gons, and trisection in order to understand how their understanding relates to geometry. In fact, Dummit and Foote (2004) wrote:

> It is an elementary fact from geometry that if two lengths *a* and *b* are given one may construct using straightedge and compass the lengths $a \pm b$, ab, and $^{a}/_{b}$. It is also an elementary geometry construction to construct $\sqrt{a}$ if *a* is given: construct the circle with diameter $1 + a$ and erect the perpendicular [line] to the diameter. The length is $\sqrt{a}$. (p. 532)

Thus, all arithmetic operations can be constructed using a compass and straightedge, and additive or multiplicative inverses, the product of two numbers, and the square root of a number are all constructible numbers. Gallian (2013) defined a *constructible number* as a real number $\propto$ in which a line segment can be drawn with length $|\propto|$ in a finite number of steps (p. 400). He then provided three ways to construct such points: intersect two lines, intersect two circles, or intersect a line and a circle (p. 401). One chief result is to show the set of all constructible numbers then forms a subfield of $\mathbb{R}$ and any constructible number must be a field extension of $\mathbb{Q}$. Another result that is especially useful for future secondary mathematics teachers is that one cannot trisect an angle using compass and straightedge.

Similarly, seven of the abstract algebra textbooks introduced symmetry groups with geometric transformations. For instance, Nicodemi et al. (2007) defined symmetry of a regular figure as "a rotation of the figure around an axis of symmetry that takes the figure congruently onto itself" (p. 196). Gallian (2013) posed it this way:

> Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. More specifically, we want to describe the possible relationships between the starting position of the square and its final position in terms of motion. (p. 31)

Thus, students can use their knowledge of high school geometry concepts of rotations, axes of symmetry, angle bisectors, and reflections in order to understand a symmetry group. Several of the textbooks illustrated the six symmetries (identity, 2 rotations, and 3 flips/reflections around the axes) for an equilateral triangle. Dummit and Foote (2004) generalized those findings for all regular *n*-gons: "There are exactly *2n* symmetries of a regular *n*-gon" and "These symmetries are the *n* rotations about the center through ${2\pi i}/{n}$ radian, $0 \leq i \leq n-1$, and the *n* reflections through the *n* lines of symmetry" (p. 24). In addition, these textbooks employed students' knowledge of function composition to discuss the operator used with symmetry groups. Gallian (2013) explained the operation order in this way, "In lower level math course function composition $f \circ g$ means *g* followed by *f*"(p. 33), meaning that the order of the symmetries moves right to left similarly to function composition.

Three of the nine textbooks discussed how the secondary school mathematics concept of function was included in the abstract algebra concept of a permutation group. For instance, Fraleigh (2003) explained a permutation as a rearrangement of the set elements that is truly a bijective function between two sets. Furthermore, all three of these textbooks stressed the sameness of secondary mathematics notion of function composition and the binary operator involved in the cycle decomposition of the product of permutations. In secondary school mathematics, students are taught the composite function $f \circ g(x)$ is read right to left, where $g(x)$ is inputted into the f function. Similarly, in abstract algebra to compute the product $\sigma \circ \tau$ in S_n requires reading the permutations from right to left, so the product $(1 \quad 2 \quad 3) \circ (1 \quad 2)(3 \quad 4)$ sends 1 to 2 in the right permutation and 2 to 3 in the left permutation resulting with the composite map 1 to 3 for a completed cycle decomposition of the product $(1 \quad 3 \quad 4)$.

Real-world application. Even though real-world application connections do not directly address my first research question given that the applications are outside the mathematical context, it is important to note that Gallian (2013) focused a great deal of attention on making real-world application connections between the abstract algebra concepts and other concepts outside of mathematics. For instance, the pyramidal molecule ammonia is the dihedral group D_3 and crystals are the dihedral group D_4. Several mineralogists and chemists have studied the symmetries of figures using group theory to motivate their work. In fact, the orbit-stabilizer theorem was used on the rotation of a soccer ball to illustrate the new carbon form called buckyballs, and the symmetry group again provided useful information about this form because the absorption spectrum of a molecule relies on these symmetries (Gallian, 2013, pp. 154–155). These few examples provide some rationale why group theory is useful outside a mathematical context.

Furthermore, two of the abstract algebra textbooks related the concept of isomorphism with a discussion about different languages. These two textbooks compared the English words for one, two, three to the German words eins, zwei, drei. In both languages these three numbers mean the same thing even though the presentation of them is different. Similarly, two algebraic structures may look different and yet have the same form, which is the definition of an isomorphism. This real-world application connection provides students a non-mathematical context for isomorphism that can aid in the development of a more robust understanding of the concept.

Summary

This chapter has detailed the results of analyzing nine abstract algebra textbooks to identify explicitly stated mathematical connections between abstract algebra and secondary school mathematics and to examine how these connections were discussed. The results of this chapter were explained by connection category: alternative representation, comparison through common features, generalization, hierarchical relationshil, and real-world application. The five most commonly stated mathematical connections found in textbooks can be seen in Table 8.

Despite the fact that previous researchers have characterized abstract algebra as the generalization of school algebra and then used this characterization as a rationale for requiring prospective secondary mathematics teachers to take abstract algebra, the results from this study show that abstract algebra textbooks more explicitly make mathematical connections of other types; namely, alternative representation, comparison through common features, and hierarchical relationship. The sheer number of different connections explicitly made in textbooks was highest for the hierarchical relationship connection category. However, the most commonly stated concept connections across textbooks were presented using a comparison of common features.

Table 8

Most Commonly Stated Mathematical Connections in Textbooks

Connection Category	Abstract Algebra Concept	Secondary School Mathematics Concept	No. of Textbooks
Alternative representation	Group	Geometric transformations, Solving linear equations	7
Comparison through common features	Algebraic structures & properties	Number systems, arithmetic operators	8
Comparison through common features	Polynomial ring	Polynomial operators and vocabulary	8
Hierarchical relationship	Compass/geometric constructions	Ruler, circles, intersection, and other geometric concepts	7
Hierarchical relationship	Symmetry group	Rotation, reflection, function composition	7

CHAPTER 5: MATHEMATICAL CONNECTIONS DISCUSSED BY MATHEMATICIANS AND MATHEMATICS EDUCATORS

The purpose of this chapter is to discuss the various ways mathematicians and mathematics educators described mathematical connections between abstract algebra and secondary school mathematics while participating in individual interviews. I first detail the ways in which the mathematicians and mathematics educators talked about mathematical connections in light of the framework used to analyze the abstract algebra textbooks. Next, I elaborate on the ways in which mathematicians and mathematics educators discussed connections in terms of the other connections perspectives found in the literature review: mathematical connections as an artifact of learning and mathematical connections as a mathematical activity. Last, I explain the other types of mathematical connections that emerged from the interview data. For instance, the participants in this study made use of the historical roots of group theory by emphasizing certain characteristics over others when talking about mathematical connections. Some participants also emphasized mathematical connections to teaching. The remainder of this chapter will elaborate upon the mathematical connections described by the participants.

Mathematical Connections that Align with Textbook Analysis

Initially, I analyzed the interview data using the analytic connection framework that was used to analyze the abstract algebra textbooks. This decision was validated when mathematicians and mathematics educators maintained that connections exist throughout the mathematics and thus are characteristic of the discipline, which was the stance from which the analytic connection framework emerged. Furthermore, the participants either used the abstract algebra textbooks analyzed in this study when teaching the course or used their own materials. The way in which the participants described mathematical connections was captured by four of the five connection categories: comparison through common features, generalization, hierarchical relationship, and real-world applications. However, the participants emphasized the impact or usefulness of each category differently in that most of the participants described connections via comparison whereas only one participant mentioned real-world applications. In the following sections I will describe these findings by connection category.

Comparison through common features. In this connection category, two concepts are compared as being similar, exactly the same, or not the same due to common features. The participants in this study not only compared concepts but more specifically compared structures. Structure was so important to many of the mathematicians and mathematics educators that they explicitly stated abstract algebra is the study of structure, so naturally these participants mentioned how the focus of this course would be on a comparison of these various structures. The ways in which these participants talked about structure, however, seemed to vary slightly. For instance, one participant described structure as, "When mathematicians say structure they generally mean in a set and some types of properties overlaid on that set." This participant then stated, "Thus, abstract algebra is about the comparing structures and their properties." Another participant mentioned, "So one of the reasons to do abstract algebra is to give a structure to all those properties you learned."

