Parts and Plurals

Essays on the Logic and Metaphysics of Plurality

by

Anthony Shiver

(Under the Direction of Charles Cross)

Abstract

This dissertation consists of four independent papers on the foundations of mereology and the logic of plural expressions. The first chapter shows that the standard formal definition of atomicity—the thesis that everything is ultimately composed of atoms—fails to exclude nonatomistic models and is therefore inadequate. It then formulates a new definition of atomicity in a plural logic framework. The second chapter argues that the bundle theory of objects is consistent with the principle of the identity of indiscernibles when the relation between properties and objects is interpreted as parthood. The third chapter develops a semantics and axiom system for a logic of compound terms. This logic extends the expressive and inferential power of standard first-order logics that leave compound terms unanalyzed or, worse, misrepresent their logical properties entirely. The final chapter argues against the ontological innocence of plural reference. It appeals to formal, epistemological, and metaphysical principles endorsed by a variety of philosophers to show that ‘some things exist’ does not entail ‘something exists.’ If this is correct, then plural quantification works very differently than most philosophers think.

Index Words: Plural Logic, Mereology, Compound Terms, Identity, Ontological Commitment, Plural Quantification, Bundle Theory
PARTS AND PLURALS
ESSAYS ON THE LOGIC AND METAPHYSICS OF PLURALITY

by

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For my wife Brandy and my sister Lesley.
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# Table of Contents

**Acknowledgments** ................................................................. v

**Chapter**

1 **Introduction and Literature Review** .......................................... 1
   1.1 Aggregation ................................................................. 1
   1.2 Plural Logic ............................................................... 4
   1.3 Mereology ................................................................. 8
   1.4 References ............................................................... 12

2 **How do you say ‘Everything is ultimately composed of atoms’?** ............ 17
   2.1 References ............................................................... 28

3 **Mereological Bundle Theory and the Identity of Indiscernibles** ............... 29
   3.1 Introduction .............................................................. 30
   3.2 Mereological Bundle Theory ............................................ 31
   3.3 Extensionality and the Identity of Indiscernibles ......................... 34
   3.4 Black’s Puzzle and a Solution ......................................... 40
   3.5 Replies to Objections .................................................. 43
   3.6 Conclusion ............................................................... 47
   3.7 References .............................................................. 47

4 **The Logic of Compound Terms** ................................................ 50
   4.1 Lists in Natural Language ............................................... 51
   4.2 Lists in Formal Languages .............................................. 54
4.3 The Syntax of FPL  
4.4 Semantics for FPL  
4.5 An Axiom System for FPL  
4.6 Soundness for FPL  
4.7 Completeness of FPL  
4.8 Model Theory for FPL  
4.9 References  

5 Much Ado about ‘Some Things’  
5.1 Introduction  
5.2 The Formal Defense  
5.3 The Epistemic Defense  
5.4 The Metaphysical Defense  
5.5 Speculative Conclusions  
5.6 References  

6 Conclusion  
6.1 Atomicity  
6.2 Mereological Bundle Theory  
6.3 Quantification into Lists  
6.4 Rethinking Assignments of Plural Variables  
6.5 References
Chapter 1

Introduction and Literature Review

1.1 Aggregation

Our everyday experience is filled with objects that are both one thing and many things. The computer I am using to type this introduction is one machine built out of many components: a keyboard, a screen, a hard drive, and so on. The hands I am using to operate the keyboard have many parts as well, as any text on human anatomy will make clear. But what about my hands and the keyboard; is there an object built out of them? It seems that there is not. We wouldn’t normally think of these things as composing an object. But when we try to explain why they do not compose an object, we immediately encounter metaphysical and logic difficulties that resist easy solution. These difficulties manifest a tension between the pragmatic benefits of positing aggregate objects (i.e., objects that collect together other objects but are nevertheless distinct from those objects) and the appeal of sparse ontologies that do without aggregate objects. In recent decades novel ways of relieving the tension have been developed. The essays in this volume address foundational issues within these relatively recent developments.

People have been theorizing about aggregation at least as far back as Plato. We skip ahead to the late nineteenth century, when aggregation began receiving serious attention as a subject of formal enquiry. The most famous and developed theory of aggregation is set theory. Contemporary set theories are so complex that it is easy to forget that set theory began as an analysis of the everyday concept of collecting individuals into a heap.

Georg Cantor’s set theory is about *Menge*, which are many things collected into units (acceptable English translations include ‘aggregates’, ‘piles’, ‘lots’, ‘heaps’, and, of course,
‘sets’.) Though contemporary debates have sharpened the meaning of ‘set’ to distinguish it from other kinds of aggregates, these distinctions came later as both philosophers and mathematicians struggled to understand the nature of the collections that set theory ultimately characterized. In the beginning, Cantor (1885) uses language general enough to subsume a variety of aggregation concepts when describing *Menge*.

By an “aggregate” (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) *M* of definite and separate objects *m* of our intuition or our thought. These objects are called the “elements” of *M*.

In signs we express this thus: *M* = \{*m*\}.

We denote the uniting of many aggregates *M*, *N*, *P*, ..., which have no common elements, into a single aggregate by (*M*, *N*, *P*, ...).

The elements of this aggregate are, therefore, the elements of *M*, of *N*, of *P*, ..., taken together.

We will call by the name “part” (*Theil*) or “partial aggregate” (*Theilmenge*) of an aggregate *M* any other aggregate *M*₁ whose elements are also elements of *M*.

If *M*₂ is a part of *M*₁ and *M*₁ is a part of *M*, then *M*₂ is a part of *M*. (Cantor 1885, 481; Jourdain trans 2007, 1137)

Cantor’s general picture of aggregation leaves open several important ontological questions about aggregates. Are they objects or logical constructions? If they are objects, are they the same kind of object as their parts (i.e., abstract or concrete)? What is the difference between an object *m* and an aggregate containing *m* but no other parts?

Answers to these questions tend to cluster. These clusters correspond roughly to standard set theory, classical mereology, and plural logic, each of which presents a distinct approach to aggregation. On the standard set theoretic picture, aggregates are abstract objects even if

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1. *Teil* in contemporary German.
their parts are not, and aggregates of one thing are numerically distinct from that thing. On the standard mereological picture, aggregates are the same kind of things as their parts. A mereological aggregate of material objects is itself a material object. This is because, unlike in set theory, it is common in mereology to think that an aggregate’s properties supervene on the properties of its parts. Plural logic takes a radically different approach to aggregation than set theory or mereology. In plural logic, aggregates are not treated as objects; instead, aggregation terms like ‘they’ or ‘them’ are taken to denote several things directly, without assuming that those things compose some further individual. Since aggregation is interpreted as a mode of reference, there is no distinction between an individual and a singleton aggregate of it.

Another important issue is whether the aggregation operation produces a hierarchy of aggregates when iterated. The familiar set-building operator generates a hierarchy: $S \neq \{S\} \neq \{\{S\}\}$. Fusion, the analogous operator in classical mereology, does not: $S = fu(S) = fu(fu(S))$. Concensus has not been reached in plural logic, with some authors claiming that so-called superplural reference (plural reference to pluralities) is unintelligible and others claiming that it is not only intelligible but is an actual feature of ordinary language.

Finally, these theories treat empty aggregation in different ways. Standard set theory posits an aggregate with no parts other than itself (the empty set). Classical mereology does not allow empty aggregates; there are no mereological fusions that lack proper parts. This is another area of plural logic in which there is no concensus. If empty plural reference is counted as reference to null individuals, then there are null aggregates in plural logic.\(^2\) Alternatively, if empty plural reference is characterized as failed reference, there are no empty aggregates in plural logic.

In what follows I will briefly introduce some of the relevant literature concerning plural logic and mereology.\(^3\) This is not meant to be a comprehensive review of the literature on

\(^2\)This view is defended in Oliver and Smiley 2013, section 5.6.
\(^3\)The discussion of the roots of plural logic draws heavily on Oliver and Smiley’s (2013) more thorough treatment, which I recommend to the reader.
these topics (which is enormous), but rather an overview that sets the stage for the essays that follow.

1.2 Plural Logic

Early set theories had a now famous flaw: they allowed sets to be defined using any predicate whatever. It is axiomatic in such systems that for any predicate $P$, there is a set that contains all and only those things satisfying $P$. This principle seems obvious enough given how sets are denoted in practice (i.e., by combining the relevant predicate with the function ‘the set of’, as in ‘the set of real numbers’). Bertrand Russell (1903) discovered that this principle leads to a contradiction. Any theory that includes it is inconsistent. To see why, consider the predicate ‘$x$ is not a member of $x$’. If there is a set $S$ whose members are all and only the sets that are not members of themselves, then it follows that $S$ is both a member of itself and not a member of itself. Hence there is no such set, and from this is follows that the naive principle is false. This result is extremely important. As Russell puts it: “[H]aving dropped [the naive principle], the question arises: Which propositional functions define classes which are single terms as well as many, and which do not? And with this question our real difficulties begin.” (1937, 103)

Russell would later develop the theory of types as a resolution to this puzzle. But an alternative theory of aggregates is hinted at in Russell’s early theory of classes. There he draws a distinction between a class as one and a class as many. A class as one is the sort of aggregate described by set theory and mereology. “The class as one may be identified with the whole composed of the terms of the class, i.e., in the case of men, the class as one will be the human race.” (1937, 76) Classes as many, on the other hand, are not individuals.

The notion of a whole, in the sense of a pure aggregate which is here relevant, is, we shall find, not always applicable where the notion of class as many applies.\footnote{Russell has in mind classes like ‘sets that are not members of themselves’ discussed above.} In such cases, though terms may be said to belong to the class, the class must not be
treated as itself a single logical subject. [...] In a class as many, the component terms though they have some kind of unity, have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from remaining many. A further reason for distinguishing wholes from classes as many is that a class as one may be one of the terms of itself as many, as in “classes are one among classes” (the extensional equivalent of “class is a class-concept”), whereas a complex whole can never be one of its own constituents. (1937, 68–69)

The notion of a class as many is difficult to discuss in English. As Russell notes in his discussion of enumerations (which he calls collections).

There is a grammatical difficulty which, since no method exists of avoiding it, must be pointed out and allowed for. A collection, grammatically, is singular, whereas A and B, A and B and C, etc. are essentially plural. This grammatical difficulty arises from the logical fact [...] that whatever is many in general forms a whole which is one: it is therefore not removeable by a better choice of technical terms. (1937, 71)

It is very natural to use singular terms (e.g., the set of integers) when discussing enumerated collections, since, as Russell notes, there is always a set of just those enumerated things. However, Russell’s own paradox shows that we should not accept, as he asserts here, that “whatever is many in general forms a whole which is one.” There are some sets that are not self-membered (namely, all of them), but as we saw above there is no set with all and only the non-self-membered sets as its members. There is apparently a class as many of these sets, but it would be a mistake to regard it as a single thing. It even seems a mistake to call it a class.

How do we talk and reason about classes as many if they aren’t individuals? The painfully obvious answer, which did not occur to Russell and apparently occurred to no one else for the better part of a century, is to use plural noun and verb constructions.
After doing a thorough job of showing that every proposed definition of ‘aggregate,’ ‘set,’ or ‘class’—including the definitions given by Cantor and Russell above—fails to elucidate its definiendum, Max Black (1971) suggests that we start over with plural reference as our primary tool for aggregation.

The most obvious ways of referring to a single thing are by using a name or a definite description: ‘Aristotle’ or ‘the president of the United States’. Equally familiar, although strangely overlooked by logicians and philosophers, are devices for referring to several things together: ‘Berkeley and Hume’ or ‘the brothers of Napoleon’. Here, lists of names (usually, but not necessarily, coupled by occurrences of ‘and’) and what might be called “plural descriptions” (phrases of the form ‘the-so-and-so’s’ in certain uses) play something like the same role that names and singular descriptions do. Just as ‘Nixon’ identifies one man for attention in the context of some statement, the list ‘Johnson and Kennedy’ identifies two men at once, in a context which something is considered that involves both of them at once. And just as ‘the President of the United States’ succeeds in identifying one man by description, so the phrase ‘the American presidents since Lincoln’ succeeds in identifying several, in a way that allows something to be said about all of them at once. (1971, 629)

One primitive use of the word ‘set’ is as a stand-in for plural referring expressions of the kinds discussed above. If I say “A certain set of men are running for office” and am asked to be more specific, then I might say, “To wit, Tom, Dick, and Harry”—or, in the absence of knowledge of their names, I might abide by my original assertion. One might therefore regard ‘set’ in its most basic use, as an indefinite surrogate for lists and plural descriptions. To know how to use the word ‘set’ correctly at this level is just to know the linguistic connections between such uses of ‘set’ and the uses of more definite multiply-referring devices. (1971, 631)
George Boolos (1984, 1985) also argues that plural reference can be used as a basis for set theory. Boolos even goes further. He argues that plural expressions can be used to better understand second-order logic. He suggests that we interpret reference to and quantification over predicates as reference to and quantification over the objects that satisfy those predicates. But, like Russell, Boolos emphasizes that plural reference lacks the ontological commitment of the many-as-one aggregates depicted by set theory.

The lesson to be drawn from the foregoing reflections on plurals and second-order logic is that neither the use of plurals nor the employment of second-order logic commits us the the existence of extra items beyond those which we are already committed. We need not construe second-order quantifiers as ranging over anything other than the objects over which our first-order quantifiers range, and, in the absence of other reasons for thinking so, we need not think that there are collections of (say) Cheerios, in addition to the Cheerios. Ontological commitment is carried by our first-order quantifiers; a second-order quantifier needn’t be taken to be a kind of first-order quantifier in disguise, having items of a special kind, collections, in its range. It is not as though there were two sorts of things in the world, individuals, and collections of them, which our first-order and second-order variables, respectively, range over and which our singular and plural forms, respectively, denote. There are, rather, two (at least) different ways of referring to the same things, among which there may well be many, many collections. (1984, 449)

Since Boolos’s essays were published most of the work on plural logic has focused on either defending or refuting plural quantification’s ontological innocence.\(^5\) For my part, I stand with those who defend the ontological innocence of plural reference and quantification. Since I think this question has been decided, I think the most important questions about plural logic

are, How does it work? What are its axioms and what are its metatheoretical properties? How much complexity do we add to classical predicate logic if we include plural variables and quantifiers that bind them? Byeong-Uk Yi (2005, 2006), Thomas McKay (2006), and Alex Oliver and Timothy Smiley (2001, 2006, 2013) have done excellent work on these issues, with the latter’s recent book *Plural Logic* representing the state of the art in plural logic research.

Chapters 4 and 5 in this volume address issues not yet covered in the literature on plural logic. None of the extant axiomatizations of plural logic include an apparatus for reference via lists, which is, as Black (1971) notes, one of our basic tools for referring plurally. Chapter 4 develops a system of plural logic that allows lists to be constructed from singular constants. Axioms for this logic are shown to be sound and complete with respect to an uncontroversial Henkin-style semantics, providing some evidence that the axioms are an adequate characterization of how lists function in ordinary reasoning. Chapter 5 argues that plural reference behaves differently than singular reference in the presence of vagueness or ontic indeterminacy. Sometimes there is no fact of the matter whether something exists, since there are cases where determinate singular reference is impossible. However, in these cases it is determinately true that some things exist, since determinate plural reference is possible even in cases where determinate singular reference is impossible. If this is right, then plural reference tracks features of the domain of discourse that singular reference does not.

1.3 Mereology

Not long after mathematical investigation in set theory began in earnest, authors began developing a rival theory of aggregates that would eventually be called *mereology* (the study of parts). Early investigations into mereology were conducted by Husserl (1901) and Leśniewski (1916, 1927–1931), with anglophone audiences getting their first introduction to the theory via Leonard and Goodman’s “The Calculus of Individuals and Its Uses” (1940). More recent
systematic work in mereology includes Simons 1987, Casati and Varzi 1999, Hovda 2009, and Varzi 2015, with the latter representing the current state of the art in the field.

Mereology was explicitly developed as a nominalistic alternative to set theory. Famously, sets are abstract objects even if their members are concrete ordinary objects like tables and chairs. But an aggregate of, say, all the chairs that currently exist, seems to be nothing more than a scattered pile of wood, metal, and plastic objects, all of which are concrete. And it also seems that a pile of concreta is itself concrete. Standing behind this intuition is a plausible ontological principle that the properties of an aggregate supervene on the properties of its parts. Unlike set theory, mereology is a theory of aggregation that respects this principle.

Like its cousin set theory, mereology is not a single formal system. Rather, it is many rival systems, each purporting to characterize the relevant notion of aggregation. Classical mereology, the first and strongest formal theory of parthood, can be represented using the following set of axioms.

**(Irreflexivity)** For all \( x \), \( x \) is not a proper part of \( x \).

**(Transitivity)** For all \( x \), \( y \), and \( z \), if \( x \) is a proper part of \( y \) and \( y \) is a proper part of \( z \), then \( x \) is a proper part of \( z \).

**(Uniqueness)** For any nonoverlapping \( x \)s, if the \( x \)s compose \( y \)\(^6\) and the \( x \)s compose \( z \), then \( y \) is \( z \).

**(Universalism)** For any nonoverlapping \( x \)s there is a \( y \) such that \( y \) is composed of the \( x \)s.

These axioms generate a powerful theory of aggregation that can do many of the same formal tasks as set theory. Notably, though, these axioms do not entail a commitment to empty aggregates, singleton aggregates, or Cantor’s hierarchy of infinities, each of which has been seen as an unwelcome consequence of the set-theoretic approach to aggregation.

\(^6\)That is, \( y \) has all of the \( x \)s as parts and anything that does not overlap (i.e., share a part with) one of the \( x \)s is not a part of \( y \).
Despite the fact that these axioms commit one to much less than do the axioms of even
the weakest set theory, many (but not all) writers on mereology think that these axioms
are too strong to characterize the ordinary conception of aggregation. Consider a statue
and the clay from which it is sculpted. These objects are often understood to constitute
a counterexample to Uniqueness, since the statue and the clay seem to be distinct objects
composed of the very same material.\textsuperscript{7} Univeralism, too, is often challenged, both because
of its ontological promiscuity and the unusual nature of some objects whose existence it
entails.\textsuperscript{8}

Just as Russell noted upon the rejection of naive set comprehension, we encounter serious
difficulties when we try to find suitably weaker principles to replace the classical axioms.
The most significant of these difficulties concern the principles governing composition and
decomposition. Peter van Inwagen’s \textit{Material Beings} (1990) is the \textit{locus classicus} for these
issues. In it, van Inwagen formulates the Special Composition Question: When is it true that
there is a $y$ such that the $x$s compose it? (1990, 22)

Van Inwagen considers several answers to this question before deciding on the Proposed
Answer.

\textbf{(PVI’s Proposed Answer)} There is a $y$ such that the $x$s compose $y$ if and only if the
activity of the $x$s constitutes a life (or there is only one of the $x$s). (1990, 82)

The Proposed Answer is not popular.\textsuperscript{9} Surprisingly, most writers on mereology seem to
prefer one of the extreme answers—Universalism (one of the axioms of classical mereology
featured above) or Nihilism.

\textbf{(Nihilism)} There is a $y$ such that the $x$s compose $y$ if and only if there is only one of the
$x$s. (1990, 73)

\textsuperscript{7}See, for example, Wiggins 1968 and Gibbard 1975.
\textsuperscript{8}For an influential critique and defense of Universalism see Lewis 1986. A more recent discussion
of the principle can be found in Sider 2001.
\textsuperscript{9}Section 4 in chapter 5 of this volume discusses some of the reasons why.
Nihilism is often paired with an error theory that makes use of plural reference. The idea is that our ordinary conceptions of objects are instantiated by many atoms (i.e., objects with no proper parts) collectively without the atoms composing a whole (a strategy very similar to that of using plural reference to avoid Russell’s paradox discussed above). It’s correct, but technically false, that there is a computer on which this essay is being typed; in reality, what we would normally call a computer is not one thing, but many mereological atoms arranged in the shape of (and collectively performing all the tasks associated with) a computer. Likewise for other purported composite objects. Thus nihilists are committed to the existence of far fewer objects than their rivals without having to deny that much is wrong with our ordinary picture of the world.\textsuperscript{10}

Despite their superiority in terms of ontological simplicity, Nihilism and van Inwagen’s Proposed Answer have a theoretical drawback: they are incompatible with the possibility of gunk. Gunk is a pattern of infinite decomposition in which every object has proper parts. Thus, gunky objects are not built out of atoms. This pattern cannot be instantiated in a world in which composition does not occur. In a world in which no things ever compose anything and there are no atoms, nothing exists.\textsuperscript{11}

Most authors are hesitant to accept the possibility of gunk even if they reject nihilism. To deny the possibility of gunk is to accept the thesis of Atomicity, i.e. that everything is ultimately composed of atoms. Chapter 2 of this volume shows that the standard formal characterization of Atomicity is only adequate given a further commitment to Universalism and the uniqueness of composition. Further, a general characterization of Atomicity requires more ideological resources than those invoked by the traditional characterization. I propose a general characterization presupposing nothing more than plural descriptions, plural quantification, and the standard mereological predicates ‘proper part’ and ‘overlap’.

\textsuperscript{10}The error theory approach is pioneered by van Inwagen 1990 and defended by nihilists in Dorr and Rosen 2002 and Sider 2013.
\textsuperscript{11}See Sider 1993.
Parthood relations can hold between material objects like a cat and its tail, but also between abstract objects like a song and its chorus. If a permissive answer to the Special Composition Question is true, there might even be mixed composites, part concrete and part abstract. Mereology is neutral with respect to the nature of composites. It tells us only how things are structured *qua* composites and is neutral on other ontological questions.

Metaphysicians have exploited this neutrality in some creative and exciting ways. David Lewis (1991), for example, uses a combination of mereology and plural logic to clarify the murky conceptual foundations of set theory; Lewis (1986) also uses mereology to develop his infamous theory of modal realism. Ted Sider (2001) argues that many of the puzzles of temporal persistence are solved by treating persisting objects as mereological aggregates of objects existing at distinct times. Kathrin Koslicki (2008) argues for a neoaristotelean metaphysics on which material objects have nonmaterial structures as proper parts. L. A. Paul (2002, 2006, 2012) goes even further in this direction, arguing that all objects are mereological composites of fundamental properties.

Chapter 3 of this volume concerns L. A. Paul’s (2002, 2006) property mereology. On her theory, ordinary objects are aggregates of properties; an object instantiates a property just in case that property is a part of the object. I argue that Paul should accept that composition is unique, and that doing so will also commit her to the Principle of the Identity of Indiscernibles. The upshot is that Paul’s mereological interpretation of instantiation provides a novel strategy to defuse putative counterexamples to the identity of indiscernibles. I describe how this strategy applies to the most discussed counterexample in the literature. I then argue that the fact that mereological bundle theory explains why counterexamples to the principle fail is an abductive reason to accept the theory over its rivals.

1.4 References


197.


Quarterly* 51: 289–306.

Logic* 35: 317–348.


MA.


Chapter 2

There seem to be exactly three options with respect to the number and compositional role of mereological atoms in the world: either (i) everything is ultimately composed of mereological atoms, (ii) nothing is ultimately composed of atoms, or (iii) some things are ultimately composed of atoms and some are not. Simons (1987), Casati and Varzi (1999), and Varzi (2012) formulate definitions for these theses as follows.\(^2\)\(^3\)

\begin{enumerate}
\item Atomicity: \(\forall x \exists y (Ay \& Pyx)\)
\item Atomlessness: \(\forall x \exists y PPxy\)
\item Mixed: \(\exists x Ax \& \exists x \forall y (Pyx \rightarrow \exists z PPzy)\)
\end{enumerate}

These definitions are inadequate. To see why, consider the following model.

Let \(\mathbb{N}\) be the set of natural numbers and \(\mathcal{P}(\mathbb{N})\) be its powerset. Construct a set \(\mathbb{N}_{\mathcal{NN}}\) as follows.

\[\mathbb{N}_{\mathcal{NN}} = \{ S \mid \text{either } S \in \mathcal{P}(\mathbb{N}) \text{ and } |S| = |\{\emptyset\}| \text{ or } S \in \mathcal{P}(\mathbb{N}) \text{ and } |S| = |\mathbb{N}| \text{ and for any } x \in \mathbb{N}, \text{ if } x \in S, \text{ then } s(x) \in S^\circ \}.\]\(^4\)

\(^2\)All three agree on the first two definitions, though only Simons explicitly formulates the ‘mixed’ option. I mention these three works because they are generally regarded as the best introductions to mereology. It should be noted, though, that the standard definition of atomicity goes back at least to Goodman (1951) and appears in numerous other works on mereology.

\(^3\)I use \(Ax\) for ‘\(x\) is an atom’, \(Pxy\) for ‘\(x\) is a part of \(y\)’, \(PPxy\) for ‘\(x\) is a proper part of \(y\)’, \(Oxy\) for ‘\(x\) overlaps \(y\)’, and \(Uxy\) for ‘\(x\) underlaps \(y\)’. \(PPxy\) is taken to be primitive, \(Pxy\) is defined as \(PPxy \vee x = y\), \(Ax\) is defined as \(\neg \exists y PPxy\), \(Oxy\) is defined as \(\exists z (Pzx \& Pzy)\), and \(Uxy\) is defined as \(\exists z (Pzx \& Pzy)\).

\(^4\)Where \(s(x)\) is the successor function and \(|X|\) is the cardinality of \(X\).
Interpret the partially ordered set (poset) \( \langle \mathbb{N}_{\text{GR}}, \subseteq \rangle \) as a model of mereology in which the singletons are atoms and the subset relation is the relation of parthood. Call this model \( \mathcal{M} \).

For the visually-oriented reader (and for ease of discussion), here is \( \mathcal{M} \) represented as an infinitely-descending Hasse-diagram.

\[
\begin{array}{c}
\mathbb{N} \\
\{1, 2, 3, \ldots\} & \{0\} \\
\{2, 3, 4, \ldots\} & \{1\} \\
\ldots
\end{array}
\]

Read the diagram as displaying the set of natural numbers as a composite object being decomposed, one by one, into its atomic parts.

It is easy to verify that \( \mathcal{M} \) satisfies (1)–every set in \( \mathbb{N}_{\text{GR}} \) is either an atom or has an atom as a proper part. However, the sequence of gapless infinite subsets of \( \mathbb{N} \) (the leftmost branch of the diagram) is a non-terminating maximal chain in \( \mathcal{M} \).\(^5\) And if a poset has a non-terminating maximal chain, it follows that the poset is not atomistic.\(^6\)

To put the point another way, even though it is true that everything in \( \mathcal{M} \) is either an atom or has an atom as a proper part, it is also true that every composite has a composite as a proper part. But if every composite has a composite proper part, then it is false that

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\(^5\)A chain is any linearly ordered subset of a poset, and a chain is maximal if it is not a subset of any other chain in the poset.

\(^6\)Every non-terminating chain is infinite, but some infinite chains terminate (e.g., the set of all real numbers in \([0,1]\) ordered by \( \leq \)). We will here only be concerned with infinite chains that fail to terminate. See Cotnoir and Bacon (2012) and Cotnoir (2013) for discussions of other kinds of infinite parthood chains.
everything is ultimately composed of atoms; it is, at least partly, composites all the way down. Hence, $\mathcal{M}$ is not atomistic even though it satisfies (1).

It is tempting to interpret the model as atomistic because of our familiarity with set theory. One might object, for example, that $\mathcal{M}$ is atomistic because each infinite set in the model is a union of the singletons of its members and the singleton of every $n \in \mathbb{N}$ is in the model. But this reasoning fails. $\mathcal{M}$ is not a model of set theory—it doesn’t have a null object, not every two sets have a union or intersection, and so on. The sets of the model are merely representations of nodes in a structure, and it would be a mistake to take set-theoretic intuitions about the structure very seriously.

If you are still unconvinced, consider the model $\mathcal{M}_\pi$ that adds an arbitrary element (in this case, $\pi$) to each infinite set in $\mathcal{M}$ but leaves out the corresponding singleton.

$$\mathbb{N}_{\mathcal{M}_\pi} = \{S \mid \text{either } \gamma S \in \mathcal{P}(\mathbb{N} \cup \{\pi\}) \text{ and } |S| = |\{0\}| \text{ and } S \neq \{\pi\} \text{ or } \gamma S \in \mathcal{P}(\mathbb{N} \cup \{\pi\}) \text{ and } |S| = |\mathbb{N}| \text{ and for any } x \in \mathbb{N}, \text{ if } x \in S, \text{ then } s(x) \in S \text{ and } \pi \in S' \}.$$ 

Interpret the poset $\langle \mathbb{N}_{\mathcal{M}_\pi}, \subseteq \rangle$ as a model of mereology in which the singletons are atoms and the subset relation is the relation of parthood. Call this model $\mathcal{M}_\pi$.

It is more obvious that this model is not atomistic, since $\{\pi\}$ is not in $\mathcal{M}_\pi$. Yet $\mathcal{M}_\pi$ is isomorphic to $\mathcal{M}$. So one of these models is atomistic just in case the other is. Hence, $\mathcal{M}$ is not atomistic.

Intuitively, $\mathcal{M}$ is a mixed case. In it, some things are ultimately composed of atoms (namely, the atoms represented by the singletons) and some are not (namely, all of the composites represented by the infinite sets). It is easily verified, though, that this model does not satisfy (3), since there is no object in the model such that all of its parts have proper parts.

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\[7\]I don’t use this alternative model in place of $\mathcal{M}$ because doing so would overly complicate the exposition of the models considered later in the paper.
So, neither (1) nor (3) expresses its target thesis. (1) does not rule out our mixed model and (3) does not rule it in. This isn’t too surprising. All (1) really says is that everything has at least one atom as a part. All (3) says is that there are some atoms and there are some atomless things. Since neither says anything about what things are ‘ultimately composed of’, the definitions guarantee much less than intended.8

A natural thought is that even though (1) fails to capture its intended theses alone, it should nevertheless exclude the troublesome model in conjunction with the other axioms of a suitable mereological system. This was likely the thinking of the authors of the definition, since it is only ever presented in a text after the strongest mereological system has been developed.

In fact, (1) is adequate given the axioms of the strongest mereological system. It fails to exclude nonatomic models in any weaker system, however, and is therefore not a definition of the thesis. I show this systematically in what follows and end by proposing a general statement of atomicity that is adequate for all mereological systems. 9

The simplest mereological system, *Ground Mereology* (M), consists in the deductive closure of axioms (4)–(6).

(4) (Irreflexivity): ∀x¬PPxx

(5) (Asymmetry): ∀x∀y(PPxy → ¬PPyx)

(6) (Transitivity): ∀x∀y∀z((PPxy & PPyz) → PPxz)

8From here on I will only be concerned with atomicity, though what is said can be applied to developing mixed mereologies as well.

9I follow Casati and Varzi (1999, ch.2) in developing the systems below and in prefixing A to any system supplemented with (1). The discussion is intentionally brief. See Varzi (2012) or Simons (1987) for detailed expositions.
Since $\mathcal{M}$ is ordered by the subset relation and the proper-subset relation is irreflexive, asymmetric, and transitive, $\mathcal{M}$ is a model of $\mathbf{M}$. Since (1) is consistent with (4)–(6), $\mathcal{M}$ is also a model of $\mathbf{AM}$, which is the system obtained by adding (1) to $\mathbf{M}$. Since $\mathbf{AM}$ countenances $\mathcal{M}$, it is not an atomistic mereology.

The axioms of $\mathbf{M}$ are quite lax—they allow counterintuitive models of parthood such as (i) composite objects with exactly one proper part, and (ii) numerically distinct objects composed of exactly the same proper parts. If we want to rule out such objects we can add (7) and (8), respectively, to $\mathbf{M}$.

(7) (Weak Supplementation): $\forall x \forall y (PPxy \rightarrow \exists z (Pzy \& \neg Ozx))$

(8) (Strong Supplementation): $\forall x \forall y (\neg Pyx \rightarrow \exists z (Pzy \& \neg Ozx))$

Call $\mathbf{AM}$ together with (7) Atomistic Minimal Mereology ($\mathbf{AMM}$) and $\mathbf{AM}$ together with (8) Atomistic Extensional Mereology ($\mathbf{AEM}$). These systems are much more powerful than $\mathbf{AM}$. Yet neither is strong enough to exclude $\mathcal{M}$. $\mathcal{M}$ satisfies (7) since every non-singleton has at least two disjoint proper subsets. $\mathcal{M}$ also satisfies (8), though this is easier to see when (8) is contraposed to (8').

(8'): $\forall x \forall y (\forall z (Pzy \rightarrow Ozx) \rightarrow Pyx)$

(8') says that if every part of $y$ overlaps $x$, then $y$ is part of $x$. But this is clearly satisfied by $\mathcal{M}$. Every numerically distinct composite in $\mathcal{M}$ is composed of a unique set of parts. So, $\mathbf{AMM}$ and $\mathbf{AEM}$ are not atomistic mereologies.
In addition to supplementation, one can strengthen the theories of mereology by adding closure principles that guarantee the existence of arbitrary binary sums and products.

(9) (Sum): \( \forall x \forall y (Uxy \rightarrow \exists z \forall w (Owz \leftrightarrow (Owx \lor Owy))) \)

(10) (Product): \( \forall x \forall y (Oxy \rightarrow \exists z \forall w (Pwz \leftrightarrow (Pwx \& Pwy))) \)

Call the result of adding both (9) and (10) to AM Atomistic Closure Mereology (ACM) and the result of adding them to either AMM or AEM Atomistic Extensional Closure Mereology, or (ACEM).\(^{10}\)

\( \mathcal{M} \) clearly satisfies (10), since every two things that overlap have a product.\(^{11}\) However, \( \mathcal{M} \) does not satisfy (9). Consider any two atoms, say \( \{0\} \) and \( \{1\} \). These atoms underlap, since they are both part of \( \mathbb{N} \). Yet there is nothing in the model such that everything it overlaps also overlaps either of the atoms. That is, there is nothing that has just the atoms, and nothing else, as its proper parts. So, adding (9) is sufficient to exclude \( \mathcal{M} \). So far so good for (ACM) and (ACEM).

