MATHEMATICAL CONNECTIONS MADE IN PRACTICE: AN EXAMINATION OF TEACHERS' BELIEFS AND PRACTICES

by

LAURA MARIE SINGLETARY

(Under the Direction of Patricia S. Wilson)

ABSTRACT

In this descriptive study, I examined the kinds of mathematical connections three secondary mathematics teachers made in their teaching practice, and I explored the teachers' beliefs about mathematics. For each teacher, my primary data sources included six in-depth, semi-structured interviews and approximately two weeks of classroom observations. I used an inductive and iterative coding process to analyze the classroom data, and I developed the Mathematical Connections Framework to describe the explicit kinds of mathematical connections teachers made in practice. To analyze the teachers' beliefs, I coded the interview and classroom data, drawing upon Green's (1971) metaphorical interpretation of the structure of a belief system and Leatham's (2006) theory of sensible systems of beliefs. These theoretical perspectives helped me understand the structure of the teachers' beliefs about mathematics and how the beliefs were held as a sensible system. I present my findings through a series of narrative cases as well as a comparison across the cases. The teachers in this study made various kinds of mathematical connections for and with their students. Examining teachers' beliefs about mathematics provided valuable insights into these teachers' practices, helping me understand some of the reasons for the variation occurring among the mathematical connections the teachers made in practice. The mathematical connections each teacher made in practice were often related to the teacher’s beliefs about mathematics and, in particular, the teacher’s beliefs about the connected nature of mathematics.

INDEX WORDS: Mathematical Connections, Teaching Practices, Beliefs, Mathematical Beliefs, Secondary Mathematics Teachers, Mathematics Education

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LAURA MARIE SINGLETARY

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LAURA MARIE SINGLETARY

Major Professor: Patricia S. Wilson

Committee: AnnaMarie Conner
Judith Preissle

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
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DEDICATION

I dedicate this dissertation to my dearest love, my husband. The words on this page cannot begin to express how very thankful I am for you. Thank you for your unending love, encouragement, and support throughout this entire process. Thank you for being my strength and my rock. Thank you for always believing in me. Thank you for rejoicing with me and for wiping the tears from my eyes. You have always been and will always be the greatest blessing in my life. Without you by my side, this journey would not have been possible. I love you so very much.

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INTRODUCTION

> The [school mathematics] curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. …Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. …they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (National Council of Teachers of Mathematics [NCTM], 2000, p. 3)

This statement details NCTM's vision for school mathematics. While this vision statement serves to encourage mathematics educators to continually pursue a goal of mathematics for all, this statement also acts as a reminder of the current state of mathematics education and the distance researchers and educators must travel to fully realize this goal. Recent movements toward educational reform followed from the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983), because this report proclaimed the dire straits of U.S. education in areas such as mathematics. Responding to the call for reform, NCTM published *Curriculum and Evaluation Standards for School Mathematics* (1989) as a coherent set of mathematics standards to become the paradigm for school mathematics (Schoenfeld, 2004).

About ten years after initiating the standards-based reform movement in mathematics education, NCTM revisited and revised their original standards effort and produced a new standards document titled *Principles and Standards for School Mathematics* (2000). This revision responded to the previous decade's developments in mathematics education research. The *Standards* continue to challenge what it means to learn and teach school mathematics. They do so by encouraging teachers to engage students in the following mathematical processes: problem solving, reasoning and proof, communication, connections, and representation. Among these processes, the literature in mathematics education often describes making mathematical connections as the process necessary for students to develop a meaningful understanding of mathematics (Boaler & Humphreys, 2005; Coxford, 1995; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Stein, Smith, Henningsen, & Silver, 2000).

Although many policy makers and mathematics educators may agree that it is important to make mathematical connections, making mathematical connections is not always an easy task or a common occurrence in practice. The study reported in the following pages describes the mathematical connections three secondary mathematics teachers made in practice and explores the teachers' beliefs that underlie the mathematical connections they made.

Mathematical Connections and School Mathematics

The increased emphasis for mathematical connections to be made within school mathematics began long before the standards-movement promoted by NCTM. As early as 1902, E. H. Moore, then President of the American Mathematical Society, petitioned members of the society to integrate mathematical topics across subjects of the school curriculum to demonstrate the connected nature of mathematics and to enhance students' understanding. In 1923, the National Committee on Mathematical Requirements responded to the many problems identified within secondary mathematics education by proposing a reorganization of the school mathematics curriculum. The committee recommended reorganizing the curriculum around unifying themes (e.g., the concept of function) to help students develop an understanding of the relationships within mathematics. The call to emphasize connections within school mathematics continued with the Commission on the Secondary School Curriculum of the Progressive Education Association. In 1940, the commission recommended students experience the "development of a unified mathematical picture" (Coxford, 1995, p. 3). In 1980, NCTM responded to the "back to basics" movement of the 1970s by producing a pamphlet titled *An Agenda for Action: Recommendations for School Mathematics of the 1980s*. NCTM recommended that problem solving should be the focus of school mathematics, describing "true problem-solving power" as requiring "a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them" (para. 5). With reform movements across the 20th century promoting the inclusion and development of mathematical connections within school mathematics, it is necessary to consider more recent discussions related to the importance and inclusion of mathematical connections within school mathematics.

Subsequent reform movements have recommended that students experience mathematical connections in a variety of ways. NCTM's (1989) original *Standards* document described ways students can experience mathematical connections and considered possible results of students making mathematical connections: (a) recognize equivalent representations of the same concept, (b) relate procedures in one representation to procedures in an equivalent representation, (c) use and value the connections among mathematical topics, and (d) use and value the connections between mathematics and other disciplines (p. 146). In 2000, NCTM simplified their characterization of connections in school mathematics. This revision focused on the expectation that a meaningful understanding of mathematics requires making connections across multiple mathematical topics, allowing students to view "mathematics as an integrated whole" (p. 354). In addition, the revision recommended making mathematical connections to other school subjects and to real world applications to provide students with an appreciation of the usefulness of mathematics outside of the mathematics classroom. Similar to NCTM's Process Standard of Connections, the National Research Council (2001) identified "conceptual understanding" as one of the five strands of mathematical proficiency–a main goal for students' mathematical learning. The authors defined conceptual understanding as students' knowledge of how mathematical ideas are interrelated and connected, and they said this kind of understanding can be seen in a student's ability "to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes" (p. 119). These descriptions of mathematical connections provide a useful way to conceptualize how mathematical connections may occur in school mathematics.

Despite attempts by reform movements to underscore the importance of making mathematical connections to enrich students' understanding, Romberg and Kaput (1999) described traditional school mathematics as a subject with "mechanistic manipulations" devoid of context or connections (p. 4). The authors continued to describe traditional instruction as the segregated development of basic skills and procedures, where this method of instruction "gives students little reason to connect ideas in today's lesson with those of past lessons or with the real world" (p. 4). Similarly, from the analysis of the Third International Mathematics and Science Study (TIMSS) video study comparing mathematics instruction in the United States, Germany, and Japan, Hiebert (1999) discussed the procedural and computational focus of much of the mathematics instruction in the United States. He noted, "Little attention is given to helping students develop conceptual ideas, or even to connecting the procedures they are learning with the concepts that show why they work" (p. 11). These descriptions of traditional school mathematics suggest that making mathematical connections is far from common in many mathematics teachers' instructional practices.

As a mathematics educator, I assume that mathematical connections can help students understand mathematics. In addition, when mathematical connections are not made, students' understanding is limited because each mathematical topic is developed in isolation (Carpenter & Lehrer, 1999). For these reasons, I began to ask questions about why traditional mathematics instruction continues to develop mathematical topics in isolation despite the many calls for reform. Romberg and Kaput (1999) claimed the traditional method of instruction continues to exist because the majority of people, many of whom are mathematics teachers, view school mathematics as a "static body of knowledge" comprised of facts and rules rather than as a connected discipline marked by problem solving. They believed this common view of school mathematics influences "the scope of the content to be covered and the pedagogy of the school mathematics curriculum" (p. 4). Similarly, in *Connecting Mathematical Ideas*, Boaler and Humphreys (2005) observed that an individual teacher's orientation toward mathematical knowledge influences the development of mathematical knowledge in the classroom. This discussion suggests that the teacher, including the teacher's views, orientations toward, or what I would describe as beliefs about mathematics, significantly influences how and to what extent mathematical connections are made in practice. Therefore, in the following section, I examine mathematical connections from the perspective of teachers and their beliefs about mathematics.

Mathematical Connections and Teachers

Some researchers have examined the mathematical connections teachers make among concepts and procedures, separate from teachers' practice. For example, Businskas (2008) and Wood (1993) investigated how secondary mathematics teachers were able to make connections among a variety of mathematical concepts and procedures. Businskas (2008) conducted interviews with nine secondary mathematics teachers and identified ways teachers associated mathematical concepts and procedures. She noted, however, that only a few of the teachers in her study explicitly described making mathematical connections in their practice. In a similar study, Wood (1993) surveyed and interviewed prospective secondary mathematics teachers to analyze the kinds of mathematical connections they made across mathematical concepts and procedures. He also examined certain factors such as gender, mathematical knowledge, and educational background and how these certain factors might be related to the kinds of mathematical connections the prospective secondary mathematics teachers described. However, in this analysis, he was unable to identify any relationships existing between the certain factors he evaluated and the kinds of mathematical connections the prospective secondary mathematics teachers made.

Discussions across the literature in mathematics education assume teachers should make mathematical connections in their instruction to support students' learning of mathematics (e.g., Boaler & Humphreys, 2005; Hiebert & Carpenter, 1992; NCTM, 2000, 2007; NRC, 2001). However, researchers have yet to examine the mathematical connections teachers make in practice. For this reason, Hiebert and Carpenter (1992) called for researchers to study "what connections become explicit during teacher-student interactions" (p. 86). Part of the purpose of this study is to address Hiebert and Carpenter's call. I wanted to extend the research base on mathematical connections from teachers' descriptions to teachers' practice. Knowing more about the kinds of mathematical connections teachers make in practice could influence the teaching and learning of mathematics with understanding as well as research investigating the use of mathematical connections in instruction. In addition, given the previous discussion of beliefs, I wanted to explore teachers' beliefs about mathematics as a method to interpret and understand these particular aspects of teachers' practice.

Researchers studying teachers' beliefs make the assumption that "what teachers believe is a significant determiner of what gets taught, how it gets taught, and what gets learned in the classroom" (Wilson & Cooney, 2002, p. 128). Furthermore, seminal research on mathematics teachers' beliefs identified complex relationships existing between teachers' beliefs about mathematics and their teaching practices (Cooney, 1985; Raymond, 1997; Thompson, 1982, 1984). I similarly assume teachers' beliefs are related to their practice, and I began to wonder what beliefs may explain the kinds of mathematical connections teachers make in practice.

Philipp (2007) broadly defined as beliefs as the "lenses that affect one's view of some aspect of the world" (Philipp, 2007, p. 259). Given this definition, it seems likely that teachers' beliefs about mathematics may influence whether teachers view mathematics as a connected or segregated discipline. Past research and discussion on teachers' beliefs about mathematics suggest that teachers' beliefs about mathematics may influence the extent to which teachers make mathematical connections, whether in descriptions or in practice. In fact, Ernest (2008) proposed a theoretical model of simplified relations between teachers' philosophical views[1] of mathematics and the corresponding image of mathematics developed in the classroom. This model suggests teachers' views of, or beliefs about, mathematics influence the extent to which teachers make mathematical connections in practice. Because making mathematical connections is an important component of understanding, researchers in mathematics education need to study

[1] Ernest (2008) used the terms views, philosophies, epistemologies, and beliefs interchangeably. I choose to use the term "philosophical views" or "views" when specifically referring to Ernest's model, because he uses these terms more often in his work. However, I consider what Ernest described as teachers' "philosophical views of mathematics" to be the same as what I mean by teachers' "beliefs about mathematics."

the kinds of mathematical connections teachers make in their teaching practice and how teachers' beliefs about mathematics may explain aspects of the mathematics constructed in the classroom. In paritcular, it is critical to understand teachers' beliefs about mathematics to understand the kinds of mathematical connections they make for and with their students.

Purpose of the Study

The purpose of this descriptive study was to examine the kinds of mathematical connections that three secondary mathematics teachers made in their teaching practice and to explore how their beliefs about mathematics might be related to the kinds of mathematical connections made in practice. The following research questions guided the study:

1. What kinds of mathematical connections do three secondary mathematics teachers make in their teaching practice?
2. What are the beliefs about mathematics of these secondary mathematics teachers?
3. What relationships, if any, exist between the kinds of mathematical connections that these secondary mathematics teachers make and their beliefs about mathematics?

In the chapters that follow, I examine relevant literature, describe the methods and design of the study, present the findings, and discuss the results and consider their implications. Across these chapters, I consider mathematical connections from the perspective of teachers' practice, and I examine the teachers' beliefs about mathematics to interpret these particular aspects of the teachers' practice.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I examine the literature relevant to my research questions. To do so, I organize this chapter in two main parts, focusing on mathematical connections first and then beliefs. In the first main section, I define mathematical connections and describe the broad perspectives used to conceptualize mathematical connections in school mathematics. In the second main section, I detail the theoretical perspectives guiding my study of teachers' beliefs about mathematics.

Defining and Describing Mathematical Connections

Although the term *mathematical connections* is often used in the mathematics education literature, it is rarely defined. Therefore, in this chapter, I begin by approaching this seemingly vague, yet rather familiar, notion of mathematical connections with a basic definition. Then, I examine mathematical connections from various perspectives relevant to discussions of practice, and I use examples to provide meaning to what is meant by mathematical connections in the mathematics education literature. The following sections provide a conceptual framework for the study of the kinds of mathematical connections teachers make in practice.

The word *connections* originates from the Latin noun *connexionem*, meaning "a binding or joining together" (Online Etymology Dictionary, 2011). An understanding of this etymology informs a basic definition of connections in which a connection is defined as a bridge, relating one thing to another. Therefore, in its most basic form, a mathematical connection is a *relationship*[2] between a mathematical entity[3] and another mathematical or nonmathematical entity. This definition provides a basis for understanding what is meant in discussions of mathematical connections. However, this concise definition is not necessarily the most appropriate way to communicate how mathematics educators think about and describe mathematical connections in school mathematics.

Businskas (2008) found that there are many layers of meaning surrounding how the construct of *connections* is used in the mathematics education literature. At times, the literature describes mathematical connections as a part of connected discipline, where connections among concepts and procedures are a defining characteristic of mathematics. Whereas, in other situations, the literature may refer to mathematical connections as products of understanding, where the connections exist within the mind of the learner. Or, mathematical connections may be viewed as part of the process of doing mathematics. Although these broad perspectives of mathematical connections are not necessarily mutually exclusive in the literature, each individual perspective provides a useful lens to focus on the different meanings applied to mathematical connections. In addition, I note that many of the descriptions of mathematical connections are located within the literature written for practitioners. In the following sections, I develop each perspective in more detail.

Mathematical Connections: Part of a Connected Discipline

Usiskin (2003) described connections as "[a] fundamental characteristic of mathematics" (p. 28). Usiskin's quote reflects a common theme in the mathematics education literature, for the

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[2] Businskas (2008) provided a similar definition of a mathematical connection, defining a connection as a "true relationship between two mathematical ideas, A and B" (p. 18)

[3] I follow Zbiek and Conner's (2006) definition of a mathematical entity, where a mathematical entity is "any mathematical object from any area of curricular mathematics" (p. 92).

literature often refers to mathematical connections as a natural part of mathematics because mathematics is a connected discipline. Descriptions of how mathematics is connected vary, characterizing mathematics as “an integrated whole,” or “a unified body of knowledge,” or even “a woven fabric rather than a patchwork of discrete topics” (NCTM, 2000, p. 354; Ernest, 1989, p. 251; House, 1995, p. vii). From this perspective, Coxford (1995) described unifying themes and mathematical connectors as methods to display the connected nature of school mathematics.

Unifying themes and mathematical connectors can be used to connect multiple topics across algebra, geometry, discrete mathematics, and calculus (Coxford, 1995). Using *change* as an example of a unifying theme, Coxford (1995) developed the following list of questions to consider some of the ways a unifying theme can connect mathematical topics across school mathematics:

> How is a constant rate of *change* related to lines and linear equations?...How does the perimeter or area of a plane shape *change* when it is transformed using isometries, size transformations, shears, or some unspecified linear transformation? …What is the instantaneous rate of *change* of a function at x_o? (p. 4)

Mathematical connectors are similar to unifying themes in that they connect a wide variety of mathematical topics. However, Coxford (1995) carefully distinguished between the two by describing mathematical connectors as the mathematical topics that “permit the student to see the use of one idea in many different and, perhaps, seemingly unrelated situations” (p. 10). He suggested the following topics could be used as mathematical connectors in school mathematics: variable, function, matrix, algorithm, graph, ratio, and transformation. As an example, he reviewed several different meanings and uses of *variable* in school mathematics, “from the unknown in a problem to the changeable argument in a function to the pure symbol found in statements such as the distributive property” (p. 11). In this way, he defined a mathematical connector as a mathematical topic that can be used in many and seemingly unlikely ways across school mathematics.

The distinctions made by Coxford (1995) seem to be unique in the literature, because mathematics educators almost always refer to these kinds of “big ideas” as unifying themes or unifying concepts (Clement & Sowder, 2003; Crowley, 1995; Hirschhorn & Viktora, 1995; NCTM, 2006; Usiskin, 2003). In addition, the literature usually discusses unifying themes in more tangible ways, demonstrating the theme through various mathematical tasks or curricular units that highlight the connections embedded in the theme. For example, NCTM (2006) presented four different tasks for teachers to explore how the unifying theme of transformations spans each strand of the secondary mathematics curriculum. Across these four tasks, NCTM developed the idea of transformations on geometric figures, transformations of functions on the Cartesian plane, transformations to fit functions of real-world phenomena, and transformations to understand how lines of best fit can be used with data sets. Viewing mathematical connections from this perspective is a common approach in mathematics education and provides a general interpretation of mathematical connections as part of a connected discipline. As I continue, I discuss mathematical connections as products of understanding.

Mathematical Connections: Products of Understanding

Many of the metaphorical descriptions of how mathematics is connected also refer to how mathematics might be connected and understood within the mind of the learner. These descriptions are often surrounded by discussions of how mathematical connections are products of understanding. For example, Hiebert and Carpenter (1992) defined understanding through the language of connections, “A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Therefore, another way to broadly conceptualize mathematical connections is to view mathematical connections as products of a learner's understanding of mathematics.

Constructing connections among mathematical ideas is a fundamental part of learning mathematics with understanding. However, it is possible for an individual to hold a disconnected knowledge of mathematics. Skemp (1976) described *instrumental understanding*[4] of mathematics as a kind of segmented knowledge of mathematics where the learner understands mathematical rules as fragmented and isolated constructs, separate from knowing why or how the separate pieces of mathematics relate and build on one another. To contrast, Skemp described *relational understanding* as knowledge of both what to do and why. He described this kind of understanding as knowing how mathematical concepts are interrelated, allowing learners to understand mathematics as a "connected whole."

Mathematical connections are often described as the result of a learner's organization of mathematical ideas into coherent schemes and networks. For example, Hiebert and Carpenter (1992) suggested connections exist within and between internal representations of mathematical knowledge. They described possible structures of an individual's internalized network of mathematical knowledge. Metaphorically speaking, they found it useful to depict these networks of mathematical knowledge as either *vertical hierarchies* or as *interconnected webs*. In the first metaphor, certain connections develop as some representations include other representations, whereas some representations are connected as special cases of a particular generalization. In the second metaphor, a network may be structured as an interconnected web, where the junctures act as internal representations of concepts and the threads between them as the connections. Hiebert

[4] Skemp (1976) said he did not initially regard instrumental understanding as a form of understanding, until he realized that many teachers considered understanding to consist of knowing rules and knowing how to use those rules.

and Carpenter provided these metaphorical interpretations of networks of mathematical knowledge, because they believed “the notion of connected representations of knowledge will continue to provide a useful way to think about understanding mathematics” (p. 67).

Hiebert and Carpenter (1992) provided an algebraic example to explain how a learner may use internalized connections in doing a school mathematics problem. Students typically learn a variety of mathematical procedures to solve algebraic expressions and equations, and a learner’s understanding of the inner workings of these procedures can be obtained from connections constructed between the symbolic system and various properties of the number system. Hiebert and Carpenter suggested, for example, that the simplification of the expression $3x + 5x$ requires a connection between symbolic notion and the distributive property. Constructing this connection also allows the learner to avoid typical misconceptions, such as $3x + 5y = 8xy$. They concluded by emphasizing the important role teachers play in helping the learner make mathematical connections among mathematical ideas.

Mathematical Connections: Part of the Process of Doing Mathematics

Many of the descriptions of mathematical connections in the literature focus on mathematical connections as part of the process of doing mathematics. The process of making mathematical connections is a significant component of mathematical work (Boaler, 2002). When making mathematical connections is conceptualized as a process, this process is often described as the byproduct of engaging in other mathematical processes, such as multiple representations, problem solving, proof, and real world applications and mathematical modeling. NCTM (2006) characterized these processes as connective processes, which teachers can incorporate into their instruction to reflect the coherence and connectedness of school mathematics.

Mathematical connections can be made through the examination and exploration of *multiple representations* of a given concept. NCTM (2000) explained, "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships" (p. 67). Mathematical concepts can be represented in a variety of ways: oral language, written words and symbols, manipulative models, realistic situations, and pictures and diagrams (Lesh, Post, & Behr, 1987). NCTM (2006) suggested one way for teachers to facilitate the process of making connections across multiple representations is to provide students with one type of representation, such as a mathematical concept depicted in a realistic situation, and then ask students to generate a result using a different type of representation, such as a visual or symbolic representation. For example, using the High Dive unit from the Interactive Mathematics Program (Alper, Fendel, Fraser, & Resek, 2000), teachers may ask students to determine when to release a diver from a platform to land safely in a cart of water. To respond to this situation, students may begin by representing the diver's height symbolically, as a function of time $h(t) = 65 + 50 \sin(9t)$. Moving from a symbolic representation to a visual one, additional connections may be made as students graph the function of the diver's height.

Engaging in *problem solving* encourages students to make and draw on mathematical connections to solve the problem. Hodgson (1995) advocated problem solving as a natural mechanism for students to establish and use mathematical connections. He described mathematical connections as "integral components of successful problem solving" (p. 18). The "handshake problem" is a classic mathematics problem[5] that allows for connections to be made among patterns generated by visual representations of discrete cases and the symbolic representation of the generalized solution. In addition, the handshake problem lends itself to a

[5] To solve the handshake problem, an individual must determine the number of handshakes that occur among a given number of people if each person shakes hands with every other person exactly once.

number of solution methods. Stein, Engle, Smith, and Hughes (2008) recommended teachers develop discussions surrounding students' various solution methods, suggesting such discussions provide opportunities for connections to be made across the various solution methods.

Cuoco, Goldenberg, and Mark (1996) described mathematical *proof* as a method of establishing "logical connections among statements…between what you want and what you know" (p. 387). Similarly, Stylianides (2007) characterized mathematical proof as a "connected sequence of assertions for or against a mathematical claim" (p. 291). Cuoco et al. provided the following proof as an example of a series of logical connections.

> If the greatest common divisor of two integers can be written as a linear combination of the two integers, then if *p* is a prime and *p* is a factor of *ab*, then either *p* is a factor of *a* or *p* is a factor of *b.* (p. 387)

Similar to the process of problem solving, creating a mathematical proof requires students to draw upon and make mathematical connections to establish a mathematical claim.

Real world applications and *mathematical modeling* provide opportunities for students to make mathematical connections to contexts outside of the mathematics classroom. Gainsburg (2008) outlined multiple ways teachers could make mathematical connections to real world contexts or applications: (a) simple analogies, (b) classic "word problems," (c) the analysis of real data, (d) discussions of mathematics in society, (e) "hands-on" representations of mathematical concepts, and (f) mathematically modeling real phenomena (p. 200). In particular, rich problems involving mathematical applications and modeling provide students with the opportunity to approach the problem from multiple directions, allowing them to see and make multiple connections (Coxford, 1995). To illustrate, NCTM (2006) provided a mathematical task, titled Growing Balloons, to highlight how developing mathematical models is a connective process. In this task, students are expected to construct empirical and theoretical models of the relationship between the number of breaths used to blow up a spherical balloon and the corresponding circumference of the spherical balloon. Throughout the task, students are encouraged to consider the possible connections existing between the models. This task provides an example of mathematical modeling as a connective process.

Connecting the Perspectives

All three perspectives provide different ways to think about mathematical connections in school mathematics. Whether mathematical connections are part of a connected discipline, are products of understanding, or part of the process of doing mathematics, each perspective contributes to an understanding of how mathematical connections are perceived by mathematics educators. After reviewing these various perspectives, the next important question becomes, Are these distinctions among the various perspectives necessary? Hodgson (1995) questioned, "Is a connection a feature of the subject matter or a feature of the learner's understanding?" (p. 7). He claimed the distinction was irrelevant. He suggested that if connections are not a feature of a student's learning, then whether or not connections inherently exist within mathematics is merely a philosophical musing. His argument resonates with the role of connections in learning school mathematics with understanding.

Combining the final two perspectives provides a better picture of what is meant by mathematical connections in school mathematics. Businskas (2008) reasoned, "In our efforts to comprehend what a mathematical connection is, sometimes we think of a connection as an object, sometimes as a process" (p. 17). Similarly, I appreciate the flexibility of being able to think about mathematical connections in different ways, sometimes as a product of understanding and at other times as part of the process of doing mathematics. The common ground between these two perspectives is associating this construct of mathematical connections with understanding in school mathematics. Reviewing these perspectives, two important implications seem to follow for teaching and learning mathematics. First, developing mathematical connections across mathematical entities is important for learners to develop a better understanding of school mathematics. Second, each perspective implicitly emphasizes the significant role of the teacher in helping students construct and make mathematical connections. Teachers are themselves learners of mathematics, and studying what teachers do requires understanding their beliefs about what they know and do. In the following sections, I discuss the frameworks necessary to conceptualize my study of teachers' beliefs about mathematics.

Defining Beliefs

Researchers interested in the study of beliefs must first give consideration to an adequate definition of beliefs. Pajares (1992) argued that researchers not only should provide a definition of beliefs but also consider how beliefs differ from related constructs. All too often, researchers do not define *beliefs* (Pajares, 1992; Philipp, 2007); and, at times, researchers use the terms *beliefs*, *affect*, *conceptions,* and *knowledge* as if these constructs were synonymous (Pajares, 1992). When researching beliefs, it is necessary to provide clarification. Therefore, to define beliefs, I begin by distinguishing between beliefs and these other constructs, and I discuss how these comparisons inform my definition of beliefs.

Researchers in mathematics education use *affect* as a general term to describe feelings (Charalambous, Panaoura, & Philippou, 2009; Goldin, 2002; McLeod 1988, 1992). Goldin (2002) outlined four dimensions that comprise the affective domain: emotions, attitudes, beliefs, and values. Distinctions among these dimensions can be rather subtle. For example, beliefs are more stable than emotions or attitudes; and, comparatively speaking, beliefs are more amenable to change or modification than values (Charalambous, Panaoura, & Philippou, 2009; Goldin, 2002). In his review of the literature, Phillip (2007) found that most research on beliefs did not attend to the role of affect or situate beliefs within the affective domain. However, some characteristics transcend the dimensions of the affective domain and thereby inform the way I think about beliefs. First, the affective domain is complex. Therefore, beliefs are complex. Second, interactions with others and cultural systems influence these dimensions. As I studied teachers' beliefs, I considered how interactions with their surroundings might have influenced their beliefs. Third, Goldin argued, "affect itself has a *representational* function…[and is] represented in and projected by the individual" (p. 60). To me, Goldin's claim implied internally held beliefs can be inferred from what an individual says and does. These characteristics constitute the beginnings of a foundation for which I build a definition of beliefs.

Some researchers have used the term *conception* as a broad term encompassing constructs such as beliefs, meanings, concepts, propositions, rules, mental images, and preferences (Phillip, 2007, p. 259; Thompson, 1992, p. 130). These researchers have characterized beliefs as a subset of conceptions. Similarly, Thompson (1984; 1992) used the terms conceptions and beliefs almost interchangeably, and she argued that the distinction between the two constructs "may not be a terribly important one" (1992, p. 130). However, other researchers, such as Pehkonen (2004), carefully defined conceptions to mean one's conscious or professed beliefs. As a researcher, I disagree with Thompson, because it is necessary to distinguish between these two constructs for the purpose of clarity. Therefore, I align my perspective with Pehkonen's description that recognizes conceptions as "the cognitive component of beliefs" (p. 3), implying one consciously holds conceptions and thereby is able to think and reflect on them. From this perspective, I view conceptions as a subset of one's beliefs. For my research, this clarification was an important one. I looked beyond what teachers said in interviews or indicated on beliefs surveys (i.e., their conceptions or professed beliefs), because I also considered their actions during classroom observations to infer their beliefs about mathematics.

Thompson (1992) contrasted beliefs with knowledge. First, she discussed how individuals hold beliefs with varying levels of intensity and confidence. This aspect of beliefs is understood when contrasted with knowledge, for “one would not say that one knew a fact strongly” (Abelson, 1979, p. 360). Second, Thompson recognized that beliefs are not universally shared, implying it is possible for others to believe differently. In comparison, knowledge is held with a level of certainty or is what Confrey (2000) described as a “justified belief” (as cited in Goldin, 2002, p. 65). These distinctions between beliefs and other similar constructs are an important step in beginning to develop a definition of beliefs.

Pajares’ (1992) review of the literature found that the construct of beliefs varies across authors and research purposes, typically lacking precision in definition, and thus results in a rather ill defined and “messy” construct. More recent literature reviews on mathematics teachers’ beliefs (Furinghetti & Pehkonen, 2002; Phillip, 2007) demonstrated the lack of consensus for a suitable definition of beliefs. In fact, Törner (2002) shared Eisenhart, Shrum, Harding, and Cuthbert’s (1988) assessment that emphasized the “definitional confusion among researchers” (p. 52). For this reason, I provide a definition of beliefs. On an intuitive level, I think of beliefs as the lens one uses to interpret the world. On an analytic level, I follow Rokeach’s (1968) definition of beliefs as “any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, ‘I believe that…’” (p. 113). In defining beliefs, I emphasize the concept that beliefs influence actions. My understanding of beliefs aligns with Rokeach’s position that “All beliefs are predispositions to action” (p. 113).

Although beliefs may not be explicitly held, it is possible to infer beliefs through a careful and detailed analysis of an individual's descriptions and actions. Therefore, using a series of interviews and observations for this study, I analyzed teachers' descriptions along with their actions to develop a more holistic understanding of their beliefs.

Sensible Systems of Beliefs

After defining what I mean by belief, I next examine how beliefs may be related to other beliefs. In *The Activities of Teaching,* Green (1971) began his chapter on beliefs by claiming that beliefs cannot be held independently, isolated from other beliefs. This description implies that beliefs develop and exist in groups or sets, residing in belief systems. As a researcher, I make the assumption that the entire set of an individual's beliefs forms a sensible system. To support my assumption, I draw from Leatham's (2006) theoretical framework of sensible systems of beliefs.

Leatham (2006) developed a framework for sensible systems of beliefs as a response and critique of past research on teachers' beliefs. Leatham argued against the positivistic approach used by researchers in the past that assumed "teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean" (p. 91). Leatham continued by discussing how past research identified inconsistencies among teachers' professed beliefs and their actions. Therefore, in his framework, Leatham assumed individuals hold beliefs in ways that make sense to them. He developed this framework for belief systems by considering Thagard's (2000) coherence theory of justification, "To justify a belief …we do not have to build up from an indubitable foundation; rather, we merely have to adjust our whole set of beliefs … until we reach a coherent state" (p. 5). This theoretical lens suggests that for a belief to exist within a system it must make sense, given the other surrounding beliefs within the system. Using Leatham's theoretical framework, I argue that either descriptions or actions alone do not provide adequate evidence to make inferences of beliefs. Rather, when a teacher's descriptions or actions seemed to contradict my inferences of his or her beliefs, as the researcher, I looked deeper to develop a better understanding of how a particular belief makes sense within a given system. To further consider properties that characterize a sensible system, I draw on Green's (1971) metaphor for how beliefs are organized and held with respect to Leatham's framework.

Structure of Belief Systems

Green (1971) provided a comprehensive commentary on the structure and organization of belief systems. The complexities inherent to belief systems are illuminated by Green's metaphorical interpretation of the structure and organization of beliefs. Green's work contributed three dimensions to understanding the structure of belief systems: the connections and relationships between beliefs, the tenacity of certain beliefs, and the ways beliefs are grouped in clusters. In the following paragraphs, I elaborate on the dimensions that form Green's theoretical perspective on beliefs, and I describe how I integrate Green's perspective with Leatham's (2006) perspective to construct the theoretical framework that guided my research of teachers' beliefs.

First, Green (1971) described a "quasi-logical" structure existing between beliefs. Beliefs are either *primary* or *derivative*. Given two related beliefs, say belief A and belief B, there exists a logical relationship between these two beliefs, where A implies B. Structured in this way, he observed, "A is seen as the reason for B, and B, in turn, as the reason for some other belief, say C" (p. 44). If A implies B, then this relationship refers to how an individual holds or structures these beliefs. Therefore, within this system, A is the primary belief and B is the derivative belief. Green viewed the quasi-logical structure as dynamic, allowing relationships among beliefs to be modified as additional beliefs are accepted within the structure. As I sought to understand each teacher's beliefs, I considered the quasi-logical relationships existing between beliefs as a method to understand and to organize how these beliefs may be viewed as sensible within the system.

Second, Green (1971) described relationships between beliefs with respect to spatial order or their psychological strength (p. 47). Along this dimension, beliefs are either *central* or *peripheral*. He characterized central beliefs as the beliefs held most strongly whereas peripheral beliefs are held with less intensity and are more likely to change. To further his description, he imagined a group of concentric circles with varying radii. The innermost circle represents the most strongly held beliefs, where such central beliefs are nearly impossible to change. The beliefs in the outer circles represent peripheral beliefs, and these beliefs more amenable to change and modification. Developing these spatial relationships was an additional means of making reasonable inferences about a teacher's beliefs. This process allowed me to consider how beliefs were structured and organized in a way that emphasized coherence within the system.

Third, Green (1971) maintained that beliefs are held in clusters, potentially isolated from other clusters of beliefs. This dimension allows beliefs to be seen as contradictory to a researcher (or observer), whereas they are not seen as contradictory to the individual holding these beliefs. This dimension allows for beliefs to be clustered by a given context, where an individual may believe one thing in a particular context and the opposite in a different context. Since beliefs may not be held explicitly, seemingly contradictory beliefs may reside within different clusters of beliefs based on a specific context to differentiate between the clusters of beliefs.

Using Leatham's (2006) framework, I assume that sensible systems of beliefs do not allow for explicit contradictions. Leatham's framework provides additional insight into the third dimension of Green's (1971) metaphor. In my opinion, a possible implicit conflict between beliefs can and will remain as long as the opposing beliefs reside within different clusters undisturbed. However, when beliefs that may be considered to be contradictory become evident in an explicit way to the individual holding these beliefs, the individual then must address and amend this conflict within the system. This process results in the individual maintaining a sensible system of beliefs. It is within these perceived contradictions, that as a researcher, I reexamined the teacher's descriptions and actions in an attempt to learn more about how the teacher's beliefs were structured in a sensible way.

Research on beliefs requires inference, because beliefs may not be explicitly or consciously held. These theoretical perspectives provided a useful framework for this study, because they required careful attention to the complexity inherent in understanding a teacher's sensible system of beliefs. As the researcher, I recognized that it was not always be possible for me to understand how a teacher's beliefs were sensible; however, these frameworks encouraged me to look beyond the past trends in research that highlighted inconsistencies between a teacher's professed beliefs and actions. Using these frameworks to support the design and implementation of my study, I viewed what I perceived as an inconsistency to be a metaphorical red flag, signifying a need for further exploration of what the teacher said and did before making inferences about his or her beliefs. In the following section, I consider how teachers' beliefs relate to their practice.

Philosophical Views of Mathematics

A philosophy of mathematics accounts for the nature of mathematics. For centuries, philosophers have developed a number of philosophical views of mathematics. Dossey (1992) observed how pervasive some of these philosophical views are in society, and he commented on the profound influence some of these views have on the teaching of mathematics. Similarly, a central thesis of Ernest's writings suggests teachers' philosophical views of mathematics "have a powerful, almost determining impact on mathematical pedagogy" (1991, p. 137). Given these claims, examining teachers' philosophical views of, or beliefs about, mathematics seems to provide a useful orientation when conducting research on beliefs and practice.

In *The Impact of Beliefs on the Teaching of Mathematics*, Ernest (1989) described three philosophical views of mathematics: the instrumentalist view, the Platonist view, and the problem solving view. He focused on these particular views because of their significance as philosophical perspectives and because they have been documented in the mathematics education literature through studies on teaching (e.g., Thompson, 1984). Ernest claimed that a teacher's beliefs about mathematics provided the basis for the teacher's philosophical views of mathematics, even though "these views may not have been elaborated into fully articulated philosophies" (p. 250). Considering Ernest's claim, I recognize that a teacher's philosophy of mathematics, whether implicit or explicit, is the way the teacher views mathematics, and thereby a teacher's personal philosophy of mathematics constitutes the teacher's beliefs about mathematics. Because these views seem particularly relevant to the beliefs teachers may hold about mathematics, I used these views to discuss the various beliefs held by the teachers in this study. In the remainder of this section, I describe each philosophical view, along with similar philosophical views, and I discuss how each view may influence the teaching and learning of school mathematics.

Ernest (1989) described the *instrumentalist view* of mathematics as one that perceives mathematics to be "an accumulation of facts, rules and skills to be used in the pursuance of some external end. Thus, mathematics is a set of unrelated but utilitarian rules and facts" (p. 250). This view recognizes mathematics as a set of useful, yet unrelated tools. Thompson (1992) considered this view of mathematics to be similar to Skemp's (1976) descriptions of an instrumental understanding and knowledge of mathematics. In school mathematics, teachers holding an instrumentalist view of mathematics often act as instructors, emphasizing the mastery of skills and correctness of procedures. Teachers present mathematics as a fixed set of plans for evaluating and solving problems and exercises. Each plan is prescriptive with a detailed list of instructions. Instrumental mathematical knowledge results in a kind of learning in which learners use an "increased number of fixed plans by which [learners] can find their way from particular starting points to required finishing points" (Skemp, 1976, p. 14). In her review of the literature on mathematics teachers' beliefs, Thompson (1992) claimed this view of mathematics is a common view held by mathematics teachers.