As a result of their focus on structure, the majority of these mathematicians and mathematics educators, when teaching abstract algebra, explicitly concentrated on the development of students' abilities to recognize structural connections. They took the time to focus on these ideas even at the expense of teaching other abstract algebra topics. Five participants, in fact, emphasized that once a structure is known, questions can be asked about its properties to determine appropriate expectations and assumptions about similar structures. Overall, twelve of the thirteen mathematicians and mathematics educators discussed mathematical connections between abstract algebra and secondary school mathematics as a comparison of structural features.

One participant defined a mathematical connection as "when two contexts that look different on the surface have the same underlying structure." For this participant, connections were more about abstract similarities than about studying surface features. Thus, he stressed that many problems in abstract algebra may appear to be different from other mathematics despite being quite similar or even the same as problems that students have seen before. Another participant mentioned that students can think about and approach problems in completely different ways despite the content and underlying structure being the same. For instance, abstract algebra provided structure to school algebra, so even though the two may look different, the mathematical content is actually the same. A third participant shared similar sentiments:

> It is the way algebraist think, when we talk about structure we immediately start thinking about, "Oh is this something I've seen before?" … Algebraists like to divide things up into chunks that behave similarly, so once you've started saying, "Does this behave like something I already know?" Oh well, let's chunk it with those things.

For these participants, mathematical connections between abstract algebra and secondary school mathematics involved recognizing structural similarities despite dealing with phenomena that initially appeared to be different.

Given the major focus placed upon structural similarities by the mathematicians and mathematics educators in this study, it was not surprising that five of these participants underscored the importance of isomorphism to an abstract algebra course. One participant explained isomorphism as, "If you have these two systems and then somebody points out, hey they are really the same, there's just a different language here." In fact, this participant shared that one of the main reasons for studying abstract algebra is isomorphism and the similarities of structures. Another participant repeatedly mentioned the time spent discussing what he called "prototypes" when teaching abstract algebra, which is how he described to his class a structure to which new concepts can and should be compared. Two examples that he provided of these prototypes are the dihedral group of order 8 for non-abelian groups and $\mathbb{Z}_n$ for infinite cyclic groups of order n. The other three participants all described the structural similarities between the exponential or logarithmic functions and the real numbers. One participant discussed this connection in this way: the exponential and logarithmic rules, $e^x e^y = e^{x+y}$ and $\log ab = \log a + \log b$, "are both from the fact that the real numbers under addition is isomorphism to the positive real numbers under multiplication and you can use either x or $\log x$ as the mapping." This same participant emphasized the role of equivalence in problem solving and explicitly detailed the various ways equivalence is presented in abstract algebra, such as two objects or structures being isomorphic.

The mathematicians and mathematics educators in this study described specific mathematical connections through the comparison of structures in various other ways. However, the majority of the participants highlighted similarities between number theory and specific algebraic structures as well as solving polynomial equations and specific algebraic structures. These similarities primarily concentrated on the operations and properties of the structures and how these structures behave similarly and different across different domains. For instance, one participant discussed:

> When you translate a problem from domain to another, that's a connection, like constructing a regular polygon with a straightedge and compass turns out to be equivalent to finding the roots of the equation $x^{17} - 1$ and being able to express those in terms of radicals. So there's structural similarities.

Another participant mentioned the parallels between the secondary school mathematics concepts *function* and *domain* and the abstract algebra concepts *operation* and *set*. Structurally, these concept pairs behave or act similarly despite seeming quite different. Additional examples of specific mathematical connections are included in the Emerging Mathematical Connections: Historical Roots of Group Theory section found later in this chapter.

Generalization. Four of the thirteen mathematicians and mathematics educators described mathematical connections as abstract algebra concepts being generalizations of secondary school mathematics concepts. An additional participant acknowledged the existence of generalization connections and provided one example, but he thinks about mathematical connections and teaches abstract algebra from another perspective.

Only one participant explicitly characterized abstract algebra as the generalization of school algebra. He defined mathematical connections as "seeing how new knowledge or general concepts in abstract algebra inform what students have learned previously or underpin what they learned previously." When discussing specific connections, he detailed how the ring and field properties are generalizations of the properties that are useful to solving polynomial equations. Similarly, another participant noted, "Each new abstract algebra concept is a generalization of number systems from the K-12 curriculum." He also elaborated on how ring and field properties are generalizations of the properties needed to solve polynomial equations. Still another participant described how binary operators are generalizations of arithmetic operators.

In addition, another participant discussed the importance of teachers knowing the generalization relationship between polynomial rings and polynomials taught in secondary school mathematics. He explained:

> So you need a halfway decent understanding of them and you don't really understand them until you see the broader context of rings. For example, something as easy as a polynomial has at most n degree roots. Well, that's not always true and once you understand why that's not always true, it gives you better insight to what is going on and I think that it is really important for a teacher to know what is going on, not only to be able to teach it well but to be able to communicate. Once you embed these specific cases into the natural broader context I think it really aids the understanding.

This participant understood the secondary school mathematics concept *polynomial* as specific cases of the more general abstract algebra concept *polynomial ring*. In fact, he stressed the need for secondary teachers to possess deep understandings of the more general polynomial ring in order to better understand and teach polynomials in secondary school.

One participant described the abstract algebra concepts such as *group* and *isomorphism* as generalizations of the representation of symmetries of regular polygons. He mentioned several activities used in his class to build on this idea. For example, he provided his students with opportunities to explore the symmetries of an equilateral triangle to get students to generalize a set of rules regarding these symmetries to determine the definition of a group. Additionally, when teaching isomorphism, he provided his students a mystery table and asked them if the table represents a group. After they determined the mystery table did in fact represent a group, the students tried to find a correspondence between it and their regular polygon symmetry tables. The goal was for them to notice that the two tables were the same but with different symbols. Students generalized this idea of "sameness" from this example to eventually deduce the definition of isomorphism.

Hierarchical relationship. Two of the thirteen mathematicians and mathematics educators briefly discussed hierarchical relationships between secondary school mathematics concepts and abstract algebra concepts in talking about specific secondary school mathematics concepts being components of or included in abstract algebra concepts. One of these participants emphasized student knowledge about polynomials to understand polynomial rings and knowledge of functions to understand the bijections on a set for the definition of a group. This participant stated, “These topics provide a firm foundation to the material learned in abstract algebra.”

The other participant described the geometric concepts that students should learn in secondary school mathematics that will influence their abilities to learn group theory. Some examples of these concepts include: rotation, reflection, axes, regular polygons, and degree measures. In addition, this participant affirmed that abstract algebra students should be very familiar with secondary school mathematics concepts such as complex numbers, properties of real and rational numbers, Euclidean algorithm, and least common multiples and greatest common divisors. He then detailed the connections of these topics to abstract algebra concepts. For instance, knowledge about least common multiples and greatest common divisors is necessary for students to understand LaGrange’s Theorem.

Real-world application. Only one participant described mathematical connections between abstract algebra and secondary school mathematics in terms of real-world applications. One connection he made was between direct products taught in abstract algebra and molecules taught in secondary school chemistry. He explained:

> When you get to direct products, you can say that you glue those things together to get a much more complicated group by taking the basic component. Just like water is made out of hydrogen and oxygen, many groups are made out of Z2 cross Z4 cross a Z8 or something like that.

This participant discussed how students want to be taught this type of mathematical connection so that they may recognize how mathematical concepts are related to concepts outside of mathematics.

Additional Mathematical Connections

After a second analysis pass through the entire data set, additional kinds of mathematical connections emerged that were not present in the textbook data. For instance, the participants discussed the mathematical connections in terms of two more connections perspectives: mathematical connections as an artifact of learning and mathematical connections as a mathematical activity. These additional connections are discussed in the succeeding sections.

Mathematical Connections: Artifact of Learning. Three of the thirteen mathematicians and mathematics educators described mathematical connections as an artifact of learning. Businskas (2008) defined this type of mathematical connection as “a process that occurs in the mind of the learner(s) and the connection is something that exists in the mind of the learner” (pp. 12–13). These participants concentrated on the mental connections developed within the students’ minds when talking about these mathematical connections.