It is possible to add more structure to \( \mathcal{M} \) and generate a new model in which every underlapping pair of objects has a least upper bound. In the context of our set-theoretic representation, this would amount to adding every finite subset of \( \mathbb{N} \) to the domain of \( \mathcal{M} \) in order to account for arbitrary sums of singleton pairs, and also adding the sum of every nonempty finite subset and gapless infinite subset of \( \mathbb{N} \). Construct the new model as follows.

Let \( \mathbb{N}_{\mathcal{M}'} = \{ S \cup T \mid S, T \in \mathbb{N}_{\mathcal{M}} \} \).

\(^{10}(10)\) and (7) together imply (8), hence (MM) becomes extensional with the addition of (10) and the closure versions of (AMM) and (AEM) are equivalent.

\(^{11}\)For the pairs of infinite sets, their product is the set further down the hierarchy; for the infinite/singleton pairs, their product is the singleton.
Interpret the poset \( \langle \mathbb{N}_{\omega^0}, \subseteq \rangle \) as a model of mereology in which the singletons are atoms and the subset relation is the relation of parthood. Call this model \( \mathcal{M}' \).

When all arbitrary binary sums are added to the model, we can pick any two disjoint objects and find an object that completely overlaps both of them (and nothing else). So, (9) is satisfied by \( \mathcal{M}' \).

The important question, though, is whether (1) is adequate on \( \mathcal{M}' \). Since the new sums are all binary sums formed from pairs of elements of \( \mathcal{M} \), a new sum will violate (1) only if a pair of elements of \( \mathcal{M} \) violates (1). But, as we saw above, none of them do. So \( \mathcal{M}' \) satisfies (1).

Is \( \mathcal{M}' \) atomistic? Clearly not. If the addition of new sums ensured that the model were atomistic, it would be because every maximal chain would be made to terminate. But because only the binary sums of members of \( \mathbb{N}_{\omega^0} \) were added, and not all sums in general, there are still maximal chains in \( \mathcal{M}' \) that do not terminate.

Let \( T \) be the maximal non-terminating chain of \( \mathcal{M} \). \( T \subseteq \mathbb{N}_{\omega^0} \), but \( T \) is not maximal in \( \mathcal{M}' \) because \( T \subset S \) where \( S = \{ X \mid X \in \mathbb{N}_{\omega^0} \text{ and } |X| = |N| \} \). But \( S \) is both maximal and non-terminating in \( \mathcal{M}' \). \( S \) is clearly non terminating (it has no minimal element), but suppose for reductio that it is not maximal. Then there is a chain \( Y \) in \( \mathcal{M}' \) such that \( S \subset Y \). Since \( S \) contains all of the infinite subsets of \( \mathbb{N}_{\omega^0} \), \( Y \) must contain finite sets.

All maximal chains of \( \mathbb{N}_{\omega^0} \) that contain finite sets terminate. Hence, \( Y \) terminates. However, supposing that \( Y \) contains a finite set implies that some member of \( S \) is a binary sum of finite subsets (since general summation is not defined in closure mereologies). But this is not so for the members of \( T \). None of them are binary sums of any finite sets, since, in general, no infinite set is the binary sum of any finite sets. So, since \( S - T \) are just the sums produced by binary unions of finite sets with members of \( S \), no member of \( S \) is a binary sum of finite sets. So, \( Y \) does not contain finite subsets. So, contrary to hypothesis, \( Y \) does not exist. So, \( S \) is maximal in \( \mathcal{M}' \). So, finally, \( \mathcal{M}' \) is not atomistic.
Let us add the axiom that generates the full strength of classical mereology.

(11) (Infinitary Sum): $\exists x \phi \rightarrow \exists z \forall y (Oyz \leftrightarrow \exists x (\phi \& Oyx))$

(11) is an axiom schema, not a single axiom. It stands in for a countable infinity of axioms (since there are only countably many formulas that can be substituted for $\phi$.) The result of adding every instance of (11) to AEM is called Atomistic General Extensional Mereology (AGEM).

$\mathcal{M}'$ is not a model of AGEM. General summation demands more structure of its models. Once we add the necessary structure to satisfy AGEM we finally have an atomistic model. Here’s why. The presence of infinitary summation and extensionality guarantees that there is a unique sum of all of the atoms (if there are any atoms). Suppose that there is a unique sum of the atoms and everything has an atom as a part (i.e., (1) is satisfied). Then everything overlaps the sum of the atoms. Suppose some $x$ is not a part of the sum of the atoms. Then, contrary to (1), there is an $x$ that doesn’t have an atom as a part. So, by reductio, everything is part of the sum of the atoms.\(^{12}\)

When ‘the sum of atoms’ is defined\(^ {13}\), (1) secures atomicity. Hence, we might express the target thesis more perspicuously as follows:

(12) (Atomicity\(^*\)): $\exists x x = \sigma y Ay \& \forall x P x \sigma y Ay$

This principle literally says that there is a sum of all atoms and everything is a part of it. So, given (12), if something exists at all then it is either an atom or part of something made exclusively of atoms (and is therefore ultimately made of atoms).

\(^{12}\)Thanks to an anonymous reviewer for pointing this out.

\(^{13}\)In general, $\sigma x \phi x$ (the unique sum, $x$, of everything satisfying $\phi$) is defined as $\tau z \forall w (Ozw \leftrightarrow \exists v (\phi v \& Ovw))$, where $\tau$ is the definite description operator.
We can confirm the adequacy of (12) by noting that it rules out the models considered earlier. When (12) is satisfied, a composite can only be a node in a (maximal) parthood chain if that chain terminates at an atom. Since every composite is a part of the sum of the atoms, none of the chains of which it is a node will consist in composites all the way down.

We now have an adequate definition of atomicity, but the cost of expressing it is quite high. (12) is strictly nonsense without extensionality, and unmotivated without a commitment to the existence of every arbitrary sum. Without extensionality, definite descriptions like ‘σy.∀y’ may fail to denote; there will be no guarantee that there aren’t several numerically distinct objects satisfying the same description (i.e., composed of the very same atoms). Without a commitment to there always being a sum of the atoms, there is nothing for the definite descriptor in our definition to denote. But why should we think that there is always a single thing composed of all and only atoms unless we already accept unrestricted summation?

It almost goes without saying that extensionality and unrestricted summation are controversial theses. Extensionality is inconsistent, for example, with the popular idea that a statue and the material from which it is constructed are numerically distinct. Unrestricted summation, either binary (as in (9)) or general (as in (11)), is inconsistent with the popular idea that there are restrictions on composition, either because composition is a causal phenomenon occurring in some circumstances but not others, or a mere fiction that never actually occurs. There are those who happily accept both extensionality and unrestricted summation, yet surely one need not be among them to say that everything is ultimately composed of atoms!

Suppose that you, like most metaphysicians, accept a mereological system weaker than GEM, but also, like most metaphysicians, think that everything is ultimately composed of atoms. How, exactly, do you express your view?

The way to do it, I think, is to say that there is a class of sums of atoms (which may only be the atoms themselves, or may include some composites made of the atoms) and everything is a part of one of its members. This doesn’t presuppose anything about how many sums of
atoms there are, whether there is a sum of all of them, whether the sums are extensionally individuated, or anything of the like. The proposal can be formalized easily using sets, but since part of the appeal of mereology is the promise of eschewing sets entirely, it is good practice to express mereological theses without invoking sets. Accordingly, the following proposal uses the resources of plural logic to denote the members of any relevant class.

Let doubled variables and constants (e.g., \(xx\)) be understood as plural variables or constants, and let \(\prec\) be the variably polyadic predicate ‘is among’. Following Oliver and Smiley (2013, 123–4), use the exhaustive descriptor ‘:’ (a notational variant of the more familiar ‘such that’ operator used to define sets by abstraction) to plurally denote the individuals satisfying some predicate distributively (e.g., \(x : \phi x\) means ‘the things that individually \(\phi\)’). Now, consider the following definitions.

\[
\text{‘}aa\text{’ (the atoms)} =_{df} \text{‘}x : A x\text{’}
\]

\[
\text{‘}S x\text{’ (}x\text{ is a sum of atoms)} =_{df} \text{‘}\exists y y (y y \prec a a \& \forall z (O z x \leftrightarrow \exists w (w \prec y y \amp O w z)))\text{’}
\]

The definition of \(aa\) is clear enough; \(aa\) is a name that refers plurally to all of the atoms and does the work that ‘the set of all atoms’ would usually do. \(S x\) is designed to mimic the notion of being a sum of the members of a set of atoms. Informally, its definition says that there are some atoms such that everything overlapping \(x\) overlaps at least one of them. The idea is that \(S x\) is satisfied by any sum of atoms, no matter how many atoms the sum includes. Given these definitions, we can state a perfectly general version of atomicity.

(13) (General Atomicity): \(\forall x \exists y (S y \amp P x y)\)

(13) says that everything is a part of a sum of atoms. When it is satisfied every maximal parthood chain terminates, since anything that exists must be a part of a sum of atoms.
and every maximal parthood chain of a sum of atoms terminates. It is general in the sense that it requires neither extensionality nor arbitrary summation to work. (13) doesn’t require extensionality because, unlike (12), it is consistent with the existence of non-unique sums of the atoms. (13) doesn’t require arbitrary summation because, unlike (12), it refers to sums of atoms without presupposing how many sums there are.

So, to wrap things up, I think (13), and not (12) or (1), is how you say ‘Everything is ultimately composed of atoms’.14

2.1 References


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Chapter 3

Mereological Bundle Theory and the Identity of Indiscernibles

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3.1 Introduction

Objects have properties — my chair is comfortable, electrons are negatively charged, the moon is spherical, and so on. This is not a controversial claim. What is controversial, though, is how the relationship between property and object is to be analyzed. L. A. Paul ([13], [14], [15]) has argued that this relationship should be understood in terms of mereological composition. On Paul’s mereological bundle theory of objects, the reason electrons have the property of being negatively charged is that electrons are composed of properties\(^2\), one of which is the property *being negatively charged*. Electrons, and all other objects for that matter, are ultimately fusions of properties; e.g., a particular electron is just a fusion of a set of properties like negative charge, mass \(m\), spin \(s\), and (depending on one’s physics) a unique spatiotemporal location \(l\).

As an ontologist with little aversion to immanent universals I find Paul’s view appealing. Not only does the view give us a bundle theory of objects with a bundling relation whose logic is well-understood, it also promises to resolve some longstanding puzzles in the metaphysics of material constitution and temporal persistence. Here I argue that mereological bundle theory can do even more for us than Paul has suggested. After showing that Paul’s version of the bundle theory entails the (in)famous Principle of the Identity of Indiscernibles (PII), I argue that the theory has the resources to provide us with a new strategy for dispelling putative counterexamples to (PII). If the new strategy that I develop for defending (PII) is consistent and plausible, friends of (PII) will have a strong reason to prefer mereological bundle theory as an approach to the general ontology of objects, and proponents of the mereological bundle theory will have one more trophy to place on their mantle.

\(^2\)By properties Paul means something like immanent universals, though she thinks that her theory can accommodate tropes as well. Throughout this essay when I refer to properties I will mean something like immanent universals as well, though it will become clear later in the paper that I do not believe these universals are multiply located in the usual sense.
3.2 Mereological Bundle Theory

Bundle theories of objects analyze the relationship between an object and its properties by identifying objects with the “bundle” of properties (or some subset of that bundle) they instantiate. As a metaontological strategy, the bundle theory has some obvious perks. By identifying objects with their properties the bundle theorist is not left with the embarrassing task of explaining what the object is *sans properties*, and what, *sans properties*, distinguishes it from other barren objects. Nor is he left with the task of explaining how we gain knowledge of these barren objects when our epistemic access seems to be limited to properties alone. But the bundle strategy has its own hurdles to clear, not the least of which is explaining the nature of the bundling relation. The bundling relation is what determines whether a set of properties constitute a unified object or a mere collection. The search for a non-arbitrary, well-understood relation to support this distinction has become a philosophical industry in itself. Paul’s recent contribution to this search is a clever twist on a familiar idea.

Mereological bundle theory improves upon traditional bundle theory by taking the primitive relation of bundling to be the more familiar relation of fusing or composing, such that objects are fusions of properties or fusions of property instances. Hence, mereological bundle theorists endorse a property mereology; a mereology where properties or property instances can be parts of objects. An advantage of the approach derives from the fact that standard mereologies take composition to be primitive or define it using a different primitive mereological notion (such as primitive parthood). Thus, taking the basic primitive of bundle theory to be composition can reduce the need for additional primitives in one’s overall ontology and substitutes a familiar type of relation relied upon elsewhere in ontology for an unfamiliar type of relation unique to the bundle theorist. [15, pgs. 1-2]
Paul’s basic insight is that classical mereology leaves open how the parthood relation is to be interpreted, and though most philosophers have thus far understood parthood as a purely spatiotemporal relation, spatiotemporal parthood is just an instance of a broader type of parthood – qualitative parthood. [14] offers the following ‘master mereology’, $M_{QP}$, according to which the parthood relation holds between properties and fusions of properties. $M_{QP}$ is characterized by axioms (A1)-(A3) and definitions (D1)-(D5).

A1. For any qualitative proper part $x$, $x$ is not a qualitative proper part of itself. (irreflexivity)

A2. If $x$ is a qualitative proper part of $y$, then $y$ is not a qualitative proper part of $x$. (asymmetry)

A3. If $x$ is a qualitative proper part of $y$ and $y$ is a qualitative proper part of $z$, then $x$ is a qualitative proper part of $z$. (transitivity)

D1. $x$ is a qualitative part of $y$ iff $x$ is a qualitative proper part of $y$ or $x$ is identical to $y$.

D2. $x$ qualitatively overlaps $y$ iff $x$ and $y$ have a qualitative part in common.

D3. $x$ is qualitatively disjoint from $y$ iff $x$ and $y$ have no qualitative part in common.

D4. $x$ partly qualitatively overlaps $y$ iff $x$ and $y$ have some but not all qualitative parts in common.

D5. $x$ is the qualitative fusion of $ys$ iff $x$ has all the $ys$ as qualitative parts and no qualitative parts disjoint from the $ys$.\(^3\)

\(^3\)These definitions are given in [14, pg. 5]
On this view, spatiotemporal parthood is not a unique kind of parthood relation. It is the parthood relation that holds between spatiotemporal properties on the one hand, and a fusion of any kind of property on the other.\footnote{Not just any properties can be fused, since being a circle and being a square are not compossible. What determines whether two properties are compossible is an important, but unanswered, question for the mereological bundle theory. Since I don’t know the answer, I will only note the question here. A related question is, When does composition occur on the mereological bundle theory? While no principled answer suggests itself, we know that the mereological bundle theorist cannot accept unrestricted composition, since this would entail fusions of properties that aren’t compossible. Here I will assume that composition is restricted by compossibility, but is otherwise unrestricted.}

If we accept that the bundling relation is to be explained in terms of qualitative parthood relations, $M_{QP}$ — or some extension of $M_{QP}$ — is a natural way to explain what objects are and how they are different from one another — an object is what it is by satisfying (D5) with respect to a set of qualitative parts. That is, objects just are qualitative fusions of their properties.\footnote{Note that by (D5) properties are objects on this theory as well, since they are qualitative parts of themselves by (D1) and not qualitatively disjoint from themselves by (D3).}

A major motivation for adopting Paul’s theory is that it can support common metaphysical intuitions that other theories cannot. For example, many people have the intuition that some ordinary material objects which are distinct nevertheless occupy a single spatiotemporal location. The statue David and the marble (call it Lump) of which it is composed, for example, seem to be two different objects occupying the same location. It is easy to see why one might hold the objects are distinct: David is essentially a work of art; Lump is not. Lump is essentially marble; David is not. What is not easy to see is how it is possible for David and Lump to be distinct objects when they are both composed of the very same material. Standard spatiotemporal mereology is extensional; that is, it satisfies the following axiom of extensionality.

\textbf{A4.} If $x$ and $y$ share all the same proper parts, then $x$ is identical to $y$. (extensionality)
Hence, if we understand mereological composition in the standard spatiotemporal sense, the fact that *David* and Lump share the same (spatiotemporal) parts entails that *David* and Lump are the very same object. It seems to some, then, that we must either give up extensionality of parts or give up on the intelligibility of distinct co-located objects. Not so, says Paul. What this dilemma shows us is that standard spatiotemporal mereology is not the mereology of ordinary objects like *David* and Lump. If we adopt the more general mereology of $M_{QP}$, we can make sense of the fact that *David* and Lump are distinct but co-located material objects. Let A be the property of being essentially a work of art, B be the property of being essentially marble, and let C be the (admittedly unnatural) property of having all of the material and spatiotemporal properties of *David*. What explains the difference between *David* and Lump? They do not completely qualitatively overlap: *David* is the fusion of A and C (and many more properties), while Lump is the fusion of B and C (and many other properties). What explains the fact that *David* and Lump are co-located? They partially qualitatively overlap: both *David* and Lump have C as a part, and this part contains all of the location properties in question.

### 3.3 Extensionality and the Identity of Indiscernibles

Paul [14] proposes similar solutions to puzzles about *de re* modality, persistence, supervenience, redundant causation, event individuation, personal identity, nonreductive physicalism in mind, and reference. Mereological bundle theory has explanatory power to burn. But notice that $M_{QP}$ does not include all of the axioms required for a fully developed mereology. An obvious omission is a supplementation principle that would make explicit the relationship between qualitative sums and identity.

> My bundle theory allows for the possibility of actual-world cases of qualitatively indiscernible objects at different locations because such objects can be individuated by their location properties, by properties of their spatiotemporal parts, or primitively. Primitive individuation does not require the acceptance of primitive
thisness or haecceities, but unless it is the property parts (instead of the whole fusion) that are primitively individuated, it does require the rejection of a mereological supplementation principle, qualitative extensionality, according to which objects [...] with the very same proper qualitative parts are identical. ... Somewhat controversially, I think qualitative extensionality holds. Acceptance of qualitative extensionality is not acceptance of what is standardly taken to be the “principle of the identity of qualitative indiscernibles.” This because by “proper qualitative parts” I mean to include many different sorts of property parts, including primitive individuating properties (if such there be), properties of having certain locations, and properties of having certain spatiotemporal parts. Whether one accepts or rejects qualitative extensionality will not affect the treatment of material coincidence I develop below. [14, pgs. 17-18]

Paul is right to suppose that she need not add a qualitative version of (A4) as an axiom of her $M_{QP}$. But while this is a nice point about property mereology in general, one cannot make use of the explanatory power of Paul’s mereological bundle theory without appropriately supplementing $M_{QP}$. To see why, recall how Paul resolves puzzles of material co-location. In the case of David and Lump discussed above, Paul helps herself to unique fusions of the properties in sets A, B, and C. But $M_{QP}$ alone does not guarantee that there are any qualitative fusions, let alone unique fusions. Hence, Paul must be invoking a stronger mereology to solve these puzzles. The question is, How much stronger than $M_{QP}$ must our property mereology be in order to support an adequate mereological bundle theory? The answer is: quite strong. If Paul’s solution to the puzzle of material co-location is a principled solution, it must tacitly invoke the Strong Supplementation Principle (SSP).

(SSP) For all x and y, if x is not a proper part of y, then there is a part of y that doesn’t overlap x.

Without (SSP) there is no guarantee that the fusion of the material and location properties that is a proper part of David is the same fusion of material and location properties that
is a proper part of Lump, even though there is only one such set of properties. That is, without (SSP) there is no guarantee that fusions with the same parts are identical, and hence no guarantee that fusions are unique. And without the uniqueness of fusions, there is no guarantee that David and Lump qualitatively overlap in the way that Paul suggests.

Hence, if we want to reap the benefits of mereological bundle theory we need to add a qualitative version of (SSP) to the axioms of \( M_{QP} \); call the resulting mereology \( EM_{QP} \). It is easy to show that \( EM_{QP} \) is qualitatively extensional.

**Theorem 3.3.0.1.** Qualitative fusions are extensionally individuated in \( EM_{QP} \).

*Proof.* Let \( P_1, \ldots, P_n \) be properties and let objects \( O \) and \( O' \) be qualitative fusions of the Ps. Suppose \( O \) and \( O' \) share all the same proper qualitative parts (i.e., for all \( P_i \in \{P_1, \ldots, P_n\} \), \( P_i \) is a proper qualitative part of \( O \) if and only if \( P_i \) is a proper qualitative part of \( O' \)). Suppose for reductio that \( O \) is not identical to \( O' \). Then by (D1), either \( O \) is a proper qualitative part of \( O' \) or \( O \) is not a qualitative part of \( O' \). The proof proceeds by cases.

**Case 1:** Suppose \( O \) is a proper qualitative part of \( O' \). Then, by weak supplementation\(^6\), \( O' \) has a proper qualitative part \( P_i \) that \( O \) lacks, contrary to hypothesis.

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\(^6\) **Lemma 3.3.0.1.** Weak Supplementation Principle (WSP): for all \( x \) and \( y \), if \( x \) is a proper part of \( y \), then there is a part of \( y \) that is disjoint from \( x \).

*Proof:* Suppose that \( x \) is a proper part of \( y \). Suppose, for reductio, that there is no part of \( y \) disjoint from \( x \). By (A2), \( y \) is not a proper part of \( x \). Hence, by (SSP), there is a part \( p \) of \( x \) that doesn’t overlap \( y \). Either \( p \) is identical to \( x \) or \( p \) is a proper part of \( x \).

**Case 1:** Suppose \( p \) is \( x \). Then by hypothesis \( p \) is a proper part of \( y \). Since \( p \) is a proper part of \( y \), by (D2) \( p \) overlaps \( y \). But \( p \) doesn’t overlap \( y \). (contradiction)

**Case 2:** Suppose \( p \) is a proper part of \( x \). By hypothesis, \( x \) is a proper part of \( y \), so, by (A3), \( p \) is a proper part of \( y \). Hence, by (D2), \( p \) overlaps \( y \). But \( p \) doesn’t overlap \( y \). (contradiction)

Either case results in absurdity. So, by reductio, there is a part of \( y \) that is disjoint from \( x \). Q. E. D.
Case 2: Suppose $O$ is not a qualitative part of $O'$. Then $O$ and $O'$ are qualitatively disjoint, and therefore share no qualitative parts. Hence, by (SSP), $O'$ has at least one proper qualitative part that $O$ lacks, contrary to hypothesis.

Either case results in absurdity. Hence, by reductio, $O$ is identical to $O'$.

\[
\begin{proof}
\end{proof}
\]

This theorem shows that a qualitative version of (A4) — call it (A4') — is true on $EM_{QP}$ (and hence on mereological bundle theory).

**A4'.** If $x$ and $y$ share all the same qualitative proper parts, then $x$ is identical to $y$. (qualitative extensionality)

In other words, any two objects sharing all the same qualitative proper parts are not two objects after all, but one and the same object. Even though this consequence bears an uncanny resemblance to the principle of the identity of indiscernibles (PII)

**PII.** For all objects $x$, $y$, if for all properties $P$, $x$ is $P$ iff $y$ is $P$, then $x = y$.

Paul asserts in the passage quoted above that these two principles are not equivalent, since qualitative extensionality includes properties excluded by a suitable qualitative version of (PII).

This issue is not as cut and dry as Paul’s brief comments would make it seem. Expanding on her comments, it seems the issue is that $M_{QP}$ leaves open whether impure (i.e., object-dependent) properties, like *being identical to Aristotle* or *being distinct from the Moon*, are
to be counted as objects in the theory of qualitative parts. If they are, then (PII) and (A4')
are equivalent. If not, then not.

Paul is clear in the passage quoted above that she counts impure properties as qualitative
parts. What she says about (PII) is less clear, but she seems to imply that impure properties
are not in the domain of (PII). But the scope of the quantifiers in both (PII) and (A4') vary
according to the context of discourse, and there is no prima facie reason to suppose that the
domain of qualitative parts should not be coextensive with the domain of qualities. We need
reasons to distinguish the domains of (PII) and (A4') - reasons that Paul does not supply. I
don’t think that there are any good reasons to do this, and the task of the remainder of this
section is to argue that (PII) and (A4') are equivalent on their most plausible interpretations.

In the extensive literature on how to interpret (PII) it is generally agreed that interpre-
tations that do not restrict the domain of properties are trivially true and uninteresting.7
Hence, if we want to entertain a necessary connection between quality and quantity, we need
to restrict the domain of the principle in such a way that our interpretation is strong enough
to be plausible, but not so strong that it is obviously false. To this end it is helpful to clearly
distinguish four kinds of property: intrinsic, extrinsic, pure, and impure. (5, 1)

1. If any (spatiotemporal) part-for-part duplicate y of an object x necessarily has property
   
   $P$, then $P$ is an intrinsic property of $x$; otherwise, $P$ is an extrinsic property of $x$.8

2. If a property is analyzed in terms of individual substances, then it is an impure property;
   otherwise, it is a pure property. (Forrest 5, 1)

No interpretation of (PII) that quantifies over impure properties is worth defending, since
such interpretations are trivially satisfied by the inclusion of impure properties (e.g. being

identical to Aristotle). For this reason, we exclude impure properties from the domain of our

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7See for example Black [1], O’Connor [12], Gale [6], Cross [3], Della Rocca [4], Rodriguez-Pereyra
[18], Hawley [8], and Forrest [5].

8There is some controversy over how to draw the distinction between intrinsic and extrinsic
properties (see Weatherson and Marshall, [20], 2.1), but I don’t think we will go too far wrong with
this definition.
interpretation of (PII). This leaves us with two options for our domain: (i) pure properties, or (ii) pure intrinsic (but not extrinsic) properties. Restricting the domain to pure properties will strengthen our principle. Restricting the domain to pure intrinsic properties will strengthen it even more. Therefore, let us call the former the Weak Interpretation and the latter the Strong Interpretation. The Weak Interpretation is not trivial. Indeed, it is the target of the counterexamples from symmetry we consider in the next section. It also seems that the Strong Interpretation is too strong, since there is no obvious reason to rule out a universe containing two independent objects that have all the same pure intrinsic properties but differ in their extrinsic properties.\footnote{The existence of a universe containing exactly three objects, two qualitatively identical spheres that differ only in their pure extrinsic relations to a cube, seems possible and unproblematic, for example.}

Hence, we adopt the Weak Interpretation of (PII).

Like (PII), \( (A4') \) is a formal principle in need of interpretation. We know that the intended interpretation is a principle governing the relations between qualitative parts and the bundles they compose. But this information doesn’t fully determine the domain of qualitative parts. We must decide how this domain is to be restricted, and on this subject I am less liberal than Paul. It seems to me that not every property instantiated by an object should be counted as a qualitative part of that object. The set of properties instantiated by an object includes impure properties while, intuitively, the set of qualitative parts of an object does not. \textit{Sitting in my lap} is a property often instantiated by my cat, but is not a part of my cat, for I would not be removing a part of my cat if I were to lift him from my lap and place him on the floor. This objection can be put in a more principled way. Including impure properties in the

\footnote{Rodriguez-Pereyra \cite{18} has a much more fine-grained analysis of the proper interpretation of (PII). He defines a non-trivial interpretation of (PII) to be one that excludes trivial properties from its domain, where a property \( F \) is trivial if and only if “differing with respect to \( F \) is or may be differing numerically.” (219) While sorting out the details of this definition he argues that not all impure properties are trivializing. Thus, since I exclude all impure properties, one might think that my analysis is incorrect. It is not clear to me, however, that this analysis is extensionally different from the one I intend to be giving, for it is the properties that involve unanalyzable numerical identity or numerical distinctness that are meant to be excluded by excluding the impure properties. Consequently, I do not disagree with Rodriguez-Pereyra, but instead interpret him as offering a more precise, albeit idiosyncratic, definition of impure property than the one relied on in this paper.}
scope of \( (A4') \) runs afoul of Hume’s Dictum, i.e. that there are no metaphysically necessary connections between distinct, intrinsically typed, entities.\(^{11}\) If an object *just is* a fusion of properties and some of these properties are impure, then it seems that an object is what it is partly because of what another, distinct entity is.\(^{12}\)

If this were true then nearly every composite object in our property ontology would violate Hume’s Dictum. But satisfying Hume’s Dictum is, it seems to me, constitutive of the very notion of being an object. So, rather than denying it, we should instead restrict the domain of qualitative parts to pure properties.

So we have good reason to adopt an interpretation of \( (A4') \) that is equivalent to (PII). If this is right, then there is no use in denying the equivalence of (PII) and \( (A4') \). I don’t see this as a surprise. The goal in formulating these principles is, first and foremost, to give an adequate principle of object individuation solely in terms of properties. The only difference between (PII) and \( (A4') \) is that they are expressed in two different languages. Since the principles will deliver extensionally equivalent answers on what objects there are, they are also truth-functionally equivalent. Hence, since Paul’s mereological bundle theory entails \( (A4') \), it also entails a suitably strong qualitative version of (PII).

3.4 **Black’s Puzzle and a Solution**

If the conclusions of the previous section are correct, any mereological bundle theory that can do the explanatory work of Paul’s theory is threatened by putative counterexamples to (PII). We will here only concern ourselves with the most famous (and most threatening)

\(^{11}\)See Wilson [21].

\(^{12}\)Obviously, distinctness has to be understood as mereological distinctness rather than numerical distinctness here, since an object does depend on its numerically distinct proper parts for its own identity, it has necessary connections to numerically distinct entities. If Hume’s Dictum is interpreted in the sense of numerical distinctness, then it seems that either composition is impossible or composition is identity. Since we’re here assuming that composition is both possible and not identity, we conclude that distinctness should not be interpreted as numerical distinctness. For more on this very interesting issue see Einar Duenger Bøhn [2].
Isn’t it logically possible that the universe should have contained nothing but two exactly similar spheres? We might suppose that each was made of chemically pure iron, had a diameter of one mile, that they had the same temperature, colour, and so on, and that nothing else existed. Then every quality and relational characteristic of one would also be a property of the other. Now if what I am describing is logically possible, it is not impossible for two things to have all their properties in common. This seems to me to refute the principle. [1, pg. 156, Black’s emphasis]

This looks like a clear counterexample to qualitative extensionality — the spheres are exactly alike in every quality, yet there are apparently two rather than the one sphere that qualitative extensionality would seem to countenance. Can the mereological bundle theorist explain away this apparent counterexample? I think so. But first we should take note of some of the available responses to Black’s putative counterexample.

Katherine Hawley [8] gives us a handy list of strategies for defending (PII): i) argue that the two spheres are really just one, multiply located sphere (the identity defense), ii) argue that there is a suitable property that distinguishes the spheres after all (the discerning defense), or iii) argue that there are not spheres in this universe, but just one scattered, non-spherical simple object that closely matches the description of being two spatially separated spheres (the summing defense).

It might seem that a bundle theorist ought to pursue the identity defense. Indeed, John O’Leary Hawthorne [9] and Hawthorne and Jan Cover [10] argue that since immanent universals can be multiply located, bundles of universals can be multiply located, too. If this is right, then, contra Black, the traditional bundle theory does not violate (PII).

Yet commentators have argued that Hawthorne’s identity defense has serious flaws. Gonzalo Rodriguez-Pereyra [17], for example, argues that Hawthorne’s defense is not effective
because it fails to show how the bundle theory can account for the genuine possibility of numerically distinct but indiscernible particulars. If Black’s universe contains two numerically distinct but indiscernible spheres, Hawthorne’s identity defense merely provides an incorrect description of this universe. What Hawthorne’s defense should do, says Rodriguez-Pereyra, is either show that numerically distinct but indiscernible particulars are impossible or at the very least show that Black’s universe is not an example of this possibility. But Hawthorne’s defense does neither of these, and so it fails to show that traditional bundle theory can explain away Black-style counterexamples from symmetry.

Rodriguez-Pereyra also points out the multiple locations of Hawthorne’s multiply located sphere are difficult to account for on a traditional bundle theory. Suppose that space is relational Black’s universe. Then it seems that the locations of the sphere are determined by relations of the sphere to itself, and since the universe is distributed symmetrically it is difficult to see how distinct relations could arise to support the existence of distinct locations. The situation is no better if we instead suppose substantival space. For each region of space in Black’s universe is qualitatively identical to every other region, and so on a bundle theory there is nothing that could distinguish distinct places in that space.

I think these criticisms are damning for the traditional bundle theory. But I also think that we can salvage much of what seems right about Hawthorne’s defense while avoiding the problems raised by Rodriguez-Pereyra. Since mereological bundle theory has access to a mereology that is more general than a strictly spatiotemporal mereology, it has access to fourth strategy which I will call the overlap defense. The overlap defense is modeled on Paul’s answer to the puzzle of material co-location, and though it might not be the first solution that comes to mind when thinking about Black’s universe, I think it is an improvement on the strategies discussed by Hawley.

The overlap defense invokes qualitative parts to distinguish two senses of the question, “How many spheres are in Black’s universe?” For there are at least two kinds of sphere on the mereological bundle theory account. There are unlocated spheres and there are the more
familiar located spheres. The fact that there are spheres with location properties in Black’s universe guarantees that there are also unlocated spheres; i.e., the qualitative remainder of the located spheres minus their location properties. In Black’s universe, this qualitative subtraction gives us one unlocated sphere. If the question is how many unlocated spheres there are, the answer is 'one'. If the question is how many located spheres there are, the answer is 'two'. Thus, in contrast with the one-, two-, and zero-sphere solutions discussed by Hawley, the overlap defense I am offering is a three-sphere solution to Black’s puzzle.