The *Platonist view* of mathematics characterizes mathematical knowledge as a "static but unified body of certain knowledge" (Ernest, 1989, p. 250). Viewed in this way, mathematics is a fixed and coherent body of objective knowledge, existing outside of time and space. Platonists believe mathematics is discovered rather than created. Thompson (1992) suggested the Platonist view was similar to the absolutist view of mathematics, where mathematics is characterized by truth and is absolutely certain (Ernest, 1991, 2008). In a mathematics classroom, this view is depicted through the teacher explaining mathematical concepts and ideas, providing a unified description of mathematics. Ernest described students' learning as students receiving mathematics knowledge from their teacher. Hersh (1997) regarded this view as the dominant philosophy of mathematics.

Ernest (1989) described the *problem solving view* of mathematics as a "dynamic, continually expanding field of human creation and invention" (p. 250). From this view, mathematics is a "dynamically organized structure," where mathematical ideas are developed and refined as result of historical progress and cultural influence (p. 250). Comparable philosophies of mathematics, such as Hersh's (1997) humanist view and Ernest's (2008) fallibilist view, emphasize a similar humanistic view of the genesis of mathematical knowledge. These similar philosophical perspectives all value the human construction of mathematics. In the classroom, mathematical learning develops through the active construction of mathematics, through problem posing and solving. Teachers facilitate students' learning, allowing students to construct their own mathematical knowledge by making connections across representations and problems.

Ernest (1989) referred to the connected nature of mathematics in each of the philosophical views he described. In particular, how mathematics is or is not connected seemed to follow from his overall description of how the particular philosophical view represents the nature of mathematics. The instrumentalist considers mathematical facts and rules to be unrelated, implying mathematics is not a connected discipline. In contrast, the Platonist recognizes mathematics as a "unified body," and the problem solver describes mathematics as a "dynamically organized structure" (p. 250). Both views acknowledge mathematics as a connected discipline. Therefore, a teacher's beliefs about how mathematics is connected seems to be a part of the teacher's beliefs about mathematics, which holds implications for the teacher's practice.

Framing Teachers' Beliefs and Practice

Research on teachers' beliefs is founded on an assumption that "beliefs are the best indicators of the decisions individuals make throughout their lives" (Pajares, 1992, p. 307). Research on mathematics teachers' beliefs has indicated strong and rather complex relationships between mathematics teachers' beliefs about the nature of mathematics and the decisions they make in the classroom (e.g., Cooney, 1985; Raymond, 1997; Thompson 1982, 1984). Drawing on past research and theory, I assume mathematics teachers' beliefs about mathematics influence their teaching practice. Therefore, in the following section, I review and critique Ernest's (2008) model of simplified relations, because this model provides a theoretical perspective that directly informs the purpose of my study. Then, I broadly interpret Ernest's model to consider how teachers' beliefs about mathematics may influence the kinds of mathematical connections teachers emphasize in their instruction.

In *Epistemology Plus Values Equals Classroom Image of Mathematics*, Ernest (2008) proposed a theoretical model of simplified relations to support his conjecture that a specific relationship exists among teachers' philosophies of mathematics, the corresponding values applied to mathematical knowledge as a result of the teachers' philosophies of mathematics, and the resulting image of school mathematics portrayed in the classroom (see Figure 1). Since the focus of my research is on mathematics teachers, I discuss only the first three levels within Ernest's model because of their particular relevance to my study (see Appendix A for Ernest's full model of simplified relations). Within the first three levels of this model, Ernest conjectured that an absolutist philosophy of mathematics could lead to applying separated values to mathematics. He then theorized that a teacher with this philosophical position would most likely teach in such a way that develops a separated image of mathematics in the classroom. In a parallel fashion, Ernest presented a similar relationship among a teacher's fallibilist philosophy of mathematics, which could lead to applying connected values to mathematics resulting in the development of a connected image of mathematics in the classroom.

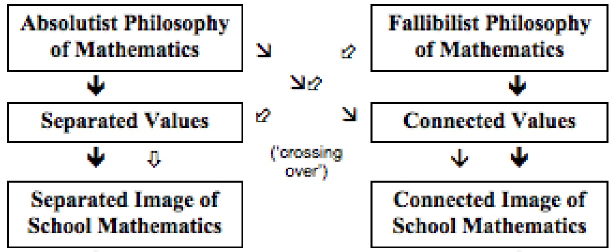

Figure 1. The First Three Levels of Ernest's (2008) Model for the Simplified Relations (p. 8). Copyright 2008 by the Philosophy of Mathematics Education Journal. Reprinted with permission.

I note that Ernest (2008) presented his model as a simplification. However, the layers within Ernest's model are complex and require adequate development to understand the relationship he suggested. Therefore, throughout the remainder of this section, I describe the relevant aspects of Ernest's model in detail and outline how his model influenced my research.

To construct the model, Ernest (2008) began by asking how teachers' philosophies and values may influence the image of mathematics developed within the classroom. To respond to this question, he outlined two dominating philosophical views[6] of the nature of mathematics. To discuss values in mathematics education, Ernest integrated Bishop's (1999) description of values and Gilligan's (1982) theory of *separated* or *connected* values. Ernest did not explicitly define what he meant by values; therefore, I looked to Bishop's description of values in mathematics education to provide additional clarity as I interpreted Ernest's model. Bishop defined values as "deep affective qualities" (p. 2) that are "likely to underpin teachers' preferred decisions and actions" (p. 2). Although there may be subtle nuances between Bishop's definition of values and my definition of beliefs, for my study, I interpret the values described in this model to be what Green (1971) defined as central beliefs and what Rokeach (1968) described as "predispositions to action" (p. 113).

[6] Thompson (1992) noted the parallelism between the "absolutist and fallibilist views and Ernest's Platonic (Platonist) and problem-solving views is readily observable" (p. 132).

Ernest (2008) described the *absolutist* philosophy as a belief system where mathematics "is the one and perhaps the only realm of certain, unquestionable and objective knowledge" (p. 3). Ernest's description of the absolutist philosophy emphasized separated values of mathematics, where mathematics is valued as a set of rules, abstractions, objectifications, and generalizations (p. 4). Given this philosophy and these corresponding values, the role of human innovation and problem solving in mathematics is significantly undervalued. In a mathematics classroom, this separated image of mathematics is depicted through students working unconnected mathematical tasks.

In contrast to the absolutist philosophy, Ernest (2008) described the *fallibilist* philosophy of mathematics by saying, "mathematical truth is corrigible, and can never be regarded as being above revision and correction" (p. 3). The fallibilist philosophy considers mathematics to be the outcome of social process and refinement. Connected values follow from the fallibilist philosophy, where mathematics is valued through emphasizing mathematical relationships, connections, processes, holism, and human-centeredness (p. 4). These values underscore mathematics as a connected body of knowledge. In the mathematics classroom, the connected image of mathematics is represented when students make connections as they work on mathematical problems that allow them to construct their own mathematical knowledge.

In this model of simplified relations, Ernest (2008) suggested possible relationships among teachers' philosophical views, corresponding values, and the resulting image of mathematics. In Figure 1, Ernest used bold vertical arrows to demonstrate what he considered to be the most direct and likely relationship. However, Ernest did recognize that such a simple dichotomy is not necessarily realistic. Ernest reflected on his model and remarked,

> Developing and applying such a model reveals layer upon layer of additional complexities that are factored out by the simplifications involved. …Needless to say

> there are many more variations of personal epistemologies and sets of values than this simple dichotomisation reveals. In addition, such interactions do not take place in isolation, but in social contexts, and these add many further layers of complexity. (p. 1)

To account for some of the complexities, Ernest speculated that such a direct relationship may not be the only possible relationship and allowed for the possibility of “crossing over.” The arrows crossing to either side of the model signify this possibility. Ernest provided an example to highlight what he meant by crossing over. He suggested that it was conceivable for a teacher to take an absolutist view of mathematics while holding connected values of school mathematics, and Ernest believed that this combination may result in a connected image of mathematics developed within the classroom. He believed this relationship was possible because various reform movements in mathematics education emphasize the importance of making mathematical connections in school mathematics.

For the purpose of this study, I used Ernest’s model merely as a guide, because it is difficult for me to conceive that the relationship between a teacher’s beliefs and teaching practice could be characterized in such a simplistic manner. For example, I found the two philosophies he provided to be too limiting to characterize the various beliefs teachers hold about mathematics; instead, I used the philosophical views outlined in Ernest’s theoretical paper from 1989. However, Ernest’s model suggests that a relationship exists, and I used this model to support my belief that the mathematical connections developed by a teacher are related, at least in part, to the teacher’s beliefs about mathematics. Guided by Ernest’s model, I examined teachers’ beliefs about mathematics to begin to understand the kinds of mathematical connections teachers made in practice.

These various theoretical and conceptual perspectives formed the foundation of my study. This combined perspective influenced my research design and methods, and it continually informed the ways in which I collected and analyzed my data. In the next chapter, I describe the research design and methods I used to address my research questions.

CHAPTER 3

RESEARCH DESIGN AND METHODS

Maxwell (2005) outlined research goals that can be achieved through qualitative research methods. Among these goals, three seemed particularly relevant to the demands of my research questions: (a) understanding the *meaning,* for participants in the study, of the events, situations, experiences and actions they are involved with or engage in; (b) understanding the particular *context* within which participants act, and the influence that this context has on their actions; and (c) understanding the *process* by which events and actions take place (p. 22). To understand the subtle ways teachers' beliefs influenced the kinds of mathematical connections they made, I required a qualitative approach to collecting and analyzing data for this study. Qualitative methods allowed me to study the problems posed by my research questions with "depth and detail" (Patton, 2002, p. 14). I used a multiple-case study design (Merriam, 1998), to address the following research questions:

1. What kinds of mathematical connections do three secondary mathematics teachers make in their teaching practice?
2. What are the beliefs about mathematics of these secondary mathematics teachers?
3. What relationships, if any, exist between the kinds of mathematical connections that these secondary mathematics teachers make and their beliefs about mathematics?

A multiple-case study design allowed me to examine the "particularity and complexity" of each case (Stake, 1995, p. xi). Miles and Huberman (1994) described the purpose of such a design,

> By looking at the range of similar and contrasting cases, we can understand a single-case finding, grounding it by specifying *how* and *where* and, if possible, *why* it carries on as it does. We can strengthen the precision, the validity, and the stability of the findings. (p. 29)

For my dissertation research, I constructed three individual case studies that are both *descriptive* and *interpretive* by nature (Merriam, 1998). After I developed the individual cases, I conducted a cross-case analysis. This analysis allowed me to suggest various naturalistic[7] generalizations as I described and interpreted my findings across the cases. In the following sections, I describe my perspective as a researcher, the selection of my participants, and the various methods that helped me address my research questions.

Researcher Perspective

Patton (2002) encouraged qualitative researchers to consider the ways the researcher's perspective influences the research, because "in qualitative inquiry, the researcher is the instrument" (p. 14). Understanding the perspective I brought to my research was important for both data collection and analysis, because I conducted classroom observations, asked interview questions, and interpreted data. Therefore, it was necessary for me to consider my beliefs related to mathematics as well as how my beliefs may have influenced my research.

As both a student and a teacher of mathematics, I adore the beauty inherent to the structure of mathematics. I believe mathematics is a connected body of knowledge. To me, mathematics is like a quilt, intricately and beautifully woven together by the relationships existing between mathematical concepts and procedures. Mathematician and Field's Medalist W. P. Thurston (as cited in Romberg & Kaput, 1999, p. 5) described mathematics in a similar way, using the following metaphor to describe the nature of mathematics:

> Mathematics isn't a palm tree, with a single long straight trunk covered with scratchy formulas. It's a banyan tree, with many interconnected trunks and branches–a banyan tree that has grown to the size of a forest, inviting us to climb and explore.

[7] Patton (2002) described naturalistic inquiry as research that observes the phenomena as it "unfolds naturally" without an attempt to change or modify the phenomena (p. 39).

At the beginning of my teaching career, Thurston’s metaphor became more apparent to me, because I recognized mathematics as an environment for my students to explore. I realized the importance of encouraging students to develop problem-posing and problem-solving skills, and my students taught me that mathematical study is dynamic. As a teacher, I made mathematical connections across various concepts, representations, and methods to help my students gain a meaningful understanding of mathematics.

As a researcher, I acknowledge that my beliefs about mathematics influenced the way I conceptualized, thought about, and continue to think about my research. My personal reflections about mathematics remind me of the complicated nature of my research topic, and I recognize the care to be taken when making inferences about a teacher’s beliefs. It is possible that my beliefs acted as a lens for the mathematical connections I noticed and the teachers’ beliefs I inferred, and I realize that I cannot fully separate my personal beliefs about mathematics from my research. For these reasons, throughout this study, I continually asked questions of my data and my findings. I relied on my theoretical framework during my data collection and analysis, and I repeatedly considered alternative hypotheses as I interpreted my data. I was aware of the perspectives of others, and I was skeptical of the perspective I brought to my study. In addition, I asked each of my participants to review and respond to my interpretation of his or her beliefs and practice. Furthermore, I regularly met with my major professor to have a person with an additional perspective review and question my research findings.

Selection of Participants

Variation across cases is one of the strengths of a multiple-case design, “for the greater the variation across the cases, the more compelling an interpretation is likely to be” (Merriam, 1998, p. 40). Following Merriam’s suggestion, I recognized that the purposeful selection of teachers was of vital importance to my study. In addition, Stake (1995) recommended that researchers should purposefully select cases to maximize the opportunity to learn. Therefore, given the purpose of my study, I selected three secondary mathematics teachers who made a variety of mathematical connections in their teaching. Because of this purposeful selection, I was able to identify variation among the kinds of mathematical connections made and beliefs held, thereby allowing for some interesting comparisons to be made in my analysis.

To begin this selection process, in the summer of 2011, I contacted the mathematics specialist for the Northeast Georgia Regional Education Service Agency and mathematics educators at the University of Georgia to identify potential participants. I specifically requested recommendations of secondary mathematics teachers who regularly made mathematical connections in their teaching. I received the names of 14 secondary mathematics teachers. I contacted each of the recommended teachers via email and asked if they were interested in participating in my dissertation research. Nine teachers responded and were willing to participate.

I continued the selection process by asking each of the nine teachers to complete the questionnaire provided in Appendix B. The questionnaire began with questions about the teacher's educational background. The final items in the questionnaire posed classroom scenarios, which required the teacher to respond in such a way that provided me with indirect access to the kinds of mathematical connections he or she may make in the classroom. I adapted these final items from a survey Wood (1993) used to identify teachers' conceptions of mathematical connections. I analyzed the teachers' responses to the classroom scenarios, looking for a variety of mathematical connections within each of the teachers' responses and across the teachers' responses. From this analysis, I narrowed the pool of possible participants to six secondary mathematics teachers.

In the fall of 2011, I conducted a full day of classroom observations in each of the six teachers' classrooms. During these observations, I attended to the kinds of mathematical connections the teacher did or did not make. This process informed my final selection of participants in two ways. First, this process helped me to select teachers who regularly make mathematical connections in their teaching. Second, I selected teachers based on the variation in the kinds of mathematical connections made during my observations. Drawing on the broad descriptions of mathematical connections in the literature, I used maximum variation (Patton, 2002) to purposefully select three secondary mathematics teachers. I chose these teachers based on my assumption that the opportunity to learn would be the greatest when variation existed among the kinds of mathematical connections made. For the three secondary mathematics teachers I invited to participate in my dissertation research, I used both the questionnaire and the initial classroom observations as additional sources of data.

I purposefully selected Rachel McAllister, Justin Smith, and Robert Boyd[8] to participate in this study. Each teacher had 10 or 11 years of teaching experience at the time of my study. In addition, each of the teachers taught integrated mathematics courses[9] on a 4×4 block schedule. However, they taught different mathematics classes in different public high schools in different school districts in northeastern Georgia. Although there were the differences in both the content

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[8] The names of the teachers and schools are pseudonyms.

[9] Beginning in 2005, the Georgia Department of Education adopted and gradually implemented the Georgia Performance Standards, a set of integrated mathematics curriculum standards. As a result, each mathematics course in the high school curriculum includes units on algebra, geometry, and statistics (Georgia Department of Education, 2006).

and the context of their teaching, each teacher made a variety of mathematical connections in the questionnaire and during my initial observation.

Rachel McAllister taught at Lincoln High, a small rural public high school. She said she enjoyed teaching, because she “goes to school” instead of work every morning (Interview 1). She earned a bachelor’s of science degree in mathematics education from the University of Georgia (UGA), and she was pursuing an educational doctorate in leadership and administration from a private institution. In the fall of 2011, she taught two classes of Accelerated Mathematics II and a Mathematics II Support class.

Justin Smith taught mathematics at Walker High, a Title I school situated just outside the suburbs of Atlanta. In college, Justin began studying computer science, until he realized how much he enjoyed working with people. He switched majors and obtained a bachelor’s of science degree in mathematics education from UGA and a master’s degree in mathematics education from Piedmont College. In the fall 2011, Justin taught two sections of Mathematics IV and one section of Accelerated Mathematics III.

Robert Boyd was one of six mathematics teachers at Parker City High. He began his teaching career as a middle school teacher, teaching classes in mathematics and English. He had a bachelor’s of science degree in middle childhood education from Georgia State University and a master’s of science degree in secondary mathematics education from Piedmont College. During the time of my observations, Robert taught two sections of Mathematics II and one section of Advanced Placement Statistics.

I selected Rachel, Justin, and Robert to participate in my dissertation research because of the diversity in kinds of mathematical connections they made during my initial observations. Rachel, for example, asked her students to consider different methods to solve for the zeros of a function. Students presented their methods to the class, and Rachel made connections as she compared the individual methods. Justin used dynamic geometry software to create a unit circle and to explore the reason why tangent is undefined at $\pi/2$. In this demonstration, he connected the reason tangent is undefined at $\pi/2$ to dividing by 0. During this lesson, he also made several connections to the real world. I observed Robert teach a lesson on the characteristics of quadratic functions. Throughout this lesson, he made connections by comparing the characteristics of quadratic functions to characteristics of linear functions. In the following section, I describe the various data sources I collected for each case study.

Data Collection

The strength of a case study is located within the researcher's ability to collect multiple sources of data as evidence of claims (Yin, 2009). In addition, Leatham (2006) described the importance of using multiple data sources when studying a person's beliefs.

> In order to infer a person's beliefs with any degree of believability, one needs numerous and varied resources from which to draw those inferences. You cannot merely ask someone what their beliefs are (or whether they have changed) and expect them to know or know how to articulate the answers. (p. 93)

Therefore, I collected multiple sources of data to develop rich case studies to address my research questions. For each case, my primary data sources were in-depth, semi-structured interviews and classroom observations.

First Interview

I conducted an initial interview (see Appendix C) with each teacher prior to the classroom observations. The purpose of this initial interview was twofold. First, the interview provided me with an opportunity to begin to develop rapport with the teacher. Building a relationship founded on trust is an essential part of a well-constructed and ethical research design (Seidman, 2006). Second, the questions I asked in this interview provided me an opportunity to understand the teacher's background with respect to teaching and learning mathematics. This background provided a context for the participant's beliefs and practice. Seidman (2006) described the importance of an initial interview to gain access to the context surrounding the case, because "people's behavior becomes meaningful and understandable when placed in the context of their lives and the lives of those around them. Without context there is little possibility of exploring meaning" (p. 16). I audio recorded each interview, and each interview lasted approximately an hour to an hour and a half.

Curriculum Materials

The collection of documents contributes to the in-depth nature of case study research (Merriam, 1998; Stake, 1995). During the first interview, I asked each teacher to provide copies of the relevant curriculum materials over the course of my scheduled classroom observations. Relevant curriculum materials included selections from the course textbook, student worksheets, and mathematical tasks. I also reviewed the relevant curriculum frameworks and standards. To catalog these materials, I made copies of the relevant materials for my later analysis.

These materials provided some additional insights into the teacher's emphasis of certain mathematical connections during a given lesson. For example, one of the curriculum standards addressed during my observations of Rachel's teaching was standard MA2G2b, "Explain the relationship between the trigonometric ratios of complementary angles" (Georgia Department of Education, 2009, p. 3). In the lesson she taught to correspond with this standard, Rachel continually emphasized this standard as she helped her students understand, "the cosine of angle is equal to the sine of its complement," which resulted in related mathematical connections (Observation, September 22). Understanding the possible influence of the curriculum materials on each teacher's practice helped me understand and make sense of how additional contextual factors, beyond the teacher's beliefs about mathematics, may be related to the kinds of mathematical connections made.

Classroom Observations

I wanted to observe each instructor teaching an entire mathematics unit. For this reason, I planned to conduct approximately two weeks of consecutive classroom observations in one mathematics class for each teacher. In addition, this observation period provided a significant window of time to understand the teacher's practice and the kinds of mathematical connections made.

I purposefully selected the unit I observed in each teacher's classroom. I made my selection based on the presence of unifying themes within the unit, because unifying themes can be used to unite and connect multiple topics across the school mathematics curriculum (Coxford, 1995; NCTM 2006). I chose to conduct my observations of units that included, at least at times, the unifying theme of functions. The content in the units I selected seemed to provide the teachers with opportunities to make multiple mathematical connections, which, in turn, afforded me with possibility to observe and document multiple kinds of mathematical connections made in practice.

In September, I began my observations in Rachel's classroom. I observed Rachel for 5 days[10] as she taught a unit on right triangles and trigonometric functions in her Accelerated Mathematics II class. Then, in October, I conducted 11 observations in Justin's Accelerated Mathematics III class. During this time, he taught his students a unit with a focus on parametric

[10] Rachel planned on teaching this unit over the course of 8 class periods. However, due to personal reasons, she was absent for a few days in the middle of the unit. Therefore, students learned about 45°-45°-90° triangles with a substitute teacher.

functions. I conducted my final round of observations in November. I observed Robert for 11 day as he taught his Mathematics II students a unit on piecewise, step, and exponential functions.

During each observation, I videotaped the teacher. I focused the video camera on the teacher at all times, because I was interested in the kinds of mathematical connections the teacher made. To collect useful video data, I asked the teacher to wear a wireless microphone to clearly capture what the he or she said on the video recording. At the end of the lesson, I captured images of any manipulatives used by the teacher. After each observation, I jotted brief notes in my research journal. I focused my journal reflections on the kinds of mathematical connections I noticed, beliefs I inferred from the teacher's practice, and conjectures I developed.

For each teacher, I conducted additional observations later in the semester. It seemed necessary to observe whether or not the kinds of mathematical connections made were dependent upon the mathematical content of the unit. Therefore, I chose to observe distinctly different mathematical content in these additional observations. I observed Rachel as she taught six lessons[11] on measures of variability, Justin as he taught three lessons on the laws of sines and cosines, and Robert as he taught two lessons on linear regression. These additional observations helped me make sense of the kinds of mathematical connections the teacher made, the teacher's beliefs about mathematics, and the possible relationships existing between the teacher's beliefs and practice.

Second Interview

The purpose of the second interview was to begin to explore the teacher's beliefs about mathematics. This interview was both in-depth and semi-structured, and I conducted this interview during the latter portion of the observation cycle. The purpose in waiting to conduct

[11] I chose to conduct more additional observations in Rachel's classroom, than in Robert's or Justin's, because of her previous absences in the original unit I observed.

this interview was to allow myself, as the researcher, time to become accustomed with the context of the classroom and to develop a list of conjectures about the teacher's beliefs about mathematics based on my observations of the teacher's practice. I believe conducting this interview during the latter portion of the observation cycle helped me avoid, at least initially, the possible influence of the teacher's descriptions on the beliefs I inferred from my observations of practice. Because the teachers' beliefs may not be explicitly held, in this interview, I asked questions to elicit detailed descriptions of the teacher's practice to infer the teacher's beliefs about mathematics (see Appendix D for the interview protocol).

Third Interview

The third in-depth, semi-structured interview continued to explore teachers' beliefs about mathematics. The interview began with a brief beliefs survey for the teacher to complete independently, adapted from Thompson's (1982) dissertation research on teachers' beliefs (see Appendix E for the survey). This beliefs survey included several dichotomous descriptions about mathematics, mathematics teaching, and mathematics learning, which were placed at opposite ends of a continuum. The survey directions asked the teacher to place an X somewhere on the continuum to adequately represent his or her belief about that particular characteristic of mathematics. After the teacher completed the beliefs survey, I asked the teacher to discuss the various selections on the survey as well as the meanings he or she applied to each of the survey items. For example, I asked Robert, "Between doubtful and certain, your [selection] is rather close to certain. Can you tell me about that?" (Interview 3).

The context of this survey provided an opportunity for the teacher to reflect on various descriptions of mathematics. Because many beliefs about mathematics are often implicitly held, the beliefs survey acted as an artifact to elicit detailed discussions of the teacher's beliefs. The beliefs survey was a useful research tool to uncover and explore such beliefs, because completing it caused the teacher to think about different characteristics of mathematics explicitly. Using a beliefs survey in an interview setting afforded me with the opportunity to better understand the various meanings, values, and beliefs the teacher ascribed to mathematics. I purposefully conducted the third interview after the observation cycle to minimize the possibility that this interview would influence the teacher's practice.

Fourth Interview

The fourth interview continued to explore the teacher's beliefs about mathematics. To begin this interview, I asked the teacher to read through and to complete a beliefs task, modified from Thompson's (1982) dissertation research on teachers' beliefs (see Appendix F for the task). The task posed statements about possible goals in teaching mathematics and asked the teacher to rank these goals. The teacher's ranked selections provided additional insight into what was valued most about mathematics and what was particularly important to teach his or her students about mathematics.

After the teacher completed the beliefs task, I asked questions to explore the teacher's reasoning behind his or her ranking of the different statements. For example, I asked Justin, "The statement that says, 'To provide the students with the opportunity to learn how to reason logically,' you gave a 5. Tell me more about that, Justin" (Interview, 4). In his response, Justin explained why he valued logical thinking over the other teaching goals provided in the task. As a result, the ranking required by the task provided an additional mechanism for me to make sense of the teacher's central beliefs (Green, 1971). I conducted the fourth interview after the conclusion of the observation cycle, and each interview lasted approximately an hour.

Fifth Interview

The purpose of the fifth in-depth interview was to understand the meanings the teacher applied to previous statements he or she made about mathematics. In this interview, I used a card sort technique (Cooney, 1985; Ryan & Bernard, 2000), a method used to capture what the teacher identified as important aspects of his or her beliefs about mathematics. To prepare for this interview, I reviewed the previous four interview transcripts and isolated statements that seemed to be related to the teacher's beliefs about mathematics. I placed each statement on an individual index card. Prior to the interview, I provided the teacher with his or her statements on the individual cards (each teacher received approximately 100 cards). I asked the teacher to read through each of the cards and to identify statements that captured important aspects of his or her beliefs about mathematics before we met for the scheduled interview.

During the interview, I asked the teacher to cluster the selected cards into categories of his or her choosing. Next, I asked the teacher to create a title for each cluster along with a brief description detailing what the cluster of statements seemed to express. After the teacher completed this card sort activity, I asked open-ended questions requesting additional descriptions about why the teacher placed specific statements within a given cluster and how the different statements seemed to be related within the given cluster. These questions helped me understand the reasoning the teacher used to select and organize the statements into the different clusters and informed my inferences about the quasi-logical relationships existing among the teacher's beliefs (Green, 1971). I also asked questions about the statements the teacher did not include in the card sort activity. The teacher's responses helped me infer which beliefs were more central beliefs and which beliefs were held as peripheral beliefs (Green, 1971).

Sixth Interview

I conducted the sixth and final interview in the spring of 2012. I conducted this interview after the previous interviews and classroom observations were fully transcribed and initially analyzed. I used this final interview as a kind of interactive member check. Therefore, I developed questions specifically tailored to each individual teacher, creating questions related to my inferences of the teacher's beliefs and the kinds of mathematical connections he or she made for and with the students (see Appendix G for the individualized interview protocols).

First, I began by asking the teacher to review excerpts of transcripts from my classroom observations. For each teacher, I chose five to six classroom episodes because I considered these episodes to be representative of the kinds of mathematical connections the teacher made. I used this technique to incorporate the teacher's perspective on the classroom data and to understand the meaning he or she applied to such a mathematical connection. In addition, this technique provided a unique space for the teacher's descriptions of practice to reflect aspects of his or her beliefs about mathematics, providing additional data to address my third research question.

Second, I included a concept-mapping activity in this interview. I provided the teacher with index cards individually labeled with what I inferred to be his or her central beliefs about mathematics. I asked the teacher to review each of the cards and to consider if any cards should be added or removed to adequately represent his or her central beliefs about mathematics. Then, I introduced smaller index cards labeled with what I inferred to be the related beliefs clustered around the teacher's central beliefs. Once again, I asked the teacher to review the cards and to add or remove any cards to represent his or her beliefs. To continue, I asked the teacher to consider how, if at all, his or her central beliefs about mathematics were related. I asked the teacher to describe the relationships existing among these beliefs. Through this process, the teacher created what Maxwell (2005) described as a concept map. Finally, I asked the teacher to review the model I developed to represent my inferences of the teacher's beliefs about mathematics and how the beliefs were held (my construction of the model is developed in the following section titled, Data Analysis). Each teacher commented on how very similar my model was to the concept map he or she created.

I used this concept-mapping activity to engage the teacher in thinking critically about his or her beliefs about mathematics. The teacher was able to reflect on his or her beliefs about mathematics before viewing the model of my inferences of the teacher's beliefs. Therefore, this activity gave the teacher a personal point of reference from to which to reflect on the model I created. I structured the concept-mapping activity in this way so that the teacher would have the opportunity to think through his or her beliefs without initially being biased by my interpretation of those beliefs.

Third, I concluded the interview by asking the teacher to develop a metaphor to describe how mathematics is connected. The process of creating a metaphor allowed the teacher to reflect on abstract ideas about the nature of mathematics and to apply meaningful images to describe this abstract concept. The teacher's metaphor provided valuable insights into how the teacher believed mathematics is connected.

This final interview afforded me with the opportunity to gain the teacher's perspective on my data and initial findings. Listening to the teacher's descriptions of practice, I gained a better understanding of how the teacher's beliefs about mathematics influenced the kinds of mathematical connections the teacher made. The comments provided during the concept-mapping activity and the teacher's metaphor of how mathematics is connected allowed me to refine my inferences of the teacher's beliefs about mathematics.

Order of Data Collected

I sequenced data collection (see Table 1) to minimize my influence as the researcher on the teacher's descriptions or actions. This structure allowed me to transcribe and initially analyze data before collecting new data. In the following section, I describe how I analyzed my data.

Table 1

Timeline of Data Collection for Each Teacher

Data Collection Strategy	Strategy Description	Timeline of Data Collection		
		Rachel	Justin	Robert
First Interview	Asked questions about teacher's background	September 12, 2011	September 21, 2011	September 23, 2011
Classroom Observations	Observed entire mathematics unit	September 14, 16, 19–21, 2011	October 7–25, 2011	October 31–November 14, 2011
Second Interview	Asked questions about mathematics	September 22, 2011	October 12, 2011	November 9, 2011
Third Interview	Asked questions about beliefs survey	October 10, 2011	October 26, 2011	November 15, 2011
Fourth Interview	Asked questions about beliefs task	November 5, 2011	October 27, 2011	November 16, 2011
Additional Classroom Observations	Observed lessons related to a different mathematical content	September 26–30, October 5, 2011	November 10–11, 14, 2011	November 28–29, 2011
Fifth Interview	Asked questions about card sort task	November 28, 2011	December 5, 2011	December 13, 2011
Sixth Interview	Interactive member check	March 6, 2012	March 5, 2012	March 12, 2012

Data Analysis

Merriam (1998) described data analysis as the activity of "making sense" of one's data (p. 178). She continued by saying that "data analysis is a complex process that involves moving back and forth between concrete bits of data and abstract concepts, between inductive and deductive reasoning, between description and interpretation" (p. 178). In this section, I describe how I made sense of my data by first using multiple rounds of coding to analyze my data and then by writing about my data. I begin by describing how I analyzed the data within the individual case studies and then discuss how I conducted the cross-case analysis.

A well-constructed case study includes a holistic description and a thorough analysis of the bounded unit (Merriam, 1998). My purpose in analyzing and developing individual case studies was to communicate an understanding of the case. I analyzed the entire body of data related to the individual case by constructing codes, categories, and themes that represented the patterns I noticed within the data. This method entailed coding and recoding data and occurred throughout the data collection and analysis process, and this method helped me develop categories and themes in response to my research questions.

My within-case analysis began with my data collection. For example, during the first interview, I made conscious analytic decisions about topics to explore and follow-up for further meaning. My theoretical perspective and my research questions guided the decisions I made in the interview setting. As I collected more data, the analysis process became more complex. I was able to make comparisons, first by comparing data within the case and then by considering comparisons across the cases. As I collected data through observations and interviews, I kept a research journal to capture these comparisons generated by the thoughts I had and conjectures I made throughout data collection and into my analysis (Roulston, 2010). This journal allowed me to make analytic memos detailing my rationale for the decisions I made during this process. During the time of data collection, I transcribed each interview prior to conducting the next interview as an additional analysis technique. This process provided me with an opportunity to reflect on what was said in the previous interview and helped me make decisions about the next steps in my data collection.

My analysis continued as I created a transcript for each classroom observation. I organized the classroom video data into fully transcribed episodes that focused on the mathematical segments of the classroom instruction. In the transcript, I included screen shots of the video from the observation. I also wrote analytic memos in the column of my transcript as a preliminary analysis of these classroom episodes.

The majority of my within-case analysis consisted of coding my data. I used the transcripts from the interviews and classroom observations as my primary sources of data for this analysis. First, I applied broad codes such as *mathematics* or *connections* to my data. I then read through the entirety of the coded data for each case to further develop my coding scheme. Merriam (1998) remarked that although this process of code construction seems rather intuitive, it is also heavily informed by one's research questions and theoretical perspective. This process of coding involved revising and reorganizing codes. In this process, I refined my initial broad codes and developed new, more refined, codes as I noticed themes within the data. Using the set of refined codes, I recoded my data. I found this process to be characterized by continual comparison, as I constantly compared units of data. I continued this coding process until it seemed that I had saturated the data with codes. I used HyperRESEARCH (ResearchWare, Inc., 2011), a qualitative data analysis software, to facilitate this coding process.

To make sense of the coded data, I wrote thematic narratives to address aspects of my research questions (Riessman, 2008). This writing process helped me to develop key themes for each case and to explain how these themes were possibly related. I relied on my theoretical perspective to help me construct these “theory-driven” narratives (Luker, 2008, p. 142). I developed narrative cases for each participant, one research question at a time. In the following sections, I describe in detail my analysis process for each research question.

Analysis of the Kinds of Mathematical Connections Teachers Made

To address my first research question, I developed the Mathematical Connections Framework (see Chapter 4 for a detailed description of the framework) to frame my discussion of the mathematical connections teacher made in practice. To develop this framework, I began by coding the classroom data using the broad code *connections*. I applied this code to a unit of data when a relationship seemed to occur in the lesson (i.e., *A* is related to *B*). Then, I examined all of the classroom data that I had coded as *connections* using the qualitative software. I arranged the data in several ways until I developed a more refined and coherent picture of the kinds of mathematical connections each teacher made. I noticed patterns existing within my data. First, I realized teachers made mathematical connections in more or less explicit ways for their students. As a result, I developed a set of refined codes to distinguish between the levels to which mathematical connections were explicitly made (see Table 2 on p. 60 for the definitions of the levels of mathematical connections made). Second, I noticed additional patterns emerging because there were different ways *A* could be related to *B*. Therefore, I developed a different set of refined codes to capture the various relationships existing between *A* and *B*. I collapsed the refined codes into meaningful categories to describe the kinds of mathematical connections teachers made in practice (see Table 3 on p. 66 for the categories of the kinds of mathematical connections). The Mathematical Connections Framework was the result of this analysis process, and this conceptual framework defines and describes the levels and the kinds of mathematical connections teachers made in practice.

To continue my analysis, I used the refined codes and categories from the Mathematical Connections Framework to recode the classroom data. I analyzed the recoded data and generated themes from each teacher's practice. Then, I wrote drafts of narrative cases to facilitate the generation and refinement of these themes. During this writing process, I incorporated significant pieces of data to justify these themes. Throughout this process, I continually questioned the data and carefully considered alternative hypotheses as I interpreted and wrote about my data.

Analysis of Teachers' Beliefs about Mathematics

I continued this coding process to understand each teacher's beliefs about mathematics. I began by coding units of interview data with the broad code *mathematics*. I used this code to capture pieces of data in which the teacher referred to mathematics. Sifting through the data coded as mathematics, I created a set of descriptive codes to describe the ways teachers talked about mathematics, such as "makes sense" or "the way the world works." Saldaña (2009) is among those methodologists who described this kind of coding as *in vivo* coding, because the researcher creates codes directly from the words used by the participant. I decided to use this coding technique to keep the teachers' words ever present in my analysis.

I used the Green's (1971) three dimensions of a beliefs system to make sense of the coded data. I used these dimensions to make inferences about the teacher's beliefs about mathematics and how they were held, determining whether the beliefs were central or peripheral, primary or derivative, and how the beliefs were clustered. I paid particular attention to the data provided by the third and fourth interviews, because these interviews required the teachers to select or rank descriptions about mathematics. Therefore, the teacher's selections and rankings provided an additional mechanism for me to make sense of the teacher's central beliefs or peripheral beliefs. I also looked for associations within and between the teacher's descriptions about mathematics. This process helped me make inferences about which beliefs were clustered around central or peripheral beliefs and which beliefs existed in a quasi-logical relationship.

Before writing about my data, I created an initial model of my inferences of each teacher's beliefs about mathematics using Green's (1971) three dimensions of a belief system. This model captured the teacher's central and peripheral beliefs about mathematics in bold font, where the size of the font reflected the "psychological strength" of the belief (p. 47). I used arrows to demonstrate the quasi-logical relationships I inferred existing between the teacher's beliefs. I included the teacher's words (which were also my *in vivo* codes) related to the central beliefs in the clusters surrounding the central belief.

I used this initial model as I continued to analyze the teacher's beliefs. I compared my initial model with the clusters the teacher created during the card sort task in the fifth interview. I noticed similarities existing between the central beliefs I inferred to the clusters the teacher created. I also compared the statements the teacher included within the cluster to my inferences of the beliefs grouped around a particular central belief. Following this comparison, I made slight adjustments to each of my initial models. For example, as I reviewed the cluster of statements Justin titled *Logic* in the card sort task, I noticed he included the following statement within this cluster, "I can see the connections, from one thing, to the next, to the next" (Interview 2). In my initial model of Justin's beliefs, I had not related this particular belief to Justin's central belief about logic. Therefore, I reanalyzed my data, and I modified Justin's initial model to include the belief "linear flow of concepts" clustered around his central belief of logic.

The modified model provided a visual synthesis of my inferences of the teacher's beliefs about mathematics. I used this modified model as a concise record of the beliefs I inferred from the teacher's descriptions to make comparisons with the teacher's practices. There were times when I perceived inconsistencies between the beliefs I inferred from the teacher's descriptions and the teacher's practice. I used these perceived inconsistencies to explore my data in more detail. Leatham (2006) explained this process of reexamining data when the researcher notices perceived inconsistencies existing between descriptions and actions.