One participant characterized these connections as “more a state of mind,” so she did not think it was as important to explicitly state every connection to her abstract algebra students but rather to teach students how to make connections for themselves. When teaching abstract algebra, she challenged students’ assumptions about previous knowledge and properties because students “aren't blank slates when they come in the classroom” and often take their algebraic knowledge for granted. Despite these beliefs about mathematical connections, she still stated that she makes a point to explicitly mention as many connections as possible when teaching. A second participant shared these sentiments by discussing mathematical connections as “process connections.” She stressed the importance of having student think deeply about previously established concept definitions for secondary school mathematics to build mental connections between them and abstract algebra concepts. Consequently, both of these participants discussed their shared belief that students learn best when starting from something familiar and building on what they already know by challenging their minds with something new to extend the knowledge.

The other participant had a slightly different perspective on this type of mathematical connection. He focused his connection talk on students' development of mathematical habits of mind and the role mathematical connections play in this development. For instance, in order for students to be able to identify patterns or make conjectures, they must first establish connections in their minds between concepts. Thus, abstract algebra students cannot make conjectures about an unknown structure without first thinking about how, for example, a specific number system learned in school mathematics relates to the unknown structure. After that connection is made, they are free to investigate the reasons why the structures are similar.

Mathematical Connections: A Mathematical Activity. While all thirteen mathematicians and mathematics educators held the belief that connections exist across mathematics, six of these participants also discussed the involvement of the learner in the activity of establishing or identifying these connections through proof writing. Several of these participants, when teaching abstract algebra, emphasized student learning of proof techniques over specific abstract algebra theorems. However, these participants also mentioned a few abstract algebra concepts that provide opportunities for enhanced proof learning. These concepts included inverses, cosets, normal subgroups, and ideals. One participant noted that the latter three concepts, in particular, introduce an extra qualifier that students need to pay attention to when developing proof theory.

While elaborating on the importance of proof writing, one participant asserted that proofs can illustrate how much or little a student knows about a certain topic. She explained, "If you really understand that, then the proof is trivial. If they have no idea how to start it, then you know they don't have the right understanding." Ultimately, she concluded that the more mathematical connections a student knows or has developed, the more sophisticated proof a student can write and understand. Analogously, another participant mentioned how mathematics was taught in the Middle Ages to teach people to think clearly about the world; correspondingly, abstract algebra students are taught to think, read, and write proofs in order to help them think abstractly about mathematics. A third participant commented, "High school teachers should be comfortable with proofs, and abstract algebra is certainly one of the best courses for that, because you have to think abstractly." Another participant acknowledged that throughout an abstract algebra course students learn several proof techniques or templates that teach them to think clearly and parse information. She further added that proofs require students think about the role of definitions and how they build on these definitions by identifying connections to establish a sequence of logical arguments.

Emerging Mathematical Connections: Historical Roots of Group Theory

For many mathematicians and mathematics educators, the historical developments of algebraic structures (groups, rings, and fields) have shaped their understandings of them. As I further analyzed the interview data, additional mathematical connection categories emerged that aligned with the historical roots of group theory. More specifically, the participants tended to highlight mathematical connections related to one of the four historical origins of abstract algebra: solving polynomial equations, generalizing solutions to number theory problems, satisfying pre-established axioms, and characterizing geometric transformations. In order to better interpret the data, I will first provide a brief description of the historical development of group theory and then detail the described mathematical connections to these historical roots.

History of Group Theory. Until the early nineteenth century, the study of algebra primarily focused on the solving of polynomial equations. Joseph Lagrange initially analyzed known solving methods for cubic and quartic equations to discover that these methods shared the common feature of reduction; namely, the resolvent equation is one degree lower than the original equation. In 1770, Lagrange used this finding, as well as permutations, to find a formula to solve fifth-degree polynomials (Hillman & Alexanderson, 1994; Kleiner, 2007; O'Connor & Robertson, 1996). Unbeknown to him, these permutations were elements of a group, so his work is now referred to as classical algebra. Evariste Galois, however, was the first to realize in 1831 that the algebraic solution of an equation was connected to group theory (O'Connor & Robertson, 1996).

Solutions to problems in number theory also played a prominent role in the development of group theory. In 1801, Karl Friedrich Gauss published *Disquisitiones Arithmeticae,* which summarized prior developments in number theory and examined the equivalence classes of various quadratic forms $ax^2 + 2bxy + cy^2$. The byproduct of this work was an algebraic structure having certain properties that we now know to be a group (Hillman & Alexanderson, 1994). Four specific algebraic structures were presented in Gauss' paper: the additive group of integers modulo *m*, the multiplicative group of integers relatively prime to m, the group of equivalence classes of binary quadratic forms, and the group of *n*th roots of unity (Kleiner, 2007). Gauss' work also eventually led to the development of finite abelian groups.

In 1854, Arthur Cayley introduced another way to understand groups; namely, through the use of a table (Hillman & Alexanderson, 1994; O'Connor & Robertson, 1996). He formulated the abstract group concept by using a set with a binary operator that satisfied certain axioms. This development was different from previous approaches in that certain axioms were first selected, and algebraic structures were developed based on those axioms, whereas the others deduced properties from known polynomial structures and number systems. Cayley's axiomatic approach to group theory guided the work of twentieth century mathematicians (O'Connor & Robertson, 1996).

In 1872, Felix Klein proposed yet another way to think about groups with his *Erlangen Program*, which studied the invariants under a group of transformations (Kleiner, 2007; O'Connor & Robertson, 1996). It was here that Klein utilized algebraic methods to abstract ideas of symmetry by classifying geometries and their underlying transformations. Several groups appeared through Klein's *Erlangen Program*: elliptic group, group of rigid motions, hyperbolic group, projective group, and symmetry group (Kleiner, 2007). This work largely influenced future mathematicians as seen in the growth in geometry transformation research in the nineteenth century. These distinct origins of group theory were visible through different participants' descriptions of abstract algebra and connections to secondary school mathematics.

Solving Polynomial Equations. Four of the thirteen mathematicians and mathematics educators in this study explained mathematical connections between abstract algebra and secondary school mathematics in terms of solving equations. Three additional participants acknowledged that this perspective was important to abstract algebra, but they did not elaborate on these connections. The four participants who did elaborate on these connections used similar descriptions to this one:

> I say to them, "Let's do solve equations." Then ask which properties they used to solve the equations, could you use fewer. Then the students came and presented on the board and say the following properties can be used for this one, these can be used for this one, etc. … If we want to solve any equation, which properties do we need, let's write a definition that captures that. That's how I introduce the definition of groups.

These participants then emphasized mathematical connections such as inverse, identity, and operators and how these properties naturally come out through solving equations. For example, an inverse is first taught as a number; namely, an additive inverse is first introduced to middle school students as the negation of a number and a multiplicative inverse is introduced as the reciprocal of a number. High school students are then taught a more operational approach to inverse in learning how to find inverse functions and inverse matrices. In abstract algebra, students learn inverse as a necessary element of a set, whether that set is shown through solving equations or in the definition of an algebraic structure. One participant noted his surprise that more abstract algebra students do not recognize the connection between the zero product property and integral domains with solving quadratic equations through factoring. The other three participants who focused on connections to or through solving equations also mentioned this connection.

Number Theory. Four of the thirteen mathematicians and mathematics educators described mathematical connections related to number theory. Two additional participants acknowledged the importance of number theory to the historical development of abstract algebra, but they did not elaborate on these connections. The four participants discussed how the known number system can act as a foundation to facilitate discussion of the properties and operations associated with algebraic structures. For instance, these participants elaborated on the similarities between integers and polynomial rings. One of these participants mentioned the similar structural nature of base 10 expansions with polynomials, whereas three of the these participants focused more on the parallels between integer long division and polynomial long division, factoring numbers and factoring polynomials, and the Euclidean algorithm and other division algorithms.

One participant also explained structural connections between complex numbers and polynomials and how they both relate to the construction of splitting fields:

> A lot of high school students, if you watch them work, they will calculate the complex numbers as if i were x, and they work on them as if they are polynomials, and when they are all done they replace i^2 with -1. Well, that's actually a very deep idea. And we develop complex numbers from that point of view, building on what high school kids typically do. This is the idea that Kroencker used to construct splitting fields from polynomials. Basically what you are doing is taking a polynomial of one variable with real numbers and reducing it modulo $x^2 + 1$, so that's the whole approach to complex numbers.