In the same way that the co-location of David and Lump is explained by the overlap of some of their qualitative parts, the apparent indiscernibility of Black’s spatiotemporally separated spheres is explained by their sharing a qualitative part; namely, the unlocated sphere composed of the qualitative parts having mass $m$, having a diameter of one mile, being made of chemically pure iron, etc. In the case of David, we eliminated the double counting of material. In this case we are eliminating the double counting of qualities. Thus, just as there is only one bit of marble in the case of David and Lump, there is only one sphere in the case of Black’s universe.

One qualitative fusion matches Black’s description of the spheres, but it is fused to two different sets of location properties. Neither location is part of it, since it is not located. But the fusion of that sphere with two distinct sets of location properties gives us two distinct objects, each spherical by virtue of having the unlocated sphere as a qualitative part. The number of spheres varies with the addition or subtraction of unique properties. Thus the overlap defense provides what Hawthorne’s identity defense does not: an explanation of how there could be numerically distinct but indiscernible particulars when particulars are qualitatively individuated. That is, it shows that Black’s universe is consistent with (PII).

3.5 Replies to Objections

Since the overlap defense is similar to both the identity and discerning defenses, our first concern should be answering extant objections to those strategies. The major objection that
Hawley discusses for the identity defense is that each sphere has a certain mass \( m \) and that if there is just one multi-located sphere, then there should be a total mass of \( m \) in Black's universe. It seems obvious, however, that the total mass will be \( 2m \), since the sphere of mass \( m \) exists in two places. This isn’t a problem for the overlap defense. The overlap defense does not assert that the sphere is multi-located. It is more subtle than that. There is just one qualitative sphere, and it isn’t located anywhere because it is not even partially composed of location properties. However, the qualitative sphere does have the property of \textit{having mass} \( m \), and its mass does get counted twice to produce a total of \( 2m \) in Black’s universe as one would expect, since the property of having mass \( m \) appears in two distinct locations in this universe. But this double-counting is legitimate. The property of having mass \( m \) is multiply instantiatable — it can be part of more than one qualitative fusion. And since one qualitative fusion can be a proper part of two distinct qualitative fusions, \textit{having mass} \( m \) can be a part of two distinct qualitative fusions by virtue of being a proper part of a proper part overlapping both of those distinct fusions. So, the overlap defense avoids the too-little mass problem.

Though his primary goal is to undermine Hawthorne’s identity defense, several of Rodriguez-Pereyra’s criticisms are aimed at the bundle theory’s ability to discern the locations of the multiply located spheres. These objections can be partially met by appeal to the notion of weak discernibility. Objects are weakly discernible if they stand in some irreflexive relation to one another [16]. In Black’s universe each sphere is one mile from the other. Yet nothing can be one mile from itself. So these relations could only be instantiated if there were more than one sphere capable of bearing relations to other spheres. Instances of these relations cannot be distinguished from one another without direct reference to one or the other of the spheres — something Black is quick to point out that we cannot do if we think there is only one sphere. But however indistinguishable, the instantiation of weakly discerning properties is sufficient for numerical distinctness. Hence, Rodriguez-Pereyra’s criticism of the traditional bundle theory’s failure to account for multiple locations does
not stop bundle theorists from making sense of the claim that there are multiple spheres in Black’s universe. The weak discernibility of the spheres guarantees this without requiring a bundle theory friendly account of location.

The major problem Hawley finds with the discerning defense is that it rests on the power of weakly discerning location properties to ground the identity of the objects that the properties weakly discern. According to Hawley, weakly discerning location properties are not sufficient for grounding the distinctness of objects, since the weak discernibility of these properties is grounded in the distinctness of the objects (or object) in question. If this is a real problem for the discerning defense, then it is a problem for the overlap defense as well. For what blocks the spatiotemporal spheres from collapsing into one spatiotemporal sphere is the fact that there is a weakly discernible location property that is a qualitative part of one sphere and not the other. If the irreflexivity necessary for distinguishing this property from its twin is grounded in the fact that there is another sphere of which it is not a part, then it seems that the account rests on a circularity.

In the case of Black’s universe Hawley’s worry is misplaced. Trouble arises when we try to build from scratch weakly discernible location properties by abstracting monadic properties from partially satisfied relational properties. Whether there are distinct but weakly discernible locations occupied by the spheres is not in question. As Rodriguez-Pereyra rightly notes, the existence of these distinct locations is built into Black’s description and to ignore it is a mistake. Black constructs his case so that there is no principled way for us to distinguish these weakly discerning location properties. But their indistinguishability is irrelevant. The existence and distinctness of these properties is sufficient (by (A4')) to ground the distinctness of the spatiotemporal spheres.  

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13Not only are these weakly discerning location properties distinct, they are also incompossible. That is, like being a circle and being a square, no object can have both of them. Otherwise, the object in question would be both some distance from itself and not some distance from itself. This also shows why there is not a co-located sphere in Black’s universe in any straightforward sense. Such an object would be the sum of the located spheres, and would thus be both some distance from itself and not some distance from itself.
Further, as Paul suggests, the mereological bundle theorist may, if she pleases, take the distinctness of location properties as primitive. This suggestion seems harmless enough to me. Indeed, brute difference between properties seems necessary for the account to be coherent at all. But it may be objected that this move flouts the very spirit of (PII). If we are willing to tolerate brute difference between qualitative simples, then why be so hostile toward brute difference between composite objects? One answer is that \( A4' \) tolerates brute difference between simples but not composites, and \( A4' \) is part of a theory of objects that solves a wide variety of metaphysical puzzles without molesting our intuitions about those puzzle cases. Hence, we already have some reason to accept \( A4' \), and therefore some reason to accept what follows from it. More importantly, theoretic simplicity demands that we get by with the fewest number of primitives possible. Hence, brute difference between qualitative simples is not \textit{ad hoc}. We could posit more brute differences to account for our observations and intuitions, but given the brute differences between properties the mereological bundle theorist can give us non-primitive difference between a wide variety of objects. Why pay more for the same?

One final consideration. Black’s universe is problematic for mereological bundle theory in part because impure properties are not to be counted as individuals on the mereological bundle theory (or so I argued above). On at least one common conception of spatial occupation, having a location is an impure property: an object occupies a location by bearing the occupation relation to it. How is it, then, that objects can be partially composed of location properties? If location properties were impure properties, then they could not be qualitative parts of the spheres in Black’s universe and so the overlap defense would be incoherent. My suggestion is that we abandon the conception of occupation as a special relation between an object and a location and reduce it to the more general relation of composition. I mentioned above that it is open to the mereological bundle theorist to treat locations as brutally distinguished individuals. We may go one step further and treat occupation itself as the fusion of location properties with the other individuals (i.e., properties) countenanced
by the theory. This is a natural move for the mereological bundle theorist and similar substantivalist fusion strategies have been suggested by David Lewis [11] and Ted Sider [19] on independent grounds.

3.6 Conclusion

These considerations are, I think, sufficient to show that the overlap defense is an interesting and consistent way to interpret Black’s Two Sphere Universe. Black’s universe contains two (weakly) discernible, spatiotemporally located spheres that share one unlocated sphere as a qualitative part. So interpreted, Black’s universe is not a counterexample to either (PII) or (A4’). Since Black’s example is the most famous attack on (PII), successfully defusing it is a major victory for the overlap defense.\textsuperscript{14} And since the mereological bundle theory provided the conceptual resources for the overlap defense, it emerges all the more plausible as a general theory of objects.\textsuperscript{15}

3.7 References


\textsuperscript{14}An anonymous reviewer notes that you might use the overlap defense to make sense of the distinctness of particles in a fermion system, which are apparently weakly discernible by their spin properties in the same way that Black’s spheres are weakly discernible by being one mile from a sphere. I am interested in whether the overlap strategy could be applied in the quantum realm, but I lack the expertise to confidently draw any conclusions on this issue.

\textsuperscript{15}Thanks to Einar Duenger Bøhn, the audience at the 2012 Arché/CSMN Graduate Conference at the University of Oslo, and three anonymous referees for helpful comments on earlier drafts of this paper.


CHAPTER 4

THE LOGIC OF COMPOUND TERMS\footnote{Shiver, A. To be submitted to \textit{Journal of Symbolic Logic}.}
4.1 Lists in Natural Language

4.1.1 Lists and Singular Terms

Consider the following inference.

Ann and Bob lifted the piano, so Bob lifted the piano. \[\text{(4.1)}\]

(1) is invalid. Pianos are heavy. We can certainly imagine a scenario in which our Ann and Bob lift the piano with their collective strength while neither would have succeeded working alone. In such a case the predicate ‘lifted the piano’ is satisfied by Ann and Bob collectively but not individually.

For all its virtues classical first-order logic (FOL) is ill-equipped to diagnose this kind of error. We might be tempted to represent (1) as

\[Pa \land Pb \therefore Pb\] \[\text{(4.2)}\]

in FOL. But this won’t do, since the resulting argument is valid and (1) is not. On the other hand, we might think it better to leave the premise unanalyzed and treat the argument as a truth-functional inference. This yields the right result, since

\[A \therefore B\] \[\text{(4.3)}\]

is invalid. But if we try this method with the valid argument

Ann and Bob lifted the piano, so Bob and Ann lifted the piano. \[\text{(4.4)}\]
we get the wrong result again.

4.1.2 Lists and Plural Terms

Plural logic allows us to represent (1) as

\[ \exists x(x \in P & ((a \prec x \& b \prec x) \& \neg \exists y(y \prec x \& (a \neq y \& b \neq y)))) \therefore Pb. \]  

where (5) is pronounced ‘Some things lifted the piano, and Ann and Bob (and nothing else) are among them. So, Bob lifted the piano.’ This representation doesn’t respect the grammar of (1) but does a fair job of modelling its content. Indeed, on every extant semantics for plural logic of which I am aware, this representation yields the correct verdict of ‘invalid’. But notice that in not respecting the grammar of (1) we represent its premise as a disguised existential sentence. This is a mistake; there is no existential quantification at work in ‘Ann and Bob’. Thus, with (5) we get the right diagnosis of ‘invalid’, but we get it for the wrong reason.

A slightly better plural representation of (1) is

\[ P_{cc} \& (a \prec cc \& b \prec cc) \therefore Pb. \]  

Where the plural term ‘cc’ stands in for the list ‘Ann and Bob’. But this won’t do, either, since the premise of (6) does not imply that ‘cc’ denotes exactly ‘a’ and ‘b’ and we must again invoke a specious quantifier embedded in ‘Ann and Bob’ to accurately represent the content of (1).

Aside from the fact that ‘Ann and Bob’ names two things at once, there isn’t much in the original argument to recommend the plural representation. The plural terms of plural logic are used to represent plural noun phrases like ‘The Cheerios (in my bowl)’ and ‘the Supreme
Court Justices’ that denote several things at once. In this sense lists function similarly to plural terms—a list is a grammatically plural phrase that denotes several things. But lists have syntactic features not found in other plural terms. The most salient feature of lists is not that they are plural terms, but that they are compound plural terms formed from, and necessarily related to, their component terms.

4.1.3 Nesting and Superplural Reference

Another distinguishing feature of lists is that they can be multiply-compound by being nested within other lists. Consider the following sentence.

Batman and Robin and The Joker and Harley Quinn are famous comic book enemies.

(4.7)

The intended interpretation of this sentence is that two pairs of comic book characters are enemies, though none are enemies individually and they are not enemies all taken together. Thus some nested lists involve apparent plural denotation of plurals, i.e. higher-level plural denotation. ²

Lists can also be formed from plural terms and nested, as in the sentence:

The X-MEN and S.H.I.E.L.D and the Brotherhood of Mutants and HYDRA defeat Thanos in the next Marvel crossover film.

(4.8)

This nested list is a third-level plural expression apparently denoting a plurality of pluralities of pluralities. In each of these two sentences the list-forming operator ‘and’ maps two or more n-level terms to an n+1-level term. Some have claimed that there are no higher-level

²Standard plural languages can represent the distinction between first- and higher-level plurals, too, though at the expense of syntactic simplicity. Cf. Rayo 2006.
plural constructions in English (e.g., McKay 2006 and Rayo 2006), and even those who recognize the existence of second-level plurals in English doubt the existence of third- and higher level plural constructions (e.g., Oliver and Smiley 2013). In principle, though, list-forming ‘and’ can be used to construct plural expressions of any level in English. Higher-level lists are no doubt cumbersome, but that is a matter of pragmatics, not syntax or semantics.

It is important to note the distinction between grammatically higher-level plurals and referentially higher-level plurals here. One can agree that the list-forming operator can be used to construct grammatically higher-level plural terms but still deny that there is such a thing as semantically higher-level plural terms. We take it that the grammatical point is settled by examples like the above sentence—there are grammatically acceptable third- (and higher) level superplurals in English. The intended interpretation also suggests that the semantics of English allows for the denotation of higher-level pluralities.

There is some reason to prefer a language that does not allow denotation (or quantification) beyond the first plural level. Some ontologically scrupulous philosophers prefer plural languages to higher-order singular languages because plural quantification (and denotation) apparently ranges over only individuals. No ‘spooky’ abstract objects are invoked by plural quantification, since collections are not treated as individuals by the first-level plural apparatus. A higher-level plural semantics threatens to do away with this perceived ontological innocence.

4.2 Lists in Formal Languages

4.2.1 Lists as Terms

Lists are plural referring expressions formed from finitely many terms. Their properties largely depend on the properties of the terms from which they are formed. For example, a list formed from individual constants denotes finitely-many objects. A list formed from plural constants, on the other hand, will denote less-than or equal to the number of things denoted
by its constituent constants or, if the plural terms denote at least one infinite plurality, the
greatest limit cardinal of the constituent terms.

The central syntactic feature of any list logic is its variably-polyadic list-forming oper-
ator, which we will represent using square brackets: ‘[...].’ Just as names can be strung
together with commas and ‘and’ to form lists in ordinary English and mathematical dis-
course, so individual constant symbols may be used to form lists by concatenating them and
surrounding the resulting string with square brackets. Thus our toy list, ‘Ann and Bob’, can
be represented as ‘[ab]’.

Because the list-forming operator is often syntactically indistinguishable from the
sentence-forming operator ‘and’ in English, sentences involving lists are sometimes ambiguous
in English. Consider, for example, ‘Ann and Bob and Carol and Dan lifted the piano.’ It
is unclear whether this sentence is equivalent to ‘Ann, Bob, Carol, and Dan lifted the
piano,’ where everyone joined in the effort, or ‘Ann and Bob lifted the piano and Carol and
Dan lifted the piano,’ where the lifting was done collectively among the members of each
two-person group but there was no cooperation between groups. A properly constructed
list logic will not have ambiguous formulas. The first case can be represented using the list
‘P[abcd];’ the second case can be represented using ‘P[ab] & P[cd].’ Ideally, these expressions
will be neither model-theoretically nor proof-theoretically equivalent.

Treating lists as terms allows the formation of nested lists like ‘[[ab][cd]].’ Nested lists can
be used to represent higher-order lists that sometimes occur in English, like ‘The joint authors
of multivolume classics on logic are Whitehead and Russell, and Hilbert and Bernays.’ As
Oliver and Smiley explain it, “the description and the nested list [in the previous sentence]
are both second-level plurals. They are not ordinary first-level plural terms that simply
denote the four men; instead they denote two pairs of men.” (2013:28) One may increase or
decrease the expressive power of a list logic by allowing or disallowing the construction of

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3We will usually treat the list-forming operator in English as a function from terms to terms. For a discussion of whether the list-forming operator is a function or a form of punctuation, see Oliver and Smiley 2013, ch.10.
nested lists in one’s formation rules, or by allowing or disallowing the collapse of lists in the axioms or semantics. One can also extend a higher-order language by allowing lists to be formed from predicates, plural terms, or combinations of individual and higher-order terms.

Lists in English have a structure similar to unordered sets. Lists are commutative—‘Ann and Bob lifted the piano’ is semantically equivalent to ‘Bob and Ann lifted the piano.’ Lists are also idempotent, since ‘Ann and Bob lifted the piano’ is semantically equivalent to ‘Ann lifted the piano’ if ‘Ann’ and ‘Bob’ corefer. When nested lists are used to refer to higher-level plurals they are not associative. Even though it is true that ‘Russell and Oliver and Smiley have written works on plural logic’ when the list braces associate to the right, it is false when the list braces associate to the left. Because of these structural similarities the semantics of lists can be represented naturally in a standard model-theoretic system.

Empty lists and singleton lists are odd limit cases that one may or may not wish to include in a list logic. Empty lists can be advantageous, since they can be used to represent co-partial functions and empty reference in general. On its ordinary meaning there is no such thing as a singleton list–lists always contain more than one name. But even if lists of one entry are allowed, it seems clear that such a list would refer to whatever its entry refers. One may therefore either block the construction of singleton lists or simply ensure that they are semantically and proof-theoretically equivalent to their entry.

4.2.2 Predication

One can simulate multigrade predication by allowing lists to combine with predicates of definite adicity. Some English predicates, like ‘is composed of,’ seem to have variable adicity, since their argument places can accept variably many terms. ‘My body is composed of my arms, legs, head and torso’ and ‘My body is composed of my left side and my right side’ are perfectly well-formed and seem to token the same predicate. FOL cannot represent these tokens as of the same type; either it must make the mistake of treating composition as a distributive binary relation and treat the list-forming ‘and’ as a conjunction (see section 1), or
it must treat the composition predicates as distinct—the 5-place, the second 3-place. Neither option is satisfactory. A list logic can treat composition as an equivocal binary relation—it can represent the first sentence as ‘Ca[bcde]’ and the second as ‘Ca[fg]’.

In addition to the list-forming operator it is convenient to include a two-place inclusion relation, which we represent using the squared horseshoe, ‘□.’ The inclusion symbol can be pronounced as ‘is/are among’ and is similar to the inclusion predicate of plural logic. Presumably, the list inclusion relation will be defined along the lines of the subset relation; i.e. if every constant occurring in a list A occurs in list B, then A is/are among B. This generates a partial ordering of the pluralities on which the semantics of the language is defined. Hence, one will not go too wrong in thinking of the inclusion relation as an analogue of the subset relation. However, in a language restricted to first-level pluralities every individual constant is included in any list in which it occurs. In such languages the inclusion predicate is better understood as a parthood relation.\(^4\)

On the most straightforward approach to the semantics of plural languages, all predicates are non-distributive by default. As we saw in the first section of the paper, the fact that a predicate is satisfied by a list in no way guarantees that the predicate is satisfied by any individual named in the list. It is often desirable to have a predicate be distributive in one or more of its places. If our language includes an inclusion predicate we may define a predicate as distributive using a sentence like the following, which says the monadic predicate \(A\) is distributive.

\[
\forall x (Ax \rightarrow \forall y (y \sqsubset x \rightarrow Ay))
\] (4.9)

Distributive predicates of greater adicity can be defined similarly.

\(^4\)The set-theoretic approach to the semantics of plural languages is somewhat controversial. Some advocates of plural logic argue that sets are not well-suited to represent pluralities as plural. These issues do not concern us here. While we think that sets can be interpreted singularly or pluraly, we also think that nothing of substance hangs on the debate with regard to formal modelling. See McKay 2006, Cotnoir 2013, and Oliver and Smiley 2013 for discussion.
4.2.3 Quantification

Languages that include lists are most naturally paired with plural quantifiers, since lists will be possible instances of quantified variables in these languages. This can have some counterintuitive consequences, depending on the kinds of pluralities the quantifiers are allowed to range over. For example, it is a theorem of FPL (the list logic presented in the next section) that it is not the case that there are exactly two things. This theorem is a consequence of letting the quantifier range over all non-empty pluralities—if there are two things \( a \) and \( b \) then there is also a plurality \( a \) and \( b \). As a result, if there is a model of an FPL sentence that says there are exactly \( n \) many things, then either \( n = 1 \) or \( n = k! + 1 \) for some integer \( k > 1 \).

There is an easy fix for when the narrower quantification of singular languages is intended. Define the singular existential quantifier as

\[
\exists_S x P =_d \exists x (\forall y (y ⊏ x \rightarrow y = x) \& P)
\]

(4.10)

where \( y \) is not \( x \) and no \( y \)-quantifier occurs in \( P \). Likewise, define the singular universal quantifier as

\[
\forall_S x P =_d \forall x (\forall y (y ⊏ x \rightarrow y = x) \rightarrow P)
\]

(4.11)

where again \( y \) is not \( x \) and no \( y \)-quantifier occurs in \( P \).

Logics equipped with an inclusion predicate and plural quantifiers have some features reminiscent of second-order languages. For example, in such a language one can represent the famous ‘nonfirstorderizable’ Geach-Kaplan sentence, ‘Some critics admire only one another’ as

\[
\exists x (Cx \& \forall_S y (Ay y \rightarrow y ⊏ x))
\]

(4.12)
which is pronounced “There are some things that are critics, and those critics are such that they admire something only if it is one of them.” If the language includes function symbols one can likewise express other sentences that apparently use second-order quantification, like the mathematical induction principle

\[ \forall x(0 \sqsubseteq x \land \forall y (y \sqsubseteq x \rightarrow s(y) \sqsubseteq x) \rightarrow \forall z z \sqsubseteq x) \] (4.13)

and the comprehension scheme

\[ \exists y \mathcal{P} \rightarrow \exists x \forall y (y \sqsubseteq x \leftrightarrow \mathcal{P}) \] (4.14)

where x does not occur free in \( \mathcal{P} \).

Not every reasonable language that allows these sentences to be constructed will count them as true. For example, the variables of FPL-, which is discussed in section 8.3 below, only range over the pluralities that can be listed using individual constants, namely the finite pluralities. So even though a list logic can express something very similar to the mathematical induction principle in (11), it will count this sentence as false if it does not countenance infinite pluralities. Similarly, the comprehension principle in (12) is contingent in many list logics, since there may be infinitely many things in a domain that satisfy \( \mathcal{P} \), in which case the \( \mathcal{P}s \) will not be listable.

4.2.4 Identity

As in FOL, the identity predicate of list logics is intended to model coreference. Both individual terms and lists can corefer. For example, since ‘Batman’ and ‘Bruce Wayne’ denote the same thing, the list ‘Batman and Bruce Wayne’ has the same denotation as ‘Bruce
Wayne and Batman,’ i.e., the same thing that the individual terms denote. Identity can also hold between lists of constants that do not corefer, as is the case in the English sentence ‘Bruce Wayne and Clark Kent are Batman and Superman.’ Many-one identity is allowed in a trivial sense. We can express identities between individual terms and lists, but such identities are only true if the terms in the list are coreferential. For example, we can express both ‘Superman and the Man of Steel are Clark Kent’ and ‘Superman and Batman are Clark Kent,’ though only the first is true.\(^5\) This feature of list logics has applications in mathematical reasoning that aren’t served by FOL or standard plural languages. For example, in a list logic that includes function symbols we can express ‘\(\sqrt{4} = 2\) and \(-2\)’ as ‘\(f(4) = [2, -2]\).’ Examples like this, together with the option of including the empty list as a term, show that list logics are especially well-suited for developing theories of functions that account for partiality, co-partiality, and plurality.

4.2.5 Metatheory

By itself, the semantic and proof-theoretic machinery necessary to support the list-forming operator does not impose any particular metatheoretic constraints on a language. Likewise, plural quantification alone does not extend the expressive power of singular formal languages. This is because, in principle, plural terms need not bear any model- or proof-theoretic relationships to one another, and so sentences involving plural quantification without interpreted predicates relating terms can be reinterpreted in languages without plural terms or quantifiers. It is only when the inclusion predicate is given its intended interpretation that the syntactic relationships between terms are made semantically explicit and the resulting language gains nontrivial expressive power over its unextended counterpart.

In the following sections we demonstrate how these considerations play out in practice. We call the logic FPL, for finite plural logic. FPL is an extension of classical first-order logic with identity that includes first-level lists, first-order plural quantification, and an inclusion

\(^5\)Thus, the kind of many-one identity that some think holds between a composite object and its parts is not captured by identities between lists. See Cotnoir 2013 for discussion.
predicate. For simplicity’s sake, FPL does not include empty lists, higher-level plural nesting, or function symbols and does not allow variables to appear in lists.

4.3 The Syntax of FPL

Def 4.3.1. The vocabulary of FPL consists in the following:

1. The predicate letters of FPL are the capitalized Roman letters ‘A’–‘Z’ with or without positive integer subscripts but in every case followed by a positive integer superscript. A predicate letter of FPL followed by superscript \( n \) is an \( n \)-ary predicate letter of FPL. (e.g., \( C_2^3 \) is a binary predicate letter.)

2. The constants of FPL are the lower-case Roman letters ‘a’–‘v’ with or without positive integer subscripts.

3. The variables of FPL are the lower-case Roman letters ‘w’–‘z’ with or without positive integer subscripts.

4. The list-forming braces of FPL are the square braces ‘[’ and ‘]’.

5. The inclusion predicate of FPL is ‘\( \sqsubseteq \)’.

6. The identity predicate of FPL is ‘\( = \)’.

7. The connectives of FPL are ‘\( \neg \)’ and ‘\( \rightarrow \)’.

8. The quantifier symbol of FPL is ‘\( \forall \)’.

9. The punctuation marks of FPL are the curved braces ‘(’ and ‘)’.

Def 4.3.2. The metalanguage for discussing FPL consists in Mathematical English and the following vocabulary:

1. The script letters ‘\( \mathcal{P} \)’, ‘\( \mathcal{Q} \)’, ‘\( \mathcal{R} \)’, ‘\( \mathcal{S} \)’, and ‘\( \mathcal{T} \)’ are metavariables ranging over strings of FPL vocabulary.
2. The script letter ‘\(\mathcal{A}\)’, with or without a subscript, is a metavariable ranging over predicate letters of FPL.

3. The bold lower-case letters ‘\(a\)’ and ‘\(b\)’, with or without positive integer subscripts, are metavariables ranging over the constants of FPL.

4. The bold lower-case letters ‘\(x\)’ and ‘\(y\)’, with or without positive integer subscripts, are metavariables ranging over the variables of FPL.

5. The bold lower-case letter ‘\(l\)’, with or without positive integer subscripts, is a metavariable ranging over the lists of FPL.

6. The bold lower-case letter ‘\(c\)’, with or without positive integer subscripts, is a metavariable ranging over the closed terms of FPL.

7. The bold lower-case letter ‘\(t\)’, with or without positive integer subscripts, is a metavariable ranging over the terms of FPL.

**Def 4.3.3.** The set of terms of FPL is \(T = V \cup C\), where \(V\) and \(C\) are the smallest sets such that

1. \(x \in V\), for every variable \(x\);

2. \(a \in C\), for every constant \(a\);

3. if \(t_1, \ldots, t_n \in C\) where \(n \geq 1\), then \([t_1 \ldots t_n] \in C\).

Variables and constants are called *simple* terms; non-simple terms are called *lists*. A list formed from \(n\)-many terms has *length* \(n\). A constant has a *height* of 0; a list of terms has a height of \(n + 1\), where \(n\) is the greatest height of any term in the list. The *depth* of a term is the number of lists the term occurs within. If \(t \in C\), then \(t\) is a *closed* term. If a term is not closed it is an *open* term.
Def 4.3.4. A *quantifier* of FPL is any string of the form ‘∀x’ or ‘∃x’, where x is a variable. Any quantifier containing the variable x is an *x-quantifier*.

Def 4.3.5. \( P \) is an *atomic formula* of FPL iff \( P \) is an expression of the form \( At_1 \ldots t_n \), where \( A \) is an \( n \)-ary predicate letter of FPL and \( t_1, \ldots, t_n \) are terms of FPL.

Def 4.3.6. The set of *formulas* of FPL is the smallest set \( F \) containing every atomic formula of FPL and satisfying the following conditions:

1. If \( P \in F \), then \( \neg P \in F \).
2. If \( P, Q \in F \), then \( (P \rightarrow Q) \in F \).
3. If \( P \in F \) and \( x \) occurs in \( P \) but no \( x \)-quantifier occurs in \( P \), then \( \forall x P \in F \).

Def 4.3.7. If \( P \) and \( Q \) are formulas of FPL, then:

1. \( (P \& Q) =_{df} \neg(P \rightarrow \neg Q) \);
2. \( (P \lor Q) =_{df} (\neg P \rightarrow Q) \);
3. \( (P \leftrightarrow Q) =_{df} ((\neg P \rightarrow Q) \rightarrow \neg(P \rightarrow \neg Q)) \);
4. \( \exists x P =_{df} \neg \forall x \neg P \).

Def 4.3.8. The *scope* of a logical operator is the shortest formula in which the operator occurs. An occurrence of a variable \( x \) in a formula \( P \) is *bound* iff it is within the scope of an \( x \)-quantifier in \( P \). A bound occurrence of a variable is *bound by* the quantifier in whose scope it falls. An occurrence of a variable is *free* in a formula \( P \) iff it is not bound.

Def 4.3.9. A *sentence* of FPL is a formula of FPL containing no free occurrence of any variable.

Def 4.3.10. Where \( P \) is a formula of FPL in which the variable \( x \) occurs free, \( P(t/x) \) is the formula that results from uniformly substituting \( t \) for every free occurrence of \( x \) in \( P \).
Convention 4.3.1. Mention quotes may be left undisplayed when a formula is mentioned using its canonical name.

Convention 4.3.2. Outer parentheses of formulas whose main operator (i.e., the logical operator with widest scope) is binary may be left undisplayed (e.g., ‘(A&B)’ may be written as ‘A&B’).

Convention 4.3.3. Superscripts of predicates may be left undisplayed when there is no danger of ambiguity.

Convention 4.3.4. Formulas of the form \( t_1 t_2 \) or \( t_1 t_2 \) may be displayed as \( t_1 = t_2 \) or \( t_1 \sqsubseteq t_2 \), respectively.

Convention 4.3.5. Formulas of the form \( \neg t_1 t_2 \) or \( \neg t_1 t_2 \) may be displayed as \( t_1 \neq t_2 \) or \( t_1 \not\subseteq t_2 \), respectively.

4.4 Semantics for FPL

Def 4.4.1. A model of FPL is an ordered triple \( \langle U, \hat{U}, v \rangle \), where

1. \( U \) is a nonempty set;

2. \( \hat{U} \subseteq \mathcal{P}(U) \) such that:
   (a) \( \hat{U} \) contains at least one singleton, and
   (b) for any singletons \( A_1, \ldots, A_n \in \hat{U} \), \( \bigcup A_1, \ldots, A_n \in \hat{U} \); and

3. \( v \) is a function that meets the following conditions:
   (a) \( v(a) \) is a singleton in \( \hat{U} \) for every constant \( a \) of FPL;
   (b) \( v(A) \subseteq \hat{U}^n \) for every \( n \)-ary predicate letter \( A \) of FPL.

If \( \mathcal{I} \) is a model of FPL and \( \mathcal{I} = \langle U, \hat{U}, v \rangle \), then \( U_\mathcal{I} = U, \hat{U}_\mathcal{I} = \hat{U} \) and \( v_\mathcal{I} = v \).
Def 4.4.2. An assignment (of values to variables) \( d \) for a model \( I \) is any function that assigns to each variable \( x \) of FPL a member \( d(x) \) of \( \hat{U} \).

Def 4.4.3. For any model \( I \), if \( u \in \hat{U}_I \) and \( d \) is an assignment for \( I \), then the assignment \( d(u/x) \) is defined as follows: for every variable \( x' \),

\[
d(u/x)(x') = \begin{cases} 
  u & \text{if } x' \text{ is the variable } x, \text{ and} \\
  d(x') & \text{otherwise.}
\end{cases}
\]

Define \( d(u_1/x_1, \ldots, u_n/x_n) = d(u_1/x_1) \ldots (u_n/x_n) \).

Def 4.4.4. Satisfaction is a relation that holds between a model of FPL, an assignment for that model, and a formula of FPL. We write \( d, I \models P \) for “\( d \) satisfies \( P \) relative to \( I \”).

1. For any term \( t \) of FPL and any model \( I \) and any assignment \( d \) for \( I \), define \( r(t, d, I) \) as follows:

\[
r(t, d, I) = \begin{cases} 
  v_I(t) & \text{if } t \text{ is a constant;} \\
  d(t) & \text{if } t \text{ is a variable;} \\
  \bigcup r(t_1, d, I), \ldots, r(t_n, d, I) & \text{if } t = [t_1 \ldots t_n].
\end{cases}
\]

If \( P \) is the atomic formula \( \mathcal{A}t_1 \ldots t_n \), where \( t_1, \ldots, t_n \) are terms of FPL, then \( d, I \models P \) iff \( \langle r(t_1, d, I), \ldots, r(t_n, d, I) \rangle \in v_I(\mathcal{A}) \).

2. If \( P \) is the atomic formula \( t_1 = t_2 \), then \( d, I \models P \) iff \( r(t_1, d, I) = r(t_2, d, I) \).

3. If \( P \) is the atomic formula \( t_1 \sqsubseteq t_2 \), then \( d, I \models P \) iff \( r(t_1, d, I) \subseteq r(t_2, d, I) \)

4. If \( P \) and \( Q \) are formulas of FPL, then:

   (a) \( d, I \models \neg P \) iff \( d, I \not\models P \);

   (b) \( d, I \models P \rightarrow Q \) iff \( d, I \not\models P \) or \( d, I \models Q \);
5. \( d, \mathcal{I} \models \forall x \mathcal{P} \) iff \( d(u/x), \mathcal{I} \models \mathcal{P} \) for every \( u \in \mathcal{U}_\mathcal{I} \).