> When a teacher acts in a way that seems inconsistent with the beliefs we have inferred, we look deeper, for we must have either misunderstood the implications of that belief, or some other belief took precedence in that particular situation. (p. 95)

This process led me to ask questions of my inferences and of my data. I reconsidered the context of the beliefs I inferred. I reexamined the definitions and meanings the teacher applied to different terms he or she used to describe mathematics, and I considered possible explanations for what I perceived to be an inconsistency. For example, in my analysis of Robert, I observed what I initially perceived to be an inconsistency. I used this perceived inconsistency to reexamine my data. Through this process, I realized Robert's beliefs differed, at times, depending upon whether he was talking about mathematics as a formal discipline or the mathematics his students studied. This process allowed me to make final modifications to the models I developed for each of the teachers (the final models are included in Chapter 5).

I used the final models as an outline to develop the narrative cases of each teacher's beliefs. Analysis continued as I wrote about the teacher's beliefs. I searched for sensible ways to characterize and construct the teacher's statements and actions related to the beliefs I inferred. Because of the complex nature of studying beliefs and the difficulty involved in understanding another's beliefs, I believe this entire analytic process was necessary for me to make reasonable inferences of the teachers' beliefs.

Analysis of Relationships Existing Between Teachers' Beliefs and Practice

To address my third and final research question, I looked across the data I coded as *connections* and the data I coded as *beliefs*. In addition, for each teacher, I outlined the themes I developed to describe the teacher's beliefs and the kinds of mathematical connections the teacher made. I used these themes to consider possible relationships existing between the teacher's beliefs and practices. After I identified possible relationships, I asked questions of my data and examined alternative explanations for the relationships I observed. I constructed narratives to help me refine my interpretations. This writing process required me to search for evidence to make reasonable claims about the relationships I perceived between the teacher's beliefs and practice.

I asked my major professor and other colleagues to review the narrative cases that I developed to make sense of each of my research questions. Their feedback was invaluable, because they asked questions of some of my assumptions and inferences. I used this feedback to clarify and further refine the descriptions I developed in each of my narrative cases.

Cross-Case Analysis

My cross-case analysis consisted of making comparisons among the cases, by looking for both commonalities and differences from case to case. Yin (2009) indicated that this level of analysis is only possible after the individual case studies have been developed. To begin my comparison, I carefully read the individual narrative cases (Stake, 2006). During this initial analysis, I followed Stake's analytic advice, "each [case] needs to be heard while the other is being analyzed” (p. 46). To facilitate this process of careful reading, I used a worksheet Stake developed to capture my thoughts while reading the individual case studies (see Appendix H).

For my next level of analysis, I used Yin’s (2009) technique of creating word tables to display the data and findings from the individual cases. Each word table provided a visual display that captured various themes related to a specific research question. For example, to create a word table that addressed my first research question, I developed a data display of the different kinds of connections made within each of the individual case studies. The visual juxtaposition of themes and findings related to my first research questions helped me begin to make comparisons across the cases. I used this visual representation as a method to look for overall patterns across the cases and to interpret my data. As I noticed patterns in the data, I used the same process of writing about my data to refine and support themes. I developed narratives to help me organize and communicate my findings with arguments supported by my data.

Trustworthiness of the Study

Freeman, de Marrais, Preissle, Roulston, and St. Pierre (2007) argued that the quality of a study is continuously constructed as the researcher interacts with the study. The authors emphasized this claim by saying, “concerns about the quality of their work are evident in discussions about formulating both research design and questions with explicit theoretical and philosophical traditions; accessing and entering settings; selecting, collecting, and analyzing data; and building a case for conclusion” (p. 27). Therefore, as I developed the design of my study, I carefully considered aspects related to the quality of my study. I situated my research problem within the literature and used the literature to guide the development of my research questions. As I began my fieldwork, I continued to consider issues related to the quality of my design to develop a trustworthy study.

Conducting a pilot case can be invaluable in developing a trustworthy study. Yin (2009) recommended using a pilot case study to refine both data collection techniques and analysis methods. Following this advice, in the spring of 2011, I conducted a pilot case study. My pilot work provided me with an opportunity to reflect on theoretical and practical elements of my study, and I revised various elements to ensure that I collected data for my dissertation that addressed each of my research questions. For example, as a result of my pilot work, I added the final three interviews to collect additional data to infer beliefs.

Pajares (1992) cautioned researchers that “beliefs cannot be directly observed or measured but must be inferred from what people say, intend, and do—fundamental prerequisites that educational researchers have seldom followed” (p. 314). Therefore, I incorporated the use of multiple data sources to answer each research question. This method of data triangulation (Patton, 2002) required multiple sources of data to support and validate findings. For example, to answer my second research question, I inferred teachers’ beliefs about mathematics from their descriptions and actions, using a variety of interview techniques to examine the teacher’s beliefs. In addition, each teacher participated in a member checking activity by reviewing classroom episodes and creating a concept map in the sixth and final interview. I included the data from this interview in the corpus of the data, and I used this data in my analysis and the refinement of my findings. Incorporating teachers’ responses from member checking as an additional data source helped to support my inferences and substantiate my findings; I believe this method was a significant aspect of constructing a trustworthy study.

Organization of Findings

My research findings are presented in the following two chapters. Chapter 4 describes the kinds of mathematical connections the teachers made in their teaching practice (Research

Question 1). Chapter 5 details my inferences of each teacher’s beliefs about mathematics (Research Question 2) and the possible relationships existing between the teacher’s beliefs and the kinds of mathematical connections he or she made in practice (Research Question 3). It was important to introduce the teachers through their teaching practice first in Chapter 4, because this perspective is necessary to understand my inferences of the teacher’s beliefs in Chapter 5. In each of these chapters, I present narrative cases of each teacher, and then I provide a comparison across the cases.

CHAPTER 4

MATHEMATICAL CONNECTIONS MADE IN PRACTICE

The purpose of this chapter is to describe the mathematical connections secondary mathematics teachers made in their teaching practice. To do so, this chapter is organized in three main parts. In the first section, I introduce the definitions, categorizations, and the framework I developed to discuss the mathematical connections teachers made in practice. The second section contains narrative cases of each participating teacher and the kinds of mathematical connections made in his or her teaching practice. The final section provides a comparison across the three narrative cases.

Framing Mathematical Connections in Practice

Mathematical connections can be examined from a variety of perspectives. This chapter considers mathematical connections specifically from the perspective of practice. To begin, I introduce the Mathematical Connections Framework. The framework describes the levels and kinds of mathematical connections teachers made in practice. First, I discuss the *levels* for how mathematical connections can be made in practice. Second, I provide categorical descriptions to distinguish among the *kinds* of mathematical connections teachers made in their teaching. I developed these constructs by making comparisons across my participants and their teaching practices. Throughout this section, I incorporate examples from practice to explain and support the constructs developed and grounded in my observations of practice.

Levels of Mathematical Connections

In its most basic form, a mathematical connection is a *relationship* between a mathematical entity and another mathematical or nonmathematical entity, where *A* is related to *B*. This broad definition allows a connection to exist in any instance in which teachers (or students) indicate the existence of a relationship between *A* and *B*. In practice, teachers made connections in more or less explicit ways for their students. As a result, it was necessary to distinguish among the levels to which mathematical connections were made. Therefore, I define the different levels existing among the ways mathematical connections were made in practice (See Table 2).

Table 2

Definitions for the Levels of the Mathematical Connections Made in Practice

Level of the Mathematical Connection	**Definition of the Level**
No mathematical connection	*No mathematical connection* was made within a given classroom episode when a relationship was not even suggested.
Suggested mathematical connection	A *suggested mathematical connection* was made when a relationship between *A* and *B* was suggested and there was a distinct cue suggesting that a connection existed: *A* and *B* are somehow related *or A* is related to something (where the something is left unsaid).
Provided mathematical connection	A *provided mathematical connection* was made when the mathematical entities and the relationship existing between them were explicitly provided: where *A* is related to *B*.
Provided-and-explained mathematical connection	A *provided-and-explained connection* was made when the relationship between the mathematical entities was provided along with an explanation detailing why the mathematical entities were related: *A* is related to *B* because of *C*.

Within a given classroom episode, it was entirely possible that *no mathematical connection* was made. Therefore, a mathematical connection did not exist within a given episode when a relationship was not even suggested. In practice, no mathematical connections usually existed during episodes in which the teacher was giving directions, listing the specific steps of a procedure, or telling students what to do next. For example, no mathematical connection was made when Justin provided feedback to a student based on the work he observed on her paper.

> Student: I am confused on what to do.
> Justin: What did you get for your total time?
> Student: That [points to answer on paper].
> Justin: No [that is not correct].
> Student: What do I do?
> Justin: All right, I would not use the quadratic formula if I didn't have to.
> Student: Ok.
> Justin: But, let me see what went wrong [looks at Student's work]. Oh, I think you, no. Where did you get, hold on let me see something, did you do 57.2(sin56)?
> Student: Mm-hmm.
> Justin: 47.42. Oh. You didn't put a minus between it.
> Student: Oh. Ok.
> Justin: Now, divide it by -32.2.
> Student: Ok. [Justin walks away] (Observation, October 20)

In this episode, Justin told the student what steps she did incorrectly in a certain procedural calculation. Given this example, it seems possible for a researcher to infer that a mathematical connection could have or should have been made even though no connection was provided. However, the purpose of this part of the framework is solely to identify the level to which a connection is or is not made, rather than recommend what mathematical connections should be made in practice.

During my observations, there were times when each teacher seemed to suggest a mathematical connection existed. A *suggested mathematical connection* occurred when a relationship between (or among) *A* and *B* was suggested. Identification of a suggested connection required a distinct cue from the teacher that a connection existed. It was possible for a teacher to suggest a connection existed in one of two ways, both of which required a level of inference from the researcher. First, a teacher may have suggested that a particular mathematical entity was related to another mathematical entity, in which the relationship between the two was not explicitly provided. In practice, this type of suggested connection took place when a teacher developed one topic directly after another, where the short period of time seemed to imply that *A* and *B* were somehow related. As an illustration, Rachel introduced measures of variability by first reviewing box and whisker plots with her students.

> Box and whisker plot review. This is not a part of our standards, but for us to do what we have to do for our standard, we have to understand this. I know you did it in Accelerated Math I and in 8^{th} grade math, but we need to make sure that you can do it. (Rachel, Observation, September 26)

In this episode, Rachel suggested that a box and whisker plot was related to the measures of variability by saying that her students needed to understand box and whisker plots to do the day's lesson on measures of variability. Although Rachel developed each topic with her students, one right after the other, the precise relationship between the two remained implicit throughout the remainder of the lesson.

The second type of suggested connection occurred when a teacher suggested that a particular mathematical entity was related to something, in which the second entity was rather unclear or left unsaid. For example, Justin asked his students to graph a set of parametric equations on their calculator $x(t) = (150\cos(18))t$ and $y(t) = -16t^2 + (150\sin(18))t + 3$. While they were graphing the parametric equations, he suggested a connection existed.

> Justin: Notice that it has sine and cosine [pointing between $x(t) = (150\cos(18))t$ and $y(t) = -16t^2 + (150\sin(18))t + 3$]. Very important thing you should recognize, *x* is [points to equation]?
> Student: Cosine
> Justin: And *y* is [points to equation]?
> Student: Sine.
> Justin: You all should recognize that. (Observation, October 7)

In this episode, he stated that his students should recognize the trigonometric functions within the individual parametric equations, suggesting this was related to something else they had learned. During a later interview, Justin carefully explained this particular suggested connection, providing the additional components within the relationship, which had not been made explicit in the classroom discussion. He explained, “The unit circle, how it is (*x*, *y*), which is the same as (cosine, sine), and when you get to vectors it is (*x*-movement, *y*-movement), and kind of the same thing when you get to parametric equations” (Interview 2). If Justin had made this statement to his students during the class discussion, this episode would no longer be classified as a suggested connection.

There were several occasions when each teacher provided the connection between mathematical entities, making the relationship explicit. Therefore, a *provided connection* took place when the mathematical entities and the relationship existing between (or among) them were explicitly provided. As such, it is possible to describe a provided connection in the following way: *A* is related to *B*. During my observations of practice, I noticed that the relationship could take on many forms. At times, the provided connection could exist rather simplistically, where *A* is the same as *B*. For example, in responding to a student’s question, Justin related the procedure a student needed to use to solve a particular problem to the procedure Justin had previously written on the board in the class notes.

> Student: How do you find the time in this?
> Justin: Same as we did on this one. Remember how we did it [points to procedure written on the board]? (Observation, October 19)

In his response, Justin noted the procedure necessary to solve the given problem was the same as the procedure he presented to the class. By providing the relationship between the procedures, he provided a connection for his student. An additional example of a provided connection took place as Rachel compared measures of variability with measures of center.

> All of the measures that we have talked about, mean absolute deviation, variance, and standard deviation, are measures of variability. Not measures of center. When you are little, you learn, mean, median, and mode. Those are describing the center of the data. (Observation, September 26)

The connection in Rachel's comparison was provided when she said measures of variability were not like measures of center, stating *A* is not the same as *B*. In each of these instances, the connections were noted, but there was no further discussion explaining why the relationship between *A* and *B* existed.

There were times when each teacher went beyond simply indicating the relationship. The teacher also gave an explanation describing the reasons why *A* and *B* were related. A *provided-and-explained connection* existed when the relationship between (or among) mathematical entities was provided along with an explanation detailing why the mathematical entities were related. *A* is related to *B* because of *C*. The following episode from Rachel's classroom is an example of a provided-and-explained connection.

> The cosine of angle is equal to the sine of its complement. Well, we would relabel it, right? So, it would be opposite/hypotenuse. But, notice that opposite and adjacent are the same thing, because we switched angles. So, *that is why this is true*, because the opposite of one angle is adjacent of the other angle. The opposite side of one acute angle happens to be the adjacent side of the acute angle, of its complement. (Observation, September 22)

In this episode, Rachel first provided a relationship by equating the sine of the angle to the cosine of the angle's complement. She then continued by explaining the reasons why the sine of an angle is equivalent to the cosine of the angle's complement. In a subsequent interview, Rachel shared her reason for providing the explanation, "I think that is more important that they *understand* that relationship between the complementary angles and that sine is just the ratio of the sides of the triangles and either one works" (Interview 6). Both Robert and Justin gave similar reasons for making this particular level of a mathematical connection in their teaching, implying they perceived this level of a mathematical connection directly related to students' understanding of mathematics.

Kinds of Mathematical Connections

A mathematical connection is defined as a relationship between *A* and *B*. However, this definition does not describe the different ways *A* and *B*, the components within the connection, can be related. In order to understand the kinds of mathematical connections teachers made in practice, it was necessary to make sense of the differences existing among the relationships. I analyzed the different relationships existing between *A* and *B*, and I provide the following categories to describe the different kinds of mathematical connections made in practice (see Table 3). In practice, some of the particular kinds of mathematical connections existed only as provided connections and not as provided-and-explained connections. Therefore, I do not claim that these categorical descriptions are complete or independent of one another. Rather, I submit them as specific examples of the different relationships teachers made in their practice.

Table 3

Categories for the Different Kinds of Mathematical Connections Made in Practice

Category of the Connection	Kind of Connection: Provided Connection	Kind of Connection: Provided-and-explained Connection
Connecting through comparison	*A* is similar to *B*	*A* is similar to *B because* of *C*
	A is the same as *B*	*A* is the same as *B because* of *C*
	A is not the same as *B*	*A* is not the same as *B because* of *C*
	A or *B* similarly defines or describes *C*	
Connecting specifics to generalities	*A* is an example of *B*	*A* is an example of *B because* of *C*
Connecting methods	*A* or *B* can be used to find *C*	*A* or *B* can be used to find *C* *because* of *D*
Connecting through a logical implication	If *A*, then *B*	If *A*, then *B because* of *C*
	If *A*, then *B* and not *C*	
Connecting to the real world	*A* is an example of *B* in the real world	*A* is an example of *B* in the real world *because* of *C*

Connecting through comparison. In each teacher's practice, the most common kind of connection was the connection that was made when the teacher made comparisons between *A* and *B*. Many times, a teacher compared similarities, stating *A is similar to B*. These comparisons often took place across different or equivalent representations of a given mathematical entity. It was also possible that the teacher compared similarities existing between seemingly unrelated concepts. For example, Rachel compared a repeated piece of data in her statistics lesson to the algebraic concept of a multiplicity, "65 occurs twice, kind of like that multiplicity stuff we talked about with the graphing" (Observation, September 26). In other episodes, the teacher made a comparison by saying, *A is the same as B*. Rachel provided this kind of connection when she compared the procedures used to isolate a variable in two different equations.

Rachel: Do you see what we should do next, if we are trying to get *y* alone [points to the equation $y(x + 2) = 3$ written on the board]? [Pause] Let me ask you this, let me do a little thought bubble. Let's think about this for a minute. What if I had just given you $2y = 3$ [writes $2y = 3$ on the side of the board]?
Student: Divide by 2.
Rachel: You divide by 2. And in this thought bubble, we would divide by 2. Well, why can't we do the same thing here [points to $y(x + 2) = 3$]. But, instead of just being 2 it is this binomial $(x + 2)$. (Observation, September 14)

This comparison allowed Rachel to convey that the same procedural operation was necessary to simplify either equation.

As teachers made comparisons, there were occasions when each teacher contrasted the differences existing between *A* and *B*. This kind of mathematical connection occurred when the teacher described the differences by saying, *A is not the same as B*. In these instances, the teacher usually explicitly stated the reason why the components within the connection were different, thus making a provided-and-explained connection. For example, Robert compared the graphs of two different exponential functions. He graphed each function on the same coordinate plane, allowing his students to notice the difference existing between the shapes of the graphs.

Robert: So, let's start with what we know. This is our parent graph, so to speak. $f(x) = 2^x$…[graphs the function on the board]. Ok. So, that is what happens with this. I am going to change the base. Let's change our base to [writes $f(x) = \left(\frac{1}{2}\right)^x$ on the board], everybody, go ahead and use your table function in your calculator, and type that in. And, we are going to see what happens when we do $f(x) = \left(\frac{1}{2}\right)^x$ [graphs the function on the same coordinate plane]. Everybody got it in?
Student: It switched them.
Robert: It switched them. (Observation, November 9)

This episode continued as Robert generated a table of values for each corresponding graph, and he used the tables to explain how the value of the base influenced the visible differences between the two graphs.

Connecting through comparison also occurred when the teacher defined or described a mathematical entity in similar ways. *A or B similarly defines or describes C*. For example, Robert described the location of the vertex of a particular absolute value function in three different ways.

> So, if I say origin, which is right there [points] or if I say the vertex, which is right there [points], or if I say on axis of symmetry, which is right there [points], I am basically saying the same thing. So, we kind of got the same idea. (Observation, October 31)

Across these comparisons, teachers made mathematical connections by comparing related mathematical entities.

Connecting specifics to generalities. Teachers provided a different kind of connection when they related a specific case to a more generalized concept or rule. In practice, this connection occurred when a teacher used examples to develop a particular concept or rule. As such, these connections were most often suggested connections, in which the teacher did not explicitly state *A is an example of B*. However, there were episodes in which the teacher directly stated the relationship, therefore providing the connection for his or her students. Justin's introduction of parametric equations was an explicit case of this kind of connection. He began the lesson by writing the formal definition of parametric equations on the board, and he used an example problem to make sense of the definition.

> Here is an example of what [parametric equations] look like, and we will come back and make sense of everything, I hope [writes example on board]. $x = t^2 - 2$, and $y = 3t$ where $-2 \le t \le 2$. …Now, reread the definition. See if it makes sense. This is honestly the way I had to learn this as well, I would do some examples, go back and look at the definition. (Observation, October 7)

The example problem, $x = t^2 - 2$ and $y = 3t$ where $-2 \le t \le 2$, provided a specific case of the general concept of parametric equations. When this kind of connection was made explicit, it usually existed as a provided-and-explained connection. It seemed necessary for the teacher to highlight the reasons why a particular example could be thought of as a case of a more general concept or rule. For example, Rachel used a set of data points as an example to help her students understand why the sum of the deviations from the mean would always equal zero. She extended the example by asking her students to consider if the same result would be true for any set of data. The example provided a context for her students to reason more generally. As a result, Rachel's students provided the reason why they could generalize from the single example, for they explained that the sum of the deviations would always be zero because the mean acted as the balancing point for the data set (Observation, September 28).

Connecting methods. Teachers provided a different kind of connection as they considered multiple methods to solve a problem. The connection between methods resided within the solution, where *A or B could be used to find C*. Most often, this connection was a provided connection, because teachers rarely explained the reasons why the methods were related beyond that the methods led to a common solution. The methods used were generally from a particular strand of mathematics. For example, Robert shared two algebraic methods to graph a linear equation with his class. First, he used the slope and the y-intercept to graph the linear equation. He then showed his students how they could arrive at the same conclusion by developing a table of values.

> [The slope] is going to be 2. How convenient. Up 2, over 1. …And, then, up 2 again, and over 1. Then, going to the left, I am going to go backwards. 2 over 1, 2 over 1. Or, if that bothers you, and you are not quite sure what I was doing there. Then what you want to do is do that [makes a table of values]. Put in some numbers less than 3, some numbers greater than 3, and you should get the same points. (Observation, October 31)

There were times when the methods used were from different strands of mathematics. Rachel, for example, asked her students to verify why the altitude of an equilateral triangle was $\frac{\sqrt{3}}{2}$ feet given each side of the equilateral triangle was 1 foot in length. One student stated that the length could be verified by using the Pythagorean theorem, while another shared a method using the scale factor for similar triangles. Rachel summarized the discussion and described the two methods, stating the first method was more algebraic in nature while the second method relied on geometric properties (Observation, September 14). In subsequent interviews, each teacher believed connecting methods portrayed the connected nature of mathematics.

Connecting through a logical implication. Logical implications provided a different kind of mathematical connection. Teachers provided the connection through the implication. *If A, then B*. Some of these connections followed from formal properties and theorems, such as, "If I have two similar scalene triangles, then corresponding angles should be the same." (Rachel, Observation, September 14). However, in practice, there were several occasions when the logical implication was less formal and acted more like a cue or a reminder to students; if this happens, then this must follow. For example, Justin had his students use parametric equations to create a passing play for a football team. He provided the following implication as a reminder, telling his students to create a new set of parametric equations whenever a football player's route changed directions. "If you change directions, [then] you need a new set of equations" (Observation, October 17).

At times, the teacher provided a slightly more complicated structure for the implication, *If A, then B and not C*. For instance, Robert used this kind of implication during his lesson on negative exponents.

> Robert: Put stars, a box, or exclamation marks, something, to warn yourself, what do negative exponents not give you?
> Students: Negatives.
> Robert: Negative numbers. Not negative numbers. Ok. Negative exponents do not change the sign of your number. They give you the reciprocal of the positive exponent. (Robert, Observation, November 8)

Although Robert's warning was not in the precise form of a logical implication, he essentially provided the followed implication: if you have a negative exponent, then the result is the reciprocal of the base with a positive exponent, which does not imply the result is a negative number. Each teacher used implications to demonstrate a relationship of dependence, where one component of the connection followed logically from another.

Connecting to the real world. Each teacher provided connections between mathematics and the real world. In this kind of connection, the phrase “real world” refers to a familiar concept or relatable context from outside the mathematics classroom. Real world contexts were often included through word problems or tasks, where *A is an Example of B in the real world*. Many times, the teacher did not explicitly state this relationship beyond the context and the words surrounding the numbers in the problem. As a result, these word problems were examples of suggested connections. To illustrate, Justin used several word problems during a lesson on parametric equations and projectile motion. “A short-range rocket is launched at 500 mph at an angle of 62° from a platform 20 feet off the ground. How long did the rocket stay air born?” (Justin, Observation, October 20). Although the mathematics in the word problem was surrounded by a context, the relationship between the real world context and the mathematical concept of parametric equations was not explicitly provided by Justin or in the problem.

There were occasions, however, when the teacher directly related a mathematical entity to a real world context, resulting in a provided connection. For example, Robert used bacterial growth as a real world example of an exponential growth function.

> So, everybody, this right here [points to $f(x) = 2^x$] is called exponential growth. Ok, can you all see why it is called exponential growth? Because it is growing exponentially. When you get sick, that is because the bacteria have experienced exponential growth in your body.” (Robert, Observation, November 9)

In this example, Robert introduced the mathematical concept before providing the real world example of exponential growth. There were times when the teacher reversed the order, introducing a real world context as a method to make sense of the mathematics. To illustrate, Rachel used a real world context to introduce the concept of measures of variability.

> Ms. McAllister has a problem and she needs your help. She has to give one math award this year to a deserving student. But she can't make her decision. Here are her test grades for her two best students. Grace: 90, 90, 80, 100, 99, 81, 98, 92. Tyler: 90, 90, 91, 89, 91, 89, 90, 90. Write down which of the two students you think should get this award just based on their test scores right there. (Observation, September 26)

In a whole-class discussion, Rachel's students noticed Grace's test grades were more variable than Tyler's. Rachel concluded this discussion by stating this real world context was an example of variability between data sets, and she continued to develop the measures of variability around the context of these students' test grades. Through this real world example, Rachel and her students provided the connection between the context and the mathematical concept. In fact, when the context was used to introduce the concept, the students usually became involved in making the connection.

Using the Mathematical Connections Framework to Analyze Practice

I developed the Mathematical Connections Framework through an inductive and iterative coding process, and I then used the framework to analyze and describe the mathematical connections teachers made in practice. The utility of the first part of the framework was that it allowed me to systematically identify the explicit mathematical connections existing within a teacher's practice. This identification provided a method of data reduction, a way to sift through classroom episodes identifying specific instances of mathematical connections. The second part of the framework allowed me to continue to make sense of the kinds of mathematical connections teachers made in practice. The categorical descriptions acted as an analytic tool to understand the differences existing among the relationships provided between the components of the mathematical connection.

Using the framework, I was able to continue the fine-grained analysis by considering additional aspects related to the mathematical connections made in practice. For example, during my analysis, it became useful to consider who contributed the components of the mathematical connection or who contributed the relationship existing between the components. It was possible for the teacher or the students or both the teacher and the students to contribute a component of the connection or to contribute the relationship existing between components. Analyzing contributions seemed to be a significant way to describe the mathematical connections made in practice because it described who contributed to the making of the mathematical connection.

In addition, I looked within the components to examine the nature of the content within each of the components. First, I considered whether the content of the individual components reflected procedural knowledge or conceptual knowledge. I used Rittle-Johnson, Siegler, and Alibali's (2001) definitions of procedural and conceptual knowledge. They defined procedural knowledge as the knowledge needed "to execute action sequences to solve problems" (p. 346). Therefore, procedural knowledge is specific to particular types of problems. They defined conceptual knowledge as the knowledge of "principles that govern a domain and of the interrelations between units of knowledge in a domain" (p. 346). The authors claimed this knowledge does not depend on specific problem types and is therefore generalizable. Second, I examined whether the content of the components came from an individual strand of mathematics or from the different strands of algebra, geometry, and statistics. Through this analysis, another qualitative difference developed. Many times, the content within the components differed based upon familiarity to the students, for there were instances when a teacher would connect a new and unfamiliar concept with a previous and more familiar concept. The combination of these lenses provided me with a meaningful way to describe the mathematical connections teachers made in practice.

Narrative Cases of Mathematical Connections Made in Practice

In this section, I discuss each teacher’s practice, moving from general to specific. Within each narrative case, I begin by presenting patterns of instruction. Then, I describe the development of a particular mathematical topic as a kind of evidence for patterns of instruction identified. I believe these descriptions are necessary to provide an appropriate foundation to situate and interpret the mathematical connections made in each teacher's practice. To conclude, I describe the mathematical connections made, supporting the discussion with evidence from the teacher’s practice.

Rachel McAllister: From Problem Solving to Practice

Across classroom observations, Rachel used mathematical tasks to introduce new topics. The tasks situated the mathematics within a context, where the context served as a vehicle for students to engage in problem solving and to make sense of the new mathematical concepts. Students regularly collaborated in small groups. Throughout the class period, Rachel strategically initiated whole-class discussions to formalize the mathematical ideas made in the small groups. In doing so, she encouraged her students to describe terms and concepts in different ways, and she expected her students to contribute the various methods they used to approach the problem posed by the task.

Rachel asked open-ended questions to extend the mathematics made within the tasks. She asked her students, “Can you explain it another way? Can you solve it a different way? What do you notice? Why do you think that?” Rachel used these questions to help her students understand why mathematical concepts and procedures work the way they do. In addition, her questions supported the development of mathematical discussions, allowing students’ thinking to be present throughout the discussions. The whole-class discussions provided an atmosphere where her students freely posed their own questions about the mathematics. Students asked questions about why things worked, or they questioned whether something would always work in a certain way. Rachel incorporated students' questions into the lesson, using their questions as opportunities to explore the mathematics in more detail.

A distinct pattern existed within Rachel's instruction. Rachel introduced mathematical concepts through mathematical tasks, because she said her main objectives were for her students to understand the particular mathematical concept and to engage in problem solving. As her students seemed to understand the concept, she used a variety of problems and methods to help her students practice the related procedures. She believed practice was necessary for her students to demonstrate mastery. She explained, "Kids can learn, just by investigating, but I don't think they have mastered it until they have practiced it over and over again" (Interview 3). Mastery, to Rachel, was an important component of the learning process. She acknowledged, "Understanding, and then mastery, is learning" (Interview 5). The relationship Rachel perceived between understanding and mastery was reflected in the development of mathematical concepts and procedures in her classroom, for she began with problem solving and ended with practice.

In the following section, I use classroom episodes to illustrate how Rachel developed the topic of 30°-60°-90° triangles with her students. The way she developed this topic was similar to the other topics she developed during my observations; therefore, I consider her development of this topic to provide insight into the patterns of instruction I identified within her practice.

Development of a topic: 30°-60°-90° triangles. Rachel used a task to introduce the topic of 30°-60°-90° triangles. In a whole-class setting, she explained that they were going to learn about the special relationship existing between the sides of a 30°-60°-90° triangle and how to use the relationship to find missing sides. She asked a student to read the first question in the task for the class,

> Adam, a construction manager in a nearby town, needs to check the uniformity of Yield signs around the state and is checking the heights (altitudes) of the Yield signs in your locale. Adam knows that all yield signs have the shape of an equilateral triangle. Why is it sufficient for him to check just the heights (altitudes) of the signs to verify uniformity? (Georgia Department of Education, 2009, p. 10)

Before delving into the question, Rachel asked her students to define certain mathematical terms used within the question. For example, after a student defined an altitude as the height of a triangle, Rachel drew examples of acute, obtuse, and right triangles on the board, asking students to locate the altitude for each kind of triangle. Looking across the examples, she asked her students, “Does anybody see a relationship between these altitudes and the bases of the triangles?” (Observation, September 14) Several students responded, claiming the altitude is always perpendicular to the base of the triangle. This episode provided an example of the kinds of questions she wove into the classroom discussions as a method to extend and connect the mathematics for her students.

The discussion continued as Rachel asked her students to consider the first question posed by the task. The following episode is an example of the discussions that developed in Rachel’s teaching.

> Rachel: Why is it sufficient to just measure the altitude? Say it out loud for everybody.
> Student 1: Because of the triangle, it should be the same.
> Student 2: [Inaudible]
> Rachel: If it is the same size triangle, the altitude is the height, and the height is the same of all of them. Is that what you saying, Student 2?
> Student 2: That is what I was getting to.
> Rachel: Let’s listen to Student 3.
> *Student 3: Is that only for equilaterals?*
> Rachel: Did you all hear Student 3’s question?
> Students: No.
> Rachel: She asked, “Is that true only for equilateral triangles?”

> Student 4: Well, as long as your triangle is the same size. It is always going to be the same. As long as it is congruent, it is the exact same, right? As long as it is a congruent triangle, it is going to be the same.
>
> Rachel: So, it doesn't have to be equilateral, if we are measuring the height. [If] we had a 3, 4, 5 triangle, we could measure the height on every one of them and it should be the same. So, we understand, on number 1, why is it sufficient for him to just check the heights? Because the heights are the same in all the yield signs. (Observation, September 14)

Rachel asked her students to consider why it was sufficient to measure the altitudes to determine uniformity. Several students responded. One student asked a related question, wondering if the claim only applied to equilateral triangles. The student's question suggested a kind of generalization, and Rachel incorporated the student's question into the discussion. I note that Rachel's students did not ask questions of each other; rather they directed their questions to Rachel. However, Rachel regularly included her students' questions into the whole-class discussions, expecting other students to respond to a student's question about the mathematics.

Rachel used the context of the yield sign to explore the special relationship existing between the sides of a 30°-60°-90° triangle. This particular context allowed her students to consider a yield sign as an equilateral triangle with sides 2 feet in length. To progress through the task, the class drew an altitude for the triangular sign, creating two congruent 30°-60°-90° triangles within the yield sign. The whole-class discussion continued as she asked her students to verify the length of each leg of the given right triangle. Throughout the remainder of the unit, Rachel referred to the yield sign to remind her students of the relationships existing among the sides of a 30°-60°-90° triangle.

Progressing through the task, Rachel asked her students to work in small groups. Students considered the smaller equilateral triangle within the yield sign, and they similarly constructed two 30°-60°-90° triangles within the smaller equilateral triangle. She asked them to prove why the long leg of the smaller right triangle was $\frac{\sqrt{3}}{2}$ feet given that each side of the smaller equilateral triangle was one foot in length. The following discussion is an example of how Rachel allowed and often expected her students to approach a mathematics problem using different methods than she may have originally anticipated.

> Student 1: It is already half.
> Rachel: Hmm?
> Student 1: Isn't this kind of like, since we already have that info, isn't this already obvious?
> Rachel: You need to verify it, or explain. How would you explain it, if you don't want to verify it?
> Student 2: Because we verified that this one is $\sqrt{3}$, so then, the new one is half. So, it would be half of $\sqrt{3}$, which is $\frac{\sqrt{3}}{2}$.
> Rachel: Do you know the small triangle is half of the big triangle?
> Students: Yes.
> Student 2: Cause the length of this 1 and this is 2.
> Rachel: You may. I haven't had anyone say it to me that way. Are the areas half? The perimeters half? That is really, I just didn't think about it that way.
> Student 2: The areas are not.
> Rachel: I just used the Pythagorean theorem. Also, because I know the special relationship. That is very interesting that you both thought of it that way?
> Student 2: It is obvious.
> Rachel: So, if we want to look at similarity. These are similar, this is half, then of course, that would be half.
> Student 2: And, everything is to scale, except for the angles.
> Rachel: Right, which is what makes them similar. So, if they are to scale, what is that scale?
> Student 2: Half.
> Rachel: Very good. Explain that in words. (Observation, September 14)

In the whole-class discussion that followed, Rachel asked students to share their different methods with the class. One student verified the length of the side using the Pythagorean theorem, while another described using the scale factor because the triangles were similar. Rachel summarized the differences between the two solution methods.

In the next class period, Rachel reviewed several homework problems with her students. The homework problems gave students an opportunity to practice applying the special right triangle relationships to find missing sides. The practice problems Rachel asked were more straightforward and procedural than the problems posed within the task. For example, she asked her students to calculate the short leg of a 30°-60°-90° triangle, given that the long leg of the triangle was 12 centimeters. To answer this problem, she briefly discussed two solution methods. She explained to the class,

> We know that this [pointing to the long leg], according to our relationship, is $x\sqrt{3}$. So we want just *x*, we are getting smaller, so we divide by $\sqrt{3}$. Or, if you think algebraically, to solve it, I divide by $\sqrt{3}$. (Observation, September 16)

Rachel used practice problems to help her students develop mastery of concepts and procedures. She viewed these problems as a safe environment for her students to apply mathematical concepts and procedures in different situations, allowing them to make mistakes and to use these mistakes as learning opportunities. These episodes demonstrate how she began with problem solving and ended with practice. The following section explores the kinds of mathematical connections Rachel made in her teaching.

Mathematical connections made in Rachel's practice. Rachel made mathematical connections as a way to develop her students' understanding of mathematics. During my observations, she actively included her students in the process of making connections by asking questions and building on their thinking. Each kind of mathematical connection was present in her instruction. Of the connections made, many of these connections were through comparisons or through different methods used to solve a problem. In addition, Rachel used mathematical tasks as a way to connect mathematics to the real world. In the following sections, I use episodes from her classroom instruction to explore the kinds of mathematical connections Rachel made for and with her students.

Contributions of mathematical connections. Many of the connections made in Rachel's teaching were made in collaboration with her students. Nearly half of the connections made included contributions from her students. Students frequently made contributions during the whole-class discussions, for the discussions created an environment for students to contribute components of the connection as they responded to Rachel's questions. In addition, Rachel asked questions expecting students not only to provide the relationship but also to explain the relationship. To illustrate from a previously mentioned exchange, Rachel used a task to develop measures of variability. The task asked students to determine which student, Grace or Tyler, should receive the annual mathematics award based on each student's test grades. From this example, Rachel asked her students to describe what they noticed about the sum of the deviations from the mean of each student's test scores.

> Rachel: What do you notice about the sum of deviations for both Grace's and Tyler's test scores? What was true about the sum of the deviation? Everybody go back and look.
> Student 1: They are both zero.
> Rachel: They are both zero. Why do you think that is the case?
> Student 2: Because each one is like 0, 0, -10, 10, everything has to cancel out.
> Rachel: Ok.
> Student 3: It is always going to be zero.
> Rachel: That is right, Student 3. Why?
> Student 3: Because it is the average, and it either went above the average or below the average, and it is going to cancel out.
> Rachel: Very good. Did you all hear what Student 3 said?
> Students: No.
> Student 3: For the equation, it is the number minus the mean, and if it above then it is just going to end up canceling out.
> Rachel: It is the number, the test score, minus the mean. So, he said you have one above and one below every time, because the average is the balance of the two. So, they are always going to have one that matches up to cancel out. So, if we just did the deviations, we would get zero every time. (Observation, September 27)

In this episode, Rachel's students claimed that the sum would always equal zero, providing the connection between the specific example of Grace and Tyler's test scores and the generalization that the sum would always be zero. Rachel asked an additional question, wanting her students to understand why the sum of the deviations would always equal zero. In responding to her question, a student explained the connection, explaining what the mean (or average) is and how it is used to calculate the deviations. Therefore, this episode is an example of how Rachel's students contributed components to a provided-and-explained connection in response to her questions.

In Rachel's classroom, students asked their own questions about the mathematics. At times, their questions, when isolated from the surrounding discussion, acted as suggested connections, because their questions often went beyond asking about how to do the next step in a procedure or to explain a particular concept. Instead, her students' questions focused on why the procedure worked the way it did or if the procedure was always going to work in a certain way. For this reason, the students' questions suggested a possible relationship existing between concepts, usually moving from a specific case to a more generalized rule. For example, during the introduction of 30°-60°-90° triangles, a student asked if the ratio of the short leg to the long leg was "always the $\sqrt{3}$?" (Observation, September 14). The student's question seemed to suggest that the problem used to introduce the topic might be a specific example of a general rule.

