OPTION PRICING MODELS AND RELATED EMPIRICS

by

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(Under the Direction of Chris T. Stivers)

ABSTRACT

Since the 1987 stock market crash, the Black-Scholes option pricing model gives rise to more significant biases. Several alternative models aim to overcome this weakness. They are presented in the first part of this thesis. However, none of these models are fully capable of explaining observed option market prices. Other explanations have been advanced and give some additional insight to the world of options. The second part of the thesis provides an overview of related empirical developments to date. A thorough understanding of option pricing models and related empirics proves to be important for different aspects of finance.

INDEX WORDS: Black-Scholes model, Option pricing models, Volatility smile, Risk premia, Implied binomial trees
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1 INTRODUCTION

This thesis aims to present the most important option pricing models and to what extent they are able to explain observed market prices. Further, I try to build a bridge between the partly controversial empirical findings and the underlying theory.

Having a good formula for option prices is important for several reasons. Market makers have to determine fair prices, financial institutions wish to hedge their exposures to options and finally academics use option approaches for pricing real investments or corporate liabilities.

An American call (put) option on an asset gives the holder the right to buy (sell) a fixed amount of the underlying at a predetermined price until a fixed expiration date. The corresponding European options can only be exercised at the expiration date. To abstract from the difficulties associated with the valuation of American options we will mostly be concerned with European options.

The option market increased tremendously in volume during the last years and is now a $70 trillion market.\(^1\) The biggest breakthrough in the theory of option pricing appeared to be the Black-Scholes (1973) model. Using their formula, one can quickly determine the theoretical value of an option. Unfortunately, the formula has statistically significant biases that have become even more severe since the 1987 market crash. They are now often referred to as the volatility smile. Models that followed Black-Scholes each relax one or more of the original assumptions. It is found that these models can help explain observed market prices. Yet, none of the models is completely capable of predicting prices. Other approaches try to recover the underlying probability distribution and use it to price options. On the other hand, empirical evidence points out the possibility that option prices contain risk premia.

\(^1\) See Chance (2003).
This thesis is divided into two parts. Chapter 2 gives an introduction to the theory of option pricing models. The binomial model in 2.1, the Black-Scholes model in 2.2, stochastic volatility and jump models in 2.3 and 2.4 respectively, stochastic interest rates models in 2.5 and finally the Bakshi, Cao and Chen (1997) model in 2.6 are presented to give a solid background for the empirical studies in the following section. Chapter 3 itself is divided further. First, in 3.1 the biases before the 1987 crash are investigated followed by the observation of the persistent volatility smile after that in 3.2. Then, in Section 3.3 the question what option pricing model best explains the observed market prices is answered. In 3.4, I investigate the role of jump premia whereas in 3.5 ways of recovering the underlying probability distributions or main characteristics are presented. I present possible other explanations at the end of Chapter 3 in 3.6. Chapter 4 finally concludes.
2 OPTION PRICING MODELS

The first part of my thesis presents the most common option pricing models to give a solid theoretical background. Starting with the relatively simple binomial and Black-Scholes model, more complicated models incorporating stochastic volatility, jumps, or stochastic interest rates are covered thereafter. Finally, I give a model bearing all features simultaneously which proves important in the empirical section of this thesis.

Important aspects are the assumptions the models require and the implications for hedging as well as for empirical testing.

2.1 The Binomial Model

Cox, Ross, and Rubinstein (1979) present a simple model for pricing options which is also quite useful in practice. Assume that the stock price follows a multiplicative binomial process over discrete periods. This means that the stock, currently valued at $S$, can take two different values at the end of the period: $uS$ with probability $q$ and $dS$ with probability $1 - q$. Thus, the stock can either go up or down. Assume further that the interest rate $r$ is constant and that market participants have unlimited borrowing and lending at this rate. Individuals face no taxes, transaction costs, or margin requirements. Lastly, short selling is allowed and there are no restrictions on the use of the proceeds. To avoid riskless arbitrage opportunities, $u > 1 + r > d$ is required.

The basic technique for valuing a call can best be seen in the one-period case which means that the time to maturity for the call is one period away.\footnote{This period can be a minute, an hour, a day, or a year.} If $C_u$ denotes the value of the call after one period when the stock price goes up and $C_d$ the value after one period when the
stock price goes down, we have:

\[ C_u = \max[0, uS - K] \quad \text{and} \quad C_d = \max[0, dS - K], \]

where \( K \) is the strike price of the option. This simple economy can also be illustrated in the following diagram:

Now, it is possible to exactly replicate the payoffs of the call in both states of the world using appropriate positions in the underlying stock, denoted by \( \Delta \), and a riskless bond, denoted by \( B \). If we now equate the payoff of the combined position in the stock and the bond with the call payoffs, we get

\[ \Delta uS + (1 + r)B = C_u \quad \text{and} \quad \Delta dS + (1 + r)B = C_d. \]

Solving for \( \Delta \) and \( B \) yields

\[ \Delta = \frac{C_u - C_d}{(u - d)S} \quad \text{and} \quad B = \frac{uC_d - dC_u}{(u - d)(1 + r)}. \]

The value of the call should be exactly equal to the value of the hedging portfolio since both produce the same payoffs, thus \( C = \Delta S + B \). After some rearrangements we finally have

\[ C = \frac{pC_u + (1 - p)C_d}{1 + r}, \quad \text{with} \]

\[ p \equiv \frac{(1 + r) - d}{u - d} \quad \text{and} \quad 1 - p \equiv \frac{u - (1 + r)}{u - d}. \]
In the case of a non dividend-paying stock and a positive interest rate, the above equation for the value of the call is always greater than \( S - K \).

Some other features are evident from the formula. Neither the subjective probability \( q \) nor the investors’ risk attitudes enter the equation. The only (implicit) assumption was that investors prefer more to less and take advantage of riskless arbitrage. In this model, the stock price movement is the only risk factor or random variable. Since \( 0 < p < 1 \), \( p \) can be interpreted as a risk-neutral probability. Hence, the value of the call can be seen as the expectation of its discounted future value in a risk-neutral world.

The formula can be extended to \( n \) periods and then becomes

\[
C = S \Phi(a; n, p') - K(1 + r)^{-n}\Phi(a; n, p), \quad \text{where}
\]

\[
p = \frac{(1 + r) - d}{u - d} \quad \text{and} \quad p' = \frac{u}{1 + r}p.
\]

Additionally, \( a \) denotes the smallest non-negative integer\(^3\) greater than \( \log(\frac{K}{Sd^n}) / \log(u/d) \) and \( \Phi \) stands for the cumulative binomial distribution function:

\[
\Phi(a; n, p) = \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j}.
\]

It should be noted that differences between the actual price of the call and the model price always lead to a sure arbitrage profit in this simple world.

Assume now that the number of periods gets larger and larger (or equivalently that the time intervals get smaller and smaller), that is \( n \to \infty \), and adjust the other parameters in the following way:

\[
\hat{r} = r^{t/n}, \quad u = e^{\sigma \sqrt{t/n}}, \quad d = e^{-\sigma \sqrt{t/n}}.
\]

Then, one can show that the Black-Scholes option pricing formula will be approached in the limit. This model will be covered in the next section.

\(^3\) If \( a > n \), \( C = 0 \).
The binomial option pricing model possesses several advantages which are the reasons that this or related models are frequently used in practice. For example, it is straightforward to incorporate dividends and also account for early exercise in a computationally efficient way. Hence, the binomial model is capable of valuing American puts or American calls on dividend-paying stocks. More complex models like multinomial trees, lattices and implied trees have been developed recently. A basic and thorough understanding of the binomial model is thus needed for the more advanced models.

2.2 The Black-Scholes Model

The most widely-known option pricing model is still the Black-Scholes formula. In their seminal paper, Black and Scholes (1973) found a simple solution to value options. Although the derivation is somewhat tedious, later papers found more elegant ways to prove the formula. I only aim to give the most crucial steps in the derivation. The following is by no means comprehensive. Assume the following conditions in the market for the stock and for the option:

- The interest rate is known and constant.
- The distribution of stock prices $S$ is log-normal. That is, the stock returns follow the diffusion process

$$\frac{dS}{S} = \mu dt + \sigma dz,$$

where $\mu$ is the drift term or instantaneous expected return for the stock, $\sigma$ is the instantaneous volatility of the underlying, and $z$ is a standard Wiener process. While $\mu$ may be stochastic or depend on other variables, $\sigma$ is nonstochastic and at most a function of time. In the above equation, $\mu dt$ can be viewed as a certain return while $\sigma dz$ denotes a stochastic term which is normally distributed. Furthermore, the stochastic part is independent from its values in other periods with mean zero and variance $\sigma^2 dt$. 
- The variance of the stock is constant.
- The stock pays no dividends.
- The option is European.
- There exist no transaction costs.
- Unlimited borrowing at the (riskfree) interest rate.
- Unlimited short selling is possible.

Under these assumptions\(^4\), it is possible to take a hedge position, using a long position in the stock and a short position in the option. This hedge is completely insensitive to stock price movements. If \(C(S, T)\) is the value of the call as a function of the stock price \(S\) and time to maturity \(T\), then we have to sell short \(1/C_1\) options against one share of stock long, where \(C_1\) denotes the first partial derivative with respect to the first argument. In the case the hedge is maintained continuously, the above approximation is exact and the hedge is riskless. It should therefore earn the riskfree rate. Thus, the value of equity in the hedge position is

\[
S - \frac{C}{C_1}
\]

and the change in the value of the equity in a short interval \(\Delta t\) is given by

\[
\Delta S - \frac{\Delta C}{C_1}.
\]

Using Ito’s formula\(^5\) yields

\[
\Delta C = C_1 \Delta S + \frac{1}{2} C_{11} \sigma^2 S^2 \Delta T + C_2 \Delta T.
\]

---

\(^4\) In the following sections some of these restrictions will be relaxed.

\(^5\) Explaining also the underlying mathematics would clearly lead us too far. Yet, excellent references for the Mathematics of derivatives are Elliott and Kopp (1999), Lamberton and Lapayre (1996), and Neftci (2000).
The return has to be \( r\Delta T \) since the hedge position is riskless. So, the change in equity is equal to the riskless return times the value of the equity:

\[
- \left( \frac{1}{2} C_{11}\sigma^2 S^2 + C_2 \right) \frac{\Delta T}{C_1} = \left( S - \frac{C}{C_1} \right) r\Delta T.
\]

This yields the following stochastic differential equation

\[
C_2 = rC - rSC_1 - \frac{1}{2}\sigma^2 S^2 C_{11}, \tag{1}
\]

with boundary conditions

\[
C(S, 0) = S - K, \quad \text{for } S \geq K
\]
\[
= 0, \quad \text{for } S < K,
\]

where \( K \) is the strike price of the option. Making an appropriate substitution, (1) becomes the heat-transfer equation of physics, which can be solved. This leads to the following well-known formula:

\[
C = SN(d_1) - Ke^{-rT}N(d_2), \tag{2}
\]

\[
d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},
\]
\[
d_2 = d_1 - \sigma\sqrt{T},
\]

and \( S \) is the current price of stock, \( K \) is the strike price of the option, \( r \) equals the continuously compounded interest rate, \( T \) is the time to maturity, \( N(d) \) is the value of the cumulative normal density function, and \( \sigma^2 \) denotes the variance rate of the return on the stock. The formula can also be verified using other ways.\(^6\)

Obviously, many of the assumptions may be violated in reality resulting in biases when tested empirically. The most severe problem with testing the model is that the volatility

\(^6\)Black and Scholes (1973) give an alternative derivation using the CAPM, Merton (1973) proves the formula using stochastic interest rates.
of the underlying stock is the only unobservable variable in the model. In addition, model option prices are very sensitive to this input. Obvious choices for the volatility are historical estimates. On the other hand, we can also back out the implied volatility when we set the observed market price equal to the Black-Scholes price and solve iteratively for the volatility. Although the implied volatility can differ for different maturities, the Black-Scholes model implies that it should be equal for different strike prices (all else fixed).

2.3 Stochastic Volatility Models

As we have already seen, the most crucial input to the Black-Scholes formula is the estimate of the spot volatility. Yet, obviously the assumption of a constant volatility is not valid in real markets. Therefore, it is only natural to develop models with stochastic volatilities.

In 1987, several papers examined this issue, e.g. Hull and White (1987), Scott (1987), Johnson and Shanno (1987) and Wiggins (1987). Later, the body of work was extended by Melino and Turnbull (1990) and Stein and Stein (1991). Yet, it was not until 1993 when finally a closed form solution was found by Heston (1993).

First I will give a brief overview of the aforementioned papers. The work by Heston (1993) will be covered in greater detail since the solution technique is important for more advanced models as well. One of the difficulties of all stochastic volatility models is, as we will see, that the partial differential equation involves risk preferences and the volatility risk premium unless either (a) the volatility is a traded asset or (b) the volatility is uncorrelated with aggregate consumption. The other difficulty consists of finding the right stochastic process followed by the variance (or volatility). Possible choices are mean-reverting processes, constant elasticity processes or other models. However, so far it is unclear what the right process is.