**Def 4.4.5.** A sentence \( \mathcal{P} \) is true on a model \( \mathcal{I} \) iff \( d, \mathcal{I} \models \mathcal{P} \) for every assignment \( d \) defined on \( \mathcal{I} \).

**Def 4.4.6.** A sentence \( \mathcal{P} \) is false on a model \( \mathcal{I} \) iff \( d, \mathcal{I} \not\models \mathcal{P} \) for every assignment \( d \) defined on \( \mathcal{I} \).

**Def 4.4.7.** A set of sentences \( \Gamma \) quantificationally entails a sentence \( \mathcal{P} \) (\( \Gamma \models \mathcal{P} \)) iff \( \mathcal{P} \) is true on every model on which all members of \( \Gamma \) are true.

**Def 4.4.8.** A set of sentences \( \Gamma \) is quantificationally consistent (\( \Gamma \not\models \)) iff there is a model on which all members of \( \Gamma \) are true.

**Def 4.4.9.** A sentence \( \mathcal{P} \) of FPL is quantificationally true (\( \models \mathcal{P} \)) iff \( \mathcal{P} \) is true on every model of FPL.

### 4.4.1 Semantic Theorems

Here we prove some useful theorems about satisfaction and truth on a model in FPL.

The first theorem says that the denotation of a list of any depth is the union of the denotations of the constants occurring at any depth in the list. Hence, the value of a list of height \( n > 1 \) is always the value of the corresponding ‘flattened’ list of height 1. (e.g., \([\text{Alice}, [\text{Bob}, [\text{Carol}]]] \) receives the same denotation as \([\text{Alice}, \text{Bob}, \text{Carol}]\).) First, we prove a lemma.

**Lemma 4.4.1.1.** If \( \mathbf{l} \) is a list containing an occurrence of a list \( \mathbf{l}' \) at depth 1, then \( r(\mathbf{l}, d, \mathcal{I}) = r(\mathbf{l}', d, \mathcal{I}) \), where \( \mathbf{l}' \) is \( \mathbf{l} \) with the outer braces of \( \mathbf{l}' \) deleted. (example: \( r([a[bc]], d, \mathcal{I}) = r([abc], d, \mathcal{I}) \).)
Proof. Suppose \( l \) is a list of length \( m \) and \([c_1 \ldots c_n]\) occurs at depth 1 in \( l \). Then \( l \) is \([c'_1 \ldots [c_1 \ldots c_n] \ldots c'_m]\). So,

\[
r(l, d, \mathcal{I}) = r([c'_1 \ldots [c_1 \ldots c_n] \ldots c'_m], d, \mathcal{I})
\]

is

\[
\bigcup r(c'_1, d, \mathcal{I}), \ldots, r([c_1 \ldots c_n], d, \mathcal{I}), \ldots, r(c'_m, d, \mathcal{I})
\]

is

\[
\bigcup r(c'_1, d, \mathcal{I}), \ldots, \bigcup r(c_1, d, \mathcal{I}), \ldots, r(c_n, d, \mathcal{I}), \ldots, r(c'_m, d, \mathcal{I})
\]

is

\[
r([c'_1 \ldots c_1 \ldots c_n \ldots c'_m], d, \mathcal{I})
\]

is

\[
r(l^*, d, \mathcal{I}).
\]

\[\square\]

**Theorem 4.4.1.1** (Collapse). For all lists \( l \), \( r(l, d, \mathcal{I}) = r(l', d, \mathcal{I}) \), where \( l' \) is the same string as \( l \) but with all but the first occurrence of ‘[’ and the last occurrence of ‘]’ deleted.

*Proof. The theorem follows by \( n \) applications of Lemma 4.1.1, where \( n \) is the number of lists occurring at any depth within \( l \).*

\[\square\]

**Corollary 4.4.1.1.** For all lists \( l \), \( r(l, d, \mathcal{I}) = \bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I}) \), where \( a_1, \ldots, a_n \) are the constants occurring at any depth in \( l \).

*Proof. It follows trivially from Theorem 4.1.1 that the denotation of any list is the union of the denotations of the constants occurring at any depth in the list. The denotation of any list therefore depends solely on the values of the constants occurring in the list.*

\[\square\]

**Theorem 4.4.1.2.** For any model \( \mathcal{I} \) and formula \( \mathcal{P} \), if \( d \) and \( d' \) are assignments for \( \mathcal{I} \) that assign the same value to every variable occurring free in \( \mathcal{P} \), then \( d, \mathcal{I} \models \mathcal{P} \) iff \( d', \mathcal{I} \models \mathcal{P} \).
Proof. By strong induction on the length of $\mathcal{P}$.

(Basis) Let $\mathcal{I}$ be a model and let $d$, $d'$ be assignments for $\mathcal{I}$ that agree on the free variables in $\mathcal{P}$. Suppose $\mathcal{P}$ is of length 0. $\mathcal{P}$ is either $\mathcal{A}t_1 \ldots t_n$, $t_1 = t_2$, or $t_1 \sqsubseteq t_2$. For each case it suffices to show that for $1 \leq i \leq n$, $r(t_i, d, \mathcal{I})$ is $r(t_i, d', \mathcal{I})$.

Case 1. Suppose $t_i$ is a constant. Then $r(t_i, d, \mathcal{I})$ is $v_{\mathcal{I}}(t_i)$. But $v_{\mathcal{I}}(t_i)$ is $r(t_i, d', \mathcal{I})$.

Case 2. Suppose $t_i$ is a variable. Then $d(t_i)$ is $d'(t_i)$. But $r(t_i, d, \mathcal{I})$ is $d(t_i)$, and $r(t_i, d', \mathcal{I})$ is $d'(t_i)$. So, $r(t_i, d, \mathcal{I})$ is $r(t_i, d', \mathcal{I})$.

Case 3. Suppose $t_i$ is a list. Then by Corollary 4.1.1, $r(t_i, d, \mathcal{I})$ is $\bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I})$ for constants $a_1, \ldots, a_n$ occurring in $t_i$. By Case 1, $\bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I})$ is $\bigcup r(a_1, d', \mathcal{I}), \ldots, r(a_n, d', \mathcal{I})$. But $\bigcup r(a_1, d', \mathcal{I}), \ldots, r(a_n, d', \mathcal{I})$ is $r(t_i, d', \mathcal{I})$. So, $r(t_i, d, \mathcal{I})$ is $r(t_i, d, \mathcal{I})$.

In each case, $r(t_i, d, \mathcal{I})$ is $r(t_i, d', \mathcal{I})$.

(Induction Step) Suppose the theorem holds for all formulas of length $k$ or less. Let $\mathcal{P}$ be of length $k + 1$. Let $\mathcal{I}$ be a model of FPL and let $d$, $d'$ be assignments for $\mathcal{I}$ that agree on the free variables in $\mathcal{P}$. There are three cases.

Case 1. Suppose $\mathcal{P}$ is $\neg Q$. Then $d$, $d'$ agree on the variables free in $Q$. Hence,
\[ d, \mathcal{I} \models P \text{ iff } d, \mathcal{I} \notmodels Q \]

iff \( d, \mathcal{I} \notmodels Q \)

iff \( d', \mathcal{I} \notmodels Q \) (by the induction hypothesis)

iff \( d', \mathcal{I} \models \neg Q \)

iff \( d', \mathcal{I} \models P \)

Case 2. Suppose \( P \) is \( Q \rightarrow R \). Then \( d, d' \) agree on the variables free in \( Q \) and \( R \). Hence,

\[ d, \mathcal{I} \models P \text{ iff } d, \mathcal{I} \models Q \rightarrow R \]

iff \( d, \mathcal{I} \notmodels Q \) or \( d, \mathcal{I} \models R \)

iff \( d', \mathcal{I} \notmodels Q \) or \( d', \mathcal{I} \models R \) (by the ind. hyp.)

iff \( d', \mathcal{I} \models Q \rightarrow R \)

iff \( d', \mathcal{I} \models P \)

Case 3. Suppose \( P \) is \( \forall x Q \) where no \( x \)-quantifier occurs in \( Q \). Then for any \( u \in \hat{U}_{\mathcal{I}} \), \( d(u/x) \) and \( d'(u/x) \) agree on the variables free in \( Q \). Hence,

\[ d, \mathcal{I} \models P \text{ iff } d, \mathcal{I} \models \forall x Q \]

iff \( d(u/x), \mathcal{I} \models Q \) for every \( u \in \hat{U}_{\mathcal{I}} \)

iff \( d'(u/x), \mathcal{I} \models Q \) for every \( u \in \hat{U}_{\mathcal{I}} \) (by the ind. hyp.)

iff \( d', \mathcal{I} \models \forall x Q \)

iff \( d', \mathcal{I} \models P \)
So, by strong induction on the length of \( P \), it follows that \( d, \mathcal{I} \models P \) iff \( d', \mathcal{I} \models P \).

**Theorem 4.4.1.3.** For every model \( \mathcal{I} \) and sentence \( P \), if \( P \) is not true on \( \mathcal{I} \), then \( P \) is false on \( \mathcal{I} \).

**Proof.** Let \( \mathcal{I} \) be a model and let \( P \) be a sentence. Suppose \( P \) is not true on \( \mathcal{I} \). Then there is an assignment \( d_0 \) such that \( d_0, \mathcal{I} \not\models P \). Let \( d \) be an assignment for \( \mathcal{I} \). Since \( P \) is a sentence, it contains no free variables. Hence, \( d, d_0 \) agree on all the free variables in \( P \). So, by Theorem 4.1.2, \( d, \mathcal{I} \not\models P \). Generalizing on \( d \), \( P \) is false on \( \mathcal{I} \).

**Theorem 4.4.1.4.** Let \( \mathcal{I}, \mathcal{I}' \) be such that \( U_\mathcal{I} = U_\mathcal{I}', \hat{U}_\mathcal{I} = \hat{U}_\mathcal{I}' \), and such that \( v_\mathcal{I} \) and \( v_\mathcal{I}' \) assign the same values to every predicate letter, constant, and list occurring in a given formula \( P \). Then if \( \mathcal{I}, \mathcal{I}' \) are models such that \( v_\mathcal{I} \) and \( v_\mathcal{I}' \) assign the same values to every predicate letter, constant, and list occurring in a given formula \( P \). Then if \( d \) is an assignment for \( \mathcal{I} \) and for \( \mathcal{I}' \), then \( d, \mathcal{I} \models P \) iff \( d, \mathcal{I}' \models P \).

**Proof.** By strong induction on the length of \( P \).

*(Basis)* Let \( \mathcal{I}, \mathcal{I}' \) be models such that \( U_\mathcal{I} = U_\mathcal{I}', \hat{U}_\mathcal{I} = \hat{U}_\mathcal{I}' \), and such that \( v_\mathcal{I} \) and \( v_\mathcal{I}' \) assign the same values to every predicate letter, constant, and list occurring in a given formula \( P \). Let \( d \) be an assignment for \( \mathcal{I} \) and \( \mathcal{I}' \).

Suppose \( P \) is a formula of length 0. Then \( P \) is either \( \forall t_1 \ldots t_n, t_1 = t_2 \), or \( t_1 \sqsubseteq t_2 \). For each case it suffices to show that for \( 1 \leq i \leq n \), \( r(t_i, d, \mathcal{I}) \) is \( r(t_i, d, \mathcal{I}') \).

**Case 1.** Suppose \( t_i \) is a constant. Then \( r(t_i, d, \mathcal{I}) = v_\mathcal{I}(t_i) \). But \( v_\mathcal{I}(t_i) \) is \( v_\mathcal{I}'(t_i) \), which is \( r(t_i, d, \mathcal{I}') \).

**Case 2.** Suppose \( t_i \) is a variable. Then \( r(t_i, d, \mathcal{I}) = d(t_i) \), which is also \( r(t_i, d, \mathcal{I}') \). So, \( r(t_i, d, \mathcal{I}) \) is \( r(t_i, d, \mathcal{I}') \).
Case 3. Suppose \( t_i \) is a list. Then by Corollary 4.1.1, \( r(t_i, d, \mathcal{I}) \) is \( \bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I}) \) for constants \( a_1, \ldots, a_n \) occurring in \( t_i \). By Case 1, \( \bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I}) \) is \( \bigcup r(a_1, d, \mathcal{I}'), \ldots, r(a_n, d, \mathcal{I}') \). But \( \bigcup r(a_1, d, \mathcal{I}'), \ldots, r(a_n, d, \mathcal{I}') \) is \( r(t_i, d, \mathcal{I}') \). So, \( r(t_i, d, \mathcal{I}) \) is \( r(t_i, d, \mathcal{I}') \).

In each case, \( r(t_i, d, \mathcal{I}) \) is \( r(t_i, d, \mathcal{I}') \).

(Induction Step) Suppose the theorem holds for all formulas of length \( k \) or less. Let \( \mathcal{P} \) be a formula of length \( k + 1 \). Let \( \mathcal{I}, \mathcal{I}' \) be models such that \( U_{\mathcal{I}} = U_{\mathcal{I}'} \) and such that \( v_{\mathcal{I}} \) and \( v_{\mathcal{I}'} \) assign the same values to every predicate letter and constant occurring in a given formula \( \mathcal{P} \). Let \( d \) be an assignment for \( \mathcal{I} \) and \( \mathcal{I}' \). There are three cases.

Case 1. Suppose \( \mathcal{P} \) is \( \neg Q \). Then,

\[
d, \mathcal{I} \models \mathcal{P} \text{ iff } d, \mathcal{I} \models \neg Q \\
\text{iff } d, \mathcal{I} \not\models Q \\
\text{iff } d, \mathcal{I}' \not\models Q \text{ (by the induction hypothesis)} \\
\text{iff } d, \mathcal{I}' \models \neg Q \\
\text{iff } d, \mathcal{I}' \models \mathcal{P}
\]

Case 2. Suppose \( \mathcal{P} \) is \( Q \to R \). Then,
\[ d, \mathcal{I} \models P \text{ iff } d, \mathcal{I} \models Q \rightarrow R \]
\[ \text{iff } d, \mathcal{I} \models Q \text{ or } d, \mathcal{I} \models R \]
\[ \text{iff } d, \mathcal{I}' \not\models Q \text{ or } d, \mathcal{I}' \models R \text{ (by the ind. hyp.)} \]
\[ \text{iff } d, \mathcal{I}' \models Q \rightarrow R \]
\[ \text{iff } d, \mathcal{I}' \models P \]

Case 3. Suppose \( P \) is \( \forall x Q \) where \( Q \) contains \( x \) but no \( x \)-quantifier occurs in \( Q \). Then,

\[ d, \mathcal{I} \models P \text{ iff } d, \mathcal{I} \models \forall x Q \]
\[ \text{iff } d(u/x), \mathcal{I} \models Q \text{ for every } u \in \hat{U}_I \]
\[ \text{iff } d(u/x), \mathcal{I}' \models Q \text{ for every } u \in \hat{U}_I \text{ (by the ind. hyp.)} \]
\[ \text{iff } d(u/x), \mathcal{I}' \models Q \text{ for every } u \in \hat{U}_{I'} \text{ (since } \hat{U}_I = \hat{U}_{I'} \text{)} \]
\[ \text{iff } d, \mathcal{I}' \models \forall x Q \]
\[ \text{iff } d, \mathcal{I}' \models P \]

So, by strong induction on the length of \( P \), it follows that \( d, \mathcal{I} \models P \) iff \( d, \mathcal{I}' \models P \). \( \square \)

**Theorem 4.4.1.5.** For any model \( \mathcal{I} \), any assignment \( d \) for \( \mathcal{I} \): \( d(r(c,d,\mathcal{I})/x), \mathcal{I} \models P \) iff \( d, \mathcal{I} \models P(c/x) \).

**Proof.** By induction on the length of \( P \).

*(Basis)* Let \( \mathcal{I} \) be a model and let \( d \) be an assignment for \( \mathcal{I} \). Suppose \( P \) is of length 0. Then \( P \) is either \( \mathcal{A} t_1 \ldots t_n \), \( t_1 = t_2 \), or \( t_1 \sqsubseteq t_2 \).
In each case at least one $t_i$ is $x$, and so $\mathcal{P}(c/x)$ is either $\mathcal{A}t'_1 \ldots t'_n$, $t'_1 = t'_2$, or $t'_1 \sqsubseteq t'_2$ where

$$t'_i = \begin{cases} 
    t_i & \text{if } t_i \text{ is not } x \\
    c & \text{if } t_i \text{ is } x 
\end{cases}$$

For each case it suffices to show that for $1 \leq i \leq n$, $r(t'_i, d, \mathcal{I})$ is $r(t_i, d(r(c, d, \mathcal{I})/x), \mathcal{I})$.

**Case 1.** Suppose $t_i$ is a constant or a list. Then,

$$r(t'_i, d, \mathcal{I}) \text{ is } r(t_i, d, \mathcal{I})$$

is $r(t_i, d(r(c, d, \mathcal{I})/x), \mathcal{I})$.

**Case 2.** Suppose $t_i$ is a variable other than $x$. Then,

$$r(t'_i, d, \mathcal{I}) \text{ is } d(t'_i)$$

is $d(t_i)$

is $d(r(c, d, \mathcal{I})/x)(t_i)$

is $r(t_i, d(r(c, d, \mathcal{I})/x), \mathcal{I})$.

**Case 3.** Suppose $t_i$ is $x$. Then $t'_i$ is $c$. So,

$$r(t'_i, d, \mathcal{I}) \text{ is } r(c, d, \mathcal{I})$$

is $d(r(c, d, \mathcal{I})/x)(x)$

is $r(t_i, d(r(c, d, \mathcal{I})/x), \mathcal{I})$. 
In each case $r(t',d,I)$ is $r(t_i,d,r(c,d,I)/x),I)$.

*(Induction Step)* Suppose the theorem holds for all formulas of length $k$ or less. Let $P$ be a formula of length $k + 1$ containing a free occurrence of $x$. Let $I$ be a model and let $d$ be an assignment for $I$. Then $P$ is either $\neg Q$, $Q \rightarrow R$, or $\forall y P$ where $y$ is not $x$.

**Case 1.** Suppose $P$ is $\neg Q$. Then $P(c/x)$ is $\neg Q(c/x)$. So,

$$d,I \models P(c/x) \text{ iff } d,I \models \neg Q(c/x)$$

$$\text{ iff } d,I \not\models Q(c/x)$$

$$\text{ iff } d(r(c,d,I)/x),I \not\models Q \text{ (by the induction hypothesis)}$$

$$\text{ iff } d(r(c,d,I)/x),I \models \neg Q$$

$$\text{ iff } d(r(c,d,I)/x),I \models P$$

**Case 2.** Suppose $P$ is $Q \rightarrow R$. Then $P(c/x)$ is either $Q(c/x) \rightarrow R(c/x)$, $Q(c/x) \rightarrow R$, or $Q \rightarrow R(c/x)$.

**Subcase 2.1.** Suppose $P(c/x)$ is $Q(c/x) \rightarrow R(c/x)$. Then

$$d,I \models P(c/x) \text{ iff } d,I \models Q(c/x) \rightarrow R(c/x)$$

$$\text{ iff } d,I \not\models Q(c/x) \text{ or } d,I \models R(c/x)$$

$$\text{ iff } d(r(c,d,I)/x),I \not\models Q \text{ or } d(r(c,d,I)/x),I \models R \text{ (by the ind. hyp.)}$$

$$\text{ iff } d(r(c,d,I)/x),I \models Q \rightarrow R$$

$$\text{ iff } d(r(c,d,I)/x),I \models P$$
Subcase 2.2. Suppose $\mathcal{P}(c/x)$ is $\mathcal{Q}(c/x) \rightarrow \mathcal{R}$, where $x$ does not occur free in $\mathcal{R}$. Then

\[d, I \models \mathcal{P}(c/x) \text{ iff } d, I \models \mathcal{Q}(c/x) \rightarrow \mathcal{R}\]

iff $d, I \nvDash \mathcal{Q}(c/x)$ or $d, I \models \mathcal{R}$

iff $d(r(c, d, I)/x), I \nvDash \mathcal{Q}$ or $d, I \models \mathcal{R}$ (by the ind. hyp.)

iff $d(r(c, d, I)/x), I \models \mathcal{Q} \rightarrow \mathcal{R}$ (by Theorem 4.1.2)

iff $d(r(c, d, I)/x), I \models \mathcal{R}$ (by the ind. hyp.)

iff $d(r(c, d, I)/x), I \models \mathcal{P}$

Subcase 2.3. Similar to subcase 2.2.

Case 3. Suppose $\mathcal{P}$ is $\forall y \mathcal{Q}$ where $y$ is not $x$ and $x$ is not free in $\mathcal{Q}$. Then,

\[d, I \models \mathcal{P}(c/x) \text{ iff } d, I \models \forall y \mathcal{Q}\]

iff $d(u/y), I \models \mathcal{Q}$ for every $u \in \hat{U}_I$

iff $d(u/y, r(c, d, I)/x), I \models \mathcal{Q}$ for every $u \in \hat{U}_I$ (by the ind. hyp.)

iff $d(r(c, d, I)/x), u/y), I \models \mathcal{Q}$ for every $u \in \hat{U}_I$ (since $x$ is not $y$)

iff $d(r(c, d, I)/x), I \models \forall y \mathcal{Q}$

iff $d(r(c, d, I)/x), I \models \mathcal{P}$

So, by strong induction on the length of $\mathcal{P}$, it follows that $d, I \models \mathcal{P}(c/x)$ iff $d(r(c, d, I)/x), I \models \mathcal{P}$.

\[\square\]
4.5 An Axiom System for FPL

Def 4.5.1. The axioms of FPL are all of the universalizations of all of the FPL sentences having one of the following schemata as their form. A sentence is a universalization of $\mathcal{P}$ only if it satisfies the following rules: (i) $\mathcal{P}$ is a universalization of $\mathcal{P}$, and (ii) if $\mathcal{Q}$ is a universalization of $\mathcal{P}$, then $\forall x \mathcal{Q}(x/c)$ is a universalization of $\mathcal{P}$.

1. Propositional Axiom Schemata

   (a) $\mathcal{P} \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})$
   (b) $(\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow ((\mathcal{P} \rightarrow (\mathcal{Q} \rightarrow \mathcal{R})) \rightarrow (\mathcal{P} \rightarrow \mathcal{R}))$
   (c) $(\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow ((\mathcal{P} \rightarrow \neg \mathcal{Q}) \rightarrow \neg \mathcal{P})$
   (d) $(\neg \mathcal{P} \rightarrow \mathcal{Q}) \rightarrow ((\neg \mathcal{P} \rightarrow \neg \mathcal{Q}) \rightarrow \mathcal{P})$

2. Quantifier Axiom Schemata

   (a) $\forall x(\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow (\mathcal{P} \rightarrow \forall x \mathcal{Q})$
   (b) $\forall x(\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow (\forall x \mathcal{P} \rightarrow \forall x \mathcal{Q})$
   (c) $\forall x \mathcal{P} \rightarrow \mathcal{P}(c/x)$

3. Identity Axiom Schemata

   (a) $c = c$
   (b) $c_1 = c_2 \rightarrow (\mathcal{P}(c_1/x) \rightarrow \mathcal{P}(c_2/x))$

4. Inclusion Axiom Schemata

   (a) $c_1 \sqsubseteq c_2$, where $c_1$ occurs in $c_2$
(b) $c_1 \sqcup c_2 \rightarrow (c_2 \sqcup c_1 \rightarrow c_1 = c_2)$

(c) $c_1 \sqcup c_2 \rightarrow (c_2 \sqcup c_3 \rightarrow c_1 \sqcup c_3)$

5. List Axiom Schemata

(a) $c = [c]$

(b) $c_1 \sqcup c_2 \rightarrow c_2 = [c_1 c_2]$

(c) $[\ldots [c_m \ldots c_n] \ldots] = [\ldots c_m \ldots c_n \ldots]$

(d) $[c_1 \ldots c_i c_{i+1} \ldots c_n] = [c_1 \ldots c_{i+1} c_i \ldots c_n]$

**Def 4.5.2.** *Modus Ponens* (MP) is the inference rule: Given $P$ and $P \rightarrow Q$, infer $Q$.

**Def 4.5.3.** A derivation $\delta$ of sentence $P$ from a set of sentences $\Gamma$ is a finite sequence $\langle P_1, \ldots, P_n \rangle$ where $P_n$ is $P$ and for all $i \leq n$, either

1. $P_i$ is an axiom,

2. $P_i \in \Gamma$, or

3. there exist $j, k < i$ such that $P_i$ follows by MP from $P_j$ and $P_k$.

**Def 4.5.4.** $\Gamma \vdash P$ ($\Gamma$ derives $P$) iff there is a derivation of $P$ from members of $\Gamma$.

**Def 4.5.5.** $\vdash P$ ($P$ is a theorem) iff $\emptyset \vdash P$.

**Def 4.5.6.** $\Gamma \vdash (\Gamma$ is inconsistent) iff $\Gamma \vdash P$ and $\Gamma \vdash \neg P$ for some sentence $P$ of FPL.

4.5.1 Metatheorems

**Metatheorem 4.5.1** (Deduction Metatheorem). If $\Delta \cup \{P\} \vdash Q$, then $\Delta \vdash P \rightarrow Q$. 
Complete proofs of the Deduction Metatheorem can be found in many mathematical logic textbooks (e.g. Enderton 2001:118-119) so we omit it here.

**Metatheorem 4.5.2** (Generalization on Arbitrary Closed Terms). If $\Gamma \vdash P(c/x)$ and $c$ does not occur in any member of $\Gamma$, then $\Gamma \vdash \forall x P$.

*Proof.* Suppose $\Gamma \vdash P(c/x)$ and $c$ does not occur in any member of $\Gamma$. The proof is by induction on the length of the derivation $\delta$ of $P(c/x)$ from $\Gamma$.

**Basis:** Suppose $\delta$ is of length 1, i.e. $\Gamma = \emptyset$ and $\delta = \langle P(c/x) \rangle$. Hence, $P(c/x)$ is an axiom. Since $P(c/x)$ contains no occurrences of $x$, it follows that $x$ is substitutable for $c$ in $P(c/x)$. Hence, $\forall x P$ is a universalization of $P(c/x)$. Hence, $\forall x P$ is an axiom. So, $\emptyset \vdash \forall x P$, i.e. $\Gamma \vdash \forall x P$.

**Induction Step:** Suppose that for any derivation of length $k$ or less of any sentence $Q(c_1/y)$ from members of $\Delta$ (where no member of $\Delta$ contains any occurrence of $c_1$) there is a derivation of $\forall y Q$ from members of $\Delta$. Suppose that $\delta$ is of length $k + 1$. The case in which $P(c/x)$ is an axiom is covered by the basis, and since $c$ occurs in $P(c/x)$ we know by hypothesis that $P(c/x)$ is not in $\Gamma$. Hence, we can assume that $P(c/x)$ is by MP and that sentences $R$ and $R \rightarrow P(c/x)$ occur earlier in $\delta$ than step $k + 1$. Hence, there are sequences $\delta_1$ and $\delta_2$ occurring within $\delta$ that are derivations of $R$ and $R \rightarrow P(c/x)$, respectively, from members of $\Gamma$.

Assume without loss of generality that $R$ does not contain an $x$-quantifier. By the induction hypothesis, there is a derivation $\delta_3$ of $\forall x (R \rightarrow P)$ from members of $\Gamma$. Construct a new derivation $\delta_4$ from the members of $\Gamma$ occurring in $\delta_1$ and in $\delta_3$ that includes the steps leading to $R$ and $\forall x (R \rightarrow P)$. Using Axiom 2(a), we may add $\forall x (R \rightarrow P) \rightarrow (R \rightarrow \forall x P)$ to the derivation. By two applications of MP, we get $\forall x P$. Hence, $\delta_4$ shows that $\Gamma \vdash \forall x P$. 
Metatheorem 4.5.3 (List Collapse). $\vdash c = [a_1 \ldots a_n]$ where $a_1, \ldots, a_n$ are the constants occurring in $c$.

Proof. By induction on the height of $c$.

Case 0: Suppose $c$ is of height 0. Then $c$ is a constant $a$. Hence, by Axiom 5(a) $\vdash c = [a]$, i.e. $\vdash c = [a_1 \ldots a_n]$.

Case 1: Suppose $c$ is of height 1. Then $c$ is $[b_1 \ldots b_m]$ for some constants $b_1, \ldots, b_m$. Suppose that each of the $b_i$s are distinct. Then $b_1 \ldots b_m$ are $a_1 \ldots a_n$, though perhaps with differences in order. By Axiom 3(a), $\vdash [a_1 \ldots a_n] = [a_1 \ldots a_n]$. By a finite number of applications of Axioms 5(d) and 3(b), the $a_i$s may be reordered to show that $\vdash [b_1 \ldots b_m] = [a_1 \ldots a_n]$, i.e., $\vdash c = [a_1 \ldots a_n]$.

Suppose that there are duplicate constants occurring in $[b_1 \ldots b_m]$. Consider any pair of occurrences of $b$. By a finite number of applications of Axioms 5(d) and 3(b), the pair may be concatenated. So, $\vdash [b_1 \ldots b_m] = [b_1 \ldots bb \ldots b_m]$. By Axiom 5(c), $\vdash [b_1 \ldots bb \ldots b_m] = [b_1 \ldots [bb] \ldots b_m]$. By Axiom 3(a), $\vdash b = b$. By Axiom 5(b), $\vdash b = b \rightarrow b = [bb]$. So, $\vdash b = [bb]$. Hence, $\vdash [b_1 \ldots [bb] \ldots b_m] = [b_1 \ldots b \ldots b_m]$. A finite number of applications of this method will remove all duplicate constants from $[b_1 \ldots b_m]$. A finite number of applications of Axiom 5(d) will show that $\vdash [b_1 \ldots b_m] = [a_1 \ldots a_n]$. It follows that $\vdash c = [a_1 \ldots a_n]$.

Induction Step: Suppose that the theorem holds for all closed terms of height $k$ or less. Let $c$ be of height $k + 1$. Then $c$ is $[d_1 \ldots d_m]$ for some closed terms $d_1, \ldots, d_m$. Each $d_i$ is of height $k$ or less, so by the induction hypothesis $\vdash d_i = d'_i$ for each $d_i$ where $d'_i$ is a list of the constants occurring in $d_i$. By repeated applications of Axiom 3(b), $\vdash [d_1 \ldots d_m] = [d'_1 \ldots d'_m]$. 

By repeated applications of Axiom 5(c), \[ \vdash \left[ \mathbf{d}_1' \ldots \mathbf{d}_m' \right] = [a_{1d_1} \ldots a_{jd_j} \ldots a_{1d_m} \ldots a_{pd_m}], \]

where \( a_{1d_1}, \ldots, a_{jd_j} \) are the \( q \geq 1 \) constants occurring in \( \mathbf{d}_i' \).

\( a_{1d_1}, \ldots, a_{jd_j}, \ldots, a_{1d_m}, \ldots, a_{pd_m} \) are the constants occurring in \( \mathbf{c} \). Applying the duplicate-constant deletion procedure described in the Case 1 and a finite number of applications of Axiom 5(d), it follows that \( \vdash \left[ \mathbf{d}_1' \ldots \mathbf{d}_m' \right] = [a_1 \ldots a_n] \). Using Axiom 3(b), it follows that \( \vdash \mathbf{c} = [a_1 \ldots a_n] \).

\[ \square \]

**Metatheorem 4.5.4** (Piecewise List Construction). If \( \Gamma \vdash a_1 = b_1 \), and \ldots, and \( \Gamma \vdash a_n = b_n \), then \( \Gamma \vdash [a_1 \ldots a_n] = [b_1 \ldots b_n] \).

*Proof.* Suppose \( \Gamma \vdash a_1 = b_1 \), and \ldots, and \( \Gamma \vdash a_n = b_n \). We proceed by induction on \( n \).

**Basis.** Let \( n = 1 \). Since \( \Gamma \vdash a_1 = b_1 \), there is a derivation \( \delta \) of \( a_1 = b_1 \) from members \( \mathcal{P}_1, \ldots, \mathcal{P}_m \) of \( \Gamma \). Suppose \( \delta \) has \( n \) steps. Construct a new derivation \( \delta_1 \) as follows.
\[ P_1 \]
\[ \vdots \]
\[ m. P_m \]
\[ \vdots \]
\[ n. a_1 = b_1 \]
\[ n+1. a_1 = [a_1] \]
\[ n+2. a_1 = [a_1] \rightarrow (a_1 = b_1 \rightarrow [a_1] = b_1) \]
\[ n+3. a_1 = b_1 \rightarrow [a_1] = b_1 \]
\[ n+4. [a_1] = b_1 \]
\[ n+5. b_1 = [b_1] \]
\[ n+6. b_1 = [b_1] \rightarrow ([a_1] = b_1 \rightarrow [a_1] = [b_1]) \]
\[ n+7. [a_1] = b_1 \rightarrow [a_1] = [b_1] \]
\[ n+8. [a_1] = [b_1] \]

\[ \delta_1 \] shows that \( \Gamma \vdash [a_1] = [b_1] \), i.e. \( \Gamma \vdash [a_1 \ldots a_n] = [b_1 \ldots b_n] \).