> Student: Is it always the $\sqrt{3}$?
> Rachel: Yes.
> Student: Always?
> Rachel: Let's do it with a variable, let's see if it is always for any triangle. (Observation, September 14).

Rachel responded to the student's question by constructing an algebraic proof to demonstrate the relationship between the short leg and the long leg of a 30°-60°-90° triangle. Her proof acted as a provided-and-explained connection, because the proof both established and explained the ratio between the short leg and the long leg of a 30°-60°-90° triangle. Rachel used the student's question to further develop and explore the relationship existing among the sides of a 30°-60°-90° triangle for her students.

Rachel's students also posed more conceptual questions. Similarly, these questions were often suggested connections, questioning whether certain relationships existed among concepts. Rachel responded to these student questions by providing the explicit mathematical connection. For example, a student wondered if there were other right triangle relationships beyond the ones developed in class (i.e., 30°-60°-90° and 45°-45°-90° triangles).

> Student: I understand there are special right triangles, but [are] there other relationships that you can use the triangle and the angle in the same way?
> Rachel: Yes, they are all a ratio. So you could memorize all angles [and relationships] if you wanted.
> Student: But what is?
> Rachel: The reason we do the ones we do, next year, when you learn the unit circle with radian measure, it is the measures you memorize of all of the increments of 30s and 45s, up to 360. So, if you don't have them memorized, when you go to turn them on the coordinate plane into all four quadrants, you won't be able to memorize them. It would be harder to memorize them. So, it is the basis of why we do the unit circle why we do. So, you need to memorize the exact ones. Then again, when you do them in Calculus, you need to know them too. So, that is the only reason why we do those, memorize them, so you can memorize and use them for the other classes. (Observation, September 22)

The student's question suggested 30°-60°-90° and 45°-45°-90° triangles were specific cases of a general concept. Rachel's response confirmed the student's suspicion. She provided the connection by describing how relationships exist among the sides and angles for every right triangle. In addition, she explained why they learned the specific relationships for 30°-60°-90° and 45°-45°-90° triangles, saying those particular relationships would be useful in future mathematics courses. In providing and explaining the connection, Rachel was also able to connect the current mathematical concept to the mathematics on the horizon.

The kinds of mathematical connections made in Rachel's teaching regularly included contributions from her students. This was a theme that cut across the kinds of mathematical connections she made in her teaching. The previous episodes are examples of how Rachel included her students in the process of making connections in whole-class discussions. First, the questions she asked elicited students' contributions of various components of the mathematical connection. Second, she incorporated students' questions into the discussion, building from their suggested connections to develop provided-and-explained connections for the class. For these reasons, as the teacher, Rachel played an integral role in supporting her students' contributions of various components in these mathematical connections, because these connections took place during teacher-student (or student-teacher) interactions.

Connecting through comparison. During my observations of Rachel's practice, the most common kind of mathematical connection she made was through comparison. Many of these connections through comparison also existed as provided-and-explained connections. She provided connections by making comparisons between mathematical concepts and procedures, and she explained connections as she explored the relationships surrounding the mathematics within the comparison. Across instances of this kind of mathematical connection, it was common for her explanation to extend the mathematics in the connection beyond a given problem type or particular procedure.

In Rachel's practice, one of the more pronounced characteristics of this kind of connection was the development of conceptual knowledge as a result of the connection. Even when she made comparisons between procedures, the explanation of the connection seemed to shift the focus from procedural to conceptual knowledge. For example, while working in small groups, Rachel noticed a student did not correctly rationalize an expression. Rather than telling the student what to do, Rachel used the opportunity to compare procedures, examining the difference between the student's procedure (multiplying by $\sqrt{2}$) and the procedure needed to rationalize the function (multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$). The following excerpt is the conclusion of her comparison, in which Rachel used the identity property to explain the difference between the procedures.

> Rachel: That is not correct. Let's talk about why you would multiply by $\frac{\sqrt{2}}{\sqrt{2}}$.
> Student: Because it gets rid of the square root on the bottom.
> Rachel: Because a square root times a square root, right. Perfect square. Ok. So we end up having a perfect square on the bottom, and you can take the square root of a perfect square. But why should you do, multiply by $\frac{\sqrt{2}}{\sqrt{2}}$? What is $\frac{\sqrt{2}}{\sqrt{2}}$?
> Student: 1.
> Rachel: You are right. It does get rid of the square. You can't just multiply by some number other than 1. Remember that identity property we talked about with the matrices.
> Student: So it keeps it the same, it just takes the square root away.
> Rachel: Makes it look a little different. We are multiplying by 1, so we aren't changing the value, we are changing the way it looks. (Observation, September 14)

During this episode, it would have been possible for Rachel to focus the discussion solely on procedures. However, her use of the identity property to explain the difference between the procedures seemed to result in a more conceptual focus throughout the remainder of the episode. As such, the mathematics within this connection seemed to be flexible and therefore more conceptual, going beyond specific problem types and capable of being transferred to other situations.

Given this kind of connection, there were times when the content of the individual components within the mathematical connection were from different mathematical strands or seemingly unrelated concepts. Rachel usually made these kinds of connections when introducing a new concept, because it was a way to relate a new concept to a more familiar concept or procedure. In the following episode, Rachel related how to "undo" the sine function with similar procedures her students learned in previous units.

> Rachel: It is a very new concept. This is sine, which is a function of *x*. Sine is actually being done. It is the verb. Ok? So, if we want to undo it though, just as we did with matrices, and with the functions, that is exactly right. To undo it, we are going to use

the inverse. Remember the inverse of multiply is divide that is why you all think we should divide. But, this isn't multiplying.

Student: That stinks.

Rachel: So, we are going to the sine inverse, doesn't that look just like what we did with matrices, doesn't that look like the same symbol? And [when] we graphed the function and its inverse. (Observation, September 16)

In this episode, Rachel made several connections through comparison. Rachel compared inverse trigonometric functions to inverse matrices and to functions and their inverses. She made an additional connection by contrasting sin(x) to the operation of multiplication, explaining to her students that these two things were not the same. She also compared the similarity existing in the symbolic notation for sine inverse, inverse functions, and inverse matrices. This example continues to demonstrate the conceptual focus of the mathematics within the connections Rachel made in her teaching. In the final interview, Rachel described her motivation for making this particular connection. She said she thought it was important to "build relevance between concepts" for her students to better understand the concept of "undoing" (Interview 6).

Connecting methods. There were several episodes in Rachel's teaching when different solution methods were provided for the same problem. To illustrate, Rachel's students constructed two 30°-60°-90° triangles within an equilateral triangle. Given that the sides of the equilateral triangle were 2 feet in length, Rachel asked her students to describe how they found the length of a short leg of a 30°-60°-90° triangle.

Student 1: You split the top side [of the equilateral triangle] in half, so 2 divided by 2 is 1 [foot].

Rachel: Very good, Student 1. He said he split it in half. Student 2 said she bisected it. Both correct. (Observation, September 14)

In this example, students provided two methods to find the length of the short leg of the 30°-60°-90° triangle. Although both methods led to the same conclusion, Rachel did not explain how dividing a length in half is the same as bisecting the length. A similar example occurred when two formulas were provided for finding the area of a square.

> Rachel: How do you find the area of a square?
> Student: Base times height.
> Rachel: Base times height, or side squared. (Observation, September 20)

In this episode, the reason why either formula worked was left unsaid. Rachel did not explain that the formulas were the same because the base and the height of a square are always the same length. Across similar episodes, Rachel usually did not explain the reasons why either method could be used. Therefore, in her instruction this kind of connection existed almost exclusively as a provided connection, rather than a provided-and-explained connection.

Because connecting methods was a common practice in Rachel's classroom, students regularly provided an additional method to solve problems without prompting from Rachel. These were student-proposed methods alternative to the regular methods developed during the course of the lesson. For that reason, she used these instances to explore the student's method, explaining how the method worked and why the student's method was related to the other methods used. To illustrate, a homework problem asked students to find the length of a leg of a 45°-45°-90° triangle given the hypotenuse had a length of *x*. To solve the problem, Rachel told the class, "We divide by $\sqrt{2}$, and then we are going to have to rationalize" (Observation, September 20). After finding the solution, a student volunteered that he had used a different method to find the length of the leg.

> Student: Do you always have to do that method? Because, with that one, I just worked it out backward.
> Rachel: Tell me what you did.
> Student: So, I found, you have to pull out a $\sqrt{2}$. So, if you take $x\sqrt{2}$, it is giving you $x\sqrt{4}$. Which is $2x$, and then you could just say do $\frac{1}{2}x$ multiplied by $\sqrt{2}$ gives you *x*.
> Rachel: Say it to me one more time.
> Student: My answer was $\frac{1}{2}x\sqrt{2}$.
> Rachel: Right, which is what we got. That is correct.

Student: I just worked it out backwards.
Rachel: I know
Student: Thinking, you have *x*, ok?
Rachel: You started as this leg was *x* or the hypotenuse was *x*?
Student: The hypotenuse was *x*. And it is what times $\sqrt{2}$ will give me that. So, I multiply $x\sqrt{2}$ times that. And that gives me $x\sqrt{4}$, which I just halfed *x*.
Rachel: Ok.
Student: So, I can work it out backwards and that will work for all of them?
Rachel: Yeah. It is the same as when somebody tries to do an algebraic equation, if I just said $3 + x = 8$ [writes equation on the board], most of you, this one is basic, most of you will subtract 3, and get 5, right? But some of you will think what do I add to 3 to get 8? And that is exactly what you [Student] were doing. (Observation, September 20)

After the student provided the additional method, Rachel explained how the student's method of working backward was similar to the method she used. She used the equation $3 + x = 8$ as a more basic context for her explanation. This episode was one of the few examples in which this kind of mathematical connection also occurred as a provided-and-explained connection. Rachel may have thought that it was necessary to explain the relationship between methods because the student's method was unfamiliar and seemingly more complicated. Perhaps the explanation in the previous examples was more readily apparent and obvious to her students, eliminating the necessity for Rachel to explain those connections.

Connecting to the real world. Rachel used tasks to connect mathematics to real world contexts. Given this kind of connection, she regularly introduced new concepts through a real world context. For example, Rachel used the aforementioned context of a yield sign to introduce the topic of 30°-60°-90° triangles (Observation, September 14) or the context of giving a mathematics award to help her students consider the importance of variation among numbers (Observation, September 26). Rachel said she purposefully began lessons with a context, because the context seemed to provide a more tangible mechanism for her students to explore and to make sense of the mathematics (Interview 6). Rachel continued to make connections to real world contexts throughout the development of a mathematical topic. For example, she used a task to help her students practice calculating variance and standard deviation (Observation, October 1). In the task, students measured the diameter of several different tennis balls and then calculated the variance and standard deviation of the data. They used this information to determine if the measure of the diameter for any of the tennis balls was an outlier and therefore should not be used in the school's tennis match. She emphasized that this kind of connection was a way to make the mathematics more relevant to her students, because they were able to see how mathematics might be useful outside of her classroom.

Justin Smith: Building the Foundation

I observed Justin teach a unit on parametric equations in his Accelerated Mathematics III class. He said he enjoyed teaching this unit because "it takes a lot of thinking" and because the unit allowed for real world applications (Interview 1). During my observations, I noticed a distinct pattern in how Justin developed mathematical topics for his students. Before introducing a new topic, he reviewed certain concepts and procedures to build the foundation necessary for his students to learn the new material. New topics received a rather formal introduction, which involved detailed lecture notes filled with definitions and corresponding example problems. He used the example problems to clarify definitions, continually referring back to the definitions as he worked through the examples. This process built the foundation for incorporating real world applications of parametric equations into his lessons, which Justin described as "interesting mathematics" (Interview 3).

Most of the mathematics developed during my observations was situated within a real world context, where the majority of class time was spent with students working in small groups to solve contextualized word problems. The initial purpose of the real world context was to help his students build their understanding about a particular mathematical topic. He explained, “Seeing where it came from, hopefully they can remember it and it sticks with them, and they know what everything is and how to use it” (Interview 1). However, the kinds of word problems he used eventually became familiar and rather repetitive. As a result, the purpose of using the real world contexts seemed to shift, because he used these problems to help his students develop procedural knowledge.

Justin actively monitored his students’ progress as they worked in small groups. He frequently asked or answered questions about mathematical procedures. He remarked, “I always ask them why, ‘Why are you saying that? An answer isn’t good enough. Tell me why you are doing what you are doing’” (Interview 4). By asking why, Justin expected his students to describe the steps necessary to solve a problem. The focus on procedures seemed to include an emphasis on using particular procedures. There were several episodes in Justin’s teaching in which he explicitly told his students to use a particular method to solve a problem. In these episodes, Justin seemed to value certain methods over others, because certain methods allowed for uniformity and efficiency.

In the following section, I use classroom episodes to illustrate how Justin introduced the topic of parametric equations. These episodes characterize how Justin developed a topic by building a foundation for his students.

Development of a topic: Parametric equations. To begin his lesson on parametric equations, Justin wrote the following starter on the board: Plot the graph of $f(x) = \pm\sqrt{x-1} + 2$ if x = 1, 5, 10, 17. He told his students, “The only thing I want you guys to be able to get out of this [starter] is being able to plug numbers in something that maybe you don’t know the shape of, and get correct answers out here” (Observation, October 7). With this starter, Justin began to build a foundation for his students, because he used this starter to indicate how he expected his students to work through problems involving parametric equations.

Justin's formal lecture notes began with the definition of a parametric curve. He presented the following definition:

> Parametric Curve: The graph of the ordered pairs (x, y) where x = f(t) and y = g(t) are functions defined on the interval I of t-values is a parametric curve. The equations are parametric equations for the curve, the variable t is a parameter, and I is the parameter interval. (Observation, October 7)

He admitted that this was "quite a hefty definition" (Observation, October 7). He reminded his students that they just found several ordered pairs in the starter. He continued by saying they could use the ordered pairs to create the parametric curve. Noticing several confused looks on his students' faces, Justin said,

> What I want to do is jump into one. I know we will come back and make sense of that definition. Let me show you what one looks like. Here is an example of what one looks like, and we will come back and make sense of everything I hope [writes an example on board]. $x = t^2$-2, and $y = 3t$ where $-2 \leq t \leq 2$. What I want you guys to do, without knowing really what they are, what we are going to do with them, or why they are useful, I want you to graph them on this interval. Reread the definition. See if it makes sense. We are going to build on this quite a bit today. (Observation, October 7)

Justin used this example problem, along with similar problems, to help his students understand and "make sense" of the definition. They worked through each example by creating a t-table and constructing the corresponding graph of the parametric curve. At the end of each example problem, Justin reread the definition and clarified components of the definition by reviewing the related aspects in the example problems.

Justin helped his students as they worked in small groups on some related exercises. During this time, he noticed a student who did not create a t-table like the one he developed in his lecture notes. The following episode demonstrates how Justin seemed to prefer particular methods.

Justin: Student, you did an almost perfect *t*-table.
Student: Do I have to have a *t*-table?
Justin: You don't have to have a *t*-table, you got it, but I really, really like a *t*-table.
Student: What did I do wrong?
Justin: Nothing wrong at all, I just had a little different *t*-table.
Student: Oh, ok.
Justin: [addressing the whole class] I know some of you I saw, found the graph, found the correct things that are happening, cause this one is kind of not so difficult right now, try to do a *t*-table like that [pointing to *t*-table on the board], cause later on they are going to get a little more, a lot more complicated. (Observation, October 7)

In this episode, Justin described the student's work as "almost perfect." This description seemed to imply that he valued certain procedures over others because he viewed certain procedures to be correct procedures. Then, when Justin addressed the class, he emphasized that his *t*-table would serve as a better organizational tool when his students encounter more complex problems.

After building the necessary foundation, Justin spent the last part of the lesson developing what he called "the big connection" (Observation, October 7). He hoped this connection would help his students understand the purpose of parametric equations and why they are useful. He asked his students to consider what kinds of real world problems might require separate equations for vertical and horizontal distance with respect to time. He then presented his students with "the Derek Jeter problem," where he asked his students to determine if Derek Jeter hit a homerun in a given situation (Observation, October 7). Given this context, students realized that they had to take both the horizontal distance of the fence as well as the height of the fence into account with respect to time to solve the problem. This problem allowed Justin's students to see the usefulness of parametric equations in a real world context.

Over the next several class periods, Justin used several word problems similar to the original baseball problem. As they learned more about parametric equations, he included subtle nuances to differentiate among the problems. However, despite the subtle nuances, the kinds of problems Justin used became familiar and procedural. To illustrate, a student asked Justin what they would be learning the following day,

> Student: What are we learning?
> Justin: What is tomorrow? We will be doing more difficult problems of this.
> Student: We are expanding?
> Justin: It is the same equations, same stuff, and different problems. (Observation, October 20)

Justin thought it was necessary for his students to work through a significant amount of problems "just to understand the concept" (Interview 2). He shared this belief with his students, "They start to get easier, the more you do. Cause it is really the same thing over, and over, and over" (Observation, October 20). Therefore, this repetition seemed to be the final method Justin used to build a foundation for his students. The following section explores the kinds of mathematical connections Justin made in his practice.

Mathematical connections made in Justin's practice. As the teacher, Justin contributed the majority of the mathematical connections made during my observations. Not every kind of connection was present within his instruction, because Justin did not make any connections between methods. Many of the connections Justin made in his teaching were through comparisons or using real world contexts. At times, the connections he made through comparison also occurred within a real world context. In addition, several suggested connections occurred in his teaching. The following sections provide additional details about the connections Justin made for his students.

***Contributions of mathematical connections*.** Justin regularly interacted with his students. However, the majority of his interactions with students were discussions of mathematical procedures, in which he asked or answered questions about how to do things. Within these interactions, few mathematical connections occurred, because the focus of these interactions usually involved Justin telling (or asking) the student what to do next. As a result, Justin's students rarely contributed components of a mathematical connection. The following episode is a typical example of the kinds of interactions that took place between Justin and his students. The episode begins with a student's question.

> Student: Ok. That is how you would do this, right?
> Justin: Divide by 3600. That is it.
> Student: All right. Now, what unit is that in?
> Justin: That is now going to be feet per second.
> Student: Ok. (Observation, October 20)

In this episode, the student directed a question at Justin, asking a specific question about a procedure. Justin reviewed the student's work and told her what to do next. This episode is an example an interaction between Justin and a student in which no connection was made.

There were, however, a few episodes when students contributed components of a mathematical connection or provided the relationship between the components. Students made contributions when asked to describe the steps needed to solve a given problem. Within these episodes, students' descriptions of procedures occurred as connections through logical implications. For example, Justin presented the whole class with a word problem about a baseball player hitting a homerun over the ballpark fence. He asked his students how they might go about solving the problem.

> Justin: This is a tough problem the first time you do it. Once we go through this together it is not as difficult. But to think of solving this on your own without me helping at all is very, very difficult….I just don't want to give it to you. All right, now the fence is 350 feet away. I like drawing this sometimes [draws a picture on the board]. Right here is 350 feet away from where this guy hits the ball. All right, and the fence is 20 feet. So, the ball is going to travel some type of [draws a parabola], Does it clear that fence? All right, what are some of the problems with finding an answer to this? Student?
> Student: Don't you have to let $x = 350$?
> Justin: Why?
> Student: Cause that is what the fence is.
> Justin: How would that help you?

Student: If you plug in 350 for *x*, and [then] you can get *t*, and then whatever your *t* equals, you can plug that in to get *y*, to see if it is greater than the fence.
Justin: Why?
Student: To see if he clears it. (Observation, October 19)

In this episode, Justin asked several questions to help the student elaborate on his original response and describe a procedure for solving the problem. Describing the steps required to solve the problem, the student provided connections through logical implications, such as, "If you plug in 350 for *x*, and [then] you can get *t*" (Observation, October 19). The procedural nature of the connections provided by this student resembled the nature of the mathematical connections contributed by Justin.

Many of the explicit connections Justin contributed took place during his formal lecture notes. These connections occurred as provided connections as well as provided-and-explained connections. Justin regularly contributed multiple connections within a given lecture. For example, during his introduction of parametric functions, Justin used graphing calculators to demonstrate how the parametric functions combined to create a parametric curve. He asked his students to graph the following parametric functions, $x(t) = -0.1(t^3 – 20t^2 + 110t – 85)$ and $y(t) = 5$ (See Figure 2). He explained his reasoning in using this particular set of functions.

Justin: By putting a constant in here, 5, this is not really going to be a parametric equation right here, when we graph these. This is only going to show you the x as the horizontal change. It is just a horizontal value. It is going to show you what the horizontal part of this looks like. Just the horizontal. Just left and right. You won't see the up and down….you get a straight line. (Observation, October 7)

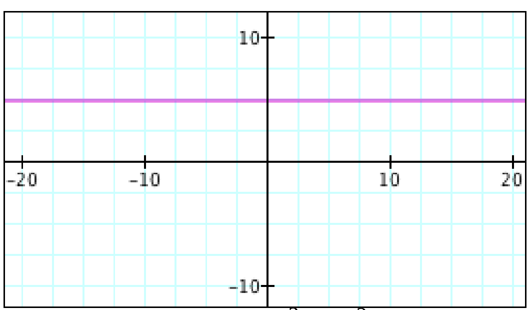

Figure 2. Graph of the $x(t) = -0.1(t^3 – 20t^2 + 110t – 85)$ and $y(t) = 5$.

Justin then asked his students to change $y(t)=5$ to $y(t)=-t$ (See Figure 3). The new graph was no longer a horizontal line. In this demonstration, he contributed a provided-and-explained connection as he compared the graphical representations.

> Justin: Hopefully this will be an "Ah-Ha!" moment.
> Student: Ah-Ha.
> Justin: Well, that straight line was boring.
> Student: I was perfectly content.
> Justin: Hopefully something interesting happens. Now watch what happens when you graph it.
> Student: Woah!
> Justin: Now this is the big difference here. All right. Now, initially, when we plotted just the *x* function, if you read the problem, it is about a guy walking on a street somewhere, I think. And, if we just look at *x* equals, he is just going like this, but what does this thing down here mean, that he did?
> Student: He turned around.
> Justin: He turned around, and walked backward, back. And, then, he turned around again, and kept on walking. Now, I don't know why he did that. But, maybe he saw something on the ground and went back to go get it. He dropped his cell phone, and he noticed it. All right. Without parametric equations, we would not have known he did this thing [he turned around], right here. (Observation, October 7)

In this demonstration, Justin contributed a provided connection as he compared the different graphical representations. He explained the connection through his use of the different sets of parametric equations. Justin also provided a connection by using a real world example to describe aspects of the parametric curve.

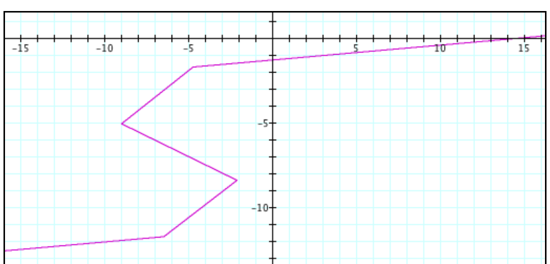

Figure 3. Graph of the $x(t) = -0.1(t^3-20t^2 + 110t - 85)$ and $y(t) = -t$.

Connecting through comparison. During my observations of Justin's teaching, the majority of the connections he made were through comparison. Many times, Justin provided a connection by comparing the procedure a student needed to solve a problem to the same procedural method Justin presented in his lecture notes. Justin also provided connections by comparing the differences existing between functions. For example, he asked his students to consider how a horizontal gust of wind might influence the parametric functions used to simulate projectile motion (Observation, October 21). His students proposed several different modifications to the original parametric functions. Justin wrote their conjectures on the board and made comparisons among the modifications they recommended. He continued this process of making comparisons until his students recognized which function correctly accounted for a horizontal gust of wind. In a similar example, Justin compared the similarities and differences existing between vector equations and the parametric functions used to simulate projectile motion (Observation, October 18). Connections through comparison were often procedural in nature, because many of these connections occurred as Justin made comparisons between particular types of procedures or equations for his students.

Throughout the unit on parametric equations, Justin rarely made connections through comparison in which the content of the individual components of the connection was from different mathematical strands or seemingly unrelated topics. Across my fourteen days of classroom observations, I saw only two examples[12] of connections where Justin related a topic from the unit on parametric equations to a seemingly unrelated topic from a previous unit. The first example of this kind took place when Justin compared the trigonometric functions within the individual parametric equations to the unit circle.

> Justin: Do you see a relationship to why *x* is cosine, can you remind them of that?
> Student: Because of the points on the unit circle?
> Justin: Yeah. Always has been, *x* has always been related to cosine, right, and *y* has always been related to sine. (Observation, October 11)

[12] Justin made each of these two connections multiple times for different groups of students.

This episode captured one of the two examples of a connection between a current topic and a topic from a previous unit. The second example of this kind occurred when Justin provided a comparison between procedures: “Remember how we adjusted the sinusoidal problems, to make it work, adjusting this [problem] in the same way” (Observation, October 18). Despite making few connections to mathematical content from outside of the unit on parametric equations, Justin included different levels of real world connections throughout his instruction.

Connecting to the real world. Justin used real world contexts in every lesson I observed. Some times he used a real world context to support the development of a mathematical topic, and in doing so he provided (and sometimes explained) a connection to the real world. For example, Justin used real world examples to make comparisons between parametric equations simulating projectile motion and vector equations.

> Justin: A vector had a speed and a direction [writes $v = \langle |v| \cos(\theta), |v| \sin(\theta) \rangle$ on board]. For example, if we talk about planes, like motorized planes, we don’t have to worry about some of the forces because the motor can overcome them. We can keep a plane at a constant speed and altitude because of the motor. So, we didn’t really have to worry about up and down motion. We just looked at like a 2-dimensional plane. *x* and *y*. At a constant altitude the whole time. Because if you have a motorized vehicle, it can overcome gravity through the power of the motor. Projectiles, they are not motorized. They are thrown, thrusted into the air, they don’t have any, like a rocket, to propel them, it is like instantly thrown, whatever the velocity is initially, that is all you have. You don’t have anything to keep pushing it. All right. So, these [equations] change a little bit. We are going to change this right here [points to the equations for the vector] into an *x* equation and a *y* equation. *x* is going to be $t \cdot v \cdot \cos(\theta)$. *t* is your time. *v* is your?
> Students: Velocity.
> Justin: *v* is your initial velocity [writes the equation $x = t \cdot v \cdot \cos(\theta)$ on the board, directly below the vector equation]. So, let’s do this out here. t = time, v = initial velocity [writes t = time, v = initial velocity on board]. Now, *x* is normally the easy one. You normally have one term right there. *y*. *y* is the tough one. It starts off the same. Time times your initial velocity times the sine of your angle. But, we have a couple of things that can come into play. Remember *y* is your vertical, your height.
> Student 1: *y* is your gravity.
> Justin: Gravity pulls against you. So, it is going to be a $-1/2\, gt^2$….Alright, but there is something else in the y that we have got to take into account.…
> Student 2: The height.

Justin: The what height?
Students: The starting. The initial.
Justin: The initial height. And, we use, in this book, they use s for initial height [writes the equation $y = t \cdot v \cdot \sin(\theta) - 1/2 \cdot g \cdot t^2 + s$ on board]. (Observation, October 18)

In this episode, Justin made multiple provided-and-explained connections. First, he described motorized planes as a real world example of vectors, because the motor is able to keep the plane at a constant speed. He then continued by describing rockets as an example of projectile motion, because without a motor to propel the rocket the initial velocity does not remain constant. By comparing these real world examples, Justin was able to provide and explain the differences existing between the parametric equations simulating projectile motion and vector equations for his students.

More commonly, however, Justin used real world contexts within word problems as a method to practice procedures. These problems consisted of similar contexts with different numbers, requiring the same procedural steps as the previous problems used to introduce mathematical topics. For example, Justin asked his students to work in groups on the following problem.

Problem written on the board: A baseball player hits a baseball with an initial velocity of 160 ft/sec from a height of 3 feet with an angle of 21 degrees from the horizontal. Will the ball clear a fence 420 feet away that is 20 feet high? If yes/no, how much/less than the fence? If the fence wasn't there, how far would the ball go?
Justin: Do you guys remember how to do this?
Student: Kind of.
Justin: All right, let me set you free. (Observation, October 19)

This problem, like several others, was almost identical to the "Derek Jeter problem" Justin used to help his students understand the purpose of parametric equations. When used in this way, these word problems acted as suggested connections, because Justin did not provide the connection between the mathematics and the real world context beyond the words surrounding the numbers in the problem. Therefore, many of the connections Justin made to the real world occurred as suggested connections.

Suggested connections. In Justin's final interview, I asked him to describe how mathematical concepts and procedures were or were not related to one another. He responded by saying, "Most of them build on each other. You have to know one before, at least you have to know one before you should go on to the next. …There are things that you have to know before you can go to the next thing." (Interview 6). His response reflected a pattern that occurred within his instruction, because he built mathematical concepts and procedures by developing related topics, one directly after another. As a result, he made several suggested connections from this pattern of instruction as he tried to build a foundation for his students.

In Justin's teaching, I observed suggested connections in that he juxtaposed concepts or procedures without explicitly providing the relationship. Therefore, the suggested connection resided within the proximity in time between developing *A* and then *B*, implying *A* and *B* were somehow related. For example, during a lesson on creating parametric equations through two given points, a student asked Justin how this topic was related to other mathematics. The student's question acted as the cue suggesting a connection may exist.

> Student: What does this like actually relate to? Anything else in math? How does it help us?
> Justin: Oh, we are getting there.
> Student: Ok.
> Justin: Remember, we are about a 1/3 of the way through our parametric unit because of midterm. So, you have learned the basics, and when we come back [from the weekend], we will actually start doing the magnitude, cos (θ), with velocity and gravity. You all are basically learning what a parametric means right now. Like, a time, and a distance. And, we haven't even gotten into height yet. So, that is where we will come back on Monday. (Observation, October 11)

In this episode, Justin responded to the student's question by suggesting that "the basics" of parametric equations were related to "magnitude, cos (θ), with velocity and gravity"

(Observation, October 11). Although Justin developed each topic, one right after the other, the relationship between the mathematical topics remained implicit throughout the remainder of the unit. This pattern of suggested connections occurred across my observations of Justin's instruction.

Robert Boyd: Lectures with Reasons

I observed Robert teach his Mathematics II students a unit on piecewise, step, and exponential functions. He said he appreciated teaching this unit because "there is a logic to [functions], and it has something to do with the way you actually write everything down [and] how it is broken into pieces" (Interview 1). During my observations, Robert consistently used the majority of the 90-minute class period to lecture on a given topic. He began by reviewing content necessary to introduce the topic of the day's lesson. The main body of his lecture included a few example problems, which he developed in great detail. The example problems he used were almost always purely mathematical. He asked questions throughout his lecture to engage his students in the lesson. He asked questions about the reasons certain procedures worked the way they did, thus developing lectures with reasons. At the end of his lecture, Robert gave his students a brief period of time to complete a few mathematical exercises. During this time, he expected his students to work individually while he answered questions one on one.

In his lectures, Robert developed new topics by relating them to topics from past lessons, previous units, or previous mathematics classes–things that his students already knew and understood. This was a defining characteristic of his instruction. With this pedagogical approach, he tried to make new mathematical concepts and procedures seem less complicated for his students.

> I am able to present it as something that is not new. "Ok, this is not some other thing that you have to know. It is what you already know." I think if the kids can get that into their

> mind, first off it becomes simpler because they are like, "Oh yeah, I already know that." It is already in their brain. They are making connections…. They are making connections with things they already know, they are building on things they already know, so it is not a brand new creation. It also takes away some of the confusion, where when they look at a function, they don't have to say, "Is it this kind of function, or this kind of function, or this kind of function?" They know that when there is a +7 out at the end, it doesn't matter which kind of function it is. They know what the +7 means. I think that just helps them so much. It saves them a lot of confusion, and I think it makes it less intimidating, which for a lot of the kids is a huge deal. (Interview 2)

He also thought this approach helped his students see "the big picture and how everything fits together" (Interview 3). Therefore, in his lectures, Robert made explicit comparisons between the old and the new as a way to help his students learn mathematics.

Across Robert's lectures, he developed the reasons behind mathematical rules and procedures. He discussed why he thought it was important to provide lectures with explicit reasons for his students:

> To understand why that happens. To understand…there is a reason. And if you understand that reason then you'll remember it better, you'll use it better, [and] it is more useful to you. I think that answering that why question is important if you can. So even when we do the Quadratic Formula, even though they don't even know completing the square yet, …if I have spare time, I always like to show it to them and just say, "Ok here is our general form. We're going to go through this method, you don't have to know the method, I'm not teaching you the method, but I want you to see where the Quadratic Formula comes from" just so they know where it comes from. Because I think they need to understand that somebody just didn't sit down and make something up. There is a reason this is what it is, and it is because that is the way the world is and that's the way it works. Not because some guy 100 years ago decided he wanted to come up with something for people to do. It is that way because that is the way it works. (Interview 2)

Robert incorporated this kind of reasoning throughout his lectures to help his students understand mathematics. He hoped this method would help his students see beyond the mechanical steps of a procedure so that they might understand how and why things work in mathematics.

In the next section, I use classroom episodes to illustrate how Robert introduced the topic of piecewise functions. These classroom episodes reflect the way he developed mathematical topics for his students. Therefore, I include these episodes because they provide evidence for the patterns of instruction I identified within his practice.

Development of a topic: Piecewise functions. Robert used a starter to begin the day's lesson on piecewise functions. He asked his students to graph an absolute value function, which was something his students learned in their previous mathematics course. He told his students why he was using absolute value functions to introduce piecewise functions, "#1. They are simple. #2. They are familiar, right? We have seen them before. And, #3, cause the state of Georgia says we have to" (Observation, October 31). After his students completed the starter, he said, "We are going to look at the absolute value graph in a different way. We are going to look at it as a piecewise function. So, this is our new topic for today, piecewise functions" (Observation, October 31). This episode is an example of how Robert developed a new topic by relating it to a more familiar one.

Robert spent approximately a third of his lecture developing a reasonable way to think about $f(x)=|x|$ as a piecewise function. First, he asked his students to consider where it would make sense to break the graph of the absolute value function into pieces.

> Robert: If you were going to break this graph [points to absolute value function on the board] into pieces, where would be the logical place to break it?
> Student 1: The origin.
> Student 2: The vertex.
> Robert: Ok, I heard the vertex, I heard the origin. Any other ideas?
> Student 3: The center.
> Student 4: The axis of symmetry.
> Robert: I am not saying anything is terribly right or wrong, ok, axis of symmetry. The center. Ok. Where is, while we are on the subject, where is our axis of symmetry?
> Student 4: (0, 0)
> Robert: (0, 0) is not an axis, it is a point. x=0 is the axis of symmetry. Ok? x=0. Otherwise known as the y-axis. That is my axis of symmetry, right down the middle [traces the y-axis on graph]. Ok. So, if I say origin, which is right there [points] or if I say the vertex, which is right there [points], or if I say axis of symmetry, which is right there [points], I am basically saying the same thing. So, we kind of got the same

> idea. And that makes sense to split it there. Can anybody put into words, why you picked that point? Somebody who said origin, or vertex, or axis of symmetry?
>
> Student 5: Cause it is in the middle. (Observation, October 31)

The episode continued as Robert drew additional absolute value functions with various horizontal and vertical shifts. Making comparisons across the different graphical representations, Robert used these additional graphs to help his students understand the reason why the vertex was the appropriate location to break the graph of the absolute value function into pieces.

> Robert: If my graph looked like this, [draws new function, shifted to the right] would you still pick here [points to the origin] [to break it into pieces]?
> Student 1: No.
> Robert: Why not?
> Student 1: Because it isn't the axis of symmetry.
> Robert: Ok, where would you pick?
> Student 1: Where it drops down.
> Robert: Here [points to the vertex of the new function]. What is special about this point, besides the fact that it has a name, and we call it the vertex?
> Student 2: It changes direction.
> Robert: It changes direction, right? This is where the graph changes. And, that is what we are going to do with these absolute value functions. We are going to split them at the vertex, and the reason we are going to split them at the vertex, is because that is where the graph actually changes. (Observation, October 31)

He continued by describing the method for writing the first piece of the function followed by the method for writing the second piece of the function. He reminded his students that the separate pieces of the absolute value function should look familiar, because, as pieces, they looked and acted like linear functions. This episode is a typical example of how Robert developed lectures with reasons to help his students remember why things worked the way they did. Also typical of Robert's instruction was the pattern of students responding to his questions with short answers. This pattern of teacher-student interaction took place throughout his instruction, in which students' questions and comments were always directed to Robert instead of their peers. The following section explores the kinds of mathematical connections Robert made in his practice.

Mathematical connections made in Robert's practice. A common theme existed across the mathematical connections Robert made in his instruction. He connected new topics to things his students already understood, "[If] I can help them put those pieces together, or [if] I can put the pieces side by side and then hope that they can put them together, because I think that really helps them a lot" (Interview 3). Each kind of mathematical connection was present in his instruction. However, of the mathematical connections he made, the majority of these connections were through comparison. In addition, he made some connections to the real world. Both kinds of connections resembled the common theme occurring in his instruction. The following sections use classroom episodes to explore in detail the kinds of mathematical connections Robert made for and with his students.

Contributions of mathematical connections. Robert's students contributed to approximately one-fourth of the mathematical connections made in his classes during my observations. Students made contributions in response to the questions Robert included throughout his lectures. Students' contributions usually occurred in certain kinds of connections, for they usually contributed components within a connection through logical implication or a connection through comparison. For example, a student contributed part of a connection during a lecture on step functions. While graphing a step function, Robert asked a question that prompted the student's contribution.

> Robert: How do I know I can't have 2 *y*-intercepts, you all, cause 2 *y*-intercepts would be right on top of each other. And, if they are right on top of each other?
> Student: [Then] it fails the vertical line test.
> Robert: It fails the vertical line test. Very good. It is not a function any more. (Observation, November 4)

In this episode, the student contributed a component of a connection by providing the latter portion of a connection through logical implication. This was the most common form of contributions from Robert's students.

Robert contributed the majority of connections made in his instruction, in which he regularly contributed provided-and-explained connections. This level of connection existed in Robert's teaching as he developed the reasons behind mathematical rules and procedures. In his final interview, Robert described why he thought it was important to develop this level of connection for his students.

> I have this desire for students to actually understand what the heck they are doing. Not to just do it, but to be able to get it. Because if they understand why it happens, then I figure they stand a better chance of transferring it to other ideas later on. (Interview 6)

Robert believed it was important to make these mathematical connections explicit for his students. For example, while examining the characteristics of a particular piecewise function, Robert contributed a provided-and-explained connection by comparing the difference between "none" and "0" as an answer for describing the constant interval of the particular piecewise function.