Ultimately, these four participants relied on students' background knowledge of number systems from secondary school mathematics to develop relevant abstract algebra concepts.

Axiomatic Approach. Only one participant discussed mathematical connections axiomatically. One additional participant mentioned this perspective, but only as a historical origin of abstract algebra. The first participant, when teaching abstract algebra, prefers to concentrate on axioms such as commutativity and associativity that he believes students should have learned in secondary school mathematics. He then builds students' understandings of algebraic structures on these properties. Much of his talk about connections was then focused on the role commutativity plays in learning about abelian groups. Additionally, he discussed the notion of losing commutativity in studying certain structures, such as moving from dimension two complex numbers to dimension four quaternions. As a result, he emphasized logically deriving definitions and proving theorems from known axioms.

Geometric Transformations. Four of the thirteen mathematicians and mathematics educators described mathematical connections between abstract algebra and secondary school geometry. Two additional participants acknowledged the existence of these connections but strongly preferred another perspective and, as a result, would not teach their students these connections. The four participants who emphasized geometric connections liked the visual nature of this approach to learning abstract algebra. These participants explicitly mentioned connections between geometric transformations such as rotation and reflection to algebraic structures. All were similar to one participant's described approach to teaching abstract algebra:

> Basically, there's a long sequence of classes in the beginning that starts with symmetries on the equilateral triangle. The students reinvent their own symbols for those symmetries, then we talk about combining those symmetries. They develop a set of rules as to why that works. This leads into the axioms of a group.

This same participant reinforced these ideas by assigning his abstract algebra students a symmetry journal for homework in which students explored the symmetries of regular polygons not already examined in class.

Two participants also discussed the mathematical connections to the combination of geometric transformations. One of these participants detailed the notions of identity and inverse in terms of these transformations. More specifically, she explained how an object reflected twice over an axis is in fact itself, so the action of two reflections acts as an identity and one reflection acts as an inverse. The other participant elaborated on using geometric transformations to explain the associative and commutative properties and parity. He noted:

> So I say look at your table of the symmetries of the square. They remember from the earlier work that a reflection and a reflection gives a rotation and a rotation and a rotation gives another rotation, if you mix them you get a reflection, so those act like evens and odds.

Thus, for these four participants, students' background knowledge of geometry from secondary school mathematics was pivotal to understanding abstract algebra concepts.

Emerging Mathematical Connections: Connections for Teaching

Three of the thirteen mathematicians and mathematics educators discussed yet another type of connection: mathematical connections for teaching. These participants emphasized that mathematical connections are important for prospective secondary school mathematics teachers to identify in order to enhance their future teaching. All three of the participants detailed how abstract algebra knowledge can be used for a teacher's preparation of lesson plans. For instance, one participant noted that secondary school mathematics teachers can employ what they know about a structure to help them plan a lesson. When a teacher is in the midst of extending an activity, he or she could quickly determine whether or not the extension is valid because it fits within the structure or stop because it is not within the structure that she is dealing with. By knowing about the structure the students are working with, the teacher could also realize the questions she may need to ask students about the problem in order to further student understanding and learning.

Another participant described how abstract algebra could inform lesson planning, such as using arithmetic of quadratic fields and rational points on conics to develop problems that work out nicely. Similarly, the other participant explained:

> You are teaching them geometry and you want to give them some examples of triangles in a plane. It would be very nice if the side lengths of the triangle were integers, because it makes it much easier to work with, and by using abstract algebra it tells you how you can construct many examples.

He also mentioned that one of the things he likes to discuss when teaching abstract algebra is the impossibility of trisecting an angle, which comes up in secondary school geometry. Another mathematical connection he described, "In high school you could take the affine group on the plane and construct an equilateral triangle with medians and apply the affine group, then you can see the properties are invariant of the affine group with midpoints." Ultimately, for this participant, to teach something effectively means the teacher needs to know why something is true, as well as be able to communicate why you need certain hypothesis or properties so that a structure does not break down. Consequently, to him, mathematical connections between abstract algebra and secondary school mathematics involve those beneficial to teaching effectively, so important connections are those that both provide necessary background knowledge and improve a teacher's ability to communicate that knowledge.

Summary

In this chapter, I detailed the various ways mathematicians and mathematics educators described mathematical connections between abstract algebra and secondary school mathematics while participating in individual interviews. The pure mathematicians that have only published in pure mathematics journals with little to no experience with the secondary school curriculum made fewer connections than the participants with mathematics education experience, especially those having several years of secondary school teaching experience. To be more specific, the pure mathematicians discussed approximately five to eight mathematical connections whereas the other participants discussed approximately ten or more connections. The results of this study were organized based on the analytic connection framework used for the textbook analysis, additional perspectives of mathematical connections described in the literature review, and emerging types of mathematical connections from the interview data, which included connections that aligned with the historical roots of group theory and connections for teaching. The final modified connection list initially created from the textbook data, used during the interviews, and modified after each interview to include additional mentioned concept-by-concept connections is shown in Table 9.

Table 9

Final Mathematical Connections List

Abstract Algebra Concept	Secondary School Mathematics Concept	Connection Category
Algebraic structures & their properties	Function and domain	Comparison Through Common Features; Hierarchical Relationship
Algebraic structures & their properties	Identity	Hierarchical Relationship
Algebraic structures & their properties	Inverse	Hierarchical Relationship
Algebraic structures & their properties	Number systems and known operators	Comparison Through Common Features; Generalization
Algebraic structures & their properties	Solving linear equations	Comparison Through Common Features; Hierarchical Relationship
Binary operator	Arithmetic operators & number systems	Generalization
Binary operator	Domain	Comparison Through Common Features
Binary operator	Function	Comparison Through Common Features
Binary operator	Function composition	Comparison Through Common Features; Generalization
Binary operator	Function transformations	Comparison Through Common Features
Commutative ring theory (localization)	Fractions	Hierarchical Relationship
Compass/geometric constructions	Geometry concepts including: points, lines, circles, regular n-gons, angles, intersection, and trisection	Hierarchical Relationship

Abstract Algebra Concept	Secondary School Mathematics Concept	Connection Category
Congruence	Solving linear equations	Comparison Through Common Features
Cyclic group	Division algorithm	Hierarchical Relationship
Cyclic group	Greatest common divisor	Comparison Through Common Features
Cyclic group	Imaginary unit $\boldsymbol{i}$	Comparison Through Common Features
Cyclic group	Rotations and periodicity	Comparison Through Common Features
Direct product	Cartesian plane and ordered pairs	Generalization
Direct product	Matrices for area and volume	Comparison Through Common Features
Equivalence	Equal sign	Comparison Through Common Features
Equivalence	Inequality	Comparison Through Common Features
Equivalence	Similarity	Comparison Through Common Features
Equivalence	Solving Equations	Hierarchical Relationship
Equivalence classes	Decimal expansions	Comparison Through Common Features
Equivalence classes	Equivalent fractions	Comparison Through Common Features
Equivalence classes	Linear functions	Comparison Through Common Features
Equivalence relation	Congruence	Comparison Through Common Features
Equivalence relation	Inequality	Comparison Through Common Features

Abstract Algebra Concept	Secondary School Mathematics Concept	Connection Category
Equivalence relation	Similarity	Comparison Through Common Features
Equivalence relation	Symmetry	Hierarchical Relationship
Extension field/splitting field	Complex numbers	Hierarchical Relationship
Extension field/splitting field	Domain	Comparison Through Common Features
Extension field/splitting field	Solving for roots of a polynomial	Hierarchical Relationship
Fundamental theorem of algebra	Roots of a polynomial	Comparison Through Common Features
Galois theory	Radicals	Hierarchical Relationship
Galois theory	Roots of polynomial equations	Hierarchical Relationship
Groups and specific types of groups	Function composition	Hierarchical Relationship
Groups and specific types of groups	Geometric transformations & symmetries	Alternative Representation; Hierarchical Relationship
Homomorphism/isomorphism	Equality	Comparison Through Common Features
Homomorphism/isomorphism	Function	Comparison Through Common Features; Hierarchical Relationship
Homomorphism/isomorphism	Infinity and finitely infinite	Comparison Through Common Features
Homomorphism/isomorphism	Invariance	Comparison Through Common Features
Homomorphism/isomorphism	Mapping	Hierarchical Relationship
Ideal	Number systems	Comparison Through Common Features
Ideal	Subset	Hierarchical Relationship