**Induction Step.** Suppose the metatheorem holds for all subscripts \( k < n \). Suppose further that \( \Gamma \vdash a_1 = b_1 \), and \( \ldots \), and \( \Gamma \vdash a_{n-1} = b_{n-1} \). Then by the induction hypothesis \( \Gamma \vdash [a_1 \ldots a_{n-1}] = [b_1 \ldots b_{n-1}] \). Hence there is a derivation \( \delta_2 \) of \([a_1 \ldots a_{n-1}] = [b_1 \ldots b_{n-1}] \) from members \( Q_1, \ldots, Q_m \) of \( \Gamma \) and a derivation \( \delta_3 \) of \( a_n = b_n \) from members \( R_1, \ldots, R_j \) of \( \Gamma \). Suppose \( \delta_2 \) has \( i \) steps and \( \delta_3 \) has \( k \) steps. Construct a new derivation \( \delta_4 \) as follows, where \( P_n \) is the sentence appearing on line \( n \) in \( \delta_4 \).
1. $Q_1$

$m. Q_m$

$n. [a_1 \ldots a_{n-1}] = [b_1 \ldots b_{n-1}]$

$i + 1. R_1$

$j + 1. R_j$

$i + k. a_1 = b_1$

$i + k + 1. [a_1 \ldots a_{n-1}]a_n = [a_1 \ldots a_{n-1}]a_n$

$i + k + 2. P_1 \rightarrow (P_{i+k+1} \rightarrow [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]a_n)$

$i + k + 3. P_{i+k+1} \rightarrow [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]a_n$

$i + k + 4. [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]a_n$

$i + k + 5. P_{i+k} \rightarrow (P_{i+k+4} \rightarrow [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]b_n)$

$i + k + 6. P_{i+k+4} \rightarrow [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]b_n$

$i + k + 7. [a_1 \ldots a_{n-1}]a_n = [b_1 \ldots b_{n-1}]b_n$

$i + k + 8. [a_1 \ldots a_{n-1}]a_n = [a_1 \ldots a_n$

$i + k + 9. [b_1 \ldots b_{n-1}]b_n = [b_1 \ldots b_n$

$i + k + 10. P_{i+k+8} \rightarrow (P_{i+k+7} \rightarrow [a_1 \ldots a_n = [b_1 \ldots b_{n-1}]b_n)$

$i + k + 11. P_{i+k+7} \rightarrow [a_1 \ldots a_n = [b_1 \ldots b_{n-1}]b_n$

$i + k + 12. [a_1 \ldots a_n = [b_1 \ldots b_{n-1}]b_n$

$i + k + 13. P_{i+k+12} \rightarrow (P_{i+k+9} \rightarrow [a_1 \ldots a_n = [b_1 \ldots b_n)$

by justification $1$ of $\delta_2$

(by steps $2$ to $m - 1$ of $\delta_2$)

by justification $m$ of $\delta_2$

(by steps $m + 1$ to $i - 1$ of $\delta_2$)

by justification $i$ of $\delta_2$

(by steps $2$ to $j - 1$ of $\delta_3$)

by justification $j$ of $\delta_3$

(by steps $j + 1$ to $k - 1$ of $\delta_3$)

by justification $k$ of $\delta_3$

by Axiom $3(a)$

by Axiom $3(b)$

by Axiom $3(c)$

by Axiom $5(c)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$

by Axiom $3(b)$
\[ i + k + 14. \mathcal{P}_{i+k+9} \rightarrow [a_1 \ldots a_n] = [b_1 \ldots b_n] \quad \text{MP } i + k + 13, \ i + k + 12 \]
\[ i + k + 15. [a_1 \ldots a_n] = [b_1 \ldots b_n] \quad \text{MP } i + k + 14, \ i + k + 9 \]

\( \delta_4 \) shows that \( \Gamma \vdash [a_1 \ldots a_n] = [b_1 \ldots b_n] \).

So, by induction on \( n \), \( \Gamma \vdash [a_1 \ldots a_n] = [b_1 \ldots b_n] \). \( \Box \)

4.6 Soundness for FPL

In this section we prove that if \( \Gamma \vdash \mathcal{P} \), then \( \Gamma \models \mathcal{P} \), i.e. that every derivation is valid according to the semantics of FPL.

The proof of soundness for FPL amounts to showing that all instances of the axioms of FPL are quantificationally true and that MP is a valid rule of inference in the semantics of FPL.

4.6.1 Validity of the Propositional Axioms

These proofs are omitted.

4.6.2 Validity of the Quantifier Axioms

These proofs are omitted.

4.6.3 Validity of the Identity Axioms

Lemma 4.6.3.1. For any closed term \( c : \models c = c \).

Proof. Let \( c \) be a closed term. Let \( \mathcal{I} \) be a model of FPL and let \( d \) be an assignment for \( \mathcal{I} \). Trivially, \( r(c, d, \mathcal{I}) \) is \( r(c, d, \mathcal{I}) \). So, by Def 4.4.2, it follows that \( d, \mathcal{I} \models c = c \).
Generalizing on $d, \mathcal{I}$, it follows that $\models c = c$. 

\[\text{Lemma 4.6.3.2.}\] For all closed terms $c_1, c_2$: $\models c_1 = c_2 \rightarrow (\mathcal{P}(c_1/x) \rightarrow \mathcal{P}(c_2/x))$.

Proof. Let $c_1, c_2$ be closed terms. Let $\mathcal{I}$ be a model of FPL and let $d$ be an assignment for $\mathcal{I}$.

Suppose that $d, \mathcal{I} \models c_1 = c_2$ and $d, \mathcal{I} \models \mathcal{P}(c_1/x)$. Then by Def 4.4.2, $r(c_1, d, \mathcal{I})$ is $r(c_2, d, \mathcal{I})$. By Theorem 4.1.5, $d(r(c_1, d, \mathcal{I})/x), \mathcal{I} \models \mathcal{P}$. So, $d(r(c_2, d, \mathcal{I})/x), \mathcal{I} \models \mathcal{P}$. Applying Theorem 4.1.5 again, it follows that $d, \mathcal{I} \models \mathcal{P}(c_2/x)$.

So, applying Def 4.4.4(b) twice and generalizing on $d, \mathcal{I}$, it follows that $\models c_1 = c_2 \rightarrow (\mathcal{P}(c_1/x) \rightarrow \mathcal{P}(c_2/x))$. 

\[\text{4.6.4 Validity of the Inclusion Axioms}\]

The inclusion predicate is a syntactic version of the subset relation. The axioms reflect this. They assert that the inclusion predicate partially orders the pluralities of FPL, i.e. the inclusion relation is reflexive, antisymmetric, and transitive. The validity of each axiom turns on the analogous property of the subset relation.

\[\text{Lemma 4.6.4.1.}\] For any closed term $c$: $\models c_1 \sqsubseteq c_2$, where $c_1$ occurs in $c_2$.

Proof. Let $c_1$ and $c_2$ be closed terms such that $c_1$ occurs in $c_2$. Let $I$ be a model of FPL and let $d$ be an assignment for $I$. 

Let $a_1, \ldots, a_n$ be the constants occurring in $c_1$ and $b_1, \ldots, b_m$ be the constants occurring in $c_2$. By two applications of Corollary 4.1.1, it follows that $r(c_1, d, \mathcal{I})$ is $\bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I})$ and $r(c_2, d, \mathcal{I})$ is $\bigcup r(b_1, d, \mathcal{I}), \ldots, r(b_m, d, \mathcal{I})$.

Since $c_1$ occurs in $c_2$, all of the $a$s occur in $c_2$. Hence, the $a$s are among the $b$s. It follows that $\bigcup r(a_1, d, \mathcal{I}), \ldots, r(a_n, d, \mathcal{I}) \subseteq \bigcup r(b_1, d, \mathcal{I}), \ldots, r(b_m, d, \mathcal{I})$. So, by Def 4.4.3, $d, \mathcal{I} \models c_1 \sqsubseteq c_2$.

Generalizing on $d, \mathcal{I}$, it follows that $\models c_1 \sqsubseteq c_2$, where $c_1$ occurs in $c_2$.

\begin{lemma}
For any closed terms $c_1$ and $c_2$: $\models c_1 \sqsubseteq c_2 \rightarrow (c_2 \sqsubseteq c_1 \rightarrow c_1 = c_2)$
\end{lemma}

\begin{proof}
Let $c_1$ and $c_2$ be closed terms. Let $\mathcal{I}$ be a model of FPL and let $d$ be an assignment for $\mathcal{I}$. Suppose that $d, \mathcal{I} \models c_1 \sqsubseteq c_2$ and $d, \mathcal{I} \models c_2 \sqsubseteq c_1$.

By Def 4.4.3, both $r(c_1, d, \mathcal{I}) \subseteq r(c_2, d, \mathcal{I})$ and $r(c_2, d, \mathcal{I}) \subseteq r(c_1, d, \mathcal{I})$. Hence, $r(c_1, d, \mathcal{I})$ is $r(c_2, d, \mathcal{I})$. Hence, by Def 4.4.2, $d, \mathcal{I} \models c_1 = c_2$.

So, by applying Def 4.4.4(b) twice and generalizing on $d, I$, it follows that $\models c_1 \sqsubseteq c_2 \rightarrow (c_2 \sqsubseteq c_1 \rightarrow c_1 = c_2)$.
\end{proof}

\begin{lemma}
For any closed terms $c_1$, $c_2$, and $c_3$: $\models c_1 \sqsubseteq c_2 \rightarrow (c_2 \sqsubseteq c_3 \rightarrow c_1 \sqsubseteq c_3)$
\end{lemma}

\begin{proof}
Let $c_1$, $c_2$, and $c_3$ be closed terms. Let $\mathcal{I}$ be a model of FPL and let $d$ be an assignment for $\mathcal{I}$. Suppose that $d, \mathcal{I} \models c_1 \sqsubseteq c_2$ and $d, \mathcal{I} \models c_2 \sqsubseteq c_3$.

By Def 4.4.3, both $r(c_1, d, \mathcal{I}) \subseteq r(c_2, d, \mathcal{I})$ and $r(c_2, d, \mathcal{I}) \subseteq r(c_3, d, \mathcal{I})$. So, $r(c_1, d, \mathcal{I}) \subseteq r(c_3, d, \mathcal{I})$. By Def 4.4.3, $d, \mathcal{I} \models c_1 \sqsubseteq c_3$.
So, by applying Def 4.4.4(b) twice and generalizing on $d, I$, it follows that $\models c_1 \sqsubseteq c_2 \rightarrow (c_2 \sqsubseteq c_3 \rightarrow c_1 \sqsubseteq c_3)$.

4.6.5 Validity of the List Axioms

Intuitively, a list is a finite sequence of terms that collectively denotes the things individually denoted by the terms it contains. This notion is formalized in Def 4.4.1 in the semantics of FPL. The axioms for lists reflect additional properties of that formulation. Axiom (a) says that a list with a single entry denotes exactly what the entry denotes. Axiom (b) says that a list names all the things that its sublists name. Axiom (c) says that iterated lists collapse, and so every iterated list denotes all and only the things that a “flattened” list of its constants denotes. Axiom (d) says that the order of a list’s entries is proof-theoretically irrelevant. Here were show that the semantics of FPL validates these axioms.

Lemma 4.6.5.1. $\models c = [c]$

Proof. Let $c$ be a closed term. Let $I$ be a model of FPL and let $d$ be an assignment for $I$. $c$ is either a constant or a list. In either case, $r(c, d, I)$ is $\bigcup r(a_1, d, I), \ldots, r(a_n, d, I)$ for the constants $a_1, \ldots, a_n$ occurring in $c$ (by Corollary 4.1.1).

By Def 4.4.1, $r([c], d, I)$ is $\bigcup r(c, d, I)$. But $\bigcup r(c, d, I)$ is just $\bigcup (\bigcup r(a_1, d, I), \ldots, r(a_n, d, I))$, i.e. $\bigcup r(a_1, d, I), \ldots, r(a_n, d, I)$. So, $r(c, d, I)$ is $r([c], d, I)$. So, by Def 4.4.2, $d, I \models c = [c]$.

Generalizing on $d, I$, it follows that $\models c = [c]$.

Lemma 4.6.5.2. $\models c_1 \sqsubseteq c_2 \rightarrow c_2 = [c_1 c_2]$
Proof. Let \( c_1, c_2 \) be closed terms. Let \( \mathcal{I} \) be a model of FPL and let \( d \) be an assignment for \( \mathcal{I} \). Suppose \( d, \mathcal{I} \models c_1 \sqsubseteq c_2 \). Then by Def 4.4.3, \( r(c_1, d, \mathcal{I}) \subseteq r(c_2, d, \mathcal{I}) \).

By Def 4.4.1, \( r([c_1, c_2], d, \mathcal{I}) \) is \( \bigcup r(c_1, d, \mathcal{I}), r(c_2, d, \mathcal{I}) \). Since \( r(c_1, d, \mathcal{I}) \subseteq r(c_2, d, \mathcal{I}) \), it follows that \( \bigcup r(c_1, d, \mathcal{I}), r(c_2, d, \mathcal{I}) \) is \( r(c_2, d, \mathcal{I}) \). So, by Def 4.4.2, \( d, \mathcal{I} \models c_2 = [c_1, c_2] \).

Applying Def 4.4.4(b) and generalizing on \( d, \mathcal{I} \), it follows that \( \models c_1 \sqsubseteq c_2 \rightarrow c_2 = [c_1, c_2] \).

Lemma 4.6.5.3. \( \models \ldots [c_m \ldots c_n] \ldots = \ldots c_m \ldots c_n \ldots \)

Proof. Suppose \( \ldots [c_1 \ldots c_n] \ldots \) is a list of length \( j \) where \( [c_1 \ldots c_n] \) is a list of length \( n \). Let \( \mathcal{I} \) be a model of FPL and let \( d \) be an assignment for \( \mathcal{I} \).

By Lemma 4.1.1, \( r([\ldots [c_1 \ldots c_n] \ldots ], d, \mathcal{I}) \) is \( r([\ldots c_1 \ldots c_n] \ldots , d, \mathcal{I}) \). So, by Def 4.4.2, \( d, \mathcal{I} \models \ldots [c_m \ldots c_n] \ldots = \ldots c_m \ldots c_n \ldots \).

So, generalizing on \( d, \mathcal{I} \), it follows that \( \models \ldots [c_m \ldots c_n] \ldots = \ldots c_m \ldots c_n \ldots \).

Lemma 4.6.5.4. \( \models [c_1 \ldots c_i, c_{i+1} \ldots c_n] = [c_1 \ldots c_{i+1}, c_i \ldots c_n] \)

Proof. Let \( c_1, \ldots, c_i, c_{i+1}, \ldots, c_n \) be closed terms. Let \( \mathcal{I} \) be a model of FPL and let \( d \) be an assignment for \( \mathcal{I} \).

Let \( t \) be \( [c_1 \ldots c_i, c_{i+1} \ldots c_n] \).

By Def 4.4.1, \( r(t, d, \mathcal{I}) \) is \( \bigcup r(c_1, d, \mathcal{I}), \ldots, r(c_i, d, \mathcal{I}), r(c_{i+1}, d, \mathcal{I}), \ldots, r(c_n, d, \mathcal{I}) \).

Let \( t' \) be \( [c_1 \ldots c_{i+1}, c_i \ldots c_n] \).
By Def 4.4.1, \( r(c',d,I) \) is \( \bigcup r(c_1,d,I), \ldots, r(c_{i+1},d,I), r(c_i,d,I), \ldots, r(c_n,d,I) \).

By the extensional property of sets, \( r(t,d,I) \) is \( r(t',d,I) \). So, by Def 4.4.2, \( d,I| = t = t' \), i.e., \( d,I| = [c_1 \ldots c_i c_{i+1} \ldots c_n] = [c_1 \ldots c_{i+1} c_i \ldots c_n] \).

Generalizing on \( d, I \), it follows that \( \models [c_1 \ldots c_i c_{i+1} \ldots c_n] = [c_1 \ldots c_{i+1} c_i \ldots c_n] \).

\[ \square \]

4.6.6 Validity of MP

**Metatheorem 4.6.1.** If \( \Gamma \models P \) and \( \Gamma \models P \rightarrow Q \), then \( \Gamma \models Q \).

**Proof.** Suppose \( \Gamma \models P \) and \( \Gamma \models P \rightarrow Q \). Let \( I \) be a model of FPL on which every member of \( \Gamma \) is true and let \( d \) be an assignment for \( I \). Then \( d,I| = P \) and \( d,I| = P \rightarrow Q \). So, by Def 4.4.4(b), \( d,I| = Q \). Generalizing on \( d, I \), it follows that \( \Gamma \models Q \). \[ \square \]

So, according to the semantics of FPL, it is safe to infer \( Q \) given premises that include \( P \) and \( P \rightarrow Q \).

4.6.7 Soundness of FPL

**Metatheorem 4.6.2** (Soundness Metatheorem). If \( \Gamma \vdash P \), then \( \Gamma \models P \).

**Proof.** Suppose \( \Gamma \vdash P \). By Def 5.5, there is a derivation \( \delta \) of \( P \) from members of \( \Gamma \). By Def 5.3, every member of the sequence \( \delta \) is either a member of \( \Gamma \), an instance of an axiom schema, or is by MP. We proceed by strong induction on the length of \( \delta \).

**Basis** Suppose \( \delta \) is of length 1, i.e., \( \delta \) is \( P \). Then either \( P \) is an instance of an axiom schema or \( P \in \Gamma \). If \( P \) is an instance of an axiom schema, then by the lemmas of sections 6.1-6.5, \( \models P \), and so \( \Gamma \models P \) by Def 4.9. If \( P \in \Gamma \) then it follows by Lemma 6.7.1 that
\[ \Gamma \models \mathcal{P}. \] In either case, \( \Gamma \models \mathcal{P} \).

**Induction Step** Suppose \( \delta \) is of length \( k \) and that \( \Gamma \models \mathcal{P}_i \) for every \( 1 \leq i < k \). \( \mathcal{P}_k \) is either a member of \( \Gamma \), an instance of an axiom schema, or is by modus ponens on \( \mathcal{P}_j \) and \( \mathcal{P}_n \) where \( j, n < k \).

**Case 1** Suppose \( \mathcal{P}_k \) is an instance of an axiom schema. Then by the lemmas in sections 6.1-6.5, \( \models \mathcal{P}_k \) and so \( \Gamma \models \mathcal{P}_k \) by Def 4.9.

**Case 2** Suppose \( \mathcal{P}_k \in \Gamma \). Then by Lemma 6.7.1, \( \Gamma \models \mathcal{P}_k \).

**Case 3** Suppose \( \mathcal{P}_k \) is by MP on \( \mathcal{P}_j \) and \( \mathcal{P}_n \) where \( j, n < k \). Then by the induction hypothesis, \( \Gamma \models \mathcal{P}_j \) and \( \Gamma \models \mathcal{P}_n \). Hence, by metatheorem 6.1, \( \Gamma \models \mathcal{P}_k \).

Hence, by strong induction on the length of \( \delta \), \( \Gamma \models \mathcal{P} \).

So, if \( \Gamma \vdash \mathcal{P} \), then \( \Gamma \models \mathcal{P} \).

\[ \square \]

### 4.7 Completeness of FPL

Here we show that if \( \Gamma \models \mathcal{P} \), then \( \Gamma \vdash \mathcal{P} \). We use the method developed by Henkin (1949) in outline and modify the proof in Bergmann, Moor, and Nelson (1998) for the details.

The proof rests on a number of lemmas, culminating in the Main Completeness Lemma from which the Completeness Metatheorem follows easily.

#### 4.7.1 Definitions

**Def 4.7.1.** A set \( \Delta \) of sentences of FPL is maximally consistent iff \( \Delta \not\models \) and for all \( \mathcal{Q} \), if \( \mathcal{Q} \notin \Delta \), then \( \Delta \cup \{ \mathcal{Q} \} \vdash \).
Def 4.7.2. A set of sentences of FPL is $\omega$-complete iff for every sentence $\neg\forall x P \in \Delta$ there is a closed term $c$ such that $\neg P(c/x) \in \Delta$.

Def 4.7.3. If $a$ is a constant with subscript $k$, then $\overline{a}$ is the constant that results from changing the subscript of $a$ from $k$ to $2k$. If $a$ is an unsubscripted constant, then $a$ is $\overline{a}$.

Def 4.7.4. If $t$ is a list, then $\overline{t}$ is the result of replacing every constant $a$ occurring in $t$ with $\overline{a}$.

Def 4.7.5. If $P$ is a formula of FPL, then $\overline{P}$ is the result of replacing every occurrence of every closed term $t$ in $P$ with $\overline{t}$.

Def 4.7.6. If $\Delta$ is a set of sentences of FPL, then $\overline{\Delta}$ is $\{\overline{P} : P \in \Delta\}$.

4.7.2 Basic Completeness Lemmas

Lemma 4.7.2.1. If $P \in \Delta$, then $\Delta \vdash P$.

Proof. Suppose $P \in \Delta$. Then by Def 5.4, the sequence $\langle P \rangle$ is a derivation of $P$ from members of $\Delta$. So, by Def 5.4, $\Delta \vdash P$. □

Lemma 4.7.2.2. If $\Delta \vdash P$, and $\Delta \subseteq \Delta'$, then $\Delta' \vdash P$.

Proof. Suppose $\Delta \vdash P$, and $\Delta \subseteq \Delta'$. Then by Def 5.4 there is a derivation $\delta$ of $P$ from members of $\Delta$. But since $\Delta \subseteq \Delta'$, $\delta$ is also a derivation of $P$ from members of $\Delta'$. Hence, by Def 5.4, $\Delta' \vdash P$. □

Lemma 4.7.2.3. If $\Delta \vdash$, then there exists a finite $\Delta'$ such that $\Delta' \subseteq \Delta$ and $\Delta' \vdash$.

Proof. Suppose $\Delta \vdash$. Then there are derivations $\delta_1$ and $\delta_2$ such that $\delta_1 = \langle P_1, \ldots, P_m, \ldots, Q \rangle$ and $\delta_2 = \langle R_1, \ldots, R_n, \ldots, \neg Q \rangle$ for some sentence $Q$ where $\{P_1, \ldots, P_m\} \subseteq \Delta$ and $\{R_1, \ldots, R_n\} \subseteq \Delta$.
Consider the set $S = \{ P_1, \ldots, P_m, R_1, \ldots, R_n \}$. By Lemma 7.2.2, $S \vdash Q$ and $S \vdash \neg Q$. So, by Def 5.7, $S \vdash$. Clearly $S \subseteq \Delta$. Further, since $\delta_1$ and $\delta_2$ are finite, $S$ is finite. So, there is a finite $\Delta'$ such that $\Delta' \subseteq \Delta$ and $\Delta' \vdash$ (namely, $S$).

\[ \square \]

**Lemma 4.7.2.4.** $\Delta \cup \{ \neg P \} \vdash$ iff $\Delta \vdash P$.

**Proof.** (L-to-R) Suppose $\Delta \cup \{ \neg P \} \vdash$. Then there is a $Q$ such that $\Delta \cup \{ \neg P \} \vdash Q$ and $\Delta \cup \{ \neg P \} \vdash \neg Q$. By Metatheorem 5.1, $\Delta \vdash \neg P \rightarrow Q$ and $\Delta \vdash \neg P \rightarrow \neg Q$. Hence, there are derivations $\delta_1$ and $\delta_2$ such that $\delta_1 = \langle P_1, \ldots, P_m, \neg P \rightarrow Q \rangle$ and $\delta_2 = \langle R_1, \ldots, R_n \ldots, \neg P \rightarrow \neg Q \rangle$, where $\{ P_1, \ldots, P_m \}, \{ R_1, \ldots, R_n \} \subseteq \Delta$. Suppose $\delta_1$ has a length of $j$ and $\delta_2$ has a length of $k$. 
Construct a new derivation $\delta_3$ as follows:

1. $P_1$
   
   $\vdots$

   $m$. $P_m$
   
   $\vdots$

   (steps $m + 1$ to $j - 1$ of $\delta_1$)

   $j$. $\neg P \rightarrow Q$
   
   by justification $j$ of $\delta_1$

   $j + 1. R_1$
   
   $\vdots$

   $j + n. R_n$
   
   $\vdots$

   (steps $n + 1$ to $k - 1$ of $\delta_2$)

   $j + k. \neg P \rightarrow \neg Q$
   
   by justification $k$ of $\delta_2$

   $j + k + 1. (\neg P \rightarrow Q) \rightarrow ((\neg P \rightarrow \neg Q) \rightarrow P)$
   
   by axiom 1($d$)

   $j + k + 2. (\neg P \rightarrow \neg Q) \rightarrow P$
   
   by MP on $j$ and $j + k + 1$

   $j + k + 3. P$
   
   by MP on $j + k$ and $j + k + 2$

Since $\{P_1, \ldots, P_m, R_1, \ldots, R_n\} \subseteq \Delta$, $\delta_3$ shows that $\Delta \vdash P$.

(R-to-L) Suppose $\Delta \vdash P$. Then by Lemma 7.2.2, $\Delta \cup \{\neg P\} \vdash P$. Also, by Lemma 7.2.1, $\Delta \cup \{\neg P\} \vdash \neg P$. So, $\Delta \cup \{\neg P\} \vdash P$.

\[\square\]

Lemma 4.7.2.5. $\Delta \cup \{P\} \vdash P$ iff $\Delta \vdash \neg P$.

Proof. This lemma follows by an argument similar to that for the previous lemma except the derivation in the left to right argument will invoke Axiom 1($c$) in place of Axiom 1($d$).

\[\square\]
Lemma 4.7.2.6. If $\Delta \vdash$ and $\Delta \subseteq \Delta'$, then $\Delta' \vdash$.

*Proof.* Suppose $\Delta \vdash$ and $\Delta \subseteq \Delta'$. Then $\Delta \vdash Q$ and $\Delta \vdash \neg Q$ for some $Q$. So, since $\Delta \subseteq \Delta'$, it follows by Lemma 7.2.2 that $\Delta' \vdash Q$ and $\Delta' \vdash \neg Q$. Hence, $\Delta' \vdash$.

□

Lemma 4.7.2.7. If $\Delta \vdash P$ and $\Delta \subseteq \Delta'$ and $\Delta'$ is maximally consistent, then $P \in \Delta'$.

*Proof.* Suppose $\Delta \vdash P$ and $\Delta \subseteq \Delta'$ and $\Delta'$ is maximally consistent. By Lemma 7.2.2, $\Delta' \vdash P$. Suppose for reductio that $P \notin \Delta'$. Then by the maximality of $\Delta'$, $\Delta' \cup \{P\} \vdash$. So, by Lemma 7.2.5, $\Delta' \vdash \neg P$. Hence, $\Delta' \vdash P, \neg P$, and so $\Delta' \vdash$ contrary to the consistency of $\Delta'$. So, by reductio, $P \in \Delta'$.

□

Lemma 4.7.2.8. If $\Delta$ is maximally consistent and $\omega$-complete, then:

1. $\neg P \in \Delta$ iff $P \notin \Delta$;

2. $P \rightarrow Q \in \Delta$ iff $P \notin \Delta$ or $Q \in \Delta$;

3. $\forall x P \in \Delta$ iff $P(c/x) \in \Delta$ for every closed term $c$.

*Proof.* Suppose $\Delta$ is maximally consistent and $\omega$-complete. Then:

1. *(L-to-R)* Suppose $\neg P \in \Delta$. Suppose for reductio that $P \in \Delta$. Then by two applications of Lemma 7.2.1, $\Delta \vdash \neg P$ and $\Delta \vdash P$. So $\Delta \vdash$, contrary to the consistency of $\Delta$. So, by reductio, $P \notin \Delta$.

*(R-to-L)* Suppose $P \notin \Delta$. Then by the maximality of $\Delta$, $\Delta \cup \{P\} \vdash$. So by Lemma 7.2.5, $\Delta \vdash \neg P$. Suppose for reductio that $\neg P \notin \Delta$. Then by the maximality of $\Delta$, $\Delta \cup \\{\neg P\} \vdash$. So, by Lemma 7.2.4, $\Delta \vdash P$. So, $\Delta \vdash P, \neg P$ and therefore $\Delta \vdash$ contrary to the consistency of $\Delta$. So, by reductio, $\neg P \in \Delta$. 

2. \(L\text{-to-}R\) Suppose \(P \rightarrow Q \in \Delta\) and that \(P \in \Delta\). Then, by MP, \(\Delta \vdash Q\). So, by Lemma 7.2.7, \(Q \in \Delta\).

\(R\text{-to-}L\) Suppose \(P \notin \Delta\) or \(Q \in \Delta\).

Case 1. Suppose \(P \notin \Delta\). Then by Lemma 7.2.8(1), \(\neg P \in \Delta\). But \(\{\neg P\} \vdash P \rightarrow Q\). So, by Lemma 7.2.7, \(P \rightarrow Q \in \Delta\).

Case 2. Suppose \(Q \in \Delta\). \(\{Q\} \vdash P \rightarrow Q\), so by Lemma 7.2.7 it follows that \(P \rightarrow Q \in \Delta\).

In either case, \(P \rightarrow Q \in \Delta\).

3. \(L\text{-to-}R\) Suppose \(\forall x P \in \Delta\). Then \(\Delta \vdash P(c/x)\) for any closed term \(c\). (The derivation will invoke Axiom 2(b).) Since \(\Delta\) is maximally consistent, \(P(c/x) \in \Delta\) by Lemma 7.2.7.

\(R\text{-to-}L\) Suppose \(P(c/x) \in \Delta\) for every closed term \(c\). Suppose for reductio that \(\forall x P \notin \Delta\). Then by Lemma 7.2.8, it follows that \(\neg \forall x P \in \Delta\) since \(\Delta\) is maximally consistent. Since \(\Delta\) is \(\omega\)-complete, there is a closed term \(c_0\) such that \(\neg P(c_0/x) \in \Delta\). Hence \(\Delta \vdash \neg P(c_0/x)\) by Lemma 7.2.1. But by hypothesis \(P(c_0/x) \in \Delta\), and so \(\Delta \vdash P(c_0/x)\) too, contrary to \(\Delta\)'s consistency. So, by reductio, \(\forall x P \in \Delta\).

\[\square\]

**Lemma 4.7.2.9.** There exists at least one enumeration of all the sentences of FPL.

**Proof.** We use the method of Gödel-numbering. Let \(g\) be a function that assigns a positive integer to each symbol in the vocabulary of FPL as follows:
We adopt the convention that when a predicate letter is immediately followed by a superscript and a subscript, the superscript occurs before the subscript. Let \( \sigma_n \) be the \( n^{th} \) symbol occurring in a sentence \( P \) consisting in \( n \)-many symbols. Let \( G \) be a unary function from sentences of FPL to the positive integers such that:

\[
G(P) = 2^{g(\sigma_1)} \cdot 3^{g(\sigma_2)} \cdot \ldots \cdot m^{g(\sigma_n)}
\]

Where \( m \) is the \( n^{th} \) prime in the sequence of primes ordered by magnitude. By the fundamental theorem of arithmetic, \( G \) is injective. Hence, the sentences of FPL may be enumerated by the magnitude of their \( G \)-values.

\[ \square \]

**Lemma 4.7.2.10.** For any model \( \mathcal{I} \) and assignment \( d \) for \( \mathcal{I} \), if for every \( u \in \hat{U}_\mathcal{I} \) there is a closed term \( c \) such that \( r(c, d, \mathcal{I}) = u \), then if every substitution instance of \( \forall x P \) is true on \( \mathcal{I} \), then \( \forall x P \) is also true on \( \mathcal{I} \).

**Proof.** Let \( \mathcal{I} \) be a model of FPL and \( d_1 \) be an assignment for \( \mathcal{I} \). Suppose that for every \( u \in \hat{U}_\mathcal{I} \) there is a closed term \( c \) such that \( r(c, d_1, \mathcal{I}) = u \). Suppose further that every substitution instance of \( \forall x P \) is true on \( \mathcal{I} \); i.e., \( P(c/x) \) is true on \( \mathcal{I} \) for every closed term \( c \).

Let \( u' \in \hat{U}_\mathcal{I} \). By hypothesis, there is a closed term \( c_1 \) such that \( r(c_1, d_1, \mathcal{I}) = u' \). Since every substitution instance of \( \forall x P \) is true on \( \mathcal{I} \), it follows that \( d_1, \mathcal{I} \models P(c_1/x) \). By Theorem
4.1.5 it follows that $d_1(r(c_1,d_1,I)/x), I \models P$, i.e. $d_1(u'/x), I \models P$.

Generalizing on $u'$, $d_1(u/x), I \models P$ for all $u \in \hat{U}_I$. So, by Def 4.4.5, it follows that $d_1, I \models \forall x P$.

Generalizing on $d_1$, $\forall x P$ is true on $I$.

Lemma 4.7.2.11. $\Delta \cup \{\neg P\} \models$ iff $\Delta \models P$.

Proof. (L-to-R) Suppose $\Delta \cup \{\neg P\} \models$. Then there is no model on which every member of $\Delta \cup \{\neg P\}$ is true. Let $I$ be a model on which every member of $\Delta$ is true. Then, by hypothesis, $\neg P$ is not true on $I$. So, by Lemma 4.1.3, $\neg P$ is false on $I$. Let $d$ be an assignment for $I$. By Def 4.6, $d, I \not\models \neg P$. It follows by Def 4.4.4(a) that $d, I \models P$.

Generalizing on $d$, it follows by Def 4.5 that $P$ is true on $I$. Generalizing on $I$, it follows that $\Delta \models P$.

(R-to-L) Suppose $\Delta \models P$. Let $I$ be a model on which every member of $\Delta$ is true. Then $P$ is true on $I$. Let $d$ be an assignment for $I$. Then $d, I \models P$. By Def 4.4.4(a), it follows that $d, I \not\models \neg P$. Generalizing on $d$, it follows by Def 4.6 that $\neg P$ is false on $I$. So, not every member of $\Delta \cup \{\neg P\}$ is true on $I$.