> This [piecewise function] does not have a horizontal part of the graph, which means this does not have a constant interval. So, for my constant interval, I will put none. If it makes you feel better, you can put empty set. There isn't one. That is our math symbol for none. Please do not put 0, 0 is a number if you are saying it is 0, then you are saying it is constant at 0, which doesn't make sense, because every point is constant. Ok. 0 doesn't change. (Observation, November 3)

In this mathematical connection, Robert pointed out that an answer of "none" is not the same as an answer of "0," because an answer of 0 would indicate that there was a constant interval located at 0. Robert seemed to contribute provided-and-explained connections specifically when working through a mathematical procedure. In his lectures, he usually gave his students detailed step-by-step instructions to carry out the procedure. However, he almost always explained the reasons why the procedure worked the way it did.

Connecting through comparison. During my observations of Robert's practice, more than two-thirds of the connections he made were through comparison. Many of these connections through comparison also occurred as provided-and-explained connections. Given this kind of connection, he made several comparisons across representations of functions, in which he usually compared a new representation to a representation that was more familiar to his students. For example, in his introductory lecture on exponential functions, Robert compared the similarities between the standard form of the exponential function and the vertex form of a quadratic function.

> Robert: An exponential function…. The x is in the exponent. So, this is what it always is going to look like [writes the standard form for the exponential function on board, $f(x) = ab^{x-h}+k$]. I am going to have some number [points to a], multiplied by my base, b, and my base has an exponent on it. That exponent might have something subtracted or added to it [points to x–h]. That whole expression might have something added or subtracted to it[(points to k]. Everybody, remember, we are harkening back to when we did our vertex form. $f(x)=a(x-h)^2+k$. [writes the vertex form of a quadratic function on the board, $f(x)=a(x-h)^2+k$]. You might notice that some of the letters in this [points to $f(x) = ab^{x-h}+k$] are the same as the letters in this one [points to $f(x)=a(x-h)^2+k$]. Some of the variables in this [points to $f(x) = ab^{x-h}+k$] are the same as the variables in this [points to $f(x)=a(x-h)^2+k$].
>
> Student 1: Why did they use those letters?
>
> Robert: Why did they use h and k? I don't know. Take a look here all. The reason I want to keep using these letters, is because we already used them here [points to $f(x)=a(x-h)^2+k$] and because they do the same thing. Over here [points to $f(x)=a(x-h)^2+k$], what does a do?
>
> Students: Distributes.
>
> Robert: No. I am talking about to my graph.
>
> Student 2: Shifts?
>
> Student 3: Stretches.
>
> Robert: It stretches. This is my stretch. Or, if it is negative, this will flip it over, right? A reflection. What do we call this part [points to h in $f(x)=a(x-h)^2+k$]?
>
> Students: Opposite land.
>
> Robert: This is opposite land, and opposite land tells me what? It is going to do the opposite, but it is going to do what to the graph?
>
> Student 4: Shifts it right or left.

Robert: Shifts it, right or left. There you go. …Now, this is my right, left, shift. This over here [points to k in $f(x)=a(x-h)^2+k$)] is what?
Student 5: Up down shift.
Robert: It is what?
Student 5: Up down shift.
Robert: It is the up down shift. The vertical, or the up down shift. B: All right. So, guess what this is going to do [points to the a in $f(x) = ab^{x-h} + k$].
Students: Stretch.
Robert: It is going to stretch it. What if it is negative?
Students: Flip.
Robert: It will flip it….All right, what is, now this is my base [points to the b in $f(x) = ab^{x-h} + k$]. The base doesn't do anything. The base, this right here, is just the parent function. b^x would be the parent function. So that number b, it is not a shift, it is not a stretch, it is just what we started with. Ok. That is it. Now, what is that up there going to do [points to the h in $f(x) = ab^{x-h} + k$].
Student 6: Opposite land.
Robert: That is opposite land, and it is what shift? Which way?
Students: Left, right.
Robert: Left, right. Ok. That is my left right shift. And, k down here [points to the k in the $f(x) = ab^{x-h} + k$] is going to be?
Students: Up down shift.
Robert: Up down shift. My vertical shift. You all, if you knew vertex form at all, you know this [standard form of an exponential function]. You know it, already. (Observation, November 9)

In this episode, Robert, with the help of his students, made several provided-and-explained connections between the symbolic representations, by comparing the parameters (what Robert called variables) in an exponential function to the parameters in a function that was more familiar to his students. For example, he provided a connection through comparison by saying the a in the exponential function was the same as the a in the vertex form of a quadratic function. He also explained this connection by describing a as the stretch for each function. Overall, Robert provided and explained the connection between the two functions, explaining the similarities between the two functions because of the parameters. In the final interview, Robert described his purpose in comparing the new function to a more familiar one, "I really wanted to start with something that I figured they would understand. …So it wouldn't be intimidating to them. Then, of course, the bonus is that you could always refer back to it over and over again" (Interview 6).

Some of the connections Robert made through comparison examined the differences existing between representations. For example, Robert made comparisons between the following exponential functions, $f(x) = 2^x$ with $f(x)=2^{x-2}$. He compared the differences between the table of values for each function as well as the differences between the graphs of the each function.

> Robert: Let's take a look at this function [points to $f(x)=2^{x-2}$]….We will graph it when we are done. Let's write our function values first. I put in 5, what do I get out [begins to create a table of values]?
> Students: 8.
> Robert: 8. Because if I put in 5, I get 2^3, which is 8. If I put in 4, I get 2^2, which is 4. And, then, I put in 3, I get 2^1, which is 2. And, I have seen this pattern somewhere before. Right? What is going to be next?
> Students: 1
> Robert: 1. Next?
> Students: 1/2.
> Robert: And, then?
> Student 1: 1/4, and 1/8, and 1/16.
> Robert: Ok. Now let us compare how this is different to $f(x) = 2^{x-2}$. My parent function would be 2^x, right? So, if I put in 0, I get 1, I put in 1, I get 2. [Fills in the table for $f(x) = 2^x$] 4, 8, 16, 32. 1/2, 1/4. This 1 [pointing to 1 in table of values for $f(x) = 2^x$], when I put in 0 into the parent function, what do I have to put in to get 1, on my new function [pointing to $f(x) = 2^{x-2}$]?
> Student 1: 2.
> Robert: I have to put in 2. So, my number that was at 0, got moved to 2.
> Robert: This is my *x*. So if what was at 0 got moved to 2, how did it get shifted?
> Student 1: [Inaudible]
> Robert: On the *x*-axis? It is not up.
> Students: Over. It is over.
> Robert: It is over which way?
> Student 2: Right.
> Robert: Right. To the right. To the right, it got moved to the right 2. Ok, we will take a look at that. Let's check that out real quick [graphs both functions]. Cause I know some folks are going, "Ok, I get that." And some other folks are going, "What?" So, I want to make sure you all understand what I am talking about here. … And, so the parent graph looks like that [graphs $f(x)=2^x$]. Now, my new graph, if I take those points, and do to 2^{x-2}, those points I get, (0, 1/4), I get (1, 1/2), I get (2, 1), (3, 2), (4, 4), (5, 8) [plots points on the graph], so it looks like this [completes the graph of the function]. And so you see this point that was here, is now here [circles the points

> when y=1 on each graph]. So, my graph was shifted to the?
> Students: Right. (Observation, November 9)

In this episode, Robert made several connections as he compared the table of values for each function, the graph of each function, and the input for each function when the output was equal to 1. The provided-and-explained connections seemed to help his students recognize the differences between the different representations of each function as well as the reason why the –2 in the exponent shifted the parent function to the right. In addition, during this episode, some of the connections he made were through logical implications, where his students, at times, contributed the latter portion of this kind of connection. This episode is an example of how Robert made provided-and-explained connections by examining the reasons behind concepts and procedures throughout his teaching.

Some of the connections through comparison took place in moments when Robert warned his students about possible misconceptions. These warnings seemed to follow from his experiences with former students and their misconceptions about mathematics. Robert usually embedded his warnings within a discussion of a mathematical procedure, where he compared the procedure to something it did not do. For example, when Robert constructed a table of values for the function $f(x)=2^x$, he warned his students about possible mistakes made when calculating powers of 2.

> Now, if I put in 3, I get 2^3. 2^3 is 2, 4, don't say 6. Everybody says, well not everybody, but people say 6 a lot, be careful, because we are used to counting by 2's. 2, 4, 6. But we are not counting by 2's. We are multiplying by 2's. So, it is 2, 4, 8. 8. Be careful. Everybody that is the #1 reason people miss problems on tests that they should never ever miss in a million years, because they do powers of 2 wrong. Careful with your powers of 2. Nobody seems to have that trouble with any of the others it is just 2's because we are used to counting, because we count by 2's sometimes. (Observation, November 8)

During this episode, Robert's warning provided a connection, because he compared powers of 2 to counting by 2. In his final interview, Robert described his purpose in making this kind of connection through comparison. He said, “I’m just trying to head off the little mistakes that make them miss problems” (Interview 6). Robert thought these warnings helped his students.

At times Robert also used connections through comparison to help his students remember characteristics of certain functions. The result of the comparison was a kind of mnemonic device, because the connection supplied his students with some sort of imagery or analogy as a trigger for their memory. For example, Robert provided a connection when he told his students that they could easily remember the shape of an absolute value function, because the shape of the graph was the same shape of the letter *V*, the letter used to begin the word *value* (Observation, October 31). In a different lesson, he realized his students were having difficulty differentiating between graphs of exponential growth and graphs of exponential decay. To help his students distinguish between the two, he used a simple analogy for them to remember the shape of an exponential growth function.

> So, you are born. The asymptote is where your parents live. So, you live with your parents. You live with your parents [tracing the graph]. Then you get to be a teenager. And, I don’t want to be by my parents so much and then you get into your twenties. Then you get married. And then, you are gone. (Observation, November 10)

This analogy was a provided connection, comparing the shape of the exponential growth function to the process of growing up. Without exception, Robert used these connections through comparison solely to help his students remember certain aspects of mathematics.

Connecting to the real world. Connections to a real world context were rare during my observations of Robert’s teaching. In fact, he only made connections to real world contexts in two of my thirteen classroom observations. Given this kind of connection, Robert used examples from the real world as a familiar context to learn new mathematical topics. For example, he used a real world context to introduce step functions. He used the context of paying for parking in a parking garage to discuss the way step functions round real numbers to integers. After he discussed parking fees in a parking garage with his students, he provided the connection when he remarked,

> Let's look at the warm-up again. It wasn't just a warm-up. It had another purpose. You all just did some math. You thought you were talking about a parking garage, but you just did some math. What you secretly did, what you did without even knowing it was a step function. (Observation, November 4)

In later observations, he referred back to this context to remind students how step functions operate. This episode is another example of how he connected step functions, which was something new, to parking fees in parking garages, which was something more familiar to his students. As a result, the content of the components within this connection was similar to the nature of the content of the components he connected through comparison.

During a lecture on exponential functions, Robert provided a connection to the real world when he ended the lesson. He used the familiar context of money and interest as a real world example of exponential growth.

> Robert: Everybody this is something you really need to understand. We talk a lot about a lot of different things in math, and everybody says, "Oh, what does this have to do with my life?" and, "We are never going to do this when I get out of high school," all right, this affects you every single day of your life, and probably will for the rest of your life. That is interest.
> Student 1: Oh?
> Robert: Interest grows exponentially. So, I heard it said once, interest is either your best friend or your worst enemy. When is it your best friend?
> Student 2: When it is your money.
> Robert: Your money that is earning interest. And it is your worst enemy when you owe money. That is exactly right. You all, if there is anything that you remember for years and years, remember to be wary of interest. Remember to make it your friend and not your enemy. If you can earn interest, awesome. Interest, compound interest, grows exponentially. (Observation, November 11)

Within this episode, Robert emphasized the importance of understanding the implications of compound interest representing exponential growth. The few connections Robert made to real world contexts seemed to reflect his dislike for what he described as "real world" problems in school mathematics. He defined those kinds of problems as problems with an unnecessary context and problems his students did not care about. Instead, he seemed to make connections to the real world rather judiciously, using contexts as a way to help his students understand the mathematics or as a way to understand mathematical principles existing in the real world.

Comparison of the Mathematical Connections Made in Practice

In this section, I make comparisons among the kinds of mathematical connections teachers made in practice. In doing so, I examine what is common across the cases as well as what is unique. I compare the ways the teachers included students in the process of making mathematical connections as well as the levels and kinds of mathematical connections the teachers made.

Students' Contributions of Mathematical Connections

Students' contributions of mathematical connections varied across the teachers' practices in three significant ways. First, variation existed in the amount of student contributions. Rachel's students made contributions to approximately half of the connections made in her instruction. Students in Robert's classroom contributed to one-fourth of the connections made during my observations of his instruction, and it was rare for Justin's students to contribute components of a mathematical connection.

The differences in the proportion of student contributions seemed to follow, in part, from the structure of the teacher's instruction as well as the content of his or her instruction. For example, many of the contributions students made in Rachel's classroom took place during whole-class discussions, which seemed to create an environment for students to contribute components of a mathematical connection or to suggest mathematical connections in the questions they asked. In comparison, whole-class discussions were not as common in Robert or

Justin's teaching. Instead, their students made contributions during lectures, which did not seem to afford students the opportunities of discussion. In addition, the few student contributions made in Justin's classes seemed to be the result of the more procedural nature of the content he developed; because during student-teacher interactions, Justin usually asked or told the students what step to do next.

The second kind of variation in students' contributions was among the levels of connections. Rachel's students regularly contributed components of provided-and-explained connections. The open-ended questions Rachel asked during the whole-class discussions prompted students to provide part of the connection, and the "Why?" questions she asked prompted students to explain the connection. In addition, Rachel's students asked questions of their own, which acted as suggested connections. To contrast, students in Robert or Justin's classroom usually contributed components of provided connections. Students' contributions followed from questions about procedures, where both Robert and Justin asked questions such as, "If this happens, then what should follow?" In each case, the level of connection contributed by the students seemed to be related to the kinds of questions teachers asked to elicit students' contributions.

Third, variation occurred among the kinds of connections students contributed in each teacher's classroom. Rachel's students made contributions across each of the five kinds of mathematical connections. Her students commonly contributed different methods to solve a problem or offered components within connections through comparison. Students in Robert's classroom contributed components within connections through comparison or connections through logical implications. When Justin's students made contributions, they usually contributed the latter component of a connection through logical implication. The variation in the kinds of connections contributed by students seemed to reflect patterns in the teacher's instruction as well as the teacher's beliefs about mathematics (see Chapter 5). For example, a common pattern in Rachel's instruction was to develop multiple solution methods for a given mathematical task. Therefore, it was not surprising that she involved students in this way, allowing them to make contributions by connecting methods.

Across the examples of students' contributions of mathematical connections, one important and common theme emerged. The teacher played a significant role in supporting or not supporting students' contributions of mathematical connections. There was a pattern of teacher-student interactions within the classroom discourse. In this pattern, the teacher often asked or encouraged students to contribute to the mathematical connection, explained the mathematical connection, or confirmed the mathematical connection. This pattern contrasts with a situation where student-student interactions establish a mathematical connection. My data indicate that the teacher was involved in *every* mathematical connection that students contributed to during my observations of practice.

Levels and Kinds of Mathematical Connections

Each of the various levels of mathematical connections occurred in each teacher's practice. In each teacher's instruction, it was more common for the connection to occur as a provided connection than as a provided-and-explained connection. Among the three teachers, Rachel made the most provided-and-explained connections, while Justin made the least provided-and-explained connections; these patterns occurred both in number and in proportion. Across the kinds of mathematical connections made in each teacher's classroom, provided connections as well as provided-and-explained connections were most frequently made as connections through comparison, and it was rare for teachers to develop provided-and-explained connections to the real world or between methods.

The most common kind of connection in each teacher's classroom was connections through comparison. However, differences emerged as I compared the other kinds of mathematical connections that each teacher made. For example, connecting methods was a kind of mathematical connection that occurred almost exclusively in Rachel's practice. She facilitated this kind of mathematical connection as her students engaged in problem solving while working through mathematical tasks. This kind of connection was rare in Robert's practice, occurring only when Robert described different ways to graph a linear equation or different ways to simplify rational expressions. In addition, this kind of connection was not present within any of my observations of Justin's teaching. I believe there are three possible reasons for the differences among the teachers' practices. First, Rachel used mathematical tasks throughout her teaching to allow her students to engage in problem solving. The investigative nature of these tasks seemed to afford opportunities for her to develop this kind of mathematical connection with her students, whereas Robert and Justin did not incorporate tasks of this nature in their teaching. Second, this kind of connection may have been absent from Justin's practice because he seemed to value particular procedures over others, which limited the development of additional methods to solve a problem. Third, the differences in the teachers' beliefs about mathematics influenced the extent to which this kind of mathematical connection occurred in each teacher's practice (See Chapter 5).

The teachers made connections to a real world context in different ways. There were times when Rachel and Robert introduced a real world context before introducing the mathematical topic (e.g., Rachel's context of giving a math award to introduce measures of variability or Robert's parking rates at a parking garage to introduce step functions). When developed in this way, Rachel and Robert used the context to help their students make sense of the new mathematical concept or procedure. In addition, each teacher provided connections to real world contexts after introducing a new mathematical concept or procedure. In their final interviews, each of the teachers said they used real world contexts in this way to help their students see how mathematics is used in the world outside of their classrooms. However, connections to the real world were rare in what I observed of Robert's teaching, whereas Justin included real world contexts every day of my classroom observations. However, many of the contexts Justin used were embedded in word problems used to practice procedures; and as a result, these word problems acted as suggested connections to the real world.

Connecting through logical implications and connecting specifics to generalities were kinds of connections more common in Justin and Robert's teaching than in Rachel's. In Justin and Robert's teaching, these connections reflected a focus on procedures. In particular, connecting through logical implications usually provided the steps necessary to work through a procedure (i.e., if this happens, then this must follow), and connecting specifics to generalities provided connections between specific examples of a procedure and the general procedure. In Justin and Robert's teaching, these kinds of connections were almost always provided connections rather than provided-and-explained connections. Rachel, on the other hand, provided connections through logical implications when citing formal theorems and properties, and she provided-and-explained connections between specifics and generalities by using examples to make a generalization about a particular concept or theorem. As a result, some of the differences among these kinds of mathematical connections the teachers made occurred in the content of the components being connected.

In each teacher's practice, some qualitative differences occurred in the nature of some of the components the teacher used to make mathematical connections. For example, at times Rachel connected an algebraic component with a geometric component for her students. She provided connections across mathematical strands when she made connections between methods or connections through comparison. In addition, Rachel made connections by using both concepts and procedures within the components of the mathematical connection. By including mathematical concepts within the connection, the focus of the connection shifted from procedural knowledge to conceptual knowledge. In Robert's teaching, the components he connected usually compared a new concept or procedure with a more familiar concept or procedure, a pattern of instruction that transcended Robert's practice. To contrast, the content of the components Justin connected was often from the day's lesson or the current unit, reflecting the way he tried to build mathematics for his students. By making mathematical connections in a linear fashion, almost all of the connections made in Justin's teaching remained within a single mathematical strand. These qualitative differences in the nature of some of the components the teachers used to make mathematical connections seemed to be related, at least in part, to the teachers' beliefs about mathematics.

CHAPTER 5

TEACHERS' BELIEFS ABOUT MATHEMATICS AND THEIR RELATIONSHIP TO THE MATHEMATICAL CONNECTIONS MADE IN PRACTICE

This chapter presents my inferences of each teacher's beliefs about mathematics and the possible relationships existing between each teacher's beliefs and the kinds of mathematical connections he or she made in practice. In this chapter, I present narrative cases of each teacher, detailing my inferences. In the final section of this chapter, I provide a comparison across the narrative cases.

To discuss each teacher's beliefs, I use the three dimensions provided by Green's (1971) metaphorical interpretation of the structure and organization of belief systems. These dimensions allow me to describe each teacher's beliefs about mathematics and how the teacher's beliefs are held. In addition, throughout this chapter, I draw upon Leatham's (2006) theoretical perspective of sensible systems of beliefs. With this theoretical perspective, I assume that for a belief to exist within a system it must make sense to the other surrounding beliefs within the system. Therefore, when a teacher's descriptions or actions seemed to contradict my initial inferences of his or her beliefs, I provide alternative explanations to try to understand how a particular belief might make sense within the given system. Lastly, I use Ernest's (1989) three philosophical views about the nature of mathematics to characterize the teachers' beliefs about mathematics, and I use the Mathematical Connections Framework to discuss the levels and kinds of mathematical connections each teacher made.

Rachel McAllister: Mathematics is Like a Spider's Web and a Big Snowball

During my first interview with Rachel, I asked her to describe why she chose to teach mathematics. Within the reasons she gave, she provided insights into her beliefs about mathematics.

> Math makes sense. Math is everyday in the real world. …I really think math is the end all, be all. It is the basis for absolutely everything. I tell my students that ask, "When are we ever going to use this again?" "Well, you may not ever do this again, but you will do something in your job that requires you to find the problem, come up with a way to solve the problem, and check your answer. At every job you do."... So, I think that is why, cause it is real. Math is real. And, then we could get into looking at nature and how math is in nature. It is like, I am religious, God is a mathematician! I know it. Because there [are] way too many coincidences for him not to have been. (Interview 1)

Across her interviews, Rachel's descriptions of mathematics were from her perspective as a teacher of mathematics. She continually used examples from her teaching and from her students' learning to reflect on the nature of mathematics. During my analysis, I realized mathematics meant school mathematics to Rachel. In the following section, I provide my inferences about Rachel's beliefs about mathematics.

Beliefs about (School) Mathematics

Rachel's most central belief about mathematics was her belief that "math is logical" (Interview 1). This belief seemed to be the foundation for many of her other beliefs about mathematics. She defined the logical nature of mathematics as "there is a reason, there is a procedure, there is a solution, and the solution makes sense" (Interview 1). In this definition, Rachel observed how solutions follow from reasons and procedures, where these reasons and procedures imply that the solution to a mathematical problem makes sense. Later in the interview, Rachel described "reasons" as the "explanations why" a procedure or solution made sense (Interview 1). When discussing how mathematics is logical, on many occasions, she characterized mathematics as making sense or as being understandable. She used these expressions almost interchangeably. Therefore, to understand Rachel's most central belief about mathematics, it was necessary for me to examine the meanings she applied to these additional expressions, because these meanings informed how she viewed mathematics as being logical.

Rachel regularly characterized mathematics as making sense. I asked her to discuss more about what she meant by mathematics making sense. She responded,

> [Mathematics] makes sense for everybody. Numbers make sense. Sometimes, I really think, wow, how did we really assign a 1 and a 2 and when we added them you got 3? You really do that. And, then it just makes sense. And, you think, well English, that doesn't make sense. There are always exceptions. … it just isn't any good. [Laughs] Math makes sense. There is a purpose, [and] there is a reason. You can prove it many different ways. (Interview 1)

Within this description, Rachel's comparison of mathematics to the English language implicitly suggested mathematics is consistent and certain. In the conclusion of her statement, she included the phrase "there is a purpose, [and] there is a reason," which was similar to the way she described mathematics as being logical. The similar phrasing provided an example of the close relationship existing between Rachel's beliefs about mathematics being logical and mathematics making sense.

The card sort task contributed additional insights into Rachel's central belief that mathematics is logical and how this belief was related to her descriptions of mathematics being understandable. She titled one of the clusters as *Understanding* (See Figure 4). She described this cluster of statements by saying, "[Mathematics] is understandable, it is very logical. That is my goal, I don't want them to see it as abstract, I want them to see it for a real tangible subject" (Interview 5). As Rachel continued, she defined abstract, in this particular sense, as being difficult to comprehend; and she associated mathematics being "a real tangible subject" with mathematics being understandable. I asked Rachel to talk about this cluster of statements. In her response, Rachel frequently used the terms "logical" and "makes sense" as she talked about what it meant to *understand* mathematics. She talked about her desire for her students to understand the mathematical reasons why concepts and procedures make sense. This cluster of statements provided further evidence to support my inference that her belief about mathematics being logical was closely related to her belief that mathematics makes sense and is understandable.

Title of the Cluster of Statements: Understanding

Statements Clustered Together:

To me, logical means there is a reason, there is a procedure, and there is a solution. And the solution makes sense. There is a reason for getting the solution.

So, I think in the whole, big picture, is the discriminant important? No. But knowing the solutions and understanding what the solutions are, so, the why, I think just gets all the details and helps them, if you can understand why, you can always, I think you just remember it better. Now, that doesn't mean they can do it any better, they just remember it better. Makes more connections in their brain.

To me, understanding is when you could go back and every step that we have done makes sense. And, you could go back and say that is what we did at that step. That is why we did that step.

Being able to solve a problem because I knew this and that and then could figure out the rest of it.

Because I want them to understand it, and not everybody understands it the same way.

Descriptive Statement: Mathematics is understandable, it is very logical. That is my goal, I don't want them to see it as abstract, I want them to see it for a real tangible subject.

Figure 4. Rachel's cluster of statements titled Understanding.

Within this cluster, Rachel included statements from her previous interviews about problem solving, understanding the bigger picture, understanding why things work, and understanding mathematics in different ways. Without exception, Rachel's descriptions about mathematics were from her perspective as a mathematics teacher, often describing her students and her desire for them to understand mathematics. As Rachel described the statements she included in this cluster, she emphasized the importance of her students having the necessary prior knowledge to understand and make sense of mathematical concepts and procedures. She described the necessary prior knowledge as the "rungs on a ladder" used to climb toward understanding (Interview 5). With this metaphor, she suggested a kind of organization within mathematics, where this organization seemed to be related to her descriptions of mathematics being logical, making sense, and being understandable.

Rachel’s most central belief about mathematics was related to her belief that mathematics is organized. She appreciated the organization existing among mathematical ideas.

> I like that [mathematics] is organized. There [are] steps, and even if you are not doing it the same way as someone beside you, you still go through step-by-step, [and] find the answer, and then you can always go back and check your answer. Which is very methodical. (Interview 2)

Rachel described this organization in two ways. First, she believed answers to mathematical problems are found using methodical (and logical) procedures or processes. Second, she allowed for the possibility that there are different ways to arrive at the same answer or conclusion. Multiple solution methods implied that relationships or connections existed among mathematical concepts and procedures. For Rachel, this organization of mathematical concepts was not necessarily linear. Rather, she compared mathematics to a spider’s web; describing mathematics as being woven together by the various connections she perceived existing among mathematical entities, which allowed various pathways to a particular solution (Interview 6).

The connections Rachel found to be characteristic of mathematics followed directly from her belief that mathematics is both logical and organized. For Rachel, the connected nature of mathematics was also held as a central belief about mathematics. She said, “I think math in and of itself is a surprise, the way it all fits together” (Interview 3). She considered connections among mathematical concepts to be more than mere coincidences, and she thought these connections demonstrated a kind of beauty within mathematics. She appreciated recognizing the connections existing within mathematics, and she fondly recalled noticing many novel mathematical connections in the beginning of her teaching career as she used the Core Plus Mathematics Project curriculum (Coxford et al., 1997). Using this integrated curriculum, Rachel remembered learning about how several seemingly unrelated mathematical topics were connected.

During the card sort task, Rachel constructed a separate cluster of statements with the title, *Connections in Mathematics* (See Figure 5). She described mathematical concepts as being interrelated, and she often talked about connections in mathematics with respect to her teaching or her students' learning.

> All of the in's and out's of math, and how it all connects, and how the algebra and geometry and everything, and even further to calculus and so forth, it puts the big picture in my mind, puts all of the pieces together for me. (Interview 2)

Rachel appreciated the connections existing across algebra and geometry. She said, "I don't think [algebra and geometry] should have ever been separated, because I don't think that mathematicians who thought it up separated it" (Interview 2). Therefore, Rachel said she tried to teach mathematics in the way it was developed, by trying to emphasize the connections existing among the strands of mathematics. She thought this method of teaching would help her students see the "big picture" of mathematics, because "it is all connected" (Interview 5).

Title of the Cluster of Statements: Connections in Mathematics

Statements Clustered Together:

I don't think [algebra and geometry] should have ever been separated, because I don't think that mathematicians who thought it up separated it.

My favorite thing about math is that we use [mathematics] everyday in everything we do.

I think when you have the bigger picture, when you can represent it, do it in different representations, I still believe they have a deeper understanding because of those connections that are made, that it does all intertwine with each other.

I believe math is problem solving.

I think math is the end all, be all. It is the basis for everything.

All of these different little connections, so if one path doesn't get you there, then the other path reminds you how to do it.

Descriptive Statement: This is my big picture category. I think math is connected to everything, other mathematics content, other classes, future jobs, and nature. It is all connected.

Figure 5. Rachel's cluster of statements titled Connections in Mathematics.

In the final interview, Rachel described her purpose in making mathematical connections for and with her students. Her explanation provided additional insights into her beliefs about the connected nature of mathematics. Within her explanation, she provided a metaphor detailing her beliefs about how mathematics is connected.

> [Making connections] builds the relevance between topics….it builds those relationships. It re-teaches. It is constantly saying, that is what I did in those units. It is keeping the snowball building rather than a bunch of little snowballs to have a fight. We are making one big snowball. (Interview 6)

I asked Rachel to further discuss the comparison she made. In particular, I asked her to describe what the snowball(s) represented in her metaphor.

> Like if we just said, "Here are the matrices." Roll them up [into a single snowball]. Done. We stick them in the igloo to have the snowball fight, and we separate them. I don't like it all being separated … when we have these little snowballs. Sometimes we do build them separately. …In the end, we squish them together. To me, that is Math 2. One big snowball. But then, the [snow]ball gets really big because then it is all high school math. It is really all together. (Interview 6)

Rachel described the necessity as a teacher to, at times, develop mathematical topics separately. She emphasized, however, that she did not appreciate mathematical topics existing separately. Therefore, in the end, she connected mathematical topics by combining them into "one big snowball." I inferred the image of the snowball resembled her beliefs about the connected nature of mathematics; because the snowflakes, when combined together, were so closely related and connected that it was not possible to distinguish among them or to separate the snowflakes within the "one big snowball."

For Rachel, knowing and understanding the "bigger picture of mathematics" allowed her to make various connections for her students, which she believed, in turn, helped her students make sense of the mathematics and problem solve. Rachel applied this central belief to her teaching: "I think when you have the bigger picture, when you can represent the different representations, I believe [students] have a deeper understanding because of those connections that are made, that it does all intertwine with each other" (Interview 2). As a result, I observed Rachel make connections across mathematical concepts and procedures throughout her teaching.

Rachel claimed problem solving was the very purpose of doing mathematics (Interview 3). She defined problem solving: "Any time we do, we read a problem, we identify what we are trying, what's given to us, what we are trying to find, we make a plan, then we follow step by step. And a lot of that is logical thinking to me" (Interview 4). Rachel believed mathematics often existed in the real world through the guise of problem solving. She described her students as mathematicians, because her students regularly engaged in mathematical problem solving. She believed *people do mathematics*. She claimed mathematics was "alive," because "math is still growing and we are still finding new stuff" (Interview 2). I inferred Rachel's beliefs about problem solving represented a central belief within her beliefs system because of her frequent references to the problem solving nature of mathematics along with her statement that problem solving was the very purpose of mathematics. From Rachel's beliefs about problem solving, which include her descriptions of her students as mathematicians and of mathematics "still growing," I classified the majority of her beliefs about mathematics to reflect Ernest's (1989) problem solving view of mathematics.

Across her interviews, she used the phrases problem solving, logical reasoning, and logical thinking synonymously. In the card sort task, she tied her belief about problem solving in mathematics to her belief about the connected nature of mathematics. She described this relationship between these beliefs. She discussed how her students relied on past mathematical connections and made new mathematical connections when they engaged in problem solving.

Rachel's beliefs about problem solving seemed to be a derivative of her belief that mathematics is characteristically logical and understandable. During the card sort interview, I asked Rachel questions about this relationship. She explained, "When they can solve a problem, not a math number, not problem #24, but when they can solve a problem, they are *understanding* [emphasis added] the mathematics that they are doing" (Interview 5). The mathematical tasks Rachel used to introduce mathematical topics echoed the importance she placed on problem solving. In addition, she considered problem solving to be an integral part of engaging in daily life. She claimed, "Problem solving is not reserved for math [class]. Math helps us solve problems" (Interview 5). With this perspective, Rachel saw mathematics existing in a variety of different contexts and settings.

Rachel's discussions about mathematics, at times, included references to correct answers. She remarked, "Math is certain, and you get something from doing it. And, when you find an answer, you can prove that it is the right answer" (Interview 3). During my analysis, I found Rachel's references to correct answers were rather infrequent, and these references were often the byproduct of her statements about mathematics being certain and consistent and therefore logical. As a result, correct answers, for Rachel, seemed to follow from the logical nature of mathematics, which allows individuals the autonomy to determine if an answer is correct. Consequently, it seemed that her belief about correct answers in mathematics was a derivative belief of her belief about the logical nature of mathematics. However, this belief existed as a peripheral belief, because it was not a focus of Rachel's descriptions or instruction.

Using Green's (1971) three dimensions of a beliefs system, I constructed a model of my inferences on Rachel's beliefs about (school) mathematics (See Figure 6). This model depicts Rachel's central beliefs about mathematics as well as the quasi-logical relationships existing among her beliefs. I included the words Rachel used to describe mathematics within the model.

As a result, this model provides a visual representation and synthesis of my inferences of her beliefs about mathematics.

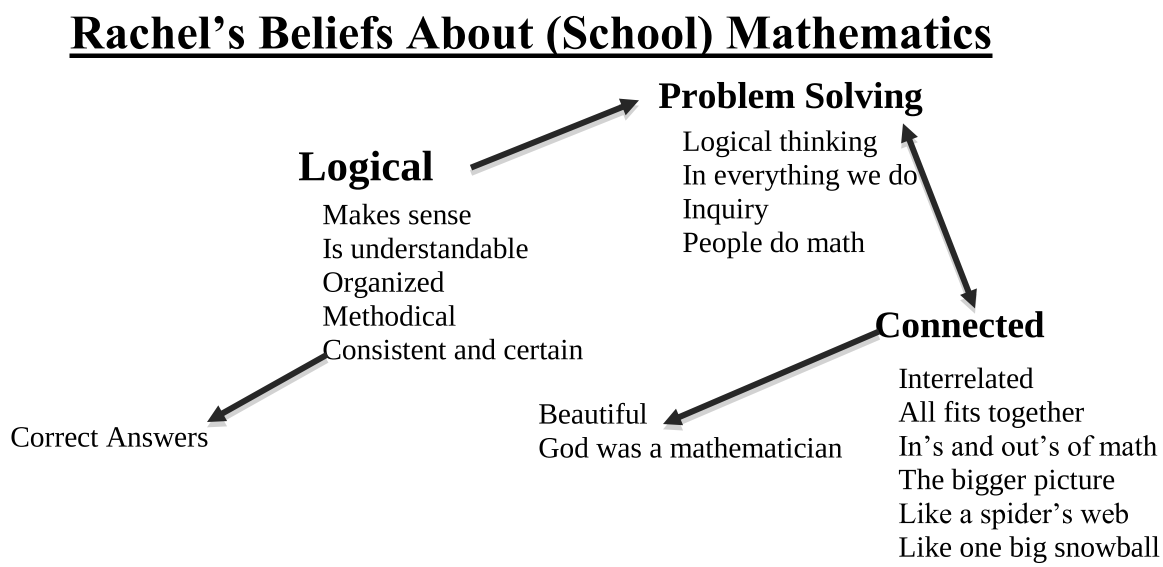

Figure 6. Rachel's beliefs about (school) mathematics using Green's (1971) dimensions of a beliefs system.

In the final interview, Rachel responded to my inferences of her beliefs in Figure 6. Her initial response to the model was "I like it all. It is what I said. I think that they're all good" (Interview 6). As I asked her questions about Figure 6, she made a slight amendment to her answer. She said the only thing she would change would be to include "connections to higher math" as part of her belief that mathematics is connected. She admitted, however, that this addition was possibly more prevalent in her current thinking because of recent training and preparation to implement the Common Core State Standards. At the end of the interview, she said this model represented her definition of mathematics.

Relating Beliefs About Mathematics to Connections Made

Rachel believed mathematics was logical in the ways that it made sense and was understandable. She believed there were reasons for why things worked the way they did in mathematics, and these reasons allowed her and her students to understand mathematics. As her most central belief about mathematics, her belief about the logical nature of mathematics seemed to transcend the levels and kinds of mathematical connections she made for and with her students. The resounding influence of this belief was found in how she made connections to support her students' understanding of mathematics.

Rachel equated understanding with learning. She defined understanding in the following way: "Understanding is when you could go back and every step that we have done makes sense. And, you could go back and say that is what we did at that step, [and] that is why we did that step" (Interview 3). She wanted her students to make sense of the mathematics and to understand the reasons for the various steps in a procedure. It seemed that she believed it was possible for her students to do these things because of the logical nature of mathematics. Rachel also described understanding through the lens of making connections, by saying, "connections are means to understanding [mathematics]" (Interview 5). Many of the connections Rachel made were to help her students understand the connections among mathematical entities. Therefore, the provided-and-explained connections Rachel made seemed to be strongly related to her belief that mathematics is logical, because the logical nature of mathematics was closely related to mathematics making sense and being understandable.

Rachel's beliefs about problem solving included her belief that people do mathematics. Her beliefs about problem solving seemed to influence how she included her students in the process of making mathematical connections. Rachel used mathematical tasks to allow her students to engage in problem solving, and whole-class discussions created opportunities for students to make their reasoning public. In addition, the components of connections her students contributed went beyond describing the next step needed to solve a problem. Rather, she asked open-ended questions that allowed her students to provide components of the connection, and she encouraged them to explain why a procedure worked or if the procedure was always going to work in a certain way. As a result, Rachel's students actively contributed provided-and-explained connections. By involving students in this way, Rachel seemed to portray her beliefs about problem solving, because her students contributed provided-and-explained connections across each of the five kinds of mathematical connections throughout my observations.

Rachel's belief that mathematics is connected was derivative of her beliefs about the logic and organization of mathematics. The connections she perceived among mathematical entities resembled a spider's web. This image characterized mathematics as being connected, where various connections exist within and between the mathematical strands. She also described mathematics as one big snowball, where mathematical entities connected in such a way that it was not possible for her to consider them as separate entities. Rachel's beliefs about the connected nature of mathematics seemed to influence the many connections she made. During my observations, she made provided-and-explained connections when she made connections between methods or connections through comparison. In these connections, the different methods developed or different components within the comparison often cut across the strands of mathematics or related seemingly unrelated concepts and procedures. The nature of these connections was, for the most part, unique to Rachel's instruction. I believe these connections reflected her beliefs about the connected nature of mathematics.