Abstract Algebra Concept	Secondary School Mathematics Concept	Connection Category
Inverse	Multiplicative reciprocal	Generalization
Inverse	Negative numbers	Generalization
Irreducible polynomial	Factoring polynomials	Generalization
Kernel	Nullspace of a matrix	Comparison Through Common Features
Lagrange’s theorem	Euclidean algorithm	Hierarchical Relationship
Lagrange’s theorem	Greatest common factor	Hierarchical Relationship
Lagrange’s theorem	Least common multiple	Hierarchical Relationship
Nilpotent	Geometric series and convergence	Hierarchical Relationship
Permutation group	Function & function composition	Hierarchical Relationship
Permutation group	Permutation	Hierarchical Relationship
Permutation group	Symmetry	Hierarchical Relationship
Polynomial ring	Operations with polynomials & polynomial long division	Comparison Through Common Features; Hierarchical Relationship
Polynomial ring	Polynomial vocabulary (degree, coefficients, roots, etc.)	Comparison Through Common Features; Hierarchical Relationship
Polynomial ring	Power series	Hierarchical Relationship
Quotient group/Quotient field	Equivalent fractions	Generalization
Quotient group/Quotient field	Fractions & operations with fractions	Generalization
Product of cycle decomposition	Composite function	Hierarchical Relationship
Quaternions	Complex numbers	Comparison Through Common Features

Abstract Algebra Concept	Secondary School Mathematics Concept	Connection Category
Sign rule in a ring	Product of two negative numbers is positive	Generalization
Subgroups	Subsets	Hierarchical Relationship
Unary operators	Negation	Generalization
Unary operators	Trigonometric functions	Generalization
Unit	Invertible matrices	Comparison Through Common Features
Zero divisors	Geometric reflections & rotations	Comparison Through Common Features
Zero divisors	Solve quadratic equations by factoring	Hierarchical Relationship

CHAPTER 6: CONCLUSIONS, IMPLICATIONS, AND FUTURE WORK

In this research study, I sought to first identify the mathematical connections between abstract algebra and secondary school mathematics explicitly stated in textbooks and discussed by mathematicians and mathematics educators. I then classified and described those connections. The primary motivation for this research followed from my assumption that students enhance their concept images and concept definitions of abstract algebra content with the recognition and understanding of these connections. However, previous research has shown that undergraduate abstract algebra students do not recognize mathematical connections between abstract algebra and secondary school mathematics. In this study I aspired to address this issue by identifying mathematical connections that could be made and investigating ways mathematicians and mathematics educators describe them. To reiterate, the following research questions guided this study:

1) What mathematical connections are explicitly stated between abstract algebra and secondary school mathematics in abstract algebra textbooks, and how are these connections discussed?
2) Which mathematical connections between abstract algebra and secondary school mathematics do mathematicians and mathematics educators identify, and how do they describe them?

Qualitative research methods allowed me to answer my research questions by enabling me to more accurately present the perspectives of written materials such as textbooks through text analysis as well as the perspectives of mathematicians and mathematics educators about mathematical connections found in abstract algebra as expressed through interviews. This study utilized two main sources of data: abstract algebra textbooks and interviews with mathematicians and mathematics educators. I selected nine abstract algebra textbooks to analyze (described in detail in Chapter 3 and seen in Table 2) and 13 mathematicians and mathematics educators involved in abstract algebra teaching or research to interview (described in detail in Chapter 3 and seen on page 32). In the first stage of research, I created a list of mathematical connections identified in abstract algebra textbooks and classified the ways in which these mathematical connections were presented. To this list, I added mathematical connections that I thought may have been implicitly stated in the textbooks or I recognized from using my own knowledge of abstract algebra. I then interviewed the mathematician and mathematics educators, altering the connections list to include those mentioned by the participants. In this final stage of research, I also classified the ways in which the participants described mathematical connections between abstract algebra and secondary school mathematics.

Even though researchers (CBMS, 2001; Cofer, 2012; Cook, 2012) have shown that undergraduate abstract algebra students are not recognizing mathematical connections between abstract algebra and secondary school mathematics, little previous work has been done to identify and discuss these connections. The findings of this study offer two major contributions to the literature. First, specific mathematical connections between concepts found in abstract algebra and those found in secondary school mathematics were identified and listed. The resulting list can serve as a teaching supplement for abstract algebra professors who may have not considered the secondary school mathematics curriculum in their teaching. Second, this study provided ways in which the mathematical connections between abstract algebra and secondary school mathematics could be discussed. As seen in this study, a connection between two concepts can be described in different ways. Mathematicians and mathematics educators who would like to talk about connections more explicitly can then utilize the classifications of connections to be more clear about the type of connection they want to emphasis. Policymakers and stakeholders can also take the results of this study to support the requirement of taking an abstract algebra course for prospective secondary mathematics teachers. In the following sections I elaborate on this study's specific conclusions and implications as well as possible future work.

Conclusions

One interesting conclusion that emerged from this research is that despite the varying mathematical connection perspectives described in abstract algebra textbooks and by the participants, there were similar key perspectives that could be categorized into specific groups. More specifically, the textbook authors presented connections in five distinct ways: alternative representation, comparison through common features, generalization, hierarchical relationship, and real-world application. Similar to the textbooks, the mathematicians and mathematics educators, when describing connections, also used the analytic framework categories: comparison through common features, generalization, and hierarchical relationship. In fact, the textbooks and participants both primarily described specific concept connections between abstract algebra and secondary school mathematics using comparison through the common features and hierarchical relationship.

To be more specific, the abstract algebra concepts *algebraic structures and properties* and *polynomial ring* compared to the secondary school mathematics concepts *number systems and arithmetic operators* and *polynomial operators and vocabulary* were the most widely explicitly stated connections across textbooks. The textbooks stated the greatest number of distinct concept connections using hierarchical relationship. Twelve of the thirteen mathematicians and mathematics educators also described mathematical connections through a comparison of structural features. These participants detailed the numerous ways two structures could be compared and identified the particular structures they typically used while making these comparisons. For instance, the participants discussed the structural similarities between the properties defining group theory and the properties needed to solve a linear equation with one operator. These results are inconsistent with previous research that has suggested the primary mathematical connection between abstract algebra and secondary school mathematics is generalization.

Not surprisingly, the authors of the abstract algebra textbooks and the participants identified various mathematical connections and prioritized certain connection types more than others. For instance, most of the textbooks introduced polynomial rings by making comparison through common features connections, whereas only one textbook discussed the hierarchical relationship connection between the abstract algebra concept *polynomial ring* and the secondary school mathematics concept *power series*. Even though polynomial ring is a standard abstract algebra concept found in all the textbooks, not all of them made the same explicit connection to secondary school mathematics or used the same connection type. Similarly, the mathematicians and mathematics educators' descriptions of connections reflected their own experiences with students when teaching abstract algebra. Three of the participants described connections as an artifact of learning by concentrating on the mental connections developed within the students' minds between concepts when learning abstract algebra. Six participants also discussed the involvement of the learner in a mathematical activity of establishing or identifying these connections through proof writing. In addition, the mathematicians and mathematics educators differed according to their individual conceptualizations of group theory. That is, participants with views of abstract algebra based on axioms, solving equations, number theory, or geometry prioritized different sets of connections.

Another interesting conclusion that emerged from this research is that the mathematical connections stated in the abstract algebra textbooks and described by the participants were linked to secondary school geometry nearly as often as secondary school algebra. These results were also inconsistent with previous research that suggested that the importance of abstract algebra lies in its mathematical connections to school algebra. To be more specific, seven of the nine textbooks connected group theory to geometric transformations through alternative representation and hierarchical relationship connections. In addition, seven textbooks connected the abstract algebra concepts *compass or geometric constructions* to school geometry concepts *angles*, *circles*, *regular n-gons,* etc. Several of the mathematicians and mathematics educators identified and discussed these same connections between abstract algebra and secondary school geometry. In fact, four participants prioritized connections to school geometry in their connection talk and when teaching abstract algebra. A comparable number of abstract algebra textbooks and participants explicitly made connections to secondary school algebra. That is, seven textbooks made an alternative representation connection between the abstract algebra concept *group theory* and secondary school algebra concept *solving linear equations*, and eight textbooks compared specific abstract algebra structures to secondary school algebra structures, operators, and vocabulary. Four participants also prioritized connections to school algebra in their connection talk and when teaching abstract algebra.