Generalizing on $I$, there does not exist a model on which every member of $\Delta \cup \{\neg P\}$ is true. So, $\Delta \cup \{\neg P\} \not\models$.

Lemma 4.7.2.12. If $\Gamma \cup \{\neg \forall x P\} \not\models$ and $c$ does not occur in $\neg \forall x P$ or in any member of $\Gamma$, then $\Gamma \cup \{\neg \forall x P, \neg P(c/x)\} \not\models$. 

Proof. We prove the contrapositive. Suppose \( c \) does not occur in \( \neg \forall x P \) or \( \Gamma \), and that 
\[ \Gamma \cup \{ \neg \forall x P, \neg P(c/x) \} \vdash . \]

Hence, \( \Gamma \cup \{ \neg \forall x P, \neg P(c/x) \} \vdash Q \) and \( \Gamma \cup \{ \neg \forall x P, \neg P(c/x) \} \vdash \neg Q. \)

So, by two applications of the deduction metatheorem, \( \Gamma \cup \{ \neg \forall x P \} \vdash \neg P(c/x) \rightarrow Q \) and \( \Gamma \cup \{ \neg \forall x P \} \vdash \neg P(c/x) \rightarrow \neg Q. \)

We may therefore construct a derivation \( \delta \) using Axiom 1(d) that shows that \( \Gamma \cup \{ \neg \forall x P \} \vdash P(c/x) \). Since \( c \) does not occur in any member of \( \Gamma \cup \{ \neg \forall x P \} \), it follows by Metatheorem 5.2 (generalization on arbitrary constants) that \( \Gamma \cup \{ \neg \forall x P \} \vdash \forall x P. \) But by Lemma 7.2.1, \( \Gamma \cup \{ \neg \forall x P \} \vdash \neg \forall x P. \) Hence, \( \Gamma \cup \{ \neg \forall x P \} \vdash . \)

\[ \square \]

4.7.3 **Canonical Models**

**Def 4.7.7.** For any set \( \Delta \) that is maximally consistent in FPL and any closed term \( c \), define \( c_\Delta \) to be the set \( \{ c' : c = c' \in \Delta \} \).

**Def 4.7.8.** For any set \( \Delta \) that is maximally consistent in FPL, define the *canonical model* \( \mathcal{I}_\Delta \) for \( \Delta \) as follows:

1. \( U_{\mathcal{I}_\Delta} = \{ a_\Delta : a \text{ is a constant} \} \);

2. \( \hat{U}_{\mathcal{I}_\Delta} = \{ \{ a_1, \ldots, a_n \} : a_1, \ldots, a_n \in U_{\mathcal{I}_\Delta} \} \);

3. \( v_{\mathcal{I}_\Delta}(a) = \{ a_\Delta \} \) for every constant \( a \);

4. \( v_{\mathcal{I}_\Delta}(A) = \{ r(t_1, d, \mathcal{I}_\Delta), \ldots, r(t_n, d, \mathcal{I}_\Delta) \} : A t_1 \ldots t_n \in \Delta \} \) for every \( n \)-ary predicate \( A \) and assignment \( d \) for \( \mathcal{I}_\Delta \).
**Def 4.7.9.** For any set $\Delta$ that is maximally consistent in FPL and any closed term $c$, define $c_\Delta$ to be the set $\{c' : c' \sqsubseteq c \in \Delta\}$.

**Lemma 4.7.3.1.** If $\Delta$ is maximally consistent then:

1. for every closed term $c$, $c \in c_\Delta$;
2. for all closed terms $c$ and $c'$, $c \in c'_\Delta$ iff $c_\Delta$ is $c'_\Delta$;
3. for all closed terms $c$ and $c'$, if $c_\Delta$ is not $c'_\Delta$, then $c_\Delta \cap c'_\Delta$ is $\emptyset$;
4. for any closed term $c$ and any assignment $d$ for $I_\Delta$, $r(c, d, I_\Delta)$ is $\{a_{1_\Delta}, \ldots, a_{n_\Delta}\}$ where $a_1, \ldots, a_n$ are the constants occurring at any depth in $c$;
5. for any closed terms $c_1$ and $c_2$ and any assignment $d$ for $I_\Delta$, $r(c_1, d, I_\Delta)$ is $r(c_2, d, I_\Delta)$ iff $c_{1_\Delta}$ is $c_{2_\Delta}$.

**Proof.** Suppose $\Delta$ is maximally consistent. Then:

1. Let $c$ be a closed term. By the first identity axiom schema, $\vdash c = c$. By $\Delta$'s maximal consistency and Lemma 7.2.7, $c = c \in \Delta$. So, by Def 7.7, $c \in c_\Delta$.

2. Let $c$ and $c'$ be closed terms.

   (L-to-R) Suppose $c \in c'_\Delta$. Then $c' = c \in \Delta$ by Def 7.7. Suppose $c'' \in c_\Delta$. Then $c = c'' \in \Delta$. But $\{c' = c, c = c''\} \vdash c' = c''$. So, $\Delta \vdash c' = c''$ by Lemma 7.2.2. Hence, by $\Delta$'s maximal consistency and Lemma 7.2.7, $c' = c'' \in \Delta$. So, by Def 7.7, $c'' \in c'_\Delta$.

So, if $a'' \in c_\Delta$, then $c'' \in c'_\Delta$. By an exactly similar argument, if $c'' \in c'_\Delta$, then $c'' \in c_\Delta$. So, $c_\Delta$ is $c'_\Delta$. 
(R-to-L) Suppose $c_\Delta$ is $c'_\Delta$. By Lemma 7.3.1(1), $c \in c_\Delta$. It follows immediately that $c \in c'_\Delta$.

3. Let $c$ and $c'$ be closed terms. Suppose that $c_\Delta$ is not $c'_\Delta$. Suppose for reductio that $c_\Delta \cap c'_\Delta$ is not empty. Then there is a $c_0 \in c_\Delta \cap c'_\Delta$. So, $c_0 \in c_\Delta$ and $c_0 \in c'_\Delta$. So, by two applications of Lemma 7.3.1(2), $c_\Delta$ is $c_0\Delta$ and $c'_\Delta$ is $c_0\Delta$. Hence, $c_\Delta$ is $c'_\Delta$, contrary to hypothesis. So, by reductio, $c_\Delta \cap c'_\Delta$ is empty.

4. Let $d$ be an assignment for $I_\Delta$. Let $a_1, \ldots, a_n$ be an enumeration of the constants occurring at any depth in $c$. Either $c$ is a constant or a list.

Case 1. Suppose $c$ is a constant. Then $c$ is $a_1$. $r(a_1, d, I_\Delta)$ is $v_{I_\Delta}(a_1)$ by Def 4.4.1, which is $\{a_1\Delta\}$ by Def 7.8.

Case 2. Suppose $c$ is a list. Then

\[
r(c, d, I_\Delta) = r([a_1 \ldots a_n], d, I_\Delta)
\]

is $\bigcup r(a_1, d, I_\Delta), \ldots, r(a_n, d, I_\Delta)$ by Corollary 4.1.1

is $\bigcup v_{I_\Delta}(a_1), \ldots, v_{I_\Delta}(a_n)$

is $\bigcup \{a_1\Delta\}, \ldots, \{a_n\Delta\}$ by Def 7.8

is $\{a_1\Delta, \ldots, a_n\Delta\}$.

In either case, $r(c, d, I_\Delta)$ is $\{a_1\Delta, \ldots, a_n\Delta\}$ where $a_1, \ldots, a_n$ are the constants occurring at any depth in $c$. 
5. Let $d$ be an assignment for $\mathcal{I}_\Delta$.

\textit{(L-to-R)} Suppose $r(c_1, d, \mathcal{I}_\Delta)$ is $r(c_2, d, \mathcal{I}_\Delta)$. By Lemma 7.3.1(4), it follows that 
\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_m\}$ where $a_1, \ldots, a_n$ are the constants occurring in $c_1$ and $b_1, \ldots, b_m$ are the constants occurring in $c_2$. Hence, for each $a_i$ there is a $b_j$ such that $a_i = b_j$. Likewise, for every $b_j$ there is an $a_i$ such that $b_j = a_i$. Hence, by Lemma 7.3.1(1) and Def 7.7, $a_1 = b_{k_1}, \ldots, a_n = b_{k_n} \in \Delta$ and $b_1 = a_{p_1}, \ldots, b_m = a_{p_m} \in \Delta$. So, by Lemma 7.2.1, $\Delta \vdash a_1 = b_{k_1}, \ldots, \Delta \vdash a_n = b_{k_n}$ and $\Delta \vdash b_1 = a_{p_1}, \ldots, \Delta \vdash b_m = a_{p_m}$.

Hence, by Metatheorem 5.4, it follows that $\Delta \vdash [a_1 \ldots a_n a_{p_1} \ldots a_{p_m}] = [b_{k_1} \ldots b_{k_n} b_1 \ldots b_m]$.

From here we may use the procedure described in the proof of Metatheorem 5.3 to add or remove duplicate constants as necessary applying Axiom 5(d) to put the constants in the correct order. By this procedure we can show that $\Delta \vdash [a_1 \ldots a_n] = [b_1 \ldots b_m]$.

By Metatheorem 5.3, $\Delta \vdash c_1 = [a_1 \ldots a_n]$ and $\Delta \vdash c_2 = [b_1 \ldots b_m]$. So, $\Delta \vdash c_1 = c_2$.

By $\Delta$’s maximal consistency and Lemma 7.2.7, $c_1 = c_2 \in \Delta$. By Def 7.7, $c_2 \in c_{1\Delta}$. So, by Lemma 7.3.1(2), $c_{1\Delta}$ is $c_{2\Delta}$.

\textit{(R-to-L)} Suppose $c_{1\Delta}$ is $c_{2\Delta}$. By Lemma 7.3.1(2) it follows that $c_1 \in c_{2\Delta}$. So, by Def 7.7, $c_1 = c_2 \in \Delta$. So, $d, \mathcal{I}_\Delta \models c_1 = c_2$. So, $r(c_1, d, \mathcal{I}_\Delta)$ is $r(c_2, d, \mathcal{I}_\Delta)$.

\[\square\]

\textbf{Lemma 4.7.3.2.} If $\Delta$ is maximally consistent then:

1. for every closed term $c$, $c \in c_{\Delta^*}$;
2. for all closed terms \( c \) and \( c' \), \( c \in c'_{\Delta} \) iff \( c_{\Delta} \subseteq c'_{\Delta} \);

3. for any constant \( a \) and closed term \( c \), \( a \in c_{\Delta} \) iff \( a_{\Delta} \in r(c, d, \mathcal{I}_{\Delta}) \);

4. for all closed terms \( c \) and \( c' \), \( c_{\Delta} \subseteq c'_{\Delta} \) iff \( r(c, d, \mathcal{I}_{\Delta}) \subseteq r(c', d, \mathcal{I}_{\Delta}) \).

Proof. Suppose \( \Delta \) is maximally consistent. Then:

1. By Axiom 4(a), \( \Delta \vdash c \sqsubseteq c \). By Lemma 7.2.7, \( c \sqsubseteq c \in \Delta \). So, \( c \in c_{\Delta} \).

2. \((L\text{-to-}R)\) Suppose \( c \in c'_{\Delta} \). Then \( c \sqsubseteq c' \in \Delta \). Suppose \( c'' \in c_{\Delta} \). Then \( c'' \sqsubseteq c \in \Delta \). By Lemma 7.2.1, \( \Delta \vdash c \sqsubseteq c' \) and \( \Delta \vdash c'' \sqsubseteq c \). By Axiom 4(c), \( \Delta \vdash c'' \sqsubseteq c \rightarrow (c \sqsubseteq c' \rightarrow c'' \sqsubseteq c') \). So, \( \Delta \vdash c'' \sqsubseteq c' \). By \( \Delta \)'s maximal consistency and Lemma 7.2.7, \( c'' \sqsubseteq c' \in \Delta \). So, by Def 7.9, \( c'' \in c'_{\Delta} \). Generalizing on \( c'' \), \( c_{\Delta} \subseteq c'_{\Delta} \).

\((R\text{-to-}L)\) Suppose \( c_{\Delta} \subseteq c'_{\Delta} \). By Lemma 7.3.2(1), \( c \in c_{\Delta} \). So, by Def 7.9, \( c \in c'_{\Delta} \).

3. \((L\text{-to-}R)\) Suppose \( a \in c_{\Delta} \). Then \( a \sqsubseteq c \in \Delta \). So, \( \Delta \vdash a \sqsubseteq c \). But by Axiom 5(b), \( \vdash a \sqsubseteq c \rightarrow c = [ac] \). So, \( \Delta \vdash c = [ac] \). By Def 7.7, \( [ac] \in c_{\Delta} \). So, by Lemma 7.3.1(2), \([ac]_{\Delta} \) is \( c_{\Delta} \). So, by Lemma 7.3.1(5), \( r([ac], d, \mathcal{I}_{\Delta}) \) is \( r(c, d, \mathcal{I}_{\Delta}) \). Since \( a \) occurs in \([ac]\), it follows by Lemma 7.3.1(4) that \( a_{\Delta} \in r([ac], d, \mathcal{I}_{\Delta}) \). So, \( a_{\Delta} \in r(c, d, \mathcal{I}_{\Delta}) \).

\((R\text{-to-}L)\) Suppose \( a_{\Delta} \in r(c, d, \mathcal{I}_{\Delta}) \). By Lemma 7.3.1(4), there is a \( b \) occurring in \( c \) such that \( a_{\Delta} \) is \( b_{\Delta} \). So, by Lemma 7.3.1(2), \( b \in a_{\Delta} \). So, by Def 7.7, \( a = b \in \Delta \). So, by Lemma 7.2.1, \( \Delta \vdash a = b \). But since \( b \) occurs in \( c \), it follows by Axiom 4(a) that \( \vdash b \sqsubseteq c \). So, by an application of Axiom 3(b), it follows that \( \Delta \vdash a \sqsubseteq c \). By Lemma
7.2.7, \( a \sqsubseteq c \in \Delta \). So, by Def 7.9, \( a \in c_{\Delta^*} \).

4. \((L\text{-to-}R)\) Suppose \( c_{\Delta^*} \subseteq c'_{\Delta^*} \). Let \( a_\Delta \in r(c, d, I_\Delta) \). It follows by Lemma 7.3.2(3) that \( a \in c_{\Delta^*} \). So, \( a \in c'_{\Delta^*} \). Hence, by Lemma 7.3.2(3) again, \( a_\Delta \in r(c', d, I_\Delta) \). Generalizing on \( a \), it follows that \( r(c, d, I_\Delta) \subseteq r(c', d, I_\Delta) \).

\((R\text{-to-}L)\) Suppose \( r(c, d, I_\Delta) \subseteq r(c', d, I_\Delta) \). Let \( a \in c_{\Delta^*} \). By Lemma 7.3.2(3), \( a_\Delta \in r(c, d, I_\Delta) \). So, \( a_\Delta \in r(c', d, I_\Delta) \). Hence, by Lemma 7.3.2(3) again, \( a \in c_{\Delta^*} \). So, generalizing on \( a \), \( c_{\Delta^*} \subseteq c'_{\Delta^*} \).

Lemma 4.7.3.3. \( \mathcal{U}_{I_\Delta} \) is countable.

*Proof.* Define a function \( G : \mathcal{U}_{I_\Delta} \longrightarrow \mathbb{N} \) such that \( G(a_\Delta) \) is the least Gödel number of any \( a \in a_\Delta \) for some fixed Gödel-numbering of the constants.

Suppose \( a_\Delta \) is \( a'_\Delta \). Then the smallest Gödel-number of a constant in \( a_\Delta \) is also the smallest Gödel number of a constant in \( a'_\Delta \). So, \( G(a_\Delta) \) is \( G(a'_\Delta) \). Hence, \( G \) is well-defined.

Suppose \( G(a_\Delta) \) is \( G(a'_\Delta) \). Suppose for reductio that \( a_\Delta \) is not \( a'_\Delta \). By Lemma 7.3.1(3), \( a_\Delta \cap a'_\Delta \) is empty. So, the least Gödel number of a constant in \( a_\Delta \) is not the least Gödel number of a constant in \( a'_\Delta \). Hence, \( G(a_\Delta) \) is not \( G(a'_\Delta) \), contrary to hypothesis. So, by reductio, \( a_\Delta \) is \( a'_\Delta \). Hence, \( G \) is injective.

Since \( G \) is injective, \(|\mathcal{U}_{I_\Delta}| \leq |\mathbb{N}| \). So, \( \mathcal{U}_{I_\Delta} \) is countable.

Lemma 4.7.3.4. For any maximally consistent set \( \Delta, I_\Delta \) is a model of FPL, i.e.
1. $\mathcal{U}_\Delta$ is a nonempty set;

2. $\hat{\mathcal{U}}_\Delta \subseteq \mathcal{P}(\mathcal{U}_\Delta)$ such that:
   - (a) $\hat{\mathcal{U}}_\Delta$ contains at least one singleton, and
   - (b) for any singletons $A_1, \ldots, A_n \in \hat{\mathcal{U}}_\Delta$, $\bigcup A_1, \ldots, A_n \in \hat{\mathcal{U}}_\Delta$; and

3. $v_{\Delta}$ is a function that meets the following conditions:
   - (a) $v_{\Delta}(a)$ is a singleton in $\hat{\mathcal{U}}_\Delta$ for every constant $a$ of FPL;
   - (b) $v_{\Delta}(A) \subseteq \hat{\mathcal{U}}^n_\Delta$ for every $n$-ary predicate letter $A$ of FPL.

**Proof.** Let $\Delta$ be a maximally consistent set of FPL sentences. Then,

1. By Def 7.8(1), $a_\Delta \in \mathcal{U}_\Delta$. Hence, $\mathcal{U}_\Delta$ has at least one member.

2. Suppose $S \in \hat{\mathcal{U}}_\Delta$. Then by Def 7.8(2), $S = \{a_1\Delta, \ldots, a_n\Delta\}$ for some $a_1\Delta, \ldots, a_n\Delta \in \mathcal{U}_\Delta$.
   Hence, $S \in \mathcal{P}(\mathcal{U}_\Delta)$. Generalizing on $S$, $\hat{\mathcal{U}}_\Delta \subseteq \mathcal{P}(\mathcal{U}_\Delta)$.
   - (a) Since $a_\Delta \in \mathcal{U}_\Delta$, it follows by Def 7.8(2) that $\{a_\Delta\} \in \hat{\mathcal{U}}_\Delta$. So $\hat{\mathcal{U}}_\Delta$ contains at least one singleton.

   - (b) Let $S_1, \ldots, S_n$ be singletons and suppose $S_1, \ldots, S_n \in \hat{\mathcal{U}}_\Delta$. Then by Def 7.8(1), $S_1, \ldots, S_n$ are $\{a_1\}, \ldots, \{a_n\}$ for some constants $a_1, \ldots, a_n$. By Def 7.8(2), $\{a_1\Delta, \ldots, a_n\Delta\} \in \hat{\mathcal{U}}_\Delta$. So, since $\{a_1\Delta, \ldots, a_n\Delta\}$ is $\bigcup S_1, \ldots, S_n$, it follows that $\bigcup S_1, \ldots, S_n \in \hat{\mathcal{U}}_\Delta$.

3. It suffices to show that $v_{\Delta}$ is a function from constants to singleton members of $\hat{\mathcal{U}}_\Delta$.
   Hence we need to show that (i) for any constant $a$ there is a singleton $S \in \hat{\mathcal{U}}_\Delta$ such that $v_{\Delta}(a)$ is $S$, and (ii) for any constant $a$, if $v_{\Delta}(a)$ is $S$ and $v_{\Delta}(a)$ is $S'$, then $S$ is $S'$. 

(a) Let \( a \) be a constant. Then by Def 7.8(3), \( v_{\Delta}(a) \) is \( \{a_\Delta\} \). Since by Def 7.8(1) \( a_\Delta \in U_{\Delta}, \{a_\Delta\} \in \hat{U}_{\Delta} \) by Def 7.8(2). Generalizing on \( a \), for any constant \( a \), there is a singleton \( S \in \hat{U}_{\Delta} \) such that \( v_{\Delta}(a) \) is \( S \).

(b) Let \( a \) be a constant. Suppose that \( v_{\Delta}(a) \) is \( S \) and \( v_{\Delta}(a) \) is \( S' \). By Def 7.8(3), both \( S \) and \( S' \) are \( \{a_\Delta\} \). Hence, \( S \) is \( S' \). Generalizing on \( a \), it follows that for any constant \( a \), if \( v_{\Delta}(a) \) is \( S \) and \( v_{\Delta}(a) \) is \( S' \), then \( S \) is \( S' \).

\[ \square \]

4.7.4 Intermediate Completeness Lemmas

**Lemma 4.7.4.1.** If \( \Delta \not\vdash \), then \( \overline{\Delta} \not\vdash \).

*Proof.* It suffices to show that if \( \overline{\Delta} \vdash \) then \( \Delta \vdash \). Suppose that \( \overline{\Delta} \vdash \). Then there are derivations \( \delta_1 \) and \( \delta_2 \) as follows: \( \delta_1 \) is a derivation of \( Q \) from members \( \overline{R}_1, \ldots, \overline{R}_m \) of \( \overline{\Delta} \), and \( \delta_2 \) is a derivation of \( \neg Q \) from members \( \overline{S}_1, \ldots, \overline{S}_n \) of \( \overline{\Delta} \). Note that \( \overline{R}_1, \ldots, \overline{R}_m, \overline{S}_1, \ldots, \overline{S}_n \in \Delta \).

Consider derivations \( \delta_1' \) and \( \delta_2' \), where step \( k \) of \( \delta_1' \) is the sentence \( P_k \), where step \( k \) of \( \delta_1 \) is \( \overline{P}_k \). Reason similarly for \( \delta_2 \). If \( \overline{a} \) is a constant occurring in \( \overline{c} \) in \( \overline{P}_k \) that has an even subscript \( 2n \), then the corresponding closed term in \( P_k \) is \( c' \), where \( c' \) is the same string as \( c \) except with subscript \( n \) in place of \( 2n \) in \( a \). For any constant \( a \) occurring in \( c \) that is odd-subscripted or has no subscript, \( \overline{a} \) is \( a \).

Suppose \( \overline{P}_k \) is \( \overline{Q} \) for some \( Q' \). Suppose for reductio that \( P_k \) is not \( Q' \). Then \( P_k \) contains a subscripted closed term \( c_j \) where \( Q' \) contains a subscripted closed term \( c_{j'} \) where \( j \) is not \( j' \). But if \( \overline{P}_k \) is \( \overline{Q} \), then \( P_k \) contains \( c_{2j} \) where \( Q' \) contains \( c_{2j'} \) and \( 2j \) is \( 2j' \). So, \( j \) is \( j' \) and we have a contradiction. So, by reductio, \( P_k \) is \( Q' \).
Hence, each step $k$ in derivations $\delta'_1$ and $\delta'_2$ will follow by the same justification as step $k$ in derivations $\delta_1$ and $\delta_2$, respectively. Derivation $\delta'_1$ shows that $\Delta \vdash Q$ and derivation $\delta'_2$ shows that $\Delta \vdash Q$. So, $\Delta \vdash$. 

\[\square\]

**Lemma 4.7.4.2.** If $\Delta' \not\vdash$ and $\Delta'$ contains no closed terms having odd-numbered subscripts, then there exists a maximally consistent, $\omega$-complete set $\Delta$ such that $\Delta' \subseteq \Delta$.

**Proof.** Suppose $\Delta' \not\vdash$ and $\Delta'$ contains no closed terms having odd-numbered subscripts. By Lemma 7.2.9 there is an enumeration of the sentences of FPL. Let $\mathcal{P}_1, \ldots, \mathcal{P}_k, \ldots$ be such an enumeration. Define a sequence of sets as follows:

$$
\Delta_1 = \Delta'
$$

$$
\Delta_{n+1} = \begin{cases} 
\Delta_n & \text{if } \Delta_n \cup \{\mathcal{P}_n\} \vdash; \\
\Delta_n \cup \{\mathcal{P}_n\} & \text{if } \Delta_n \cup \{\mathcal{P}_n\} \not\vdash \text{ and } \mathcal{P}_n \text{ is not of the form } \neg \forall x Q; \\
\Delta_n \cup \{\neg \forall x Q, \neg Q(c/x)\} & \text{if } \mathcal{P}_n = \neg \forall x Q \text{ and } \Delta_n \cup \{\mathcal{P}_n\} \not\vdash \text{ and } c \text{ is an odd-subscripted closed term not occurring in } \Delta_n \text{ or in } \mathcal{P}_n.
\end{cases}
$$

Let $\Delta$ be $\bigcup\{\Delta_k : k \geq 1\}$.

We now prove the following claims about $\Delta$.

**Claim 1** For each $k \geq 1$, $\Delta_k \not\vdash$.

**Claim 2** For each $i, j \geq 1$, if $i \leq j$ then $\Delta_i \subseteq \Delta_j$. 
Claim 3 If $P_k \in \Delta$, then $P_k \in \Delta_{k+1}$.

Claim 4 $\Delta \not\vdash$.

Claim 5 If $Q \not\in \Delta$, then $\Delta \cup \{Q\} \vdash$.

Claim 6 If $\neg \forall x Q \in \Delta$, then for some closed term $c$, $\neg Q(c/x) \in \Delta$.

1. Proof is by strong induction on $k$

   (Basis) $\Delta_1 \not\vdash$ since $\Delta_1$ is $\Delta'$ and by hypothesis $\Delta' \not\vdash$.

   (Induction Step) Suppose $\Delta_n \not\vdash$ for all $n \leq k$. Either $\Delta_k \cup \{P_k\} \vdash$ or $\Delta_k \cup \{P_k\} \not\vdash$.

   Case 1. Suppose $\Delta_k \cup \{P_k\} \vdash$. Then $\Delta_{k+1}$ is $\Delta_k$. So, by the induction hypothesis, $\Delta_{k+1} \not\vdash$.

   Case 2. Suppose $\Delta_k \cup \{P_k\} \not\vdash$. $P_k$ is either $\neg \forall x Q$ or it isn’t.

   Subcase 2.1. Suppose $P_k$ is not $\neg \forall x Q$. Then $\Delta_{k+1}$ is $\Delta_k \cup \{P_k\}$. So, $\Delta_{k+1} \not\vdash$ by the hypothesis of Case 2.

   Subcase 2.2. Suppose $P_k$ is $\neg \forall x Q$. Then $\Delta_{k+1}$ is $\Delta_k \cup \{\neg \forall x Q, \neg Q(c/x)\}$, where $c$ is an odd-subscripted closed term and does not occur in $\Delta_k$ or $P_k$. Since $\Delta_k \cup \{P_k\} \not\vdash$ by hypothesis, it follows by Lemma 7.2.12 that $\Delta_k \cup \{P_k, \neg Q(c/x)\} \not\vdash$, too. Hence, $\Delta_{k+1} \not\vdash$.

So, in all cases, $\Delta_{k+1} \not\vdash$. 
So, by strong induction on $k$, $\Delta_k \not\models$ for all $k \geq 1$.

2. Proof is by strong induction on $j$ for a fixed $i$.

(Basis) Suppose $i \leq j$ and $j = 1$. Then $i = 1$ and $\Delta_i$ is $\Delta_j$, and so $\Delta_i \subseteq \Delta_j$.

(Induction Step) Suppose $\Delta_i \subseteq \Delta_k$ for every $k$ such that $i \leq k \leq j$. By the induction hypothesis, $\Delta_i \subseteq \Delta_j$. But by the definition of $\Delta_{j+1}$, $\Delta_{j} \subseteq \Delta_{j+1}$. So, $\Delta_i \subseteq \Delta_{j+1}$.

So, by strong induction on $j$ holding $i$ fixed, $\Delta_i \subseteq \Delta_j$.

3. Suppose $P_k \in \Delta$. Then $P_k \in \Delta_j$ for some $j \geq 1$. Either $j \leq k$ or $k < j$.

Case 1. Suppose that $j \leq k$. Then by Claim 2, $\Delta_j \subseteq \Delta_k \subseteq \Delta_{k+1}$. Since $P_k \in \Delta_j$, $P_k \in \Delta_{k+1}$.

Case 2. Suppose that $k < j$. Suppose for reductio that $P_k \not\in \Delta_{k+1}$. Then $\Delta_k \cup \{P_k\} \not\models$. But since $k < j$, $\Delta_k \subseteq \Delta_j$ by Claim 2. So, $\Delta_k \cup \{P_k\} \subseteq \Delta_j \cup \{P_k\}$. By Lemma 7.2.6, it follows that $\Delta_j \cup \{P_k\} \not\models$. But $P_k \in \Delta_j$. So, $\Delta_j \not\models$, contrary to Claim 1. So, by reductio, $P_k \in \Delta_{k+1}$.

In either case, $P_k \in \Delta_{k+1}$.
4. Suppose for reductio that $\Delta \vdash x$. By Lemma 7.2.3 there is a $\Theta \vdash x$ such that $\Theta \subseteq \Delta$ and $\Theta$ is finite. Hence, there is a $k_0$ such that $k_0 = \max\{k : \mathcal{P}_k \in \Theta\}$.

By Claim 3, $\mathcal{P}_k \in \Delta_{k+1}$ for each $\mathcal{P}_k \in \Theta$. So by Claim 2, $\Delta_{k+1} \subseteq \Delta_{k_0+1}$ for each $k$ such that $\mathcal{P}_k \in \Theta$. Hence, $\Theta \subseteq \Delta_{k_0+1}$. So by Lemma 7.2.6, $\Delta_{k_0+1} \vdash$, contrary to Claim 1. So, by reductio, $\Delta \not\vdash x$.

5. Suppose $Q \not\in \Delta$. $Q$ is $\mathcal{P}_{k_1}$ for some $k_1 \geq 1$. So, $\mathcal{P}_{k_1} \not\in \Delta$. Hence, $\mathcal{P}_{k_1} \not\in \Delta_{k_1+1}$. Hence, $\Delta_{k_1} \cup \{\mathcal{P}_{k_1}\} \vdash$. But $\Delta_{k_1} \subseteq \Delta$, so $\Delta_{k_1} \cup \{\mathcal{P}_{k_1}\} \subseteq \Delta \cup \{\mathcal{P}_{k_1}\}$, i.e., $\Delta \cup \{Q\}$. So, by Lemma 7.2.6, $\Delta \cup \{Q\} \vdash$.

6. Suppose $\neg \forall x Q \in \Delta$. Then $\neg \forall x Q$ is $\mathcal{P}_{k_2}$ for some $k_2 \geq 1$. By Claim 3, $\mathcal{P}_{k_2} \in \Delta_{k_2+1}$.

Since $\mathcal{P}_{k_2}$ is $\neg \forall x Q$, $\Delta_{k_2+1}$ is $\Delta_{k_2} \cup \{\mathcal{P}_{k_2}, \neg Q(c/x)\}$ for some odd-subscripted closed term $c$ not in $\Delta_{k_2}$ or $\mathcal{P}_{k_2}$. Hence, $\neg Q(c/x) \in \Delta_{k_2+1} \subseteq \Delta$.

Claims 4–6 show that $\Delta$ is maximally consistent, $\omega$-complete, and $\Delta' \subseteq \Delta$.

\[\square\]

**Lemma 4.7.4.3.** If $\Delta$ is $\omega$-complete and maximally consistent, then $\Delta \not\models x$.

**Proof.** Suppose $\Delta$ is $\omega$-complete and maximally consistent. Consider the canonical model of $\Delta$, $\mathcal{I}_\Delta$. Recall that by Lemma 7.3.4 $\mathcal{I}_\Delta$ is a model. Hence $\mathcal{U}_{\mathcal{I}_\Delta}$ and $\hat{\mathcal{U}}_{\mathcal{I}_\Delta}$ are not empty and $\mathcal{I}_\Delta$ has at least one assignment $d_0$.

**Claim** For any sentence $\mathcal{P}$, $\mathcal{P}$ is true on $\mathcal{I}_\Delta$ iff $\mathcal{P} \in \Delta$.
We prove the claim by induction on the length of $\mathcal{P}$.

**Basis.** Suppose $\mathcal{P}$ is of length 0. Then either (i) $\mathcal{P}$ is $A c_1 \ldots c_n$ where $A$ is an $n$-ary predicate letter and $c_1, \ldots, c_n$ are closed terms, or (ii) $\mathcal{P}$ is $c_1 = c_2$, where $c_1$ and $c_2$ are closed terms, or (iii) $\mathcal{P}$ is $c_1 \sqsubseteq c_2$, where $c_1$ and $c_2$ are closed terms.

**Case 1.** Suppose $\mathcal{P}$ is $A c_1 \ldots c_n$ where $A$ is an $n$-ary predicate letter and $c_1, \ldots, c_n$ are closed terms.

(L-to-R) Suppose $\mathcal{P}$ is true on $I_\Delta$. Then $d_0, I_\Delta \models A c_1 \ldots c_n$. Hence, $\langle r(c_1, d_0, I_\Delta), \ldots, r(c_n, d_0, I_\Delta) \rangle \in v_{I_\Delta}(A)$. So, by Def 7.8(3), $A c_1 \ldots c_n \in \Delta$. So, $\mathcal{P} \in \Delta$.

(R-to-L) Let $d$ be an assignment for $I_\Delta$. Suppose $\mathcal{P} \in \Delta$, i.e., $A c_1 \ldots c_n \in \Delta$. Let $d$ be an assignment for $I_\Delta$. Since $\mathcal{P} \in \Delta$, $\langle r(c_1, d, I_\Delta), \ldots, r(c_n, d, I_\Delta) \rangle \in v_{I_\Delta}(A)$. Hence, $d, I_\Delta \models A c_1 \ldots c_n$, i.e., $d, I_\Delta \models \mathcal{P}$. Generalizing on $d$, it follows that $\mathcal{P}$ is true on $I_\Delta$, i.e. $\mathcal{P} \in \Delta$.