Rachel's belief about correct answers was peripheral in her beliefs system. This belief seemed to exist in her practice as she tried to move her students toward mastery of different concepts and procedures, because she believed her students needed to not only understand mathematics but also to demonstrate mastery. She said, "If you are going to be a mathematician, you don't need to just understand it, you need to be a master at it" (Interview 5). However, her belief about correct answers, while present among her beliefs, did not seem to limit the kinds of connections she made because it existed as a peripheral belief. For that reason, this belief did not imply that she privileged certain methods over others. Rather, in her practice, Rachel valued and expected her students to contribute provided-and-explained connections, usually through comparison or between methods. Therefore, the way she held her belief about correct answers, which followed from her beliefs about the certainty and consistency of mathematics, seemed to support the kinds of connections she made, because it did not hinder the development of specific kinds of mathematical connections in her teaching.

The logical nature of mathematics seemed to influence how Rachel connected mathematics to real world contexts, because she believed real world contexts helped her students make sense of and understand mathematics. During the card sort task, Rachel expressed a desire for her students to see and to understand mathematics as a "real tangible subject" (Interview 5). To make mathematics more tangible for her students, she often situated mathematical topics within a context (both initially and throughout the development of the topic). This real world context acted as a mechanism for her students to make sense of and to understand the mathematics.

Justin Smith: Mathematics is Like Building a Car

Justin's beliefs about mathematics primarily focused on mathematics being logical. This was his most central belief about mathematics. He described logic as a kind of reasoning and sense making about the answers to mathematical problems and exercises. Consequently, the logic in mathematics directly implied Justin had the ability to determine if an answer to a mathematical problem was right or wrong. Related to his beliefs about logic, Justin believed mathematical concepts and procedures build on one another in a distinctly linear fashion. Justin also appreciated real world applications of mathematics, because he thought real world applications demonstrated the usefulness of mathematics outside of his classroom. During his interviews, he consistently used examples from his teaching to discuss mathematics, and many of Justin's descriptions about mathematics were from his perspective as a teacher. For these reasons, I inferred mathematics meant school mathematics to Justin.

As I compared Justin's descriptions about mathematics, I initially perceived an inconsistency in the beliefs I inferred. Guided by Leatham's (2006) theoretical perspective, I used the inconsistency I perceived as an opportunity to explore my data with more depth. As I made sense of Justin's beliefs about mathematics, I realized some of his descriptions about mathematics, specifically his descriptions about the connected nature of mathematics, differed based on the context surrounding his descriptions. His descriptions differed because, at times, he talked about mathematics from his experiences teaching mathematics based on an integrated mathematics curriculum; and, at other times, he described mathematics from his experiences teaching from a subject-specific mathematics curriculum sequence (i.e., teaching algebra, geometry, and data analysis as separate subjects and classes). As I reexamined my data, I used the differences in his descriptions to refine my inferences of his beliefs. In the following section, I detail my inferences of Justin's beliefs about mathematics, and I consider a possible explanation for the inconsistency I initially perceived.

Beliefs About (School) Mathematics

Justin regularly referred to mathematics as being logical. However, he was careful to specify that he did not mean logical "like the logic in truth tables" (Interview 1). Rather, Justin defined logic in mathematics as a kind of "real world common sense" (Interview 1). In the first interview, I asked him to share more about what he meant about mathematics being logical. He responded with a recent example from his teaching.

> When a kid tries to type in their calculator 2 – 7, which we know logically is –5. They type something wrong, and they get something like 20, and they write it down anyway. They don't think about it. So trying to get them to think, "Does that make sense to you? Logically can it happen?" (Interview 1)

In this particular example, Justin seemed to describe logic as a kind of number sense. As such, he related the logical nature of mathematics to being able to make sense of the answer to a problem (or exercise), because the answer to a problem should make sense given the numbers or the context within the problem. In addition, I inferred from this example that Justin implicitly related logic to a set of consistent rules and procedures that guide mathematical activity.

Justin used the phrases "makes sense" and "problem-solving skills" across his interviews when describing the logic in mathematics. For example, Justin described how he expected his students to use their problem-solving skills to make sense of the answers they found.

> That relates back to the problem-solving skills that I harp on, because if they just put something down, I don't want them to say, "Whoop, I have my answer, done." I want them to go back and look, "Does that answer make sense for that problem?" And if the answer is wrong, "Where did I go wrong?" Try to figure it out. (Interview 4)

This example is similar to the previous example Justin provided to describe the logical nature of mathematics. The questions he included in this example were often present in his mathematics teaching. Justin said he used questions of this kind to help his students understand how mathematics is logical (Interview 5).

The card sort task provided additional insights into Justin's belief that mathematics is logical. He titled one of the clusters as *Logic* (See Figure 7). He introduced this cluster of statements by saying, "I mean, you kind of can figure out that my big thing is thinking and [being] logical" (Interview 5). This comment supported my inference that this belief was his most central belief about mathematics. Justin characterized this cluster of statements by saying,

> Making sense of it. Thinking it out. Like once you get an answer, ask if it makes sense. Could that actually happen? Did we do anything wrong in here? That kind of thing.…I guess that is what I would mean by logical. (Interview 5)

In his description of logic, Justin included the phrases "making sense of it" and "thinking it out." These phrases seemed to compare logic to reasoning though a mathematical procedure or an answer to a problem. However, across Justin's descriptions of logic, he tended to be rather specific when describing logic, because he usually described logic as reasoning or sense making about the answer to a particular mathematical problem or exercise. During the card sort task, Justin synthesized many of his thoughts about logic. Many of these statements were related to his other beliefs about mathematics.

Title of the Cluster of Statements: Logic

Statements Clustered Together:
You can tell if you are right or wrong.

Teaching kids that they need logic everywhere you go on your job. You may not be solving equations, but saying, "Is this correct or is that correct?" That kind of logic, real world common sense logic.

By understand it, what I think I mean is that I can see the connections from one thing, to the next, to the next.

It is just good to know why.

Descriptive Statement: Making sense of it. Thinking it out. Like once you get an answer, ask if it makes sense. Could that actually happen? Did we do anything wrong here? That kind of thing.

Figure 7. Justin's cluster of statements titled Logic.

Justin's descriptions about the logic in mathematics regularly implied he had the ability to recognize when an answer was right or wrong. In fact, many of his descriptions of logic also included using logic to determine if an answer was right or wrong. This belief was a derivative of his belief about the logical nature of mathematics. He related knowing whether an answer is right or wrong to the certainty of mathematical knowledge.

> Math should normally have an answer that has a solution you should be certain about. …In math, you are normally right or wrong. It may be doubtful for a while, but someone

> normally proves something right or wrong. That is one of the main things I like about it, so I would say it is certain. (Interview 3)

The certainty of mathematics allowed Justin to know why an answer was true or to justify why a mathematical claim was true. Justin claimed mathematics is not subjective like other school subjects. He remarked, “You are not interpreting [in mathematics], you are actually finding an answer” (Interview 2).

During the card sort task, Justin constructed a separate cluster of statements that he titled, *Right or Wrong* (See Figure 8). The description he used to characterize this cluster of statements implicitly referred to the certainty of mathematics and mathematical knowledge.

> It is nice that almost every time in math you can, you can really tell if you are wrong, but most of the time is the way you check and see if you are correct, if you have the time. (Interview 5)

I inferred this belief was also one of Justin’s central beliefs about mathematics. I made this inference because of how often he referred to mathematics in this way. In addition, during the card sort task, he described this particular cluster of statements as “one of the main things I kind of like about math” (Interview 5). Justin’s beliefs about the certainty of mathematical knowledge corresponded to Ernest’s (1989) Platonist view of mathematics.

Title of the Cluster of Statements: Right or Wrong

Statements Clustered Together:

Generally, you know if you are right or wrong. I definitely know if you are wrong. And, if you have enough curiosity, you normally can figure it out somehow.

There is no, “I don’t like the way you worked this equation,” because it is right or wrong.

I’m not the one that came up with the rules.

Correct thoughts, correct statements, which is precision. I make them show good work and I tell them that there is not always one way to work it as long as you can justify what you’ve done is correct math.

Descriptive Statement: It is nice that almost every time in math you can, you can really tell if you are wrong, but most of the time is the way you check and see if you are correct, if you have the time.

Figure 8. Justin’s cluster of statements titled Right or Wrong.

Within this cluster of *Right or Wrong*, Justin included a statement about “correct math” and “good work,” suggesting these descriptions were related to this particular belief about mathematics. Later, he described this kind of mathematics as “good math,” which he defined as “emphasizing the steps, doing correct math” (Interview 5). Justin believed mathematical procedures required precision and precise steps. Following from this statement and my observations of his practice, I inferred that doing “correct math” implied there were certain ways to do correct mathematics. Justin’s focus on correct mathematics, which included a focus on correct procedures, led me to classify these beliefs as reflecting Ernest’s (1989) instrumental view of mathematics. I made this classification because Justin’s emphasis on correct procedures and correct mathematics often resembled what Ernest described as “a set of unrelated but utilitarian rules” (p. 250).

In my analysis of Justin’s descriptions during the card sort task, I carefully examined the link between Justin’s beliefs about the logic in mathematics and his beliefs about the connections existing among mathematical concepts. Within his cluster of statements he titled *Logic*, Justin included a statement about the connections he saw in mathematics, which suggested his beliefs about logic in mathematics were somehow related to his beliefs about the connected nature of mathematics. I noticed he often talked about the linear flow of mathematics. He said he recognized the “connections [existing] from one thing, to the next, to the next” (Interview 2). He appreciated the ways mathematical concepts and procedures built on one another. I asked Justin to describe the connections he perceived in mathematics. He responded by providing a list of geometric concepts to describe what he meant by the linear flow of mathematics.

> The ordering of geometry, and how it relates. Like you build logical statements that you use to prove that triangles are congruent. Then you build from triangles to polygons. Then you can actually do the right triangle trig in some geometry so that you can see why you need it…Why it’s necessary….it is more related to something you might actually use outside of school. (Interview 2)

With this list, Justin described geometrical concepts and procedures as building on one another. He thought these connections reflected the logical structure of mathematics. In the third interview, I asked Justin to elaborate on his descriptions of the logical structure of mathematics. He said,

> It is important to know the logical structure, because you kind of figure out if you are right or wrong again….Back to the sequence of events and the connections, and it makes sense if you really learn it. It makes sense why this next thing is happening. (Interview 3)

With this description, I inferred that Justin's beliefs about the connections within mathematics seemed to follow from his beliefs about mathematics being logical and making sense.

To understand Justin's beliefs about the connected nature of mathematics, I asked him to use a metaphor to describe how mathematical concepts and procedures may or may not be related to one another. After thinking about it for a while, Justin compared mathematics to building a car. He explained,

> Everything in a car makes sense. And as long as you know *the order in which you are supposed to put it together* [emphasis added], everything works nice and smoothly, but if you leave off one little thing, the engine will not run as smoothly as it should. (Interview 6)

Justin described the particular order needed to build a car. From this metaphor, I inferred that Justin also believed mathematical entities were related in a particular order (this description was similar to his previous description of the "sequence of events" in mathematics). I asked him to talk more about how building a car was related to the connections he perceived in mathematics.

> Most of them build on each other. You have to know one before, at least you have to know one before you should go to the next. You don't necessarily have to know the Pythagorean theorem in order to do the trig identities, but they build up to [it]. (Interview 6)

Justin's response provided additional evidence for my inferences about his beliefs about the connected nature of mathematics. His metaphor described a particular order in mathematics. In addition, by saying “you have to know one before you should go to the next,” Justin implied relationships between mathematical entities develop in a linear order.

Justin emphasized the importance of connecting mathematics to real world applications. Some of the statements he made about connections to applications reflected his beliefs about mathematics. He believed these connections provided evidence that mathematics is useful, and he thought it was helpful to explain why mathematics is useful to his students.

Justin’s appreciation of the usefulness of mathematics seemed to explain his preference for certain mathematical strands. For example, he considered algebra to be a boring subject, because he did not see any practical applications of the rational root theorem or the fundamental theorem of algebra. However, because Justin believed mathematical concepts and procedures build on one another, he recognized the necessity of algebra. He described algebra as the “wrench and the hammer to get to where you can use [mathematics]” (Interview 2). Unlike algebra, he defined geometry as a “thinking subject” because of the proof and reasoning developed in geometry (Interview 1). He believed geometry encouraged logical thinking and problem-solving skills, which he believed to be very useful and applicable outside of the mathematics classroom. In addition, he did not consider statistics to be “true” mathematics. He explained, “That is not always math, because you are kind of manipulating the data, and I don’t think you can manipulate true math” (Interview 5). Justin characterized applied mathematics as “interesting mathematics” (Interview 3). He enjoyed mathematics courses that include mathematical connections to applications,

> I like the classes where you are actually applying math more like differential equations. We did a lot of good stuff in there. And when you get to Trig[onometry] you get to apply a lot of math. And Calculus is fun because you can do the related rate problems. (Interview 1)

He believed mathematical connections to real life applications exemplify why mathematics is necessary and useful.

Justin often made the claim, “Math is connected” (Interview 3). However, during the beliefs survey and interview, Justin said mathematics was more segregated than connected. I asked him to tell me more about why he made that selection. He responded by talking about the organization of the new integrated mathematics curriculum standards:

> I’m biased because I mean I see the connections in math, the way the new GPS (Georgia Performance Standards) is done, it is not connected right now. You know, they are doing like segment of Algebra, segment of Geometry, segment of Statistics. Then a year later they go back and do the same things again, the same things again so. That is why I was saying it is segmented in high school. In Math I, II, and III. Or in Accelerated Math I, Accelerated Math II. If we taught it the way we used to teach it Algebra 1, Algebra 2, Geometry, Trig, Calculus. Much more connected then because you could see the flow of it. I can still see the connections even though we teach it segmented, but the students have a hard time. Because they jump from one day they are talking about parent functions and then all of a sudden they are talking about chords in circles and then they are talking about Mean Absolute Deviation, which I had never heard of before they came out with this. I see it as segmented the way we are supposed to teach it. Now if we could go back to the GPS Algebra or the Common Core Algebra or whatever, it may be better. (Interview 3)

Initially, I regarded Justin’s response as a perceived inconsistency among my inferences of his beliefs about the connected nature of mathematics. However, as I took a closer look, I realized that he was describing mathematics as segregated specifically from his experiences teaching mathematics from Georgia’s newly adopted integrated mathematics curriculum standards. From this curricular perspective, Justin taught units that developed a “segment of algebra, segment of geometry, segment of statistics,” which he did not consider to be as connected as “the way we used to teach it Algebra 1, Algebra 2, Geometry, Trig, Calculus.” Describing mathematics from this perspective provided me a better picture of his beliefs about the connected nature of mathematics. Within his response, Justin emphasized the “flow” of mathematics. This description of mathematics illustrated his belief that mathematics is more connected within a single strand of mathematics than between strands. In addition, I realized that his past descriptions of how mathematics was connected were always contained within a discussion of a single strand of mathematics (e.g., his description of "ordering of geometry" on p. 135). As a result, I realized that it seemed more likely that Justin would consider an algebra course or a geometry course to be more connected than the integrated mathematics courses he was currently teaching because of his beliefs about the connected nature of mathematics.

Using Green's (1971) three dimensions of a beliefs system, I constructed a model of my inferences of Justin's beliefs about (school) mathematics (See Figure 9). Because many of Justin's descriptions about mathematics were from his perspective of a teacher, this model represents his beliefs about school mathematics. This model captures his central beliefs about mathematics as well as the relationships existing among his beliefs about (school) mathematics.

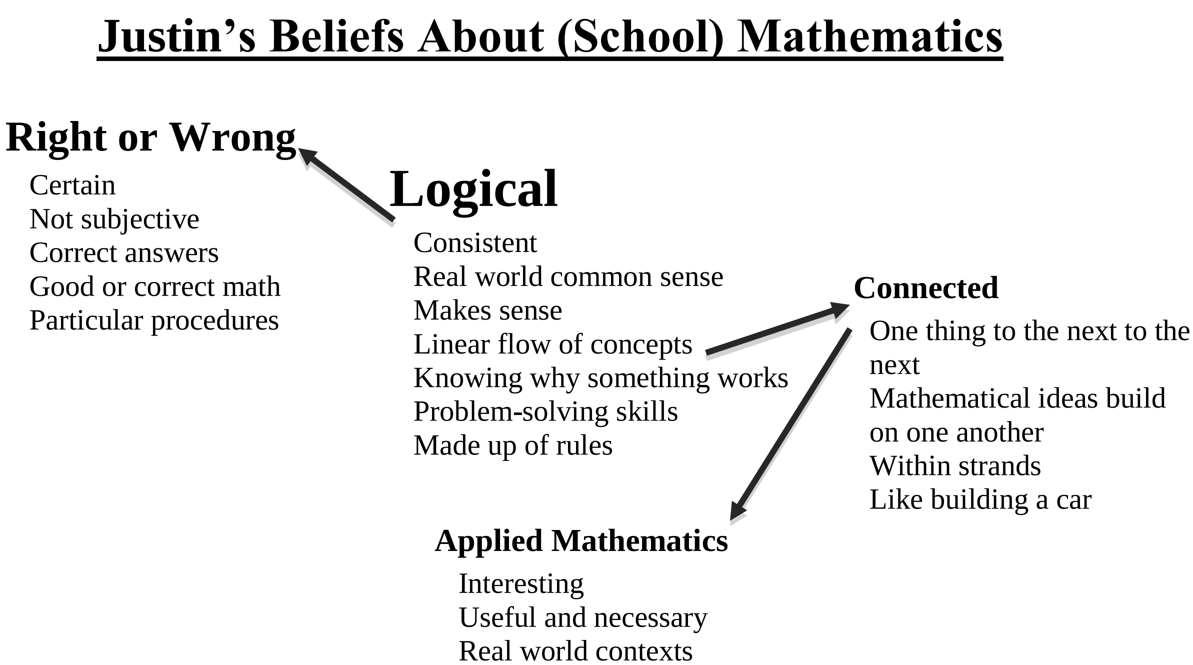

Figure 9. Justin's beliefs about (school) mathematics using Green's (1971) dimensions of a beliefs system.

In the final interview, Justin responded to my inferences of his beliefs in Figure 6. I asked him if he would make any changes to the model. He replied, "I was trying to think if there is particular one that I wanted to add, and I don't disagree with any of those" (Interview 6). Later, Justin commented that he was impressed that the model I had created prior to the interview was the same as the model he created during the interview. He said that it was like I figured out what he was thinking before he had ever thought about it (Personal communication, March 30).

Relating Beliefs About Mathematics to Connections Made

Justin believed the logic in mathematics implied the existence of a logical organization of mathematical entities, wherein mathematical entities build on one another in a distinctly linear fashion. His beliefs about the logical and linear organization of mathematics related to the levels and kinds of connections he made for his students. For example, the suggested connections he made usually occurred as he developed one mathematical topic after another (where he suggested two mathematical entities were related but did not provide the relationship). In addition, his beliefs about the logical and linear organization of mathematics influenced the content of the components he connected through comparison. The content of the components he connected almost exclusively resided within the content of the day's lesson or the current unit on parametric equations. Unlike Rachel, Justin rarely made connections through comparison by relating components from different strands of mathematics or seemingly unrelated topics.

Justin's beliefs about mathematics seemed to influence the kinds of connections he did or did not make in his teaching. Mathematics, to Justin, was about finding and then evaluating answers to problems. During my observations, Justin frequently described particular procedures as "good mathematics" to his students. Given this belief, he seemed to privilege certain procedures over others. Justin's focus on particular procedures and "good mathematics" seemed to influence the kinds of mathematical connections he made in two ways. First, his emphasis on particular procedures over others seemed to be part of the reason he did not develop any connections between methods. Second, his focus on developing procedures did not create many opportunities for his students to contribute components of a mathematical connection, because many of Justin's interactions with his students involved asking or answering a question about the next step in a particular procedure.

Justin appreciated developing connections to real world applications, because he thought this kind of connection demonstrated the usefulness of mathematics outside of his classroom. He described connections to real world applications as "interesting math." Justin said he enjoyed teaching Accelerated Mathematics III because he liked classes "where you are actually applying math more" (Interview 1). With the frequency of his comments about the importance of applying mathematics, I did not find it surprising that Justin provided (and sometimes explained) connections to real world applications or that he suggested connections through his use of contextualized word problems throughout his unit on parametric equations.

To make sense of Justin's beliefs and practices, I continually asked myself if I perceived any inconsistencies in his data. I asked this question to continue to explore my data. As a result, I realized that there seemed to be a disconnect between some of his descriptions and actions. Justin often emphasized the importance of problem-solving skills in his descriptions of mathematics. However, this description did not seem consistent with his teaching practices because of his focus on developing particular procedures. In addition, procedural knowledge regularly existed within the components of the connections he made for his students, which reflected his focus on procedures. However, as I reviewed his descriptions of problem-solving skills, I realized I was applying my understanding of what I think it means to problem solve in my analysis. Justin defined problem-solving skills as following the steps to solve a problem and determining if the solution is correct. His definition of problem-solving skills focused on developing procedures and finding correct answers. Therefore, I refocused my lens. I began to look at my data through Justin's definition of problem-solving skills, and I realized his definition of problem-solving skills related to the focus he placed on developing procedural connections, finding correct answers, and determining if an answer is "right or wrong."

Robert Boyd: Mathematics is Like a Human Body and a Brick Wall

Mathematics, to Robert, was characterized by both certainty and a logical organization of concepts and procedures. These beliefs comprised his most central beliefs about mathematics. He considered these characteristics of mathematics to be so closely related that he said he did not know how to separate the two (Interview 2). These central beliefs also acted as primary beliefs about mathematics. Following from his beliefs about the certainty and logical organization of mathematics, he believed the very purpose of mathematical study was to engage in logical thinking and reasoning, and his belief about the ability to reason logically implied mathematics could be used to describe the way the world works.

Guided by Leatham's (2006) theoretical perspective, I recognized Robert's beliefs about mathematics differed based on context, whether he was describing mathematics in a more formal sense from his university experiences with mathematics or when he was describing his experiences teaching school mathematics. Some of these distinctions between mathematics and school mathematics were evident in Robert's card sort task. During this task, Robert constructed a separate cluster of statements titled, *What Math is Like* (See Figure 10). Within this cluster, Robert included statements contrasting the discipline of mathematics with the mathematics he taught his students. In addition, I found further evidence of the distinctions Robert made between mathematics and school mathematics when I noticed what seemed to be an inconsistency as I compared the beliefs I initially inferred from Robert's descriptions with his classroom practices.

Title of the Cluster of Statements: What Math is Like

Statements Clustered Together:
Math is pretty certain.

They do the math, and the math doesn't fit. But the problem is that the math is not lying. The problem is not the math, the problem is that their model is wrong, and then they use the math. It's just amazing how much they can learn just by calculating things out.

I think a lot of the truths in mathematics and mathematics are fairly set, I don't know them all. And so I'm always finding new connections between things that I didn't notice before.

As we get higher in mathematics in school it gets a little further away from reality.

I realize that there are still things being learned and added onto mathematics. That is way beyond kind of where we are in high school though. I think most of what we are doing in the high school is pretty established and pretty standard and we all know what it is.

In the past I might have put it further towards segregated but since they had the integrated curriculum and I started doing all these different things instead of, I mean before that, I basically taught Algebra and that was it, but when you start doing all the different pieces, you start to see how this, Algebra is really a part of Geometry. And Geometry can be used to explain the Algebra and so they are all mixed and overlapped quite a bit.

Descriptive Statement: I think these are all just quotes that are describing math, how it is as a discipline. Like math in and of itself, as opposed to how it is used, as opposed to any, kind of outside of me. This is what math is like.

Figure 10. Robert's cluster of statements titled What Math is Like.

In the following sections, I detail my inferences of Robert's beliefs about mathematics as well as his beliefs about school mathematics. In the first section, I describe Robert's beliefs about mathematics, noting that many of these beliefs were also beliefs he held about school mathematics. In the second section, I describe Robert's beliefs about school mathematics that were distinctly different than his beliefs about mathematics as a formal discipline.

Beliefs about Mathematics

Robert believed mathematical concepts and procedures existed in a kind of logical organization. This organization was closely related to his belief about mathematics being certain (or true). This belief was one of his central beliefs about mathematics, which transcended his discussions of mathematics and school mathematics. He described the relationship he perceived between the logical organization in mathematics and certainty of mathematical knowledge.

> There is an organization to [mathematics]. This piece goes with this piece, and this piece goes with this piece. And if I go from here to here to here to here, I can show or I can see why something is true. (Interview 1)

He appreciated the "beauty of how all things work and fit together," and he thought mathematical concepts and procedures fit together like pieces in a puzzle (Interview 4). Furthermore, he shared the enjoyment and satisfaction he felt when he was able to problem solve and put the pieces of a mathematical puzzle together.

There were subtle differences in the ways Robert described the organization of mathematics. Each description added additional depth and detail. He believed a logical structure existed within the organization, "where one thing follows logically from the other" (Interview 3). He also characterized mathematics as being interrelated. Expanding upon this notion, he said, "Each piece of math, that we think about as separate pieces of math, really are all related to each other" (Interview 5). For Robert, this belief about mathematics was rather new and a byproduct of teaching from an integrated mathematics curriculum. He remarked that he would have described mathematics as more segregated than connected prior to the using the new integrated mathematics curriculum. However, through his experiences with this integrated mathematics curriculum, he recalled noticing interesting connections existing among mathematical ideas. He broadly described these connections by saying, "Algebra is really a part of geometry, and geometry can be used to explain the algebra. So they are all mixed and overlapped quite a bit" (Interview 3). In these characterizations, Robert's beliefs about the logical organization of mathematics emphasized relationships existing among mathematical concepts and procedures.

To make sense of Robert's descriptions of the organization of mathematics, I asked him to develop a metaphor to represent his beliefs about the organization of mathematics. His response compared mathematics to the human body. He explained, "It has all of these pieces. They are separate pieces, but all of those separate pieces rely on each other. And, they all have their perhaps slightly different uses, but each one is dependant on the other" (Interview 5). Robert's metaphor seemed to build on his previous descriptions about how mathematics is logically organized and connected. He characterized mathematical entities as being separate from one another, which followed from his original belief about the segregated nature of mathematics. He viewed the connections among the "separate pieces" as the ways these pieces of mathematics depended on and related to one another, which reflected his more recent appreciation and belief about the connected nature of mathematics.

Robert shared an additional metaphor to characterize the organization he perceived within mathematics. This second metaphor compared the development of mathematics to a brick wall. He used this second metaphor to argue that understanding the relationships existing among separate mathematical ideas provides a more powerful understanding of mathematics.

> You start with the simple, and you just, as you build, it gets deeper, and more complex, and more powerful as you build. But by connecting all of those, by connecting all of these things, it makes it, well, if you are not connecting them, then I guess you are not building anything. Because if you put bricks on top of each other, you are not really building a wall, unless they are connected to each other somehow. So, otherwise it all falls apart. I think it is the same idea. It is not nearly as strong or nearly as useful unless all of the ideas are connected together. (Interview 5)

This second metaphor reflected his beliefs about the logical organization of mathematics, because he described the ways mathematical ideas develop and connect. To Robert, mathematical ideas develop as individuals understand how ideas (or bricks) build on and connect to the surrounding ideas (or bricks). He described building the wall by starting with the simple and building toward a more complex structure. Similar to the previous metaphor, Robert described separate pieces of mathematics coming together to form a “bigger picture” of mathematics (Interview 1).

Robert described mathematics as being absolutely certain. This belief was a central belief for Robert, and this belief was present in his descriptions of mathematics and school mathematics. In his interviews, he often referred to the certainty of mathematics as mathematical truths. He described certainty by saying, “I have this number, and this number, and this number. And I do this with them, and that with them, and I get this answer, and by golly, that is the answer” (Interview 1). For Robert, mathematical consistency existed within the certainty of mathematics. Consistency implied mathematical equations and procedures “work every time” (Interview 1).

Robert believed the certainty and consistency of mathematics allowed him to find correct answers and to discover mathematical truths. Given these beliefs, I classified Robert’s beliefs about mathematics to represent Ernest’s (1989) Platonist view of mathematics. He described how the certainty of mathematics allows for the discovery of truth. To do so, he used his knowledge of science to explain how scientists use mathematics to vet scientific theories and conjectures.

> [Scientists] have this picture of how they think the universe works. Then something happens, and they do the math, and the math doesn’t fit. But the problem is that the *math is not lying*. The problem is not the math, the problem is that their model is wrong…. It’s just amazing how much they can learn just by calculating things out. (Interview 2)

In this example, Robert considered the certainty of mathematics as a means for explaining various characteristics of the physical world. Therefore, this central belief paved the way for his belief that mathematics explains the way the world works.

Robert considered logical thinking and reasoning to be the very purpose of learning mathematics. Often, Robert described logical thinking as problem solving or deductive reasoning. He believed the logical organization and the certainty of mathematics provided him with the means necessary to engage in logical thinking and reasoning.

> I always tell students at the beginning of the year that this skill may or may not help them, but it is the process of thinking logically. You will be able to reason through… I think the power of deduction and being able to think through a situation logically is one of the big benefits of learning mathematics. (Interview 4)

Robert emphasized logical thinking and reasoning in his teaching by telling his students where mathematical rules and formulas came from and the meaning behind them. In doing so, he demonstrated how simple mathematical concepts and procedures combine together to form more complex mathematical concepts and procedures (similar to his metaphor of a brick wall).

Robert believed logical thinking could extend beyond the study of mathematics into other elements of everyday life. He saw logical thinking as a means to understand and make sense of the world.

> I think the value of problem solving is even though [students] might not be doing this type of math in their everyday life, but there are lots of times when they are just going to have to look at things and use their powers of deduction to think about it and come to the right answer. (Interview 5)

Logical thinking allowed for inquiry into the nature of things. He related logical thinking to the ability to ask and to answer questions about the world. Therefore, Robert's belief about logical thinking was also associated with his belief that mathematics could describe the way the world works.

Robert described mathematics as the basis for everything. He believed it was possible to use mathematics to describe and characterize the physical world. Expressing this belief, he said, "So much of what we do and so many of the things that we enjoy in life are based on–it is all math. The world works by math" (Interview 1). This aspect of mathematics made mathematics powerful. To Robert, this power of mathematics implied, "You can figure things out. I like that not just when you do math you are solving math problems, but I like that math can solve problems. You know you can learn things by doing the math" (Interview 2). Robert's belief about logical thinking and problem solving was also a derivative of his belief that mathematics was certain, because Robert claimed the certainty of mathematics allowed people to engage in logical thinking and problem solving to discover mathematical truths. In this way, Robert's beliefs about problem solving continued to reflect Ernest's (1989) Platonist view of mathematics, because he believed problem solving allowed for the *discovery* of mathematical truths.

Beliefs about School Mathematics

As Robert talked about mathematics, distinctions about mathematics arose based on the context of his descriptions. Many of his descriptions about mathematics were from his perspective as a mathematics teacher. Through this lens, he often described mathematics through the experiences of his students, his teaching, and the school mathematics curriculum. Although his most central beliefs about mathematics (his beliefs about the logical organization and certainty of mathematics) were not dependent on context, some of his more peripheral beliefs did depend on context. Therefore, to understand Robert's beliefs as a sensible system of beliefs, it was necessary to consider his beliefs about school mathematics, particularly how some of his beliefs about school mathematics were distinctly different from his beliefs about mathematics.

Robert described mathematics as "a puzzle, a game, and a challenge" (Interview 4). He enjoyed the problem solving nature of mathematics. Although he personally enjoyed engaging with mathematics in this way, he described the common view of school mathematics as scary. Within the context of school mathematics, he described certain formulas and equations as looking "ugly and intimidating" (Interview 1). He believed his students often reacted to school mathematics as if it were some kind of punishment. Robert believed school mathematics was scary for many of his students.

> One of the things that I think that is important for me to do with my kids is to get them to not be afraid of [mathematics]. Just don't, you know, stop looking at it as this thing that is going to kill you or it is going to, that it is a burden that you have to carry, that it doesn't have to be scary and it doesn't have to be boring. That it can actually be interesting and fun. And even if it isn't fun, at least you don't have to be intimidated by it. I say, "It is just numbers, numbers on paper. Don't you know it is not going to hurt you? It is not going to hurt you. Just look at it. Look at what it says. Think about what you need to do to solve it." (Interview 4)

Robert thought most students were afraid of mathematics because of the more procedural and skill-based nature of school mathematics. He equated the procedural and skill-based nature of school mathematics to grunt work or digging ditches, because he said too often his students did not understand the mathematics. Without understanding, his students were forced to try to remember many different mathematical skills and procedures, which was rather difficult for them to do. Therefore, in his practice, he tried to move beyond rote procedures by explaining how and why things work the way they do in mathematics. He hoped this approach helped to make mathematics a little less scary for his students.

Robert described school mathematics as highly abstract and removed from his students' everyday reality. Although he believed mathematics was inherently useful in describing the ways the world works, he did not believe this characteristic of mathematics was present within the school mathematics curriculum. He explained by sharing the tension he experienced by teaching mathematics in a way that does not demonstrate the usefulness of mathematics within the school mathematics curriculum.

> There is so much you can do with [mathematics]. Mathematics is just so useful. And, that really is, I think, a struggle that I have with our [school mathematics] curriculum. I mean, it is the problem the curriculum has always had. Forever. You know, we are doing quadratic equations, well, quadratic equations are great, but people don't often use them. So, the kids don't really see how useful the, they don't see how useful the math is.

> Because any situation they come up with, for example, we are doing quadratic equations, and one of the tasks talks about someone throwing a protein bar into the air, and you are predicting the height of the protein bar. And, I am like, Nobody cares. I mean, is it real? Yeah, I guess it is real, but nobody cares. (Interview 1)

He perceived an incongruity between mathematics and school mathematics because of the procedural nature of school mathematics. Robert's characterization of school mathematics often focused on procedures and skills. He credited Georgia's curriculum standards and standardized assessments for this particular characterization of school mathematics. With the focus on procedures along with the pressures involved with high stakes testing, Robert felt that the beauty and power of mathematics was often lost. He said it was difficult to engage his students in problem solving. In addition, Robert thought it was important for his students to understand that mathematical ideas often develop because people need new ways to describe how the world works. However, Robert said this was difficult to do regularly because of what he perceived to be the procedural constraints of school mathematics. Therefore, these beliefs were not included in his system of beliefs about school mathematics.

I constructed models of my inferences about Robert's beliefs about mathematics (See Figure 11) and school mathematics (See Figure 12). I present both models for the purposes of comparison. His two most central beliefs were not dependent upon context. However, his beliefs about school mathematics did seem to have a more direct influence on his pedagogical practices and, in particular, the kinds of connections he made for his students. To differentiate, I represent his beliefs unique to school mathematics in italics. As a result, these models provide a visual representation and comparison of his beliefs about mathematics and school mathematics.

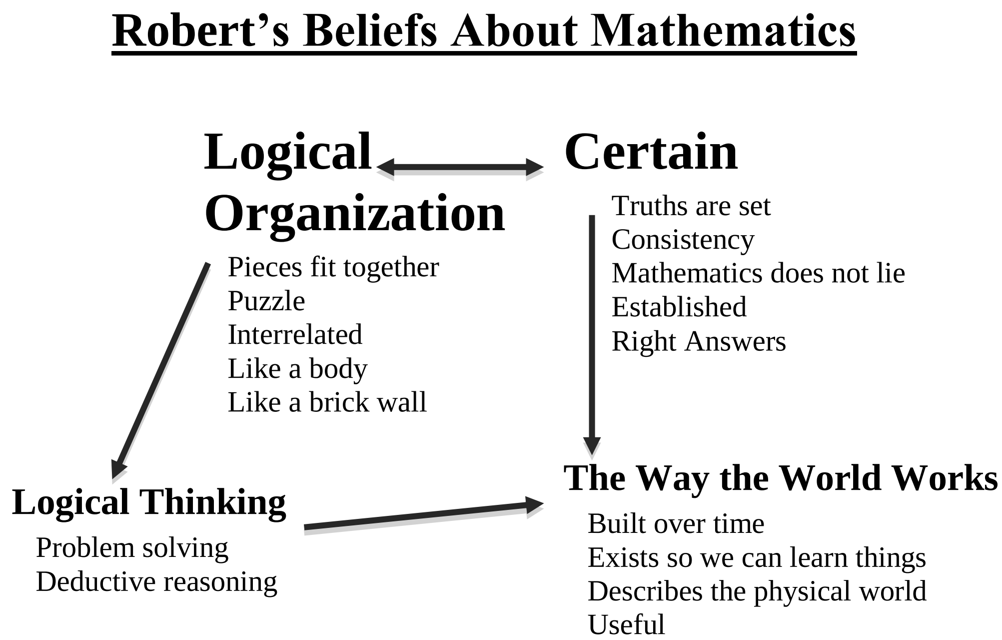

Figure 11. Robert's beliefs about mathematics using Green's (1971) dimensions of a beliefs system.

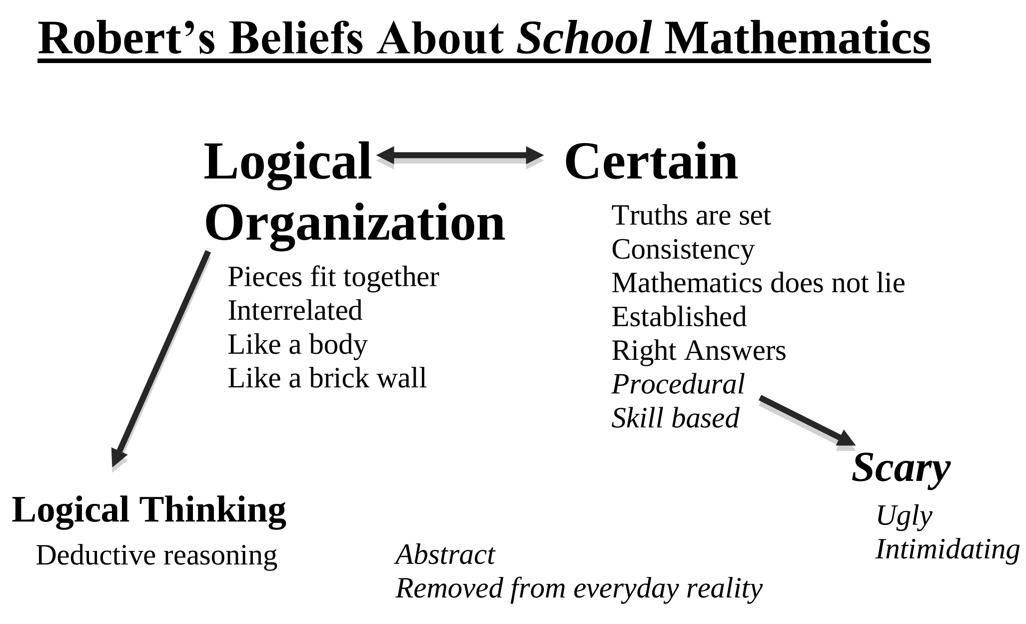

Figure 12. Robert's beliefs about *school* mathematics using Green's (1971) dimensions of a beliefs system.