---

7 See for example Schmalensee and Trippi (1978) or Poterba and Summers (1986).
8 See Section 2.6.
9 Of course, there is a similar problem for the stock returns.
10 See for example the recent paper by Jones (2003).
Hull and White (1987) consider the following stochastic processes for the security price $S$ and the instantaneous variance $V = \sigma^2$:

$$dS = \phi S dt + \sigma S dw$$

(3)

$$dV = \mu V dt + \xi V dz,$$

(4)

where the parameter $\phi$ can depend on $S$, $\sigma$ and $t$, while the other two variables $\mu$ and $\xi$ are only dependent on $\sigma$ and $t$. Additionally, $dw$ and $dz$ are two Wiener processes with correlation $\rho$. In this section, it is assumed that the interest rate is constant. The differential equation to solve becomes

$$\frac{\delta f}{\delta t} + \frac{1}{2} \left( \sigma^2 S^2 \frac{\delta^2 f}{\delta S^2} + 2 \rho \sigma \xi S \frac{\delta^2 f}{\delta S \delta V} + \xi^2 V^2 \frac{\delta^2 f}{\delta V^2} \right) - rf = -rS \frac{\delta f}{\delta S} - \mu \sigma^2 \frac{\delta f}{\delta V}$$

when it is simultaneously assumed that the volatility is uncorrelated with aggregate consumption. After the further assumption that $\rho = 0$, that is the volatility is uncorrelated with the stock price\(^{11}\), and by using the risk-neutral valuation procedure\(^{12}\), the authors show that the option value is given by

$$f(S_t, \sigma^2_t) = \int C(\bar{V}) h(\bar{V} | \sigma^2_t) d\bar{V}.$$  \hspace{1cm} (5)

In (5), $\bar{V}$ denotes the mean variance over the life of the derivative security defined by

$$\bar{V} = \frac{1}{T-t} \int_t^T \sigma^2_d d\tau$$

and $C(\bar{V})$ denotes the Black-Scholes price for the mean variance:

$$C(\bar{V}) = S_t N(d_1) - Ke^{-(T-t)} N(d_2),$$

where

\(^{11}\)This is quite a restrictive assumption since all empirical evidence indicates that $\rho$ is in fact negative. Usually, higher stock prices tend to come along with lower variances.

\(^{12}\)That is, discount the final payoffs with the appropriate factor.
\[ d_1 = \frac{\ln(S_t/K) + (r + \bar{V}/2)(T - t)}{\sqrt{V(T - t)}} \quad \text{and} \quad d_2 = d_1 - \sqrt{V(T - t)}. \]

Also, \( h(\bar{V}|\sigma^2_t) \) is a conditional density function. Thus, (5) states that the option price under stochastic volatility and the prevailing set of assumptions is the Black-Scholes price integrated over the distribution of the mean volatility. Yet, since it is impossible to find an analytic form for this distribution Hull and White use a Taylor series approximation for their results.

In the case that \( \rho = 0 \) it can be shown that the Black-Scholes price tends to overprice at-the-money options and underprice deep in-the-money and deep out-of-the-money options. The authors investigate cases where \( \rho \neq 0 \) and where \( \xi \) and \( \mu \) may depend on \( \sigma \) and \( t \) by using Monte Carlo simulation. For a positive correlation between the stock price and its volatility, out-of-the-money options are underpriced by the Black-Scholes formula, while in-the-money options are overpriced. The effect is reversed for a negative correlation. Furthermore, there is a time to maturity effect, longer-term near-the-money calls exhibit a lower implied volatility than shorter-term calls.

Scott (1987) develops a very similar model. He considers a mean-reverting process for the volatility. Noting that a riskless hedge portfolio must involve two options, he is unable to find a unique option pricing function using only arbitrage arguments. However, he also relies on an equilibrium asset pricing model and uses the Monte Carlo method. The volatility risk premium \( \lambda^* \) and the correlation between the stock returns and the volatility are constrained to be zero. Scott estimates the parameters of the variance process using the method of moments and Karman filter models. The overall conclusion is that the stochastic volatility model performs marginally better at explaining actual option prices.

Wiggins (1987) derives a partial differential equation by using only equilibrium arguments claiming that perfect arbitrage is not possible since the volatility is not a traded asset in reality. However, by assuming a logarithmic utility for agents, the author, in fact, sets the volatility risk premium equal to zero for options on the market portfolio and constant otherwise. To solve for the differential equation a finite-difference approximation is used. Wiggins’ overall conclusion is that out-of-the-money calls decrease in value compared to in-the-money values, if the correlation between stock returns and volatility is negative. The
effect is reversed for a positive correlation coefficient. Further, the less persistence there is in volatility shocks (that is, the higher the mean-reverting parameter), the less Black-Scholes prices differ from stochastic volatility model prices. Empirically, Black-Scholes prices show almost no biases for options on individual stocks, but for index options the Black-Scholes formula overvalues out-of-the-money calls compared to in-the-money calls. Thereby, the correlation coefficient is significantly negative. The economic significance is however dubious.

Johnson and Shanno (1987) assume the existence of an asset which is perfectly correlated with volatility allowing the pricing via arbitrage. This way they find a differential equation which they solve via Monte Carlo simulation. Out-of-the-money (in-the-money) calls increase (decrease) in value with $\rho$.

Stein and Stein (1991) derive a closed form solution for the distribution of stock prices when the stock volatility follows an Ornstein-Uhlenbeck (AR1) process. Although they can incorporate a nonzero volatility risk premium, the model assumes a zero correlation. Their model shows a U-shape pattern for the implied volatility when the strike price is varied, while the implied volatility is lowest for options at-the-money. The authors also show that stochastic volatility can be responsible for fat tails in the underlying distribution.

Heston (1993) presents a very important way of obtaining closed form solutions for option pricing using characteristic functions. Assume that the underlying asset $S$ and the volatility $v$ follow the stochastic processes

$$dS(t) = \mu Sdt + \sqrt{v(t)}Sdz_1(t) \quad (6)$$

and

$$d\sqrt{v(t)} = -\beta \sqrt{v(t)}dt + \delta dz_2(t). \quad (7)$$

Using Ito’s Lemma, the process for the variance can thus be written

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma \sqrt{v(t)}dz_2(t), \quad (8)$$
where $z_1(t)$ and $z_2(t)$ are Wiener processes which have correlation $\rho$, $\kappa$ is the speed of adjustment, $\theta$ denotes the long-run mean, and $\sigma$ denotes the variation coefficient of the diffusion volatility. As usually, the interest rate $r$ is assumed constant for the present.\textsuperscript{13} As seen before, the partial differential equation for the value of any asset $f(S, v, t)$ involves the price of volatility risk $\lambda(S, v, t)$:

$$
\frac{1}{2} v S^2 \frac{\delta^2 f}{\delta S^2} + \rho \sigma v S \frac{\delta^2 f}{\delta S \delta v} + \frac{1}{2} \sigma^2 v^2 \frac{\delta^2 f}{\delta v^2} + r S \frac{\delta f}{\delta S} + \{\kappa[\theta - v(t)] - \lambda(S, v, t)\} \frac{\delta f}{\delta v} - r f + \frac{\delta f}{\delta t} = 0. \tag{9}
$$

The author assumes that the volatility risk premium is proportional to $v$, namely $\lambda(S, v, t) = \lambda v$.\textsuperscript{14} A European call option has to satisfy several boundary conditions which can be used to make the solution of the PDE unique. Heston guesses a solution similar to the Black-Scholes one:

$$
C(S, v, t) = SP_1 - KP(t, T)P_2, \tag{10}
$$

where $P_1$ and $P_2$ can be interpreted as risk-neutral probabilities and where $P(t, T)$ is the discount factor. By writing the PDE (9) in terms of the natural logarithm of the stock price and by defining

$$
u_1 = \frac{1}{2}, \quad \nu_2 = -\frac{1}{2}, \quad a = \kappa \theta, \quad b_1 = \kappa + \lambda - \rho \sigma, \quad b_2 = \kappa + \lambda,
$$

the probabilities can be written in terms of the characteristic functions $f_1(x, v, T; \phi)$ and $f_2(x, v, T; \phi)$. The solution is

$$
P_j(x, v, T; \ln[K]) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[ \frac{e^{-i\phi \ln[K]} f_j(x, v, T; \phi)}{i\phi} \right] d\phi, \tag{11}
$$

\textsuperscript{13}This assumption is relaxed later in the Heston paper.
\textsuperscript{14}Of course, the risk premium can have a completely different and more difficult form, but at least it is not simply assumed to be zero.
where

\[ f_j(x, v, t; \phi) = e^{C(T-t;\phi)+D(T-t;\phi)v + \phi x}, \]

\[ C(T-t;\phi) = r\phi_i(T-t) + \frac{\alpha}{\sigma^2} \left\{ (b_j - \rho \sigma \phi i + d)(T-t) - 2 \ln \left[ \frac{1 - ge^{d(T-t)}}{1 - g} \right] \right\}, \]

\[ D(T-t;\phi) = \frac{b_j - \rho \sigma \phi i + d}{\sigma^2} \frac{(1 - e^{d(T-t)})}{(1 - ge^{d(T-t)})}, \]

and

\[ g = \frac{b_j - \rho \sigma \phi i + d}{b_j - \rho \sigma \phi i - d}, \]

\[ d = \sqrt{(\rho \sigma \phi i - b_j)^2 - \sigma^2(2u_j \phi i - \phi^2)}. \]

Although looking cumbersome, this formula can easily be implemented on a computer and is solvable in a millisecond. The model can also be extended to price bond and currency options for example.

In the very interesting last part of his paper, Heston examines the effects of the stochastic volatility model compared to the Black-Scholes formula. Thereby, he uses the risk-neutralized volatility process

\[ dv(t) = \kappa^*[\theta^* - v(t)]dt + \sigma \sqrt{v(t)}dz_2(t), \]

with

\[ \kappa^* = \kappa + \lambda \quad \text{and} \quad \theta^* = \frac{\kappa \theta}{\kappa + \lambda}. \]

In the case that the mean-reversion speed coefficient \( \kappa^* \) is positive, the resulting distribution for the spot returns is normal over long periods.\(^{15} \) Since the risk-neutralized process is used, the long-run mean variance \( \theta^* \) is not necessarily the same as the variance of the true process.

\(^{15} \)Therefore, the Black-Scholes formula should give a good approximation for long-term options.
The reason for this is the volatility risk premium. The sign of this premium determines if the mean variance is larger or smaller than the real variance. However, it is possible to estimate $\theta^*$ through the observed market prices. The true process can give estimates for $\kappa$ and the real variance on the other hand. Lastly, the author proposes that $\lambda$ can be estimated by using average returns of delta-hedged option positions.

It is important to notice for the following empirical tests that the parameters of the volatility process change the underlying spot asset distribution. So, the correlation coefficient $\rho$ has a positive influence on the skewness. For example, a positive correlation means that when the value of an asset increases the variance tends to be high. However, when the stock price is low, the variance tends also to be small. This effect creates a fat right tail and a thin left tail in the underlying distribution. Thus, out-of-the-money calls increase in value (compared to the Black-Scholes with similar volatility) since the probability of ending in-the-money is enhanced. On the other hand, the prices of in-the-money calls are depressed. If the correlation coefficient is negative, the effect is completely reversed.

The so-called volatility of volatility $\sigma$ effects the kurtosis. Increasing $\sigma$ means creating fat tails on both sides. Intuitively, increasing the variance of the variance increases also the possibility of extreme outcomes. Thus, both deep-out and deep-in-the-money options have a higher value while near-the-money options are worth less. Overall, increasing $\sigma$ means increasing far-away-from-the-money options relative to near-the-money options while positive correlation between the volatility and the spot asset produces a positive skewness. A positive skewness, in turn, increases the prices of out-of-the-money calls compared to in-the-money calls.

Thus, stochastic volatility models can produce many different pricing effects for options compared to the Black-Scholes model. What we have not considered yet are jumps in stock returns. It will be shown in the next section that incorporating jumps can produce similar results.

---

16 That is the third moment around the mean.
17 That is the fourth moment around the mean.
18 Or equivalently by the put-call parity, a positive skewness increases the prices of in-the-money puts compared to out-of-the-money puts.
### 2.4 Jump Models

This section considers the possibility that the stochastic process of the underlying stock return is discontinuous. In contrast, the Black-Scholes model assumes continuous sample paths and leaves no possibility for so-called jumps. In reality, the presence of jumps in stock returns is accepted fact. Important announcements often lead to sudden large changes in stock prices. Therefore, the incorporation of jumps into an option pricing model can be a worthwhile feature. The jump process allows “a positive probability of a stock price change of extraordinary magnitude, no matter how small the time interval between successive observations” (Merton (1973), page 127). Thus, the complete dynamics for the stock returns can be written as a standard geometric Brownian motion and a jump process.\(^{19}\)

Merton (1976) considers the following process for the stock returns:

\[
\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dq. \tag{12}
\]

In (12), \(\alpha\) is the instantaneous expected return on the stock, \(\sigma\) is the instantaneous volatility conditional on no jumps, \(dZ\) is a Wiener process. Furthermore, \(q(t)\) denotes an independent Poisson process, \(\lambda\) denotes the mean number of jumps per unit of time, and \(k \equiv E(Y - 1)\) where \(E\) is the usual expectation operator for the random variable \(Y\) and \((Y - 1)\) gives the percentage change in the stock price in the case a Poisson event occurs. Also, \(dq\) and \(dZ\) are assumed to be independent. Of course, the occurrence of a jump is the Poisson event. Jumps are independently and identically distributed (i.i.d.). Considering a time interval of length \(h\), the probability of observing a jump can be written as

\[
P(\text{nojump}) = 1 - \lambda h + O(h),
\]

\[
P(\text{onejump}) = \lambda h + O(h),
\]

\[
P(\text{morethanonejump}) = O(h).
\]

\(^{19}\)The most-widely used jump process is a Poisson process.
Here, $\lambda$ is defined as before and $O(h)$ is the asymptotic order symbol.\textsuperscript{20} If a jump occurs, the immediate percentage stock return is drawn from a distribution. That means that the stock price jumps to $Y_S$. It is natural to assume that $Y \geq 0$.\textsuperscript{21} Furthermore, the $Y$’s themselves are also i.i.d. Thus, the stock return process (12) falls into two parts: the normal movements $\sigma dZ$ and the occasional abnormal jumps $dq$. Although being continuous most of the time, the stochastic process for the stock returns contains jumps of differing sign and magnitude at certain points in time.

The problem for the derivation of the correct option price on a stock exhibiting jumps is that the jump risk can not simply be hedged. Following the Black-Scholes riskless hedge, one can see that the jump risk is still present. However, if an investor is long the stock and short the option, he will earn more than the expected return $\alpha_P$ on the hedge portfolio most of the time. Yet, in the case of a jump, the investor will suffer a large loss. On average, these effects will cancel out and he will gain the expected return. However, the no arbitrage way can not be used to derive the option price.

To find a solution, Merton assumes that the Capital Asset Pricing model (CAPM) is correct. Then, the arrival of new information that causes the non-marginal jumps is most likely firm-specific and hence non-systematic risk. As already seen, the Black-Scholes hedge portfolio bears only jump risk and has according to the CAPM a zero beta. Yet, that means that this hedge should yield an expected return equal to the risk-free rate $r$. This reasoning leads to the differential-difference equation

$$\frac{1}{2} \sigma^2 S^2 F_{SS} + (r - \lambda k) SF_S - F_r - rF + \lambda E[F(SY, \tau) - F(S, \tau)] = 0,$$

\textsuperscript{(13)}

subject to usual boundary conditions

$$F(0, \tau) = 0 \quad \text{and} \quad F(S, 0) = \max[0, S - E].$$

In (13), subscripts denote partial derivatives and $F(S, \tau)$ is the option price as a function of

\textsuperscript{20}That is, $\lim_{h \to 0} O(h)/h = 0$.

\textsuperscript{21}The case $Y = 0$ can be imagined as a firm announcing bankruptcy.
the stock price and time. It can be seen that the presence of jumps does effect the option price and the solution is different from the Black-Scholes one in general. To get a closed-form solution, the distribution for $Y$ has to be specified. However, if $X_n$ denotes the product of the random variables $Y_1, \ldots, Y_n$, then the solution to (13) is

$$F(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \tau} (\lambda \tau)^n}{n!} E[F(SX_n e^{-\lambda \tau k}, \tau; K, \sigma^2, r)].$$

Here, $E$ is the expectation operator with respect to $X_n$ and $K$ is the strike price of the option. Merton gives two special cases where the above equation can be simplified.

Naturally, the implication of jumps changes the way options are priced. For deep-out-of-the-money options the probability of ending in-the-money is enhanced and hence, the price is higher than in the Black-Scholes world. Similarly, deep-in-the-money and short maturity options will sell for more than their Black-Scholes counterpart.

Cox and Ross (1976) examine a variety of different jump models without the Wiener diffusion part $\sigma dz$. However, they use a different technique to derive the option price. Assuming risk-neutral preferences the value of an option $P(S, t)$ on a stock $S$ can be written as

$$P(S, t) = e^{-r(T-t)} E[h(S_T)|S_t] = e^{-r(T-t)} \int h(S_T) dF(S_T, T|S_t, t),$$

where $h(S_T)$ is the boundary value $P(S, T)$ of the option and $F(S_T, T|S_t, t)$ is the distribution function of the stock at time $T$ given the stock price at time $t$. The authors use this technique to derive solutions for some special cases, e.g. pure birth process without and with drift. Furthermore, it is assumed that the jumps are non-random, that is they are not drawn from a distribution.

Out of statistical considerations, Ball and Torous (1983) present a Bernoulli jump process. To see their point, consider the probability mechanism specified by Merton. Let $N$ denote the number of jumps in a time interval of length $t$ and divide this interval into $n$ equal

\footnote{In the case $\lambda = 0$, one gets the Black-Scholes solution.}
subintervals of length $h$. This can also be written as $h = t/n$ where $n$ is an arbitrary integer. Furthermore, $X_i$ denotes the number of jumps in interval $i$. This means that the sum of all $X_i$'s, which are i.i.d., equals $N$. We have

$$P(X_i = 0) = 1 - \lambda h + O(h)$$

$$P(X_i = 1) = \lambda h + O(h) \quad \text{for} \quad i = 1, \ldots, n$$

$$P(X_i > 0) = O(h),$$

where $O(h)$ is defined as before. Since each $X_i$ is approximately Bernoulli distributed for large $n$, $N$ possesses almost surely a binomial distribution with $p = \lambda t/n$:

$$P(N = k) \approx \binom{n}{k} \left( \frac{\lambda t}{n} \right)^k \left( 1 - \frac{\lambda t}{n} \right)^{n-k}, \quad k = 0, 1, \ldots, n.$$  

It can easily be shown that the binomial distribution approximates the Poisson distribution for $n \to \infty$. If $t$ is now chosen to be small, one gets as an approximation for $N$:

$$P(X = 0) = 1 - \lambda t \quad \text{and}$$

$$P(X = 1) = \lambda t.$$  

The important difference now between this and the Merton model is that, for a given $t$, either no jump or exactly one jump occurs. Ball and Torous use this for statistical purposes and claim to have a more efficient model to find estimates for jump parameters on 47 stocks. They conclude that the found parameters are consistent and that 78 percent of the stocks show statistically significant jumps at the 1 percent level.

Naik and Lee (1990) acknowledge that there is no pricing by arbitrage possible if the market portfolio follows a diffusion-jump process. Using a general equilibrium framework they study the price of options and the hedging performance of replicating options. In their economy, substantial amounts may have to be put in the hedge if jumps occur.
Bates (1991) produces an option pricing model with an asymmetric jump-diffusion model. The new feature is the introduction of asymmetric jumps, i.e. the jumps can have a nonzero mean. In turn, a positive mean can generate a positive skewness in the underlying security return while the opposite is true for a negative mean. Considering S&P 500 index options the jump risk is systematic. Bates comes up with an option pricing formula by using the risk-neutral measure. Additionally, he assumes that optimally invested wealth follows a jump-diffusion process and that the representative investor has time-separable power utility. For American options a quadratic approximation is found.

As in the case of stochastic volatility, the incorporation of jumps changes the moments of the underlying probability distribution. Bates (1991) model allows an effect on the skewness, whereas the presence of jumps itself results in a larger kurtosis.

2.5 Stochastic Interest Rate Models

Why should the additional feature of stochastic interest rates be useful given the insensitivity of the Black-Scholes option price to interest rates? The answer is that prices of longer-term options are quite sensitive to interest rates. Further, foreign exchange options depend heavily on domestic and foreign interest rates. Thus, a more realistic model for interest rates can improve an option pricing model. Since the processes interest rates follow and the term structure of interest rates are whole theories themselves, it is impossible to give a comprehensive overview. Two of the most-widely used interest rate models in option pricing are probably the Cox, Ingersoll, and Ross (1985) and the Heath, Jarrow, and Morton (1992) models.

Shortly after the original Black-Scholes model, Merton (1973) incorporates stochastic interest rates. Besides the stock price dynamics

\[
\frac{dS}{S} = \alpha dt + \sigma dz,
\]
the author assumes the following process for the interest rates:

$$\frac{dP}{P} = \mu(\tau) dt + \delta(\tau) dq(t; \tau).$$  \hspace{1cm} (14)$$

In (14), \(P(\tau)\) is the price of a riskless bond which pays one dollar \(\tau\) years from now, \(\mu\) denotes the instantaneous expected return, \(\delta^2\) is the instantaneous variance and \(dq(t; \tau)\) is a standard Wiener process for maturity \(\tau\). It is important to note that \(\mu(\tau)\) may be stochastic through the influence of \(P\) while \(\delta^2\) is nonstochastic and independent of the level of bond prices. Furthermore, while the bond processes can be less than perfectly correlated for different maturities, Merton assumes that there is no serial correlation among the unanticipated returns of all assets, that is

$$dq(s; \tau) dq(t; T) = 0 \quad \text{for} \quad s \neq t \quad \text{and}$$

$$dq(s; \tau) dz(t) = 0 \quad \text{for} \quad s \neq t.$$

In this setting, the Black-Scholes assumptions of a constant interest rate is obtained as a special case by letting \(\delta = 0\) and \(\mu = r\). By standard arguments, the change in the option price \(H(S, P, \tau; K)\) with strike price \(K\) satisfies the stochastic differential equation

$$dH = H_1dS + H_2dP - H_3dt + \frac{1}{2} H_{11}\sigma^2 S^2 dt + H_{12}\rho \sigma \delta SP dt + \frac{1}{2} H_{22}\delta^2 P^2 dt,$$

where, as before, subscripts denote partial derivatives and \(\rho\) denotes the instantaneous correlation between the stock and the bond returns. One of the main ideas in the way of finding the final formula is that a self-financing strategy in the stock, the option, and the riskless bonds exists. After some rearrangements and changes in variables, the author finds the option pricing formula

$$y(x, T) = \frac{1}{2} [xf(h_1) - f(h_2)],$$  \hspace{1cm} (15)
where \( f \) is the tabulated error complement function, \( y(x, T) = h(x, \tau) = H(S, P, \tau; K)/KP \), and

\[
\begin{align*}
    h_1 &= -\frac{\log x + 1/2T}{\sqrt{2T}}, \\
    h_2 &= -\frac{\log x - 1/2T}{\sqrt{2T}}.
\end{align*}
\]

Merton shows that the only possible riskless bond must have the same maturity as the option to form the hedge.

One other option pricing model incorporating stochastic interest rates is given by Amin and Jarrow (1992) which is also presented and tested in Rindell (1995). They use a special version, namely a one factor, constant volatility, term structure model, of the Heath, Jarrow and Morton (1992) type. Assume that the two factor diffusion process for the stock price \( S(t) \) is given by

\[
\frac{dS}{S} = \alpha dt + \sigma_1 dw(t) + \sigma_2 dv(t),
\]

where \( \alpha \) is the instantaneous expected return on the stock, \( \sigma_1 \) and \( \sigma_2 \) are constants with the latter one positive. Further, \( w(t) \) and \( dv(t) \) denote independent Wiener processes. Let \( f(t, \tau) \) denote the forward interest rate at time \( t \) for borrowing or lending at a future time \( \tau \geq t \). The forward rates follow the process

\[
df(t, \tau) = \mu(t, \tau) dt + \delta dw(t),
\]

where \( \mu \) denotes the expected change in the interest rate and \( \delta \) is a positive constant. It is important to note that the Wiener process \( w(t) \) affects both the stock price and the forward rates process. Since the correlation \( \rho \) between stock returns and changes in the interest rate are given by

\[
\rho = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}},
\]
the sign of $\sigma_1$ determines also the sign of the correlation coefficient. The price of a discount bond $B(t, \tau)$ at time $t$ and maturing at time $\tau$ is thus given by

$$\frac{dB(t, \tau)}{B(t, \tau)} = \gamma(t, \tau)dt - \delta(\tau - t)dw(t),$$

where $\gamma$ denotes the instantaneous expected return of the bond. Then the price of a European call on the stock $C(0, K, \tau)$ at time zero maturing at time $\tau$ with strike price $K$ becomes

$$C(0, K, \tau) = S(0)N(h) - KP(0, \tau)N(h - \xi),$$

with

$$h = \frac{1}{\xi} \ln \left( \frac{S(0)}{KP(0, \tau)} \right) + \frac{1}{2} \xi^2, \quad \text{and}$$

$$\xi^2 = [ (\sigma_1^2 + \sigma_2^2) \tau + \sigma_1 \delta \tau^2 + \frac{1}{3} \delta^2 \tau^3 ].$$

As usually, $N(\cdot)$ denotes the cumulative standard normal distribution function. The above equation yields the Black-Scholes formula as a special case if $\sigma_1 = \delta = 0$.

After the separate introduction of stochastic volatility, jumps, and stochastic interest rates, it seems desirable to include all three features in one model.

2.6 Putting it all Together - The Bakshi, Cao and Chen Model

Scott (1997) presents a model containing jumps, stochastic volatility, and stochastic interest rates. By using a Fourier inversion method he obtains a closed-form solution allowing the volatility and the stock price plus the interest rates and the stock price to be negatively correlated.

However, I have chosen the Bakshi, Cao, and Chen (1997) model since the paper is also covered in the review of related empirics of this thesis. It is convenient to present their model here for directly relating their empirical findings later. The model has the advantage
of possessing a closed-form solution and containing almost all considered models as special cases. Assume that under the risk-neutral measure\textsuperscript{23} the nondividend-paying stock price $S(t)$, the diffusion component of return variance (conditional on no jump occurring) $V(t)$, and the instantaneous spot interest rate $R(t)$ follow the processes

\[
\frac{dS(t)}{S(t)} = [R(t) - \lambda \mu_J]dt + \sqrt{V(t)}d\omega_S(t) + J(t)dq(t),
\]

(16)

\[
dV(t) = [\theta_V - \kappa_V V(t)]dt + \sigma_V \sqrt{V(t)}d\omega_V(t),
\]

(17)

and

\[
dR(t) = [\theta_R - \kappa_R R(t)]dt + \sigma_R \sqrt{R(t)}d\omega_R(t),
\]

(18)

for any time $t$. In the above equations, $\lambda$ is the frequency of jumps in one year, $\mu_J$ is the unconditional mean of the jump size, and $J(t)$ is the percentage jump size (conditional on a jump occurring) that is lognormally, and i.i.d. over time. Since the standard deviation of $\ln[1 + J(t)]$ is $\sigma_J$, it follows that $\ln[1 + J(t)]$ is $N(\ln[1 + \mu_J] - 1/2\sigma_J^2, \sigma_J^2)$ distributed. Also, $d\omega_S(t)$ and $d\omega_V(t)$ are standard Brownian motions with correlation $\rho$ and $q(t)$ denotes a Poisson process with

\[
P[dq(t) = 1] = \lambda dt \quad \text{and} \quad P[dq(t) = 0] = 1 - \lambda dt.
\]

Here, $dq(t) = 1$ signals that a jump occurs. Furthermore, $q(t)$ and $J(t)$ are assumed to be uncorrelated with each other or with the Brownian motions $d\omega_S(t)$ and $d\omega_V(t)$. Finally, $\kappa_V$ ($\kappa_R$) denotes the speed of adjustment, $\theta_V/\kappa_V$ ($\theta_R/\kappa_R$) the long-run mean, and $\sigma_V$ ($\sigma_R$) the volatility of the diffusion volatility $V(t)$ (of the interest rate process). The standard Brownian motion $d\omega_R(t)$ is assumed to be uncorrelated with any other process in the economy.\textsuperscript{24} Under the assumed stochastic processes, the total return variance can be split into two parts,\textsuperscript{23}This is equivalent to the absence of arbitrage opportunities.\textsuperscript{24}The authors argue that this simplification is justified because, for any correlations between stock return, volatility and interest rates, the performance of the resulting formula is not enhanced.
namely the diffusion component of return variance and the instantaneous variance of the jump component $V_J(t)$:

$$\frac{1}{dt} Var_t \left( \frac{dS(t)}{S(t)} \right) = V(t) + V_J(t),$$

where

$$V_J(t) = \frac{1}{dt} Var_t [J(t)dq(t)] = \lambda [\mu_J^2 + (e^{\sigma_J^2} - 1)(1 + \mu_J)^2].$$

It is also important to notice that there are three types of risk, namely volatility risk, interest rate risk, and jump risk, present in this economy. Because the stochastic processes are assumed to be risk-neutral ones, risk premia do not appear in the formulas but are, in fact, internalized in the stochastic structure by $\kappa_V$ and $\kappa_R$.

Turning back to the original problem, the differential equation to solve for the time-$t$ price $C(t, \tau)$ of a European call option written on a stock with strike price $K$ and time to maturity $\tau$ becomes by using Ito’s Lemma

$$\frac{1}{2} V S^2 \frac{\delta^2 C}{\delta S^2} + [R - \lambda \mu_J] S \frac{\delta C}{\delta S} + \rho \sigma_V V S \frac{\delta^2 C}{\delta S \delta V} + \frac{1}{2} \sigma_V^2 V \frac{\delta^2 C}{\delta V^2}$$

$$+ \left[ \theta_V - \kappa_V^2 \right] \frac{\delta C}{\delta V} + \frac{1}{2} \sigma_R^2 R \frac{\delta^2 C}{\delta R^2} + \left[ \theta_R - \kappa_R^2 \right] \frac{\delta C}{\delta R} - \frac{\delta C}{\delta \tau} - RC$$

$$+ \lambda E[C(t, \tau; S(1 + J), R, V) - C(t, \tau; S, R, V)] = 0,$$

(19)

with the usual boundary condition

$$C(t + \tau, 0) = \max[S(t + \tau) - K, 0].$$

Although appearing quite tedious, a closed-form solution for the above differential equation can be obtained by using a technique similar to Heston (1993). The option price must satisfy

$$C(t, \tau) = S(t) \Pi_1(t, \tau; S, R, V) - KB(t, \tau) \Pi_2(t, \tau; S, R, V),$$

(20)
where

\[ B(t, \tau) = \exp[-\varphi(\tau) - \varrho(\tau)R(t)], \quad \text{with} \]

\[ \varphi(\tau) = \frac{\theta_R}{\sigma_R^2} \left\{ \left( \varsigma - \kappa_R \right) \tau + 2 \ln \left[ 1 - \frac{(1 - e^{-\varsigma \tau})(\varsigma - \kappa_R)}{2\varsigma} \right] \right\}, \]

\[ \varrho(\tau) = \frac{2(1 - e^{-\varsigma \tau})}{2\varsigma - (\varsigma - \kappa_R)(1 - e^{-\varsigma \tau})}, \quad \varsigma = \sqrt{\kappa_R^2 + 2\sigma_R^2} \]

gives the current price of a zero-coupon bond that pays $1 in \tau periods from time \( t \) and \( \Pi_1 \) and \( \Pi_2 \) denote risk-neutral probabilities. Equation (20) looks very similar to the Black-Scholes formula. The exact form of the probabilities is however more complex:

\[ \Pi_j[t, \tau; S(t), R(T), V(T)] \]

\[ = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-i\phi \ln(K)}f_j(t, \tau, S(t), R(t), V(t); \phi)}{i\phi} \right] d\phi, \]

for \( j = 1, 2 \). The formulas for the characteristic functions \( f_j \) are given by

\[ f_1(t, \tau) = \exp \left\{ -\frac{\theta_R}{\sigma_R^2} \left[ 2 \ln \left( 1 - \frac{(\xi_R - \kappa_R)(1 - e^{-\xi \tau})}{2\xi_R} \right) + \left[ \xi_R - \kappa_R \right] \tau \right] \right\} + \frac{2i\phi(1 - e^{-\xi \tau})}{2\xi_R - (\xi_R - \kappa_R)(1 - e^{-\xi \tau})} R(t) \]

\[ + \lambda(1 + \mu_j)\tau \left[ (1 + \mu_j)^2 e^{(i\phi/2)(1+i\phi)\sigma^2} - 1 \right] - \lambda i\phi \mu_j \tau \]

\[ + \frac{i\phi(\phi + 1)(1 - e^{-\xi \tau})}{2\xi_V - (\xi_V - \kappa_V)(1 + i\phi)\rho \sigma_V(1 - e^{-\xi \tau})} V(t) \]
and

\[
f_2(t, \tau) = \exp\left\{ -\frac{\theta_R}{\sigma_R^2} \left[ 2 \ln \left( 1 - \frac{(\xi_R^* - \kappa_R)(1 - e^{-\xi_R^* \tau})}{2 \xi_R^*} \right) + [\xi_R^* - \kappa_R] \tau \right] \\
- \frac{\theta_V}{\sigma_V^2} \left[ 2 \ln \left( 1 - \frac{[\xi_V^* - \kappa_V + i\phi \rho \sigma_V](1 - e^{-\xi_V^* \tau})}{2 \xi_V^*} \right) \right]
\]

\[
+ [\xi_V^* - \kappa_V + i\phi \rho \sigma_V] \tau + i\phi \ln[S(t)] - \ln[B(t, \tau)]
\]

\[
+ \frac{2(i\phi - 1)(1 - e^{-\xi_R^* \tau})}{2 \xi_R^* - (\xi_R^* - \kappa_R)(1 - e^{-\xi_R^* \tau})} R(t)
\]

\[
+ \lambda \tau [(1 + \mu_J)^i e^{(i\phi/2)(i\phi - 1) \sigma^2 V} - 1] - \lambda i\phi \mu_J \tau
\]

\[
+ \frac{i\phi (i\phi - 1)(1 - e^{-\xi_V^* \tau})}{2 \xi_V^* - (\xi_V^* - \kappa_V + i\phi \rho \sigma_V)(1 - e^{-\xi_V^* \tau})} V(t) \}
\]

where

\[
\xi_R = \sqrt{\kappa_R^2 - 2\sigma_R^2 i\phi}, \quad \xi_V = \sqrt{[\kappa_V - (1 + i\phi) \rho \sigma_V]^2 - i\phi (i\phi - 1) \sigma^2 V},
\]

\[
\xi_R^* = \sqrt{\kappa_R^2 - 2\sigma_R^2 (i\phi - 1)}, \quad \text{and} \quad \xi_V^* = \sqrt{[\kappa - i\phi \rho \sigma_V]^2 - i\phi (i\phi - 1) \sigma^2 V}.
\]

By setting the appropriate parameters equal to zero, we can generate several special cases. For example, the Black-Scholes model is obtained by setting \( \lambda = 0 \) and \( \theta_V = \theta_R = \kappa_V = \kappa_R = \sigma_V = \sigma_R = 0 \). Also, this rich stochastic structure can generate very different underlying distributions. The correlation \( \rho \) and the average jump size \( \mu_J \) determine the skewness, while the kurtosis is influenced by \( \sigma_V, \lambda, \) and \( J(t) \). The jump coefficients can produce almost any desired skewness and kurtosis levels in the short run, the diffusion volatility process \( V(t) \) however can only generate short-run skewness and kurtosis to a certain extent since the process is by definition continuous.

Lastly, the three sources of risk can be hedged by the following ratios:

\[
\Delta_S(t, \tau; K) = \Pi_1,
\]

\[
\Delta_V(t, \tau; K) = S(t) \frac{\delta \Pi_1}{\delta V} - KB(t, \tau) \frac{\delta \Pi_2}{\delta V},
\]

27
and

\[ \Delta_R(t, \tau; K) = S(t) \frac{\delta \Pi_1}{\delta R} - KB(t, \tau) \left[ \frac{\delta \Pi_2}{\delta R} - \phi(\tau) \Pi_2 \right], \]

where, for \( g = V, R \) and \( j = 1, 2 \),

\[ \frac{\delta \Pi_j}{\delta g} = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ (i\phi)^{-1} e^{-i\phi \ln(K)} \frac{\delta f_j}{\delta g} \right] d\phi. \]

After these theoretical foundations for the pricing of options, we can turn to the empirical studies which are trying to determine if the models can explain market option prices, which models perform better than others, and how the biases observed in actual markets may be explained. Yet, we will see that there does not exist just one explanation which can solve the puzzling problem of the so-called volatility smile.
3 RELATED EMPIRICS

We now turn to the other side of option pricing. Since one already has a good understanding of the theoretical option pricing models, it is worthwhile to have a look on the overall empirical performance. We will see that the Black-Scholes model does a good job until 1987. Then, alternative models are needed to explain observed option prices. The persistence of volatility poses a problem for even the most advanced models. Therefore, other methods are offered, e.g. determine the risk premia, the underlying probability distribution function, or buying and selling pressure.

3.1 Biases of the Black-Scholes Model Before the 1987 Crash

The early empirical work does not refer to volatility smiles but investigated certain differences between the Black-Scholes model prices and the observed market prices. One of the main problems with this early work is that the option market was in its beginnings at that time, so that sufficient and comprehensive data was hard to find. Also, it has been common to make possibly dangerous simplifications.\footnote{See for example Harvey and Whaley (1991).}

Black and Scholes (1972) themselves provide the earliest investigation of the empirical validity of their model. They use data obtained from a diary of an option broker for the years 1966 to 1969. In their model, hedging existing option positions should not yield any excess returns. Besides the finding of a non-stationary variance, they claim that the model works well with the right volatility input. Using a past data variance estimate, the authors find that the Black-Scholes model overprices options on high variance stocks and underprices options on low variance stocks. However, by using the variance which actually applied over
the option period, the excess returns on the hedges are greatly reduced. Additionally, since the market for options is very small, high transaction costs are observed, effectively ruling out all possible excess profits.

Black (1975) notes that there are systematic differences between the option model prices and the observed market prices. While far out-of-the-money options tend to be overpriced, the reverse is true for far in-the-money ones which bear almost no time value. Also, shorter-maturity options tend to be overpriced.

Call options on six different stocks for the year 1976 are investigated by Macbeth and Merville (1979). Circumventing the problem of estimating the variance using past stock data, they compute the implied volatility for their studies. That is the volatility that equates the Black-Scholes model price with the observed actual market price. The authors find that call options on these stocks have different implied volatilities for different strike prices which is not compatible with the Black-Scholes model. In particular, the implied variance rate declines as the exercise price increases. Also, there appears to be a relationship between the implied variances and time to maturity. In-the-money options with short maturity (less than 90 days) tend to have a higher $\sigma$ than the same corresponding options with a longer time to maturity. Exactly the opposite is observed for out-of-the-money options. Using the at-the-money implied volatility of longer maturity options as the true one, a certain pattern becomes obvious. The Black-Scholes model consistently undervalues in-the-money options and overvalues out-of-the-money options. Yet, this is exactly the opposite of what Black (1975) states. The total dollar differences tend to be generally smaller for shorter-maturity than for longer-maturity options. Furthermore, besides short-maturity out-of-the-money options, the underpricing (overpricing) of in-the-money (out-of-the-money) options is increasing with the extent to which the options are in-the-money (out-of-the-money) and it is also increasing for a longer time to maturity. Lastly, the authors claim that it is justified to use the Black-Scholes formula for their studies although the options they consider are American ones. As is already known, American options leave the possibility of early exercise for the dividend-paying stocks. However, Macbeth and Merville investigate this and come
to the conclusion that the influence of early exercise is not present for two of the six stocks and that it is very likely to be negligible for the others.

The earliest really comprehensive study is probably conducted by Rubinstein (1985). This paper sets some standards for the important task of deleting all possible errors or biases inherent in the data. Because this proves so important for empirical work I will give an overview on how to find and use the best inputs following the standards introduced by Rubinstein.

First, the data is taken from the Market Data Report of the Chicago Boards of Exchange (CBOE). It contains almost all trades, quotes, and volume starting from August 23, 1976 to August 31, 1978. Thus, the study includes a very large sample. During a period of stable stock prices, the records for each series of options are aggregated so that the actual data base becomes smaller. Then, the following requirements must be satisfied simultaneously for options prices to be accepted for the empirical investigation to reduce statistical noise:

- Previous and following different stock prices must either both go up or down by one tick ($1/8). This shall ensure a relative stock price stability.
- The occurrence of the previous different stock price is no longer than 500 seconds ago. Obviously, this means that we are considering times of sufficient liquidity.
- The constant stock price interval must be at least 300 seconds long. Since market participants in the option market need time to react to the new price this criterion is introduced.
- Records do not lie either in the first or last 1,000 seconds of a trading day (9:00 am to 3:00 pm). Option prices at the end and at the beginning of a day are often influenced by abnormal price movements. Therefore, this requirement has a purpose.
- During a constant stock price interval the difference between high and low option prices must be less or equal than $1/4. This argument ensures that there is less uncertainty about the true market price.
- The total contract volume and the total number of raw data records must be at least 3 for obtaining a sufficient trading volume in the options.

- During the first five minutes of the constant stock price interval at least one bid-ask record must be obtained. This requirement means that the option price is either related to the old previous stock price or the new bid-ask spread.

Rubinstein claims that the computation of differences between model and observed prices requires knowledge of the option market price, the simultaneous underlying stock price, time to expiration, striking price, the interest rate, future dividends and ex-dividend dates before expiration, and the volatility of the underlying security. The last point seems to be the most challenging, but the author uses a trick to circumvent this difficulty. By the comparison of pairs of options which differ only by strike price or time to maturity, the two implied volatilities are compared. However, for this procedure to work, one needs to know the other inputs. It is obvious that the time to expiration and the strike price of the option can be measured perfectly. The interest rate is obtained by taking the average of a lending rate which is the yield to maturity on a T-bill maturing as close as possible to the expiration date and a borrowing rate which is estimated to be the lending rate plus 1/2 percent plus the difference between the current broker call loan money rate and the yield on a two-week bill. Two measures are applied for the estimation of future cash dividends prior to expiration. The first estimate is simply the dividend paid one year ago and the second assumes that dividends are forecasted perfectly. Only if the ordering of the implied volatilities for two options is the same under both estimates, the options are accepted for the nonparametric tests. Future ex-dividend dates are treated as forecasted perfectly. As is well known, ex-dividend dates near to the expiration date make an early exercise more likely. Two firms are hence eliminated from the sample. Also, all puts and calls for which early exercise is likely are taken out.\footnote{Calls are eliminated if the Black-Scholes value for the call with an (imaginary) expiration date just before the ex-dividend date is greater than the normal value.} The variation in the bid-ask spread of the underlying stock price is accounted for by adjusting the stock price upward (downward) by three cents if the previous and following stock price is higher (lower). Rubinstein argues that the actual option price is
the most difficult variable to measure since during a constant stock price interval oftentimes several different option prices are found. To partly alleviate this problem a weighted average using the relative trading volume is taken. The author acknowledges that this still leaves a significant amount of noise, but it seems improbable that this procedure introduces statistical biases.

After screening the data, each option price is put in exactly one of 25 categories. There exist five categories differing in time to maturity and five categories differing in level of moneyness. Then, option pairs are matched. Two records are time to maturity matched if the records are measured on the same day during the same constant stock price interval for the same underlying security and the same strike price. Each option record can only appear in one matched pair. For the strike price matching, one only has to change time to maturity and strike price in the above definition. In total, Rubinstein obtains 13,114 time to expiration and 9,704 strike price matched pairs. Now, one can compare the implied volatilities of two matched pairs, e.g. one can observe the influence of moneyness by taking one out-of-the-money and one in-the-money pair for a fixed maturity. If the assumption is that the Black-Scholes formula gives a valid description of reality, then we expect half of the implied volatilities for out-of-the-money calls to be greater and half to be smaller than the implied volatilities for the in-the-money calls. Using the standard normal distribution one can now either accept or reject the null hypothesis if the actual observation is different from the half-half relationship. Besides this statistical test a test of economic significance is also performed. This is done by searching the implied volatility that makes the percentage deviation between the market and the model prices of the option pair equal. As a formula, find $\sigma$ so that

$$\frac{C_1(\sigma) - P_1}{P_1} = \frac{P_2 - C_2(\sigma)}{P_2} \equiv 0.01\alpha,$$

---

27 These are: Very near maturity (21 - 70 days to maturity), near maturity (71 - 120 days), middle maturity (121 - 170 days), far maturity (171 - 220 days), and very far maturity (more than 221 days).

28 These are: Deep out-of-the-money (The current stock price divided by the strike price lies in between 0.75 and 0.85), out-of-the-money (0.85 - 0.95), at-the-money (0.95 - 1.05), in-the-money (1.05 - 1.15), and deep in-the-money (ratio is greater than 1.15).
where $P_i$ denotes the observed market price and $C_i$ the Black-Scholes value for call $i = 1$ or $i = 2$. Also, $\alpha$ can be seen as a lower bound of the percentage difference since the pricing deviation is obtained over all implied volatilities. Again, it is unnecessary to compute variances. Splitting the data in two periods, namely period one from August 23, 1976 to October 21, 1977 and period two from October 24, 1977 to August 31, 1978, the author can confirm both Macbeth and Merville’s and Black’s findings. The main results are:

- For out-of-the-money calls: the shorter the time to expiration, the higher its implied volatility. This finding is statistically significant in both periods and for all stocks.

- In period one the lower the striking price, the higher the implied volatility (confirms Macbeth and Merville), in period two the higher the striking price, the higher the implied volatility for calls with the same time to maturity (confirms Black). However, this finding is not statistically significant for very near maturity (deep) out-of-the-money calls.

- Also, there is a weaker relationship for at-the-money options. The longer the maturity, the higher the implied volatility in the first period and the lower the implied volatility in the second period.

Rubinstein finally concludes that other option pricing models are also incapable of explaining the reversal of the strike price bias. Although these biases are highly statistically significant, no clear results can be made about the economic significance. The concerns that dividends may explain the biases appear highly unlikely since the same conclusion can be drawn about a subsample with options on non-dividend or low-dividend paying stocks.

Sheikh (1991) uses almost exactly the same method with minor differences to investigate S&P 100 index options. He divides the sample into 3 subperiods (July 5, 1983 to February 29, 1984; July 1, 1985 to September 30, 1985 and October 1, 1985 to December 31, 1985) since the market has been stable, increasing, and decreasing in these intervals. Also, the author finds different relations for different time periods. In period one, a positive relation between implied volatilities and in-the-money depth exists. In the second period, a U-
shaped pattern is observed with a minimum of the implied volatility for at-the-money calls. However, in the last subperiod Sheikh finds that, except for deep out-of-the-money calls, implied volatilities first decline with depth in-the-money, achieving a minimum for in-the-money options, but finally reaching a maximum for deep in-the-money calls. The term structure is also completely different in the subperiods. While for periods one and three, shorter time to maturity tends to come together with a higher implied volatility, the second period yields U-shaped term structures for out-of-the-money calls and increasing implied volatilities for longer time to maturity when at-the-money options are considered. Like Rubinstein, Sheikh concludes that the mispricings are highly statistically significant, but that they are also economically significant.

As a conclusion on the early work, one can say that striking price biases from the Black-Scholes values are statistically significant and that these biases have the same direction for most underlying stocks at a particular point in time, but that the biases themselves are changing their direction over time. Yet, overall the Black-Scholes formula gives quite a good fit before 1987. In the next section I will show that this picture changes dramatically after the big crash.

3.2 The Volatility Smile After the 1987 Crash

While the mispricings of options compared to the Black-Scholes formula are mild before the crash, everything changes after the "Black Monday" (October 19, 1987). On that day, the S&P 500 index dropped by approximately 20 percent, the futures on the index lost even more.

Again it is Rubinstein (1994) who finds that the stable relationship between Black-Scholes model prices and market prices deteriorates significantly. The author uses the same procedure for the measurement of economic significance as in his earlier paper which he now calls minimax dollar error. Also, a measurement of the percentage error, namely the minimax percentage error, is introduced. What Rubinstein finds is quite remarkable. Percentage
errors for 3 % to 9 %\textsuperscript{29} out-of-the-money calls on the S&P 500 index increased from -0.3 percent in 1986 to -14.2 percent in 1992. For -9 to -3 percent in-the-money calls the error increased from -0.3 to -4.9, for -3 to 3 percent at-the-money calls from -0.5 to -8.8 percent. The percentage error for comparing 9 percent out-of-the-money with 9 percent in-the-money calls also exploded from -0.7 to -15.3 percent. So, besides the obvious statistically significant mispricings, now there are also economically significant biases present. Furthermore, remember that these numbers only indicate the minimum price error. After the crash, in-the-money calls (and hence out-of-the-money puts) become much more expensive. Finally, low strike price options possess significantly higher implied volatilities than options with higher strike prices. This also implies a complete shift of the implied volatility curve. After 1987, there is a negative relationship between the implied volatilities and the strike prices, so the volatility is a decreasing function of moneyness. Rubinstein suspects a “crash-o-phobia”, that means investors are willing to pay more for out-of-the-money puts in the fear of further market crashes.

The literature does not suggest however that the shape of the implied volatility function is the same at every point in time or for options with differing maturities. Bates (2000) for example finds a different shape of the implied volatility functions for options on the S&P 500 index with December or November options. Also, he acknowledges that the shape of the functions is vastly different after the 1987 crash than before that date. Bollen and Whaley (2004) find a U-shaped implied volatility curve for American options on stocks but a negatively sloped curve for the S&P 500 index. Bakshi, Cao, and Chen (1997) note that the implied volatility curve exhibits different patterns for options with different maturities. For short-term options, the curve is U-shaped (a smile) with the largest implied volatilities obtained for deep in-the-money calls or deep out-of-the-money puts. As the time to maturity of the options increases, the curve becomes more and more an upward-sloping curve, indicating that a higher level of moneyness creates higher implied volatilities for calls (the reverse is true for puts). Overall, through the impact of the put-call-parity, implied volatility functions of a given time to maturity category possess a similar shape for calls and puts.

\textsuperscript{29}This means that the strike price is 3 to 9 percent less than the concurrent index.
Clearly, the observed volatility smile or smirk is not compatible with the Black-Scholes model and is thus one of the most difficult problems to solve. Can the shape of the implied volatility functions be explained by introducing other features in option pricing models, such as stochastic volatility or jumps? In the next section we will see if these other models can provide a better fit to the observed market prices and if they are able to create the implied volatility functions.

3.3 The Search for the Best Model

A whole body of work has been attributed to the empirical testing of the Black-Scholes or alternative option pricing models. Almost all of the existing work concludes that alternative models describe option prices better than the Black-Scholes model. Of course this is expected because the newer models add complexity by relaxing one or more Black-Scholes assumptions. Since I already described their model, a detailed analysis will be given only for the Bakshi, Cao, and Chen (1997) article. Other studies are briefly presented.

Rindell (1995) tests the Amin and Jarrow (1992) stochastic interest rate model against the Black-Scholes model for European index options in the Swedish market using the generalized method of moments. He concludes that the stochastic interest rate model clearly outperforms the Black-Scholes one. Also, time to maturity biases almost completely disappear for the more complex model. However, the model suffers from unreasonable parameter values.

Corrado and Su (1998) study the different ability of the Black-Scholes and the Hull-White stochastic volatility model to price S&P 500 index options. Volatility parameters implied by the option prices are obtained by minimizing the squared sum of errors. The authors find that the Hull-White stochastic volatility model significantly reduces the proportions of model option prices lying outside the market bid-ask spreads. While this proportion is always bigger than 0.90 for the Black-Scholes model, it is halved for the stochastic volatility model. Also, moneyness related biases nearly vanish.
The most comprehensive analysis of option pricing models is probably given by Bakshi, Cao, and Chen (1997). They investigate three different aspects by using intra-daily S&P 500 options data obtained from the Berkeley Option Database from June 1, 1988 to May 31, 1991: internal consistency of the implied parameters, out-of-sample pricing and hedging. In the empirical tests only the last reported options bid-ask quote before 3:00 pm is taken. Further, the estimate of the interest rate is obtained by the average of the bid-ask Treasury Bill discounts for the two bonds surrounding the maturity of the option. The S&P 500 index is adjusted for dividends by subtracting the present value of the daily dividends $D(t)$ from the current spot price:

$$\bar{D}(t, \tau) \equiv \sum_{s=1}^{\tau-t} e^{-R(t,s)s} D(t+s),$$

where $R(t, s)$ denotes the $s$-period yield-to-maturity. The authors also apply a screening procedure which leads to the loss of 624 observations (1.3 percent of the whole sample). Options are divided into six moneyness categories, where $S/K$ ranges from less than 0.94 to more than 1.06 in 3 percent steps, and three time to expiration categories, namely short-term (less than 60 days), mid-term (between 60 and 180 days) and finally long-term (more than 180 days) options. The total sample consists of 38,749 call option observations.

First consider the structural parameter estimation and in-sample performance for the Black-Scholes (BS), the stochastic volatility (SV), the stochastic volatility with stochastic interest rates (SVSI), and the stochastic volatility with jumps (SVJ) model. Jump and volatility parameters are obtained by backing them out from the observed actual market data. For the estimation of the implied parameters the summed squared errors of the difference between the model prices $C_n(t, \tau_n; K_n)$ and market prices $\hat{C}_n(t, \tau_n; K_n)$ are minimized. Here is how it works: Take $N$ option prices on the same stock from the same day and let $\tau_n$
denote the time to maturity and $K_n$ the strike price of the $n$-th option, where $n = 1, \ldots, N$. With the information from the market,

$$
\epsilon_n(V(t), \Phi) \equiv \hat{C}_n(t, \tau_n; K_n) - C_n(t, \tau_n; K_n)
$$

is a function of the parameter space $\Phi \equiv \{\kappa_R, \theta_R, \sigma_R, \kappa_V, \theta_V, \sigma_V, \rho, \lambda, \mu_J, \sigma_J\}$ and $V(t)$. The next step consists of minimizing this for every day in the sample:

$$
SSE(t) \equiv \min_{V(t), \Phi} \sum_{n=1}^{N} |\epsilon_n(V(t), \Phi)|^2.
$$

The main problem with this estimation of the implied spot variance and the structural parameters is that more weight is put on expensive options on the cost of cheaper options. However, applying this procedure it is seen that the implied volatilities for the BS, SV, and SVSI model do not differ much while the SVJ model possesses the largest implied volatility (e.g. 1.15 percent higher than for the BS model for all options). This ranking is true for all categories. Parameters are obtained for all options, short-term options and at-the-money options. Remember that the underlying probability distribution is controlled by the different volatility and jump parameters. Therefore, it appears interesting that, for all options, the SVJ model has the highest speed of adjustment parameter $\kappa_V$,\footnote{Namely 2.03 compared to 1.15 for the SV and 0.98 SVSI model.} the lowest correlation coefficient $\rho$,\footnote{Namely -0.57 compared to -0.64 for the SV and -0.76 for the SVSI model.} and the lowest variation coefficient of the volatility process $\sigma_V$. Jumps have an average size $\mu_J$ of -5 percent with 7 percent uncertainty $\sigma_J$ and occur with a mean frequency $\lambda$ of 0.59 times per year. So, the SVJ model probably has the most reasonable parameter estimates, while the SVSI model requires the most stringent estimates. This impression is also confirmed by looking at the squared pricing errors (SSE). Clearly, the BS model performs worst with an SSE of 69.60, vastly outperformed by the SVSI with 10.68 and the SV model with 10.63. The introduction of jumps gains some additional fit leading to an SSE for all options of only 6.46 for the SVJ model. Thus, while the incorporation of a stochastic volatility drives the improvement of the fit and the jump feature further enhances...
this, the possibility of stochastic interest rates has no effect. That short-term options are the most difficult to price can be seen from the fact that almost all parameter values increase if only these types of options are considered for the estimation. For all option pricing models, $\kappa_V$ and $\sigma_V$ increase and for the SV and the SVSI $\rho$ also increases. To bring the jump parameters in line with the observed market prices, unreasonable coefficients are necessary: The average jump size is -9 (!) percent with uncertainty factor 12 percent and a doubled instantaneous volatility of the jump component of 12.3 percent. For at-the-money options there exist two major differences to the all option analysis: For the SV and the SVSI model, the correlation coefficient is higher, for the SVJ model, jumps occur more frequently (0.68 times per year) with a smaller magnitude (-4 percent). Fortunately, the separated estimation for short-term and at-the-money options leads to significantly lower values for the SSE, with the SVJ model performing best followed by the SVSI, the SV, and the BS model. On the other hand, for all models considered, the parameter values should be the same regardless of what type of options are used. Since the option parameters differ however, every option pricing model is misspecified.

The degree of misspecification is revealed in two ways. First, the authors compare implied volatility graphs. This is done for the period from July 1990 to December 1990. Therefore, the implied parameter values from the previous day are used to back out the implied volatility from today. Then, average values are computed for each moneyness-maturity category. Although the implied volatility graphs form a U-shaped pattern for all models, the SVJ model indicates the least misspecification followed by the SVSI, the SV, and lastly by the BS model. For mid-term and long-term options, the smile almost vanishes for the three stochastic volatility models while it is still present for the BS model. On average, the SVJ model yields higher implied volatilities by 1.5 percent compared to the SVSI and the SV model, with the latter having nearly identical volatilities. Further, implied volatilities are equal for the BS, the SV, and the SVSI model for at-the-money options in all maturity categories. Secondly, implied parameters are compared with time-series data. Better fit of the implied parameters to the actually observed time-series path leads also to a more consistent specification. The
BS model has the assumption of a lognormal distribution for the underlying stock (or in this case index) returns, yielding a skewness of zero and a kurtosis of three. Since the observed skewness is -0.43 and the kurtosis is 6.58 over the whole sample period the authors reject the BS model. Thus, the actual observed parameters do not share the same risk-neutral distribution as the implied counterparts for the other models and this creates a problem. However, under certain assumptions they are nearly equal so that the implied parameters can be compared to the time-series data coefficients. While some parameters have nearly the same values for both data sets, the alternative models have problems in explaining the time series data as well because the real correlation coefficients are much lower (from -0.23 to -0.28). Additionally, a maximum-likelihood method leads to the conclusion that the volatility of volatility is 4 times higher under implied options data than under time-series data. For the SVSI model, interest rates parameters are also misspecified. Bakshi, Cao, and Chen find inconsistent differences for $\rho$, $\sigma_V$, and all interest rates parameters (if applicable) and conclude that all models are not completely able to explain the observed time-series data.

The second way of judging the usefulness of the models under consideration consists of measuring out-of-sample performance. Instead of improving the performance, additional parameters can lead to an overfitting and a worse performance for this procedure. Average absolute and percentage pricing errors are computed by taking the difference between today’s observed and model prices where the previous day’s implied parameter and volatility values are used. The mean is obtained by taking the average for each day and every call, separately for all models. In general, the out-of-sample performance based on all options (as well as maturity and moneyness based\textsuperscript{35}) sees the SVJ model first, followed in increasing order by the SVSI, the SV, and the BS model for absolute and percentage pricing errors. Interesting exceptions where the SVSI model performs best are long-term deep in-the-money calls, but also deep out-of-the-money calls for every maturity. Again, the incorporation of stochastic volatility leads to a first order improvement of the pricing errors\textsuperscript{36}, while jumps and stochastic interest rates may sometimes further enhance the forecast, though by only a

\textsuperscript{35}This means that only the implied parameters implied by the corresponding options in the same maturity or moneyness category are used.

\textsuperscript{36}Especially for out-of-the-money and in-the-money calls.
slight amount. Absolute pricing errors typically increase for options with longer maturity while the percentage errors decrease, especially for out-of-the-money calls. Further, for percentage pricing errors the BS model clearly shows maturity and moneyness biases. However, the alternative models only have biases for short-term options. One possibility for the bad performance considering short-term options may be the error function (21) which assigns more weight to more expensive derivatives, like long-term options. All models share the weakness that they generally overprice out-of-the-money and underprice in-the-money calls for the short- and mid-term category, although by a largely differing amount. By considering only the implied values by options from the same maturity category, percentage errors are reduced significantly, notably for out-of-the-money options, for the SVJ, but also to a smaller extent for the SVSI and the SV model. The moneyness based procedure results also in lower percentage errors, with the BS model benefiting most. The overall ranking of the models remains however unchanged.

To relate the percentage pricing errors to cross-sectional pricing biases or changing market conditions the following regression is implemented:

\[
\epsilon_n(t) = \beta_0 + \beta_1 \frac{S(t)}{K_n} + \beta_2 \tau_n + \beta_3 SPR_n(t) + \beta_4 SLO(t) + \beta_5 LV(t - 1) + \eta_n(t),
\]

where \( SPR_n(t) \) is the percentage bid-ask spread for the \( n \)-th option at time \( t \), \( SLO(t) \) denotes the difference of the yield for 30-day and one year Treasury bills, and \( LV(t - 1) \) is the standard deviation of the previous day’s index returns over 5-minute intervals. For every model, all variables possess some explanatory power, although of differing magnitude. As expected, the BS model is much more sensitive to almost all parameters. In general, increases in the volatility yesterday or the yield differential lead to increased pricing errors. Furthermore, increases in the bid-ask spread result in lower errors. Going to deeper in-the-money calls, errors are increased for the BS model but decreased for the others. A maturity-based treatment deletes the significance of all explanatory variables for the stochastic volatility models with the \( R^2 \) falling to zero whereas leaving the coefficients (and the \( R^2 \)) for the BS model almost unchanged. On the opposite, using the moneyness-based criterion, the sensi-
tivity of the BS model to the regression variables is greatly improved, while it does not help the other models. The reason for this finding is simple: Stochastic volatility models show large percentage mispricings for short-term options but the errors are unrelated to maturity. Exactly the reverse is true for the BS model since the most severe mispricings are found in the out-of-the-money category. Thus, the maturity-based implied parameters should help the alternative models whereas the moneyness-based implied volatility should help the BS model.

The last performance measure the authors consider is the dynamic hedging ability. Therefore, instead of the S&P 500 futures contract, the spot index itself is used. First let us consider single instrument hedges, that means hedges consist only of the underlying asset. Of course, the number of shares used to hedge an existing short position in the underlying differs between the models, e.g. for the SVJ model the hedge ratio takes the stochastic volatility and possible occurrence of jumps into account. The minimum variance hedge can only be reviewed discretely at time intervals of length $\Delta t$. The self-financing portfolio consists of going long $X_S(t)$ stocks and parking the rest $X_0(t)$ in a riskfree and instantaneously maturing bond. The hedging error at time $t + l\Delta t$ is computed as follows:

$$H(t + l\Delta t) = X_S(t)S(t + l\Delta t) + X_0(t)e^{R(t)\Delta t} - C(t + l\Delta t, \tau - l\Delta t),$$

where $R(t)$ denotes the time $t$ instantaneous spot interest rate and $l = 1, \ldots, M \equiv (\tau - t)/\Delta t$. Now, the absolute or the average dollar-value hedging error can easily be obtained. Again, the authors take the implied volatility and parameter values from the previous day to construct the hedge on the next day. For every call and each day, determine the error of the hedge either after one day or after five days. The result is the same for both hedging revision frequencies: For absolute errors, the SV model takes the first place, followed by the SVJ, the SVSI, and the BS model whereas the SVSI performs best for out-of-the-money and long-term in-the-money calls. Yet overall, differences between the BS model and the alternative choices occur only for out-of-the-money options. Otherwise the models possess almost identical (and small) hedging errors. The incorporation of jumps does not help the hedging performance at
all since it is unlikely that a jump actually occurs during the time of the hedge but the hedge itself takes the jump risk into account. Thereafter, delta-neutral hedges are examined. This means that, in addition to the stock price movement risk, for all SV models, the volatility risk is also hedged with a second position in an option on the same stock with a different maturity. Also, in the case of stochastic interest rates, the term delta-neutral implies that interest rate risk is controlled for with a position in a bond. Due to stochastic jump sizes the authors do not hedge jump risk. Using an equivalent procedure like in the hedge exercise before, all four models and a BS model with delta-vega hedge are investigated. This gives the BS model an equal chance since any improvements can also be due to the introduction of a second option and not to model misspecification. The results are intriguing. While the errors decrease in size for all stochastic volatility models, the BS model produces errors which are usually 2 or 3 times higher than the errors for the other model. Further, errors increase dramatically when the hedge is revised only every 5 days for the BS model whereas there is only a slight increase for all other models. However, the performance of the other models is equally good, also for the BS delta-vega strategy. Only for deep in-the-money options the alternative models perform slightly better. This implies that the introduction of a second option leads to the overall improvement since the volatility risk is neutralized and the gamma of a usual hedge is also very close to zero. Stochastic volatility results only in a second order improvement. Lastly, the BS model yields negative hedging errors most of the time whereas the signs of the errors for the alternatives are randomly distributed.

Bakshi, Cao, and Chen (2000) extend this work by considering extremely long-term options, so-called LEAPS. Although they find that short-term and long-term options contain differential information, the overall conclusions are almost the same as in their earlier paper. Even for long-term options incorporating stochastic interest rates does not lead to any improvement over the stochastic volatility model. Therefore, modelling stochastic volatility drives the main improvement.

Bates (2000) tests two possible models which are consistent with the highly negatively skewed post-crash distributions inferred from S&P futures option prices: Stochastic volatility
models with and without time-varying jumps. Considering data from 1988 to 1993 he reports that the volatility of volatility parameters for the stochastic volatility model are too high given the time-series properties and that the incorporation of jumps significantly improves the model’s fit. Yet, he concludes that the absence of further sharp drops in the index from 1988 to 1993 is inconsistent with the stochastic volatility and diffusion jump model since a 10 percent weekly decline should have occurred at least once with a probability of 90 percent according to the model.

In order to overcome the misspecifications associated with all models considered so far, Eraker, Johannes, and Polson (2003) test stochastic volatility models with jumps in returns and in volatility. There is strong evidence for the presence of jumps both in volatility and returns which are responsible for a considerable amount of the variation in spot volatility and returns. For the S&P 500 and the Nasdaq 100 index, their findings suggest that the incorporation of both features are necessary, otherwise misspecified models will result. Furthermore, their models give a good approximation of the observed volatility smile due to higher implied volatilities for deep in-the-money and out-of-the-money options when jumps in returns and volatility are modelled. However, the authors acknowledge that the risk of volatility and parameter uncertainty can posses a rather big influence on option prices.

The overall findings remain true also for options on other assets. Jorion (1988) finds that discontinuous jumps occur in the foreign exchange market and that they are much more significant than in the stock market. Therefore, incorporating jumps into a currency option pricing model leads to a big improvement in terms of explaining the empirical biases. For warrants, Lauterbach and Schultz (1990) find that the dilution-adjusted Black-Scholes model possess the disadvantage of assuming a constant volatility. A CEV (Constant Elasticity of Variance) model clearly performs better. Melino and Turnbull (1995) look at the hedging errors for foreign currency options. The common method of using the implied volatility derived from a short-term option for valuing long-term options does a poor job. Incorporating stochastic volatility results in significant improvements. For Deutsche Mark options, Bates (1996) investigates stochastic volatility with and without jumps. To explain the observed
volatility smile, parameter values for the stochastic volatility process must take unreasonable values whereas the incorporation of jumps yields more plausible parameters.

A completely different and interesting approach is followed by Buraschi and Jackwerth (2001). They do not test which model performs best but rather if deterministic volatility models, like the Rubinstein (1994) implied tree model which will be covered later, or the already presented stochastic models possess the general right properties. The difference between these two types of models is testable because, for the deterministic models, options are redundant securities whereas for the stochastic models other risks are also priced in equilibrium. This results in the fact that options are needed for spanning purposes. In turn, these implications can be used to test if the pricing kernel satisfies certain martingale restrictions. Buraschi and Jackwerth run unconditional and conditional tests employing the generalized method of moments and the so-called $\delta$-metric. While options appear to be redundant assets before the 1987 crash, this picture changes after the crash. Both in- and out-of-the-money options play an important role for spanning purposes. Thus, risk factors, such as volatility or jump risk, present a worthwhile feature for all option pricing models. However, the $\delta$-metric test points out that the riskfree interest rate is redundant leading to the conclusion that introducing stochastic interest rates may not be of first order importance.

Overall, modelling stochastic volatility and jumps results in significant improvements for explaining observed option prices. However, these models still have inconsistencies. Jumps in volatility or multifactor models may enhance the fit further. A different possibility is to estimate the risk premia inherent in option prices. This will be the next topic.

3.4 Risk Premia

The empirical fact that none of the models can completely explain the observed option prices has led to several studies concerning the implicit risk premia. We have already seen in Chapter 2 that these risk premia occur in the differential equations which are solved for the option price. Thus, they may possess an important influence on prices. For example, a
negative volatility risk premium will increase option prices. Therefore, it is only natural to investigate which risks are priced in equilibrium and to what extent.

Pan (2002) examines time-series data from January 1989 to December 1996 for the S&P 500 index and at-the-money options. Only one near-the-money option with approximately 30 days to expiration is taken each day. The option pricing model is the one used by Bates (2000) whereas the method of estimation uses an implied state general method of moments approach. For the goodness-of-fit test, the data clearly rejects models with a volatility or no risk premium, whereas the model incorporating a jump risk premium can be accepted for a high statistical significance level. The author concludes that the mispricing for the stochastic volatility model is mainly due to the overpricing of long-dated options. Also, the jump risk premium quickly responds to a higher spot volatility while the volatility risk premium increases only slightly. Yet, the estimation for a model including both types of risk premia does not completely rule out the possibility that both types of premia may be important. Interestingly, the author finds that the excess mean rate of return (for the diffusion process) on the S&P 500 is 5.5 percent, but that the excess return for jump risk is 3.5 percent per year although jump are responsible for less than 3 percent of the total return variance. Hence, jump risk premia do possess economic significance, too.

A different approach is taken by Bakshi and Kapadia (2003a). They investigate if the market prices volatility risk. Yet, they examine delta-hedged gains, that is one is long an option and short the stock (or index) so that the net investment earns the riskfree rate. In this study, hedges are revisited only discretely at the end of each day. In general, if one hedges an option $C_t$ at time $t$ discretely $N$ times over the life of the derivative and rebalances the hedge at times $t \equiv t_0, t_1, \ldots, t_{N-1}$ then the delta-hedged gains $\pi_{t,t+\tau}$ are given by

$$\pi_{t,t+\tau} = \max(S_{t+\tau} - K, 0) - C_t - \sum_{n=0}^{N-1} \Delta_{tn} (S_{t_{n+1}} - S_{t_n}) - \sum_{n=0}^{N-1} r (C_t - \Delta_{tn} S_{t_n}) \frac{\tau}{N},$$

where $\Delta_{tn} \equiv \delta C_{tn}/\delta S_{tn}$ is the option’s delta computed as the Black-Scholes hedge ratio.

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37 Also he uses one in-the-money call with a strike-to-spot ratio of approximately 0.95.
38 However, the author concludes that the jump risk premium is by far more important.
39 The authors run a simulation trying to estimate the induced bias if volatility is in fact stochastic and
and $t_N \equiv t + \tau$. What has this to do with a volatility risk premium? In the Black-Scholes world, these hedges clearly should have an expected value of zero. Also, the authors show that this is true for a stochastic volatility model\(^{40}\) as long as volatility risk is not priced even when the hedge is revisited only discretely. Yet, in the case the market prices volatility risk, the theoretical prediction is that, as the vega is positive, a negative volatility risk premium implies, on average, negative delta-hedged gains. Also, the absolute value of the hedge gains are maximized for at-the-money options whereas they decline for strikes away from the money since at-the-money options possess the highest vega and the risk premium increases with vega. At times of higher volatilities the underperformance of hedges should increase. For the empirical tests, Bakshi and Kapadia use S&P 500 index options from January 1, 1988 to December 30, 1995. Volatility is estimated by the usual sample standard deviation and a GARCH(1,1) model.\(^{41}\) The vast majority of delta hedges yields significantly negative returns supporting the existence of a negative volatility risk premium. An exception consists of in-the-money calls. Yet, the authors believe that this may be induced by the relative illiquidity and show that, for out-of-the-money puts, delta hedges also significantly underperform zero. Further observations are that the hedge losses are maximized for at-the-money options, that they increase as the time to maturity is extended and that during times of higher volatility the returns become even more negative. Thus, the empirical findings confirm the existence of a negative volatility risk premium. For the cross-sectional relationship, two regressions are estimated verifying the influence of vega. These tests confirm that the delta hedged gains are decreasing with vega and thus maximized for at-the-money options. Also, the other regression points out that call prices respond positively to volatility and that options may serve as hedging instruments against market declines in general. The findings remain stable under robustness checks and even accounting for jump fears can not explain the observed statistically and economically significant negative delta hedge gains.

In a related paper, Bakshi and Kapadia (2003b) find that the negative volatility risk premium is also present in prices of individual stocks and that it is negatively correlated

\(^{40}\) To be exact, in this case $E_t(\pi_{t,t+\tau}) = O(1/N)$.

\(^{41}\) See for example Bollerslev (1986).
with market volatility but not with idiosyncratic volatility. However, for individual stocks the volatility risk premium appears to be much smaller. The authors conclude that this can provide an explanation why differences between implied and realized volatilities are also smaller for stocks than for indexes.

We can conclude that there is at least some risk premia present in the market option prices and that this may help explain the observed irregularities. However, it is not certain if volatility or jump premia are more important. More research is thus needed to explicitly distinguish between these two.

3.5 Implied Binomial Trees and Underlying Probability Distributions

In the last sections we have found that different stochastic models for the stock price and the volatility imply different underlying probability distributions. This can be seen as one side of the coin. We pick a particular model and try to fit it to the data so that we also get implicit information about the underlying probability distribution. Yet, is it possible to do the opposite? Can we use the observed option prices to back out the implied underlying probability distribution? Theoretically, if we had a continuum of European options on a stock or index with same expiration date and spanning strike prices from zero to infinity, Breeden and Litzenberger (1978) show that it is an easy task to derive the risk-neutral underlying probability distribution exactly: We simply have to compute the second derivative of the option price with respect to the strike price. Unfortunately, in reality there exist only options with discrete and distant strike prices. Usually they start well above zero and they also possess an upper boundary. Therefore, interpolation or extrapolation techniques must be employed.42

I will present only two possible methods to back out the underlying distribution. The first one uses the well-known binomial trees, the second one tries to identify main characteristics, the so-called moments, of the probability distribution. For additional models, I refer the interested reader to Jackwerth (1999) or Skiadopoulos (2001).

42 See for example Shimko (1993).
Rubinstein’s (1994) method presents more or less an extension to the binomial model. Take the Black-Scholes implied volatilities of the two nearest-the-money calls to build an \( n \)-step binomial tree in the usual way. Let \( P_j' \) denote the ending nodal risk-neutral probabilities and \( S_j \) the ordered (from lowest to highest) underlying prices after the last step in the tree, where \( j = 0, \ldots, n \). As has already been shown, in the limit \( n \to \infty \), the probability distribution approaches the log-normal one. The actual risk-neutral probabilities should be as close as possible to this prior guess with the additional requirements that the discounted payout-adjusted stock and call prices fall in between their respective bid-ask prices, denoted by \( S^b, S^a, C_i^b \) and \( C_i^a \) for \( i = 1, \ldots, m \) and \( m \gg n \). The calls are assumed to be observed simultaneously and to mature at the end of the tree. The implied posterior risk-neutral probabilities \( P_j \) solve the following quadratic program:

\[
\min_{P_j} \sum_j (P_j - P_j')^2 \quad \text{such that}
\]

\[
S^b \leq S \leq S^a, \quad \text{with} \quad S = \frac{(1 + \delta)^n \sum_j P_j S_j}{(1 + r)^n}
\]

\[
C_i^b \leq C_i \leq C_i^a \quad \text{with} \quad C_i = \frac{\sum_j P_j \max[0, S_j - K_i]}{(1 + r)^n} \quad i = 1, \ldots, m
\]

\[
P_j \geq 0 \quad \text{and} \quad \sum_j P_j = 1 \quad \text{for} \quad j = 0, \ldots, n.
\]

In the above equations, \( r \) denotes the riskless interest rate and \( \delta \) denotes the asset payout rate over each binomial period. If no arbitrage opportunities exist in the economy, this algorithm will converge. However, to derive the implied tree further assumptions are introduced: The asset return follows a binomial process, the tree is recombining, the ending nodal values are ordered, we have a constant interest rate, and all paths leading to the same ending node possess the same risk-neutral probability. Then, the tree is solved by backward induction. Use the information at time \( t \) of the terminal nodal probabilities to calculate the path and transition probabilities as well as the returns at time \( t - 1 \). Use this method until one reaches the beginning of the tree.
Jackwerth and Rubinstein (1996) apply this procedure on options on the S&P 500 index from April 2, 1986 to December 31, 1993. However, the optimization method is a little bit different: The constraints do not always have to hold exactly. In the case of a violation of a bid-ask constraint, the squared difference is multiplied by a large penalty term and added to the normal objective function. Further, the authors advise zero probabilities to extreme outcomes. They call this clamping down. Also, multimodalities may be obtained. The empirical study results in the remarkable fact that the underlying probability distribution has completely changed its form after the crash. Being nearly log-normal before the crash with a small negative skewness and negative kurtosis, the probability distribution until then is more negatively skewed (fairly stable around -1 from 1988 on) and has a positive kurtosis (again stable around 1.2 from 1988). Furthermore, the probability of very large movements is more likely under the new distribution, for example a 4 standard deviation decline is 100 times more probable than under the log-normal and 10 times more probable than under the pre-crash distribution. The results remain stable for different objective functions.

Dumas, Fleming, and Whaley (1998) provide an empirical test of the proposed deterministic volatility model with respect to out-of-sample and hedging performance. Contrary to the implied binomial tree approach, the authors impose some structure on the volatility process followed by the S&P 500 index: They test three different models depending only on the underlying asset $X$ and time $T$:

$$\sigma = \max[0.01, a_0 + a_1 X + a_2 X^2],$$

$$\sigma = \max[0.01, a_0 + a_1 X + a_2 X^2 + a_5 T + a_5 X T], \quad \text{and}$$

$$\sigma = \max[0.01, a_0 + a_1 X + a_2 X^2 + a_3 T + a_4 T^2 + a_5 X T].$$

The sample period extends from June 1988 to December 1993. The conclusion is that these relatively parsimonious models can accurately explain the observed market prices. Since the volatility function should not change through time a simple test consists of looking to the implied option values one week ahead. Although the different volatility models reduce
the prediction errors compared to the Black-Scholes formula, they actually perform worse compared to an ad-hoc Black-Scholes formula which uses the implied volatility of options in the same moneyness class. Further, the Black-Scholes model yields the best hedging performances for all models. The results are robust for other functional forms of the volatility process, for different estimation ranges, and in subperiods. Thus, although deterministic volatility models are capable of fitting the observed volatility smile exactly they do not provide accurate numbers for prediction or hedging purposes. Nonetheless, the class of deterministic volatility models now presents a whole field of study which I only scratched at the surface.

The difference between the underlying probability distributions implied by options on individual stocks and on indexes is investigated by Bakshi, Kapadia, and Madan (2003). The underlying idea consists of the possibility that all payoff functions can be replicated by a continuum of out-of-the-money puts and calls. Therefore, one can recover the volatility, skewness and kurtosis of the underlying risk-neutral return distribution from out-of-the-money options. The risk-neutral distribution for an index can exhibit negative skewness if (a) the physical distribution itself is left skewed or (b) the kurtosis is in excess of three and investors are risk averse. For individual stocks, assume that the return of the asset can be divided into a market and an idiosyncratic component. In turn, this means that one can also link the other moments to these parts. Then theory predicts that the returns of individual options are less negatively skewed than for the index. Of course, options are also priced differentially and this will affect the shape of the volatility smile. In particular, implied volatility curves possess a smaller negative slope for stocks than for index options and the absolute slope increases with skewness. Fatter tails in the risk-neutral distribution create flatter smiles when the returns are skewed. Using options on 30 large stocks and the S&P 100 index from January 1, 1991 to December 31, 1995, the authors find that the implied volatility curves have statistically significant negative slopes and that the absolute slope for the index is much higher. Translated in option prices, this means that out-of-the-money puts

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43 See Bakshi and Madan (2000).
44 However for individual options, sometimes the slope changes signs and becomes positive.
are consistently higher priced than out-of-the-money calls. The difference appears smaller for individual stocks. The cross-sectional analysis shows that a higher negative skew leads to a steeper volatility smile whereas kurtosis works in the opposite direction. However, kurtosis provides only a second order effect on implied volatility curves. Further, if skewness is omitted as an explanatory variable, the kurtosis coefficient changes the sign and in this case it results also in a steeper curve. Thus, the firm-specific component of the return is far less negatively skewed than the market component. The authors conclude that risk aversion and fat-tailed physical distributions are the main factors driving equity option prices.

Using almost the same way of measuring the skewness, Dennis and Mayhew (2002) investigate the determinants of the risk-neutral skewness implied by individual stock options in more detail. They take options data ranging from April 7, 1986 to December 31, 1996 for over 1,400 underlying stocks. In the cross-sectional analysis, it can be seen that the following firm-specific factors play a role in explaining the observed skewness. The higher the beta and the larger the firm, the more negatively skewed is the risk-neutral density. Additionally, in times of higher implied volatility, leverage or volume, the underlying distribution tends to be less negatively skewed. The effect of put-call ratio as a proxy for market sentiment appears negligible. Panel data investigation points to the importance of incorporating an unknown firm-specific component which possesses more explanatory power than all other variables together. On the other hand, risk-neutral distributions for stocks are more negatively skewed in times of a higher negative skew for the index and in periods of increased market volatility. Overall, the authors conclude that firm-specific factors are important in explaining the skewness of individual stocks whereas the influence of market factors is also given although to a smaller extent.

### 3.6 Other Explanations

Since none of the models and explanations considered so far provide a completely satisfying way of explaining option prices, other possible solutions have been advanced. I will
present two of them: (1) buying or selling pressure or (2) the leverage effect may have an influence on option prices. While the investigation of buying pressure has been recent, academics have shown little interest in the leverage effect.

Bollen and Whaley (2004) propose that net buying (or selling) pressure may be responsible for the observed volatility smile. The data consists of S&P 500 index options from June 1988 to December 2000 and options on 20 large individual stocks from January 1995 to December 2000. In this study, the implied volatility function is estimated in a slightly different way to most other articles. Although the way of measuring the Black-Scholes implied volatility stays the same, options are not divided into different moneyness but delta categories. The authors use the dividend-adjusted Black-Scholes delta because they believe that this incorporates more accurately the likelihood that an option will end up in the money. Further, Bollen and Whaley have the opinion that underlying stochastic processes or skewness and kurtosis characteristics can not explain the different shape of the implied volatility functions for stocks and the index in view of the very similar empirical cumulative distribution functions. Two alternative hypotheses are considered: the limits to arbitrage hypothesis and the learning hypothesis. The first explanation states that investors can only take on arbitrage trades to a certain extent for a variety of reasons. Therefore, liquidity suppliers face increasing hedging costs as their positions get larger. In this situation, buying pressure will result in higher prices and higher option implied volatilities. An excess of seller-motivated trades will track prices down. The learning hypothesis says that market makers receive information about the underlying stochastic process from trading activity and that this is incorporated continuously into option prices. Empirically, it is possible to differentiate between these two alternatives. The limits to arbitrage explanation predicts that there is a negative serial correlation in changes in implied volatility whereas the correlation should be zero under the alternative. Also, demand for at-the-money options provides the

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45 There are 5 delta categories for calls (puts): In category 1 which corresponds to deep in-the-money calls (deep out-of-the-money puts) delta ranges from 0.875 to 0.98 (-0.125 to -0.02), in category 2 from 0.625 to 0.875 (-0.375 to -0.125), in 3 from 0.375 to 0.625 (-0.625 to -0.375), in 4 from 0.125 to 0.375 (-0.875 to 0.625), in 5 corresponding to deep out-of-the-money calls (deep in-the-money puts) from 0.02 to 0.125 (-0.98 to -0.875). For the calculation of delta, the realized return volatility over the last sixty trading days is used.
driving factor for implied volatilities under the learning hypothesis while other options may have an influence under the limits to arbitrage hypothesis. Net buying pressure is defined as the difference between “the number of contracts traded during the day at prices higher than the prevailing bid/ask quote midpoint (i.e., buyer-motivated trades) and the number of contracts traded during the day at prices below the prevailing bid/ask quote midpoint (i.e., seller-motivated trades) times the absolute value of the option’s delta” (p. 729). The daily change in implied volatility in a given delta-ness category is regressed on security returns, trading volume, and varying buying pressure proxies. For the S&P 500 index, net buying pressure for puts is the driving factor for the change in implied volatility (and the downward sloping volatility curve) whereas it is the net buying pressure for calls for most options on individual stocks. Furthermore, changes in implied volatilities tend to reverse partly the next day and they respond to the option’s own buying pressure instead of at-the-money options in general. This evidence is supportive of the limits to arbitrage hypothesis. Significant profits from selling options and delta hedging with the underlying, however, are only achieved for the index and not for stocks. The use of out-of-the-money puts as a way of portfolio insurance is supported by the fact that these kind of options yield the highest profits. Yet, if transaction costs and vega hedging are taken into account, all profits vanish and become even largely negative which may give rise to a volatility risk premium.

One other explanation relates the capital structure of a firm directly to the stock’s option price. Geske (1979) derives a formula for firms which are partly financed with equity and partly with debt. As a result, the volatility of the stock return is not deterministic anymore, but a function of the stock price itself. If the stock price falls, the firm’s debt to equity ratio raises which leads to an increase in the volatility. Therefore, this model confirms the empirical fact that the variance of the return tends to rise if the stock price decreases. This leaves the possibility that the formula is capable of explaining the volatility biases. Of course, the model also yields different hedge ratios.

Although academics have shown little interest in the leverage effects, Toft and Prucyk (1997) present a more difficult option pricing model for leveraged firms which face taxes and
bankruptcy costs. The solution and the technique leading to it are very similar to the one for barrier options. It is important to note that this model is able to incorporate endogenous as well as exogenous bankruptcy. Overall the model yields the following testable predictions:

- The implied volatility skew is more negatively skewed for firms with higher leverage (Leverage Effect Hypothesis).

- Firms which have covenants protected debt possess a more negative skew than firms which do not have these covenants (Covenant Effect Hypothesis).

- In the class of firms which have covenant protected debt, the higher the leverage the more negative the skew (Stratified Covenant Effect Hypothesis).

The proxy for the leverage is defined as the book values of total debt plus preferred stock, divided by the sum of total debt, preferred stock, and the market value of outstanding common equity. As a proxy for protective net-worth covenants, long-term debt maturing in one year plus notes payable is divided by total debt. After a screening procedure the data consists of 138 firms for the years 1993 and 1994. In the regression analysis, all theoretical predictions can be confirmed; both higher leverage and the existence of covenant protected debt result in a more pronounced volatility skew although the estimated parameters are too high for the estimation from at least 100 individual implied volatility skews. Further, the stratified covenant effect hypothesis is also accepted; for firms with low leverage the coefficient on the covenant proxy is insignificant whereas it is statistically significant for highly leveraged firms within the class of covenant protected debt companies. The results are robust even if firm size, the book-to-market value of equity, industry-specific factors, or the level of implied volatility are included.
4 CONCLUSION

The existence of biases in the Black-Scholes option pricing model has been well documented over the last 30 years. Since then, many new formulas and explanations have been developed to price options more accurately. This thesis presents the most important of these models. We have seen that

- the incorporation of stochastic volatility or jumps fits observed option prices better
- modelling stochastic interest rates adds little value
- all models sometimes suffer from unreasonable parameter values.

A further improvement of the fit can be obtained by considering explicitly risk premia. However, more research is needed to determine how large the jump risk and the volatility premia are. Another way of getting information about option prices is to find the underlying probability distribution function or the most important characteristics like skewness and kurtosis. Lastly, I presented other explanations where the introduction of net buying pressure seems to be a worthwhile feature. Yet, none of these refinements can fully explain the observed market prices. Is the option market hence inefficient? The answer is probably not. The main problem is that all models have to assume simplifications of reality in some form. For example, it is especially difficult to incorporate trading costs. Yet exactly these kind of market microstructure effects may drive prices to a considerable amount. Bid-ask spreads of $0.50 or even $1 do not seem to be negligible at all. In empirical tests, most authors work with the midpoint of the bid-ask spreads, assuming that this is the real price. However, given net buying or selling pressure, it is possible that market makers do not set real prices
as the bid-ask midpoint. These kind of structural difficulties make it virtually impossible to fit an exact model.\(^{46}\)

However, it would be interesting to see tests on more recent data after the three-year market decline. Have the underlying risk-neutral densities again switched or have they been stable? What about the skew of the implied volatility function? Do out-of-the-money put options show even more severe mispricings now?

In future research, it may prove useful to develop equilibrium option pricing models incorporating volatility or jump risk premia. Also, a more exact study of market microstructure effects may help to understand how options are priced and how risks can be more accurately hedged.

\(^{46}\)See for example Harvey and Whaley (1991), Hentschel (2003) or Christoffersen and Jacobs (2004) for other difficulties.
REFERENCES