**Case 2.** Suppose $\mathcal{P}$ is $c_1 = c_2$, where $c_1$ and $c_2$ are closed terms.

(L-to-R) Suppose $\mathcal{P}$ is true on $I_\Delta$. Then $d_0, I_\Delta \models c_1 = c_2$. Hence, $r(c_1, d_0, I_\Delta)$ is $r(c_2, d_0, I_\Delta)$. So, by Lemma 7.3.1(5), $c_1 \Delta$ is $c_2 \Delta$. So, by Lemma 7.3.1(2), $c_2 \in c_1 \Delta$. By Def 7.7, $c_1 = c_2 \in \Delta$, i.e. $\mathcal{P} \in \Delta$.

(R-to-L) Let $d$ be an assignment for $I_\Delta$. Suppose $\mathcal{P} \in \Delta$. Then $c_1 = c_2 \in \Delta$. So, by Lemma 7.3.1(2), $c_1 \Delta$ is $c_2 \Delta$. So, by Lemma 7.3.1(5), $r(c_1, d, I_\Delta)$ is $r(c_2, d, I_\Delta)$. So, $d, I_\Delta \models c_1 = c_2$. Generalizing on $d$, it follows that $c_1 = c_2$ is true on $I_\Delta$, i.e. $\mathcal{P}$ is true on $I_\Delta$. 
Case 3. Suppose $\mathcal{P}$ is $c_1 \sqsubseteq c_2$, where $c_1$ and $c_2$ are closed terms.

(L-to-R) Suppose $\mathcal{P}$ is true on $\mathcal{I}_\Delta$. Then $d_0, \mathcal{I}_\Delta \models c_1 \sqsubseteq c_2$. So, $r(c_1, d_0, \mathcal{I}_\Delta) \subseteq r(c_2, d_0, \mathcal{I}_\Delta)$. By Lemma 7.3.2(4), $c_{1\Delta^*} \subseteq c_{2\Delta^*}$. But $c_1 \in c_{1\Delta^*}$ by Lemma 7.3.2(1), so $c_1 \in c_{2\Delta^*}$. So, by Def 7.9, $c_1 \sqsubseteq c_2 \in \Delta$, i.e. $\mathcal{P} \in \Delta$.

(R-to-L) Suppose $\mathcal{P} \in \Delta$. Then $c_1 \sqsubseteq c_2 \in \Delta$. Let $d$ be an assignment for $\mathcal{I}_\Delta$. Then $c_1 \in c_{2\Delta^*}$. So by Lemma 7.3.2(2), $c_{1\Delta^*} \subseteq c_{2\Delta^*}$. So, by Lemma 7.3.2(4), $r(c_1, d, \mathcal{I}_\Delta) \subseteq r(c_2, d, \mathcal{I}_\Delta)$. Hence, $d, \mathcal{I}_\Delta \models c_1 \sqsubseteq c_2$. Generalizing on $d$, it follows that $c_1 \sqsubseteq c_2$ is true on $\mathcal{I}_\Delta$, i.e. $\mathcal{P}$ is true on $\mathcal{I}_\Delta$.

In each case, $\mathcal{P} \in \Delta$ iff $\mathcal{P}$ is true on $\mathcal{I}_\Delta$.

Induction Step. Suppose the claim holds for all sentences of length $k$ or less. Let $\mathcal{P}$ be a sentence of length $k + 1$. There are three cases.

Case 1. Suppose $\mathcal{P}$ is $\neg Q$

(L-to-R) Suppose $\mathcal{P}$ is true on $\mathcal{I}_\Delta$. Then $d_0, \mathcal{I}_\Delta \models \mathcal{P}$, i.e., $d_0, \mathcal{I}_\Delta \models \neg Q$. So, $d_0, \mathcal{I}_\Delta \not\models Q$. Hence, $Q$ is not true on $\mathcal{I}_\Delta$. So, by the induction hypothesis, $Q \notin \Delta$. So, by Lemma 7.2.8(1), $\neg Q \in \Delta$, i.e., $\mathcal{P} \in \Delta$.

(R-to-L) Suppose $\mathcal{P} \in \Delta$, i.e. $\neg Q \in \Delta$. Then $Q \notin \Delta$ by Lemma 7.2.8(1). By the induction hypothesis, $Q$ is not true on $\mathcal{I}_\Delta$. Hence, by Lemma 4.1.3, $Q$ is false on $\mathcal{I}_\Delta$. Let $d$ be an assignment on $\mathcal{I}_\Delta$. Then $d, \mathcal{I}_\Delta \not\models Q$. Hence, $d, \mathcal{I}_\Delta \models \neg Q$, i.e., $d, \mathcal{I}_\Delta \models \mathcal{P}$. Generalizing on $d$, $\mathcal{P}$ is true on $\mathcal{I}_\Delta$. 


Case 2. Suppose \( P \) is \( Q \to R \).

(L-to-R) Suppose \( P \) is true on \( I_\Delta \). Then \( d_0, I_\Delta \models P \), i.e., \( d_0, I_\Delta \models Q \to R \). So, either 
\[ d_0, I_\Delta \not\models Q \text{ or } d_0, I_\Delta \models R. \]
Hence, either \( Q \) is not true on \( I_\Delta \) or \( R \) is not false on \( I_\Delta \). So, by Lemma 4.1.3, either \( Q \) is not true on \( I_\Delta \) or \( R \) is true on \( I_\Delta \). So, by the induction hypothesis, either \( Q \not\in \Delta \) or \( R \in \Delta \). Hence, either \( Q \) is not true on \( I_\Delta \) or \( R \) is false on \( I_\Delta \). So, by Lemma 7.2.8(2), \( Q \to R \in \Delta \), i.e., \( P \in \Delta \).

(R-to-L) Suppose \( P \in \Delta \), i.e. \( Q \to R \in \Delta \). Then either \( Q \not\in \Delta \) or \( R \in \Delta \) by Lemma 7.2.8(2). By the induction hypothesis, either \( Q \) is not true on \( I_\Delta \) or \( R \) is true on \( I_\Delta \). Let \( d \) be an assignment on \( I_\Delta \). Then either 
\[ d, I_\Delta \not\models Q \text{ or } d, I_\Delta \models R. \]
Hence, \( d, I_\Delta \models Q \to R \), i.e., \( d, I_\Delta \models P \). Generalizing on \( d \), \( P \) is true on \( I_\Delta \).

Case 3. Suppose \( P \) is \( \forall x Q \).

(L-to-R) Suppose \( P \) is true on \( I_\Delta \). Then \( d_0, I_\Delta \models P \), i.e., \( d_0, I_\Delta \models \forall x Q \). So, 
\[ d_0(u/x), I_\Delta \models Q \text{ for all } u \in \hat{U}_{I_\Delta}. \]
Hence, \( d_0(r(c, d_0, I_\Delta)/x), I_\Delta \models Q \) for every closed term \( c \). So, by Lemma 4.1.5, \( d_0, I_\Delta \models Q(c/x) \) for every closed term \( c \). Hence, \( Q(c/x) \) is not false on \( I \). Hence, by Lemma 4.1.3, \( Q(c/x) \) is true on \( I_\Delta \). By the induction hypothesis, \( Q(c/x) \in \Delta \) for all closed terms \( c \). So, by Lemma 7.2.8(3), \( \forall x Q \in \Delta \), i.e., \( P \in \Delta \).

(R-to-L) Suppose \( P \in \Delta \), i.e. \( \forall x Q \in \Delta \). Then \( Q(c/x) \in \Delta \) for all closed terms \( c \) by Lemma 7.2.8(c). By the induction hypothesis, \( Q(c/x) \) is true on \( I_\Delta \). Let \( d \) be an assignment on \( I_\Delta \). Then \( d, I_\Delta \models Q(c/x) \) for all \( c \). Hence, by Lemma 4.1.5, \( d(r(c, d, I_\Delta)/x), I_\Delta \models Q \) for all \( c \), i.e., \( d(u/x), I_\Delta \models Q \) for all \( u \in \hat{U}_{I_\Delta} \). So, \( d, I_\Delta \models \forall x Q \). Generalizing on \( d \), \( P \) is true on \( I_\Delta \).
The claim follows by strong induction on the length of $P$.

By the claim, every member of $\Delta$ is true on $I_\Delta$. Define $d_0(x) = \{a_\Delta\}$ for every variable $x$. Let $P \in \Delta$. By the claim, $d_0, I_\Delta \models P$. Hence, $P$ is not false on $I_\Delta$. So, by Lemma 4.1.3, $P$ is true on $I_\Delta$. Generalizing on $P$, it follows that $I_\Delta$ is a model of $\Delta$. Hence, $\Delta \not\models$.

\begin{proof}
Lemma 4.7.4.4. If $\Delta \not\models$ and $\Theta \subseteq \Delta$, then $\Theta \not\models$.

Proof. Suppose $\Delta \not\models$ and $\Theta \subseteq \Delta$. Then there is a model $I$ on which every member of $\Delta$ is true. But since $\Theta \subseteq \Delta$, every member of $\Theta$ is true on $I$. So, $\Theta \not\models$.
\end{proof}

Lemma 4.7.4.5. If $\Delta$ has a model $I$, then $\Delta$ has a model whose domain is $U_I$.

Proof. Suppose $I$ is a model of $\Sigma$. Define a new model $I'$ as follows: $U_{I'} = U_I$, $\hat{U}_{I'} = \hat{U}_I$, and $v_{I'}$ assigns the same values as $v_I$ to predicate letters. Let $v_{I'}(a)$ be $v_{I'}(\bar{a})$.

We prove the lemma using a series of claims.

Claim 1 $r(t, d, I')$ is $r(\bar{t}, d, I)$ for any term $t$.

Let $t$ be a term. Then $t$ is either a variable, a constant, or a list.

Case 1. Suppose $t$ is a variable. Let $d$ be an assignment for $I$, $I'$. Then,

$$r(t, d, I') = r(t, d, I)$$

is $d(t)$, since $t$ contains no closed terms

is $r(\bar{t}, d, I)$.
Case 2. Suppose \( t \) is a constant. Let \( d \) be an assignment for \( I, I' \). Then,

\[
\begin{align*}
    r(t, d, I') &\text{ is } v_{I'}(t) \\
     &\text{ is } v_I(t) \\
     &\text{ is } r(t, d, I).
\end{align*}
\]

Case 3. Suppose \( t \) is a list where \( a_1, \ldots, a_n \) are the constants occurring at any depth in \( t \). Then,

\[
\begin{align*}
    r(t, d, I') &\text{ is } \bigcup r(a_1, d, I'), \ldots, r(a_n, d, I') \text{ (by Corollary 4.1.1)} \\
     &\text{ is } \bigcup v_{I'}(a_1), \ldots, v_{I'}(a_n) \\
     &\text{ is } \bigcup v_I(a_1), \ldots, v_I(a_n) \\
     &\text{ is } \bigcup r(\bar{a}_1, d, I), \ldots, r(\bar{a}_n, d, I) \\
     &\text{ is } r(\bar{t}, d, I) \text{ (by Corollary 4.1.1)}.
\end{align*}
\]

So, in any case, it follows that \( r(t, d, I') \) is \( r(\bar{t}, d, I) \).

Claim 2  For any formula \( P \) and any assignment \( d \) for \( I, I' : d, I \models \bar{P} \text{ iff } d, I' \models P \).

We prove Claim 2 by induction on the length of \( P \).

(Basis) Suppose \( P \) is length 0. Let \( d \) be an assignment for \( I, I' \). Then \( P \) is \( \mathcal{A}t_1 \ldots t_n \), \( t_1 = t_2 \), or \( t_1 \sqsubseteq t_2 \) for terms \( t_1, \ldots, t_n \).

Hence, \( \bar{P} \) is either \( \mathcal{A}\bar{t}_1 \ldots \bar{t}_n \), \( \bar{t}_1 = \bar{t}_2 \), or \( \bar{t}_1 \sqsubseteq \bar{t}_2 \). In each case it follows from Claim 1 that \( d, I \models \bar{P} \text{ iff } d, I' \models P \).
(Induction Step) Let \(d\) be an assignment for \(\mathcal{I}, \mathcal{I}'\). Suppose \(d, \mathcal{I} \models \overline{\mathcal{P}}\) iff \(d, \mathcal{I}' \models \mathcal{P}\) for all \(\mathcal{P}\) of length \(k\) or less. Suppose \(\mathcal{P}\) is of length \(k + 1\). Then \(\mathcal{P}\) is either \(\neg Q\), \(Q \rightarrow R\), or \(\forall x Q\).

Case 1. Suppose \(\mathcal{P}\) is \(\neg Q\). Then,

\[
d, \mathcal{I}' \models \mathcal{P} \text{ iff } d, \mathcal{I}' \models \neg Q \\
\quad \text{ iff } d, \mathcal{I}' \not\models Q \\
\quad \text{ iff } d, \mathcal{I} \not\models \overline{Q}, \text{ by the induction hypothesis} \\
\quad \text{ iff } d, \mathcal{I} \models \neg \overline{Q} \\
\quad \text{ iff } d, \mathcal{I} \models \mathcal{P}.
\]

Case 2. Suppose \(\mathcal{P}\) is \(Q \rightarrow R\). Then,

\[
d, \mathcal{I}' \models \mathcal{P} \text{ iff } d, \mathcal{I}' \models Q \rightarrow R \\
\quad \text{ iff } d, \mathcal{I}' \not\models Q \text{ or } d, \mathcal{I}' \models R \\
\quad \text{ iff } d, \mathcal{I} \not\models \overline{Q} \text{ or } d, \mathcal{I} \models \overline{R}, \text{ by the induction hypothesis} \\
\quad \text{ iff } d, \mathcal{I} \models \overline{Q} \rightarrow \overline{R} \\
\quad \text{ iff } d, \mathcal{I} \models \mathcal{P}.
\]

Case 3. Suppose \(\mathcal{P}\) is \(\forall x Q\). Then,

\[
d, \mathcal{I}' \models \mathcal{P} \text{ iff } d, \mathcal{I}' \models \forall x Q \\
\quad \text{ iff } d(u/x), \mathcal{I}' \models Q \text{ for all } u \in \mathcal{U}_{\mathcal{I}'} \\
\quad \text{ iff } d(u/x), \mathcal{I} \models \overline{Q} \text{ for all } u \in \mathcal{U}_{\mathcal{I}}, \text{ by the ind. hyp.} \\
\quad \text{ iff } d, \mathcal{I} \models \forall x \overline{Q} \\
\quad \text{ iff } d, \mathcal{I} \models \mathcal{P}.
\]
In either case, \( d, \mathcal{I} \models \overline{\mathcal{P}} \) iff \( d, \mathcal{I}' \models \mathcal{P} \).

So, by induction on the length of \( \mathcal{P} \), Claim 2 follows.

Returning now to the main argument, let \( \mathcal{P} \in \Delta \). Let \( d \) be an assignment for \( \mathcal{I}' \). Then \( \overline{\mathcal{P}} \in \overline{\Delta} \), and \( d \) is an assignment for \( \mathcal{I} \). Hence, \( \overline{\mathcal{P}} \) is true on \( \mathcal{I} \), and \( d, \mathcal{I} \models \overline{\mathcal{P}} \). By Claim 2, it follows that \( d, \mathcal{I}' \models \mathcal{P} \).

Generalizing on \( d \), \( \mathcal{P} \) is true on \( \mathcal{I}' \). Generalizing on \( \mathcal{P} \), \( \mathcal{I}' \) is a model of \( \Delta \). So, \( \Delta \) has a model whose domain is \( U_\mathcal{I} \).

\[ \square \]

4.7.5 Main Completeness Lemma

**Lemma 4.7.5.1.** If \( \Delta \not\models \), then \( \Delta \not\models \).

*Proof.* Suppose \( \Delta \not\models \). By Lemma 7.4.1, \( \overline{\Delta} \not\models \). But \( \overline{\Delta} \) contains no odd-subscripted constants. So, by Lemma 7.4.2, there is a maximally consistent and \( \omega \)-complete \( \Delta' \) such that \( \overline{\Delta} \subseteq \Delta' \).

By Lemma 7.4.3, \( \Delta' \not\models \). By Lemma 7.4.4, \( \overline{\Delta} \not\models \). By Lemma 7.4.5, \( \Delta \not\models \). \[ \square \]

4.7.6 Completeness for FPL

**Metatheorem 4.7.1** (Strong Completeness Metatheorem). If \( \Gamma \models \mathcal{P} \), then \( \Gamma \vdash \mathcal{P} \).

*Proof.* Let \( \Gamma \) be a set of FPL sentences. Suppose \( \Gamma \models \mathcal{P} \). By Lemma 7.2.11, \( \Gamma \cup \{\neg \mathcal{P} \} \models \).

By the Main Completeness Lemma, \( \Gamma \cup \{\neg \mathcal{P} \} \vdash \). By Lemma 7.2.4, \( \Gamma \vdash \mathcal{P} \). \[ \square \]
4.8 Model Theory for FPL

4.8.1 Compactness

**Metatheorem 4.8.1** (Countable Compactness). Every finite subset of $\Gamma$ has a model in FPL iff $\Gamma$ has a model in FPL.

*Proof. (L-to-R)* We prove the contrapositive. Suppose there is no model of $\Gamma$, i.e., $\Gamma \notmodels$. Then $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$ for any $\mathcal{P}$. So, by Metatheorem 7.1, $\Gamma \vdash \mathcal{P}$ and $\Gamma \vdash \neg \mathcal{P}$ for any $\mathcal{P}$. Hence, $\Gamma \vdash$. So, by Lemma 7.2.3, there is a finite $\Gamma'$ such that $\Gamma' \vdash$. So, there is a $\mathcal{Q}$ such that $\Gamma' \vdash \mathcal{Q}$ and $\Gamma' \vdash \neg \mathcal{Q}$. So, by Metatheorem 6.2, $\Gamma' \models \mathcal{Q}$ and $\Gamma' \models \neg \mathcal{Q}$. It follows that $\Gamma' \models$, i.e., there is no model of $\Gamma'$.

*(R-to-L)* Suppose $\Gamma$ has a model $\mathcal{I}$. Let $\Delta \subseteq \Gamma$. Then $\mathcal{I}$ is a model of $\Delta$.

\[ \square \]

4.8.2 Löwenheim-Skolem Theorems

**Lemma 4.8.2.1.** If $\Gamma \notmodels$, then there is no sentence $\mathcal{P}$ such that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$.

*Proof. Suppose $\Gamma \notmodels$. Suppose for reductio that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$ for some $\mathcal{P}$. Since $\Gamma \notmodels$, there is a model $\mathcal{I}$ of $\Gamma$. Hence, $\mathcal{P}$ and $\neg \mathcal{P}$ are both true on $\mathcal{I}$. Let $d$ be an assignment for $\mathcal{I}$. Then $d, \mathcal{I} \models \mathcal{P}$ and $d, \mathcal{I} \models \neg \mathcal{P}$. Since $d, \mathcal{I} \models \neg \mathcal{P}$, $d, \mathcal{I} \notmodels \mathcal{P}$. (contradiction) Hence, there is no $\mathcal{P}$ such that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$. \[ \square \]

**Lemma 4.8.2.2.** If there is no sentence $\mathcal{P}$ such that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$, then $\Gamma \notmodels$.

*Proof. Suppose there is no sentence $\mathcal{P}$ such that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$. By Metatheorem 6.2, it follows that there is no sentence $\mathcal{P}$ such that $\Gamma \vdash \mathcal{P}$ and $\Gamma \vdash \neg \mathcal{P}$. So, $\Gamma \notmodels$. \[ \square \]

**Metatheorem 4.8.2** (Downward Löwenheim-Skolem). If $\Gamma$ has an infinite model, then $\Gamma$ has a countable model.
Proof. Suppose $\Gamma$ has an infinite model. Then $\Gamma \not|\not\models$. By Lemma 8.1.1, there is no sentence $\mathcal{P}$ such that $\Gamma \models \mathcal{P}$ and $\Gamma \models \neg \mathcal{P}$. So, by Lemma 8.1.2, $\Gamma \not|\models \mathcal{P}$. By Lemma 7.4.1, $\Gamma \not|\models$. By Lemma 7.4.2, $\Gamma \subseteq \Delta$, where $\Delta$ is maximally consistent and $\omega$-complete. By Lemma 8.1.4, $I_\Delta$ is a model of $\Delta$. Since $\Gamma \subseteq \Delta$, $I_\Delta$ is a model of $\Gamma$. By Lemma 7.4.5, $\Gamma$ has a model whose domain is $U_{I_\Delta}$. By Lemma 7.3.3, $U_{I_\Delta}$ is countable. Hence, $\Gamma$ has a countable model.

**Metatheorem 4.8.3** (Upward Löwenheim-Skolem). If $\Gamma$ has a countably infinite model, then $\Gamma$ has an uncountable model.

Proof. The proof is omitted, but the idea is to extend FPL with set of uncountably many new constants and extend $\Gamma$ so that its canonical model includes equivalence classes for the new constants. It will follow that the extended, uncountable model is a model of $\Gamma$. For details, see Chang and Keisler 1990:67–68.

4.8.3 Failure of Compactness for FPL-

Consider a variant of FPL that only quantifies over pluralities that can be named by lists, but is otherwise identical to FPL. The properties of such a logic might be of interest to someone working with databases, since it would allow one to distinguish between singular and plural predication in query statements. Call this language FPL-. One might reasonably guess that FPL- is expressively weaker than FPL, but this is not the case. Restricting the domain of quantification in this way actually increases the expressive power of FPL by allowing the language to detect whether the domain is finite. It follows that FPL- is not compact and therefore not complete. We will now prove these results.

Let FPL- be the logic that results from modifying the definition of a model of FPL such that $\hat{\mathcal{U}}$ contains only finite subsets of the domain $\mathcal{U}$. Then:

**Lemma 4.8.3.1.** For any maximally consistent set of FPL- sentences $\Delta$, $I_\Delta$ is a model of FPL-.
Proof. Since we have already show in Lemma 7.3.4 that $\mathcal{I}_\Delta$ is a model of FPL for any maximally consistent set $\Delta$ of FPL sentences, it suffices to show that if $S \in \hat{U}_I$ then $S$ is finite. But if $S \in \hat{U}_I$ then by Def 7.8(2) $S$ is a set of $n \text{-} many$ members of $\mathcal{U}_\Delta$. Hence, trivially, $S$ is finite. So, $\mathcal{I}_\Delta$ is a model of FPL-.

$\square$

Lemma 4.8.3.2. $\hat{U}_{I_\Delta}$ is countable.

Proof. By Lemma 7.3.3, $\mathcal{U}_I$ is countable. But the cardinality of any set of finite subsets of a set $S$ is either finite (if $|S|$ is finite) or $|S|^6$. So, $\hat{U}_{I_\Delta}$ is countable.

$\square$

Lemma 4.8.3.3. $\mathcal{U}_I$ is finite iff $\exists x \forall y \sqsubseteq x$ is true on $\mathcal{I}$.

Proof. (L-to-R) Suppose $\mathcal{U}_I$ is finite for some model $\mathcal{I}$. Suppose for reductio that $\exists x \forall y \sqsubseteq x$ is false on $\mathcal{I}$. Since $\mathcal{U}_I$ is nonempty, there is at least one assignment $d_0$ defined on $\mathcal{I}$. So, $d_0, \mathcal{I} \not\models \exists x \forall y \sqsubseteq x$. So, $d_0, \mathcal{I} \models \neg \exists x \forall y \sqsubseteq x$. Hence, $\neg \exists x \forall y \sqsubseteq x$ is not false on $\mathcal{I}$. So, by Lemma 4.1.3, $\neg \exists x \forall y \sqsubseteq x$ is true on $\mathcal{I}$.

Let $d$ be an assignment for $\mathcal{I}$. Then $d, \mathcal{I} \models \neg \exists x \forall y \sqsubseteq x$. So, $d, \mathcal{I} \not\models \exists x \forall y \sqsubseteq x$, i.e. $d, \mathcal{I} \not\models \forall x \neg \forall y \sqsubseteq x$. So, $d, \mathcal{I} \models \forall x \neg \forall y \sqsubseteq x$. It follows by Def 4.4.3 that for every $u_1 \in \hat{U}_I$, there is a $u_2 \in \hat{U}_I$ such that $u_2 \not\subseteq u_1$. Note that $\hat{U}_I$ is partially ordered by ‘$\subseteq$’, so every member of $\hat{U}_I$ is a member of a $\subseteq$-chain. It follows that no $\subseteq$-chain in $\hat{U}_I$ has a maximal element. Hence, $\hat{U}_I$ is infinite. But as noted in the proof of Lemma 8.3.2, $\hat{U}_I$ and $\mathcal{U}_I$ are either both finite or equicardinal. So, $\mathcal{U}_I$ is infinite, contrary to hypothesis. Hence, by reductio, $\exists x \forall y \sqsubseteq x$ is true on $\mathcal{I}$.

(R-to-L) Suppose $\exists x \forall y \sqsubseteq x$ is true on $\mathcal{I}$. Let $d$ be an assignment on $\mathcal{I}$. Then $d, \mathcal{I} \models \exists x \forall y \sqsubseteq x$, i.e., $d, \mathcal{I} \models \forall x \neg \forall y \sqsubseteq x$. So, $d, \mathcal{I} \not\models \forall x \neg \forall y \sqsubseteq x$. Hence, $d(u_1/x), \mathcal{I} \not\models \neg \forall y \sqsubseteq x$.

\footnote{See Enderton 1977, pg.165 exercise 32.}
for some $u_1 \in \hat{\mathcal{U}}_I$. So, $d(u_1/x), I \models \forall y y \sqsubseteq x$ for some $u_1 \in \hat{\mathcal{U}}_I$. Hence, for some $u_1 \in \hat{\mathcal{U}}_I$, $d(u_1/x, u_2/y), I \models y \sqsubseteq x$ for every $u_2 \in \hat{\mathcal{U}}_I$. So, Def 4.4.3, there is a $u_1 \in \hat{\mathcal{U}}_I$ such that for every $u_2 \in \hat{\mathcal{U}}_I$, $u_2 \subseteq u_1$. It follows that $\hat{\mathcal{U}}_I$ is finite. Hence, $\mathcal{U}_I$ is finite.

\hfill \Box

**Metatheorem 4.8.4.** FPL- is not compact.

*Proof.* Consider the sentences of FPL- that express that there are at least $n$ individuals in the domain. These sentences are constructed in the usual way using singular quantifiers (e.g. ‘$\exists x \exists y y \neq y'$ expresses ‘there are at least two individuals’). Call this set $N$. Every proper subset of $N$ is satisfied by a model with a finite domain, but $N$ can only be satisfied by a model with an infinite domain.

Now consider the set $N \cup \{ \exists x \forall y y \sqsubseteq x \}$. Since $N$ is only satisfied by infinite models and $\exists x \forall y y \sqsubseteq x$ is only satisfied by finite models, it follows that $N \cup \{ \exists x \forall y y \sqsubseteq x \}$ does not have a model. However, for any $\mathcal{P} \in N$, $\{ \mathcal{P} \} \cup \{ \exists x \forall y y \sqsubseteq x \}$ has a model. It follows that for every finite subset $K$ of $N$, $K \cup \{ \exists x \forall y y \sqsubseteq x \}$ has a model. Hence, there is a set of FPL- sentences that is unsatisfiable, every finite subset of which is satisfiable. This shows that FPL- is not compact.

\hfill \Box

**Corollary 4.8.3.1.** FPL- is not complete.

*Proof.* None of the soundness lemmas for FPL make reference to the size of any plurality, so it follows by similar reasoning to Metatheorem 6.2 that FPL- is sound. But if FPL- is sound and complete then it is compact, and FPL- is not compact. So, FPL- is not complete.

\hfill \Box

4.9 References


Chapter 5

Much Ado about ‘Some Things’\textsuperscript{1}

\textsuperscript{1}Shiver, A. To be submitted to \textit{Journal of Philosophy}. 
5.1 Introduction

Quine famously proposed the following criterion of ontological commitment:

A theory is committed to those and only those entities to which the bound
variables of the theory must be capable of referring in order that the affirmations
made in the theory be true. (Quine 1948, 33)

This criterion is complicated by the fact that English has at least two modes of existential
quantiﬁcation.2 When we say that ‘Some dogs are friendly,’ we say that at least one individual
exists such that it is both a dog and friendly. This is called singular quantiﬁcation. Singular
quantiﬁcation binds variables that range over one individual at a time. In this case, the
singular pronoun it is playing the role of the variable. But when we say ‘Some dogs are
surrounding the camp,’ we do not quantify over one individual at a time. If we did we
would be saying something false, since a single dog cannot surround a camp. Instead, we’re
saying that some things exist such that each one of them is a dog and they, together, are
surrounding the camp. This is called plural quantiﬁcation. Plural quantiﬁcation binds plural
variables (e.g. ‘they’, ‘them’, ‘us’, and ‘we’) which range over many individuals at once.

In a formal language the distinction between singular and plural variables is determined
by the language’s denotation function, which distinguishes plural variables by mapping them
one-to-many on the domain of quantiﬁcation. Singular variables are singular because they
are mapped one-to-one. Arguably, then, the real distinction between singular and plural
quantiﬁcation is a distinction in mode of reference.3 In either case, however, the referent(s)
of a singular or plural variable is (are) in the domain of quantiﬁcation. If we’re talking about
some thing, we’re talking about an individual; if we’re talking about some things, then we’re
talking about individuals. As George Boolos (1984, 1985) noticed, sets, classes, and fusions
that aggregate individuals do not seem to enter into the equation.

2We will ignore other complications like quantiﬁer variance and deflationary interpretations of
the quantiﬁer.

3See Oliver and Smiley 2013 for more on plural reference and its role in quantiﬁcation.
Boolos thought this showed the ontological innocence of plural expressions relative to expressions that explicitly refer to aggregate entities. Plural reference involves only reference to individuals, not to higher-order objects. Hence, many philosophers take plural logic to be a nominalistically respectable logic with the descriptive and inferential power of higher-order logics.\(^4\) It is reasonable to suppose, then, that Quine’s criterion of ontological commitment delivers equivalent answers no matter which kind of variable is being bound. Either way, individuals are the subjects of quantification.

One way that plural quantification could fail to satisfy the Quinean picture of ontological commitment would be to fail to be distributive, i.e., if it could be true that some things exist and false or indeterminate that any one of them exists. This would show that the plural quantifier tracks something that the singular quantifier does not.

Any reason to deny the following inference, then, would be a reason to doubt the nominalistic \textit{bona fides} of the plural quantifier:

\textbf{(The Inference)} Some things exist, so at least one of them exists.

Denying this inference may seem crazy. Indeed, three reasons to accept the inference immediately suggest themselves.

\textbf{(The Formal Defense)} Every formal system of plural logic validates The Inference, and there is no obvious, let alone motivated, way to modify the formalism to invalidate the inference.

\textbf{(The Epistemic Defense)} In order to rationally accept that some things exist, one must have evidence for the existence of each one of them. So, you can only have reason to believe the premise if you also have reason to believe the conclusion.

\textbf{(The Metaphysical Defense)} There can only be things (plural) if there are individual things. Individuals are the truth-makers of plural expressions. Hence, counterexamples to The Inference are metaphysically impossible.

\(^4\) Though a few remain suspicious. See Parsons 1990 and Linnebo 2003.
Our task is to point out weaknesses in these arguments and suggest counterexamples to them. If we fail, then all the better for common sense. We hope to show, though, that these arguments are not good reasons to buy into the innocence of plural quantification. For in some cases it seems appropriate to bind ‘they’ but not ‘it’, and this is a sign that ‘they’ is doing more than just referring to many individuals at once.

5.2 The Formal Defense

5.2.1 Models

First, we must contend with the fact that The Inference is valid in all standard systems of plural logic. To see why, we need only look at the set of structures standard plural logics use in their models.

(Plural Structure) A nonempty set $S$ is a plural structure iff there is at least one $x \in S$ such that $x$ is an urelement and for any two or more urelements $y y$ in $S$, there is a $z \in S$ such that $z = \{w : w$ is among $yy\}$.\(^5\)

Given this notion of plural structure, we can describe the semantics of plural logic as follows. A model of plural logic has a plural structure as its domain of quantification and a function that assigns constants to members of the domain and $n$-ary predicates to subsets of the $n^{th}$ Cartesian product of the domain. The denotation function assigns individual variables to the domain’s urelements and assigns plural variables to the domain’s sets. The inclusion predicate, represented using the trumpet ‘≺’ and read ‘is/are among’, is given the obvious interpretation on every model: some thing(s) is (are) among $xx$ iff either it is a member or a subset of $xx$. Identity and the truth-functional operators get the usual treatment. Most important for the present discussion is that an existentially quantified formula is true iff there is an assignment for the bound variable that satisfies the open formula to which the quantifier is prefaced.

\(^5\)Doubled constants and variables (e.g., $yy$) are to be understood as plural constant and plural variables, respectively.
Note that plural structures are constructed so that whenever there is an assignment for a plural variable, there is also an assignment for individual variables. That means the inference ‘\(\exists x xx = aa \therefore \exists x x \prec aa\)’ is valid. For every model on which there is a set named ‘\(aa\)’, there is an urelement among the things represented by that set. This is why quantification distributes in plural logic.

One reason to oppose the possibility of nondistributive quantification is the fear that accepting it would require far-reaching changes to the plural apparatus itself. But this is not so. All it requires is a generalization of what counts as a plural structure. The following modification does the job.

(General Plural Structure) A nonempty set \(S\) is a general plural structure iff for every \(x \in S\), either \(x\) is an urelement or a nonsingleton set of urelements.

Every plural structure is a general plural structure, but not vice versa. This is because, unlike plural structures, a general plural structure need not contain every urelement contained in its subsets and, crucially, need not contain any urelements at all. The structure \(\{\{1, 2\}\}\), for example, is a general plural structure for which there is no assignment validating the distributive inference.

5.2.2 Many-One Identity

The previous section showed that the semantics of plural logic can be modified in a way that blocks The Inference. That modification is simple and effective, but it is not motivated.\(^6\) However, there is a motivation to alter the semantics of plural logic in a way that casts doubt on The Inference. It is the notion that an object is, literally, identical to its parts taken collectively. Call this the Composition as Identity thesis.\(^7\)

\(^6\)Though, as we will see when we take up the metaphysical defense below, we can imagine strange worlds that are better modeled using general plural structures than with the structures of standard plural systems.

\(^7\)This thesis is usually marked as the Strong Composition as Identity thesis in the literature, but since the Weak Composition as Identity thesis (i.e., that composition is a lot like identity) does not motivate changes in the logic of reference and identity, we will ignore it here.
It is easy to see why composition as identity has had so many adherents.\textsuperscript{8} Composition is an intimate relation that behaves like identity in many respects. Where an object goes, so go its parts. Where the parts go, so goes the object. To paraphrase Ross Cameron (2014), you can’t leave your parts at home while you make a quick trip to the store. The properties of your parts also seem to be closely tied to your properties, especially your additive properties like mass and volume. If composition is identity, then these facts have a straightforward explanation.

Common sense, too, seems to be on the side of composition as identity. Consider the following arguments given by Donald Baxter (1988a).

Someone with a six-pack of orange juice may reflect on how many items he has when entering a ‘six items or less’ line in a grocery store. He may think he has one item, or six, but he would be astonished if the cashier said ‘Go to the next line please, you have seven items’. We ordinarily do not think of a six-pack as seven items, six parts plus one whole. (579)

Suppose a man owned some land which he divides into six parcels... He sells off the six parcels while retaining ownership of the whole. That way he gets some cash while hanging on to his land. Suppose the six buyers of the parcels argue that they jointly own the whole and the original owner now owns nothing. Their argument seems right. But it suggests that the whole was not a seventh thing. (585)

Composition as identity has a lot going for it, but it also faces difficulties. The standard semantics for identity cannot accommodate the claim that a collection of things are identical to one thing. Identity in classical logic is always one-one. Thus composition as identity

\textsuperscript{8}The idea has an impressive pedigree. Plato, Leibniz, Kant, and Frege each made proposals to the effect that a whole is identical to its parts. See Cotnoir 2014 for the relevant quotes and references.
theorists must break with logical orthodoxy or else explain away the apparent conflict with the standard semantics for identity.

Much of the recent work on many-one identity takes the latter approach and argues that many-one identity is a feature of how we count. One way to make sense of the identity of the six-pack and the six cans is to say that counting is always relative to a sortal concept. When I count the six-pack using the concept ‘pack’, I find that I have one thing. When I count the very same thing using the concept ‘can’, I find that I have six things. Thus, the cans are the pack because what I counted as six things completely overlaps, in some sense, what I counted as one thing.

Megan Wallace (2011a, 2011b) takes this schema to hold of all counting.

The suggestion is that we can think of thing(s) in various different ways—e.g., as cards, decks, complete sets of suits, etc.—and depending on these various ways of thinking about thing(s), we can yield different numbers or counts in answer to the question how many? We can talk about how many Fs or Gs there are, where F and G stand in for specific sortals, concepts, or kinds. But one can only take a count relative to these sortals, concepts, or kinds; we can never take a count tout court. (Wallace 2011b, 819-20, her emphasis)

Wallace suggests that when we talk about a whole’s parts we’re also talking about the whole, just under a sortal concept that counts it as multiple things. Wallace is not suggesting that identity itself is relative to sortals. Identity is not sensitive to how we carve things up conceptually. Composition as identity theorists are thoroughgoing metaphysical realists who would deny that what exists is at all contingent on our representations. Instead, count-based theories of composition as identity are meant to explain how the world grounds truths about the relations between ways of conceiving of the world. Composition as identity theorists hold

9See Wallace 2011a, 2011b and the references therein for a thorough list of works in this area. See Baxter and Cotnoir 2014 for a recent collection of articles for and against the composition as identity thesis.

10Contra, for example, Gibbard 1975.
that there really are objects that are both one and many while simultaneously denying that
the objects in question have this dual character independent of our representations.

If we take the distinction between plural and singular reference to depend on our rep-
resentational practices and not on any deep features of the world, then we have a reason
to modify our logical apparatus to allow plural and singular variables to share assignments,
albeit under distinct conceptions of the referent. The assignment of a plural variable to a
portion of a model will indicate that the referent is being treated as many things, a singular
variable indicates the referent is being treated as one thing. Neither will be taken to entail
that the referent really is plural or singular outside the context of the assignment.\(^{11}\)

This approach lets us do some strange things with plural expressions. Namely, it allows
us to refer to extended simple objects (if such things there be) by referring plurally to their
fictitious parts. All we need to do is conceive of the simple object as many things and, so
long as our plural term covers the whole object, we denote it just as we would if we were
using a singular term.\(^{12}\)

Let me clarify this thought with a familiar example. Peter van Inwagen (1981) thinks it
is problematic to infer the existence of an arbitrary portion of a human body simply because
it seems that we can refer to it as though it were an individual. Doing so, he suggests,
commits us to the Doctrine of Arbitrary Undetached Parts, or DAUP.

\textbf{(DAUP)} For every material object \(M\), if \(R\) is the region of space occupied by \(M\) at time \(t\),
and if \(\text{sub-}R\) is \textit{any} occupiable sub-region of \(R\) \textit{whatever}, there exists a material object

\(^{11}\)This approach is suggested by David Lewis (1991) and explicitly pursued in Cotnoir 2013.
Cotnoir develops a plural logic that uses sets of partitions of a given domain as the structures
for his models, allowing the same “portion” of the model to be denoted by a plural term in one
partition and a singular term in another partition.

\(^{12}\)This do not commit the champion of extended simples to the absurd view that extended simples
are composite objects (see the previous paragraph). It just means that we can manage to refer to
a simple object by trying, and failing, to refer to all of its parts at once. We fail the refer to the
parts because the simple does not have parts. We succeed in referring to the simple because if it
did have parts we would have succeeded in referring to all of them.
that occupies the region sub-R at t. (van Inwagen 1981, 75 *his emphasis*)

Van Inwagen discusses the following case while arguing against DAUP.

Consider Descartes and his left leg... If DAUP is true, then at any moment during Descartes’s life, there was a thing (problems of multiplicity aside) that was his left leg at that moment. . . . There also existed at that moment, according to DAUP, a thing we shall call *D-minus*, the thing that occupied . . . the region of space that was the set-theoretic difference between the region occupied by Descartes and the region occupied by [Descartes’s left leg]. (1981, 82)

Van Inwagen goes on to argue that if Descartes’s leg were annihilated, then D-minus and Descartes would be at one time distinct and at another time identical. But identity is not temporary, so DAUP is false. Hence, there are no arbitrary undetached parts; D-minus and the leg do not exist at any of the times that Descartes does.

By the composition as identity theorist’s lights, if ‘Descartes’ denotes a single thing, then the plural term ‘D-minus and Descartes’s left leg’ denotes, too, but it does so by representing Descartes as many things. If this is right, then D-minus and Descartes’s left leg existed (in the plural sense) if Descartes did, for the compound term ‘D-minus and Descartes’s left leg’ is coreferential with ‘Descartes’. And this seems right even if, like van Inwagen, we think that D-minus and Descartes’ undetached leg never existed.

Our interest in this case should now be obvious. While it’s true that some things exist—D-minus and Descartes’s leg—none of the individual things being picked out by the singular components of the compound term exist. The Inference fails because the ‘things’ in the plural conception of the case aren’t there. Taken together, they exist—they’re Descartes. But, if we follow van Inwagen in rejecting DAUP, there are no assignments for singular variables within the plural conception of Descartes. D-minus and Descartes’s leg are a referential package deal. It’s true that they exist, but not true that any one of them exists.
So, there is at least one motivated approach to the logic of plural expressions that invalidates The Inference.

5.3 The Epistemic Defense

One mark of a good inference is that compelling evidence for its premises also counts as compelling evidence for its conclusion. In this case, it seems natural to think that if you have evidence for the existence of some things then you automatically have evidence for the existence of each one of them. It would be absurd to believe the plural claim if you didn’t already have compelling evidence for the singular claims.

This intuition is supported by the idea that plural reference and quantification are shorthand for conjunctions of singular referential or quantificational expressions. To say some things exist is to say that the first one exists, and the second one exists, and etc. If this were the case we could derive strong support for the inference via the very plausible principle $M$ of epistemic modal logic.\(^\text{13}\)

\[ (M) \ K(\phi \& \psi) \rightarrow (K\phi \& K\psi) \]

In English: if you know a conjunction then you know each of its conjuncts.

Unfortunately, things are not that simple. Suppose I tell you that Ann and Bob got engaged. If you were to reply ‘Wonderful; two weddings!’ we would strongly suspect that you did not understand the meaning of my utterance. The sentence doesn’t mean that Ann got engaged and Bob got engaged. It means those people who are the referents of the plural term ‘Ann and Bob’ did something together, namely got engaged. A defense of the inference by appeal to the clearly correct principle $M$ would likewise betray a misunderstanding of how plural reference operates in general. Plural reference is not reducible to singular reference and conjunction.

One need not have attitudes about any of the individuals denoted by a plural expression in order to have attitudes toward them collectively. If I’m at a picnic I may regard a swarm of

\(^{13}\)See Holliday 2014.
gnats as annoying and thereby believe that ‘Those gnats are annoying’. I need not regard any of the individuals as being annoying. In fact, I need not regard any of the individuals at all. Similarly, I can think the Seattle Seahawks are a great football team without knowing who any of the players are or whether they are good at their individual positions (a feat I have accomplished personally by half-heartedly watching their last two Super Bowl appearances). I can believe that water is pooling under my refrigerator without having any beliefs about the individual H₂O molecules. It is just not plausible to think that attitudes directed toward some things supervene on or depend in any way on having attitudes toward any one of those things.

A more plausible tack is to argue that you can only have reason to accept the existence of some things if you have reason to accept the existence of each one of them. But even this principle seems to fail. We can see this by drawing an analogy between The Inference and certain cases where it is reasonable to accept some propositions even though it is unreasonable to accept any of them alone. Some propositions exhibit a kind of explanatory coherence that makes them acceptable only as a lot.

A standard (but controversial) example is the case of independent, unreliable witnesses that each report the same fact. For any unreliable witness reporting that P, it is unlikely that she is right that P. But, depending on how many witnesses there are and how initially (in)credible the fact being reported is, the conjunction ‘Witness 1 is right that P, and Witness 2 is right that P, and . . ., and Witness n is right that P’ can be rationally acceptable. As C. I. Lewis put it:

For any one of these reports taken singly, the extent to which it confirms what is reported may be slight. . . . But congruence of the reports establishes a high probability of what they agree upon, by principles of probability determination which are familiar: on any other hypothesis than that of truth-telling, this agreement is highly unlikely. (Lewis 1946, 346)

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14See van Cleve 2011 for a thorough analysis of these kinds of cases.
A similar kind of mutual support seems to hold between an explanandum and explanans:

[T]here can be mutual reinforcement between an explanation and what it explains. Not only does a supposed truth gain credability if we can think of something that would explain it, but also conversely: an explanation gains credibility if it accounts for something we suppose to be true. (Quine and Ullian 1970, 79)

If, for example, you hear a tapping noise coming from the engine of your car and the mechanic tells you that your lifters are shot, the mechanic’s being right makes it more reasonable to believe you actually heard the tapping sounds. But that you heard the tapping also makes it more reasonable to think the mechanic has the right diagnosis.

Better still is Susan Haack’s brilliant example of crossword puzzles.

How reasonable one’s confidence is that a certain entry in a crossword puzzle is correct depends on: how much support is given to this entry by the clue and any intersecting entries that have already been filled in; how reasonable, independently of the entry in question, one’s confidence is that those other already filled-in entries are correct; and how many of the intersecting entries have been filled in. (Haack 1993, 82)

These are examples of epistemic package deals, where some things can be fit for belief in a way that none of them are individually. A similar package deal can be found in particle physics. A boson system is many things, and therefore rightly referred to using a plural pronoun like ‘they’. But bosons can inhabit identical quantum states, making it impossible to distinguish them even numerically via irreflexive relations. It is mysterious how one could manage to refer to one boson and not another when they are mutually indiscernible. At the very least, reference to bosons in identical states is indeterminate.

15See Hawley 2009.
Though they don’t tie the point to reference, Simon Saunders (2006) and Katherine Hawley (2009) take indiscernibility as a reason to refrain from quantifying over bosons, opting instead to treat the system as a single object without proper parts. This is because systems of bosons do explanatory work that does not seem to depend on any explanatory work done by the individual bosons. And, all things being equal, we should not posit the existence of several entities if just one will do.

This principle of parsimony assumes that you can only be committed to the existence of some things if you are committed to the existence of each individually, which is exactly the thesis at issue. It also obscures the fact that a singularized boson system would be a very strange individual—a mereological simple occupying discontinuous locations at some times and a continuous location at others. Another option is available. Instead of denying that bosons exist, we might instead proceed along the lines of our epistemic package deals above. We could hold that the existence of the bosons is rationally acceptable while assigning less credence to the existence of each individual boson than their collective existence. This is reasonable because the bosons play an explanatory role together that they do not play individually. The explanatory work they do together gives us reason to believe they exist, even if it gives us little reason to posit the existence of any one of them.

5.4 The Metaphysical Defense

It would be foolish to look for a counterexample to The Inference in which there were things to quantify over plurally but no individuals, period. If there is a metaphysically possible counterexample to be had it will have to be one in which it is not definitely true whether there is an individual but definitely true that there are some individuals. The conditions of the case will have to be such that from a broad perspective it is obvious that some things exist, but attempt to zoom in for a close look at any one of them and it won’t be found. Here we construct two cases that fit this model, being careful to found them on metaphysical principles that at least some philosophers accept.
5.4.1 Organic Gunk

In *Material Beings*, Peter van Inwagen famously argues that the only things that exist are mereological simples and organisms. Less famous is his discussion of the consequences of that view, particularly that composition is a vague predicate and existence is indeterminate.

*Material Beings* develops and attempts to answer the *Special Composition Question*, i.e. Under what conditions do some things, the *xxs*, compose a thing *y*? The two extreme answers, ‘None’ and ‘All’ (called *nihilism* and *universalism*, respectively), do harm to our ordinary intuitions about what there is, since we recognize some composite objects but not others. Van Inwagen’s proposed answer is, of course, that some things compose another just in case they participate in an activity that constitutes a life. But since it can be vague whether something is one of the things participating in a life, and vague whether something is alive at all, there is sometimes no fact of the matter about whether composition occurs. And if this is right, there is sometimes no fact of the matter about whether a composite object exists in a particular region. More troublesome is that this is a consequence of any causal answer to the special composition question, since “any causal relation must be vague, in the sense that it will be possible for there to be objects that constitute a borderline case of objects standing in that relation.” (1990, 272)

Van Inwagen deals with ontic indeterminacy by proposing a distinction between between “full” and “borderline” objects.

Intuitively, a full object is definitely there or definitely exists and a borderline object dwells in the twilight between between the full daylight of Being and the night of Nonbeing. Now, we have already said several times that the idea of such an object makes no sense at all, but, by a familiar paradox, ideas that make no sense at all can have enormous heuristic value, a value that justifies using them to give intuitive force to elements of a formal semantics. . . . I hereby promise eventually to redeem the promissory note that I have issued by calling certain objects in my universe of discourse “borderline objects.” (1990, 274)
The formal semantics van Inwagen offers sorts the borderline cases of predicate satisfaction from the cases of definite satisfaction, assigning the former to the “frontier” of the predicate and the latter to the “extension” of the predicate. (Presumably, Captain Picard would be assigned to the frontier of ‘x is bald’, Walter White to its extension.) We then evaluate the operator ‘indef φ’ (it is indefinite whether φ) as true iff there are only borderline objects in the extension of φ or the extension of φ is empty but the frontier of φ is not. Hence, if a predicate admits of only borderline cases on a given interpretation, either by having only borderline objects in its extension or by having an empty extension and full or borderline objects in its frontier, it is indefinite whether there is anything that satisfies the predicate on that interpretation; in van Inwagen’s formalism: ‘indef ∃xφ’.

Let us now redeem our promissory note. . . . There are no borderline objects. Not really. . . . What there really are, however, are sets such that it is not definitely true and not definitely false that their members compose anything. (277)

Van Inwagen can get away with this proxy for borderline objects because he has full objects on which to lean. The vagueness of composition cannot contribute to the indeterminacy of mereological simples. Such objects have no proper parts, so they have no proper parts for which it could be vague whether they are caught up in the activity of a life. If there are always simples, then it is always definitely true that something exists.

It is a well-known criticism of van Inwagen’s proposed answer to the special composition question that it is incompatible with the possibility of gunk. As Ted Sider (1993, 286) explains:

[L]et us say that an object is made of ‘atomless gunk’ if it has no (mereological) atoms as parts. If something is made of atomless gunk then it divides forever into smaller and smaller parts—it is infinitely divisible. However, a line segment is infinitely divisible, and yet has atomic parts: the points. A hunk of gunk does not even have atomic parts ‘at infinity’; all parts of such an object have proper parts.
Sider goes on to argue that the metaphysical possibility of gunk is a counterexample to van Inwagen’s view. In a gunky world there are no simples, and it certainly seems possible for a gunky world to contain no living things. But if we consider a gunky world with no living things we find that, given van Inwagen’s view, no material objects exist. But surely some things (lots of things!) exist in such a world. So, van Inwagen’s theory is incompatible with gunky worlds. Further, the argument goes, answers to the special composition question are metaphysically necessary if true. Hence, if van Inwagen’s view is correct, gunky worlds containing material objects are impossible. But gunky worlds containing material objects are possible. So, the argument concludes, van Inwagen’s view is false.

Arguably, though, the spirit of van Inwagen’s view does allow for the possibility of gunky worlds. Imagine an organism that is extremely large and complex. So complex, in fact, that we discover its proper parts are also organisms (all of which are caught up in the activity of the life of the larger organism). Now imagine an organism that is infinitely organically complex, such that at each level of decomposition we find that its proper parts are organisms and there is no inorganic base layer. Call this organic gunk. If organic gunk is possible, then the organicist answer to the special composition question is compatible with the possibility of gunk.

Consider a world in which organic gunk is the only thing that exists (i.e., everything is a part of some organic gunk). Suppose that the conditions for life-constituting activities disappear over some short span of time, effecting a quick extinction of all things. Moreover, suppose that this gradual death starts, progresses, and stops at the same rate for all organisms in the system. Because ‘being caught up in activity constituting a life’ is vague, there will be some interval in this metabolic slowdown during which it is indeterminate whether any of the parts are alive. Hence, it will be indeterminate whether anything exists during this interval. However, it seems right, even in this extreme scenario, to say that some things exist during that interval. I suggest that until it is definitely true that everything in this

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16He has since recanted. See Sider 2013.
world is dead, it is definitely true that some things exist. There is just no fact of the matter which do and which don’t. This is because, as the case is described, the metabolic slowdown is gradual, which means that there is metabolic activity in the system until everything is definitely dead. This activity could not be present unless there were things constituting it through their relative motions.

So while it is indeterminate whether any individual exists (since it is indefinite whether, for any things, their activity constitutes a life), it is still definitely true that some things exist. Otherwise, there would be no causal activity and it would be definitely false that something exists.

Conclusion: these moments between life and death in the organic gunk world look to be counterexamples to The Inference. Some things exist, but it is not definitely true for any one of them that it exists.

5.4.2 Hume’s Dictum and Lewisian Individuals

Some philosophers think that individuality demands a kind of “modal looseness,” such that the identities of individuals do not depend on the existence of other things. This doctrine is called Hume’s Dictum, by reference to a provocative claim in *A Treatise of Human Nature*.

It is easy to observe, that in tracing this relation, the inference we draw from cause to effect, is not derived merely from a survey of these particular objects, and from such a penetration into their essences as may discover the dependance of the one upon the other. *There is no object, which implies the existence of any other if we consider these objects in themselves, and never look beyond the ideas which we form of them.* Such an inference would amount to knowledge, and would imply the absolute contradiction and impossibility of conceiving any thing different. But as all distinct ideas are separable, it is evident there can be no impossibility of that kind. When we pass from a present impression to the idea of any object, we might possibly have separated the idea from the impression,
and have substituted any other idea in its room. (A Treatise of Human Nature, Book I, Part III, VI my emphasis)

Hume’s point is clearly epistemological. Here he is arguing that we have no evidence for anything like ontological dependence between objects. The reason is that all we ever experience that would justify inference to a dependence relation is constant conjunction of the ideas representing both objects; we notice the two things correlated in our experience. While constant conjunction is some evidence of a dependence relation, it is certainly no guarantee of dependence. Further, even if constant conjunction is reliable evidence of a dependence relation, it isn’t evidence for which direction the dependence runs or the strength of the dependence. The reason constant conjunction is weak evidence, according to Hume, is that there is no contradiction in one of the objects existing without the other. It is fairly clear what Hume means here with regard to causal connections. There seems to be no contradiction in the joint existence of the event of my turning the ignition key to my car this morning and the event of the car exploding immediately after, though in fact the first event was not accompanied by the second. Less dramatically, there is no contradiction in the joint existence of my turning the ignition key and the car not starting, though in fact the car did start. Hence, it doesn’t seem like the event of my turning my car key this morning necessitated the event of my car starting. Logic does not rule out (or rule in) the existence of any event given the existence of another.

This version of Hume’s Dictum is uncontroversial. Rather than taking the principle to bar certain inferences, though, some have taken the principle to bar certain relations between entities. As Jessica Wilson puts it, some metaphysicians like to interpret Hume’s Dictum as saying that “there are no metaphysically necessary connections between distinct, intrinsically typed, entities.” (2010, 595)

David Lewis (1986), for example, adopts a very strong version of Hume’s Dictum, called the Modal Recombination Principle, within his theory of modal realism.
Roughly speaking, the principle is that anything can coexist with anything else, at least provided they occupy distinct spatiotemporal positions. Likewise, anything can fail to coexist with anything else. Thus, if there could be a dragon, and there could be a unicorn, but there couldn’t be a dragon and a unicorn side by side, that would be an unacceptable gap in logical space, a failure of plenitude. And if there could be a talking head contiguous with to the rest of a living human body, but there couldn’t be a talking head separate from the rest of a human body, that too would be a failure of plenitude. (1986, 88)

Lewis goes on to specify the principle, saying that the duplicate of an object can exist with any other (provided there is enough room for them both).

Duplication is a matter of shared properties, but differently situated duplicates do not share all their properties. In section 1.5, I defined duplication in terms of the sharing of perfectly natural properties, then defined intrinsic properties as those that never differ between duplicates. That left it open that duplicates might differ extrinsically in their relation to their surroundings. Duplicate molecules in this world may differ in that one is and another isn’t part of a cat. Duplicate dragons in different worlds may differ in that one coexists with a unicorn and the other doesn’t. Duplicate heads may differ in that one is attached to the rest of a human body and the other isn’t. (1986, 89)

It is easy to see why Lewis and others find this principle appealing. To be an individual is, in part, to be independent. If two things depend on one another to the degree that neither can even be duplicated without duplicating the other, then it is difficult not to regard them as one thing rather than two.

Grounding individuality in the existence of duplicates is a strange way to cash out Hume’s Dictum, though. Notice that within Lewis’s system, whether something is an individual depends on whether it has duplicates and whether those duplicates are found in worlds
sufficiently different from one another. Consider a particular dragon. Is it an individual? That depends on whether it can be duplicated, and whether it can be duplicated depends on whether it is duplicated in some world or other. Supposing that the dragon is an individual, there are (lots of) duplicates of it. There is a clear sense, then, in which the existence of any one of the dragons depends on the existence of the others. They are what make it so that each one of them is modally loose enough to be an individual rather than a nondescript region of the property soup that individuals inhabit.\footnote{Presumably this modal claim is made true by the fact that a relevantly similar combination of dragon-properties lacks a counterpart in some accessible world.} Since the recombination principle is about all individuals, this reasoning generalizes—every individual $x$ is an individual because there are intrinsic duplicates of $x$ with worldmates that are not duplicates of $x$’s worldmates.

Here is a description of the situation: some things exist, but whether any one of them exists depends on whether it has appropriate duplicates. If we were to judge this picture using the spirit of Hume’s Dictum rather than Lewis’s formulation of it, we might well say that there are no individuals. That’s not to say that there are no things. They’re just all dependent on other things in a way that rules out their individuality, which is necessary to ground the fact that each one of them exists.

In this case the identities of our would-be individuals are so intertwined that they cannot be regarded as distinct entities. Yet it seems perfectly natural to refer to them plurally. They are intertwined. They are too dependent on one another for any one of them to be picked out determinately by ‘it’. But there they are nonetheless. As in the previous case, it seems that we should conclude that plural reference can succeed even in cases where there is too much indeterminacy for any singular reference to stick.

5.5 Speculative Conclusions

Our goal was to investigate the relationship between plural and singular reference by looking for ways to resist The Inference. The reader may judge whether this resistance has been suc-
cessful. In this final section I want to speculate on the role of plural reference in metaontology given the issues raised above.

First, consider what it would mean for the composition as identity theorist to be right about plural reference. In that case, plural variables can always be replaced by singular variables under an alternative conception of the subject matter. We might therefore conclude that plural logic is indeed as “innocent” as its singular counterpart. But it also means that plural logic is less interesting that Boolos might have hoped. On this interpretation our plural expressions amount to little more than complicated paraphrases of our singular expressions.

Few will buy this consequence, however. Plural expressions can be used to formulate second-order arithmetic and other theories that are decidedly not first-order. More likely is that a variably plural logic like that proposed by Cotnoir (2013) taints its singular expressions with the higher-order power of plural expressions. Allowing singular terms to corefer with plural terms means allowing any term to denote portions of the domain that cover several individuals; this endows the singular apparatus with more control over its models than it would otherwise have had. If innocence means strict adherence to first-order notions, then the logic of many-one identity is criminal through and through.

More interesting is the interpretation suggested by the psychological realities of plural reference. In our discussion of the epistemic challenge it seemed as though plural reference had more to do with a Gestalt impression of the domain than it had to do with the individuals denoted. When I say ‘There sure are a lot of freckles on Molly Ringwald’s face,’ I don’t refer to the freckles. Not really. I couldn’t tell you how many there are, though I might be able to tell you that there are more of them on her nose than on her forehead. Plural reference picks out many things at once, but not by, as it were, pinning a name tag on each one of them. We can, and usually do, succeed in referring plurally even when the reference is indeterminate for any one of the individual referents.

If this is right, then, barring artificial measures, we should expect bound plural variables to play a less central role in ontological investigations than their singular cousins. That
doesn’t mean plural quantification has no role to play. The case of nearly-dead organic gunk discussed above is an example where it is handy to have a quantifier that can tolerate a good deal of referential indeterminacy. Since ‘they’ can refer to some things without referring to any one of them in particular, it can be bound even in cases where assignments for singular variables are elusive. If I didn’t already think that plural reference were a natural fit for this role I would suggest that we invent a new mode of reference to do the job.

5.6 References


[4] Cameron, Ross (2014). Parts generate the whole, but they are not identical to it. In Aaron Cotnoir & Donald Baxter (eds.), *Composition as Identity*. Oxford University Press.


Chapter 6

Conclusion

These essays have argued for novel conclusions in the foundations of mereology and plural logic. Rather than rehash those conclusions here, I will instead briefly highlight some new questions raised by these investigations.

6.1 Atomicity

There has been a flurry of research on the dual notions of fundamentality and grounding in recent years. Grounding relations are structurally very similar to parthood relations; both are partial orderings that seem to obey at least weak (if not strong) supplementation principles. The analogue of atoms in systems of grounding are fundamentalia—either fundamental facts or fundamental objects, depending on how the system is interpreted. Hence, to claim that everything is ultimately grounded in fundamentalia is to say something strikingly similar to Atomicity. Future work on grounding should, I think, ask the analogue to the title of Chapter 2: How do you say ‘Everything is ultimately grounded in fundamentalia’?

6.2 Mereological Bundle Theory

I noted in Chapter 3 that Universalism cannot be the correct answer to the Special Composition Question given an ontology of properties. This is because some properties are not composable. There can be no squared circles (nothing composed of both is a square and is a circle). There can be no even primes that are also greater than 2. And so on.

It seems to me that the mereological bundle theorist has two options. She could accept that facts about which properties and composable are brute, and therefore follow Markosian
It is just a brute fact about *being a square* and *being a circle* that they do not combine. Alternatively, she could accept Universalism and the impossible objects that it entails, but hold that these impossible objects inhabit impossible worlds. She might even claim that impossible worlds exist *because* Universalism is true. The trick here will be to explain why there are worlds that *don’t* have squared circles or even primes greater than 2, even though both of the properties in these pairs exist. Of these options the first seems the most plausible (but also the least interesting).

6.3 *Quantification into Lists*

The system of plural logic presented in Chapter 4 features a list-forming operator that combines singular constants to form plural constants. The list-forming operator is based on the English list-forming ‘and,’ which generates lists of names like ‘Ann, Bob, and Carol’. The language of FPL lacks an important feature of English list-formation. It does not allow lists to be formed from variables. In English, it makes perfect sense to say that ‘Ann, Bob, and someone else went to the store’. One might assert it when one isn’t sure whether the third person in the group was Carol (perhaps it was Donna or Ethan). The principle concern in modeling lists of variables is how to handle nested quantification. Order of quantification matters, since it determines the order in which values are to be assigned to variables when a sentence is evaluated with respect to a model. Some questions that need to be answered are: Does the order of quantification affect the semantic role of a list of quantified variables? If so, how? A list of individual constants can always be replaced *salva veritate* with a coreferring plural constant (*e.g.*, ‘George, John, Paul, and Ringo’ corefers with ‘The Beatles’); is there any important sense in which a list of variables can be replaced by a plural variable? Under what conditions can a list of bound singular variables be replaced *salva veritate* with a bound plural variable? Answers to these questions would give us important insight into the
relationship between singular and plural reference. They would also suggest approaches to the axioms and semantics for a more robust system of list logic than FPL.

Another wish-list item for list logic is a semantics for iterated lists. In FPL iterated lists collapse both semantically and proof-theoretically. A list of lists is always equivalent to a first-level list of all the constants appearing at any level in the iterated list. Things appear to be more complicated in English. As I discussed in Section 1.3 of Chapter 4, lists of lists are sometimes used in English to effect superplural reference, i.e. plural reference that distinguishes between groupings of the things denoted. I would one day like to construct a list logic that does not automatically collapse iterated lists. Such a logic would be much more expressive than FPL. As of now I do not know how to generalize the models of FPL to accommodate higher-level plural reference. My guess is that, just as a modification of Henkin’s semantics for second-order logic proved useful in representing first-level plurality in FPL, a Henkin-style approach to higher-order logic would let us investigate axiomatizations of iterated list logics without the distractions that come with a type-theoretic approach.

6.4 Rethinking Assignments of Plural Variables

The goal of Chapter 5 was to make room for a new way of understanding how values are assigned to plural variables. Plural denotation is usually represented by a many-one denotation function that maps several individuals to a single plural expression. I argued that this is not an accurate model of how plural denotation works in English. In English, several things can be denoted by a plural expression even if there is some indeterminacy involved in which (or how many) things are being denoted.

Two alternative approaches to representing plural denotation suggest themselves. Both are strange. First, we might consider a function that assigns portions of the domain, rather than individuals, to a plural expression. The idea of a ‘portion’ of a domain is suggested in Lewis (1991) and developed in some depth by Cotnoir (2013). These authors are interested in portions because they think that the number of individuals in question is dependent on
how we’re conceiving of those individuals. We can think of a portion of a domain as one thing or many things. However we wind up representing the portion, as one thing or as many things, it will be appropriate to refer to the portion as ‘them’ and the facts about division and individuation within the portion fall as they may.

Another approach, not necessarily at odds with the first, would be to treat plural variable assignments as superassignments in the same way that vague expressions can be interpreted supervaluationally. Supervaluation is a way to preserve logical relations between sentences containing vague predicates. It relies on the notion of precisifying a predicate. A predicate is precisified when it is given a definite extension. For example, the vague predicate ‘is bald’ can be precisified by assigning to it the extension of the predicate ‘has less than 10 hairs on its head.’ Sentences containing the vague predicate are then evaluated based on all of the available precisifications. Such sentences are valued as super true if they come out true on all precisifications; they are valued super false if they come out false on all precisifications. Thus, all of the usual logical relationships that hold between tautologies, contradictions, and contingencies (which are threatened by the inclusion of indeterminacy-inducing vagueness) can be retained under the new labels super-truths, super-falsities, and indeterminacies. Likewise, in cases of ontic indeterminacy discussed in Section 4 of Chapter 5, we might seek precisifications of singular reference (ways of fixing reference, perhaps) and count plural reference as successful (i.e., as having an available assignment) if there are singular referents for every precisification, and unsuccessful if there is no precisification on which singular reference succeeds.

6.5 References