In the sixth and final interview, I asked Robert to review the two models I created to demonstrate my inferences of his beliefs. I asked him if he would make any changes to either model. Responding to Figure 11, he said, “These are the things that I appreciate about math, for sure” (Interview 6). He described the model as an accurate reflection of his beliefs. As he examined Figure 12, he said, “These italicized things [pointing to procedural and skill-based] are what makes it scary…I think because of the outside pressure, it just becomes that way. Or, because of testing, it becomes that way whether you want it to or not” (Interview 6). The statements Robert offered in response to the models provided evidence to support my inferences of his beliefs.

Relating Beliefs About Mathematics to Connections Made

Central beliefs Robert held about mathematics and school mathematics seemed to influence the kinds of mathematical connections he made for his students. He believed the logic in mathematics implied an organization among mathematical concepts and procedures, which corresponded to the ways he developed and connected mathematical concepts and procedures for his students. Robert held another central belief about mathematics, for he believed mathematics was characteristically certain. However, this belief did not seem to influence the kinds of connections he made for his students.

To make sense of Robert’s beliefs and practices, it was necessary to consider how his beliefs unique to school mathematics also seemed to influence the kinds of connections he made. For example, Robert believed school mathematics had a distinct focus on procedures. Consequently, his teaching focused on developing procedures for his students, and therefore many of the connections he made seemed to be related to developing procedural fluency within his students. The focus on developing procedures seemed to result in procedural knowledge that existed within components of the connections Robert made. This particular belief about school mathematics seemed to combine with his belief about logical thinking in mathematics; and, as a result, I believe Robert made provided-and-explained connections to teach his students these procedures. The remainder of this section considers the relationships between Robert's beliefs about mathematics or school mathematics and the levels and kinds of connections he made for his students.

Robert's beliefs unique to school mathematics seemed to influence the kinds of mathematical connections his students contributed. Students made contributions during the development of procedural knowledge, in which students provided the latter component of a connection through logical implication or they contributed to a connection through comparison. The focus on procedures in his students' contributions reflected the focus on procedures in his teaching, which was consistent with his belief that school mathematics is characterized by the development of procedures.

Robert strongly believed school mathematics was scary for many of his students. This belief about school mathematics influenced some of the connections he provided, hoping these connections made mathematics a little less scary for his students. There were times when he provided connections through comparison to help his students remember certain things about mathematics. For example, Robert provided connections through comparison as warnings about possible misconceptions. It seemed that he used this kind of connection through comparison to help his students avoid confusion. In addition, Robert provided connections through comparison that acted as a kind of mnemonic device for his students. Robert made this kind of connection through comparison to help his students remember specific characteristics of certain functions.

A combination of Robert's beliefs seemed to influence the nature of the connections he made through comparison. The combination of his belief about logical organization of mathematics along with his belief that school mathematics is scary related to how he developed new topics by comparing and connecting them to things his students already knew and understood. Robert discussed the reasons for why he connected the new with the old.

> I think it is really important to connect new stuff with old stuff. I think the more familiar we make it, the less scary it is, and the less scary it is, not only does it make it easier for the kids intellectually, but I think it makes it easier for them emotionally to take it on. They are less likely to shut down, because they are going to feel more confident in what they are doing, but also, it just flat out makes it easier for them to remember, because hey, this is like, this other thing that I already know. And, so, because of that, I think it makes them more successful. And, that is really what I am trying to do when I teach. (Interview 5)

The combination of these beliefs influenced the kinds of connections Robert made when he related the new with the old. These connections also reflected Robert's metaphor of a brick wall to describe the connected nature of mathematics, because he described starting with the simple and making connections to build a more complex and powerful understanding of mathematics. In addition, these beliefs seemed to influence the level of connections he made through comparison. Robert said,

> I think [it] is so important to understanding what you are doing. Just doing it doesn't help you understand it. It helps you understand how to do it, but it doesn't help you understand it, and what it is, and why it is. Which I think is part of the point of teaching and doing mathematics. (Interview 3)

Robert believed that if he developed topics in this way, by explaining the why while comparing the new with the old (provided-and-explained connections through comparison), then his students would understand and appreciate the logical organization of mathematics. He thought that if they understood the explanation about how one thing follows logically from another, then mathematics would not be as complicated, confusing, or scary.

Robert described concepts in algebra, geometry, and statistics as interrelated. However, during my observations, Robert rarely made provided (or explained) connections in which the components were from different mathematical strands. In addition, Robert made few connections between methods. Initially, the rarity of connections of this kind seemed to be inconsistent with the beliefs I inferred. Therefore, it was necessary to reexamine my data. In so doing, I realized Robert's beliefs about the interrelated nature of mathematics were rather new, a byproduct of teaching from an integrated mathematics curriculum. Therefore, the "newness" of this belief may explain why Robert made minimal connections across mathematical strands. It also may be possible that the nature of the content of the unit I observed or his knowledge of the content did not allow him to make connections between methods or connections across strands on a regular basis. In addition, within his descriptions of the logical organization of mathematics, specifically within the metaphors provided, Robert referred to "separate pieces of math" as being "related to each other" (Interview 5). Perhaps the characterization of "separate pieces of math" was a stronger belief than his belief that these pieces were "related to each other." Therefore, what I initially perceived as an inconsistency made sense when I considered the possible reasons related to Robert's beliefs and the corresponding kind of connection he rarely made.

Robert believed mathematics was the basis for everything. He thought it was possible to use mathematics to describe and characterize the physical world. However, during my observations, Robert rarely made connections to the real world. This seeming inconsistency made sense when I considered how Robert's beliefs about school mathematics differed from his beliefs about mathematics in this respect. Robert described school mathematics as highly abstract and removed from his students' everyday reality. Although he believed mathematics was inherently useful in describing the way the world works, he did not believe this characteristic of mathematics was present within the school mathematics curriculum. Instead, he believed school mathematics was more procedural by nature. In addition, Robert disliked what he described as "real world" problems within the school mathematics curriculum, because they often had an unnecessary context. Therefore, Robert was selective when incorporating "real world" problems into his instruction.

There were a few times during my observations when Robert provided connections to the real world. He carefully selected contexts that would help his students understand the mathematics or to understand mathematical principles existing in the real world. In these connections, it was possible to see how Robert viewed mathematics as the basis for everything. He used contexts such as interest rates, the stock market, and examples of scientific principles to show his students how mathematics could describe and characterize the physical world.

Comparison of Teachers' Beliefs and Their Relationship to Practice

In this section, I compare the teachers' beliefs about mathematics. First, I compare the teachers' beliefs about mathematics using Ernest's (1989) philosophical views of the nature of mathematics. Second, I focus my comparison on the teachers' beliefs about problem solving and then on the teachers' beliefs about how mathematics is connected. In this comparison, I attend to how the teachers' beliefs about mathematics seemed to be related to the mathematical connections they made for and with their students.

Teachers' Views of Mathematics

In this study, the teachers' beliefs spanned Ernest's (1989) three philosophical views of the nature of mathematics. In addition, one of the three teachers held beliefs about mathematics that combined various aspects of the views Ernest described. In particular, Justin's beliefs about the certainty and organized nature of mathematics corresponded to an overall Platonist view of mathematics. However, some of his descriptions and actions focused on developing particular procedures for his students, which seemed to provide an image of mathematics as "facts, rules and skills to be used in the pursuance of some external end" (p. 250). These particular beliefs Justin held aligned with an instrumental view of mathematics. Unlike Justin, Rachel and Robert's beliefs consistently resembled a particular view of mathematics. Rachel's beliefs about mathematics indicted she held a strong problem solving view of mathematics, and Robert's beliefs resembled a Platonist view of mathematics.

In her review of literature on teachers' beliefs, Thompson (1992) claimed that it seemed rather likely for a teacher's beliefs about mathematics to consist of various aspects from Ernest's (1989) philosophical views of mathematics, describing this variation as "conflicting beliefs" held in isolated clusters[13] of the teacher's beliefs system (p. 132). It is also possible, and perhaps more useful, to recognize that many more nuances exist among teachers' beliefs than Ernest's three views suggest. Therefore, it was necessary to apply Ernest's categorizations with care, realizing that the complexities of a teacher's beliefs about mathematics rarely fit within a single category. In addition, given the complexity of beliefs, it was necessary to examine teachers' beliefs about mathematics as sensible systems of beliefs to understand how the teacher's beliefs influenced many of the mathematical connections he or she made in practice.

Teachers' Beliefs about Problem Solving

In my analysis, while some similarities existed among specific beliefs held by the teachers, differences occurred in the ways these specific beliefs seemed to influence the mathematical connections the teachers made for and with their students. To understand how

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[13] Using Leatham's (2006) framework, I assume that sensible systems of beliefs do not allow for explicit contradictions. However, a possible implicit conflict between beliefs can and will remain as long as the opposing beliefs reside within different clusters undisturbed.

these seemingly similar beliefs might have influenced practice in different ways, I realized that I could not always look at specific beliefs in isolation from other beliefs within the system. Therefore, I examined how these seemingly similar beliefs related to the other beliefs within the sensible system. I also reviewed the meanings teachers applied to these seemingly similar beliefs about mathematics. This analytic method allowed me to compare how these beliefs, within the sensible system of beliefs, influenced the differences that emerged in the mathematical connections the teachers made in practice. As an example, I discuss Rachel and Justin's beliefs about problem solving to illustrate how seemingly similar beliefs about mathematics are not necessarily as similar as they would seem.

Each teacher held beliefs about the role of problem solving in mathematics. When I examined the teachers' beliefs about problem solving with their surrounding beliefs, distinct differences seemed to emerge in the ways the teachers, and in particular Justin and Rachel, included students in the process of making connections. For example, Justin emphasized the importance of problem-solving skills in his instruction. He expected his students to use problem-solving skills to make sense of the answers they found. His belief about problem-solving skills was closely related to his beliefs about "good work," "correct mathematics," and "right or wrong." The combination of these beliefs seemed to influence Justin's interactions with his students, because Justin regularly asked or answered questions about the steps in a particular procedure or if a student's solution made sense. As a result, students' contributions were often limited to providing the latter component of a connection through implication.

In comparison, Rachel's beliefs about problem solving included her belief that people do mathematics. She considered her students to be mathematicians. Unlike Justin, her beliefs about problem solving focused on the process of solving a problem and understanding the mathematics within the problem, rather than a focus on whether a particular answer was right or wrong. She used mathematical tasks to allow her students to engage in significant problem-solving experiences. Her belief about problem solving was closely related to her belief that mathematics is "connected." During whole-class or small-group discussions, students contributed components of mathematical connections as they shared their thinking about the mathematical concepts and procedures developed within the tasks. For these reasons, Rachel's beliefs about problem solving, along with her surrounding beliefs, seemed to influence the environment she created for her students to share their thinking and, in doing so, to contribute components across each of the five kinds of mathematical connections made in her teaching practice.

Teachers' Beliefs about *how* Mathematics is Connected

Each of the teachers described mathematics as a connected discipline. However, differences existed in the ways the teachers described *how* mathematics is connected. Therefore, in the following paragraphs, I explore the differences existing among the teachers' beliefs by examining the metaphors the teachers used to describe the connected nature of mathematics. Then, I consider how the differences in these beliefs may have influenced some of the differences in the kinds of mathematical connections the teachers made in practice.

Rachel's descriptions painted a vivid picture of her beliefs about how mathematics was connected. She compared the connected nature of mathematics to a *spider's web* (Interview 6). Rachel described mathematics as being woven together by the various connections she perceived existing among mathematical concepts and procedures. These connections provided various pathways to a particular conclusion. This particular belief about the connected nature of mathematics was unique to Rachel and was similar to Hiebert and Carpenter's (1992) description of mathematical knowledge structured like a web. In addition, she compared mathematics to a *big snowball*. In this comparison, mathematical entities, as the snowflakes within the snowball, were so closely related that it was not possible to distinguish among or to separate the mathematical entities (Interview 6). This description was similar to her claim that algebra and geometry should not be taught as separate subjects, because when combined together they demonstrate “mathematics as an integrated whole” (NCTM, 2000, p. 354). These beliefs existed within her problem solving view of mathematics, a view that recognizes mathematics as a “dynamically organized structure” (Ernest, 1989, p. 250).

Like Rachel, Robert described mathematical connections across mathematical entities. However, unlike Rachel, he referred to parts of mathematics as separate pieces of mathematics (Interview 5). Robert also believed mathematical entities existed within a logical organization. These beliefs fit well within his Platonist view of mathematics, which describes mathematics as a “unified body of certain knowledge” (Ernest, 1989, p. 250). To Robert, mathematics is connected in the ways that simple, separate pieces of mathematics combine together to form a more complex and powerful structure (Interview 5). He compared the connected nature of mathematics to *building a brick wall*. Given this comparison, he explained how mathematical connections develop when learners understand how separate pieces of mathematics build on and relate to surrounding pieces of mathematics. For this reason, Robert’s beliefs about the connected nature of mathematics differed from the structures Hiebert and Carpenter (1992) considered to represent networks of mathematical knowledge. Instead, Robert’s beliefs suggest a connected network of mathematical knowledge that expands both horizontally and vertically, beginning as a simple structure and capable of developing into something more complex for his students.

Similar to Robert and Rachel, Justin believed mathematics entities exist in a logical organization. However, Justin's beliefs about mathematics differed in that he emphasized the need to build mathematical entities in particular order. Metaphorically speaking, Justin compared building mathematical entities to the particular order needed to *build a car*, because "you have to know one before you should go to the next" (Interview 6). For Justin, mathematical connections develop as mathematical entities build on one another in a linear fashion. Following from this belief, he perceived mathematics as more connected within the individual strands of mathematics rather than between the strands. Consequently, his beliefs about the connected nature of mathematics are similar to Hiebert and Carpenter's (1992) depiction of a vertical hierarchy as a structure for a network of mathematical knowledge. These beliefs fit within the combination of Justin's instrumental and Platonist views of the nature of mathematics, implying an "accumulation of facts, rules and skills" exists within a "unified body" of mathematical knowledge (Ernest, 1989, p. 250).

The teachers held varying beliefs about mathematics. The variation among their beliefs, in particular their beliefs about how mathematics is connected, seemed to influence some of the differences in the nature, levels, and kinds of mathematical connections made in practice. For example, Rachel made several connections by connecting methods, a kind of connection that I observed almost exclusively in her practice. These connections seemed to reflect her beliefs about problem solving and how mathematics is connected like a spider's web. Rachel also made connections by relating an algebraic component with a geometric component for her students. The nature of these connections seemed to follow from her belief that mathematical entities are very closely related and connected. Similar to Rachel and Justin, many of the connections Robert made were connections through comparison. However, what was qualitatively different about this kind of connection in Robert's practice was the nature of the components within the connection. Robert regularly compared new topics to things his students already knew and understood. The nature of these connections corresponded to his beliefs about how mathematics was connected, because like building a brick wall, he began with simple pieces used to build something more complex. Many of the connections Justin made focused on developing particular procedures. Within these connections, Justin related a new procedure with a procedure previously developed (usually from the previous day's lesson or from the current unit), making mathematical connections in a linear fashion. In addition, the suggested connections within Justin's teaching reflected his beliefs about how mathematics is connected, because he would often suggest that what was being taught in today's lesson is somehow related to mathematics in tomorrow's lesson. This linear fashion of building and connecting mathematical entities resembled Justin's metaphor of the order used when *building a car* (Interview 6). The similarities within each teacher's beliefs and practices suggest relationships exist between the teacher's beliefs about mathematics and the mathematical connections they made in practice.

Comparing teachers' beliefs to their teaching practices is never a straightforward and simple task. The relationships I inferred between teachers' beliefs and practices were both subtle and complex. However, by looking within and between the narrative cases, I could explain many of the differences in the kinds of mathematical connections the teachers made, at least in part, by the differences existing in their beliefs about mathematics. Therefore, examining teachers' beliefs about mathematics as sensible systems provided me with a useful lens from which to interpret aspects of the teachers' practice and, in particular, the kinds of mathematical connections they made for and with their students. Examining the mathematical connections made in practice through the lens of teachers' beliefs about mathematics helped me understand some of the reasons for the variation occurring in the teachers' practices. In the next chapter, I consider the mathematical connections made in practice from multiple perspectives provided by the literature.

CHAPTER 6

EXAMINING MATHEMATICAL CONNECTIONS MADE IN PRACTICE FROM MULTIPLE PERSPECTIVES

The purpose of this study was to describe mathematical connections from the perspective of practice. Given this perspective, I asked, “In what ways do the mathematical connections teachers make in practice resemble the descriptions of mathematical connections in the literature?” To address this question, I revisited the literature and noticed several descriptions that echoed the mathematical connections my participants made. In this chapter, the mathematical connections described by the literature are compared with the mathematical connections made in practice. I orient my discussion by examining each of the three broad perspectives, in the literature, used to conceptualize mathematical connections: (1) Mathematical connections are part of a connected discipline, (2) Mathematical connections are products of understanding, and (3) Mathematical connections are part of the process of doing mathematics. I focus on aspects that are relevant to discussions of teachers’ practice and pay particular attention to the perspectives that conceptualize mathematical connections as products of understanding and as part of the process of doing mathematics. Examining the mathematical connections made in practice from the perspectives found in the literature provides the foundation necessary for mathematics educators to identify what steps must be taken to continue to move toward a vision of teaching and learning mathematics with understanding.

Mathematical Connections: Part of a Connected Discipline

One of the broad perspectives in the literature describes mathematical connections as a natural part of mathematics because mathematics is a connected discipline. From this perspective, *unifying themes*, such as functions or data, have been used to demonstrate the connections among multiple topics across the school mathematics curriculum (Clement & Sowder, 2003; Coxford, 1995; Hirschhorn & Viktora, 1995; NCTM, 2006; Usiskin, 2003). I purposefully selected the unit I observed in each teacher's classroom based on the presence of unifying themes within the unit. Consequently, unifying themes were present in each of the lessons I observed. Many of the mathematical connections teachers made were related to specific aspects of a unifying theme. Robert, for example, made several connections as he compared representations of specific kinds of functions, and Justin connected real world contexts to specific parametric functions. At other times, teachers also used a unifying theme in a more general way to make mathematical connections. For example, Rachel made a connection as she compared trigonometric functions to what it means to be a function. In this episode, Rachel used the definition of a function to arrive at a more general, and possibly more conceptual understanding, of what it means to be a trigonometric function.

Implications follow from this comparison for both research and practice. In this study, the very presence of a unifying theme within the content of the lesson supplied each teacher with the opportunity to make various kinds of mathematical connections related to the unifying theme, allowing the teacher multiple opportunities to demonstrate how mathematics is a connected discipline. Further research is needed to determine how the presence of unifying themes in a curriculum influences the mathematical connections made in practice. This research would likely characterize unifying themes as a useful way to organize a mathematics curriculum in that it provides teachers and students with opportunities to experience mathematics as a connected discipline and, in turn, provides students with opportunities to develop a more connected understanding of mathematics.

Mathematical Connections: Products of Understanding

The second broad perspective in the literature describes mathematical connections as products of understanding. Hiebert and Carpenter's (1992) definition of understanding corresponds to this particular perspective, for they defined understanding in mathematics as "making connections between ideas, facts, or procedures… understanding involves recognizing relationships between pieces of information" (p. 67). They claimed this definition of understanding was a common theme in the mathematics education literature, citing several classic works in the literature to support their assertion (e.g., Polya, 1957; Hiebert, 1986). This common theme suggests a relationship exists between mathematical connections and understanding, in that "making connections builds understanding" (NCTM, 2000, p. 274).

The relationship between mathematical connections and understanding, however, does not imply that all mathematical connections lead to a student's meaningful[14] understanding of mathematics. Hiebert et al. (1997) cautioned, "Not all connections are equally useful. Some provide real insights and others are quite trivial" (p. 5). Because Hiebert and his colleagues did not provide descriptions to differentiate among the kinds of mathematical connections that may be more or less useful, I examine the mathematical connections made in practice from this perspective using Skemp's (1976) descriptions of instrumental understanding and relational understanding to frame my discussion. In the paragraphs that follow, I recognize that the

[14] I consider a meaningful understanding of mathematics to reflect what Skemp (1976) defined as relational understanding of mathematics, which includes the mathematical knowledge of both what to do and why.

mathematical connections made by the teacher can only supply opportunities for students to develop understanding, which does not imply students develop understanding. Therefore, as I examine the relationship between mathematical connections and understanding, I consider the opportunities for understanding that the mathematical connections seemed to supply.

Some mathematical connections teachers made in practice only provided knowledge of rules and procedures. For this reason, these connections seemed to reflect what Skemp (1976) described as an *instrumental understanding* of mathematics, because the knowledge provided by these connections did not include the reasoning behind the rules or procedures. For example, at times, the teacher made a connection through comparison by telling a student that the procedure needed to solve a specific exercise was the same as the procedure presented during the lecture. These provided connections did not include an explanation detailing the reasons why the procedure was relevant or needed to solve the specific exercise. A second example follows from some of the connections teachers made through logical implications. In these instances, the provided connection usually acted as a cue or a reminder about the steps involved in following a particular procedure–if this happens, then this must follow. Similar to the previous example, these provided connections did not explain the reasons why one step followed the other. In each of these examples, the opportunity these provided connections supplied was to help students remember certain aspects of mathematical rules and procedures without knowing why.

At other times, the mathematical connections extended the mathematics beyond simply knowing rules and procedures. These connections included the reasons why a particular rule or procedure worked. Without exception, these connections existed as provided-and-explained connections. These connections resembled Skemp's (1976) description of *relational understanding*, because the explanation gave the reasons why a particular rule or procedure worked. Some of the explanations why were specific to problem types. For example, Robert made a provided-and-explained connection by comparing two specific functions, $f(x) = 2^x$ with $f(x)=2^{x-2}$ (Observation, November 9). He compared the differences between the table of values for each function, the graph of each function, and the input for each function when the output was equal to 1. Through these comparisons, Robert explained the reasons why the graph of $f(x)=2^{x-2}$ was shifted two units to the right of the graph of $f(x) = 2^x$. Some of the explanations within a connection did not depend on specific problem types and were therefore more generalizable. For example, Rachel connected how to “undo” the sine function with similar procedures her students had learned in previous units (Observation, September 16). In this episode, Rachel’s explanation did not focus on specific problem types; rather, she focused on what it meant to undo a mathematical operation. In the final interviews, each of the teachers related knowing why things worked to understanding mathematics.

From this perspective, at first glance, it may seem that provided-and-explained connections are those that “provide real insights” and provided connections are “quite trivial” (Hiebert et al., 1997, p. 5). Or, perhaps provided connections only supply opportunities for students to develop an instrumental understanding of mathematics, while provided-and-explained connections always afford students opportunities to develop a relational understanding of mathematics. However, these assumptions are rather hasty. In practice, many times simply providing the connection between two mathematical entities seemed to be sufficient for the student’s understanding of how a particular mathematical entity related to another mathematical entity. For example, Rachel provided a connection through comparison by comparing the idea of a mathematical average of a data set to the balancing point of the numbers in the data set (Observation, September 26). This provided connection did not seem trivial because it provided a connection between ideas, which resembled Hiebert and Carpenter's (1992) broad definition of understanding. In addition, there were times that, as a researcher, I assumed the explanation why things worked was implied or already understood by the class. In these instances, a provided-and-explained connection may have been tedious or redundant. For example, recall that Rachel did not explain the connection between the two formulas provided for finding the area of a square, "Base times height, or side squared" (Observation, September 20). In this episode, the explanation why the formulas were equivalent seemed to be apparent to her students, negating the necessity for Rachel to explain the connection between the methods. These examples of provided connections seemed to supply sufficient opportunities for the students to understand mathematics.

Pirie (1988) cautioned researchers to be careful when applying labels of understanding, because the researcher must make inferences about the student's thinking. For this reason, not all provided-and-explained connections might have supplied a student with the opportunity to understand mathematics in a relational way. It was entirely possible that a student did not comprehend the teacher's explanation in a provided-and-explained connection. For example, Justin reminded his students, "Remember $2 \cdot 3 \cdot 5$ is the same as $3 \cdot 5 \cdot 2$, [because] multiplication is commutative" (Observation, October 19). Given this provided-and-explained connection, the student's opportunity to comprehend the "why" was dependent on the student's knowledge of the commutative property. Without knowledge of the commutative property, this provided-and-explained connection made by the teacher did not necessarily exist as a provided-and-explained connection in the mind of the student. The student may not have developed any additional understanding as a result of the mathematical connection provided and explained by the teacher.

No simple cause-and-effect relationship occurs between the levels and kinds of mathematical connections made by the teacher and the opportunities for students to develop understanding. The hierarchy among the levels of connections within the framework does not indicate which levels of mathematical connections are more or less useful for helping students to develop a relational understanding of mathematics. In addition, no direct relationship exists between the kinds of connections and how they may or may not lead students to a relational understanding of mathematics. Sometimes connections through comparison seemed to supply opportunities for students to develop an instrumental understanding, and at other times this kind of connection seemed to supply opportunities for students to develop a relational understanding of mathematics. It becomes necessary to ask, “What can be said about mathematical connections and understanding?” First, making mathematical connections does not always mean these connections will lead to students understanding mathematics in meaningful ways. Second, when making mathematical connections, of any level or kind, it is necessary to think about how the mathematical connections relate to students developing an instrumental or relational understanding of mathematics. In particular, if mathematics educators want to create opportunities for students to understand mathematics, they must consider what mathematical connections may lead to a more relational understanding of mathematics. In addition, researchers should examine the levels and kinds of mathematical connections teachers make in practice from the perspective of the student and of the student’s understanding of mathematics. This line of research could inform the teaching and learning of mathematics with understanding, which may influence how teachers engage their students in the process of making mathematical connections.

Mathematical Connections: Part of the Process of Doing Mathematics

The third broad perspective in the literature describes mathematical connections as part of the process of doing mathematics. The process of making mathematical connections is often found within the intersection of other mathematical processes, such as multiple representations, problem solving, proof, and real world applications and mathematical modeling (Coxford, 1995; Cuoco, Goldenberg, & Mark, 1996; Hiebert et al., 1997; NCTM, 2000, 2006; Stein, Engle, Smith, & Hughes, 2008). The literature often refers to this perspective in rather tangible ways, using a variety of examples to describe what making mathematical connections means and how making mathematical connections is the process necessary for learners to develop an understanding of mathematics. The examples and descriptions of mathematical connections in the literature fit nicely within the categorical descriptions of the kinds of mathematical connections within the Mathematical Connections Framework. When compared to the kinds of mathematical connections teachers made in practice, however, a few notable differences were apparent.

Connecting through a logical implication provides the connection through the implication, if *A*, then *B*. This kind of mathematical connection is the same as what Cuoco, Goldenberg, and Mark (1996) characterized as a logical connection. They described using a series of logical connections to construct a mathematical proof. In contrast, during my observations of practice, connections through logical implications rarely occurred within the context of a mathematical proof. Rather, this kind of mathematical connection was more common within the steps of a mathematical procedure. For example, in practice, the connection, "If you plug in 350 for *x*, and [then] you can get *t*" described the procedure necessary to solve a particular problem (Justin, Observation, October 19). The difference between the descriptions from the literature and observations of practice is not surprising, because Knuth (2002) claimed the role of proof in secondary school mathematics is "peripheral at best, usually limited to the domain of Euclidean geometry" (p. 379).

Connecting methods provides a mathematical connection through the various methods used to solve a problem, where *A* or *B* could be used to find *C*. This kind of mathematical connection is similar to the kind of mathematical connections Stein, Engle, Smith, and Hughes (2008) recommended teachers make in discussions surrounding students' responses to high cognitive demand tasks. They argued that the connections made in such discussions should focus on why the methods are related to one another, rather than allowing the discussion to only consist of the separate methods used to solve the particular problem. Essentially, Stein et al. considered connections between methods to be much more valuable when the connections occurred as provided-and-explained connections. In my observations of practice, most often, this kind of mathematical connection was a provided connection. Teachers rarely explained the reasons why the methods were related beyond that the methods led to a common solution. One possible reason for this notable difference was the frequent absence of high cognitive demand tasks during my observations of practice. Or, it is also entirely possible that teachers thought the explanation of this kind of connection was often more readily apparent and obvious to students, eliminating the necessity for the teacher to explain the connection between methods.

Connecting to the real world provides a connection between a mathematical entity and a real world concept or context, where *A* is an example of *B* in the real world. Gainsburg's (2008) review of the literature identified a variety of descriptions related to this kind of connection: (a) simple analogies, (b) classic "word problems," (c) the analysis of real data, (d) discussions of mathematics in society, (e) "hands-on" representations of mathematical concepts, and (f) mathematically modeling real phenomena (p. 200). Each of these descriptions of real world connections occurred during my observations of practice. Word problems or mathematical tasks were the most common method teachers used to make a mathematical connection to the real world, whereas connections to the real world through “hands-on” representations of mathematical concepts or mathematical modeling occurred only once during my observations. Gainsburg suggested that some of these methods, such as mathematical modeling, take more time within a lesson and are therefore rarely used in secondary mathematics instruction.

Although the teachers made mathematical connections in practice that were similar to those described in the literature, it seems fair to say that the kinds of mathematical connections they made did not always resemble the kinds of mathematical connections described in the literature. Some of the possible reasons for the differences noted between the mathematical connections made in practice and the mathematical connections described by the literature are the content and focus of the mathematics curriculum as well as the logistical constraints of teaching school mathematics. These differences capture the important and practical realities of teaching. Consequently, mathematics educators must consider how to help make these connections more manageable for teachers. In particular, researchers should examine what aspects of the school curriculum (e.g., a minimal emphasis on proof or a prevalence of low cognitive demand tasks in the curriculum) and what logistical constraints (e.g., limited time) hinder teachers from making these particular connections in practice. With this knowledge, mathematics educators could provide more focused support for teachers, whether through additional curriculum materials and tasks or pedagogical assistance, to help teachers include these mathematical connections in their practice.

In conclusion, it is possible to examine mathematical connections from many perspectives. The mathematical connections described in this study are from the perspective of practice, focusing on the connections made by the teacher. Such an orientation is a necessary foundation and basis from which researchers and educators may consider how the perspectives from the literature combine together with the perspective of practice to influence the learning and teaching of mathematics with understanding. Examining the mathematical connections made in practice through the lenses provided by these perspectives in the literature suggests several implications for next steps to be taken by both researchers and educators. Next steps require studying how the presence of unifying themes in a curriculum influences the mathematical connections made in practice, how particular levels and kinds of mathematical connections help students develop a meaningful understanding of mathematics, and what curricular or pedagogical supports are necessary to help teachers make meaningful mathematical connections for and with their students. In the final chapter, I discuss the findings of this study and the implications that directly follow from these findings.

CHAPTER 7

DISCUSSION AND IMPLICATIONS

In this study, I sought to explore the kinds of mathematical connections teachers make in practice. My interest in this avenue of research followed from my assumption that mathematical connections support students' understanding of mathematics and because researchers had yet to investigate mathematical connections from the perspective of practice. Knowing about the kinds of mathematical connections teachers make in practice may influence the teaching and learning of mathematics with understanding as well as research investigating the use of mathematical connections in instruction. I also examined teachers' beliefs about mathematics, because teachers' beliefs about mathematics could provide valuable insights into the kinds of mathematical connections teachers make. To reiterate, the following research questions guided this study:

1. What kinds of mathematical connections do three secondary mathematics teachers make in their teaching practice?
2. What are the beliefs about mathematics of these secondary mathematics teachers?
3. What relationships, if any, exist between the kinds of mathematical connections that these secondary mathematics teachers make and their beliefs about mathematics?

Qualitative research methods allowed me to answer my research questions with depth and detail. I selected three secondary mathematics teachers to participate in the study– Rachel, Justin, and Robert. For each teacher, my primary data sources included six in-depth, semi-structured interviews and approximately two weeks of classroom observations. I used an inductive and iterative coding process to analyze my data, and I constructed narrative cases to make sense of the coded data and to address each of my research questions.

I developed the Mathematical Connections Framework to describe the levels and kinds of mathematical connections these teachers made in practice. The framework is a direct response to Hiebert and Carpenter's (1992) call for researchers to identify and study "what connections become explicit during teacher-student interactions" (p. 86). To respond to their call, first, I analyzed observation data to determine what comprised an *explicit* mathematical connection in practice. During my analysis, I noticed that teachers made mathematical connections in more or less explicit ways for their students. Therefore, I defined levels to distinguish among the implicit or explicit nature of these mathematical connections (See Table 2 on p. 60). I continued to address Hiebert and Carpenter's call by examining *what* kinds of mathematical connections teachers made in practice. I developed categorical descriptions to distinguish among the kinds of mathematical connections these teachers made (See Table 3 on p. 66). The definitions and descriptions of the levels and kinds of mathematical connections combine together to construct a framework grounded in the teaching practices of my participants.

I used the framework to reanalyze the mathematical connections made by these teachers. The teachers in this study made various levels and kinds of mathematical connections for and with their students. Examining teachers' beliefs about mathematics provided valuable insights into these teachers' practices, helping me understand some of the reasons for the variation occurring among the mathematical connections the teachers made in practice. In each case study, the mathematical connections the teacher made were related to the teacher's beliefs about mathematics and, in particular, the teacher's beliefs about the connected nature of mathematics. In the sections that follow, aspects of the study are developed in more detail, and the implications that follow from the findings of this study are discussed.

Discussion

Each teacher in this study was caring, well prepared, and worked diligently to offer his or her students a demanding mathematics curriculum. I selected Rachel, Justin, and Robert to participate in the study because all indicators suggested that these teachers regularly incorporated mathematical connections into their instruction. I also selected these teachers because indicators suggested that these teachers represented a variety of mathematical connections. This method of purposeful selection was one of the strengths of this research design, because the variation existing among these teachers, who regularly made mathematical connections, provided me a significant opportunity to study the mathematical connections made in practice and how teachers' beliefs about mathematics related to the mathematical connections made in practice.

Mathematical Connections

Rachel, Justin, and Robert regularly made mathematical connections for and with their students. Given this common theme, it was surprising to find that each teacher provided a distinctly different picture of how mathematical connections can be made in practice. Variation occurred in the levels and kinds of mathematical connections each of these teachers made, and variation occurred in how they included students in the process of making mathematical connections. The variation among the mathematical connections these teachers made suggests that the students in each of their classes had different opportunities to learn mathematics with understanding. In the paragraphs that follow, certain core features of the teachers' instruction are discussed that seemed to support or hinder students' opportunities to participate in the process of making mathematical connections.

First, the structure of the teacher's instruction and the corresponding classroom discourse influenced the extent to which students were able to participate in the process of making mathematical connections. Small-group and whole-class discussions often provided an environment for students to freely contribute components of a mathematical connection. During these discussions in Rachel's classroom, she asked questions that prompted her students to make conjectures, elaborate on ideas, build on others' ideas, discuss solution methods, and justify their reasoning. As a result, the collaborative nature of these discussions allowed students to reflect on and to communicate their thinking and, in doing so, to contribute to all five kinds of mathematical connections. In comparison, lectures provided students with limited opportunities to contribute to the connection making process, because students' contributions were often restricted to short answers provided in response to the teacher's questions woven throughout the lecture. For example, in any given lecture, both Justin and Robert asked students questions about the next step in a procedure or to provide the reasoning for a given step in a procedure or a mathematical rule. Given this instructional structure, students provided short responses and their contributions of connections were often limited to particular kinds of connections (e.g., connecting through a logical implication). The importance of instructional structure and the corresponding classroom discourse is echoed throughout the literature (e.g., Boaler & Humphreys, 2005; Cobb, Yackel, & Wood, 1992; Hiebert et al., 1997). To include students in the connection making process, students must be able to make connections in collaboration with their teacher and fellow students. For this reason, instructional structure and classroom discourse should not solely be confined within the bounds instituted by a teacher's lecture, rather it is useful to incorporate frequent and purposeful small-group or whole-class discussions to allow students to engage in the process of making mathematical connections by sharing and building on ideas and methods.

Second, similar to the structure of the teacher's instruction and the corresponding classroom discourse, the nature of the mathematical tasks teachers used also influenced how students were able to participate in the process of making mathematical connections. During observations of practice, problem-solving tasks, such as those Rachel used to introduce a new mathematical concept, provided students with opportunities to construct new understandings from various methods developed to solve the given task, allowing students opportunities to make connections of various levels and kinds. The tasks Robert used provided students with opportunities to connect new knowledge to previous knowledge, supplying students with opportunities to make connections through comparison. Often, tasks that were used to practice procedures did not provide many opportunities to make mathematical connections. Other researchers (e.g., Doyle, 1988; Hiebert et al., 1997; Stein, Smith, Henningsen, & Silver, 2000) have similarly emphasized the important role tasks play in determining the mathematical work students do. In particular, Hiebert and colleagues (1997) described the importance of tasks in making mathematical connections, stating certain tasks focus on rote memorization of procedures whereas others allow students to spend time reflecting on "the way things work, on how various ideas and procedures are the same or different, on how what they already know relates to the situations they encounter, [then] they are likely to build new relationships" (p. 17). In the next section, teachers' beliefs about mathematics are discussed.

Teachers' Beliefs about Mathematics

On the surface, similarities existed among the teachers' beliefs about mathematics. Each teacher referred to mathematics as being logical, connected, and appreciated the problem solving nature of mathematics. However, when looking beyond the surface, these seemingly similar beliefs were not as similar as they seemed. Teachers' beliefs about mathematics varied in rather significant ways when the content and the structure of their beliefs were examined. For example, although each teacher described mathematics as a connected body of knowledge, differences existed in the content of this seemingly similar belief, because the each of the teachers provided different descriptions about how mathematics is connected. For a second example, each teacher held beliefs about the role of problem solving in mathematics. Yet, when examined within the system, the combination of the surrounding beliefs informed the differences existing among how these teachers viewed the role of problem solving in mathematics. Therefore, researchers studying beliefs must focus on not only what teachers' believe but also how they believe it.

Some researchers (e.g., Hoyles, 1992; Raymond, 1997) have described inconsistencies existing between teachers' beliefs and practice. In this study, however, the notion of consistency is emphasized in the inferences and descriptions of the teachers' beliefs. Drawing on Leatham's (2006) philosophical perspective of sensible systems of beliefs, I view what other researchers have described as "inconsistencies" as perceived inconsistencies. When viewed in this way, perceived inconsistencies act as metaphorical red flags, signifying a need to reexamine what was initially perceived to be an inconsistency between a teacher's descriptions and actions. For example, from this perspective, I recognized that Robert's beliefs about mathematics differed based on context, whether he was describing mathematics in a more formal sense or when he was describing his experiences teaching school mathematics. This contextual distinction helped me understand that his beliefs about school mathematics were more related to the mathematical connections he made in practice. This philosophical perspective of sensible systems of beliefs provides useful information for researchers and mathematics teacher educators, because "exploring and explaining apparent inconsistencies rather than pointing out inconsistencies lends itself to developing a deeper understanding of the nature of beliefs and how they are held"

(Leatham, 2002, p. 242). In the next section, I discuss how teachers' beliefs related to their practice.