Implications

Stakeholders and policymakers' recommendations and previous research have often characterized abstract algebra as the generalization of school algebra. However, the findings of this study revealed a discrepancy between these held beliefs and the actual mathematical connections described in abstract algebra textbooks and by mathematicians and mathematics educators with expertise in abstract algebra. In fact, other connections and connection types were discussed with greater frequency in this study. Abstract algebra can no longer be considered simply as the generalization of school algebra but rather it should be regarded as an extension of previous mathematical knowledge from algebra and geometry. This study's results revealed that textbook authors and participants identified and discussed mathematical connections between abstract algebra and secondary school geometry nearly as often as those connections to secondary school algebra. Thus, abstract algebra provides prospective secondary mathematics teachers knowledge that is important to their understandings of school geometry as well as their understandings of school algebra.

Furthermore, the mathematical connections between abstract algebra and secondary school algebra in addition to the connections between abstract algebra and secondary school geometry are not simply generalizations. In fact, I discovered through this research that abstract algebra textbook and mathematicians and mathematics educators mentioned connections of other types more frequently; namely, connections of the types comparison of common features, hierarchical relationship, and alternative representations. The rationale for requiring prospective secondary mathematics teachers to take an abstract algebra course should then include abstract algebra as a further study of familiar mathematical ideas through studying structural comparisons, building upon previous mathematical concepts, and using alternative representations of algebraic concepts. In other words, a change must occur in the way in which we explain why abstract algebra is required for prospective secondary mathematics teachers.

Another implication that can be drawn from the results of this study is that a variety of mathematical connections between abstract algebra and secondary school mathematics can be made, and these connections can be described in various ways. It is important for abstract algebra professors to recognize that not all abstract algebra textbooks identify the same mathematical connections nor do all the textbooks discuss mathematical connections in the same way. For instance, not all of the abstract algebra textbooks analyzed introduced polynomial rings using comparison of common features even though the majority of the texts did. By identifying the specific mathematical connections found in their assigned textbook, one can build on these connections to help students develop more accurate concept images and concept definitions of abstract algebra content. For instance, professors can discuss connections with students in the classroom analogously to the ways in which connections are described in the assigned text. Coherently discussing mathematical connections may enhance students' understanding of important connections or help students build new understandings from previous knowledge. Further, professors should be aware of the identified connections and connection types omitted from their assigned textbook so that they can discuss omitted connections in class or through supplementary materials. Abstract algebra professors may also want to talk about connections with students in ways that are not found in the textbook but would be beneficial to learning. For example, a professor may want to introduce alternative representations of a group when the assigned textbook only presents one representation of a group.

From the interview data, in particular, another implication that can be drawn; namely, mathematicians and mathematics educators' individual conceptualizations of group theory influence their descriptions of mathematical connections between abstract algebra and secondary school mathematics. Thus, abstract algebra professors should be aware of how their individual conceptualizations of abstract algebra influence their teaching. That is, a professor who focuses on certain connections or connection types over others can limit the understandings of their students by not providing them opportunities to identify a set of connections. For instance, a professor who favors the axiomatic approach to group theory may fail to discuss the geometric connections between group theory and secondary school mathematics. As a result, students, and especially prospective secondary mathematics teachers, will not access the full benefits of this course because they will not see the connections to secondary school geometry. Therefore, abstract algebra professors should consciously consider their individual conceptualizations of abstract algebra and how it affects their students learning.

Future Work

This study provides the groundwork for future research investigating the mathematical connections between abstract algebra and secondary school mathematics. I see two logical steps for future research: classroom observations to explore the mathematical connections discussed in abstract algebra courses and teaching experiments with undergraduate mathematics students enrolled in an abstract algebra course. In the former, examining the mathematical connections being made by the professor and students will provide a more complete picture of which and how mathematical connections are being presented in practice, if any. I anticipate either several explicit connections are being made in the classroom setting or very few. In the latter research direction, I will conduct a teaching experiment with a group of undergraduate mathematics students in which students will engage in activities that challenge them to consider the various mathematical connections between abstract algebra and secondary school mathematics. The purpose of this study would be to understand student learning of abstract algebra through identifying and making use of mathematical connections and to determine teaching approaches to enhance learning. An extension of this study can be conducted by following the teaching experiment participants into their teaching profession to examine which connections and connection types are valuable for teaching and how these connections influence their teaching. Both research directions would provide a more complete picture of mathematical connections in the area of abstract algebra.

In conclusion, many stakeholders and educational movements have emphasized the importance of recognizing mathematical connections between mathematical ideas to build students' understanding of mathematics. Although previous research has examined abstract algebra learning as well as mathematical connections from a variety of perspectives, this study provided the first exploration of the explicit mathematical connections between abstract algebra and secondary school mathematics. The textbook analysis and individual interviews with mathematicians and mathematics educators provided a list of mathematical connections and descriptions of these connections. Identifying and characterizing connections between abstract algebra concepts and secondary school mathematics concepts offers abstract algebra professors additional information that can be used to enhance undergraduate students' understandings of abstract algebra in addition to providing the vocabulary to discuss these mathematical connections.

REFERENCES

Asiala, M., Brown, A., Kleiman, J., & Mathews, D. (1998). The development of students' understanding of permutations and symmetries. *International Journal of Computers for Mathematical Learning*, *3*, 13–43.

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., & Oktaç, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior, 16*(3), 241–309.

Boaler, J. (2002). Exploring the nature of mathematical activity: Using theory, research and "working hypotheses" to broaden conceptions of mathematics knowing. *Educational Studies in Mathematics*, *51*, 3–21.

Brown, A. (1990). Writing to learn and communicate mathematics: An assignment in abstract algebra. In A. Sterrett (Ed.), *Using writing to teach mathematics* (pp. 131–133). Washington, DC: Mathematical Association of America.

Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups and subgroups. *Journal of Mathematical Behavior, 16*(3), 187–239.

Bukova-Güzel, E., Ugurel, I., Özgür, Z., & Kula, S. (2010). The review of undergraduate courses aimed at developing subject matter knowledge by mathematics student teachers. *Procedia: Social and Behavioral Sciences*, *2*, 2233–2238.

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections*. (Unpublished doctoral dissertation). Simon Fraser University, Burnaby, BC, Canada.

Capaldi, M. (2012) A study of abstract algebra textbooks. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 364–368). Portland, OR: Mathematical Association of America.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19–32). Mahwah, NJ: Erlbaum.

Chappell, M. F., & Strutchens, M. E. (2001). Creating connections: Promoting algebraic thinking with concrete models. *Mathematics Teaching in the Middle School, 7*(1), 20–25.

Charmaz, K. (2000). Grounded theory: Objectivist and constructivist methods. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 509–535). Thousand Oaks, CA: Sage.

Cofer, T. (in press). Mathematical explanatory strategies employed by prospective secondary teachers. *Research in Collegiate Mathematics Education*.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers* (Issues in Mathematics Education, Vol. 11). Providence, RI: American Mathematical Society.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II* (Issues in Mathematics Education, Vol. 17). Providence, RI: American Mathematical Society.

Connection. (2014). In *Oxford Dictionary online*. Retrieved from http://www.oxforddictionaries.com/us/definition/americanenglish/connection?q=connection

Cook, J. P. (2012). *A guided reinvention of ring, integral domain, and field*. (Doctoral dissertation, University of Oklahoma, Norman, Oklahoma). Available from ProQuest Dissertations and Theses database. (UMI No. 3517320).

Cook, J. P., & Stewart, S. (2014) Presentation of matrix multiplication in introductory linear algebra textbooks. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*. Denver, CO: Mathematical Association of America.

Coxford, A. F. (1995). The case for connections. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 3–12). Reston, VA: National Council for Teachers of Mathematics.

Cuoco, A. (2001). Mathematics for teaching. *Notices of the American Mathematical Society, 48*(2), 168–174.

Cuoco, A., & Rotman, J. J. (2013). *Learning modern algebra: From early attempts to prove Fermat's last theorem.* Washington, DC: Mathematical Association of America.

Czerwinski, R. (1994). A writing assignment in abstract algebra. *PRIMUS, 4,* 117–124.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*, 267–305.

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (3rd ed.). Hoboken, NJ: Wiley.

Fey, J. Fukawa-Connelly, T., Howell, K., Birky, G., & Napp, C. (2009). *Lecture notes for Math 761*. Personal Collection of J. Fey, University of Maryland, College Park, MD.

Findell, B. (2001). *Learning and understanding in abstract algebra*. Unpublished doctoral dissertation, University of New Hampshire.

Fraleigh, J. B. (2003). *A first course in abstract algebra* (7th ed.). Reading, MA: Addison-Wesley.

Freedman, H. (1983). A way of teaching abstract algebra. *American Mathematical Monthly, 90*(9), 641–644.

Gallian, J. A. (1976). Computers in group theory. *Mathematics Magazine, 49*(1), 69–73.

Gallian, J. A. (1990). *Contemporary abstract algebra* (2nd ed.). Lexington, MA: D. C. Heath.

Gallian, J. A. (2013). *Contemporary abstract algebra* (8th ed.). Boston, MA: Brooks/Cole, Cengage Learning.

Harel, G. (1987). Variations in linear algebra content presentation. *For the Learning of Mathematics, 7*, 29–32.

Hart, E. (1994). Analysis of the proof-writing performances of expert and novice students in elementary group theory. In J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning* (MAA Notes No. 33, pp. 49–62). Washington, DC: Mathematical Association of America.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71–90.

Hazzan, O., & Leron, U. (1996). Students' use and misuse of mathematical theorems: The case of Lagrange's theorem. *For the Learning of Mathematics, 16*(1), 23–26.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York, NY: Macmillan.

Hillman, A. P., & Alexanderson, G. L. (1994). *Abstract algebra: A first undergraduate course* (5th ed.). Prospect Heights, IL: Waveland Press.

Hodge, J. K., Schlicker, S., & Sundstrom, T. (2014). *Abstract algebra: An inquiry-based approach.* Boca Raton, FL: Taylor & Francis.

Hodgson, T. R. (1995a). Connections as problem-solving tools. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 13–21). Reston, VA: National Council of Teachers of Mathematics.

Hodgson, T. R. (1995b). Reflections on the use of technology in the mathematics classroom. *PRIMUS, 5,* 178–191.

Huetinck, L. (1996). Group theory: It's a SNAP. *Mathematics Teacher, 89*(4), 342–346.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. In National Council of Teachers of Mathematics & Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K–14 curriculum: Proceedings of a national symposium*. Washington, DC: National Academies Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: Macmillan.

Kleiner, I. (2007). *A history of abstract algebra*. New York, NY: Birkhauser Boston.

Larsen, S. (2004). *Supporting the guided reinvention of the concepts of group and isomorphism: A developmental research project*. (Unpublished doctoral dissertation). Arizona State University, Phoenix, AZ.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *Journal of Mathematical Behavior*, *28*(2-3), 119–137.

Larsen, S. (2010). *Lecture notes for Teaching Abstract Algebra for Understanding*. Personal Collection of S. Larsen, Portland State University, Portland, OR.

Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *Journal of Mathematical Behavior*, *32*(4), 726–742.

Leganza, K. (1995). Writing assignments in an abstract algebra course. *Humanistic Mathematics Network Journal, 11,* 29–32.

Leitzel, J. R. C. (Ed.). (1991). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics* (MAA Reports No. 3). Washington, DC: Mathematical Association of America.

Leron, U., & Dubinsky, E. (1994). *Learning abstract algebra with ISETL*. New York, NY: Springer-Verlag.

Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly, 102*(3), 227–242.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*, 153–174.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Navigating through mathematical connections in Grades 9–12*. Reston, VA: Author.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Author.

National Research Council. (1998). *The nature and role of algebra in the K–14 curriculum.* Washington, DC: National Academy Press.

Nicholson, W. K. (2012). *Introduction to abstract algebra* (4th ed.). Hoboken, NJ: Wiley.

Nicodemi, O. E., Sutherland, M. A., & Towsley, G. W. (2007). *An introduction to abstract algebra with notes to the future teacher.* Upper Saddle River, NJ: Pearson Prentice Hall.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meaning: Learning cultures and computers* (Vol. 17). Dordrecht, the Netherlands: Kluwer Academic Publishers.

O'Connor, J. J., & Robertson, E. F. (1996). *The development of group theory.* Retrieved from http://www-history.mcs.st-andrews.ac.uk/HistTopics/Development_group_theory.html

Papick, I., Beem, J., Reys, B., & Reys, R. (1999). Impact of the Missouri Middle Mathematics Project on the preparation of prospective middle school teachers. *Journal of Mathematics Teacher Education, 2*, 301–310.

Patton, M.Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage.

Pedersen, J. J. (1972). Sneaking up on a group. *Two-Year College Mathematics Journal, 3*(1), 9–12.

Renze, J., & Weisstein, E. W. (n.d.). *Abstract algebra*. Retrieved from MathWorld--A Wolfram Web Resource: http://mathworld.wolfram.com/AbstractAlgebra.html

Robitaille, D. F., & Travers, K. J. (1992). International studies of achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 687–709). New York, NY: Macmillan.

Roulston, K. (2010). *Reflective interviewing: A guide to theory and practice*. Thousand Oaks, CA: Sage.

Selden, A. & Selden, J. (1987). Errors and misconceptions in college level theorem proving. *Proceedings of the Second International Seminar on Misconceptions in Science and Mathematics*, (Vol. 3, pp. 457-470). Ithaca, NY: Cornell University.

Shifrin, T. (1996). *Abstract algebra: A geometric approach.* Upper Saddle River, NJ: Pearson Prentice Hall.

Singletary, L. M. (2012). *Mathematical connections made in practice: An examination of teachers' beliefs and practices*. (Unpublished dissertation). University of Georgia, Athens, GA.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Harmondsworth, England: Penguin.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151–169.

Taylor, S. J., & Bogdan, R. (1984). *Introduction to qualitative research methods: The search for meanings.* New York, NY: Wiley.

Thrash, K. R., & Walls, G. L. (1991). A classroom note on understanding the concept of group isomorphism. *Mathematics and Computer Education, 25*(1), 53–55.

Usiskin, Z. (1974). Some corresponding properties of real numbers and implications for teaching. *Educational Studies in Mathematics, 5*, 279–290.

Usiskin, Z. (1975). Applications of groups and isomorphic groups to topics in the standard curriculum, grades 9–11. *Mathematics Teacher*, *68*(3), 99–106, 235–246.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A.F. Coxford (Ed.), *The Ideas of Algebra, K–12: NCTM 1988 yearbook* (pp. 8–19). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (2001). Teachers' mathematics: A collection of content deserving to be a field. *Mathematics Teacher*, *6*(1), 86–98.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293–305.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*, 101–119.

Zazkis, R. (2000). Factors, divisors, and multiples: Exploring the web of students' connections. *CBMS Issues in Mathematics Education, 8*, 210–238.

Zazkis, R., & Hazzan, O. (1999). Interviewing in mathematics education research: Choosing the questions. *Journal of Mathematical Behavior, 17*(4), 429–439.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263–281.

APPENDIX A

Interview Protocol

Hello, my name is Ashley Suominen and I am a PhD student in mathematics education at University of Georgia. I am currently conducting a research study about the mathematical connections between abstract algebra and secondary school mathematics.

Before we begin, I would like to discuss my plan on identifying your involvement in this research. In my methodology section of my dissertation I plan on listing all of the participants that helped with this project. However, I will not use direct quotes or identify anything you say or do during the interview to you personally. The reason for this decision is to provide additional creditability to the resulting formulated list of mathematical connections to future readers. If you would not like to be acknowledged in my dissertation, please let me know that as well. If at any point you feel uncomfortable, you may refuse to answer a question or stop the interview without any penalty. The interview should last approximately an hour. Do you have any questions before we begin?

Experience Questions

We will begin the interview by discussing your background in abstract algebra.

- Tell me a little bit about when you teach abstract algebra.
    - How many years/semesters have you taught the course?
    - What textbook do you use?
    - What topics do you include in the course?

- How do you typically structure or teach the course?
- What do you want your students to learn from the course?

- *If they research abstract algebra:* Tell me about your research interests in abstract algebra.

Switching gears slightly, let's discuss your background in secondary mathematics education.

- Have you ever taught mathematics at the high school level?
  - If so, what did you teach?
  - If not, what experiences do you have regarding secondary mathematics curriculum?

Mathematical Connections Questions

Next, we will explore content connections between abstract algebra and secondary school mathematics.

- First, how would you define mathematical connections? *If clarification is needed:* What does it mean for a connection to exist between two concepts?
- Given your knowledge of abstract algebra as well as the mathematics taught at the secondary level (6^{th}-12^{th} grades), what mathematical connections do you recognize between abstract algebra and secondary school mathematics?
  - *If the participant is not familiar with the secondary school mathematics curriculum, I will provide the list of curriculum topics*[1] *(Table A1).* Given your knowledge of abstract algebra as well as this list of curricular topics, what mathematical connections do you recognize between abstract algebra and secondary school mathematics?
  - *For each identified connection:* Tell me a little bit about why you identified this connection.

[1] Even though I offered two participants the option to view this document, neither participant wanted to look at the curriculum list, so none of the participants actually saw this document.

When the participant is finished identifying mathematical connections,

Through analyzing textbooks and interviewing other mathematicians and mathematics educators, I have a working list of mathematical connections between abstract algebra and secondary school mathematics. Please take a few minutes to look over this list. *Give the participant the list of mathematical connections.*

- Starting at the top of the list, do you agree with each of these connections?
    - *If so*, could you explain why you agree with the connection?
    - *If not*, could you explain why you disagree with the proposed connection?
- Now that you have seen a formulated list of mathematical connections, are there any additional mathematical connections that are missing from this list?

Closing:

I want to thank you for coming in for this interview. Do you have any other thoughts about mathematical connections that you would like to share? If I have any other follow-up questions, may I contact you again? Thank you and have a great day!

Table A1

Secondary Mathematics Curriculum Topics

Grade/course	Curriculum Topics
6^{th} grade	▪ Area, surface area, volume ▪ Greatest common factor ▪ Multiply and divide fractions ▪ One variable equations and inequalities ▪ Rational numbers ▪ Understand and use ratios
7^{th} grade	▪ Angle measure, area and circumference of a circle ▪ Understand and use proportions
8^{th} grade	▪ Congruence and similarity ▪ Definition of functions ▪ Irrational Numbers ▪ Linear equations and pairs of linear equations ▪ Radicals ▪ Volumes of cones, cylinders, and spheres
Algebra 1	▪ Arithmetic and geometric sequences ▪ Linear, quadratic, and exponential equations ▪ Piecewise functions ▪ Systems of linear equations
Geometry	▪ Circle theorems, arc length, areas of sectors of circles ▪ Conic sections ▪ Distance formula ▪ Geometric constructions ▪ Prove geometric theorems involving congruence and similarity ▪ Transformations in a plane ▪ Trigonometric ratios using right triangles
Algebra 2	▪ Complex numbers ▪ Factors and zeros of polynomials ▪ Logarithms ▪ Transformations of functions ▪ Trigonometric functions using unit circle ▪ Trigonometric identities
Fourth course	▪ Domain of trigonometric functions ▪ Prove trigonometric identities ▪ Vectors and matrices

APPENDIX B

Mathematical Connections Found in the Textbooks

Abstract Algebra Concept	Secondary School Mathematics Concept
Algebraic structures (Group, Ring, Integral Domain, Field) & their properties	Number systems and known operators; Solving linear equations
Binary operator	Arithmetic operators & number systems
Compass/geometric constructions	Geometry concepts including: points, lines, circles, regular n-gons, angles, intersection, and trisection
Congruence	Solving linear equations
Cyclic group	Division algorithm
Direct product	Cartesian plane and ordered pairs
Extension field/splitting field	Complex numbers; Roots of a polynomial
Fundamental theorem of algebra	Roots of a polynomial
Galois theory	Radicals; Roots of polynomial equations
Groups and specific types of groups	Function composition; Geometric transformations & symmetries
Homomorphism/isomorphism	Function
Inverse	Multiplicative reciprocal; Negative numbers
Irreducible polynomial	Factoring polynomials
Nilpotent	Geometric series and convergence
Permutation group; Product of cycle decomposition	Function and function composition; Permutation
Polynomial ring	Operations with polynomials & polynomial long division; Polynomial vocabulary (degree, coefficients, roots, etc.); Power series
Quotient field	Fractions & operations with fractions
Quaternions	Complex numbers
Sign rule in a ring	Product of two negative numbers is positive
Unit	Invertible matrices
Zero divisors	Solve quadratic equations by factoring

APPENDIX C

Initial Mathematical Connections List for Interviews

Abstract Algebra Concept	Secondary School Mathematics Concept
Algebraic structures (Group, Ring, Integral Domain, Field) & their properties	Number systems and known operators; Solving linear equations
Binary operator	Arithmetic operators & number systems
Compass/geometric constructions	Geometry concepts including: points, lines, circles, regular n-gons, angles, intersection, and trisection
Congruence	Solving linear equations
Cyclic group	Division algorithm; Imaginary unit ***i***
Direct product	Cartesian plane and ordered pairs
Extension field/splitting field	Complex numbers; Roots of a polynomial
Fundamental theorem of algebra	Roots of a polynomial
Galois theory	Radicals; Roots of polynomial equations
Groups and specific types of groups	Function composition; Geometric transformations & symmetries
Homomorphism/isomorphism	Function
Ideal	Number systems; Subset
Inverse	Multiplicative reciprocal; Negative numbers
Irreducible polynomial	Factoring polynomials
Kernel	Nullspace of a matrix
Nilpotent	Geometric series and convergence
Permutation group; Product of cycle decomposition	Function and function composition; Permutation

Polynomial ring	Operations with polynomials & polynomial long division; Polynomial vocabulary (degree, coefficients, roots, etc.); Power series
Quotient field	Fractions & operations with fractions
Quaternions	Complex numbers
Sign rule in a ring	Product of two negative numbers is positive
Subgroup	Subsets
Unit	Invertible matrices
Zero divisors	Solve quadratic equations by factoring

APPENDIX D

Mathematical Connections List After Interviews

Abstract Algebra Concept	Secondary School Mathematics Concept
Algebraic structures (Group, Ring, Integral Domain, Field) & their properties	Function and domain; Identity; Inverse; Number systems and known operators; Solving linear equations
Binary operator	Arithmetic operators & number systems; Domain; Function; Function composition; Function transformations
Commutative ring theory (localization)	Fractions
Compass/geometric constructions	Geometry concepts including: points, lines, circles, regular n-gons, angles, intersection, and trisection
Congruence	Solving linear equations
Cyclic group	Division algorithm; Greatest common divisor; Imaginary unit $\boldsymbol{i}$; Rotations and periodicity
Direct product	Cartesian plane and ordered pairs; Matrices for area and volume
Equivalence	Equal sign; Inequality; Similarity; Solving equations
Equivalence classes	Decimal expansions; Equivalent fractions; Linear functions
Equivalence relation	Congruence; Inequality; Similarity; Symmetry
Extension field/splitting field	Complex numbers; Domain; Roots of a polynomial
Fundamental theorem of algebra	Roots of a polynomial
Galois theory	Radicals; Roots of polynomial equations
Groups and specific types of groups	Function composition; Geometric transformations & symmetries

Homomorphism/isomorphism	Equality; Function; Infinity and finitely infinite; Invariance; Mapping
Ideal	Number systems; Subset
Inverse	Multiplicative reciprocal; Negative numbers
Irreducible polynomial	Factoring polynomials
Kernel	Nullspace of a matrix
Lagrange’s theorem	Euclidean algorithm; Greatest common factor; Least common multiple
Nilpotent	Geometric series and convergence
Permutation group; Product of cycle decomposition	Function and function composition; Permutation; Symmetry
Polynomial ring	Operations with polynomials & polynomial long division; Polynomial vocabulary (degree, coefficients, roots, etc.); Power series
Quotient group/Quotient field	Equivalent fractions; Fractions & operations with fractions;
Quaternions	Complex numbers
Sign rule in a ring	Product of two negative numbers is positive
Subgroup	Subsets
Unary operators	Negation; Trigonometric functions
Unit	Invertible matrices
Zero divisors	Geometric reflections & rotations; Solve quadratic equations by factoring