Relating Beliefs to Practice

Drawing upon Rokeach's (1968) definition of beliefs as well as past research on teachers' beliefs, I assume that teachers' beliefs are related to their instructional practices. Given this assumption, I questioned whether these teachers' beliefs about mathematics, a subset of their beliefs, were related to the mathematical connections they made, a subset of their instructional practices. In addition, some researchers (e.g., Skott, 2001) have observed that teachers' beliefs about mathematics are not always the beliefs that most influence their instructional practices. However, in this study, these teachers' beliefs about mathematics provided reasonable explanations for much of the variation occurring among the mathematical connections the teachers made in practice. For this reason, these teachers' beliefs about mathematics seemed to be directly related to the mathematical connections made in practice. It is likely that this relationship exists because making mathematical connections is a particularly mathematical instructional practice, and therefore beliefs about mathematics were directly related to this particularly mathematical instructional practice.

The teachers' beliefs about mathematics explained, at least in part, the levels and kinds of mathematical connections teachers made in practice. For each teacher, the *levels* of mathematical connections made in practice were related the teacher's beliefs about mathematics. For example, Justin made several suggested connections during my observations of his practice. He often suggested two mathematical topics were related without explicitly providing the relationship for his students, making suggested connections such as "this is related to what we are doing tomorrow" or "we need this to do what is coming up next" (Observation, October 20). Justin believed mathematics is connected in that mathematical entities build on one another in a distinctly linear fashion. This belief corresponded to the prevalence of suggested connections in his practice. Similarly, the *kinds* of mathematical connections made in practice were related to the teacher's beliefs about mathematics. For example, Rachel made several connections by connecting methods, a kind of mathematical connection that occurred almost exclusively in her practice. This kind of mathematical connection reflected her beliefs about how mathematics is connected like a spider's web, viewing mathematics as being woven together by the various connections among mathematical entities that allowed for her students to use various pathways to arrive at a particular solution. For a second example, Robert provided several connections through comparison to help his students remember certain things about mathematics or to avoid possible misconceptions about a particular concept or procedure. These connections followed from his belief that school mathematics was scary for many of his students, and therefore he used these connections through comparison to make mathematics a little less scary for his students. These examples demonstrate how the teachers' beliefs about mathematics influenced and supported the levels and kinds of mathematical connections teachers made in practice.

Examining mathematical connections through the lens of teachers' beliefs also suggests that some of the teacher's beliefs might have hindered the teacher from making certain levels or kinds mathematical connections in practice. For example, Justin's beliefs about "good mathematics" and "particular procedures" held implications for how his students were allowed to solve problems. In practice, these beliefs could be seen as Justin encouraged and expected his students to use the particular procedures he developed during his lecture. As a result, throughout my observations of Justin's teaching, I did not find a single instance of a connection between methods. The absence of this kind of mathematical connection from Justin's practice did not seem surprising, because his focus on particular procedures did not allow multiple methods to be considered or used to solve a problem. Consequently, it is likely that a teacher's beliefs about mathematics can hinder the teacher from making certain levels or kinds mathematical connections in practice.

The relationship between these teachers' beliefs about mathematics and the mathematical connections made in practice support a rather broad interpretation of Ernest's (2008) theoretical model of simplified relations among philosophical views of mathematics, values, and images of mathematics in school (see Chapter 2 for description of the model). First, the teachers' beliefs about how mathematics is connected (what Ernest described as connected or separated values) followed from or corresponded to their beliefs about mathematics (what Ernest described as philosophical views of the nature of mathematics). Second, the teachers' beliefs about mathematics, and in particular their beliefs about how mathematics is connected, carried significant implications for the mathematical connections they made in practice (what Ernest described as the connected or separated image of school mathematics). These empirical findings provide support for the main ideas Ernest proposed within the first three levels of his theoretical model, suggesting teachers' beliefs about mathematics and how mathematics is connected influence the mathematical connections made in practice.

Ernest (2008) claimed that certain views of mathematics would likely influence the extent to which the image of mathematics portrayed in the classroom represents mathematics as a connected or separated discipline. The findings in this study provide empirical evidence to support his claim. Each of the teachers in this study believed mathematics is connected and appreciated the connected nature of mathematics. Many of their beliefs corresponded to philosophical views of mathematics that recognize mathematics as a connected discipline (i.e, the Platonist view and the problem solving view of mathematics). These beliefs about mathematics supported and influenced the mathematical connections these teachers made in practice. However, Justin held certain beliefs about mathematics that resembled aspects of an instrumental view of mathematics, a view that considers mathematics to be a collection of unrelated facts, rules, and procedures (Ernest, 1989). These particular beliefs may well have hindered Justin from making certain kinds mathematical connections in practice. This finding, when combined with Ernest's claim, suggests that is possible for certain beliefs about mathematics to support or hinder the levels and kinds of mathematical connections teachers make in practice.

Other factors, in addition to teachers' beliefs about mathematics, may have also influenced the mathematical connections these teachers made in practice. First, it is possible that teachers' mathematical knowledge played an influential role in the connections made. For example, I observed Justin teach a unit on parametric equations, in which many of the connections he made focused on developing procedural knowledge for his students. Justin admitted the topic of parametric equations was relatively unfamiliar territory for him, stating that he had to relearn much of the content the year before when he taught the Accelerated Mathematics III for the first time (Observation, October 7). It is possible that the "newness" of the topic implied Justin did not have the knowledge necessary to make many connections of a more conceptual nature, because his own knowledge of the topic might have been rather procedural. Second, it is possible that the content and the structure of the curriculum and curriculum standards influenced some of the mathematical connections these teachers made. For example, one of Rachel's lessons focused on a particular curriculum standard, "Explain the relationship between the trigonometric ratios of complementary angles" (Georgia Department of

Education, 2009, p. 3). In this lesson, Rachel continually emphasized the standard as she helped her students understand, "the cosine of angle is equal to the sine of its complement," which resulted in related mathematical connections (Observation, September 22). Similarly, Robert made several connections through comparison as he used absolute value functions to introduce the topic of piecewise functions because "the state of Georgia says we have to" (Observation, October 31). Research is needed to explore how these additional factors such as teacher's knowledge or curriculum influence the mathematical connections teachers make in practice. In particular, researchers should focus on how teachers' beliefs along with these additional factors influence the mathematical connections teachers make in practice.

Thompson (1992) challenged the assumption suggesting the relationship between teachers' beliefs about mathematics and their teaching practices is "a simple linear-causal one" (p. 140). In a similar manner, I argue that there are too many layers, complexities, and additional factors involved to directly apply the "simple dichotomisation" of beliefs and practices provided by Ernest's (2008) theoretical model (p. 1). More research needs to be done to inform how Ernest's model should be modified to include additional complexities as well as additional influential factors in order for the model to be used to frame studies of teachers' beliefs and practices.

Implications for Research and Practice

The findings of this study provide two main contributions to the literature. First, I contribute the Mathematical Connections Framework as a tool for researchers and educators to examine and to learn more about the mathematical connections made in practice. Second, I contribute detailed descriptions of teachers' beliefs about mathematics and how these beliefs relate to practice. Throughout this section, I consider possible implications that follow from the findings of this study for both research and practice.

The Mathematical Connections Framework

Grossman and McDonald (2008) called for the development of frameworks to "parse teaching" in ways that provide researchers and teacher educators with the necessary "tools to describe, analyze, and improve teaching" (p.185). They claimed such frameworks can allow for the careful examination of the various "core components" of teaching to understand how they combine together to form this complex system of instruction (p. 191). Among these core components are the ways teachers connect mathematical entities for and with their students. For this reason, researchers and teacher educators could use the Mathematical Connections Framework to influence the teaching and learning of mathematics. In the following paragraphs, I describe possible uses of the framework for research and practice.

The Mathematical Connections Framework provides researchers with a useful analytic tool to examine practice, because it synthesizes many of the descriptions in the literature characterizing mathematical connections as part of the process of doing mathematics. In my own research, I found the framework useful for several reasons. The detailed definitions and descriptions within the framework supplied me with a robust coding scheme to make sense of my classroom data. First, the coding scheme allowed me to identify the individual mathematical connections teachers made within a given mathematics lesson, in which I was able to capture the mathematics connections teachers made within a brief discussion or over the course of the lesson. Second, I was able to identify the mathematical connections teachers made in a variety of instructional contexts, such as a teacher's lecture, a problem-solving task, and a whole-class or small-group discussion. Third, I was able to not only examine the levels and kinds of mathematical connections but also the nature of the content within the components of the mathematical connection. The coding scheme was flexible in that it did not limit the components within the mathematical connection to a particular strand of mathematics or a particular kind of mathematical knowledge. Using the framework, I analyzed the mathematical connections made in practice from a fine-grained perspective, which allowed me to construct narrative cases for each of my teachers to describe and differentiate among the mathematical connections each teacher made in practice.

The Mathematical Connections Framework provides a tool for researchers to use when building on this study and exploring additional aspects related to the mathematical connections made in practice. For example, using the framework to identify mathematical connections, researchers can then explore how teachers support students' contributions of mathematical connections. This examination can include an analysis of who is contributing the various components of the mathematical connection or an analysis of the kinds of questions used to elicit components of the connection. For a second example, the framework can be used to support research related to teaching and learning mathematics with understanding. Although the framework does not suggest which mathematical connections may be more or less effective for helping students develop a more meaningful understanding of mathematics, the framework does provide a coding scheme for researchers to generate additional descriptive cases of the mathematical connections made in practice. An examination across additional cases is necessary for researchers to learn more about what levels and kinds of mathematical connections may lead to students' understanding in particular situations. Such knowledge from this line of research may result in a set of instructional practices for teachers to consider when purposefully planning for and making mathematical connections in their teaching. This line of research may establish practices for making productive and purposeful mathematical connections to support students' understanding of mathematics.

Teacher educators can use the framework with both prospective and practicing teachers to support both teacher learning and reflection. First, teacher learning occurs when teachers are able to "reflect on and refine instructional practice–during class and outside class, alone and with others" (NCTM, 2000, p. 19). To improve their instruction, teachers must be able to critically analyze classroom practices. Grossman and McDonald (2008) emphasized their need for a "common vocabulary" to describe and to analyze the various components that comprise teaching (p. 187). The Mathematical Connections Framework provides prospective and practicing teachers with a common vocabulary from which they may examine and reflect on the mathematical connections made in practice. When used in this way, the framework has significant potential to influence practice. For example, knowledge of the different levels mathematical connections may influence a practicing teacher to examine and reflect on the mathematical connections he or she made within a given lesson. In this examination, the teacher may find that a particular connection did not include the explanation of the relationship and may determine that it is necessary to explain the connection in a following lesson. Second, Ball and Cohen (1999) suggested, "Teachers' everyday work could become a source for constructive professional development" (p. 6). With this in mind, teacher educators could use the framework with prospective or practicing teachers, asking teachers to examine and analyze the practice of others. For example, using the common vocabulary provided by the framework, teacher educators can stimulate dialogues among either prospective or practicing teachers about the mathematical connections made in an episode of classroom instruction. Teachers can observe a particular episode and then discuss the mathematical connections made within the given episode.

In this discussion, teachers can examine what mathematical connections were made and consider what mathematical connections could have been made within the given episode. In turn, these dialogues among teachers can influence practice, causing the teachers to personally examine, reflect on, and possibly change their own teaching practices.

Relating Teachers' Beliefs and Practices

In her review of the literature on teachers' beliefs, Thompson (1992) found that teachers' beliefs about mathematics influenced the teachers' patterns of instruction. This present study contributes additional empirical evidence to support Thompson's claim. This study extends the literature base on teachers' beliefs and practice with a particular focus, providing detailed descriptions of how secondary mathematics teachers' beliefs about mathematics were related to the mathematical connections they made in practice. In addition, teachers' beliefs about mathematics not only influenced many of the mathematical connections they made, but also, at times, seemed to hinder certain levels or kinds of mathematical connections from being made in practice.

This study contributes detailed descriptions of how these teachers believe mathematics is connected. Each teacher held markedly different beliefs about how mathematics is connected. Given the variation among their beliefs, several important questions follow. To what extent do these teachers' beliefs map the terrain of how teachers believe mathematics is connected? What are the beliefs of teachers who do not believe mathematics is connected? What are the beliefs of teachers who do not regularly make mathematical connections in practice? How do these teachers' beliefs vary? These questions emphasize a limitation of this study, for the descriptions of teachers' beliefs and practices in this study represent teachers who both believe mathematics is connected and regularly make mathematical connections. Therefore, these descriptions of beliefs and practice do not provide insights about the majority of teachers who do not believe mathematics is connected or do not regularly make mathematical connections in practice. Research is needed to describe these teachers' beliefs about mathematics and their related teaching practices. Knowledge of these teachers' beliefs would be useful in developing opportunities for these teachers to examine, reflect, and possibly modify their beliefs and practices.

The literature in mathematics education suggests that many teachers believe mathematics is merely a collection of unrelated facts and procedures (e.g., Romberg & Kaput, 1999; Thompson, 1992). Given the findings of this study, it seems reasonable to suggest that a teacher holding this particular belief about mathematics would rarely make mathematical connections in practice. For this reason, researchers and teacher educators should provide opportunities for prospective and practicing teachers to explicitly examine their beliefs about how mathematics is connected or separated. Within these opportunities, teachers should first consider how they believe mathematics is or is not connected and then consider other descriptions of how mathematical entities may be related. Subsequent opportunities can allow teachers to experience mathematics as a connected discipline. These opportunities may lead many teachers to personally redefine mathematics in a way that no longer views mathematics as separated or segregated but as an inherently connected discipline. Such opportunities could not only lead to a change in beliefs but possibly a change in practices.

In conclusion, many reform movements have emphasized the necessity of making mathematical connections to build students' understanding of mathematics. Although researchers and educators have examined mathematical connections from a variety of perspectives, this study provides the first examination of mathematical connections from the perspective of practice. Teachers' beliefs about mathematics were also studied and provided explanations for many of the mathematical connections these teachers made in practice. Describing and characterizing mathematical connections from the perspective of practice provides researchers and teacher educators with a framework that offers a common vocabulary for discussion of mathematical connections, and knowledge that can be used to help teachers develop instructional practices that emphasize teaching and learning mathematics with understanding.

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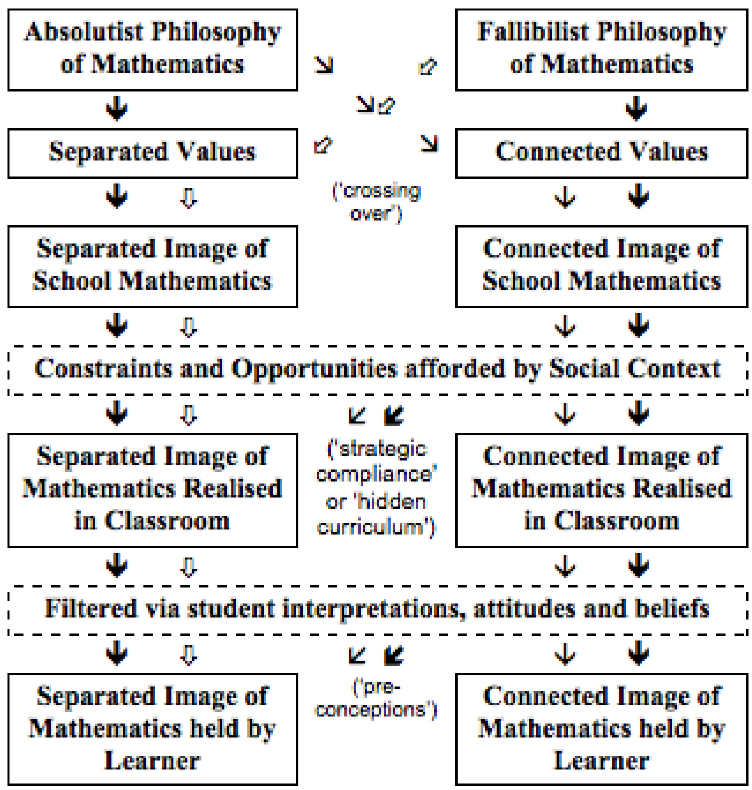
APPENDIX A

ERNEST'S (2008) FULL MODEL OF SIMPLIFIED RELATIONS



APPENDIX B

LETTER WITH QUESTIONNAIRE TO POSSIBLE PARTICIPANTS

Dear _______________,

You are invited to participate in a study for my dissertation research. The focus of my dissertation research is how teachers' beliefs influence their teaching practices. All information collected will be treated confidentially.

The attached background questionnaire consists of items asking for information regarding your background and teaching practices. Please take this opportunity to share your views on teaching and your practice with me. Thank you for your help.

Sincerely,
Laura M. Singletary
Doctoral Candidate
Mathematics Education
University of Georgia

Background Questionnaire.
Background Information

1. Name:

2. How many years have you taught mathematics at the secondary level (Grades 6–12)?

3. How long have you taught in this school district? How long have you taught in this school?

4. What mathematics classes have you taught?

5. What mathematics classes are you currently teaching?

6. What classes will you be teaching in the fall?

7. What is your highest degree?

8. List the degree(s) you have earned, the area(s) of concentration, and the institution(s) you attended (e.g. B.S. in mathematics education, University of Georgia)

9. Do you prefer to teach from a discrete course sequence (such as Algebra 1, Geometry, Algebra 2, etc.) or do you prefer to teach from a more integrated sequence (such as Mathematics I, Mathematics II, Mathematics III, etc.)? What reasons influence your selection?

Classroom Scenarios

Directions: Each scenario necessitates a response from you as the teacher. Please imagine that you are the teacher in each situation and describe how you might respond. Please articulate your thoughts carefully. Note that there are no “right” or “wrong” answers to a given situation. It is helpful for me to learn more about what you would think about and say in each of these situations.

Scenario 1: Yesterday, you taught your students to multiply two binomials by using the distributive property. Although most of the students seem to understand what you have taught, a couple still do not understand how to multiply binomials. How would you explain the problem $(x+7)(x+3)$ to such a group of students?

Scenario 2: The following conversation takes place between a teacher and a student.

Student: It doesn't make sense.

Teacher: What doesn't make sense?

Student: Yesterday, we learned that $a^0 = 1$.

Teacher: Yes, is there something you didn't understand?

Student: Well, I thought I understood until I started to think about it.

Teacher: Explain what is confusing you.

Student: Well, yesterday, we learned that x^4 meant that you had four x's, x cubed meant that you had three x's, and x squared meant that you had two x's, didn't we?

Teacher: Yes.

Student: So, if you have x^0, doesn't that mean you have zero x's and that the answer should be zero?

Teacher: Hmm…

How would you respond to this student?

Scenario 3: You just taught a lesson introducing the composition of functions. The following conversation develops as you are teaching the lesson. Please assume the student sincerely does not understand why it is necessary to learn how to compose functions.
Student: This math is awful!
Teacher: Why do you say that?
Student: Well, nobody would ever "compose functions" or use this stuff for anything if they were not math teachers!

How would you respond?

APPENDIX C

FIRST INTERVIEW PROTOCOL

Today, I would like to learn more about you, as a former student of mathematics, and now as a mathematics teacher. I have several questions to ask you, and I am looking forward to hearing your responses to these questions.

1. Tell me about your experiences with mathematics when you were a student.
2. What aspects of mathematics did you appreciate the most as a student of mathematics? Which were least attractive to you?
3. Describe your motivations/reasons for becoming a mathematics teacher.
4. What about mathematics made you want to teach mathematics instead of science or English?
5. Tell me about one of your best lessons as a mathematics teacher.
    a. What did you do?
    b. What did your students do?
    c. What did your students learn?
6. I have a list of math concepts here. I would like to know how you would develop a lesson about each of them.
    a. For example, if you had to teach a lesson about the distance formula, what would you do?
    b. For example, if you had to teach a lesson about the unit circle, what would you do?
    c. For example, if you had to introduce functions, what would you do?

APPENDIX D

SECOND INTERVIEW PROTOCOL

Today we are going to talk about mathematics, teaching, and learning. I have several questions to ask you, and I am looking forward to hearing your responses to these questions.

1. What do you like most about mathematics?

2. What do you dislike about mathematics?

3. Is it possible for a student to get the right answer to a mathematics problem and still not understand the problem? Please explain.

4. In your teaching what are some of the best ways you have found for students to learn math?

5. As the classroom teacher, in what ways do you have an impact on students' learning of mathematics?

6. What are some of the particularly effective ways you have found to teach students mathematics?

7. What do you consider to be the three most important characteristics of good mathematics teaching?

APPENDIX E

BELIEFS SURVEY

Directions: Place an X on the continuum that adequately represents your opinion about mathematics, mathematics teaching, and mathematics learning.

Mathematics is –

Dynamic ______________________________ Fixed

Predictable ______________________________ Surprising

Absolute ______________________________ Relative

Doubtful ______________________________ Certain

Segregated ______________________________ Connected

Correct procedures ______________________________ Multiple solution methods

Applicable ______________________________ Aesthetic

Good mathematics teaching entails, or depends on –

A good textbook ______________________________ Use of manipulatives

Teacher direction ______________________________ Student participation

Teacher effort ______________________________ Student effort

Explicit planning ______________________________ Flexible lessons

Helping students ______________________________ Helping students see
to like mathematics mathematics as useful

Learning mathematics requires mostly –

Practice______________________________ Insight

Independent work______________________________ Group work

Good teachers______________________________ Strong students

Trying hard______________________________ Being good at math

Memorizing ______________________________ Understanding

APPENDIX F

BELIEFS TASK

Directions: In teaching mathematics, how important, in your opinion, are the following goals? Circle the appropriate number as follows:

5 Much more important than most of the other goals listed
4 Somewhat more important than most of the other goals listed
3 Equally as important as most of the other goals listed
2 Somewhat less important than most of the other goals listed
1 Much less important than most of the other goals listed

Keep in mind that the average of your responses should be about 3.

To provide the students with the opportunity to learn how to reason logically	5	4	3	2	1
To provide the basis for additional (more advanced) study of mathematics	5	4	3	2	1
To provide a basis for studying and understanding science	5	4	3	2	1
To acquire basic skills essential to every day living	5	4	3	2	1
To contribute to the development of the individual	5	4	3	2	1
To develop appreciation of beauty in the geometrical forms of nature, art, and industry	5	4	3	2	1
To develop an attitude of inquiry	5	4	3	2	1
To develop understanding of logical structures, precision of statements and of thought	5	4	3	2	1
To provide students with the opportunity to learn how to discriminate between what is true and what is false	5	4	3	2	1
To develop an appreciation of the structure and connected nature of mathematics	5	4	3	2	1
To develop an appreciation for the beauty inherent to mathematics	5	4	3	2	1

APPENDIX G

INDIVIDUAL PROTOCOLS FOR THE SIXTH INTERVIEW

Rachel McAllister Interview 6

Today we are going to talk about some of your classroom practices as well as the way you think about mathematics. First, let us talk about your classroom practices.

1. It seems that the many of the mathematics problems used in your teaching were surrounded by real world contexts. Can you talk to me about the reasons you included mathematics problems that were situated in real world contexts in your teaching?

2. Please review the following excerpt of a transcript from your teaching. Then, talk to me about what you try to do when you interact with your students in this way.

   The following episode is an example took place when Rachel responded to a student who asked if the length of the long leg of a 30°-60°-90° right triangle would always include a square root of 3.
   Student: So, it is always going to be the square root of 3?
   Rachel: What do you mean by that?
   Student: Like 10.
   Rachel: So, I have a 10 for my short leg?
   Student: Yes.
   Rachel: Can you answer everything else? What is this (the altitude) going to be?
   Student: 10 square roots of 3.
   Rachel: And, this is going to be (points to the hypotenuse)?
   Student: 20.
   Rachel: 20. So, yes, we are always multiplying by the square root of 3. What if this (the short leg) starts out being 2 square roots of 3?
   Student: Then you would, (quietly) then it is going to be 6.
   Rachel: Very good. We are still multiplying by the square root of 3 to get bigger, we are multiplying. When I multiply this by the square root of 3, it becomes 2 times 3, which is 6. (Observation, September 14)

3. Please review the following excerpts of a transcript from one of my classroom observations. Then, talk to me about what you try to do when you interact with your students in this way.

   A student wondered if there were additional right triangle relationships existing beyond the ones developed in class (i.e., 30°-60°-90° and 45°-45°-90° right triangles).
   Student: I understand there are special right triangles, but [are] there other relationships that you can use the triangle and the angle in the same way?
   Rachel: Yes, they are all a ratio. So you could memorize all angles if you could.
   Student: But what is

Rachel: The reason we do the ones we do, next year, when you learn the unit circle with radian measure, it is the measures you memorize of all of the increments of 30s and 45s, up to 360. So, if you don't have them memorized, when you go to turn them on the coordinate plan into all four quadrants, you won't be able to memorize them. It would be harder to memorize them. So, it is the basis of why we do the unit circle why we do. So, you need to memorize the exact ones. Then again, when you do them in Calculus, you need to know them too. So, that is the only reason why we do those, memorize them, so you can memorize them for the other classes.

4. Talk to me about what you were trying to accomplish in this situation.

Rachel: The sine of an angle is equivalent to the cosine of the angle's complement. Well, we would relabel it, right? So, it would be O/H. But, notice that O and A are the same thing, because we switched angles. So, that is why this is true, because the opposite of one angle is adjacent of the other angle. The opposite side of one acute angle happens to be the adjacent side of the acute angle, of its compliment." (Observation, September 22)

5. Talk to me about what you were trying to accomplish in this situation.

In the following episode, Rachel contrasted sin(x) to the operation of multiplication, explaining to her students that these two things were not the same.

Rachel: It is a very new concept. This is sine, which is a function of *x*. Sine is actually being done. It is the verb. Ok? So, if we want to undo it though, just as we did with matrices, and with the functions, that is exactly right. To undo it, we are going to use the inverse. Remember the inverse of multiply is divide that is why you all think we should divide. But, this isn't multiplying.

Student: That stinks.

Rachel: So, we are going to the sine inverse, doesn't that look just like what we did with matrices, doesn't that look like the same symbol? And [when] we graphed the function and its inverse. (Observation, September 16)

6. Can you tell me more about what you meant when you say, "God is a mathematician." in the following interview transcript?

Rachel: Math makes sense. Math is everyday in the real world. …I really think math is the end all, be all. It is the basis for absolutely everything. I tell my students that ask, "When are we ever going to use this again?" "Well, you may not ever do this again, but you will do something in your job that requires you to find the problem, come up with a way to solve the problem, and check your answer. At every job you do."... So, I think that is why, cause it is real. Math is real. And, then we could get into looking at nature and how math is in nature. It is like, I am religious, **God is a mathematician**! I know it. Because there [are] way too many coincidences for him not to have been. (Interview 1)

7. Beliefs Concept-Mapping Activity: The large index cards represent what I think are big ideas in the ways you think about and describe mathematics. The smaller index cards

seem to be related to or ways to describe the big ideas. I want you to organize these cards in the way you think they are related*.* *Now, you may not agree that these are all big ideas or you may think some are missing.* You also *may not agree that these are small idea are necessary or you may think some are missing.* **What do you agree with? What do you disagree with? What would you add? Then, do are any of these ideas related?**

8. Let me show you my inferences of your beliefs about mathematics. What would you add or change?

9. Can you think of a metaphor to describe how mathematical concepts and procedures are related (or connected)?

Justin Smith Interview 6

Today we are going to talk about some of your classroom practices as well as the way you think about mathematics. First, let us talk about your classroom practices.

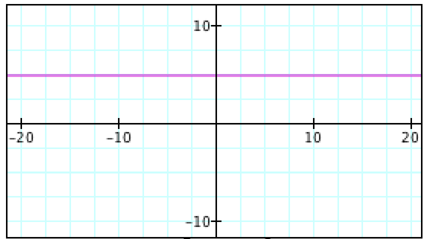
1. It seems that the many of the mathematics problems used in your teaching were surrounded by real world contexts. Can you talk to me about the reasons you included mathematics problems that were situated in real world contexts in your teaching?

2. Please review the following excerpt of a transcript from your teaching. Then, talk to me about what you try to do when you interact with your students in this way.

   Student: How do you find the time in this?
   Justin: Same as we did on this one. Remember how we did it (points to the board)?
   Student: (shakes head no)
   Justin: Do you have your equations yet?
   Student: My equations?
   Justin: I mean like over here with the numbers plugged in.
   Student: Oh.
   Justin: Do that for me, write your x equals, and your y equals.
   Student: I don't know what to do.
   Justin: Pick out your numbers from there.
   Student: I got it.
   Justin: Yeah, just plug everything in, get me those (Observation, October 19)

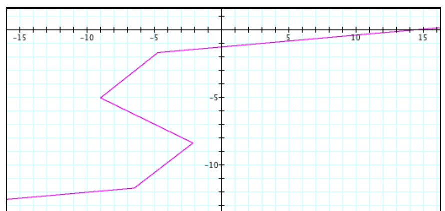
3. Please talk to me about the way you structured this lesson.

   Observation Day 1:
   Starter: Plot the graph of $f(x) = \pm\sqrt{x-1} + 2$ if x = 1, 5, 10, 17
   Definition written on board: Parametric Curve: The graph of the ordered pairs (x, y) where $x = f(t)$ and $y = g(t)$ are functions defined on the interval I of t-values is a parametric curve. The equations are parametric equations for the curve, the variable t is a parameter, and I is the parameter interval.
   Justin: "What I want to do is jump into one. I know we will come back and make sense of that definition. Let me show you what one looks like. Here is an example of what one looks like, and we will come back and make sense of everything I hope (writes and example on board). $x = t^2 - 2$, and $y = 3t$ where $-2 \leq t \leq 2$. What I want you guys to do, without knowing really what they are, what we are going to do with them, or why they are useful, I want you to graph them on this interval. Reread the definition. See if it makes sense. We are going to build on this quite a bit today. (They work through the problem)…Now, reread the definition. See if it makes sense. …This is honestly the way I had to learn this as well, I would do some examples, go back and look at the definition, two years ago when I was doing this. All right. It says, "The graph of ordered pairs (x, y), where $x = f(t)$ and $y = g(t)$, and they are defined on an interval of t values." All right, does that make a little more sense, now? Student: Yes."
   The Derek Jeter Baseball Problem
   Final Problem of the Lesson:



Graph of the $x(t) = -0.1(t^3 - 20t^2 + 110t - 85)$ and $y(t) = 5$

Justin: By putting a constant in here, 5, this is not really going to be a parametric equation right here, when we graph these. This is only going to show you the *x* as the horizontal change. It is just a horizontal value. It is going to show you what the horizontal part of this looks like. Just the horizontal. Just left and right. You won't see the up and down….you get a straight line. Now, that is not all that interesting to me. Because it, mine went from right to left.


Graph of the $x(t) = -0.1(t^3 - 20t^2 + 110t - 85)$ and $y(t) = -t$.

Justin: All right. Now, initially, when we plotted just the x function, if you read the problem, it is about a guy walking on a street somewhere, I think. And, if we just look at *x* equals, he is just going like this, but what does this thing down here mean, that he did?

Student: He turned around.

Justin: He turned around, and walked backward, back. And, then, he turned around again, and kept on walking. Now, I don't know what he did that. But, maybe he saw something on the ground and went back to go get it. He dropped his cell phone, and he noticed it. All right. Without parametric equations, we would not have known he did this thing (turned around), right here.

4. Beliefs Concept-Mapping Activity: The big cards represent what I think are big ideas in the way you think about and describe mathematics. The smaller cards seem to be related to or ways to describe the big ideas. I want you to organize these. *Now, you may not agree that these are all big ideas or you may think some are missing. You* also *may not agree that these are small idea are necessary or you may think some are missing.* **What do you agree with? What do you disagree with? What would you add? Then, do are any of these ideas related? Let me show you my inferences of your beliefs about mathematics. What would you add or change?**

5. Can you think of a metaphor to describe how mathematical concepts and procedures are related (or connected)?

Robert Boyd Interview 6

Today we are going to talk about some of your classroom practices as well as the way you think about mathematics. First, let us talk about your classroom practices.

1. Talk to me about what you were trying to accomplish in this situation?

   Robert: Let's look at the warm-up again. It wasn't just a warm-up. It had a other purpose. You all just did some math. You thought you were talking about a parking garage, but you just did some math. What you secretly did, what you did without even knowing it was a function. (Observation, November 4)

2. Talk to me about what you were trying to accomplish in this situation?

   Robert: So, you are born. The asymptote is where your parents live. So, you live with your parents. You live with your parents (tracing the graph). Then you get to be a teenager. And, I don't want to be by my parents so much and then you get into your twenties. Then you get married. And then, you are gone. (Observation, November 10)

3. Talk to me about this interaction with one of your students.

   When graphing piecewise functions, a student expressed confusion about how to graph an open point. It seemed that she did not understand how the graph could seemingly touch a point that was not included.

   Student: I thought you were saying it couldn't touch the circle.

   Robert: Yeah, it could touch the circle. It is the circle just says, hey you have gone up to this point, but you are not touching it. You can get as close as you want, but do not touch.

   Student: But you are touching it.

   Robert: Ok. The open point says that that point is, ok. (Goes to the board) I am blowing this (the open point) up, magnifying it. Ok. This point is (5, –1) Ok, so my line comes in, and it includes everything, all the way up to (5, –1) , but (5, –1) is empty.

   Student: So if you have a greater than or equal to or a less than or equal to, it is an open circle?

   Robert: No. So, here is the thing. Ok, hang on. I think I can get you there. Ok? Actually I am there, I think I can get you there. Are you ready? Stay with me. All of this is less than 5 (gestures to the part of the graph where $x < 5$) Yes, no? Am I ok? So, (traces the graph with his finger) less than 5, I am less than 5, I am less than 5, I am less than 5, and I can go all the way up until 4.99999 as many 9's as I feel like, right? And, I am still less than 5.

   Student: So, can you touch the point thing or not?

   Robert: You touch it. The line touches it, the reason this is empty though is because that point is not really there, because it is not included.

   Student: Ok.

   Robert: Did that work?

   Student: Yep.

   Robert: Great. (Observation, November 1)

4. Talk to me about what you were trying to accomplish in this situation?

Robert compared the two functions: f(x) = 2^x and f(x)=2^{x-2}.

Robert: Let's take a look at this function (points to f(x)=2^{x-2}). Before we go through this real quick, would anybody like to predict what the -2, is going to do to my graph? Student 1?
Student 1: Cut your slope in half
Robert: No, not going to change my slope. What do you think Student 2?
Student 2: Move your graph down 2.
Robert: No, not going to move my graph down 2. Where would it be if it was going to move my graph down 2?
Students: (inaudible)
Robert: It would be out here, right (pointing to indicate it would be f()=$2^x - 2$)? That would move it down 2. Ok, Student 3, what were you thinking? Was it one of those or something else?
Student 3: It is going to make the 5 go down 2.
Robert: Ok, it makes the 5 go down 2. Yeah. What does it make the graph do?
Student 3: It is going to make it go up.
Robert: Hmm.
Student 4: Space it out?
Robert: Hmm, I guess we will have to find out. We will graph it when we are done. Let's write our function values first. I put in 5, what do I get out (begins to create a table of values)?
Students: 8.
Robert: 8. Because if I put in 5, I get 2^3, which is 8. If I put in 4, I get 2^2, which is 4. And, then, I put in 3, I get 2^1, which is 2. And, I have seen this pattern somewhere before. Right? What is going to be next?
Students: 1
Robert: 1. Next?
Students: 1/2.
Robert: And, then?
Student 5:1/2, and 1/8, and 1/16.
Robert: Ok. Now let us compare, this is 2^{x-2}. My parent graph would be 2^x, right? So, if I put in 0, I get 1, I put in 1, I get 2. (Fills in the table) 4, 8, 16, 32. 1/2 1/4. This 1 (pointing to 1 in 2^x column), when I put in 0 in the parent graph, what do I have to put in to get 1, on my new graph?
Student 5: 2.
Robert: I have to put in 2. So, my number that was at 0, got moved to 2. This is my x. So if what was at 0 got moved to 2, how did it get shifted?
Student 5: (inaudible)
Robert: On the x-axis? It is not up.
Students: Over. It is over.
Robert: It is over which way?
Student 6: Right.
Robert: Right. To the right. To the right, it got moved to the right 2. Ok, we will take a look at that. Let's check that out real quick (graphs the two functions). Cause I know

some folks are going, Ok, I get that, and some other folks are going, What? So. I want to make sure you all understand what I am talking about here. So, here is my graph, here is my *y*-axis, here is my *x*-axis, and first thing I am going to write is 2^x. If I put in 0, I get 1. If I put in 1, I get 2. If I put in 2, I get 4. If I put in 3, I get 8. And then on the negative side, I get 1/2, 1/4, 1/8, 1/16, we did all of this yesterday. And, so the parent graph looks like that (graphs $f(x) = 2^x$). Now, my new graph, if I take those points, and do to 2^{x-2}, those points I get, (0, 1/4), I get (1, 1/2), I get (2, 1), (3, 2), (4, 4), (5, 8), so it looks like this (graphs). And so you see this point that was here, is now here (circles the points when $y = 1$) So, my graph was shifted to the?

Students: Right.

Robert: Right 2.

5. Talk to me about what you were trying to accomplish in this situation?

   Robert: Now, if I put in 3, I get 2^3. 2^3 is 2, 4, don't say 6. Everybody says, well not everybody, but people say 6 a lot, be careful, because we are used to counting by 2's. 2, 4, 6. But we are not counting by 2's. We are multiplying by 2's. So, it is 2, 4, 8. 8. Be careful. Everybody that is the #1 reason people miss problems on tests that they should never ever miss in a million years, because they do powers of 2 wrong. Careful with your powers of 2. Nobody seems to have that trouble with any of the others it is just 2's because we are used to counting, because we count by 2's sometimes. (Observation, November 8)

6. Can you tell me more about how mathematics is certain and logically organized? How are those two things related?

7. Beliefs Task: The big cards represent what I think are big ideas in the way you think about mathematics. The smaller cards seem to be related to or ways to describe the big ideas. I want you to organize these. *Now, You may not agree that these are all big ideas or you may think some are missing. You also may not agree that these are small idea are necessary or you may think some are missing.* **What do you agree with? What do you disagree with? What would you add? Then, do are any of these ideas related?**

8. Let me show you my inferences of your beliefs about mathematics. What would you add or change?

9. Can you think of a metaphor to describe how mathematical concepts and procedures are related (or connected)?

APPENDIX H

STAKE'S (2006) WORKSHEET FOR ANALYZING A CASE STUDY

Synopsis of case:

Uniqueness of case situation for program/phenomenon:

Case Findings:

Relevance of Case for Cross Case Themes:

Possible Excerpts for Cross-case Analysis:

Commentary: