STUDENTS' CONSTRUCTION OF FRACTIONAL KNOWLEDGE THROUGH MODIFICATION OF THEIR GENERALIZED NUMBER SEQUENCE (GNS)

by

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(Under the Direction of JOHN OLIVE)

ABSTRACT

The purpose of this study was to understand how eight grade students constructed knowledge of fraction multiplication and [measurement] division, and further Rational Numbers of Arithmetic (RNA) based on their abstract whole number sequences [Generalized Number Sequences] in interaction with a teacher-researcher.

The second-order models for students' constructions of fractional knowledge established by Steffe's and Olive's (1990) *The Fraction Project* guided the present study as *mathematics of children* and played a role as my major theoretical basis in constituting *mathematics for children* for the two participating students during the teaching experiment of this study.

As a teacher-researcher, I taught two eighth graders at a rural middle school in Georgia in a constructivist teaching experiment from October 2008 to May 2009. All teaching episodes were videotaped with two cameras—one to capture the students' works and one to capture the whole interactions among the students and the teacher-researcher [me].

In retrospective analysis of the videotapes, I constructed second-order models that explained the changes in the students' mathematical ways of operating and how the students constructed their mathematical knowledge in the context of fraction multiplication, fraction division, and multiplicative transformation between two fractions.

The students' whole number knowledge [GNS] was significant in that the students conducted their partitioning activities by modifications of their GNS such as recursive partitioning operations, distributive partitioning operations and common partitioning operations to cope with the posed tasks throughout the teaching experiment. In addition, the students demonstrated modifications of their unit-segmenting schemes as fraction measurement division situations became complicated.

The reported struggles of the two participating students, due to the lack of their interiorized use of a Fractional Connected Number Sequence (FCNS) for further mathematical activities involving fractions, also suggests that the curriculum in school mathematics for students' fraction learning needs to be revisited and reorganized to take into account the importance of students' construction of a multiplicative relationship of unit fractions to a referent whole.

INDEX WORDS: Fraction Multiplication, Fraction Division, Unit-Segmenting Scheme, Units-Coordinating Scheme, Distributive Partitioning Operation, Common Partitioning Operation, Rational Numbers of Arithmetic (RNA), Generalized Number Sequence (GNS), Fractional Connected Number Sequence (FCNS), Radical Constructivism, Scheme Theory, Teaching Experiment

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DEDICATION

To the companion in my whole life, Soo Jin, and my family for their love and support

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Page

CHAPTER I

INTRODUCTION

Background

Fractions have been considered to be the most intricate numbers to deal with in arithmetic and there have been tremendous efforts in mathematics education research for investigating children's learning of fractions since the claim of "the learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure." (Davis, Hunting, & Pearn, 1993, p. 63). Pitketly and Hunting (1996) identified two groups of researchers concerning initial fraction concepts based on their interpretations of research findings. Among them, one group of researchers claimed that the equidivision of a unit into parts, the recursive division of a part into subparts, and the reconstruction of the unit are essential for developing rational number meaning. Thus, a flexible concept of the unit, that is, the ability to switch focus from individual items to a grouping or uniting of items, depending on the person's goals, seems fundamental to all later children's rational number interpretations (cf. Kieren, 1992; Mack, 1990, 2000, 2001; Olive, 1999, Steffe & Olive, 1990).

Especially, Steffe and Olive (1990) conducted *The Fraction Project* to investigate children's construction of operations that generate children's fraction schemes by means of accommodations of their operations that produce their number sequences. That is, the goal of the project was to identify the accommodations children make to their whole number counting sequences to construct fraction schemes. For them, children's abstracting the result of iterating for producing a partitioned whole seemed to be a relevant goal because iteration is a constitutive aspect of their number knowledge (Biddlecomb, 1994; Olive, 1999; Steffe, 1992a; Steffe & Olive 2010; Steffe & Tzur, 1994).

From the *Fraction Project,* a *partitive fractional scheme* and an *iterative fraction scheme* were developed as second-order models for children's genuine concept of fraction numbers and fractional quantities (Tzur, 1999; Steffe, 2002). Olive (1999) and Steffe (2003) proposed a *commensurate fractional scheme* and a *fraction adding scheme* for fraction addition [or possibly for fraction subtraction] and *fractional composition scheme* for fraction multiplication. Their longitudinal teaching experiment was conducted using computer microworlds, the Tools for Interactive Mathematical Activity (TIMA), in which children played mathematically. The objects and operations of the microworld were designed to represent the objects and operations generated by children's mathematical constructs. The situations involved discrete units (TIMA: Toys), continuous linear quantities (TIMA: Sticks) and rectangular regions (TIMA: Bars) that could be partitioned in various ways. The children were encouraged to generate a class of situations for their development of fractional schemes and created the new fractional schemes as modifications of their existing whole number schemes. In contrast to the view that whole number knowledge interferes with learning fractional knowledge, their findings corroborated that the concepts and operations represented by children's natural language should be used in their construction of fraction knowledge (Pitkethly & Hunting, 1996).

As a final form of construction of children's fraction knowledge, Steffe and Olive (1990) introduced a scheme, called the *Rational Numbers of Arithmetic* (RNA), in which fractions have become abstracted operations. A child can be judged to have constructed the RNA when "the child is aware of the operations needed not only to reconstruct the unit whole from any one of its parts but also to produce any fraction of the unit whole from any other fraction" (Olive, 1999, p.

281). For the construction of RNA, Olive (1999) argued that a child integrates the operations of his or her *generalized number sequence* (GNS) with the operations that produce fractions as measurement units. He also argued that the process of children's construction of the RNA should be investigated because the operations that produce the RNA are also those operations that undergird the division of fractions (Olive, 1999).

Nevertheless, their study for children's construction of RNA still calls for further investigation. First, unlike a unit fraction composition that requires children's construction of a recursive partitioning operation, Steffe & Olive (2010) argues that children do not construct a fraction composition scheme as a functional accommodation[1] of the unit fraction composition scheme. Rather, the fraction composition scheme is primarily constructed as an accommodation of children's splitting scheme for connected numbers, where the accommodation involves embedding a distributive partitioning scheme. However, none of the participating children in *The Fraction Project* were reported to interiorize their distributive partitioning operations for the construction of a common fraction composition scheme during the teaching experiment period. Thus, the study of children's actual construction of a general fraction composition scheme for fraction multiplication is left to be explored.

Second, *The Fraction Project* documents the harbinger for construction of the rational numbers of arithmetic (RNA), which might play an important role in children's construction of fraction division knowledge. A fifth-grade participant, Nathan constructed a co-measurement unit for two given unit fractions [a fractional unit that can be used to exactly measure each unit fraction] for adding the two fractions. Producing a co-measurement fraction, say, finding a common partitioning unit fraction, one-sixth of a bar so that both one-half and one-third are

[1] A functional accommodation of a scheme occurs in the context of using the scheme.

multiples of the unit fraction, seems essential for children's construction of RNA. However, Nathan was not asked fraction division problems during his three years in the teaching experiment and thus there was no chance to investigate how his construction of a co-measurement fraction for two given fractions through a common partitioning operation contributed to the development of his fraction division knowledge.

Last, Nathan's transformative activity between two unit fractions was insufficient for the indication of the construction of RNA in that the RNA is "the construction of abstracted fractions as an ensemble of operations of which the child is *explicitly* aware through the interiorization process of such an activity" (Steffe & Ulrich, 2010, p. 266). Nathan did not seem to develop a scheme, the purpose of which is to *multiplicatively transform a unit fraction into another unit fraction*. I would attribute the construction of such a scheme to a child if the child is able to transform a unit fraction into any other unit fraction, and if the child is explicitly aware of the involved fractional operator. In this sense, children's constructive itinerary to find out the fractional operator for their multiplicative transformation activities from a unit fraction to the other unit fraction and further from any fraction to any other fraction emerges as one of the issues to be investigated.

Research Questions

In line with the issues mentioned above, the present study intended to explore how children constructed their fractional knowledge mainly in the context of fraction multiplication and fraction division situations. In *The Fraction Project*, although explicitly nested number sequences (ENSs) were attributed to most of the participating children at the initial stage of their teaching experiment, Olive and Steffe (2010) also reported the constructive trajectories of fraction schemes by two children, Nathan and Arthur, who apparently had already constructed a generalized number sequence (GNS). They found that the two children's operations that produced the GNS opened different possibilities for their construction of fraction schemes from those of the ENS children. Especially, Nathan developed schemes of operations with fractions that allowed him to add fractions with unlike denominators, find a fraction of a fraction, rename fractions, and simplify fractions to the lowest terms. Nathan seemed to surpass the ENS children in that he constructed such fraction schemes in his first year of the teaching experiment whereas the ENS children constructed the corresponding fraction schemes throughout the three years.

Therefore, for an extended investigation of children's construction of fractional knowledge following *The Fraction Project*, I conducted a year-long teaching experiment with two eighth-grade students, who were regarded to have constructed a GNS. First of all, my initial concern was to examine the two students' current fractional knowledge, specifically related to the construction of a fractional connected number sequence (FCNS) because construction of a multiplicative relationship of unit fractions to the referent whole and the availability of such a scheme in the assimilating part of a problem situation were fundamental factors to the construction and expansion of any sort of fractional schemes that the two students can construct. The second interest was to research how their GNS were modified and embedded as a part of their fractional knowledge mainly in the context of fraction multiplication and division problems. By way of the investigation of children's actions and operations around such interests, I would desire to propose a form of explanatory models for the following questions,

- How do two eighth-grade students who were credited with the construction of a generalized number sequences construct necessary schemes and operations for fraction multiplication and division?

1. What sort of actions and operations are necessary for the construction of a general fraction composition scheme? Does the distributive partitioning operation play an important role in fraction multiplication? If so, how do they develop such distributive thinking? If not, which other operations are critical for fraction multiplication?
2. What sort of actions and operations are used or newly emerge in the processes of the two students' solving fraction division problems? Specifically, how do the students' distributive partitioning operations and common partitioning operations contribute to the construction of a fraction division scheme?

- Are there any implications for students' construction of rational numbers of arithmetic?

CHAPTER II

LITERATURE REVIEW AND THEORETICAL ORIENTATIONS

Literature Review

Since the present study will be conducted on the basis of the findings of *The Fraction Project* (Steffe & Olive, 1990), this section consists of four parts for helping understand scholarly issues around the project and how this study is to be situated along the stream. The first part addresses children's whole number knowledge, which would be the foundation of children's fraction knowledge. The second part introduces the *Reorganization Hypothesis* and explicates children's construction of a fractional connected number sequence (FCNS) based on the hypothesis. The third part provides the main research results of *The Fraction Project* to see how children, especially with an ENS or a GNS, modified their abstract whole number sequences for construction of their fraction knowledge. Last, I review other studies for children's learning fraction knowledge, attention of which is mainly focused on the studies of children's fraction multiplication and division learning.

Children's Whole Number Knowledge

Steffe, Cobb, and von Glasersfeld (1988) conducted a teaching experiment with young children for their development of whole number knowledge. They identified three successive number sequences from children's construction of mathematical schemes and operations: the initial number sequence (INS), the tacitly nested number sequence (TNS), and the explicitly nested number sequence (ENS). A *number sequence* is "the recognition template of a numerical counting scheme, that is, its assimilating structure. A number sequence is a discrete numerical structure; it is a sequence of arithmetical unit items that contain records of counting acts" (Steffe, 2010b, p. 27). Each new number sequence is the result of a reinteriorization of the previous number sequence and generates more abstract units with which the child can operate (Olive, 1999). That is, a gradual decrease in children's dependence on their immediate experimental world can characterize the learning stages of number sequences and it is the operations that children can perform using their number sequences that distinguish among distinct stages of the number sequences (Steffe, 2010b). Later, the notion of the generalized number sequence (GNS) ensued while seeing how children who had constructed the ENS might use that number sequence to construct schemes to solve situations that can be regarded as multiplying and dividing situations (Steffe, 2010a). For this report, I will mainly focus on the explication of the ENS and the GNS, which most of children in *The Fraction Project* began their teaching experiment with.

A crucial step for the construction of an ENS is the establishment of an abstract unit item "one" as an iterable unit (Olive, 1999). The iterable one can be produced through repeatedly applying the "one more item" operation when double counting. After the construction of an iterable unit item, a child can engage in part-whole reasoning. When the unit of one is iterable, a number word refers to a composite unit containing a unit which can be iterated the number of times indicated by the number word. This iterability of one "opens the possibility for a child to 'collapse' a composite unit into a unit structure containing a singleton unit, which can be iterated so many times." (Steffe, 2010b, p. 42) This characteristic of the ENS enables children to establish multiplicative schemes that involve two levels of units. That is, the ENS provides children with the necessary operations to engage in multiplicative reasoning. Further, they can generate a numerical composite of composite unit items as a result of those operations, but they have yet to interiorize or symbolize them so that the numerical composite of composite unit items can be used as given input for further operations (Olive, 1999). A representative example of construction of the ENS is where a child has to find the number to be added to, say, 15 to make 23. If the child double counts from "fifteen" up to and including "twenty-three" to find the number, this would be an indication that the child has constructed the ENS. In sum, Steffe (2010b) highlights that "Iterating a unit item and disembedding a numerical part from a numerical whole are two principal operations of the ENS." (pp. 46-47)

The reinteriorization of the ENS results in iterable composite units. When children have constructed composite units as iterable, they can be regarded as at least in the process of reorganizing their ENS into the GNS (Steffe, 1992b). In other words, the GNS is a generalization of the operations on units of the ENS to composite units. "Speaking metaphorically, children are in a 'composite units' world rather than a 'units of one' world." (Steffe, 2010b, p. 43) In a GNS, a composite unit is iterable, that is, any composite unit can be taken as the basic unit of the sequence. For a composite unit to be judged as iterable, a child should be able to represent and combine iterations of the composite unit prior to activity. That is, "the child must have constructed a composite unit containing another composite unit that can be iterated so many times, a structure that is strictly analogous to the numerical structure the child constructs in the case of the iterable unit of one" (Steffe, 2004, p. 247). For instance, Nathan, in the first year of *The Fraction Project* teaching experiment (Olive, 2003), showed an indication of the construction of a GNS. When Nathan was asked to find out how many strings of three toys and strings of four toys would be needed to make a string of 24 toys within the context of the TIMA: Toys computer environment, he explained that "Three and four is seven; three sevens is 21, so three more to make 24. That's four threes and three fours!" As the result of iterating seven three times Nathan could produce 21, consisting of iterable threes and iterable fours and then saw 24 as a partitioned unit with two sub-partitions: three fours and four threes. Nathan's assimilating operations to produce a composite unit 24 were the corroboration of his construction of the GNS.

Children's interiorization process from their ENS to the GNS arises when they operate with composite units to solve complex multiplicative problems that require recursive applications of their units-coordinating operations to the results of those operations (Olive, 1999). Therefore, children who have constructed a GNS can take units of units of units rather than simply units of units as given. Hackenberg (2005) also differentiated between the ENS and the GNS in children's mathematical operations as students who coordinate two levels of units prior to operating often engage in strategic multiplicative reasoning, but coordinating three levels of units prior to operating is even more sophisticated. Therefore, "the GNS supersedes the explicitly nested number sequence in that it can be used in all of the ways that the explicitly nested number sequence can be used as well as ways in which the ENS cannot be used." (Steffe, 2010b, p. 43)

Reorganization Hypothesis and Fractional Connected Number Sequence (FCNS)

When the situations constructed by children are extended from discrete to continuous, some of the most important modifications of their number sequences emerge (Steffe & Wiegel, 1994). The *Reorganization Hypothesis* is the view that children's fraction schemes are generated through modifications of their abstract whole number sequences (Biddlecomb, 1994; Olive, 1999; Steffe, 1992a; Steffe & Tzur, 1994) in contrast to the *Interference Hypothesis* that whole number knowledge interferes with the learning of fractions (Post, Cramer, Behr, Lesh, & Harel, 1993; Streefland, 1991). However, Steffe (2010a) does not assume that continuous units are produced by children's use of number sequences. Rather, he argues that children had already constructed continuous units alongside of their construction of the discrete units of their number sequences and when the situations of the counting scheme involves a connected but segmented quantity through an awareness of figurative length and figurative density, a unification of discrete and continuous quantity begins. Further, constructing connected numerical composites opens the way for the construction of a connected number sequence, which is "a number sequence whose countable items are the elements of a connected but segmented continuous unit" (Steffe, 2010a, p. 56). For the construction of a connected number sequence (CNS), a child should build awareness of indefinite length as well as of indefinite numerosity as quantitative properties of a connected number, which means the incorporation of a notion of unit length into the abstract unit items of their ENS (Olive & Steffe, 2002b). Thus, children's construction of a CNS plays a crucial role in making sense of fractions. It enables children to "use their discrete adding, subtracting, and multiplying schemes to find unknown lengths using known lengths, and thus establish part-whole relations in the context of continuous quantities" (Olive & Lobato, 2008, p. 9).

Although a partitioning operation, mentally projecting a concept of a whole number into an unmarked line segment, is fundamental to children's development of fraction knowledge, the *Fraction Project* indicated there needs to be several distinctions among children's partitioning operations for detailed descriptions of their constructive itinerary of fraction schemes. First of all, a child's fragmenting of a continuous unit cannot be judged as an *equi-partitioning* until "the operating child intends to fragment the continuous unit into equal sized parts and can use any one of these equal sized parts in iteration to produce a connected but segmented unit of the same size as the original unit" (Steffe, 2010a, p. 68). The modification of the equi-partitioning scheme entails a *partitive fractional scheme*, which is regarded as the first genuine fractional scheme. With the partitive fractional scheme, a child can disembed any subcollection of elements from the original partitioned whole without destroying it and constitute a composite unit in its own right by uniting them together. This establishes the classical numerical part-to- whole operation that serves as a fundamental operation in the construction of fractional schemes (Steffe & Olive, 2010). However, the limited understanding of fractions as parts of a specific partitioned whole constrains children's construction of a multiplicative relation between the sizes of the unit fraction and the referent whole. In other words, for the construction of "thirteen-twelfths" as a fractional quantity children should transcend the part-whole meaning of fractions, which requires the construction of a *splitting operation*. The splitting operation is qualitatively different than the operations carried out in the equi-partitioning scheme. The splitting operation is a simultaneous composition of partitioning and iterating whereas in the equi-partitioning scheme the two operations are performed sequentially. With the splitting operation, a unit fraction, say, one-twelfth becomes a fractional number freed from its containing whole and available for use in the construction of thirteen-twelfths. For instance, the multiplicative relationship between the referent whole (a 1-meter bar) and a 1/12-meter bar maintains implicit in the construction of the improper fractional quantity, a 13/12-meter bar. This enables a child to inject the whole (the 12/12-meter bar) into the 13/12-meter bar and reconstitute what was formerly a part in his or her partitive scheme (the 13/12-meter bar) into a composite unit containing the original whole unit (the 12/12-meter bar) and another unit (the 1/12-meter bar). The partitive fraction scheme, upon the emergence of the splitting operation, can be regarded as an *iterative fraction scheme* that can be used to produce improper fractions (Steffe, 2010d). The possible result of the iterative fractional scheme is a *fractional connected number sequence* (FCNS), a connected number sequence in which unit fractions are the units of the connected numbers (Steffe, 2002). The construction of such fractional numbers is made possible because their fractional meaning would no longer be directly dependent on its relation to the whole of which it is part (Steffe, 2010d). Analogously, the iterability of unit fractions with a FCNS is on a par with that of a unit, one with an ENS. That is, children can use, say, one-eleventh as they use the unit of one and it can be operated with in a way that is analogous to how the child operates with the ENS involving the unit of one (Steffe, 2002).

Construction of Diverse Fraction Schemes

The construction of the explicitly nested number sequence is important for children's partitioning operation in that it opens the possibility for the children to regard the whole as invariant and the sum of the parts to equal the original whole when they are involved in partitioning operations. The children also construe the parts as units in their own right. However, the construction of the generalized number sequence is also required for children's more advanced partitioning operations like subdividing those parts further. GNS children can take a three-levels-of-units structure as a given and use this unit structure in establishing a relation between any one of the subparts produced on the second subdivision and the original whole whereas ENS children can produce a unit of units of units in action (Steffe & Olive, 2010). Likewise, the ENS children are able to produce units-coordinating operations in action, say, inserting the unit of three into each unit of four to produce four threes to find the product of four and three, but the GNS children can mentally conduct such operations prior to actual activity. According to Olive (1999), ENS children could engage in recursive partitioning and construct unit fractions of unit fractions (a composition of unit fractions). However, they have yet to form any fraction of any other fraction as a fraction of the original whole.

A *unit fraction composition scheme* emerges when recursive partitioning, that is, a novel use of a units-coordinating scheme, is embedded in a reversible part-whole fraction scheme. The emergence of the scheme is on the basis of a child's construction of a unit of units of units as an assimilating structure whose units can be used as material in further operation. That is to say, attributing to a child a unit fraction composition scheme is to see whether partitioning, say, one-fourth of a stick into three equal parts symbolizes partitioning each one of the four-fourths into three parts (Steffe, 2010e). On the other hand, the *fraction composition scheme* does not emerge simply as an accommodation of the unit fraction composition scheme. Rather, "the fraction composition scheme is primarily constructed as an accommodation of children's splitting scheme for connected numbers, where the accommodation involves embedding the distributive partitioning scheme and the recursive partitioning scheme into the splitting scheme" (Steffe & Olive, 2010, p. 333). Olive and Steffe (2010) reported three steps which seemed to contribute to the construction of a fraction composition scheme: (1) "making distributive partitioning operations explicit," (2) "taking the results of distributive partitioning as input for recursive partitioning," and (3) "assimilating the results of what was a sharing task that involved three levels of units using his iterative fraction scheme for connected numbers" (pp. 313-314). The explicit use of distributive partitioning requires having abstracted a three-levels-of-units multiplicative structure in assimilating a situation prior to activity (Hackenberg, 2007). If a child with distributive reasoning forms a goal of a *distributive partitioning scheme*, say, sharing four identical candy bars equally among five people, the child can partition each candy bar into five parts, distribute one part from each of the four candy bars to each of the five people with understanding that the share of one person can be replicated five times to produce the whole of the four candy bars. The child also knows that four-fifths of one candy bar is identical to one-fifth of all of the candy bars.

The construction of a *unit commensurate fraction scheme* can be confirmed if a child could make unit fractional parts of a composite unit in the form of a connected number and transform these unit fractional parts into commensurate fractions (Steffe & Olive, 2010). Steffe and Olive argue that taking three levels of units as a given is necessary for the construction of commensurate fractions. For instance, establishing four as an iterable unit in the context of the connected number, say, twelve makes possible for a child to be aware that a 4/12-stick can be iterated three times to make a 12/12-stick before repeating it. Then the child can transform the fraction, four-twelfths into one-third and set the commensurate relation between the two fractions. Another fraction scheme that necessitates children's construction of a three-levels-of-units structure is a *common partitioning fractional scheme* since it requires units-coordinations at three levels of units that was a coordination of two iterable composite units. For example, in order to find a common partition for making both thirds and fifths of a bar, Nathan, one of the children in the teaching experiment (Olive, 1999), coordinated his number sequences for fives and for threes until he found a common multiple. The three levels of units were the composite unit of 15 that was the composition of both five threes and three fives, the composite units of fives and of threes, and the singleton units that constituted these composite units. This common partitioning fractional scheme became a basis of the construction of co-measurement units for fractions with which a child can produce any fraction from any other fraction, that is, an indication of RNA (Olive, 1999, 2003)

In sum, the difference between two and three levels of units as assimilating structures produces distinctly different stages in the construction of fraction schemes. Two levels of units produce only the partitive fraction scheme whereas three levels of units as an assimilating structure produce the recursive partitioning, the splitting operation, the iterative fraction scheme, the unit commensurate fraction scheme, and the unit fraction composition scheme. Further, using the three-levels-of-units structure as given for further operations also opens the ways for children to construct a more general fraction composition scheme, which entails distributive operations, and to find a co-measurement unit fraction through common partitioning operations, which is fundamental to the construction of the RNA (Olive & Steffe, 2010).

Other Studies for Children' Fractional Knowledge

Fractions have been considered to be among the most complex mathematical concepts that children encounter in their primary education. Kieren (1992) believes that partitioning might be a cognitive precursor to fraction numbers. Also, rational numbers are fundamental to measuring continuous quantities in representing and controlling part-whole situations and relationships (Pitkethly & Hunting, 1996). Thus, children's mature understanding of fractions should be viewed as a synthesis of their understanding of multiplication, division, and ratio via measurement (Thompson & Saldanha, 2003).

Mack (2000, 2001) conducted a two-year study with children from fifth to sixth grade the purpose of which was to examine the long-term effect of learning with understanding in multiplication of fractions. Especially, students' ability to reconceptualize and partition different types of units was the focus of the study because it was believed to be essential for students to determine the appropriate unit to be partitioned in a problem situation as well as the unit upon which the results of partitionings are based. The findings proposed distinctive mental processes related to viewing the unit to be partitioned and the results of their partitionings as fractional amounts of a referent whole as well as related to different types of problems involving multiplication of fractions (see Table 2.1). However, her analysis was framed on the basis of different types of problem situations that her students encountered where the perceived relationship between the denominator of the multiplier and the numerator of the multiplicand is varied, rather than the suggested levels of mental processes.

Table 2.1

Building Phase, Mental Processes and Corresponding Situations (Mack, 2001, p. 279)

Building Phase/Mental Process	Corresponding Mathematical Expression	Corresponding Contextual Situation
Focusing on Fractional Amounts		
Conceptualizing results of partitionings as unit fractions	$a \div b = n\ {}^{c}/_{d}$	Share 10 cookies between 4 people.
Considering what it means to partition a unit, in general, into a fractional amount	${}^{a}/_{b} \times nb$	Find one third of of 12 cookies.
Reconceptualizing and Partitioning Composite Units		
Conceptualizing fractional amounts as embedded within a composite unit and not partitioning the unit	${}^{a}/_{b} \times {}^{b}/_{d}$	Find two thirds of three fourths of one whole pizza.
Reconceptualizing a composite unit by repartitioning the unit	${}^{a}/_{nb} \times {}^{b}/_{d}$	Find three fourths of two thirds of one whole pizza.
Reconceptualizing a composite unit by grouping unit pieces	${}^{a}/_{b} \times {}^{nb}/_{d}$	Find two thirds of nine tenths of one whole pizza.
Reconceptualizing a composite unit by repartitioning the unit and grouping resulting unit pieces	${}^{a}/_{b} \times {}^{c}/_{d}$, where b and c are relatively prime	Find three fourths of seven eighths of one whole pizza.

On the other hand, Lamon (1999) sees the operational use of rational numbers as functions, taking some set or region and mapping it onto another set or region. That is, the operator notion of rational numbers is about shrinking and enlarging, contracting and expanding, enlarging and reducing, or multiplying and dividing. She argues that the composition of operators leads very naturally to fraction multiplication. For example, 2/3 times 3/4 implies “take 2/3 of 3/4 of a unit.” However, Lamon does not seem to consider a fractional operator as a single functional object. In other words, she asserts that “2/3 of” can be interpreted as a rule for composing the operations of multiplication of 2 and division of 3. Although this interpretation might imply a sort of adaptation of whole number knowledge to a fraction multiplication situation, I would not consider such mathematical activity as an indicator of the construction of a scheme to use a fraction as an operator for multiplication because students’ construction of fractional number concepts such as FCNS is not considered.

Lamon (1999) also introduced two types of division: partitive division and quotitive division. The former involves partitioning or determining equal parts or shares whereas in the latter the question is how much of a quantity can be measured out of the other quantity. The difficulties that children might encounter in the latter would be to identify the divisor as a new unit of measure. Bulgar (2003) conducted a year-long teaching experiment with fourth-grade students for understanding children’s solving fraction division problems prior to introduction of algorithmic instruction. The task was to determine how many bows of each size (mainly unit fractional quantities) could be made from each (whole number) length of ribbon. As a result, Bulgar reported that three distinct solution methods emerged: (1) justification involving natural number, (2) involving measurement, and (3) involving fractions. She also documented that all methods were related to children’s counting and they had difficulty with division involving a non-unit fraction divisor.

In terms of partitive division, Empson, Junk, Dominguez and Turner (2005) analyzed children’s coordination of two quantities (number of people sharing and number of things being shared) in their solutions to equal sharing problems to see what extent this coordinating was multiplicative. Unlike most research in sharing problems, they included number combinations with a common factor (e.g. sharing 8 pizzas among 12 people) to make explicit the multiple routes for the construction of equivalence. Their intent was to offer an analysis that reframes prior research by de-emphasizing partitions of single units and the geometry of such partitions and focusing instead on children's conceptualizations of multiplication, division, and ratios of quantities in the context of sharing multiple units among multiple shares. However, they did not deal with children's multiplication with fractional quantities, which means that general constructive itinerary of children's fraction division had yet to be explored. Nonetheless, they gave an important implication for research in children's construction of fraction schemes by suggesting that with various number combinations, children's mathematical activities in equal sharing problems involve whole number knowledge constructs such as multiplicative reasoning.

Theoretical Orientations

Radical Constructivism: A Guiding Theory

The guiding theory for my study is radical constructivism, a theory of rational knowing that von Glasersfeld (1995) developed and is based on Piaget's genetic epistemology. Radical constructivism is situated entirely on the opposite side to metaphysical realism in terms of the relation between knowledge and reality. That is to say, whereas true knowledge can (or should) be made a direct correspondence to or be matched up with the 'Reality' in metaphysical realism, radical constructivism sees knowledge as an adaptation in the functional sense (von Glasersfeld, 1984). Therefore, from a radical constructivist view, objective ontological reality is unknowable regardless of the existence.

> Any cognitive structure that serves its purpose in current time more or less proves that it has done what was expected of it. Logically, that gives us no clue as to how the objective world might be, it merely means that we know one viable way to a goal that we have chosen under specific circumstances in our experiential world (von Glasersfeld, 1984, p. 4).

Since we, as radical constructivists, have no absolute, invariant truth in our experiential world, it requires us to provide quite a different answer from the perspective of a realistic view of the world for the question: If our experience can not teach us the ontological nature of things in themselves, how can we explain that we experience a (quite) stable and reliable world in many respects? (von Glasersfeld, 1982)

What gives cognitive structures a first, primitive and relative durability is simply repetition. This confers a preliminary, tenuous permanence to the link between action and result (action scheme) as well as to the perceptual signals that are coordinated to groups forming the trigger and the result of the scheme respectively (von Glasersfeld, 1982). Von Glasersfeld also indicated that successful repetition turns perceptual compounds into items that can be re-cognized as experiential invariants and, eventually, externalized as objects that exist on their own. Their recurrence yields a first notion of reality. He explicated two keystones in Piaget's theory within the framework of action schemes and of his analysis of cognitive development: *assimilation* and *accommodation*. For example, suppose that the elements a, b, and c constitute a sequence of experience. Regardless of the inclusion of another element, x, when the experience consisting of a, b, c, and x can be regarded the same as the previous one, this is the principle of *assimilation*, which perceives and categorizes experience in terms that are already known. The situation, however, changes if an object, in spite of the fact that it manifests a, b, and c, turns out to behave in a way that is different from the behavior that is expected of a-b-c-objects. If that happens, it causes a disturbance (perturbation) that can lead to the examination of other properties or components. That opens the way towards a discrimination of the disturbing object on the basis of some disregarded element x. This is the principle of *accommodation,* which brings forth change in an existing structure or the formation of a new one. Thus, epistemology becomes the study of how the mind operates, of the ways and means it employs to construct a relatively regular world out of the flow of its experience (von Glasersfeld, 1984).

Therefore, according to radical constructivism, learning is likely to happen when we realize that what we already know is not sufficient to deal with a problem that we actually want to solve, which indicates one of Piaget's principles, *reflective abstraction*. Von Glasersfeld (1995) elaborated the notion of reflection on mental operations, and provided a model for how it operates in conjunction with abstraction and generalization. He asserted that only a reflective mind, a mind that is looking for order in the baffling world of experience, can be aware of one's experience, and reflect upon it. "We have no idea what it is that gives us this internal awareness and the power to reflect. But we know that we have it. We may call it awareness or consciousness." (p. 7) When experiential invariants are formed out of sensory material, a higher level of reality can be achieved by the actor's awareness of the structure of its own scheme. This awareness is the result of *reflective abstraction* that enables the actor to separate the patterns of action from the actual experiential content with which they were enacted, to transfer them to other circumstances, to homogenize them and make them compatible with one another, and eventually to shape them into operational invariants that can serve not only in action but also in prediction and explanation (Piaget, 1973, as cited in von Glasersfeld, 1995). Therefore, a judgment of true or false is possible only in the context of reviewing action, not in the context of action itself. Finally, more or less reliable experiential confirmation through sequential successes in prediction and control, which establishes the notion of viability, leads us to consider our cognitive structure 'true' (von Glasersfeld, 1982).

In conclusion, based on radical constructivism, students' knowledge and reality exclusively reside in terms of elements within the students' experience. The conceptual structures that I consider to be 'knowledge' of students are the products of themselves who shape their thinking to fit the constraints they experience. In other words, I will attribute student's construction of knowledge not to passing my structure to theirs, but to their own operational system of transformations. Therefore, the best we can do as teachers is to establish a second-order model for particular students and provoke a form of perturbations possibly perceived by them based on the model in order to orient the children toward the epistemic model of the teachers.

A Model of Mathematical Learning: Scheme Theory

In formulating a model of children's mathematics within a domain of their mathematical activity, I will use the concept of *scheme* (Olive & Steffe, 2002a; von Glasersfeld, 1981). To fully appreciate scheme theory, it is necessary to understand the notion of first-order knowledge/models and second-order knowledge/models. First-order mathematical knowledge, which are the models an individual constructs to organize, comprehend, and control his or her experience, is their own mathematical knowledge whereas second-order mathematical knowledge are the models which observers may construct of the observed person's knowledge in order to explain their observation (Steffe, 2010c). Distinguishing between first-order and second-order models is crucial because, under radical constructivism, which would be the basis of scheme theory, we attribute mathematical realities to students that are independent of our own mathematical realities (Steffe & Thomson, 2000). We, as researchers and teachers, have no way to directly access students' mathematical knowledge and construct first-order models of students' mathematical knowledge. What we can best do is to build our scheme of the students' mathematical knowledge, that is, our second-order models of students' mathematical knowledge.

In other words, scheme is an observer's concept and, in the case of schemes that are mathematical, it refers to children's mathematical language and actions (Steffe, 2010c).

In order to differentiate the level of order in mathematical knowledge, Olive and Steffe (2002a) used the phrase '*children's mathematics*' to mean whatever constitutes children's mathematical realities (children's mathematical schemas) and '*mathematics of children*' to mean the researchers' models of children's mathematics (their schemas of children's mathematical knowledge). Further, they argued that second-order models should be referred to as social knowledge because those are constructed through social processes, and the first-order models that constitute the children's mathematics do not necessarily correspond piece-by-piece to what are established as second-order models (Olive & Steffe, 2002a). On the other hand, '*mathematics for children'* consists of mathematical concepts and operations that children might learn (Steffe, 1988) and it is important to note that those concepts are not predetermined as a part of teachers' own mathematical knowledge. Rather, mathematics for children must be experientially abstracted from the observed modifications children make in their schemes and necessarily turns to be second-order social knowledge because they can be known only through interpreting changes in children's mathematical activity. Therefore, in a teaching experiment, which is a research methodology based on scheme theory, the mathematics for current children with respect to a particular mathematical scheme is initially determined by the modification that the teacher-researchers have observed other children make in the particular scheme. This is called a *zone of potential construction* and as a result of actually interacting with the particular children, the zone of potential construction may be reconstituted as a *zone of actual construction* (Olive & Steffe, 2002a). However, Olive and Steffe also pointed out that we should not overlook the role of our first-order models of mathematics in formulating the second-order models, mathematics *of* children because they are likely to play an important role in orienting us as we formulate mathematics *for* children and how to interact with them.

In Piaget's model, knowledge was seen as a collection of schemes of action and models of thinking that allow us to live and move in the world as we experience it. Piaget, as a biologist, imported the concept of adaptation into the study of cognition from the theory of evolution. Based on this notion, Piaget has provided a detailed analysis of the process by means of which the cognitive organism generates relatively invariant objects from its experience and externalizes them into a framework of space, time, and causality which is itself the result of experiential coordination (von Glasersfeld, 1982). According to von Glasersfeld (1995), a scheme consists of three parts: an *experiential situation* which is activated or recognized by the child, the specific *activity* associated with the conceived situation, and a certain *result* of the activity engendered by the child's prediction. The first part of a scheme consists of a 'recognition template', which contains records of operations used in past experience. Then the activity of a cognitive scheme may consist of an implementation of the assimilation operations, and the result of the scheme may contain an experienced situation (Olive & Steffe, 2002a). In addition to the three components of a scheme, to take into account the goal of an activity is inevitable because, in radical constructivism, all cognitive activity takes place within the experiential world of a goal-directed consciousness. The generated goal has to be associated with the situation of the scheme, the scheme's activity is directed toward that goal, and the results of the scheme are compared to the goal (see Figure 2.1). Finally, if the newly formed result satisfies the goal, then the scheme is closed. Conclusively, the action of a scheme is not sensory-motor action, but interiorized action by reflective abstraction with the most minimal sensory-motor indication (Olive & Steffe, 2002a).

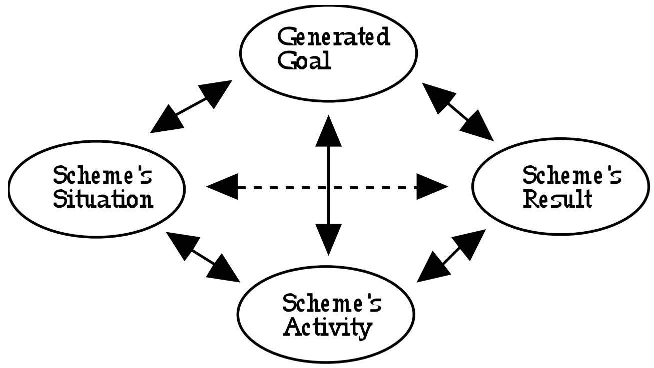


Figure 2.1 A diagram for the structure of a scheme (Steffe & Olive, 2010, p. 23)

I also agree with Olive's and Steffe's view that students' mathematical knowledge consists of schemes of action and operations that are functioning reliably and effectively at the moment. Thus, in my study, I will attempt to identify such schemes in the context of fraction multiplication and division problems, which might be related to the construction of the RNA, through observing children recurrently using a goal-directed activity on several different, but closely related occasions from my first-order mathematical perspective. I believe that my efforts to build models for mathematics of children would provide mathematics educators a useful tool as a language for their communication just as Olive and Steffe (2002a) asserted:

> Without such models, children's mathematics remains situated within the contexts of observation and can be at most described. Without the explanations that we are attempting to build through conceptual analysis, children's mathematics stands little chance of becoming taken-as-shared in the community of mathematics educators, including teachers (p. 127)

CHAPTER III

METHODOLOGY

Teaching Experiment Methodology

The teaching experiment is a methodology for conducting scientific research on mathematics learning. A primary purpose for using a teaching experiment methodology is for researchers to experience students' mathematical learning and reasoning. In other words, the assumption is that there would be no ground for understanding the mathematical concepts and operations that students construct without the experiences benefitted by teaching (Steffe & Thompson, 2000). Researchers who do not engage in teaching of children run the risk that their models might be biased to reflect their own mathematical knowledge (Cobb & Steffe, 1983).

The teaching experiment methodology is deeply rooted in radical constructivism in the sense that researchers in teaching experiments attribute mathematical realities to students that are independent of their own mathematical realities and, therefore, a primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics. Steffe and Thompson (2000) argued that the goal of establishing living models is sensible only when the idea of teaching is predicated on an understanding of human beings as self-organizing and self-regulating. The teaching experiment methodology is also in accordance with my conceptual framework, scheme theory, in the sense that learning involved in a teaching experiment is to be regarded as accommodation in the context of scheme theory. That is, what students learn is defined in terms of the modifications of their current schemes using available operations in a new way rather than in terms of the mathematical knowledge of the researchers. Therefore, the attention would be focused on understanding the students' assimilating schemes and how these schemes might change as a result of their mathematical activity. According to Steffe and Thompson (2000), the teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics and that kind of justification is different from the impersonal, universal, and ahistorical justification. In other words, mathematics should be regarded as a product of the function of human intelligence (Piaget, 1980, as cited in Steffe and Thompson, 2000) rather than as a product of impersonal, universal, and ahistorical reason. From my point of view, this belief about mathematics supports the contention that the teaching experiment methodology can be conducted in accordance with the perspective of radical constructivism.

A teaching experiment consists of a sequence of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what happens during each episode (Steffe & Thompson, 2000). The important duty of the teacher-researcher in the teaching experiment is to attempt to put aside his or her own concepts and operations and not to insist that the students learn what he or she knows. The research hypotheses one formulates prior to a teaching experiment usually guide the initial selection of the students and the researchers' overall general intentions, but also new hypotheses are to be generated and tested during the teaching episodes. However, it should be noted that the researchers do their best to 'forget' these hypotheses during the course of the teaching episodes and to remain aware of the students' contributions to the trajectory of teaching interactions because students' unanticipated ways and means of operating force the teacher-researcher to abandon prior hypotheses while interacting with the students and to create new hypotheses and situations on the spot. Through generating and testing hypotheses, boundaries of the students' ways and means of operating can be formulated (Steffe & Thompson, 2000).

My Teaching Experiment

Data Collection

The data for my research were collected from a year-long constructivist teaching experiment, in which I taught a pair of eighth-grade students at a rural middle school in north Georgia from October 2008 to May 2009. The experiment was part of the larger, longitudinal study called the Ontogenesis of Algebraic Knowing (OAK)[2], whose purpose is to bring forth and understand middle school students' algebraic reasoning. Rosa, one of the two participants, was chosen after an individual selection interview conducted in October of 2008. Carol, the other participant, had been chosen in October of 2007 and was paired with Rosa for her second year of the teaching experiment. The criterion for selection of the two students was the ability to use composite units as iterable units, which is an indicator of the GNS. During the teaching experiment, we met once or twice a week in about 40-minute teaching episodes in which I participated mostly as a teacher-researcher, or sometimes as a witness-researcher. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. One camera usually captured the whole picture of interactions among the pair of students and the teacher-researcher, and the other camera followed the students' written or computer work with the aid of two witness-researchers. The role of the witness-researcher was not only assisting in video recording but also providing other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and the

[2] Dr. Leslie P. Steffe oversaw and participated in all aspects of the study. The rest of the research team consisted of the following graduate students—myself, Catherine Ulrich, Ronnachai Panapoi, and Soo Jin Lee—who contributed during different portions of the two-year study.

retrospective analysis of the videotapes. Among the collected data during a year of the teaching experiment of 2008, twenty teaching episodes were retrospectively analyzed and part of them were transcribed for the present study.

Participants: Carol & Rosa

As mentioned above, two eighth-grade participants for the present study were chosen based on their ability to use composite units as iterable units as an indication of the GNS. Carol joined our teaching experiment from her seventh grade and thus the teaching experiment with Rosa was the second year of her experience with our research team. During the first year we posed various problems in several mathematical topics such as basic combinatorial problems, calendar problems related to modular arithmetic, and cooking recipe problems for proportional reasoning. Related to the issues in the present study, Carol demonstrated that she had constructed recursive partitioning operations but was yet to construct distributive partitioning operations in the context of fraction multiplication. The following teaching episode is a representative example of her mathematical knowledge in terms of distributive partitioning operations. The problem was as follows: "Ron has a gallon of water. He used 2/11 of his water for cleaning his desk. After clean-up, he felt thirsty and drank 2/13 of what was left. Using JavaBars construct what he left and find how much of a gallon of water he has left."

Protocol 3.1: Carol's lack of a distributive partitioning operation[3]

(Using JavaBars, Carol makes an 11-part bar and breaks it into eleven parts so that she can take two parts away from the other nine parts. Then she cross-partitions each of nine parts into thirteen parts, breaks them into one hundred and seventeen pieces and moves

[3] In the protocols, R stands for Rosa, C for Carol, T for the teacher-researcher (myself), and W for a witness-researcher. Comments enclosed in parentheses ( ) describe students' nonverbal actions or interactions from the teacher-researcher's perspective. Words enclosed in square brackets [ ] indicate dialogue that was not spoken but I have inferred from the context. Ellipses (…) indicate a sentence or an idea that seems to trail off. Four periods (….) denote omitted dialogue or interaction. When a word is in caps, it refers to an action in the computer program.

two columns of thirteen pieces away from the other seven columns of thirteen pieces. See Figure 3.1)
T: Can you explain? Carol, first.
C: Me? What I did was, I have my gallon and I first divided it into eleven pieces and I took out two of the eleven cause that's he already drank. And here's what's left of the gallon and I took out two of the thirteen because that would be two-thirteenths.
T: Two rows of thirteen?
C: Um-hm.
T: Rows or columns?
C: Columns.
T: Columns? How many columns are there?
C: One, two, three, four, five, six, seven. (Carol counts the number of the leftover.) Seven. I did it in a wrong way. Didn't I? (Inaudible)
T: What was your intention to do?
C: I was thinking that the columns... because... I was supposed to take out these (pointing out a row of the collection of leftover.)
T: That rows, you mean?
C: Yes. I just realize that.

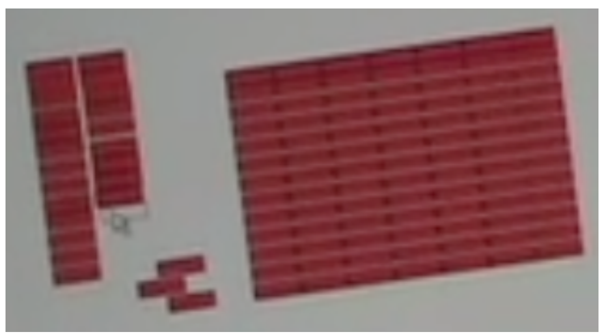
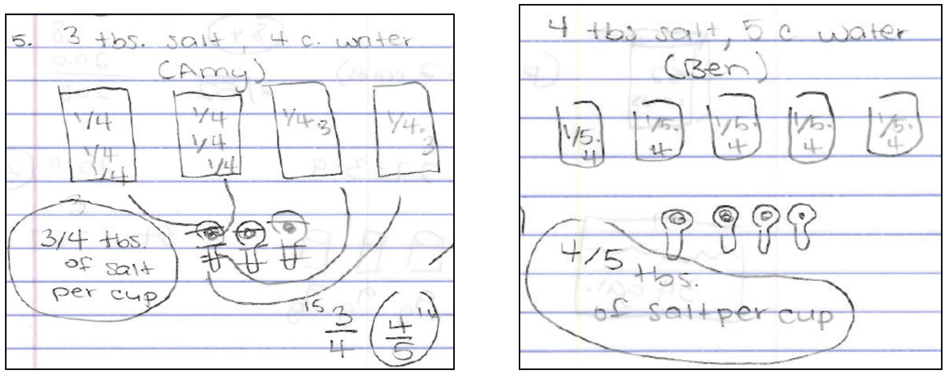


Figure 3.1: Carol's construction for 2/13 of 9/11

At a first glance, Carol's partitioning activity seemed to be on par with a distributive partitioning operation because she partitioned each of nine parts of the 9/11-gallon bar into thirteen parts to construct 2/13 of the 9/11-gallon. However, it turned out that she was yet to interiorize her distributive partitioning operation [if she had constructed it before] in that she did not pull out one-thirteenth of each of the 9/11-gallon bar, which amounted up to a row of her collection of bars (cf. Figure 3.1). Somehow, Carol knew that she needed to take out 2/13 of her 9/11-gallon bar but she ended up with taking two columns of her collection, which was 2/9 of her 9/11-gallon. Although she was able to correct it by herself by reflecting on her partitioning activity, it was retrospective accommodation, rather than anticipatory mathematical reasoning. Further, when she asked how much of a gallon her construction was, she struggled with figuring out how much it was in terms of a gallon. If she had been explicitly aware that each 1/13 from the 9/11-gallon bar was 9 of 1/13 of 1/11-gallon when she was taking it or prior to the taking activity, she could have calculated how much of a gallon her construction was with no difficulty. Therefore, at this moment, construction of a distributive partitioning operation could be attributed to her, at most, *in action*, not prior to actual action.

However, in the pre-interview conducted on October of 2008 for Carol's second year of the teaching experiment, Carol demonstrated that she had constructed a distributive partitioning operation. One of the posed problems in the pre-interview was a density comparison problem: "Amy has 3 tablespoons of salt with 4 cups of water. Ben has 4 tablespoons of salt with 5 cups of water. Who's water tastes saltiest?" Immediately, Carol drew four cups and three tablespoons for Amy and five cups and four table spoons for Ben on a paper. She put three marks on each of three tablespoons for Amy to divide each table spoon into four parts and drew matching lines between each quarter part of a table spoon and each of four cups. Then she wrote down three '1/4's or '1/4*3' on each of four cups even without matching lines for the other two tablespoons (see Figure 3.2a). For Ben, she did not even need to put partitions on each of four tablespoons. She just knew that each of Ben's cups contains four-fifths of a tablespoon by abstracting her distributing activities for Amy (see Figure 3.2b). It was possible by her distributive reasoning that distributing one-fifth of each of Ben's tablespoons to each of the five cups with understanding that the amount of each cup can be replicated five times to produce the whole of the four tablespoons. She also knew that four-fifths of one tablespoon was identical to one-fifth of all of the tablespoons. It was surprising for the research team because she did not show an indication of such stable use of her distributive reasoning until the last day of her first year of the teaching experiment.



Figures 3.2a & 3.2b: Carol's distribution for Amy (Left) and Ben (Right)

On the other hand, Rosa, one of the two participants, was chosen after an individual selection interview conducted in October of 2008 for her first year of the teaching experiment. Although the research team asked the mathematics teacher in the middle school to recommend a student who might be on a par with Carol's mathematical knowledge, there was little known about Rosa's mathematical knowledge, especially regarding fractions. On the initial pre-interview, she demonstrated some flexibility in whole number multiplication in combinatorial problems, but it was the first day of the teaching experiment with the two students that Rosa indicated that she had constructed a GNS.

Protocol 3.2: Finding multiples between 1 and 48 on 11/07/08

T: The first problem is how many multiples of three are there from one to forty-eight?
C: Do you want us write all the multiples down?
T: Ye, you can write down or just…
C: Can I do tallies?
T: Sure, you can do whatever you want.
R: Okay. (Rosa seems to count the number of multiples of three by folding her fingers down. After thirty seconds,) sixteen?
T: Sixteen? Okay. Rosa, can you explain your answer?
R: Okay. What I did is that I knew that there was, um, ten multiples so far, if you, for thirty, if you take thirty, so what I did was, um... I knew that there was eighteen left and so, um, three times six equals eighteen so I did ten plus six to get forty eight. That was confusing.
T: Good. I think you are doing very well. Can you explain, Carol?

> C: If I was gonna do tallies, but I figured that it was gonna take too long. So then I decided forty eight divided by three with, um, with see how many times, how many multiples would be.
> T: Why did you divide?
> C: Because it shows how many times that three can go into forty eight.

Rosa seemed to count multiples of three from 3 to 48 with an aid of using her fingers. However, she finally solved the problem by disembeding 30 and 18 from 48 in her re-presentation, counting the multiples of three in each number, and recomposing them together. That is, she already knew that 48 consists of 30 and 18 and also she was able to see 30 as ten units of 3 and 18 as six units of 3. In order to count the number of multiples of three between 1 and 48, she decomposed the composite unit [48] into two composite units [30 and 18], each of which has a three-levels-of-units structure based on the iterable composite unit [3] and then composed the two three-levels-of-units-structures [ten 3s and six 3s] to find the total number of multiples in the original composite unit [48]. It was possible by her ability to use three levels of units based on an iterable composite unit of three prior to actual construction. Such strategic use of the iterable composite unit [3] was an indication of her GNS. On the other hand, Carol started to mark tallies for counting all multiples of three, but at some point she erased all tally marks and divided 48 by 3. She seemed to realize that she could use a three-levels-of-unit structure for the composite unit of 48 based on the iterable unit of 3 to count the total number of multiples from 1 to 48. We already knew that she was a GNS student from the data of the first year of teaching experiment in her seventh grade but such quick use of her whole number division knowledge in this problem corroborated that she had constructed a GNS.

Data Analysis

The first type of analysis was ongoing analysis that occurred by watching videos of the teaching episodes and debating and planning future episodes. For the most part, the resources from two cameras were mixed for a single, digitalized video file on the day of each teaching episode. In this way, I created a restored view of my teaching experiment. “A restored view is a wider view of activity than can typically be captured with an individual camera, but is still a selective view that reflects the researchers’ perspective of the recorded lesson” (Olive & Vomvoridi, 2006, p. 21). Then a sequence of summaries for the teaching episodes were created in consecutive time, each of which provided not only a written description of students’ mathematical activities and interactions with the teacher, but also emerging key points in students’ thinking and learning that were taken into account for the next teaching episode. The second type of analysis, which has to be conducted later, is retrospective analysis. The purpose of the retrospective analysis of the sequence of teaching episodes is to make models of students’ ways of operating mathematically through conceptual analysis of students’ mathematical activities. First of all, a researcher is involved in understanding what the students’ actions are and hypothesizing why the students acted in such ways. Then researcher’s construction of the schemes attributed to the students can be drawn from the process. After construction of such schemes, the researcher revisits the recorded teaching episodes and consciously goes back and forth over the records to see how those schemes are related to the other mathematical schemes and structures. These models (as well as the hypotheses) are subject to revisions until the researcher’s model is not countermanded by further observations (Tunç-Pekkan, 2008). In the modeling process, core concepts in scheme theory are used such as assimilation, accommodation, cognitive and mathematical play, interaction, mental operation, self-regulation, zone of potential construction, and others. It is much more labor-intensive than the activity of teaching (Steffe & Thompson, 2000).

CHAPTER IV

RESULTS OF DATA ANALYSIS

Overview

In this chapter, I present the results of the data analysis. Basically, the present study is a portrayal of the constructive itineraries of two eighth-grade students' fractional knowledge. Although I have established an overarching goal for a year of the teaching experiment with the participant students before the teaching experiment was initiated [related to fraction multiplication, division and construction of RNA], constitution of detailed teaching directions and proposed problems for every teaching episode totally relied on the ongoing analysis of the students' mathematical behaviors and operations observed in the previous episode. There was no way to understand the mathematical concepts and operations that the students constructed without the experiences obtained by actual teaching. Therefore, note that the three phases, which will be described in this chapter, were framed as results of retrospective, conceptual analysis of the two students' mathematical schemes and operations that emerged during the teaching experiment, not pre-structured as an analyzing framework before the teaching experiment as in a structured clinical interview.

When the teaching experiment began, my initial concern was to investigate the two students' emerging schemes and operations necessary for multiplicative transformations between fractional quantities. However, when I posed the first problem [how many times is 3 meters contained in 5 meters?] in relation to their fractional knowledge on December 3rd of 2008, the two students' ways of assimilating the problem situation were different from my expectation. For

Carol and Rosa, the problem was a measurement division problem where the divisor [3 meters] did not evenly divide the dividend [5 meters]. That is, the problem was a situation for them to evoke their unit-segmenting scheme for whole number division. Since mathematical schemes and operations necessary for measurement divisional situations seemed quite different from those for multiplicative transformations among fractions, from my point of view at that moment, I decided to put their multiplicative transformation activity aside for a while and focus on exploring the students' mathematical actions and operations in fraction measurement division situations, following their ways of assimilation of the first problem. Therefore, Phase I is the report about how Carol and Rosa modified their unit-segmenting schemes when the measurement division situations became complex.

Table 4.1[4]

Overview of the Problems in Phase I: Fraction Measurement Division

Protocol	Date	Problem
4.1	2008/12/03	How many times is 3 meters contained in 5 meters?
4.2		Make a bar that is seven-fifths times as long as 5 meters
4.3		If I empty a water tank that holds 6 gallons of water using a container that holds 3/4 gallon of water, how many full containers of water would I get?
4.4		How many times is 3/4 gallons of water contained in 4 gallons of water?
4.5	2008/12/05	How many times is 3/5-meter is contained in 1 meter?
4.6		What part of 7/5-meters is contained in 1 meter?
4.7		How many times is 7/5-meters contained in 2 meters?
4.8		How many times is 5/3-meters contained in 176 meters?
4.9	2008/12/15	Make a 1-meter bar from a 3-part 3/11-meter bar and find how many times to the 3/11-meter must be used to make the 1-meter bar.
4.10	2009/05/06	Measure an 11/19-meter bar with a 4/19-meter bar.
4.11		Measure a 1/3-meter bar with a 1/7-meter bar.

[4] Note that all protocols were described in a chronological order, but there was a gap of five months between the Protocol 4.9 and 4.10. The explanation for this time gap will be given later.

Among 4 days of teaching episodes in relation to fraction measurement division, 11 protocols were chosen to document the two students' constructive processes in the context of fraction measurement division situations through modification of their unit-segmenting schemes (see Table 4.1).

Table 4.2

Overview of the Problems in Phase II: Fraction Multiplication

Protocol	Date	Problem
4.12	2009/02/10	A string one foot long was cut into eleven equal parts and one of these pieces was then cut into five equal parts. How much of the string was one of these five parts?
4.13		Make 3-meter bar and make a bar 5/2 times as long as the bar.
4.14		Take 1/3 of a 2-part bar without erasing a partitioning line.
4.15		Construct a bar for one person when 6 persons share 5 candy bars.
No Protocol		Make a bar that is 11/4 times as long as a 3-meter bar.
4.16		Make a bar that is 3/4 times as long as a 3-meter bar.
4.17	2009/02/16	A candy bar costs 63 cents. How much does eleven-sevenths of a candy bar cost?
4.18		Make a 6/6-bar. Use that bar to make a 23/18-bar without erasing the marks on the 6/6-bar.
4.19		Make a 3/3-bar. Use that bar to make a 5/4-bar without erasing the marks on the 3/3-bar.
4.20		Make a bar and pretend that it is 19/11 meters long. Make a bar that is 7/19 times as long as that bar. How long is your constructed bar?
4.21	2009/02/23	Make a bar and pretend that it is 17/15 meters long. Make a bar that is 1/2 times as long as that bar. How long is your constructed bar?
4.22		Find how much rubber is needed for one basketball if 3 pounds of rubber makes 11 basketballs.

The main issue during Phase II was "Fraction Multiplication." Before examining the students' multiplicative transformation activities with fractions, it seemed pretty natural to explore the students' knowledge for fraction multiplication first, because some multiplicative transformation situations of fractions can be interpreted as inverse situations of fraction multiplication[5]. Therefore, the aim of the teaching experiment during Phase II was, first of all, to investigate what mathematical operations the students had constructed or could construct in the context of fraction multiplication situations.

In addition, we knew little about Rosa's fractional knowledge because it was the first year of her teaching experiment with our research team. Thus, the teaching experiment during Phase II produced meaningful data not only for preparation of the next teaching experiment period [multiplicative transformation activities between two fractions], but also for investigation of Rosa's fractional knowledge for multiplication itself. For the report in Phase II, 3 days of teaching episodes were analyzed and 11 protocols are provided for detailed descriptions of the students' mathematical actions and operations (see Table 4.2).

During Phase III, 11 protocols were chosen among 9 days of teaching episodes (see Table 4.3). Although the duration of the teaching experiment in Phase III was longer than in any other phase, there were a lot of trials and errors from my point of view. Actually, over the whole teaching experiment, the most difficult challenge for me was to devise appropriate tasks for the students' multiplicative transformation activities and also to provide the students with a relevant environment for the activities. We assumed that a scheme for a multiplicative transformation between two fractions requires being explicitly aware of the fractional operator to be used in the transformation as well as the transformation activity itself. However, the students' transformation activities in JavaBars did not seem suitable for construction of such a scheme because often the students [especially, Carol] found a multiplicative operator through establishment of a part-whole relationship between the given and the result, rather than reflecting

[5] For example, to construct 5/3 times 3 meters, a student needs to find one-third of the 3 meters and multiply the result by five to get the answer, 5 meters. Transformation of 3 meters into 5 meters can be regarded as an inverse situation of the multiplication because the student needs to construct the multiplicative operator based on the result of the multiplication [5 meters].

on the processes of their transformation activities. Therefore, I decided to pose such transformation problems in the milieu of Geometer's Sketchpad (GSP) using the dilation option. In contrast to the program, JavaBars, where a student can produce her transformation activities step by step, the dilation option in GSP involves generating a scale factor for the transformation. In that sense, I believed that the dilation operation could provide an occasion for the students to reflect on and abstract their mathematical activities for the transformation, which might lead them to construct the scale factor, that is, the number to be used in multiplication [the multiplier].

Table 4.3

Overview of the Problems in Phase III: Multiplicative Transformation Between Two Fractions

Protocol	Date	Problem
4.23	2009/02/23	Using JavaBars, find a fraction of 37/31-meter to make 1/31-meter and find a fraction of 37/31-meter to make 31/31-meter.
4.24	2009/02/27	What fraction of 5/7-meter amounts to one-meter?
No Protocol	2009/03/11	What would you have to dilate twice of a segment by to get one-third of segment?
No Protocol	2009/03/23	Make 1/2A1/2B[6] as many ways as possible using dilation.
4.25	2009/03/30	Transform a 1/4-meter bar into a 1/10-meter bar.
4.26	2009/04/01	Transform a 1/5-meter bar into a 7/5-meter bar and find a scale factor for the transformation.
4.27		Transform a 7/5-meter bar into a 1/15-meter bar and find a scale factor for the transformation.
4.28	2009/04/13	Label 7/4-meter$_{14}$[7] bar (Identifying 14/8-meter with 7/4-meter.)
4.29		Compare a 3/2-meter bar and a 6/4-meter bar.
4.30	2009/04/15	Find a scale factor for transformation of 1/7-meter into 1/3-meter.
4.31		Construct two-step dilation between 1/2-meter (1/13-meter) and 1/3-meter (1/11-meter).
4.32	2009/04/24	Transform 1-meter into 1/24-meter using two steps in JavaBars.
4.33		Transform 1/3-meter bar into a 1/5-meter bar in JavaBars.

[6] The two students and I (the teacher-researcher) made an agreement about naming for the constructed segments. That is, 1/2A1/2B was a name for the segment, which was made from a given segment AB dilated by 1/2. This strange notation was adopted to indicate the effect of the dilation on each end point of the segment AB.

[7] The subscript denotes the number of partitions in which the bar is partitioned into. In this case, the 7/4-meter bar in the problem has 14 partitions.

However, such a desire turned out to be unsuccessful because of the abstractness of the GSP environment. That is, figuring out a multiplicative operator for transformation prior to actual dilating activity in GSP seemed to require too abstract mathematical thinking for Carol and Rosa in that their mathematical schemes and operations needed to be performed in re-presentation prior to actual activity so that they could anticipate the result of their actions and operations. Finally, in the context of two teaching episodes, I coordinated both programs [GSP and JavaBars] on the spot and tried to utilize them appropriately in response to the students. At any rate, throughout the two and a half months of Phase III of the teaching experiment, I realized that the students' modifications or coordination of their GNS, such as recursive partitioning operations, distributive partitioning operations and common partitioning operations, and their lack of interiorization of the iterability of unit fractions based on a FCNS, played important roles in their transformation activities.

Phase I: Fraction Measurement Division

Measuring-out a Whole Number Quantity with a Remainder

For a situation to be established as divisional, it is always necessary to establish at least two composite units, one composite unit to be segmented and the other composite unit to be used in segmenting. The goal is to find how many times one can use the measuring unit [the unit to be used in segmenting] with a given unit to be segmented. Steffe (1992b) reported that an ENS child, Johanna was able to establish her unit-segmenting scheme as anticipatory and the units to be used in segmenting were available to her as iterating units prior to operating. However, when the composite unit to be segmented is not completely measured out by the unit used in segmenting [when producing a remainder], the divisional situation might be assimilated as novel and lead to perturbation in a student's use of her unit-segmenting scheme because the result of a unit-segmenting scheme produces a fractional quantity in terms of the segmenting unit. Since Carol and Rosa were eighth-grade students, we already knew that they had a certain amount of fractional knowledge. Through a year of teaching experiment (with Carol) and pre-interview (with Rosa), I attributed to the students construction of partitive fraction schemes[8]. Although it might be reasonable to attribute to them a higher level of fractional schemes such as iterative fraction schemes[9], in the following analyses I will use 'partitive fraction schemes' as minimal fraction schemes that they constructed because 1) I did not have enough evidence to argue that they constructed higher level of fraction schemes than partitive fraction schemes at the time and 2) I conjecture that association of a partitive fraction scheme with a unit-segmenting scheme was enough to enable the students to deal with the problems in the following protocols. We decided to begin this phase of the teaching experiment with a whole number division problem that produces a remainder. As mentioned in chapter III, in all of the protocols, R stands for Rosa, C for Carol, T for the teacher-researcher (myself), and W for a witness-researcher.

Protocol 4.1 on 12/03/08: Finding how many times 3 meters is contained in 5 meters.

T: How many times is three meters contained in five meters?
(Both students write down the problem on their own paper. Rosa divides five by three using a division algorithm and gets $1.\dot{6}$ as an answer.)
C: Hmm… this is hard.
R: You want this in fraction form?
T: Yeah, I prefer to fraction form.
R: Okay.
T: You don't necessarily calculate in decimal form.
R: Yeah. I don't, how would, hmm…in decimal, okay. (Rosa draws a 5-part bar, shades three parts out of the five parts and writes down '3/5' under the circled three parts and '1

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[8] A partitive fraction scheme is the first scheme to be a genuine fractional scheme (Steffe, 2002). It enables a child to establish a substantial but limited understanding of fractions as parts of a specific partitioned whole (Tzur, 1999).
[9] In constructing the iterative fraction scheme, the child abstracts the invariant, multiplicative relation between the sizes of the unit fraction and the reference whole (Tzur, 1999). Fractional Connected Number Sequence (FCNS) is a possible result of an iterative fractional scheme (Steffe, 2002)

time' over the shaded three parts. Then, she writes 2/5 under the left two parts. See Figure 4.1a). (To Carol) how did you work it out? I don't know.
C: I divided it.
R: I divided it too. But you have to show in fraction form.
C: I got point six.
R: You got point six? I got one point six because three…
C: Because five goes into thirty six times and then I plotted a decimal point.
R: Yeah, but three can go into five one time.
C: But your… but it's not three going into five. It's three meters in five meters.
R: So you do five divided by three.
C: Oh, wait. I did it backwards. You're right. Don't listen to me.
R: Okay. (Rosa resumes her work. She divides each part of the whole 5-part bar into two and writes down '6/10' right to the next of '3/5' and '4/10' to the next of '2/5')
T: Carol, you don't necessarily calculate in decimal form. Just think about in fraction form.
C: Okay.
R: In fraction form.
T: Yeah.

Carol and Rosa assimilated the problem as a divisional situation. Such assimilation of the problem situation activated their division algorithm although Carol conducted division calculation in a reverse way, that is, three divided by five rather than five divided by three. However, since they already knew that I (the teacher-researcher) always prefer a fraction form to a decimal form as an answer[10], Rosa's attempt to find her fractional answer prompted her to think about the part-whole relationship between five meters and three meters and she drew a 5-part bar with three parts shaded. Rosa seemed to try to find her decimal answer, $1.\dot{6}$, in her drawing by dividing each of the five parts into two (cf. Figure 4.1a), which gave her two fractions with a denominator, '10' [6/10 for 3/5 and 4/10 for 2/5] for each part. However, she could not connect her decimal answer $1.\dot{6}$ to her drawing because she took the five parts as a referent whole rather than the shaded three parts. Considering their multiplicative reasoning with

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[10] I was expecting a fractional answer based on the two students' mathematical operations rather than a decimal result of their numeric calculation of division. For example, they could get 5/3 as an answer by taking one-third of the 3 meters and iterating the result [1 meter] five times to get five meters.

whole numbers, they must have been able to find the answer very easily for a simple whole number division problem, say, to find how many 3 meters are contained in 6 meters. Therefore, the struggle that the students demonstrated above indicates that to find a fractional answer for a whole number division problem involving a remainder was a novel situation for them, which led them to experience perturbation in using their unit-segmenting scheme.

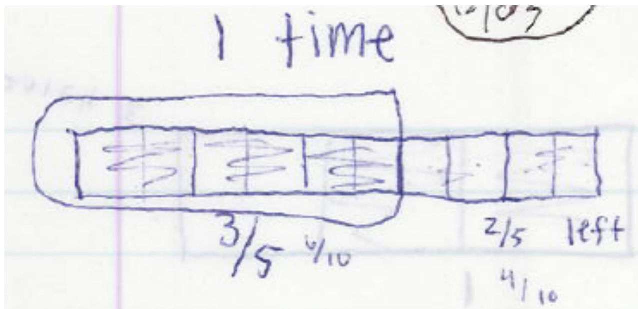


Figure 4.1a: Rosa's drawing to find her fraction answer

Protocol 4.1: (Cont.)

R: It's gonna be one and then something fraction.
C: This is hard.
T: It's not an easy problem.
R: All right. Let me try something.
W: (Witness-researcher intervenes.) Can you draw five meters, Carol?
C: I am.
W: Okay. (Carol draws two different-sized bars and divides the smaller bar into three parts and the larger bar into five parts.)
R: Is that it? One and six-tenths?
T: One and six-tenths?
R: Is that what you're looking for?
W: (Witness-researcher points out '2/5' on Rosa's paper.) You have two-fifths here.
R: See. What I did just um…I doubled it. So I did six meters in ten meters because ten can go into a hundred and then percents can go into decimals or fractions or….
W: I want you, (Witness-researcher points out Rosa's 5-part bar) I want you to use this up here, not (inaudible) decimal.
R: Well if you…
T: Use these numbers in a similar way.
C: Would it be one and two-thirds?
T: One and two-thirds?
C: Yeah.
T: How did you figure out it?
C: Um, because it goes in once and there is two-thirds of that leftover (see Figure 4.1b). Is that right?

R: I don't know how you...
T: Yeah, can you explain it to Rosa?
C: It's like what you are doing except that I compared the two, like if you have three and you have five (pointing out the 3-part bar) it goes in once and since it was three pieces there is two of three pieces left.
R: There is two-thirds, okay I see now. All right. Okay.
W: What is that two-thirds out there?
R: Two-thirds of the total three-thirds that are (inaudible). Okay. I got it now. I see. I got it.

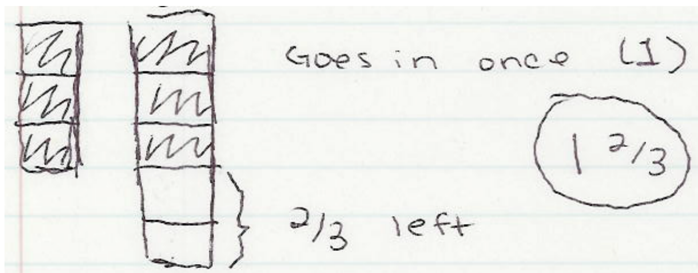


Figure 4.1b: Carol's drawing to find her fraction answer

Carol seemed to be stuck with the problem indicated by the repeated comments, "This is hard" However, Rosa' comment, "It's gonna be one and then something fraction." seemed to activate Carol's unit-segmenting operation as an assimilating operation for this problem and lead her to draw a 3-part bar and a 5-part bar to compare two quantities (cf. Figure 4.1b). I make this conjecture based on the fact that Carol was already drawing her two bars on her paper when the witness-researcher attempted to help her by asking "Can you draw five meters?" Once Carol set the goal of measuring the 5-meter bar with the 3-meter bar, she realized the answer, one and two-thirds and her explanation indicated that she constructed a three-levels-of-units structure as a result of her unit-segmenting operation. In other words, the 5-meter bar was not only five units of 1- meter, but also one unit of a 3-meter and two-thirds of the 3-meter for Carol. It can be viewed as similar to construction of a multiplicatively nested three-levels-of-units structure, say, six as two units of three, but it should be more than that because five was not a multiple of three, which means it necessarily produced a fractional remainder as a part of the measuring unit, three. I conjecture that her partitive fraction scheme contributed to her construction of such a three-levels-of-units structure. That is, when she got 2 meters as a result of her unit-segmenting operation by 3 meters from 5 meters, the result of the unit-segmenting operation turned into a situation for her partitive fraction scheme, which led her to view 2 meters as two-thirds of 3 meters. However, her construction of three levels of units was inherited from the iterability of *one*, rather than of a *unit fraction*, say one-third. This is why I cannot attribute an iterative fraction scheme to her at this time. At any rate, I hypothesize that the goal of measuring a quantity that produces a remainder, can be fulfilled by a generalizing assimilation of students' unit-segmenting scheme. When assimilating operations of a unit-segmenting scheme are modified and contain a partitive fraction scheme as a subscheme, the unit-segmenting scheme is generalized to include the records of the operation of constructing a part-whole fractional relationship with any two quantities. In other words, a student becomes aware of the need to measure the remainder of the division problems by establishing a part-whole relationship between the leftover and the measuring unit [the unit used in segmenting].

On the other hand, Rosa, although she initiated Carol's unit-segmenting scheme, did not seem to associate the result of her unit-segmenting scheme and her fraction scheme. Since Rosa joined our teaching experiment as a new partner of Carol, we did not have enough information about her construction of fraction schemes at this point in the teaching episode, although I hypothesized that she had constructed a splitting operation based on her pre-interview. What can be conjectured from this protocol is that her attempt to find a fraction answer enticed her to take the biggest quantity (five meters) as a referent whole. Since she already knew that $1.\dot{6}$ was an answer, Rosa kept trying to convert her drawing into a decimal form regardless of my insistence of a fraction form. Until Carol provided her explanation, Rosa did not realize that the 3 meters should be a referent whole nor associate her unit-segmenting scheme and her fraction scheme.

Even though Rosa immediately assimilated Carol's explanation later, whether the relation that Rosa established between her unit-segmenting scheme and her partitive fraction scheme was an embedding of her partitive fraction scheme into her unit-segmenting scheme or just a sequential chain of associations was still to be investigated. She may have constructed an associative chain of schemes, where any scheme in the chain was triggered by the results of the scheme immediately preceding, but she was unable to independently use her scheme.

When Carol was asked a similar question, "How many times is five meters contained in seven meters?" right after the protocol 4.1, she demonstrated that her partitive fraction scheme was embedded in the assimilating part of her unit-segmenting scheme. She said, "it's gonna be the same" and immediately drew a 5-part bar and a 7-part bar to get one and two-fifths. It also indicated that Carol anticipated that the remainder could be dealt with by using her partitive fraction scheme prior to conducting an actual unit-segmenting operation. In contrast to Carol, Rosa used the division algorithm for her answer again and got 1.4. Although she easily converted 1.4 to one and four-tenths and then to one and two-fifths, whether her division algorithm was used as a part of assimilating operations for her unit-segmenting scheme was uncertain at that time.

Converting a Divisional Situation to a Situation to Find a Multiplicative Operator

Protocol 4.1: (Second cont.)

W: You got it. Good. What could you have, take three meters times to get five meters?
R: Three times one point six to get five?
W: You just figured it out. Can you use what you figured out? You just figured it out. You already answered my question.
C: Two-thirds?
W: What?
C: Two-thirds? No?
W: You have to take, gotta make five meters out of three meters, take something times three meters to get five meters.
R: one point six?

C: point six? We're just guessing. We have to think about [it].
R: I know. Okay, what is he asking, Carol?
C: He is asking how many times this three can go into…
R: Into five?
C: One and two-thirds? Oh!
R: Oh!
C: You're asking for the answer?
W: What's the answer?
C: One and two-thirds.
W: Why?
T: Why, why do you think like that?
C: Because you have that going into once and that's two-thirds
R: I had the answer all along. Because point six, six, six is two-thirds. Oh, gosh. Okay, that was... Okay.

Although Carol demonstrated that she constructed a unit-segmenting scheme embedding her partitive fraction scheme in the assimilating part of her unit-segmenting scheme, the above protocol indicates that construction of necessary operations for finding a fractional operator is another problem and remains to be investigated. When the witness-researcher asked the question, "What could you have, take three meters times to get five meters?" right after Carol's providing the right answer, both of them did not realize that the answer should be same as the one obtained for the previous problem. This might be because the witness-researcher's question induced them to think the fractional operator to transform 3 meters to 5 meters, which might require them to use a unit fraction as an iterable unit. That is, I hypothesize that for the students to multiplicatively transform 3 meters to 5 meters, they should be able to partition 3 meters into three to get a unit of 1-meter, which is one-third of 3 meters. Then it can be iterated five times to make 5 meters, which leads to producing five of one-third of 3 meters, that is, five-thirds as the final multiplicative operator for the transformation. However, only after Carol assimilated the question using her unit-segmenting scheme again as in her comment, "He is asking how many times this three can go into…" she exclaimed that her previous answer, one and two-thirds was the answer. The re-presentation of a part-whole relationship between 3 meters and 5 meters seemed to enable her to re-assimilate the witness-researcher's question by using her unit-segmenting operation.

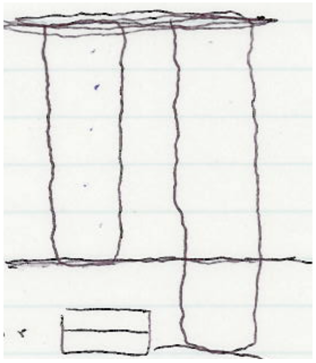
Inversion Between Situation and Result

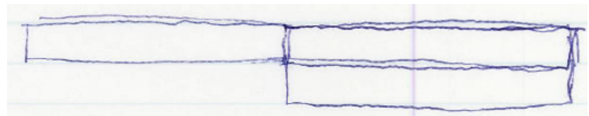
The fraction multiplication in the Protocol 4.2 was posed right after Carol and Rosa solved the fraction division problem, "How many times is 5 meters contained in 7 meters?" As mentioned above, Carol easily solved the problem indicating her construction of a unit-segmenting scheme for whole number division problems with a remainder. Therefore, we wanted to explore whether Carol could take the result of her unit-segmenting scheme as a situation for fraction multiplication although the overarching goal in the teaching experiment at the time was to investigate what schemes and operations were used in fraction divisional situations. Members of the research team thought that posing the same situation for Rosa would help us gain important information about her fractional knowledge, even though we were not sure of Rosa's construction of a unit-segmenting scheme with a remainder. Thus we posed the following problem for both students in Protocol 4.2.

Protocol 4.2 on 12/03/08: Making seven-fifths times as long as 5 meters.

(The problem is "Given a 5-meter bar, make a bar that is 7/5 times as long as the bar.)
W: I want you make a new bar that is five meters long. (Rosa draws a bar on her paper.) Got it? Don't mark it. Just pretend that's five meters.
R: Okay.
W: Don't mark the bar. (To Carol) can you do that? Just make a bar that is five meters long.
C: Can I make mine like long ways?
T: Yeah, I think you can do that. Yeah. (Both students finish drawing an unmarked bar for a five-meter bar.)
W: Okay. I want you to take, make a bar that is seven-fifths times as long as that.
C & R: Seven-fifths?
R: So, one and two-fifths longer that this? That's like another bar.
C: That is just like the question.
W: (Rosa draws another bar the same size as her original bar connected to the right of the original bar.) I want you to make it exactly now because I don't want you to get this in a guess. (To Rosa) so you're guessing

R: I'm guessing?
W: You're guessing. You don't know how long it's gonna be unless you have to make it.
R: Unless you…
W: You're just estimating. How could you tell more for sure?
R: Well, if you double it like if you go down, (Rosa draws one more bar right beneath the added bar. See Figure 4.2b). Then, no. You can take that (pointing out one of the added bars) to that over there (pointing out the empty space under the original bar). I don't really know.
C: Can I, can I explain to her?
R: I see, I see what you did because you used the marks.
C: It's just like that. Because if you have seven-fifths which is one and two-fifths, then you have one bar and then extra two pieces of the leftover so just two extra ones not just...
R: I see what you did because you used the little blocks to count. See. I didn't use blocks to count (cf. Figure 4.2a.)
W: Okay. So you used marks. Okay so why are those marks are so important, Carol?
C: What?
W: Why are those marks are so important?
C: So you can get the exact amount.
W: So you know that exactly, right? So, how many, what would you have to divide the first bar into, how many pieces?
R: Five.
C: Five.
W: Five? And take one of those pieces. What would you do it then?
C: Add it on.
R: I guess.
T: How many times? Add it on?
C: Two.
R: Two.
W: How many times you have to use it in all the way to seven-fifths?
T: Seven-fifths, to make seven-fifths. How many times you have to add?
C: You have to add two extras.
T: Two extras.
R: Right.
W: Okay.





Figures 4.2a & 4.2b: Carol's (Left) and Rosa's (Right) drawings for seven-fifths times as long as 5 meters

There was a crucial difference between Rosa's and Carol's mathematical thinking when they were assimilating this problem situation. For Rosa, seven-fifths was one and two-fifths, somehow longer than one. However, under the constraint of not putting marks on the bar by the witness-researcher, she could not mentally partition the 5-meter bar into five and use a part as one meter [one-fifth of 5 meters,] a unit to be iterated seven times to make a bar seven-fifths times as long as 5 meters. In other words, for Rosa seven-fifths was only the amount that was longer than one because it has one and more extras, but she could not figure how much the extras should be in her re-presentation.

On the other hand, Carol immediately connected the witness-researcher's question to her drawing of the 5-meter bar and the 7-meter bar from the previous question as saying "That is just like the question." The association of the result of her unit-segmenting scheme with a remainder with this fraction multiplication situation seemed to enable her to re-present making a 7-meter bar as an answer prior to actual activity. Also, ruled lines on her paper seemed to help her to compare 5 meters and 7 meters without putting marks on the bars. Nevertheless, I thought Carol's construction of splitting operations for improper fractions should be investigated more because she did not explicitly indicate the use of one-fifth of 5 meters for the construction of seven-fifths of 5 meters by iterating seven times. It was not surprising that a student who had constructed a partitive fraction scheme assimilated an improper fraction as separated into a whole number part and a proper fraction part, say, one and two-fifths for seventh-fifths.

Although Rosa could not even partition the 5-meter bar into five parts to add two more extra pieces to make a bar seven-fifths times as long as 5 meters, whether necessary schemes and operations for the mathematical activity were lacking in Rosa's mathematical reasoning was yet to be explored because this was her first episode of the teaching experiment and she might have never encountered such a construction problem related to fractional multiplication. In sum, we confirmed that Carol was able to use the result of her unit-segmenting scheme as a situation for the fraction multiplication problem. However, the witness-researcher's effort to induce them to use one meter [a fifth of 5 meters] as an iterating unit seemed to turn out to be a failure because the visualized part-whole relationship between the two drawings for 7 meters and 5 meters on their paper seemed to prevent them from feeling the necessity to construct an iterable unit, one-fifth of five meters to make seven-fifths consisting of seven units of one-fifth.

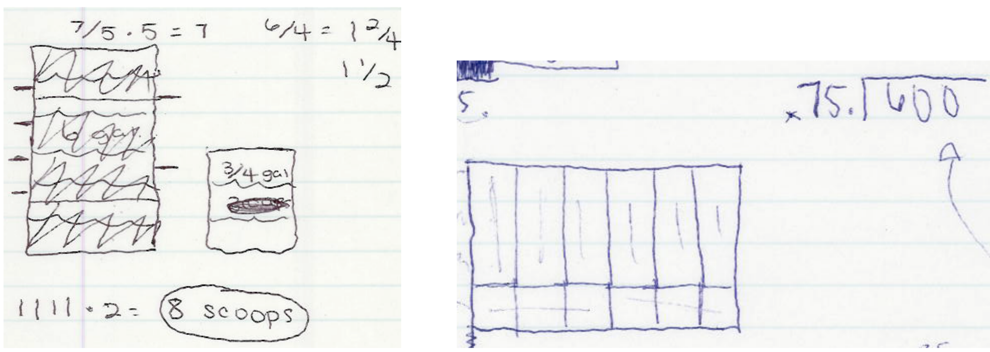
Measuring out an Evenly Divisible Quantity with a Fractional Quantity

Coming back to the fraction divisional situation, we began with a division problem involving a fractional quantity. We hypothesized that they should be able to solve it because the whole number quantity to be segmented was evenly divisible by the fractional quantity used in segmenting, the result of which did not produce a remainder.

Protocol 4.3 on 12/03/08: Measuring 6 gallons of water with 3/4 gallon of a container.

(The problem is "If I empty a water tank that holds 6 gallons of water using a container that holds 3/4 gallon of water, how many full containers of water would I get?" After verbal interactions for clarification of the problem between the teacher and the students, both students immediately begin to solve the problem on their own paper and get the right answer, eight.)
T: Can we share answer, Carol?
R: I got eight.
C: I got eight scoops.

T: You got it right. Both of you.
C: My way is kind of confusing.
R: Well, what I did was first I just, I was gonna go ahead and just divide because three-fourths is point seventy five. And then it's the same way as finding it. But what I did is I made four markers, one, two, three, four (counting vertically the number of parts in one of her six gallons, which is partitioned into four by ruled lines of the given paper. See Figure 4.3b) because three-fourths and fourth is a whole. So here's three right here, three here, three, three, three, and then we have to do two of the bottom to get the other...
W: Is that what you did, Carol, same thing?
C: No, mine was kind of different. What I did is I look, if you have three-fourths, and then you do two scoops of three-fourths. It would be six-fourths or one and a half gallons (see Figure 4.3a).
R: That's what I got. See I did seventy five times two just um... one fifty so...that's just,
C: Yeah, see if you have one and a half gallons so then what I did, I marked half, one and half. And I got four and then since it was two scoops, I times two and I got eight.
T: Great.



Figures 4.3a & 4.3b: Carol's (Left) and Rosa's (Right) drawings for measuring 6 gallons with 3/4 of a gallon

Carol's unit-segmenting operation was a progressive integration operation, which takes the results of a prior integration operation as an input element for another integration (Steffe & Olive, 2010). In order to construct 6 gallons of water with 3/4 of a gallon, she doubled a three-fourths to make one and a half gallons and took the result of the prior doubling operations to make six gallons. Also, in the sense that Carol solved the problem using her unit-segmenting scheme with no indication of a modification of the activity of the scheme, it could be claimed that the assimilation to Carol's unit-segmenting scheme was generalizing[11]. Therefore, there must be a modification of the assimilating operations prior to action. Carol's unit-segmenting operation with whole number quantities was modified to include a fractional quantity in the situation where a given whole number quantity to be segmented was evenly divisible by the given fraction used in segmenting. That is, the operation of unit-segmenting could be applied to the situation of measuring-out 6 gallons with 3/4 of a gallon in the same way as it could be applied to the situation of measuring-out 6 gallons with a whole number that evenly divides six, say, 2 gallons. On the other hand, Rosa started with her division algorithm to get the answer and she was immediately able to connect the result of her numerical calculation to her unit-segmenting scheme when I asked her to find an answer in a fraction form. In that sense, her use of division algorithm symbolized the assimilating operation of her unit-segmenting scheme, at least at this point.

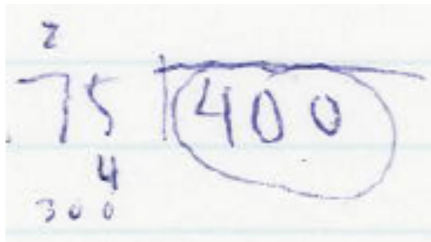
Measuring a Whole Number Quantity with a Fractional Quantity, which does not Evenly Divide the Whole Number Quantity

Protocol 4.4 on 12/03/08: Measuring 4 gallons of water with 3/4 gallon of a container.

T: I have only four gallons of water, how many times do I take out?
R: How many three-fourths?
T: Yeah, with three-fourths.
(Rosa starts writing numerical expressions of a conventional division algorithm, but seems to hesitate to go on with her calculation process. On the other hand, Carol draws a 4-part bar for 4 gallons on her paper and put a line for every one and a half gallons and shades two of one and a half gallons on her 4-part bar. See Figures 4.4a & 4.4b).
R: Okay, going to have to be...
C: Is it four and two-thirds?
R: See, that's what I'm about to get. Cause I know four scoops is three hundred which is like...

[11] An assimilation is generalizing if the scheme involved in assimilation is used in a situation that contains sensory material or conceptual items that are novel for the scheme but the scheme does not recognize it (Steffe & Olive, 2010).

T: That's very close.
C: (Disappointedly) Oh... (Carol and Rosa resume their solving activity.) It's three and two-thirds? (Teacher answers in the negative.) Oh...
T: (To Carol) What are you trying to do?
R: Is it five and one-third?
T: Five and one-third.
R: No, it's not five. It's no way.
C: It could be five.
T: I didn't say no.
C: It could be five. Because one, two, three, four (Carol seems to count the number of three-fourths in her 4-part bar for 4 gallons of water.)
T: Five and one-third is right.
R: It is? Oh, do you want me to show you how I did it? I have to do everything like in a problem situation like I can't just use pictures. (To Carol) do you want me to show how I did it?.
....
C: (Twenty seconds elapse) yeah, that's what I got. Five and one-third. I'll tell you what I did after you're done.
R: Okay, what I did is I knew that four hundreds was a whole so I just did four over one and multiplied it times four over three, which is like the... which is basically four over one divided by three-fourths. I did four hundreds divided by point seventy five. And then I got sixteen over three (see Figure 4.4a) and I just made it...
T: Why did you multiply four-thirds?
R: It's what I was intending to do was four over one divided by three-fourths but you have to change it to four over one and make it it's like multiplied by its reciprocal.
C: What I did is, I have four gallons and then I did like, I couldn't [do what I] did with the last one, and divided it into one and a half on each but then I just started off doing I found two whole ones and times two and I got four and two-thirds [of one and a half.] Then I was thinking what you were saying like how you get. And then I realize you can have an extra one because you have an extra. Then I got six, then there is the two extra leftover one from the cup. So I minused two-thirds from the six from the leftovers, and I got five and one-third.
T: Can you say that again? How did you figure out the last extra one?
C: The left extra one? Because you only had so much left of the top of the gallon you could scoop out, then you have left over two-thirds.
T: Two-thirds? Two-thirds of what?
C: Two-thirds of a little cup.
R: I see what you're saying, but I'm just don't understand your drawing, but I know what you're saying.



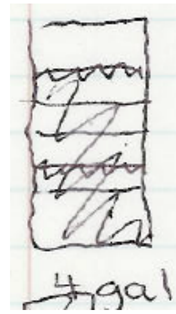


Figure 4.4a: Rosa's division algorithm (Left) & Figure 4.4b: Carol's drawing for measuring out 4 gallons with 3/4 of a gallon (Right)

As indicated in the previous protocol, Rosa associated a measuring-out situation with a division problem situation. However, she seemed stuck when she realized that four cannot be evenly measured out by three-fourths. Unlike the previous problem situation where the measuring quantity evenly divided the measured quantity, she seemed to feel perturbation associating the result of her numerical division calculation with that of her unit-segmenting scheme when the result produced a remainder. It corroborated that the relationship between her unit-segmenting scheme and her partitive fraction scheme was just an association, not that her partitive fraction scheme was embedded into the first part of her unit-segmenting scheme.

In other words, the fact that Rosa independently could not deal with the remainder of her numerical division calculation result indicated that her partitive fraction scheme (if she had constructed one) was not embedded as an assimilating operation of her unit-segmenting scheme. Rather, the result of her unit-segmenting scheme was possibly associated to the situation of her partitive fraction scheme by the additional cue from an external source like Carol's mathematical behavior. Her comment, "That's what I'm about to get. Cause I know four scoops is three hundred, which is like" also corroborated that her use of a division algorithm still symbolized her unit-segmenting operations with a fractional quantity. However, the realization that her unit-segmenting scheme did not work for this problem induced Rosa to try another division algorithm, that is, the invert-and-multiply algorithm for fraction division problems (cf. Figure

4.4c). However, even though Rosa found the answer, five and one-third, her fraction division algorithm seemed disconnected to her unit-segmenting scheme. That is, the result of her numeric calculation no longer seemed to symbolize her unit-segmenting operations. Her insecurity about the answer right after she provided it, “No, it’s not five. It’s no way.” and her justification about the answer depending only on the invert-and-multiply algorithm process for fraction division corroborated that her unit-segmenting scheme was yet to undergo an accommodation whereby her partitive fraction scheme was embedded in it as a subscheme. Further, Rosa’s comment at the end of the protocol after Carol explained the answer using her drawing, “I see what you’re saying, but I’m just don’t understanding your drawing, but I know what you’re saying.” indicated that such accommodation was not a simple process for her.

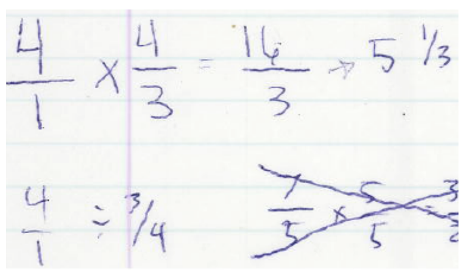


Figure 4.4c: Rosa’s invert-and-multiply algorithm

Interestingly, Rosa’s answer of division calculation using an invert-and-multiply method helped Carol reflect her unit-segmenting operation and find the right answer. Carol’s drawing definitely indicated that she conducted her unit-segmenting operations. However, her first answer was four and two-thirds rather than five and one-third. Her mistake appears to have come from the conflation of units used in segmenting. As seen in Figure 4.4b, her unit to be used in measuring out 4 gallons of water was 1 and 1/2 gallons of water, rather than 3/4 of a gallon given in the problem. The reason she chose 1 and 1/2 gallons as a segmenting unit seemed due to the convenience of actual drawing on the paper, that is, putting a line for a half was much easier than finding a mark for 3/4 of a gallon. She measured 3 gallons as twice of 1 and 1/2 gallons and doubled it because her unit was originally 3/4 of a gallon. However, the change of her segmenting unit caused a conflation of units in segmenting when dealing with the remainder as a result of her unit-segmenting operations. She accidently measured the remaining 1 gallon with 1 and 1/2 gallons, rather than measuring it with 3/4 of a gallon, which led her to get four and two-thirds for the final answer. Even with the inaccurate use of her unit-segmenting operations, I do not believe that the mistake alleviated my conjecture that Carol's partitive fraction scheme was already embedded in her unit-segmenting scheme and she had constructed a newly modified unit-segmenting scheme at a higher level because she immediately self-corrected her answer from Rosa's new answer. In spite of Rosa's uncertainty about the answer, Carol independently assimilated Rosa's answer as a new possibility. Rosa's answer seemed to help Carol re-conceive her segmenting unit and re-initiate her unit-segmenting operations. Then she realized that three of 1 and 1/2 gallons of water, which also were six of 3/4 gallon of water, covered more extras in addition to 4 gallons of water. This time, she explicitly measured the surplus with 3/4 of a gallon as indicated by her comment, "Two-thirds of a little cup." And she finally got five and one-third by subtracting two-thirds from six[12].

Rosa's Construction of a Unit-Segmenting Scheme with a Remainder through Retrospective Accommodation

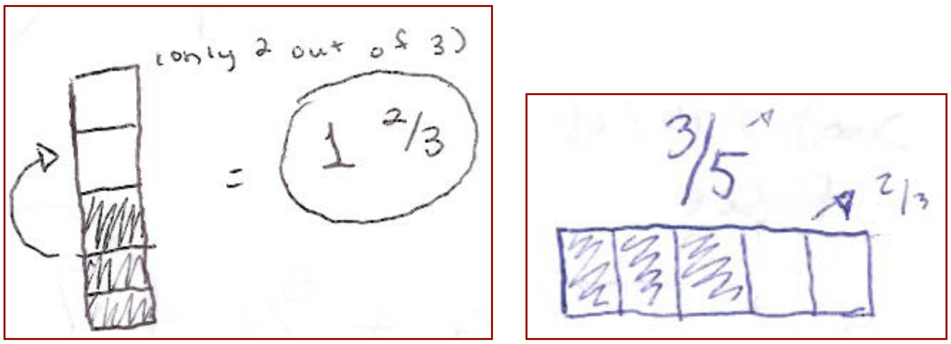
Protocol 4.5 on 12/05/08: Finding how many times 3/5-meter is contained in 1 meter.

T: How many times is three-fifths meter contained in one meter?
R: Three-fifths of a meter?
T: Um-hm.

[12] Note that there is no evidence that Carol and Rosa were explicitly aware of 4 gallons of water as sixteen of 1/4 gallon of water. If they could, they might be able to solve the problem as in the same way as they measure out 16 gallons with 3 gallons. This aspect can be explicated in relation to the iterability of a unit fraction, say, one-fourth, which will be mentioned in details later.

(Both students draw a 5-part bar and shade three parts of the 5-part bar on their own paper. However, Rosa starts writing numerals for calculation of fraction multiplication as '5/5x5/3=25/15')
C: One and two-thirds?
T: (Teacher nods his head) one and two-thirds.
R: Hold on.
T: (To Rosa) that's fine. Take your time.
R: Did you get five and three-thirds?
C: No.
T: Five and three-thirds?
R: (Rosa compares her answers with Carol's) or one and two-thirds. That's the same thing.
T: Yeah, one and two-thirds is right, but um… Can you explain your way, Carol?
C: Um… I just started up by drawing a bar, divided it into fifths. And then I have three of the fifths, which is what I was asked for. So I already knew I had one and there are two leftover out of the three pieces so I have two-thirds.
T: So two pieces is… Can you say that again why two pieces is two-thirds?
C: Two pieces is two-thirds because, if that (three parts)'s one, and it's three-thirds or three over one. Then you only have two. It would be two-thirds.
T: Okay.
R: Okay. I do, I have to do everything like with an equation. So I just did five over five which is the one meter times five over three or just divided by three-fifths and I just reduced it down and got one and two-thirds.
T: Um-hm. Did you see...
R: I know, I know how she did it. Because this (three parts) is one and then there is two here. So there is two out of that three. I see how she did it, but I would like this anyway.
T: So, what do you think, this one (two unshaded parts) is two-thirds of what?
R: Two-thirds of this way here (three shaded parts).
T: Yeah, two-thirds of three-thirds. Right?
R: Yeah.
W: Rosa, can you tell us how you thought about that?
R: First I started drawing it. But then I thought, you know, it's three-fifths into one, five-fifths. So I just thought it is a division problem and um… I just did five over five divided by three-fifths. You have to multiply and make the reciprocal. Then I just reduced it down and that's how many times three-fifths is in one meter.

In Protocol 4.1 [How many times is 3 meters contained in 5 meters?], Rosa seemed to construct *a unit-segmenting scheme with remainder* or at least associate a result of her unit-segmenting scheme with a situation for her partitive fraction scheme. At that time, although the construction was initiated by Carol, Rosa exclaimed that she understood what Carol did. However, this protocol demonstrated that Rosa had yet to construct a unit-segmenting scheme with a remainder as an anticipatory scheme. Even though there was no difference between the two students' drawings of a 5-part bar for one meter and three shaded parts for 3/5-meter (see Figures 4.5a & 4.5b), Rosa failed to conceive the remaining two-fifths of a meter as two-thirds of the segmenting unit, 3/5-meter and went back to relying on an invert-and-multiply algorithm for fraction division calculation in contrast to Carol's immediate realization of the answer from her drawing.



Figures 4.5a & 4.5b: Carol's (Left) and Rosa's (Right) drawings for measuring 1-meter with 3/5-meter

When Rosa got '25/15' and '5/3' in her calculation, she did not know the meaning of the result of her fraction division calculation in a quantitative sense. This was corroborated by her comments, "Did you get five and three-thirds?" when Rosa was asking Carol for the confirmation of her answer. I do not believe Rosa was unable to distinguish five-thirds from five and three-thirds. Rather, even though she got the right answer using an invert-and-multiply algorithm for fraction division, she did not seem to realize that the answer [five-thirds] times the 3/5-meter should be 1-meter quantitatively because the answer was what she got by dividing 1-meter by 3/5-meter. That is, the result of her calculation did not stand in for the multiplicative relationship between 1-meter and 3/5-meter that 1-meter could be constructed by multiplying by five-thirds the 3/5-meter even though she was actually taking perceptual information from her drawing of a 5-part bar with three parts shaded. Therefore, such lack of confidence for her answer gave her a temporary confusion in identification of the answer, and the confusion was not eliminated until Rosa compared her answer with Carol's answer as in her comments, "or one and two-thirds. That's the same thing." As indicated in the Protocol 4.1, Rosa quickly assimilated Carol's way of construction and was able to explain with her own words. Since we knew that she had already constructed necessary conceptual elements for a unit-segmenting scheme with remainder, that is, a unit-segmenting scheme and a partitive fractions scheme, somehow I can attribute construction of a unit-segmenting scheme with a remainder to Rosa. However, if the conceptual elements were selected and used by herself only as a result of interactive communication with Carol, I would attribute to her the construction of a unit-segmenting scheme with a remainder through *retrospective accommodation*[13].

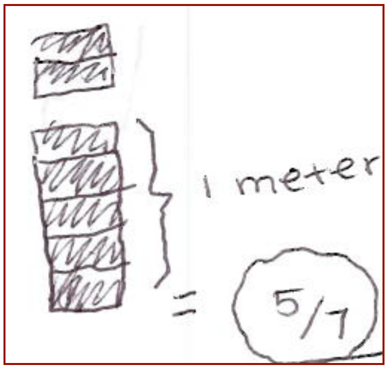
Measuring a Smaller Quantity with a Larger Quantity

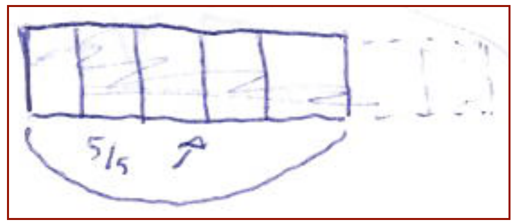
> Protocol 4.6 on 12/05/08: Finding what part of 7/5-meters is contained in 1 meter.
>
> T: Yeah, then let's go to this part. How much part is seven-fifths meters contained in one meter?
> (Carol draws a 5-part bar and two more parts separately and shades them all. Rosa also draws a 5-part bar and adds two more parts on the right end of the 5-part bar using dotted lines. However, she turns to numeric calculation again. She writes '7/5 x 5/5 = 35/25' and '7/5' but crosses a line through the numeric expressions as she feels something wrong with her calculation. See Figures 4.6a and 4.6b.)
> C: (Carol counts five parts and seven parts several times in turn and makes a face.) (Without a confidence,) would it be five-sevenths?
> T: Um-hmm.
> R: See, I remember this yesterday cause she got it and I didn't get it.
> T: Yeah, that's the same problem.
> R: How did you get it, Carol?
> C: Um… I'm not really sure but I was looking at it if you have one, two, three, four, five, that's the whole.
> R: (At the same time with Carol) that's the one.

---

[13] A retrospective accommodation involves selecting and using conceptual elements already constructed. From the student's perspective, a retrospective accommodation is self-initiated in that it is the student who must select and use the concept. From an observer's perspective, the conceptual elements may by selected as result of interactive communication (Steffe & Wiegel, 1994)

C: And you have the extra two and then I would make it seven but then you have the extra one meter which is what you have.
R: Oh~ I see how she did it. I see. Okay. I totally see how she got it.
T: Okay. So, (Teacher turns to look at Rosa' work.)
R: Five out of the seven pieces. This (her five parts drawn in solid lines) is the one and here is the extra two for the seven. So this (five parts drawn in solid lines) is one. So five over total seven pieces because it's seven. I see, I see it now.





Figures 4.6a &4.6b: Carol's (Left) & Rosa's (Right) drawings for 7/5-meter and 1-meter

Originally, the problem was "How many times is 7/5-meters contained in 1-meter" However, the two students were totally at a loss with the problem. To construct a smaller quantity by multiplying [times] a larger quantity by a number seemed to them a sort of an unimaginable situation because the word 'times' had always been used for increasing a quantity. Thus, I decided to replace 'how many times' with 'how much of' in the problem hoping that the students could attend to the larger unit as a referent unit and compare two different quantities.

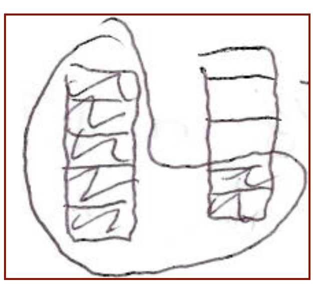
However, even when the students assimilated this problem as a situation for a unit-segmenting scheme, they struggled with it because the segmenting unit [7/5-meter] was larger that the unit to be segmented [1-meter], which was a novel situation for them to use their unit-segmenting schemes. Rosa also drew a 5-part bar and added two more parts on the right end of the 5-part bar using dotted lines but suddenly she turned to numeric calculation again. She wrote down '7/5 x 5/5 = 35/25' and '7/5' and then crossed a line over the numeric expression as she felt that something went wrong. On the other hand, it was Carol who found the answer first. She drew a 5-part bar and two more parts separately and shaded them all. After counting five parts and seven parts several times in turn, she provided an answer, "five-sevenths." Her comment, "I was looking at it if you have one, two, three, four, five, that's the whole. And you have the extra two and then I would make it seven but then you have the extra one meter which is what you have" indicated that her answer came from the part-whole relation based on the visual sensory-motor information through comparing the sizes of her five parts with her seven parts. At this time, what I would conjecture is that a situation where the unit used in segmenting was larger than the unit to be segmented would be another epistemological obstacle for the students and further might be a reason to inhibit them from expanding their measuring-out activity using a unit-segmenting scheme. Carol and Rosa needed to expand the range of assimilating situations of their unit-segmenting scheme so that it could include a situation where a smaller quantity was to be measure with a larger quantity. Until the students realize that the problem can be solved in the same way as they solve problems when measuring a quantity with a remainder (i.e. by the use of a unit-segmenting scheme with a remainder) the epistemological obstacle observed in this protocol will remain as a main cause of their perturbation when using their unit-segmenting scheme.

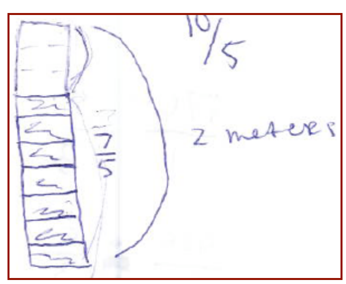
Unit-Segmenting Scheme with a Remainder Involving an Improper Fraction

Protocol 4.7 on 12/05/08: Finding how many times 7/5-meters is contained in 2 meters.

W: How many times is seven-fifths contained in two meters? Seven-fifths meters contained in two meters.
C: Would it be twice?
T: Twice of what?
C: Twice seven-fifths.
R: So it's seven-fifths in ten-fifths, right?
W: This is better one.
C: Okay. Two meters. (Carol draws two 5-part bars and shades the whole parts of one 5-part bar and two parts of the other 5-part bar. Rosa draws a 7-part bar vertically and adds three more parts on top of the bar using dotted lines. See Figures 4.7a and 4.7b. Carol

writes down '1 2/7' on the paper.) Is that the answer? (Witness-researcher shakes his head.) Wait. (Carol writes '2/7+3/7=5/7' and '1 5/7') Right? (Witness-researcher shakes his head again.)
R: Is that five-sevenths?
C: One and five-sevenths?
R: It has to be more than itself.
T: (To Carol) can you explain it to me?
C: Um, I think I did it wrong. I think I added too many, but if you have it one time (pointing out the shaded seven parts) it goes into the second one and you have one and two-sevenths. But then you still have the extra three-sevenths. Then it can go into… even if you don't have the whole one.
T: Sorry, it's hard to see. Can you, yeah, seven-fifths is…
C: Wait! No, no, no, no.
R: (At the same time with Carol) is it one and three-sevenths? One and three-sevenths, one and three-sevenths. I'm sorry.
C: Yeah, it's one and three-sevenths. Because I added those two (the shaded 5-part bar and the shaded two parts of the other 5-part bar) cause that's one right there. (Carol draws a circle holding the shaded seven parts.) This equals one.
R: (At the same time with Carol) I miscalculated. It's one and three-sevenths.
W: (Witness-researcher laughs.) All we know is how to get a hard problem, don't we?
R: It's kind of obvious, like there's three, three-sevenths left because this (seven parts of the 10-part bar) is the whole.
C: Yes, that kind of that is two-sevenths, one that should've been one whole.
R: See, I got fifty over thirty five.
T: (To Carol) yeah. I got you, I got you. Now and (to Rosa) your explanation, same? How did you get it?
R: I see how she got it. She, but I, I did,
T: What's your way?
R: Ten over five which is the two meters times five over seven and I got fifty over thirty five which is one and three-sevenths because this three-sevenths (the unshaded three parts of her 10-part bar) which is the ten-fifths of, [what] you want is out of here (the shaded seven parts of her 10-part bar.) So, that's gonna be three-sevenths, right? Yes… yes, yes, yes.
T: Three-sevenths. Okay.





Figures 4.7a &4.7b: Carol's (Left) & Rosa's (Right) drawings for 7/5-meter in 2-meter

If Rosa's comment, "seven-fifths in ten-fifths" stood in for her conception that seven-fifths consists of seven one-fifths and ten-fifths consists of ten one-fifths, any one-fifth of which can be iterated seven times and ten times to make seven-fifths and ten-fifths, she might have immediately solved this problem as in the same way as to measure 10 meters out with a unit of 7 meters. In other words, the construction of a fractional connected number, say, 10/7, as a composite unit containing one composite unit consisting of seven one-fifths and another composite unit consisting of three one-fifths can be crucial to solve this problem. Although Rosa and Carol constructed bars for 7/5 meters and two [10/5] meters on paper, there was no evidence that they assimilated the 7/5-meter bar and the 10/5-meter bar based on the construction of interiorized fractional connected numbers. Once again, for this problem Rosa demonstrated her reliance on a division algorithm, which was a typical behavior of Rosa whenever she got stuck with her drawing to figure out the answer. However, in that she independently corrected her answer and established a relation between her division algorithm with her drawing in terms of measuring out with 7/5-meter, she retrospectively accommodated her concept of fraction measurement division using conceptual elements of unit-segmenting scheme and partitive fraction scheme.

On the other hand, Carol's writing of '2/7 + 3/7 = 5/7' and '1 5/7' revealed an interesting aspect of her fractional knowledge because it indicated her conflation of units when measuring out a quantity with an (improper) fractional quantity. Somehow she felt the necessity of measuring out the whole two meters each meter of which was partitioned into five parts. She could measure out the remaining three parts [3/5-meter] using seven parts [7/5-meter] as a segmenting unit and get three-sevenths of the segmenting unit for the remainder parts. However, she conflated two units [a given unit of a meter and a segmenting unit, 7/5-meter] when she measured seven parts consisting of one 5-part bar and two parts of the other 5-part bar. Her confusion of the two measuring units was indicated by her comments, "if you have it one time (pointing out the shaded seven parts) it goes into the second one and you have one and two-sevenths. But then you still have the extra three-sevenths." Such confusion of units induced her to identify seven parts as one and two-sevenths, which led her to get one and five-sevenths as a final answer. My conjecture is that, as in the case of Rosa, if Carol had assimilated the problem situation based on the iterability of a unit fraction [1/5], she might have found the answer more easily. In other words, in the previous problems such as "measuring 5 meters with 3 meters" or "measuring 1 meter with 3/5 meter" the students did not indicate any evidence that they used a multiplicative relationship of a unit fraction with a whole unit. The former might require constructing a three-levels-of-unit structure in order to deal with the remainder [2 meters], where 5 meters was measured by 3 meters and thus 3-meters should emerge as another level of unit in relation to the given two units [1 meter and 5 meters]. The construction of such a three-levels-of-units structure, however, was inherited from the students' construction of an iterability of one, rather than a unit fraction. Likely, in the latter problem although it involved a fractional quantity [3/5-meter] and required the students' construction of three levels of units to see the remainder [2/5-meter] as two-thirds of the newly constructed unit of 3/5-meter, the relationship of a unit fraction [one-fifth] with the whole one was implicit just as the unit fraction in a partitive fraction scheme has an implicit iterability not transcending the one whole.

In contrast, in order to construct a three-levels-of-units structure involving 7/5-meter in the present problem, I would argue that the construction of a multiplicative relationship of one-fifth with one whole was essential because the students should have conceived seven-fifths as a unit of seven units of one-fifth, any of which could be iterated five times to make a referent whole [five-fifths] and also seven times for seven-fifths. As described above, both students could not establish such relationships with the given 7/5 meters and 2 meters based on a unit fraction of 1/5 for a while, even with two drawings [seven parts for 7/5-meter and ten parts for 2-meter]. Also, the fact that they were able to reflect on their mathematical operations and self-corrected the answer by themselves with two drawings implied that perceptual information for the given quantities was still one of the critical factors for them to conduct their mathematical [unit-segmenting] operations.

Carol's Construction of a Student-Generated Algorithm for Fraction Measurement Division

Since the two students got the answers for the two problems: How much of 7/5 meters is contained in 1 meter? and How many times 7/5 meters is contained in 2 meters? as a result of their unit-segmenting scheme with a remainder, we wanted to know whether the students could strategically use those results as material for the other measuring-out situations. We were especially interested in situations in which a relatively large quantity should be measured, that is, when to make drawings to get perceptual sensory-motor information would be very difficult. When the students were asked about how many times 7/5 meters is contained in 10 meters, both of them easily found the answer, fifty-sevenths by multiplying five to the previous result, ten-sevenths, which was the answer for the problem "How many times is 7/5 meters contained in 2 meters?" Further, when I increased the length to be measured to 31 meters, they easily wrote down '155/7' on their own paper by calculating '5/7 x 31/1' with an aid of the witness-researcher's reminding question, "How many times is 7/5 meters contained in one meter?" and encouraging the students to use the result for their solution. Therefore, the concern for the next teaching episode was whether such a strategic use of the result of their unit-segmenting schemes with a remainder remains permanent so that it can be used in a different but similar problem.

When they were asked to find how many times 5/3 meters is contained in 176 meters, it was Rosa that first multiplied one-hundred-seventy-six by three-fifths. However, it was not a strategic use of her unit-segmenting scheme with a remainder. Rather, it was reemergence of the invert-and-multiply algorithm for a fraction division problem whenever Rosa assimilated a problem as a division situation. Such lack of her mathematical reasoning was indicated by the uncertainty of her answer right after finishing calculation of her fraction multiplication as "Is that right? It doesn't look right." Although I eagerly attempted to induce her to use the other way related to what she did before, Rosa could not use her unit-segmenting scheme strategically for this problem. I had to directly ask her how much of five-thirds was contained in one, but she could not figure out three-fifths of five-thirds was one. Her answer was two-thirds emphasizing the difference between five-thirds and one whole. Upon my request of verification for her answer, she wrote down '5/3 × 3/3.' She seemed to use an invert-and-multiply algorithm for the division of '5/3 ÷ 3/3', rather than '3/3 ÷ 5/3.' This kind of lacuna in her mathematical reasoning coincided with what she had indicated in the solving process for the previous measuring-out problem when the measuring quantity was larger than the measured quantity. On the other hand, Carol was catching up with the present problem with an aid of the witness-researcher because she spent more time in converting her answer into a decimal form for the previous problem.

> Protocol 4.8[14] on 12/05/08: Finding how many times 5/3 meters is contained in 176 meters.
>
> (Carol is working on the problem, "How many times is 5/3 meters contained in 176 meters?")
> W: Remember, Carol. The problem is how many times is five-thirds contained in…
> C: One hundred seventy six.

[14] Note that a part of Rosa's verbal expressions and communications with the teacher are omitted because the intention of this protocol is to show Carol's construction of an algorithm for fraction measurement division and almost all parts of activities were done individually. However, brief descriptions of Rosa's work are also described to help understand the context.

W: One hundred seventy six. (When Carol writes down ‘5/3 x 176/1’ for calculation, the witness-researcher intervenes.) Wait, you missed a step.
C: Oh, wait. I have to find how many times it goes into the one. (Carol draws two 3-part bars and shades all three parts of the first bar and two parts of the second bar. Then she writes down ‘5/6’) Five-sixths? Is that how many times it goes into one? No?
T: No…
W: Make it five-thirds.
C: Oh, it goes in three-fifths times, wouldn’t it?
T: Yes.
W: Right, three-fifths.
C: Three-fifths. Then it would times... (Carol writes down ‘3/5 x 176/1’ and accidentally got ‘628/5’ by miscalculation. On the other hand, Rosa struggles to find how much of five-thirds is contained in one.)
C: I got it!
T: Okay, hold a second.
(Rosa insists that her answer should be two-thirds rather than three-fifths. Upon the teacher’s request of verifying her answer, she writes down ‘5/3 × 3/3’ and gets ‘5/3.’ She looks bewildered by the unexpected answer.)
C: Have you seen the pattern? Like, um…
R: Sometimes it works, sometimes it doesn’t.
C: Rosa, on these (pointing her drawings for the problem in the Protocol 4.6. See Figure 4.6a) if it was seven-fifths, it turned into five-sevenths. There is kind of a pattern. And then on the other one, it was five-sevenths and that ended up being seven-fifths. And if you look at this one (back to the current problem), it’s five-thirds and it ends up, if you recognize the pattern, it would be…
R: One and two-thirds?
C: No, three-fifths.
T: How much part of... Carol, can I ask a question? How much part of five-thirds [is] contained in one meter?
C: One? Three-fifths.
R: If the three is the total, okay, I think I get it.
T: (To Rosa) can you see the three-fifths…
R: Okay, so this is, okay. I have to draw it over here. So here, (Rosa draws a 5-part bar.) There is three or five. Okay, so this (three parts of the 5-part bar) is one right here and this (the whole 5-part bar) is five. And if you want to know how many times is this (three parts) in the total right?
T: Um-hm.
R: Okay, um… (Rosa writes down ‘5/3’) and it’s three-fifths because it’s three out of the total five. (The teacher nods his head.) Okay, like I see how she is doing it but then I forget what I do, what I do.
C: Isn’t there also the pattern? Like I said five-thirds ends up being three-fifths? And the other one was seven-fifths ends up being five-sevenths.
R: So it’s just the reciprocal of it. If you want to find it in one, it’s the reciprocal.
C: Yeah.
T: So can you see why the reciprocal works together?
R: Yeah, I see how it works together.

> ....
> T: Okay. Carol, can you explain from the start so that Rosa and I share?
> C: Okay. I did from the five-thirds in a hundred-seventy-six. And then I found out the three-fifths for one like one whole, it was three-fifths of one whole. Then I times that by a hundred-seventy-six because that was the number in the problem. I got six-hundred-twenty-eight over five. Denominator multiplication. And then I simplified it.

Unlike Rosa, Carol's numeric notations of '3/5 x 176/1' did not come from the conventional invert-and-multiply algorithm for fraction division. If so, the order of writing for fraction multiplication should have been reversed like '176/1 x 3/5' for the division problem of '176/1 ÷ 5/3' as Rosa usually did. Carol apparently formed the goal for activating her unit-segmenting scheme as in her comment, "Oh, wait. I have to find how many times it goes into the one." Further, the final comment of Carol's to reflect her solving processes indicated that her unit-segmenting operation was fundamental in her further mathematical operations for the present problem as "I did from the five-thirds in a hundred-seventy-six. And then I found out the three-fifths for one like one whole, it was three-fifths of one whole. Then I times that by a hundred-seventy-six because that was the number in the problem." Actually, her mathematical operations also involved units-coordinating operations. That is, she distributed three-fifths over each of one-hundred-seventy-six. However, the difference from her units-coordinating operation in her GNS, was that a fraction was distributed over a whole number. She constructed the fraction, three-fifths as an iterating unit to get one-hundred-seventy-six of three-fifths like getting fifteen by iterating a unit of three five times. This was strong evidence for Carol having constructed an iterative fraction scheme. It also means that the assimilating situations of Carol's units-coordinating scheme were expanded and included a proper fraction.

More importantly, she abstracted her unit-segmenting operations from the previous two problems, "How much of seven-fifths is contained in one or two meters?" and found a pattern for fraction measurement division problems while saying:

> If it was seven-fifths, it turned into five-sevenths. There is kind of a pattern. And then on the other one, it was five-sevenths and that ended up being seven-fifths. And if you look at this one (back to the current problem), it's five-thirds and it ends up, if you recognize the pattern.

Therefore, for Carol to find the answer for how many times five-thirds is contained in one-hundred-seventy-six, she could just flip five-thirds to make three-fifths and multiply it by one-hundred-seventy-six, which exactly coincided with the conventional invert-and-multiply algorithm for fraction division. I would call Carol's construction a *student-generated algorithm for fraction measurement division*. "Child-generated algorithms as they are manifest in notation are nothing but records of operation, and these records serve the function of constructive generalization" (Steffe & Olive, 2010, p. 274). The flipping pattern was an abstracted record of her unit-segmenting operations, especially when measuring a unit whole with a fractional quantity more than the whole. On the other hand, even with Carol's explanation, Rosa did not seem to understand what Carol was trying to say. Rosa's invert-and-multiply algorithm was a *procedure*. "A procedure is a scheme in which the activity is only connected to rather than contained in the first part of the scheme" (Steffe & Olive, 2010, p. 214). In Rosa's case, the first part of her procedure was constituted by the words "how many times, contained." Her activity of dividing using the invert-and-multiply algorithm can be regarded as her meaning for the words.

Rosa's Failure to Associate her Invert-And-Multiply Algorithm to her Unit-Segmenting Scheme

Carol was absent from the teaching episode held on the 15th of December 2008. This situation permitted Rosa to be the primary actor in solving the situations of learning and thus turned out to be a good chance to reveal the lacuna in Rosa's mathematical reasoning with fractions.

> Protocol 4.9 on 12/15/08: Making a 1-meter bar from a 3-part bar of 3/11-meter and finding how many times the 3/11-meter must be used to make the 1-meter bar.

T: Make a bar, which is composed of three equal parts. (Rosa draws a 3-part bar on the paper.) Suppose that bar is three-elevenths. Suppose that is three-elevenths of a meter bar.
R: Of a meter bar.
T: Of a meter. Three-elevenths of a meter. (Rosa writes down '3/11 of a meter') Then using that bar, how can you make a one meter bar? Can you make, using that bar, can you make one meter bar? This is, suppose this is three-elevenths of a meter.
R: So you can... It's about three, three times with a fraction. (Rosa writes down '11/11 × 11/3 = 121/33') I think I did something completely wrong.
T: Then before calculating, can you think of... Can you construct the bar with that one? Can you construct a bar with that three-elevenths of a bar?
R: Um... With this? Just three? Um...You have to like double, or triple it almost. Triple... Then you triple it. Um... Two-thirds. So it's like three and two-thirds... of the time that you can...
T: Three and two-thirds?
R: Um-hm. I think. If it goes into eleven, this (Rosa writes down '11/3') Well, I think this is wrong. (Rosa puts a strikeout on the numeric sentence. See Figure 4.8.) Because if you do three over eleven times three over one (writing down '3/11 × 3/1 = 9/11') it's nine over eleven. And you still have two-elevenths left, of the three-elevenths that you're using. So...Three...
....
R: Oh, this is three and two-thirds. That's how many times three-elevenths can go into eleven, or one meter.
T: Right. Definitely.
R: Because it's two pieces of this (the three-elevenths bar), because this is the, the measurement that you're using and it's two out of the total three pieces. So it's three times and two-thirds because of the three pieces.
....
T: So, what did you, what fraction do you have to multiply three-elevenths by?
R: Three and two-thirds.
T: Three and two-thirds. What's the improper fraction for that?
R: Eleven over three. Oh, I was right! I was in the beginning!

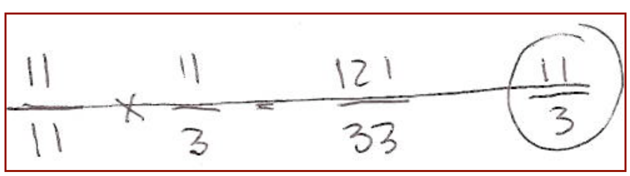


Figure 4.8: Rosa's numeric calculation to make 1-meter from 3/11-meter

Although the problem was just to construct one whole meter bar from a given 3/11-meter bar, which might require a student's reversible[15] partitive fraction scheme, Rosa assimilated this

[15] For a scheme to be a reversible scheme means that any result of the scheme can be taken as a situation of the scheme and that the activity of the scheme can be reversed to produce a result of the scheme that is a possible situation (Steffe & Olive, 2010)

problem as a situation for her unit-segmenting scheme at first as indicated by her comment, "It's about three, three times with a fraction." However, in a moment she converted it into a situation for using a procedural fraction division algorithm in the sense that she calculated '11/11 * 11/3,' which seemed to be derived from '11/11 ÷ 3/11' using an invert-and-multiply algorithm. Even though she got a right answer of '121/33' on the paper, she seemed stuck with it because she could not figure out what her answer meant in the problem situation. Upon my request to make a one-meter bar with the given 3-part bar of 3/11-meter, she was able to re-present a one-meter bar as a 11-part bar of 11/11-meter, which led her to re-assimilate the situation using her unit-segmenting scheme and found three and two-thirds. Therefore, Rosa's invert-and-multiply algorithm, unlike Carol's, could not entail the records of her mathematical operations in her unit-segmenting scheme. That is, Rosa's algorithm was not generated from carrying out her unit-segmenting operations prior to activity and thus could not symbolize such operations.

There are two interesting aspects observed in Rosa's mathematical behavior in this protocol. First, even though Rosa activated her unit-segmenting scheme and found three and two-thirds as an answer, she seemed embarrassed when she realized that the remainder was two-elevenths, not two-thirds as indicated by her comments, "Because if you do three over eleven times three over one (writing down '3/11×3/1=9/11') it's nine over eleven. And you still have two-elevenths left, of the three-elevenths that you're using. So...Three..." Secondly, until I asked her what the improper fraction was for three and two-thirds, Rosa could not realize that '3 2/3' is equal to '11/3' or '121/33' that she wrote at the beginning when she conducted numeric calculations using an invert-and-multiply algorithm.

What then made her behave in such a weird way (from an adult's point of view)? My hypothesis is that the lack of interiorization of an iterable unit fraction could explain her behavior. Re-presentation of an 11-part one-meter bar, together with my guiding question to make a one-meter bar, might have enabled her to compare the given 3-part 3/11-meter bar with the re-presented 11-part one-meter bar. This comparison let her to activate her unit-segmenting scheme. However, it was not a unit-segmenting scheme for measuring out one-meter with the 3/11-meter bar. Rather, the situation for her unit-segmenting scheme was to measure eleven parts with three parts. Thus, when she realized that the remainder, two parts were actually 2/11-meter, not 2 meters, she might have been surprised by it because that was not the anticipated result of her unit-segmenting scheme for measuring eleven with three. If Rosa had been aware that three-elevenths meters consists of three units of one-eleventh and one meter consists of eleven units of one-eleventh, any one of which can be iterated eleven times to make one whole, she could have identified the situation of measuring eleven with three with the situation of measuring one with three-elevenths, and dealt with the remaining two parts as two units of an iterable one-eleventh that constitutes 2/3 of the iterated 3/11. Such an awareness, however, requires the interiorization of an iterable unit fraction, which she apparently lacked.

The second feature can also be interpreted in terms of her lack of iterability of unit fractions. For Rosa, three and two-thirds was totally different from eleven-thirds until I asked her what was the improper fraction for three and two-thirds. Such behavior was one of the typical features when a student was in the process of constructing an iterative fraction scheme. The meaning of ‘eleven-thirds’ needs to transcend a part-to-whole meaning of fraction. A student has to take the three-thirds as a unit containing hypothetical parts each of which can be iterated three times to produce the one unit. In this way of thinking, a unit fraction becomes a fraction number freed from its containing whole and available for use in the construction of eleven-thirds by the iteration eleven times of a unit fraction, one-third. Although three problems in the Protocols 4.6, 4.7, and 4.8 involved improper fractions as measuring units, as I mentioned earlier, there was no evidence that Rosa had demonstrated an interiorized use of a unit fraction as an iterable unit because almost all explanations were initiated by Carol and thus it was hard to witness Rosa's independent use of mathematical concepts in the previous teaching episodes. Based on Rosa's mathematical behavior indicated in the present protocol, I would say she was in the process of, or close to, the interiorization of iterable unit fractions for further mathematical activity, especially, in the context of fraction measurement division.

Construction of a Fractional Unit-Segmenting Scheme

The following two protocols were extracted from a teaching episode held on May 6 of 2009, almost five months after the previous Protocol 4.9. Although the overarching goal of the teaching episode was to investigate the students' mathematical actions and operations emerging in their transformation activities between two (fractional) quantities at that time I, as a teacher-researcher, also desired to attempt several fraction measurement division problems before the academic semester was over. Throughout the teaching experiment during the academic year of 2008-2009, the students' ability to use two different kinds of three-levels-of-units structures as given material emerged as important mathematical knowledge components for their advancement of fractional knowledge. These structures are interiorization of 1) three levels of units for improper fractions based on FCNS and 2) modification (or coordination) of three levels of units based on GNS such as recursive partitioning, distributive partitioning, and common partitioning. Therefore, I was eager to know how such mathematical knowledge components would emerge in fraction measurement division situations through accommodations of their current unit-segmenting schemes as the situations became more complicated.

Protocol 4.10 on 05/06/09: Measuring an 11/19-meter bar with a 4/19-meter bar.

(The problem is "If you measure an 11/19-meter bar with a 4/19-meter bar, how many 4/19-meter bars are contained in the 11/19-meter bar?" Rosa already wrote down her answer for the problem on paper.)
T: Rosa, you said that the answer was...
R: Oh, it's two and three-fourths.
T: Why do you think like that? Can you tell me why...
R: Okay, um... I, when I took its numerators, how many times four go into eleven, and that's two times with three remaining...
T: Um-hm.
R: Actually, it's not three-fourths. It's three-nineteenths.
C: Yeah, because there is three pieces of a... We got the four because yours just...
T: Three-nineteenths?
C & R: Yeah, three-nineteenths.
T: Why did you change?
C: Because nineteenth is the denominator from all of them.
R: Yeah, and that's what you're measuring with, not fourth.
T: So, if you measure four with, measure eleven-nineteenths with four-nineteenths...
R: If you get two and three-nineteenths...
T: Two and three-nineteenths of what?
R: That's how many times four-nineteenths can go into eleven-nineteenths.
T: So, you said that three and, three-nineteenths of...four-nineteenths is contained in eleven-nineteenths?
R: No, because if you take four-nineteenths times two over one, it's eight-nineteenths and there is three left to get to eleven-nineteenths. So, it's gonna be two and three-nineteenths. That's how many times goes in.
T: So, how many times goes,
R: Two and three-nineteenths.
T: Two and three-nineteenths?
R: Um-hm.
....
T: Okay, let me pose another question and we will back to this problem. So, can you see the numbers here? Seven-seventeenths, yeah, seven over seventeen and sixteen over seventeen. What number do you have to multiply by to seven over seventeen to get sixteen over seventeen?
R: Two... and... two seventeenths?
T: Two and ...
R: Two-seventeenths?
T: Two-seventeenths?
C: Cause seven times two is fourteen and there is two leftover.
T: Two-seventeenths? Can you confirm your answer? Using paper and pencil or whatever, using JavaBars.
C: Can we use the GSP?
T: Uh, I don't think I have. I need a CD. You can, then can you calculate, by calculation can you confirm your answer? What was your answer? Seven and... Two and what?

C: Two and two-seventeenths.
T: Two-seventeenths. So...
(Rosa makes some bars in JavaBars and Carol tries to use calculation for confirmation of her answer.)
C: Wait, that will be hard to reduce it.
R: Copy. Pull out. (To Carol) did we say two-seventeenths?
C: Can I do it on JavaBars? This is hard to write it down.
T: Okay.
C: So do we start with seven right?
R: (Rosa has two 7-part bars and one 2-part bar on the screen now. See Figure 4.9a.) You know, it might be two-sevenths.
C: (Carol has three 7-part bars on her screen.) Because you do...
R: Cause the seven pieces is what you're starting with.
C: That's what I was thinking eleven-fourths.
R: See, that's what I was thinking four right here. That's what I thought. Two and three-fourths. But, then I was like, but it's four-nineteenths. So I was thinking three-nineteenths. But maybe I was right at the first time.
(Carol now has two 7-part bars and one 2-part bar on the screen.)
R: Because it's two out of the seven pieces. Because you have only seven pieces, not all seventeenths. Right?
....
C: We know it's two and two-sevenths.
T: Two and two-sevenths?
C: Seven-seventeenths to get sixteen-seventeenths.
T: Why do you think seven, two and two-sevenths?
C: Because I was just thinking... (Carol clicks and drags the 7-part bar to measure the 16-part bar by moving the 7-part bar along the 16-part bar from the leftmost part. See Figure 4.9b.) There is one whole, two whole, and then two-sevenths leftover. I'm trying to figure out.

Figures 4.9a & 4.9b: Rosa's (Left) and Carol's (Right) constructions to measure 16/17-meter with 7/17-meter

Rosa's first answer was two and three-fourths, which was correct. However, Rosa changed her answer into two and three-nineteenths. She seemed to reflect on her unit-segmenting operations while explaining her answer to me. However, when she got three pieces leftover as a result of her unit-segmenting operations in reflection, she conflated her unit to be used in segmenting [4/19] with a unit [1] given in the problem to measure the leftover [3/19]. Carol immediately agreed with Rosa's changed answer. It indicated that she also conflated units in using her unit-segmenting scheme as Rosa did, which was identified in her comment, "Because nineteenth is the denominator from all of them."

In response to the students' conflation of units in dealing with the leftover on the basis of their unit-segmenting schemes, I, as a teacher-researcher, posed another similar question. My intention was to check whether their conflation was lasting because I knew that Carol and Rosa had already constructed unit-segmenting schemes with a remainder where the whole number divisor does not evenly divide the whole number dividend. When I asked them to measure 16/17-meter with 7/17-meter, their answer was two and two-seventeenths, not two and two-sevenths. Until both Carol and Rosa constructed two 7-part bars and one 2-part bar and conducted their unit-segmenting operations with those perceptual materials, they could not realize that the two leftover should be measured in terms of the 7-part bar [7/17-meter], not measured as a length with regard to the given referent whole [1-meter]. Perceptual material [two 7-part bars and one 2-part bar] on JavaBars and implementation of unit-segmenting operations with them obviously helped the students evoke their units-segmenting schemes with a remainder, which worked properly for this problem situation.

My conjecture is that Carol's and Rosa's conflation of units might be due to the evocation of their unit-segmenting schemes but without iterability of unit fractions [1/19 and 1/17] in assimilating the fraction measurement division situation. In other words, if they had been able to see 7/17 as seven units of 1/17 each of which can be iterated seven times to make 7/17-meter and also sixteen times to make 16/17-meter prior to activity, they could have assimilated the problem as a situation of their unit-segmenting schemes with remainder as '16 ÷

7'. Actually, they were able to eliminate such confusion through construction of perceptual materials for their unit-segmenting operations using JavaBars. Rosa finally seemed to be explicitly aware of such a relationship between 7/17 and 16/17 with regard to 1/17 by her comments, "Because only you only have the seven pieces. It's two out of the seven pieces (pointing at her 7-part bar) although the seven pieces is out of the seventeen." That is, she knew that each piece of seven parts was 1/17-meter because "seven pieces is out of the seventeen". Moreover, she also realized that the leftover two parts should be measured in terms of seven-seventeenths because "you only have the seven pieces." Similarly, Carol manifested her unit-segmenting operations with her 7-part bar using JavaBars [She clicked and dragged the 7-part bar to measure the 16-part bar by moving the 7-part bar along the 16-part bar from the leftmost part.]

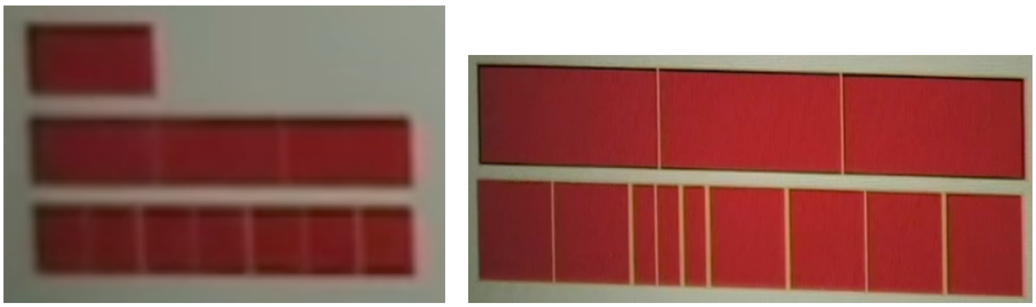
In sum, comparing with a whole number division problem where the divisor does not evenly divide the dividend, this sort of problem involving two fractions [16/17 and 7/17] seemed to require the students to conduct their unit-segmenting scheme with a remainder based on their use of FCNS as given material. Therefore, when the iterability of a unit fraction is interiorized and embedded in the assimilating part of a unit-segmenting scheme with a remainder, I would call such a modified scheme a *fractional unit-segmenting scheme* in the sense that assimilating situations of the scheme include fraction measurement division situations involving fractional numbers.

Two Students' Measuring a Unit Fraction With Another Unit Fraction

Protocol 4.11 on 05/06/09: Measure a 1/3-meter bar with a 1/7-meter bar.

T: Carol, can you read the problem and can you tell us how will, we can deal with? Measure a 1/3-meter bar with a 1/7-meter bar.

C: Okay. What I would probably do is, I would make one-third and a whole thing, like I can see the whole. And then I would clear and then make it seven pieces to compare the two.
T: Using JavaBars?
C: Yeah.
T: Okay, go ahead.
(Carol makes a bar, copies the original bar three times and joins them. Then she makes a copy of the 3-part bar, clears partitions on the 3-part bar and then divides it into seven to make a 7-part bar. See Figure 4.10a.)



Figures 4.10a & 4.10b: Carol's constructions to measure 1/3-meter with 1/7-meter

C: Hm... Let's see... One-seventh... (After twenty seconds, Carol divides the third part of the 7-part bar into three to compare with the 3-part bar. See Figure 4.10b.) It doesn't match at all. I have to work it out. It goes in... It goes in two times. It's hard to be precise. There's nothing to line up with. So have to be math... It looks like two and one-third.

From my point of view, the present problem [to measure 1/3-meter with 1/7-meter] was the most complex form of fraction measurement division problem that was posed during the whole teaching experiment. Conceptually, there are two critical steps in students' construction of necessary unit-segmenting schemes for this problem. Once a student assimilates this problem as a situation of her unit-segmenting scheme, it might require a student to coordinate one-third and one-seventh in the same re-presented bar with an awareness of a co-measurement unit fraction [one twenty-first] of both unit fractions through a common partitioning operation. That is, a student should be able to see 1/3-meter consisting of seven units of 1/21-meter and at the same time 1/7-meter consisting of three units of 1/21-meter. The second is taking the results of coordination of 1/3-meter [7/21-meter] and 1/7-meter [3/21-meter] as input for their fractional unit-segmenting scheme to conceive the problem situation as equivalent as the whole number division problem of '7÷3'. Therefore, I hypothesize that when a student's interiorized common partitioning operation is automatically associated with the student's fractional unit-segmenting scheme (i.e. the common partitioning scheme is activated by the situation of the fractional unit-segmenting scheme and its result is re-assimilated by the fractional unit-segmenting scheme) then the fractional unit-segmenting scheme is modified and expanded to include two unit fractions, which are not multiples of each other, as part of the assimilating situations of the fractional unit-segmenting scheme.

In that sense, Carol's struggle in the above protocol was not surprising because there has been no evidence of Carol's construction of a common partitioning operation throughout the teaching experiment. Her separate construction of a 1/3-meter and a 1/7-meter bar and her struggle to find an answer by visual comparison of the size of the two bars on the screen did not seem to indicate construction of any direct quantitative relationship between the two bars. In other words, she could not coordinate one-third and one-seventh in her re-presentation with regard to a co-measurement unit fraction [one twenty-first] as mentioned above. If I had asked her to construct a 1/7-meter bar from her 1-meter$_3$ bar, she could have easily made a 1/7-meter bar by partitioning each part of the 1-meter$_3$ bar into seven and pulling out one part from each of the three parts of the 1-meter$_{3,}$ and joining them on the basis of her distributive reasoning. Further, she could have found the answer by establishment of a part-whole relationship between 1/3-meter and 1/7-meter with the result of her constructions. However, her assimilation of the problem as a situation for fraction division by measuring activity did not provoke her distributive partitioning operation in the way that it was evoked in fraction multiplication. Therefore, the next concern was whether Rosa was able to demonstrate such coordination of two three levels of units in the context of fraction measurement division situation because she had constructed a common partitioning operation in the context of transformation between two unit fractions (cf. Protocols 4.25 and 4.31).

Protocol 4.11[16]: (Cont.)

T: You can explain what you did.
(Rosa already made her construction in JavaBars to explain her answer obtained by calculation. See Figure 4.11.)

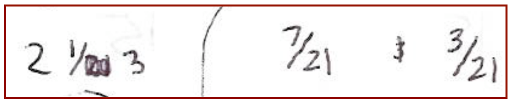
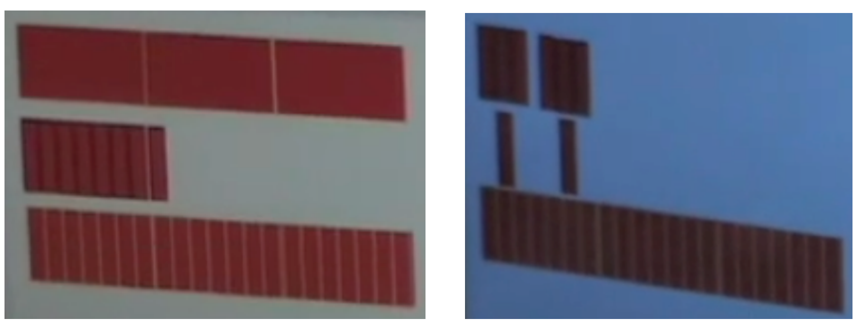


Figure 4.11: Rosa's calculation for measuring 1/3-meter with 1/7-meter

R: I converted in seven twenty-one and three twenty-one. And so, I took the three right here, um... three pieces (three 1/3-meter parts of 1-meter_3 bar) and then I just divided seven in each of the one-third (Rosa accidently pulled out eight pieces of 1/21-meter part although the camera did not capture it. See Figure 4.12a). And here's one, two, three (counting 1/21-meter pieces one by one in the middle row.) There is one (pointing out three pieces of 1/21-meter bars.) This is the other two-thirds. Um? One-third. Okay, I think I grouped in three (See Figure 4.12b). I don't know. That's supposed to go over there. I know I'm right, I know I'm right. I just forgot how I do it. I just lose a step and that messed me up.



Figures 4.12a & 4.12b: Rosa's constructions for confirmation of her calculation for measuring 1/3-meter with 1/7-meter

T: Then where is one-third meter bar?
R: Right here. Right there (pointing out one of the 1/3-meter parts of her original 1-meter bar.)
T: Okay, then where is one-seventh meter bar in your construction?
R: Right here one of these (pointing out the smallest piece, which is 1/21-meter.) No, no, no, no, no, no. Um... okay there is twenty-one right here. The seven, seven is one-third.
T: Yeah, I mean one-seventh meter bar. Yeah, seven is one-third.
R: Yeah, three. Three of these.

---

[16] Note that parts of Carol's verbal expressions and communications with the teacher are omitted because the intention of this protocol is to describe Rosa's mathematical actions and operations.

T: Three of these? (Rosa rearranges her eight of 1/21-meter bars) How many pieces will... This (Rosa's original 3-part bar) is one meter bar, right?
R: Yes. So, how many, what was your question?
T: How many pieces will it be if you take one-seventh meter bar.
R: Three, three pieces.
T: Three pieces.
R: Yes, three.
T: How long is the smallest piece in your construction?
R: One twenty-one?
T: One twenty-one, okay.
R: I know what I did. I just lost it.

First of all, her ability to calculate in dealing with fractions by manipulating numerals seemed to enable her to easily get two and one-third as an answer. Unlike in the Protocol 4.10, Rosa did not show any indication of conflation of units in her unit-segmenting operations even without perceptual information for the two fractional quantities. It seemed that construction of her fractional unit-segmenting scheme in the Protocol 4.10 was interiorized as a permanent scheme at that time. However, when I requested her to confirm the answer with constructions in JavaBars, she got confused by her constructions of eight 1/21-meter parts, which were accidently pulled out from the 21-part bar of 1-meter. She tried to group those eight pieces in three for explanation but found out that the leftover was two-thirds of a group, which did not coincide with her answer [1/3] on paper. Accidental construction of eight parts of 1/21-meter for 1/3-meter bar led her to re-assimilate the problem situation and reflect on her construction processes for a solution. This revealed the fact that her result of calculation for this problem was not symbolizing her coordination of 1/3 and 1/7 on the basis of her common partitioning operations.

From my perspective, her embarrassment indicated that her self-monitoring and feedback system for her mathematical operations, which were activated for the problem situation, did not work well. As mentioned above, in order for her fractional unit-segmenting scheme to be provoked, coordination of 1/3 and 1/7 on the basis of a co-measurement unit fraction [1/21] of the two unit fractions was presupposed. Somehow, she demonstrated that she regarded 1/3 as 7/21 and 1/7 as 3/21 with the aid of numeric calculation. Her common partitioning operations, however, may not have been conducted at a high enough level of abstraction to connect with her mathematical calculations, which may have led to her self-monitoring system being poorly executed. Her confusion, "Right here one of these (pointing out the smallest piece, which is 1/21-meter.) No, no, no, no, no, no." when I asked her to find a 1/7-meter bar in her constructions, corroborated that she had yet to interiorize the structure of her constructions by coordination of 1/3 and 1/7 based on 1/21 by common partitioning operations. Unfortunately, we did not have more opportunities to investigate the students' modification of their unit-segmenting schemes because the 2008-2009 academic semester was over. Although neither of the students constructed a fractional unit-segmenting scheme where common partitioning operations were associated with the fractional unit-segmenting scheme, the two students' mathematical behaviors indicated in this protocol cast valuable information in tracking a constructive itinerary for fraction measurement division problems through modification of students' unit-segmenting schemes.

Summary of Phase I: Fraction Measurement Division

During Phase I, Carol and Rosa demonstrated that they modified their units-segmenting schemes to deal with various fraction measurement division situations. First of all, with a whole number division problem with a remainder, Carol constructed a *unit-segmenting scheme with a remainder* by embedding her partitive fraction scheme in the first part of her unit-segmenting scheme as a subscheme. It was made possible by her construction of three levels of units with a singleton unit, and two other composite units that were relatively prime, [for instance, 1, 3, and 5]. On the other hand, Rosa was able to associate the result of her unit-segmenting scheme with a situation of her partitive fraction scheme and finally constructed a unit-segmenting scheme with a remainder by retrospective accommodation through interactive communications with Carol.

When the students were given the problem of measuring a whole number quantity that was a multiple of the given fractional measurement unit, the assimilating parts of their unit-segmenting schemes were expanded in that fractional quantities, as well as whole number quantities, could be used in unit-segmenting operations as measuring units. However, when the students measured a whole number quantity with a fractional quantity, which does not evenly divide the whole number quantity, they experienced difficulty in measuring the leftover part of the whole number quantity in terms of the fractional quantity to be used in segmenting. After Carol constructed perceptual materials [drawings on paper] to operate on, she was able to deal with a remainder of the division problem in terms of the fractional measuring unit in the problem. Her solution was also made possible by conceiving the problem situation in term of three levels of units involving two composite units that were relatively prime, as mentioned above. That is, she could solve '4 ÷ 3/4' as she might solve '16 ÷ 3' with three levels of units [1, 3, and 16]. However, the fact that her unit-segmenting operations were carried out using perceptual materials implied that perceptual representations of given quantities played a critical role in her problem solving processes [in her unit-segmenting operations].

Even though both students constructed unit-segmenting schemes with a remainder, a situation where measuring a smaller quantity with a larger quantity caused a perturbation that blocked their unit-segmenting operations. This perturbation was another epistemological obstacle that they had to overcome in order to expand the assimilating part of their unit-segmenting scheme to include measuring a smaller quantity with a larger quantity.

It is worthy of notice that Carol generated her own algorithm for fraction measurement division problems, (which was on par with an invert-and-multiply algorithm,) by using the result of her unit-segmenting scheme with a remainder for further activity. In other words, she was able to find how many times 7/5-meter was contained in 176 meters by figuring out how much of 7/5-meter was contained in 1 meter and then multiplying the result by 176. It was an example of how conventional algorithms can be meaningfully constructed through students' own constructive processes.

Five months later, when I posed a fraction measurement problem with two fractions that had the same denominator but their numerators were relatively prime, the students demonstrated that the problem could be solved by construction of a *fractional unit-segmenting scheme*. It was a modified unit-segmenting scheme with a remainder, involving a FCNS as given material used in the first part of the scheme. For a division problem between two unit fractions whose denominators were relatively prime [e.g. 1/3 and 1/7], it turned out that common partitioning operations needed to be involved in the solving process so that the students could convert the problem situation into a situation for their fractional unit-segmenting schemes[17].

Phase II: Fraction Multiplication

Considering the two students' mathematical behaviors in relation to fraction measurement division problems presented during Phase I [except for Protocols 4.10 & 4.11 held on May 6 of 2009], I decided to investigate their mathematical operations emerging in the context of fraction multiplication because Rosa demonstrated difficulty in using the result of her unit-segmenting scheme as given material for a situation of fraction multiplication (cf. Protocol 4.2). I also judged that construction of fraction multiplication knowledge would be preliminary

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[17] Detailed discussion about the students' constructive processes of their unit-segmenting schemes for fraction measurement division problems will be provided in chapter V.

steps for construction of a scheme for multiplicative transformations between two fractions. The first problem presented in Phase II was designed to investigate the students' construction of recursive partitioning operations, which I consider to be fundamental mathematical operations involved in fraction multiplication.

Construction of a Recursive Partitioning Operation

With a set of problems composed of multiplication problems involving (improper) fractions, I decided to use JavaBars for them to solve those problems. Although Carol already had one year of experience with JavaBars through the teaching experiment in the previous year, it was the first time for Rosa to use JavaBars in the teaching experiment[18]. For about six minutes, I let the students to explore various functions provided in JavaBars such as COPY[19], ERASE, JOIN, FILL, PARTS and PULLOUT[20] so that they (mainly Rosa) would be familiar with the computer program.

> Protocol 4.12 on 02/10/09: Finding the length of a part of 1-foot string when it is recursively partitioned by eleven and five.
>
> T: A string one foot long was cut into eleven equal parts and one of these pieces was then cut into five equal parts. How much of the string was one of these five parts? (Both students draw a bar and cut into eleven equal parts and then partitioned one part of the eleven parts into five.)
> C & R: (At the same time) would it be five fifty fives?
> T: The question is..
> R: So one-eleventh.
> T: Just one piece. The smallest piece.
> R: (Immediately) one fifty-fifths?
> T: One fifty-fifths.
> C: Oh~

[18] Although Rosa used JavaBars in the Protocol 4.10 and 4.11, note that those protocols were held on May 6 of 2009.
[19] When a word is in caps, it refers to an action in the JavaBars.
[20] Using PULLOUT, a student can activate that action button by clicking on it and then click on one or more parts of a stick. The student can then deactivate the action button and drag copies of the parts out of the stick while leaving the stick intact.

R: Because there is fifty-five, five times eleven.
T: How did you get it?
C: There is fifty-five pieces for each of those eleven things. Like, the whole one foot would be fifty-five pieces and then in one little block there is five. So it would be one fifty-fives, then one block would be five fifty-fives.
T: Okay, Rosa, do you agree?
R: Yeah.
T: So, how did you know that all the, all total number is fifty-five?
R: Because there is eleven equal pieces and one of the pieces has five. So you gonna do five times eleven equal to see how many are in the one foot.

A recursive partitioning operation is one of the crucial mathematical operations that undergirds fraction multiplication. Recursive partitioning is based on a student's construction of a unit of unit of units as a structure whose units can be used as material in further operation (Steffe & Olive, 2010). That is, recursive partitioning is a sophisticated form of units-coordination that is activated in the context of students' novel situations involving fractions in that the three levels of units are constructed inward (parts within parts of a referent unit), rather than outward (a composite unit of composite units).

When Carol and Rosa set their goal to find how much one-fifth of one-eleventh of a string was of the 1-foot string, they established this situation as a situation of their units-coordinating scheme. This activated the abstracted operations of partitioning into five parts and distributing these operations over the remainder of the eleven parts while monitoring how many parts were produced as a result of the distribution. Such ability enabled Carol and Rosa to realize that the total number of the smallest parts in a 1-foot string would be fifty-five immediately after they partitioned only one part of the eleven-elevenths string into five. Such quick responses from them also implied that Carol's and Rosa's partitioning of one-eleventh of a stick into five equal parts symbolized partitioning each of the eleven-elevenths into five parts as in Rosa's comment, "Because there is eleven equal pieces and one of the pieces has five. So you gonna do five times eleven equal to see how many are in the one foot." That is, they established their partitioning operations as *symbolic operations* attributed to recursive partitioning (Steffe & Olive, 2010).

Since Carol and Rosa were credited with the construction of a GNS at the beginning of the teaching experiment, it was not surprising that they could take a composite unit [fifty-five] containing another composite unit [five] that can be iterated so many times as a given prior to partitioning. Nevertheless, this protocol was very informative for the research team in the sense that we could confirm that both students [especially, Rosa] had constructed recursive partitioning based on their GNS.

Multiplication of a Whole Number by an Improper Fraction (1)

Having confirmed that both students had constructed recursive partitioning operations and also having confirmed (in Phase I of the teaching experiment) their ability to work with improper fractions, I decided to pose a multiplication problem involving an improper fraction with a whole number length.

Protocol 4.13[21] on 02/10/09: Making a bar 5/2 times as long as a 3-meter bar.

T: Make a bar that is five-half[22] times as long as that bar. Pretend this is three-meter bar again. And make a bar which is five-half times as long as that three-meter bar.
R: (Rosa makes a bar for three meters and partitioned it into three parts.) Okay, this is the original three-meter bar and make a bar five and a half times as long as that.
T: Yeah, that's the fourth problem.
R: That's ten... (She pauses for five seconds.) I have to make that so much longer. (After twenty seconds, Rosa starts to copy her original 3-meter bar five times and arranges the five 3-meter bars together in a row.) Is that right?
T: So, how long do you think is that (five copies of her 3-meter bar)?
R: Um... Fifteen pieces?
T: Fifteen pieces? So how long is that? How long is the one piece in your construction?
R: Three meters. The original one.
T: Yeah, so how long is the small piece (pointing out 1-meter part)?

[21] Note that part of Carol's verbal expressions and communications with the teacher are omitted.
[22] I intentionally leave 'five-half' grammatically wrong because the teacher-researcher's (my) wrong pronunciation of 'five-half' might be a reason for the students' misunderstanding of the problem.

R: One meter?
T: Then that should be how many meters?
R: Fifteen meters?
T: Fifteen meters, so does it confirm your answer?
R: (Rosa talks to herself) what's half of fifteen? You doubled it, so you have to half this.
T: Can you say the product that you did? Basically, you have three meters, right? So you multiplied five-half times of that, right? So if you're right, the product for this construction, what will be?
R: I gonna need a half of fifteen. (Rosa waves the cursor over the middle of the five 3-meter bars that she has lined up).
T: Of what? Yeah, five half of what?
R: Of three meters?
T: Right, that's basically the problem.
R: So three over one times five over two. Fifteen over two. And half of fifteen is seven and a half... bars.
T: Um-hm, but your constructed bar is... How meters is that?
R: (Rosa does not seem to listen to the teacher's question and erases two 3-meter bars of the five 3-meter bars.) I want to... No, I want to pull out. I have to make half of this. (The camera is capturing Carol's screen, and the teacher is also interacting with Carol about her work on the screen. So Rosa's screen is not visible.)
R: (Two minutes later) okay, I got seven and a half.
T: Seven and a half. Okay. Right. Yeah, answer is right basically.
R: Mine is right? Or...
T: Seven and a half. Fifteen divided by two is same.
R: Yeah, because if you do three over one times five over two is fifteen over two, and you divide that to get the five two that you want of the three meters.
T: Then, without calculation, was it possible to construct the bar? If you do not know the answer first, you already know, by the calculation you already know the answer, right? But if, just what if you don't know the answer, just using the original bar, can you make the bar five half times of that bar?
R: No, because there is no half of three.
T: No half of three.
R: It's one point five. So you have to break one of your bars in half of the three.
T: Okay. (To Carol) what about...
C: Yeah, I was just thinking if we got a small portion, but... You have to get your half of the bar.

When I asked Rosa to make a bar that was 'five-half' times as long as her 3-meter bar, her first response was "this is the original three-meter bar and make a bar five and a half times as long as that." After twenty seconds, Rosa made five copies of her original 3-meter bar, joined them together and argued that the 15-part bar was the answer. It was a units-coordinating operation to distribute a composite unit [three] over each of another composite units [five] to multiply three by five. In my opinion, she might have misunderstood the problem as finding '5 and 1/2' times 3 meters, not '5/2 times.' This conjecture is based on her comment, "make a bar five and a half times as long as that" and my wrong pronunciation of 5/2 as 'five-half.' However, even after she properly re-assimilated the problem as indicated in "what's half of fifteen? You doubled it, so you have to half this" she used her 15-part bar to construct 5/2 of her 3-meter bar.

Unfortunately, my request to check the length of all five copies of her 3-meter bar curtailed Rosa's potential operations to solve the problem and led her to turn to numeric calculation. In order to confirm that her answer was right, she relied on numeric calculation of fraction multiplication as indicated by her comments "because if you do three over one times five over two is fifteen over two, and you divide that to get the five twos that you want of the three meters." Therefore, even though Rosa constructed seven and a half of one-meter part as her final answer later, it was just the implementation of the result of her calculation '3/1 * 5/2' (see Figure 4.13). Nevertheless, in that Rosa kept using her 15-part bar and took a half of it by erasing seven and a half of one-meter bars to construct her final answer, I conjecture that Rosa's conception of '5/2 times' might be two sequential mathematical operations: to take five times the 3-meter bar and then to take one-half of the obtained result, rather than a multiplicative operation of one fractional quantity, that is, conceiving 5/2 as five units of 1/2.

Figure 4.13: Rosa's construction for 5/2 times a 3-meter bar

Carol copied her 3-meter bar, which was divided into three parts, four times, joined these four copies together to make a 12-part bar, filled pairs of parts with different colors to create 6 pairs of parts in her 12-meter bar, then broke these parts apart and erased one pair, leaving her with five pairs of different colored parts (see Figure 4.14). She seemed to accidently assimilate 'five halves times' as five groups of two. While listening to the conversation between Rosa and I, Carol started to measure her ten one-meter bars with her original 3-meter bar by moving it along above the ten bars from the most left end. Then she added five more one-meter bars to the ten bars and said, "Don't you have to get half of fifteen?" Her way of assimilating this problem to think of '5/2 times' was also a combination of two separate operations, '5 times 'and '1/2 times.'

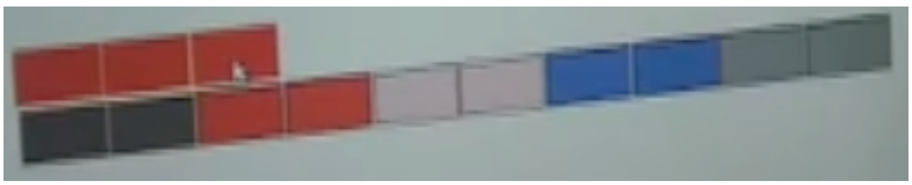

Figure 4.14: Carol's initial construction for 5/2 times a 3-meter bar

On account of Rosa's disposition of relying on numeric calculation whenever she encountered perturbation in the use of her current scheme, it was worthy of noticing Rosa's comments, "No, because there is no half of three." If her not being able to take one half of three parts (meters) was the main cause of her perturbation that led to relying on numerical calculation for fraction multiplication of '3/1 * 5/2,' the next problem for Rosa needed to be a construction problem related to distributive partitioning activities, because taking a half of a 3-meter bar might require her to use distributive partitioning operations. That is, in order to take one-half of a 3-meter bar, a student can take one-half of each meter of the 3-meter bar, with an awareness that the combination of all one-halves from each of the 3 meters equals to one-half of the total 3-meter bar.

For Carol, even with her way of assimilating '5/2 times' as two sequential operations, a distributive partitioning operation might be necessary to construct a bar '5/2 times' as long as a 3-meter bar. That is, on account of Carol's way of interpreting '5/2' times, the first step would be to construct a bar five times as long as a 3-meter bar by copying the 3-meter bar five times, which would amount to fifteen one-meter parts. Then she might somehow feel the necessity to take one half of the 15-meter bar and such necessity might induce her to divide each part of the 15-part bar into two parts and pull out one part from each of the divided 15 parts, in order to construct a half of the total 15-part bar. Although Carol did not indicate such mathematical behavior in this protocol, it seems reasonable for us to expect Carol to engage in a distributive partitioning operation when it seems necessary to solve a posed problem because construction of distributive partitioning operations was already attributed to Carol in the pre-interview conducted on October of 2008 (cf. Chapter III: Participants). Therefore, to pose a problem, which might require a distributive partitioning operation and to investigate the two students' mathematical behaviors emerging in the process of solving it, would be appropriate for the next teaching episode.

Carol's Distributive Partitioning Operation and Rosa's Assimilation of Carol's Partitioning Operation

> Protocol 4.14 on 02/10/09: Taking one-third of a 2-part bar without erasing a partitioning line.
>
> T: Make a bar (Carol and Rosa make a bar on their own screen). Divide that bar into two pieces equally. (Both students partition their own bar into two parts). Now, the problem is, take one-third of that bar without erasing that yellow line (referring to the partitioning line of each 2-part bar.) How can get...
> (Carol immediately divides each part of her 2-part bar into three and pulls out one part from each part of the 2-part bar.)
> R: How many? Take one out of the half bar?
> T: Take one-third of this original bar (Rosa's 2-part bar.)
> R: (Rosa glances at Carol's screen) Can you make more lines on here?
> T: Yeah, you can put more lines but you cannot erase it.
> C: I got it.
> (Rosa accidently erase her original 2-part bar. She makes another 2-part bar and divides each part of the 2-part bar into three and pulls out the two parts from one part of the 2-part bar, each of which is partitioned into three.) Is that right?
> T: Can you explain, Rosa, first?

R: I knew that um... There is two and you wanted to take one-third out. So I multiplied it times three...
T: Times three?
R: Um-hm. Two times three, which is six and then that's gonna give me six pieces and just take two because um.. six divided by three is two. So, you just take two of the six pieces, which equals one-third.
T: Okay. Good. Carol?
C: I did the same thing. Like I took my bar and then after I divided it into two and I divided it into three in the opposite way (horizontally). Then I just took out two pieces because it's just basically like one, two, three (referring to three rows of two pieces in her 2-part bar, which is partitioned into six.)

Carol's distributive partitioning operation was explicit in the sense that she divided each part of her 2-part bar into three with no hesitation. She also knew that taking one piece from each part of the 2-part bar equals to one-third of her 2-part bar, "Then I just took out two pieces because it's just basically like one, two, three (referring to three rows of two pieces in her 2-part bar, which is partitioned into six.)" On the other hand, the nature of Rosa's partitioning operation was not clear in that her partitioning activity was initiated by the assimilation of Carol's partitioning activity and she did not seem to be explicitly aware that the pulled-out two pieces were the result of taking one piece from each part of the original 2-part bar. Rather, the result of Rosa's assimilation of Carol's partitioning operation seemed to induce Rosa to re-assimilate this problem as a situation for a partitive unit fraction scheme for connected numbers.

Basically, in order for a partitive unit fraction scheme for connected numbers to be activated, partitions whose number is divisible by the denominator of a given unit fraction are to be provided. Rosa's partitioning activity produced '6' partitions divisible by the denominator of a unit fraction '1/3' so that Rosa could split six parts into three units of two, each of which can be iterated three times to make the whole composite unit, six. Rosa was able to use a three-levels-of-units structure as a given for her further activity by considering six as consisting of three iterable composite units of two. Since Rosa was chosen as a GNS student through a selection interview before the teaching experiment, there was a possibility that she had constructed a partitive fraction scheme for connected numbers although we did not have direct evidence for her construction of such a scheme at that time.

However, even with an assumption that Rosa had constructed a partitive fraction scheme for connected numbers, this problem was a novel situation possibly causing perturbation for her because the number of parts of the given 2-part bar was not divisible by the denominator of '1/3.' It was fortunate (from my point of view) that Rosa had a chance to see Carol's partitioning activity. There was no doubt that a glance at Carol's partitioning activity initiated Rosa's mathematical reasoning to solve the given problem. Nevertheless, the nature of Rosa's partitioning operation seemed quite different from Carol's distributive partitioning operation. That is, Carol's partitioning each part of her 2-part bar into three enabled Rosa to eliminate her perturbation by finding a number, six, which could be divisible by three, and pulling out two parts as one-third of six parts, but she did not seem to assimilate the distributive property of Carol's partitioning operation. To get an appropriate number that could be evenly divided by three was enough for Rosa to eliminate her perturbation. Her explanation corroborated this hypothesis as in "two times three, which is six and then that's gonna give me six pieces and just take two because um.. six divided by three is two. So, you just take two of the six pieces, which equals one-third." Rosa's quick assimilation to deal with the given situation was possibly due to her ability to use a three-levels-of-units structure as given material based on her GNS, but her construction of a distributive partitioning operation remained to be investigated.

Distributive Partitioning Operations in a Sharing Situation

To further explore Rosa's distributive partitioning operations, a sharing problem was posed to the students, which might induce them to conduct distributive partitioning operations.

Protocol 4.15 on 02/10/09: Constructing a bar for one person when six people equally share five candy bars.

T: Make five candy bars.
C & R: Five bars?
T: Yes, pretend that each one is a candy bar. (Both students make a bar and copy it four times to end up with five equal-sized bars.) You have five pieces of candy bars. With these candy bars, think about the situation that six people want to share those five candy bars equally. Can you find the amount of one people [person] can take with these five candy bars?
C: Five-thirtieths?
R: I was gonna say.
T: Five what?
C: Five-thirtieths.
T: Five-thirtieths?
R: Because six and five are divisible into thirty.
C: If you divide each piece into six pieces and give them one of each.
T: Yeah, just construct the bar.
R: Oh, one big bar?
T: For one piece, for one people [person] I mean. If you want to share those five pieces among six people, make a bar so that one people [person] can take.
R: (Rosa partitions each of her five bars into six parts.) Okay, and I make the bar one person can take?
T: Um-hm.
R: One-sixth. Hmm... (To Carol) you just make the same bar, won't you?
C: If you have five pieces and six people...
(Rosa pulls out five parts one by one from the fifth bar partitioned into six parts and then starts counting the total number of small parts in the five 6-part bars.)
R: So, you want one bar that everyone, one person get?
C: Um-hm.
T: Then how much is it, how much of a bar is it?
R: How much of one of the bar?
T: of one candy bar is it.
R: Five-sixths.
T: Five-sixths of a candy bar.
R: (Rosa nods her head.) Of a candy bar.
T: All right. Carol, did you finish it?
C: Yeah, I was just kind of messing around.
T: So, just for confirmation. Let me hear your explanation.
R: Mine? Okay, you have six people and you have five candy bars. So, you gonna divide each candy bar by six. So each person gonna get one-sixth of the candy bar and since there is five total candy bars, you gonna get five pieces, five one-sixth pieces of um... of all five candy bars. And that's gonna have five pieces, so basically one person's gonna get five-sixths of one whole candy bar.
T: Okay, how about you, Carol?

> C: That's what I did. I had the five pieces and divided them, each into six separate pieces and you take one out from each. And then what you do, I just join all that together and make the one large candy bar and then I started to color for each piece of the people get. There is one, two, three, four, five, six pieces from five.

Carol engaged in a distributive partitioning in a sharing situation. Upon the request to find one person's share when six persons equally share five candies, Carol made five bars for five candies on the screen, divided each bar into six parts and pulled out one part from each of the five bars. This was a typical indication of a student's construction of a distributive partitioning scheme. That is, she formed a goal of a distributive partitioning scheme, say, to share five identical candy bars equally among six people. She then partitioned each candy bar into six parts and distributed one part from each of the five candy bars to each of the six people with understanding that the share of one person could be replicated six times to produce the whole of the five candy bars. She also knew that five-sixths of one candy bar was identical to one-sixth of all of the candy bars.

On the other hand, Rosa also similarly drew five candy bars on the screen, partitioned each bar into six parts and took five parts out of the thirty little parts. Her comment, "So each person gonna get one-sixth of the candy bar and since there is five total candy bars... basically one person's gonna get five-sixths of one whole candy bar" corroborated her construction of distributive partitioning operations in the context of such a sharing situation. However, whether she interiorized her distributive partitioning operations was to be explored because this kind of sharing situation explicitly asked the students to share (distribute) each of the discretely spread candy bars among (to each of) six persons. For the indication of interiorization of distributive partitioning operations, such distributive partitioning operations should occur with a non-distributive partitioning goal as in the previous protocol [Taking 1/3 of a 2-part bar without erasing a partitioning line.] When a student constructs distributive partitioning operations in a non-distributive situation, distributive reasoning can be attributed to the student.

Multiplication of a Whole Number by an Improper Fraction (2)

Given the observation of Carol's explicit use and Rosa's novel use of distributive partitioning operations, the research team decided to attempt more multiplication questions of a whole number by an (improper) fraction. The posed problem for this protocol was "Make a bar and pretend that it is 3 meters long. Make a bar that is 11/4 times as long as that. How long is your constructed bar?" On account of the availability of Carol's distributive partitioning operation in non-sharing situations, I expected that her distributive partitioning operations would emerge in a certain way. That is, she might try to solve the problem in the same way as she did in the previous fraction multiplication problem (cf. Protocol 4.13). She might interpret '11/4 times' as two separate multiplicative operations as '11 times' and '1/4 times.' If so, she might copy her 3-meter bar eleven times, combine them together and then take one-fourth of the 33-meter bar using her distributive partitioning operation[23].

Unfortunately, when they were asked, Rosa went back to relying on numeric calculation '3/1*11/4' and found the answer as '33/4.' Upon my request of construction for the answer on the screen using JavaBars, she copied six more 3-meter bars on her screen (giving her a total of seven 3-meter bars) and seemed stuck with the problem. She then went back to numeric calculation and tried to get an answer in a decimal form. Carol could not solve the problem either, but for a different reason than Rosa, which seemed very interesting. Based on her 3-part bar for three meters, she divided each part of the 3-meter bar into four pieces. After counting

[23] Of course, the ideal solution that I was expecting was to partition each of the 3-meter bar into four parts, pull out a part from each of the 3-$meter_{12}$ bar to make one-fourth of the total 3-$meter_{12}$ bar and iterate them eleven times to make '11/4 times' the 3-$meter_{12}$ bar.

twelve partitioned pieces four of which were in one meter as “one, two, three, four, one, two, three, four, one, two, three, four,” she told herself that “we need eleven.” Then she copied her 3-meter bar, which had four partitions in each one meter, three times and broke the last copy of the 3-meter bar to take off one meter part of the broken 3-meter bar before arranging them (see Figure 4.15), which led her to construct ’11 of 4/4 of one meter’, not ‘11/4 times’ a 3-meter bar.

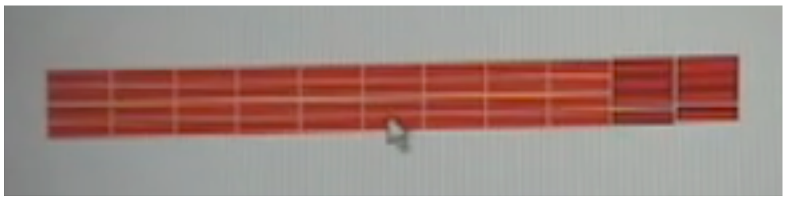

Figure 4.15: Carol’s construction for 11/4 times a 3-meter bar

Considering Carol’s explicit use of a distributive partitioning operation, she would have taken one-fourth of a 3-part bar by taking one-fourth of each part of the 3-part bar and join them together if she was directly asked to take one-fourth of the 3-meter bar. Actually, the partitioning activity of each of her original 3-meter bar into four parts indicated that somehow she felt a necessity to construct a fourth of her 3-meter bar. However, she ended up with a construction of 11 of 4/4 of one meter as mentioned above. This was not her final solution to the problem as she went on to compare this construction with her original 3-meter bar (moving the first 3-meter bar underneath the remaining two 3-meter bars and the two 4-part one meter bars.) Unfortunately, I posed a follow-up question at this time, thus curtailing Carol’s potential solution to the problem.

My hypothesis is that her difficulty in completing a solution was possibly due to her failure to anticipate what to do next with the result of her distributive partitioning activity (making 4/4 of each part of the 3-meter bar), which seemed to be related to her conception of an improper fraction, in this case 11/4. I anticipated that Carol could conceive 11/4 of one meter as eleven of 1/4 of one meter. In order to construct 11/4 of 3 meters, her distributive partitioning operation was activated with an intention of first creating 1/4 within 3 meters. The solution, however, was not found in the result of her partitioning activity because 11/4 of 3 meters was greater than 3 meters. Carol attempted to overcome the perturbation by creating eleven of something. I hypothesize that if the solution to the problem had been present within the result of her distributive partitioning operations, she would have been able to solve the problem.

Nevertheless, her mathematical behavior was in contrast to that in the previous multiplication problem. If she had been able to take one-fourth of the 3-meter bar by pulling out one piece from each part of the 3-meter bar and joining them together, and then iterate it eleven times to make '11 times 1/4 of a 3-meter,' I could have claimed that it was an indication of an interiorization of her distributive partitioning operations in that her distributive partitioning operations emerged in the context of a fraction multiplication problem with a non-sharing goal.

In sum, the students did not demonstrate appropriate mathematical constructions for '11/4 times' a 3-meter bar. Since the construction of a distributive partitioning operation seems crucial for the two students to expand their schemes and operations for fraction multiplication more generally, I improvised a multiplication problem of a whole number by a proper fraction, which might evoke distributive partitioning operations in their mathematical activities. Since conceiving a proper fraction might be very different from conceiving an improper fraction in further mathematical activities (such as fraction multiplication), posing a multiplication problem by a proper fraction would provide a good opportunity to check that the reason for Carol experiencing difficulty in the present problem was due to the involvement of an improper fraction, requiring her use of a FCNS as given material.

Multiplication of a Whole Number by a Proper Fraction

Protocol 4.16 on 02/10/09: Making a bar that is 3/4 times as long as a 3-meter bar.

T: Suppose this (a 3-part bar) is a three-meter long bar again. Let's make three-fourths times of this bar this time.

(Rosa partitions each part of her 3-part bar into four pieces in the opposite direction, but does not seem to know what to do next with those pieces. Carol also divides each part of her 3-part bar into four pieces. She pulls out the bottom piece from each of the original 3 vertical parts and repeats this process with the second row from the bottom, giving her six individual bars, each of which is 1/12 of the original bar. The next step is not captured by the camera because the camera moves to Rosa's screen, but Carol pulls out three more parts from another row of her partitioned 3-meter bar, which results in nine pieces out of twelve pieces of the 3-meter bar. See Figure 4.16)

Figure 4.16: Carol's construction of 3/4 times a 3-meter bar

R: (Fifty seconds later, Rosa pulls out the first three pieces from her 3-part bar, which was divided into twelve pieces) Is that it?
T: So, how long is that?
R: I took the three pieces and divided one of them by four. So I divided each of them by four and so I am left with twelve and twelve divided by four is three. So one of the pieces is gonna be three. We wanted three-fourths right? So I'm gonna have to have...
T: Three-fourths of what?
R: Three-fourths of the three meters.
T: Of the three meters, right? Is that (three pieces) three-fourths of three meter?
R: No, this is one-fourth.
C: Would it be nine-twelfths?
R: Yeah, it's gonna be nine-twelfths because this is one of them and you have to multiply this times three.
T: Nine-twelfths? Okay, can you explain, Carol?
C: I had the three pieces and I divided them, each into four. Then, since this would be three-fourths, I took all those (nine pieces) out.
T: So how many meters is your constructed [bar]?
C: Um, it would be... Let's see. (Carol arranges nine pieces by four. See Figure 4.17) Two and one-fourth meters?

Figure 4.17: Carol's final construction for 3/4 times a 3-meter bar

T: Two and one-fourth because...
C: You got two whole one and one-fourth from a whole one.
T: How long is your small piece?

C: This one? One-fourth?
T: One-fourth, okay. (To Rosa) did you get it?
R: Um-hm.
T: Can you explain it?
R: Okay, I took the three pieces and divided one-third...
T: One-third?
R: Into four pieces and so I have twelve total pieces and twelve divided by four is three. So, one-fourth is gonna to equal three pieces of the twelve. And you got have to multiply that times three because you want three-fourths. One-fourth, one over four times three over one is gonna three-fourths which is gonna equal nine pieces of the twelve pieces.
T: Nine pieces of twelve pieces, so...
R: Nine-twelfths is... yeah.
T: Three-fourths of your original bar, right?
R: Um-hm.
T: So how long is do you think, how long is that your constructed bar. This bar, I think you are right.
R: Nine-twelfths, nine-twelfth. Um... I have to divide that. Nine... Three...
T: How long is your small piece?
R: This one (nine pieces)? Nine-twelfths.
T: I mean, how long is your smallest piece?
R: The smallest piece? One-twelfth?
T: One-twelfth? One-twelfth meter, you mean?
R: Yes... No, one-twelfth of the three meters.
T: One-twelfth of the three meters. So how long is that?
R: One-twelfth of the three meters? (Rosa seems to be stuck.)
C: Of one meter. How long is that of one meter?
R: One-fourth?
C: Tell him.
R: One-fourth?
T: Yes, one-fourth. So how many pieces did you divide into one meter?
R: There is four.
T: Four, so the smallest piece is...
R: Is one-fourth
T: Of a meter, right?
R: Um-hm.
T: So how long is your total constructed bar?
R: Three or the small one, this one?
T: Total. I mean your all constructed bar. How long is that?
R: Two and one-fourth meters?
T: Two and one-fourth meters.
R: Um-hm, right. Cause there is nine and there is eight, eight divided two is four, so there is two whole meters right there and there is one-twelfth left which is one-fourth of one meter. So that’s gonna be two and one-fourth.
T: Right.

When a proper fraction was involved in a multiplication problem of a whole number, Carol exactly indicated the expected behavior that a student, who had constructed distributive reasoning, should demonstrate. That is, she partitioned each part of her 3-meter bar into four pieces and pulled out one piece from each of the 3-meter bar three times, which ended up with nine small pieces from the total twelve pieces of the 3-meter bar. Unlike her conflation of units in the previous problem involving an improper fraction, she easily took three-fourths of the 3-part bar and identified the length of the pulled-out nine pieces as two and one-fourth of a meter. Her explicit use of distributive partitioning, that is, explicitly taking one-fourth of each part [one meter] seemed to help her to retain the length of the smallest piece [one-fourth] taken from a part of the 3-part bar as well as the length of one part [one meter] of the 3-part bar.

On the other hand, Rosa divided each part of her 3-meter bar into four pieces but was thinking something for about fifty seconds without doing anything. Finally, after pulling out three small pieces from twelve pieces of the 3-meter bar, she asked that those were the answer for three-fourths of the 3-meter bar. I conjecture her partitioning operation was an *imitation*[24] of Carol's distributive partitioning operation because there was an agent [Carol] initiating the experienced situation [partitioning activity.] Imitation is a basic element without which one could not adequately understand peer interactions (Sinclair, 1990) nor interactive mathematical communication more generally (Cobb, Wood, & Yakel, 1990). I do infer Rosa imitated Carol because if she had independently set a goal for her partitioning operation or had assimilated Carol's distributive partitioning operation, inferring her goal of partitioning prior to the activity, there would be no reason for Rosa hesitating for almost fifty seconds before her next step.

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[24] Piaget (1962) defined imitation as the primacy of accommodation over assimilation "If the subject's schemes of action are modified by the external world without his utilizing this external world… the activity tends to become imitation" (p. 5)

After she pulled out three pieces, which was one-fourth of the 3-meter bar partitioned into twelve and while explaining her answer to me, she had a chance to reflect on her partitioning activity based on the given problem. The reflection seemed to lead her to realize that the problem was to get three-fourths of a 3-meter bar, not one-fourth of a 3-meter bar. At this moment, my hypothesis is that the situation for Rosa turned into a situation of her partitive fraction scheme for connected numbers. That is, she was able to take three-fourths of a 3-meter bar by taking three-fourths of the total twelve pieces of the 3-meter bar. Since we knew that she was a GNS student and had constructed a recursive partitioning operation, it could be hypothesized that she was able to take three pieces as an elemental unit and three groups of three pieces as three-fourths of the 3-meter bar partitioned into twelve pieces.

However, Rosa's way of assimilation and its outcome using her partitive fraction scheme for connected numbers seemed to be insufficient for this problem to be completed. After she constructed three-fourths of her 3-meter bar, she experienced a difficulty in articulating the length of her construction in terms of a meter. Even if she seemed to know that the one piece was one-twelfth of a 3-meter bar, it was not identified as one-fourth of one meter. Until Carol got her to pay attention to one meter by asking "Of one meter. How long is that of one meter?" Rosa was not focusing on the number of pieces in a part [one-meter] of her 3-part bar to figure out the length of her construction.

Then why did Rosa indicate such difficulty in finding the length of her construction in contrast to Carol's ease? My conjecture is that it might be due to Rosa's way of assimilation of Carol's distributive partitioning operations. Although Rosa constructed three-fourths of a 3-meter bar using her partitive fraction scheme for connected numbers, her situation began with twelve unidentified pieces, not from three units of four pieces each of which was one-fourth meter long. On the other hand, Carol knew that one-fourth of each part of her 3-meter bar was one-fourth meter long prior to her partitioning activity because her distributive partitioning operation enabled her to explicitly take one-fourth of *each* part of the 3-meter bar for obtaining one-fourth of all of the 3-meter bar. For Rosa, however, that was not her conception. Rather, the result of assimilating Carol's partitioning activity activated Rosa's partitive fraction scheme for connected numbers to solve the present problem. Nevertheless, note that this was not possible unless she was able to associate two three-levels-of-units structures and use them sequentially. That is, although twelve pieces originated from three units of four pieces, her strategic use of GNS made it possible for her to change the structure of the twelve pieces as four units of three so that she could get one-fourth of twelve, and further getting three-fourths of twelve. Since the fact that the length of one piece was one-fourth of one part of the 3-meter bar was not her concern, she had to reflect on her construction processes to find the length of one smallest piece on the screen.

Rosa's Construction of an Iterative Fraction Scheme for Composite Units in Action

> Protocol 4.17 on 02/16/09: A candy bar costs 63 cents. How much does eleven-sevenths of a candy bar cost?
>
> T: Just think about without paper and pencil. A candy bar costs sixty-three cents. How much does eleven-sevenths of a candy bar cost? Can you figure it out?
> R: Eleven-sevenths, so that's more than one candy bar.
> C: Yeah, so it's eleven-sevenths. That would be, that would be seven-elevenths of the whole candy bar.
> R: (At the same time with Carol) seven, seven divided into sixty-three, nine cents. That would be seventy-two cents?
> T: Seventy-two cents? How did you get it?
> R: Okay, I know one candy, I know eleven-sevenths, eleven-sevenths is more than one.
> T: Right.
> R: Okay. Sixty-three and then seventh. So seven divided by sixty-three is nine and then nine plus sixty-three is seventy-two?
> T: Nine plus...
> R: Sixty-three plus nine... is seventy-two, (to Carol) right?
> T: Nine plus sixty-three? Why did you add nine to sixty-three?

R: Because I already... Seven-sevenths is sixty-three cents and then
C: (To Rosa) because that's one-seventh. Because that's one of the sevenths.
R: That's right.
T: One-seventh?
R: Seven-sevenths is sixty-three cents.
T: Right.
R: And then you have...
C: You have eleven...
R: You have four-sevenths left.
T: Um-hm.
R: And so seven... Oh, no. You have to do twenty-eight. So twenty-eight plus sixty-three.
C: Where did you get twenty-eight?
R: Cause sevenths. Seven divided into sixty-three is nine. So that's one-seventh.
C: Yeah...
R: So nine times... four is thirty six. So it's gonna be ninety-nine cents? (The teacher nods his head). Yeah, (to Carol) do you see how I got it?
C: Yeah, because it's like two candy bars and you have to add extras, thirty-six?
R: Yeah, because if I divide sixty-three into seven, it's nine. That's only one sevenths and since we already did seven-sevenths, that's four-sevenths left. So, nine times four is thirty-six. So, it's gonna be ninety-nine.
C: Yeah, ninety-nine cents.
R: (With confidence) ninety-nine cents.
T: Ninety-nine cents? Do you agree all? So, how many one-sevenths in eleven-sevenths? One-sevenths are in eleven-sevenths?
C: Would it be eleven?
R: Eleven.
T: So, if you know, once you know one-seventh. Can you calculate eleven-sevenths?
R: Oh, nine time eleven, nine times eleven?
T: So, what is one-seventh of a candy bar in your problem?
R: Eleven cents.
T: One-seventh of a candy bar.
C: Yeah, cause ninety-nine divided by eleven.
T: How much is one-seventh of a candy bar in your problem?
C: It's not eleven.
R: Nine cents.
T: Nine cents.
C: Sixty-three divided by, oh...
T: Carol, did you get it? Then one-seventh is nine cents, right? Then eleven-sevenths is...
R: Ninety-nine cents.
T: Ninety-nine because...
C: Cause you have nine of the eleventh.

This protocol was a good chance to investigate their mathematical knowledge about a fraction scheme for composite units. This time it was Rosa who led the interactions among the teacher [me] and the two students and provided mathematical explanations. There were two aspects to be noted in relation to Rosa' fraction scheme with composite units. First, she was able to use her GNS for a situation of a whole number multiplication by an improper fraction (although it did not need distributive reasoning) through assimilating it as a situation for her fraction scheme for composite units. When Rosa was asked that eleven-sevenths of sixty-three, her goal turned to finding one-seventh of sixty-three so that she could get eleven-sevenths from one-seventh. Her immediate response, "seven, seven divided into sixty-three, nine cents" indicated that she had already constructed sixty-three as a three-levels-of-units structure consisting of seven units of nine, each of which can be iterated seven times to make a whole seven-sevenths [sixty-three] and was able to use the three-levels-of-units structure as given material to find one-seventh of sixty-three cents.

Second, Rosa's fraction scheme for composite units was yet to be completed in the sense that she was yet to construct an iterative fraction scheme for composite units that would enable her to successfully iterate a composite unit (e.g. 9) that constituted, say, 1/7 of 63, beyond the whole 63 to find 11/7 of 63. For Rosa, at least on her first assimilation of the problem, eleven-sevenths constituted a fractional quantity more than one whole only because the numerator was more than the denominator and such a conception about eleven-sevenths led Rosa to get seventy-two as her first result, indicated by her comment "Sixty-three and then a seventh. So seven divided by sixty-three is nine and then nine plus sixty-three is seventy-two?" Only after she realized that eleven-sevenths was one and four-sevenths, could she find ninety-nine as her result by adding thirty-six [four-sevenths of sixty-three] to sixty-three [seven-sevenths of sixty-three]. Nevertheless, that does not mean she has constructed an iterative fraction scheme for composite units. If Rosa found nine cents as one-seventh of sixty-three cents and used it to get eleven-sevenths by iterating nine cents [one-seventh] eleven times, I could have attributed to her the construction of an iterative fraction scheme for composite units. However, neither Rosa nor Carol realized that they could iterate one-seventh eleven times to get eleven-sevenths until I asked them how many one-sevenths were contained in eleven-sevenths. In the sense that they realized that nine cents could be iterated eleven times to get eleven-sevenths of sixty-three cents with the help of my guiding question, their iterative fraction schemes for composite units were being constructed in action, but were not given structures used prior to the activity.

Construction of a Three-Levels-Of-Units Structure Involving Two Unit Fractions: 1/6 and 1/18

On the same line with investigating Carol's and Rosa's fractional knowledge in relation to multiplicative situations, the aims of the problems in the present protocol were two fold. First, since both students already demonstrated that they had constructed recursive partitioning operations (cf. Protocol 4.12), we wanted to explore whether they could use the result of their recursive partitioning operations as given material for further mathematical activities. Second, we also wanted them to attempt various problems involving improper fractions while anticipating their use of a FCNS as a given structure, which we considered to be one of the most important components for vertical learning[25] in their fractional knowledge.

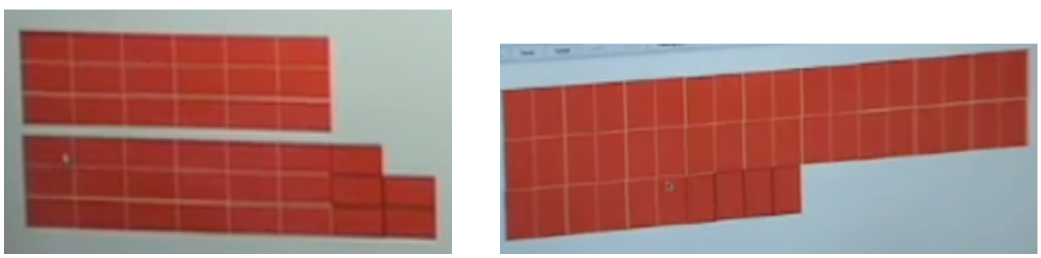
> Protocol 4.18 on 02/16/09: Using a given 6/6-bar, make a 23/18-bar without erasing the marks.
>
> T: Make [a] 6/6-bar[26] and use that bar, make a 23/18-bar without erasing the marks on the bar.
> R: Okay. So are we gonna have to copy it?
> T: You can copy. Okay, you can copy it, you can add partition marks. But you can not erase original marks on the bar.

[25] Vertical learning refers to a reorganization of schemes at a level that is judged to be higher than the preceding level. New ways of operating are introduced that are not present at the preceding level (Steffe & Olive, 2010).
[26] The teacher and the two students made an agreement that a 6/6-bar means a bar equally partitioned into six parts.

C: (Carol partitions each part of her 6/6-bar into two.) Six-sixths. You have to make twenty three out of eighteen.
R: I know that eighteen... Okay so six times three...
C: Eighteen and eighteen is thirty-six.
R: So twenty three, that's gonna be forty six pieces.
C: (Carol copies her 6/6-bar_{12} and pulls out five parts from the bar to arrange them with the 6/6-bar.) I think that's right. (Carol seems to check her answer by counting the number of parts that she constructed.) Can I go back to the problem real quick?
T: Yeah. Sure.
C: (Carol re-reads the problem on paper.) Would that be right?
T: Let's see. (The teacher is looking at Carol's construction without any response for five seconds.)
C: No.
T: Why do you think no?
C: I messed up.
T: Why do you think something's wrong?
C: Can I start it over?
T: Sure. Okay.
C: I did a wrong division thing.
T: Before do that, what was your first idea?
C: I accidently have twelve instead of eighteen pieces.
T: Okay. Okay. You can go ahead.
C: (Carol partitions each part of the 6/6-bar into three and now has 18 partitions in the 6/6-bar.) Three times six is eighteen. There we go. (Carol pulls out five parts from the 6/6-bar_{18} and arranges them with the 6/6-bar. Then she counts the number of total parts for checking her answer. See Figure 4.18a)
R: I think I did it wrong.
C: I don't know. Did you combine both of them? See, cause, what I did is, I don't know this is right, but since you have original, you have divided it into six pieces and then if you divide each of the six pieces into three pieces, and you get eighteen pieces from the whole, from the regular six, because six times three is eighteen. And then I just copy that and add that extra ones (five parts) since it's twenty three.
R: Twenty over eighteen. Oh...Okay. See, I took the six pieces as one and then... (See Figure 4.18b.)
T: Where is your original bar? What is your original bar?
R: Right there (Rosa clicks one 6/6-bar and drags it away from the collection of bars.) And then I did eighteen times, um... two. So you get thirty-six, so it's divisible by six. And eighteen is two, (inaudible) by three. And then I did forty-eight[27], so I just did six pieces and just made into forty-eight. That's forty-eight over thirty-six, which is the same thing as twenty three over eighteen.
T: So I cannot follow your reasoning. Why did you multiply six times... what?

[27] Rosa mistakenly said that the number of parts was forty-eight although it was forty-six. It was probably due to the miscalculation in her head when she converted 23/18 into 46/36 by multiplication of both the denominator and the numerator of 23/18 by two.

R: Okay. I made eighteen. The twenty three over eighteen. And I multiply that times two over two...
T: Two over two?
R: So I'd get forty-eight over thirty-six. And um... So I just took my six pieces and made forty of them.
....
T: So how long is one piece would be?
R: Six, six pieces.
T: One-sixth? I mean the smallest piece.
R: Yeah. One-sixth.
T: One-sixth of all... You said forty-eight over thirty-six?
R: Um-hm.
T: If this is forty-eight over six, over thirty-six, what would be the smallest piece?
R: Oh, it has to be one-thirty-sixth.
T: One-thirty-sixth. But the original bar of one piece is...
R: One-sixth.
T: One-sixth. Right?
R: Oh... See I'm thinking, I think I did it wrong.



Figures 4.18a & 4.18b: Carol's (Left) & Rosa's (Right) constructions of a 23/18-bar

At first, Carol made a 6-part bar and divided each part into two pieces and pulled out five pieces out of the whole bar to make a 23/18-bar, [actually it was 17/12-bar]. Although she reconstructed a 23/18-bar correctly later when she counted the number of total parts that she made to check her answer, her not being aware of the amount of the smallest piece [1/12] at the first attempt indicated that she assimilated twenty-three eighteenths as consisting of one whole and five extras, but it was not her concern that the one whole is either eighteen units of one-eighteenth or twelve units of one-twelfth. For Carol, the most important activity was to add five more parts to the one whole because the difference between the numerator [twenty-three] and the denominator [eighteen] was five. If she had assimilated twenty-three eighteenths as consisting of twenty-three units of one-eighteenth so that one-eighteenth could be iterated twenty-three times to make a 23/18-bar, her attention could have been on the construction of one-eighteenth in the 6/6-bar without erasing marks on the bar. Therefore, this protocol indicated that Carol was yet to use the multiplicative relationship of a unit fraction [1/18] with the referent whole as a given structure *when assimilating the improper fraction* [23/18] in the context of making a 23/18-bar from a 6/6-bar.

On the other hand, this protocol also revealed a crucial difference in the two students' accommodation of their GNS for construction of a three-levels-of-units structure involving the two unit fractions 1/6 and 1/18. First of all, it was possible that Carol knew that one-sixth consists of three-eighteenths so that she could partition each part of the 6/6-bar into three parts to make an 18/18-bar, (which she did eventually). Why she began by only partitioning each part into two parts is still an open question, however. Knowing that one-sixth consists of three-eighteenths means that she would be able to re-present one whole as six units of one-sixth and at the same time eighteen units of one-eighteenth. Such reasoning may have been made possible by accommodation of her GNS for whole numbers [one, three, and eighteen] for this situation and using the accommodated three levels of units [one-eighteenth, one-sixth, and one] prior to the activity. In other words, she may have established an equivalent relationship between 3/18 and a unit fraction, 1/6 by taking as given the results of her recursive partitioning operations (Steffe, 2010g).

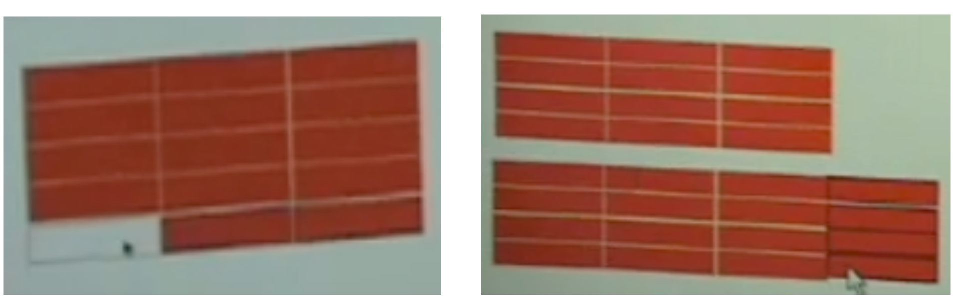
In contrast to Carol, Rosa constructed forty-six parts (even though she said "48") each of which was one-sixth of the 6/6-bar by copying her 6/6-bar six times, pulling out four more parts and joining her original 6/6-bar together with the rest of them. The reason she made forty-six parts was because she had multiplied 23/18 by 2/2. However, such an equivalent relationship between 46/36 and 23/18 was the result of her numerical calculation, which she might have learned in school. She did not see the quantitative relationship between one-sixth and one-eighteenth. Therefore, unlike Carol, it can be claimed that Rosa did not take the three-levels-of-units structure comprised by 1, 1/6, and 1/18 as a given structure to construct a 23/18-bar from a 6/6-bar. Rosa's mathematical behavior was surprising to me because I knew that she had constructed an iterable composite unit [three] and used it to make another composite unit [eighteen] by iteration in whole number multiplication and further she also demonstrated a recursive partitioning operation in Protocol 4.12. Moreover, she had constructed (at least) a partitive fraction scheme for composite units (cf. Protocol 4.17), which means she could have taken three parts as one-sixth of eighteen if she were directly asked to take one-sixth of an 18/18-bar. However, her construction for a 23/18-bar on the computer screen (Figure 4.18b) did indicate that she was yet to interiorize the three-levels-of-units structure involving two unit fractions [1/6 and 1/18] in relation to a referent whole. When teaching her, I did not realize that such a lacuna in Rosa' mathematical thinking with fractions restricted her available mathematical operations in the context of multiplicative transformations between two fractions at that time.

Rosa's Abstraction of Patterns for Partitioning Activity & Carol's Conception of Improper Fractions

The problem in this protocol was posed right after the problem in Protocol 4.18. My attention was still on the investigation of the nature of Rosa's partitioning operations and Carol's (as well as Rosa's) conception of an improper fraction. Although the formats of the two problems were very similar to each other, I expected different mathematical behaviors [possibly a distributive partitioning operation] because an interiorized three-levels-of-units structure might not be enough to solve the present problem, that is, not enough to make five-fourths using a 3/3-bar without erasing the marks.

Protocol 4.19 on 02/16/10: Using a given 3/3-bar, make a 5/4-bar without erasing the marks on the 3/3-bar.

T: Make a 3/3-bar. And using that bar, make a 5/4-bar.
R: Five over four. Divided by four. (Rosa partitions her 3/3-bar into four parts in the opposite way [horizontally] and after ten seconds pulls out one row [three parts] from the 3/3-bar$_{12}$ and arranges them below the original bar. See Figure 4.19a) I think I got it.
C: I don't think I have it yet.
R: You got it.
C: Then I need to do one more right?
R: No.
C: Yeah, because there is (pointing out each column one by one) one, two, three, four, four pieces and you need five of them, don't you? (See Figure 4.19b).
R: Um...
C: That's hard. I think I need one more. You've got one, two, three, four (counting the number of parts of the most left column in her 16-part bar) or one, two, three, four (counting the number of parts of the bottom row in her 16-part bar.)
T: Okay, we can share.
R: I think we got it. What I did is, I took that the bar that's divided in three pieces and divide that into four pieces long ways. Then, um... then I just added three more for each one for the five over four.
T: Why did you add one more? One more...
R: Because it's five over four and so this (twelve parts) is the whole bar.
C: Oh, I did it in a wrong way. I think I need a... and I only need three of them, don't I?
R: Um-hm.
C: To make the extra row.
R: Um-hm. Yeah, the extra row. Cause it's just long, just long one. If you ignore these two lines (vertical lines partitioning her bar) here, just one long is one-fourth. So if you add one more, if you pull out just one more, one-fourth bar is gonna equal five-fourths.
T: Carol, are you done with yours?
C: Yeah, I was just missing. I did the same thing she did. But I had on going to the right rather than down. So I just (inaudible) extra one you don't.
T: Why did you say, you need to ask, you need to add one more before?
C: I was looking instead of having them go down. I was thinking of like you know you got the one-third, and one-third, one-third and I was thinking have that five instead of three.



Figures 4.19a & 4.19b: Rosa's (Left) & Carol's (Right) constructions of a 5/4-bar

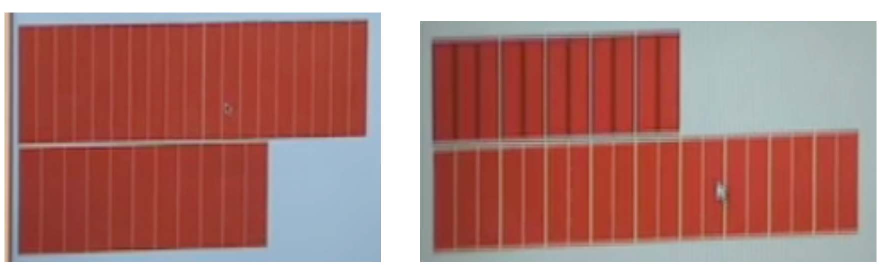
Rosa immediately partitioned each part of her 3/3-bar into four parts horizontally and then pulled out one horizontal row [three parts] from her 3/3-bar_{12} after the partitioning activity. Her construction was clean and ahead of Carol's construction. Nevertheless, I conjecture Rosa's partitioning activity had no clear goal at the moment of partitioning. As indicated in the previous teaching episode [cf. Protocols 4.14 and 4.16] Rosa's partitioning activities were not initiated independently. Rather, her partitioning operations were an imitation of Carol's partitioning activities and re-assimilated by herself in a different way from the nature of Carol's (distributive) partitioning operations. Although it was notable that she initiated her partitioning operation by herself in this protocol, the fact that it took ten seconds for Rosa to resume pulling out one row [three parts] from her 3/3-bar_{12} after the partitioning activity indicated that she had to re-assimilate the problem situation with the 12-part bar which was the result of her partitioning operation. That is, she just abstracted the pattern of Carol's (as well as her own) partitioning operation and utilized it because it has always worked. Therefore, the result of Rosa's partitioning operation [12-part bar] was a novel situation to be re-assimilated, not the anticipated result prior to the partitioning activity. Given the 12-part bar, she re-assimilated the problem as a situation for her fraction scheme for connected numbers, which enabled her to easily pull out three parts from the 12-part bar as one-fourth of twelve units. Since five-fourths was one and one-fourth for her, she could just add three parts to the 12-part bar to construct five-fourths of the whole 12-part bar as indicated by her comments, "If you ignore these two lines (vertical lines partitioning her bar) here, just one long is one-fourth. So if you add one more, if you pull out just one more, one-fourth bar is gonna equal five-fourths." From now on, I will use 'iterative' rather than 'partitive' for Rosa's fraction scheme for connected numbers because the fact that Rosa conceived 15 parts as 5/4 of 12 parts indicated the assimilating part of her fraction scheme for connected numbers included improper fractions.

On the other hand, Carol failed to construct a 5/4-bar from a 3/3-bar at her first attempt because she added one-third of her 3/3-bar [four parts of 12-part bar], rather than one-fourth (cf. Figure 4.19b). I think Carol's failure clearly indicated her way of assimilation of an improper fraction again as appeared in the Protocol 4.18. When she was assimilating '5/4' in the problem, her conception of '5/4' as '1' and '1/4' might force her to attend only to the leftover fractional part one [the numerator of 1/4] to be attached to the whole disregarding how much of the referent whole the pulled-out parts were. If she had conceived '5/4' as five units of '1/4' each of which can be iterated five times to make the '5/4' and felt the necessity of pulling out one-fourth of the 3/3-bar with a goal of iteration, her available distributive partitioning operation could have enabled her to pull out three parts as one-fourth of the 3/3-bar_{12} because she had already used a distributive partitioning operation with a non-distributive partitioning goal like in a fraction multiplication situation. Therefore, Carol's as well as Rosa's interiorization of the iterability of a unit fraction in further mathematical activities and posing an appropriate task to them became one of the main concerns for the continuing teaching episode.

Rosa's Lack of a Reversible Iterative Fraction Scheme

> Protocol 4.20 on 02/23/09: Making a bar that is 7/19 times as long as a given 19/11-meter bar and finding how long of a meter it is.
>
> T: Make a bar and pretend that this is nineteen-elevenths meters long. With this bar, make a bar that is seven-nineteenths times as long as that bar.
> (Rosa draws a bar, partitions it into eleven parts and pulls out eight parts from the 11-part bar to arrange all together in a row.)
> R: Okay, and make seven-nineteenths longer than this (indicating the combined 11/11-bar and the eight 1/11-bars that she pulled out of the 11/11-bar)?
> T: (The teacher intervenes.) But, hold a second. Let's try from the start. Erase all. I'll pose one more constraint. (Both students erase their bars.) Make a bar. (Both students make a new bar on their own screen.) Assume this bar is nineteen-elevenths.

R: Nineteen-elevenths. Oh~
C: Oh~ That one is nineteen-elevenths. Okay.
T: Okay? So, in the previous case, you just divided it as a whole, right?
R: Okay, this is more than one.
T: Yeah, assume your given bar is nineteen-elevenths. Okay? So with that bar make seven-nineteenth times as long as that bar.
R: Get another one. (Rosa makes another shorter bar under the original blank 19/11-meter bar by visual estimation based on the size of the unpartitioned 19/11-meter bar and partitions the new shorter bar into eleven parts. cf. Figure 4.20a) I'm not sure. Probably not even close. Okay. This is more than one (pointing to her original blank bar) and that's nineteen-elevenths and make one that is seven-nineteenths times as long as that bar. So you multiply then. Nineteen-elevenths times seven-nineteenths.
C: Wouldn't you just pull out seven from the one meter?
R: What would, what would you do? Would you make this one nineteen (meaning her original blank bar)?
C: What I did is, I made nineteen and I just pulled out eleven, but I don't know where I'm going with it.
R: (Rosa partitions her original 19/11-meter bar into nineteen parts.) See, that what I did but... Did you make, see, you would have to make it eleven and then you gonna have to pull out eight and add it on to make that whole one bar.
C: What?
R: (Rosa smiles awkwardly.) Okay, this one (11-part bar) is eleven pieces. So would you pull out eight more to make it nineteen-elevenths? (See Figure 4.20a)
C: (Carol does not seem to understand Rosa's explanation.) This is like your original bar, nineteen-elevenths, right (pointing to her original 19-part bar)?
R: Um-hm.
C: And what I did is, I pulled out eleven, which would be your one-meter, but this is still your original bar (pointing again to her 19-part bar. See Figure 4.20b).



Figures 4.20a & 4.20b: Rosa's (Left) & Carol's (Right) constructions of a 19/11-meter bar and a 1-meter bar[28]

When both students made a bar for one-meter, partitioned it into eleven parts, and pulled out eight parts from it to make a 19/11-meter bar to begin with, I stopped them. I then asked them to erase all bars on the screen and indicated that the first drawn bar should be a 19/11-meter

[28] Note that Rosa's one meter bar was not pulled out from her 19/11-meter bar.

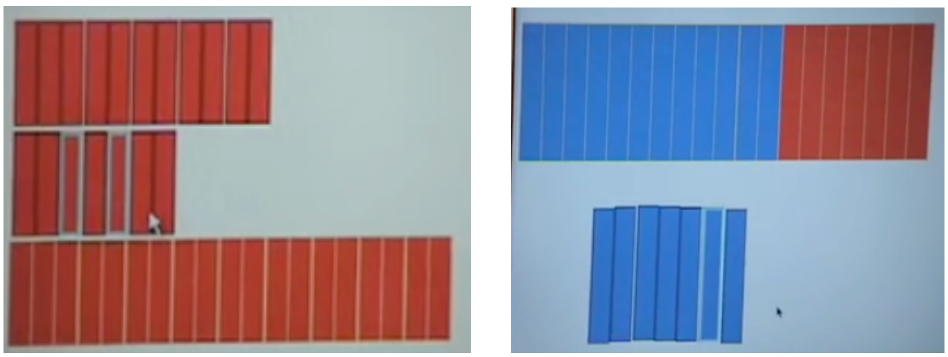
bar to make a bar '7/19 times' as long as that first bar. There were two reasons that I put such a restriction on them. First of all, since it was the first year for Rosa to join the teaching experiment with our research team, we did not have enough information about Rosa's fraction knowledge. We did know, however, that she could easily calculate and find answers for fraction multiplication and division using the methods [mainly procedural, numeric calculations] that she learned in school. So, I wanted to check whether Rosa (as well as Carol) had constructed an iterative fraction scheme and could use it reversibly. Thus I decided to pose such a constraint because to construct a 1-meter from a 19/11-meter requires the students' reversible iterative fraction scheme, that is, students' ability to assimilate the result of an iterative fraction scheme [19/11-meter] as a situation to construct the one whole that was a situation of the iterative fraction scheme. Second, if Rosa had not constructed a reversible iterative fraction scheme, I expected that to begin with the improper fractional quantity [19/11-meter] might help her establish a multiplicative relationship of a unit fraction [1/11] to the whole [11/11]. She then might be able to conceive a 19/11-meter as nineteen units of 1/11-meter, each of which can be iterated nineteen times to make the 19/11-meter, as well as iterated eleven times to make a whole 1-meter.

However, my efforts turned out to not be productive. Rather than partitioning the first drawn bar into nineteen parts to make an 11-part bar for one meter, by pulling out eleven parts from the 19/11-meter bar, as Carol demonstrated, Rosa made another new bar for 1-meter that had no relation to her original blank 19/11-meter bar with no partitioning activity. Her 1-meter bar was just a visual approximation by comparison with the size of the unpartitioned 19/11-meter bar. Although Rosa demonstrated that she could have constructed an improper fractional quantity from one whole, as indicated by her comment "see, you would have to make it eleven and then you gonna have to pull out eight and add it on to make that whole one bar," the fact that she was unable to reverse her construction processes from the improper fractional quantity [19/11-meter] to the whole [1-meter] indicated that she was yet to interiorize the multiplicative relationship between a unit fraction and a whole for further use.

Protocol 4.20: (Cont.)

T: Carol, what...
C: I'm not done yet, but I'm just...
T: Yeah, what is your first bar? What is your original bar?
C: Right here (pointing at the bigger bar below. See Figure 4.20b)
R: But you can't divide it into nineteen.
C: Huh?
R: Yeah, you can. Yes you can.
C: I'm gonna color the whole now.
R: I'm gonna have to color mine.
C: Let's see. Then we have to make seven-nineteenths. Didn't we just pull out seven? (Carol pulls out seven parts from her 19-part bar. See Figure 4.21a) That would be seven-nineteenths of it, wouldn't it? That would be...(Carol scrolls her mouse over one meter bar, which is partitioned into eleven. She seems to count the number of parts of it.) Be seven-elevenths of one meter. Is that right?
R: (Rosa does not seem to be sure of what she is doing.) Did you say as long or longer than?
T: As long... as seven-nineteenths times.
C: Think simpler.
R: Simpler than this? (Rosa pulls out seven parts from her 19-part bar.)
C: Yeah, here we go.
T: Let's listen to Carol's explanation.
C: This is my original bar, which is nineteen-elevenths. And then this is, I just pulled out eleven pieces from the nineteen-elevenths, which is one meter. Then I just pull out um... seven from these nineteenths. And it's, what I said, I think it's seven-elevenths?
T: Um-hm.
C: Yeah, seven-elevenths.
T: So why did you divide it into nineteen from the first?
C: Why did I take eleven from the nineteen?
T: I mean, divided it by nineteen you first right?
C: Yeah, just I would like to see how long one meter would be to compare with.
T: Okay.
R: But the middle one is your answer, right?
C: Yeah, this is my answer.
R: Yeah, looks like I am wrong.
T: So, what were you trying to do?

R: Okay, I did the same thing as she did except I just colored mine and she pulled hers out. This is nineteen and this is eleven (see Figure 4.21b) but this is all one bar. And then this is seven-nineteenths of that whole bar because this is, one of these is one-nineteenth, all seven.
T: So, can you guess how long is this? How long of a meter is this?
R: Um... seven-elevenths?
T: Seven-elevenths.
R: Cause eleven is one meter.
T: Right, so the smallest piece is one-eleventh...
R: of a meter, right.



Figures 4.21a & 4.21b: Carol's (Left) & Rosa's (Right) final constructions of 7/19 times as long as a 19/11-meter bar

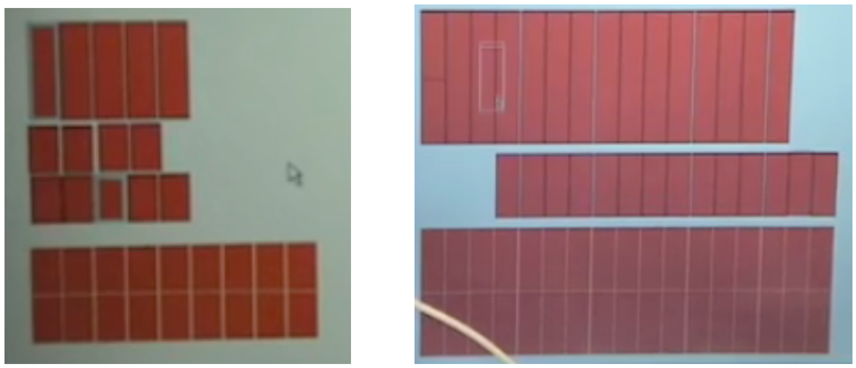
It was Carol who first partitioned her 19/11-meter bar into nineteen and pulled out eleven parts to make a 1-meter bar. Once she realized that her 19/11-meter bar consists of nineteen units of a 1/11-meter, she could pull out seven parts from her original 19/11-meter bar to make the result bar, that is a bar '7/19 times' as long as her original 19/11-meter bar. Also, she easily found out the length of her result bar as seven-elevenths of one meter because she already knew that one part of the 19/11-meter bar could be iterated eleven times to make a whole 1-meter.

On the other hand, Rosa, even though she was looking at Carol's construction of a 1-meter bar from the 19/11-meter, did not seem to understand why the partitioning activity with nineteen was necessary. Such a conjecture was corroborated by her comment, "But you can't divide it into nineteen" and by the fact that she could not immediately pull out seven parts from her 19-part bar after partitioning her 19/11-meter bar into nineteen parts. She had to re-assimilate the problem situation to utilize the perceptual materials on the screen as indicated by her question “Did you say as long or longer than?” Until Carol explained her construction processes and results, Rosa did not seem to realize why such partitioning was necessary and how it was to be meaningful in the context of dealing with an improper fraction. That is, Rosa’ partitioning operation was not reversible in the context of improper fractions in that 19/11-meter was not conceived as nineteen units of 1/11-meter, any one of which could be iterated eleven times to make a referent whole [1-meter]. Based on the assumption of such a lacuna in her fractional knowledge for the concept of an improper fraction, it was not surprising to see her first attempt to construct a 1-meter bar with no relation to her original 19/11-meter bar and the continuing struggles with the improper fraction even after assimilating Carol’s work using her current schemes.

Rosa’s Internal Constraint Due to the Lack of Interiorization of Recursive Partitioning in Fraction Multiplication

The context of Protocol 4.21 was a situation where I asked the two students to pose a fraction multiplication problem for each other similar to that in Protocol 4.20. When Rosa posed a problem to make a bar that was half as long as a 9/5-meter bar, Carol made a 9-part bar, partitioned each part of the bar into two and pulled out a part from each of the 9-part bar to arrange them in a row. After she rearranged the pulled-out nine parts from the 18-part bar of 9/5-meter in two rows under her 5-part bar of 1-meter (see Figure 4.22a) to compare with her referent whole, she told that it would be nine-tenths because there were ten pieces in one meter. In Carol’s turn, she posed a similar problem to make a bar that was half as long as a 17/15-meter bar. Rosa’s response seemed very similar to what Carol provided. Rosa immediately made a 17-

part bar, partitioned each part of the 17-part bar into two and pulled out seventeen pieces from the 34-part bar of 17/15-meter. Protocol 4.21 began at this point in the teaching episode.



Figures 4.22a & 4.22b: Carol's (Left) construction of half as long as a 9/5-meter bar & Rosa's (Right) construction of half as long as a 17/15-meter bar

Protocol 4.21 on 02/23/2009: Making a bar that is 1/2 times as long as a given 17/15-meter bar.

C: (Pointing at Rosa's construction) and then what fraction is this from only one meter? (Rosa seems to count the number of parts that she pulled out for the answer.)
R: Eight and a half?
T: Eight and a half meters?
R: No, from one meter. That's fifteen-fifteen right?
T: So how long of a meter is that (pointing to her resulting strip of 17 halves of 1/15-parts)?
R: (Rosa begins rearranging her resulting half-parts over the unit parts of her 15-part bar for 1-meter. See Figure 4.22b) cause if you put it over...
C: If that's one meter. (Rosa keeps positioning each of her resulting parts on the 15-part one meter bar, but the teacher intervenes.)
T: Okay, wait. Can we guess how long of your constructed [bar] is without this moving, movement?
R: Oh, eight and a half?
T: Eight and a half meter, you mean?
R: Um-hm. No, eight and a half of the fifteen.
T: So how long of a meter is that?
R: Um... eight and a half?
T: How can we guess how long of a meter, how long of a meter is your constructed bar without moving this stuff? I'm also asking you, Carol.
C: Um, I just took it. I took, do you want me to tell you how I did it?
T: Um-hm, so can we guess how long is the result?
C: Can I tell you what I did? I took out two [of] them since it was, or I took off two of the little pieces because it was from the extra two like instead of fifteen-fifteenths, it was seventeen-fifteenths.
R: Right. Um-hm.

C: So then I just took off two and I knew that there were thirty pieces of one and two like in half all of it. Then I added two pieces back. Cause it was one half of the meter and added two. (Rosa puts her head as if she does not understand Carol's explanation.)
T: Can you say that again? I'm not following...
C: Can I just tell you what I got? I got seventeen-thirtieths.
T: Okay, seventeen-thirtieths. How could you get that result?
R: Wait. What was your question? What fraction is this from only one-meter bar. One meter is fifteen, right?
C: Yeah, but I'm talking about the halves. Like if you add halves together.
R: We got the same answer, but yours is just...
C: Right, bigger...
R: Bigger than mine times two. Cause eight and a half times two is seventeen and fifteen times two is thirty.
T: So how many pieces did you get for your answer?
R: I got eight and a half of one meter.
T: I mean how many pieces, smallest pieces did you get?
R: Um. I got seventeen... right?
T: Seventeen? This one, before moving this stuff. Seventeen.
R: Yeah, that's how many I got.
T: Then how long is the smallest piece is that?
R: One-thirtieths.
T: One-thirtieths. So, how long is of a meter your total pieces?
R: Seventeen over thirty?
T: Okay.
R: Oh, that's what you're asking like without moving them. Okay.

Although taking '*a half*' of what they started with by partitioning each of 17 parts into two parts horizontally seemed to make the problem simpler, to figure out how much the result of taking a half was not simple for Rosa. Her first answer was eight and a half, which was a half of the number of parts of her 17-part bar. When I asked her to find how long of a meter her construction was, she attempted to arrange each part of her construction as Carol did in her previous problem. However, I interrupted Rosa's attempt to position her resultant parts on the referent one-meter bar because I wanted to know that she could find the answer without making a visual part-whole comparison. Somehow she knew that her fifteen parts amounted to one meter as indicated by her comment, "No, from one meter. That's fifteen-fifteen right?" and such awareness led her to argue that eight and a half of the fifteen as an answer. However, she insisted that the answer be eight and a half of a meter. Until I asked her the length of her smallest part, which was one-thirtieth of a meter, she did not realize that her construction was a collection of seventeen units of one-thirtieth of a meter.

My conjecture for Rosa's struggle with identifying the length of the bar she established was based on two possibilities: 1) her lack of distributive partitioning operations and 2) the failure to take the results of her partitioning operations as input for recursive partitioning. First of all, the nature of Rosa's partitioning operations was not distributive. Although the partitioning activity was executed by herself, the aim of her partitioning operations was just to take a half portion of the 17-part bar by putting a line horizontally in the middle of the 17-part bar (cf. Figure 4.22b). When Rosa put partitions on each part of the 17 parts of the 17/15-meter bar, she did not seem to be explicitly aware that taking one part from each of the divided parts of the 17/15-meter bar equaled to a half of the total 17/15-meter bar. Further, even with her result of seventeen parts (each a 1/30-meter bar), she did not engage in recursive partitioning operations to find how long the smallest part [1/30-meter] was of one meter. Even though she was a GNS student who had constructed units-coordinating schemes, recursive partitioning operations were not easily provoked in this situation (Rosa had used recursive partitioning in solving the problem in Protocol 4.19: 5/4 of a 3/3-bar). The problem situation was assimilated using her iterative fraction scheme for connected numbers without an identification of the length of a smallest part. Her construction was perceived as a half of 17 units, rather than a half of 17/15-meter and such different conceptions about her construction led her to conflate units to get eight and a half of a meter [not of 1/15-meter]. When I explicitly asked Rosa the length of her construction in terms of a meter, she began to realize that one meter consisted of fifteen parts and responded 'eight and a half of fifteen.' However, it was only a part-whole comparison between her construction [eight and a half parts] and the whole one-meter [fifteen parts]. She was yet to realize that the length of a part in her eight and a half parts was one-fifteenth of a meter and further the smallest piece was one-thirtieth of a meter (her recursive partitioning operation had not yet been provoked).

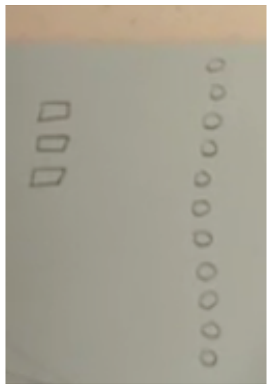
In sum, if Rosa had constructed distributive partitioning operations, that is, she had been able to realize that a half of 17/15-meter could be obtained by taking a half of *each* part of the 17/15-meter prior to activity, she could have easily found the length of the smallest part [a half of 1/15-meter] of her construction and retained it as available information for further use. Her inability to do so also indicates that she was yet to interiorize recursive partitioning operations at three levels of units. This was an internal constraint that she experienced in construction in the context of fraction multiplication situations.

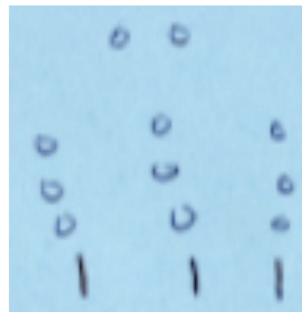
Confirmation of Carol's Distributive Reasoning

The nature of partitioning operations (indicated through both students' mathematical activities) emerged as one of the most important issues in the teaching experiment. Depending on the availability of a certain kinds of partitioning operations, the participating students demonstrated different mathematical behaviors. So far, Carol demonstrated that she had constructed her distributive partitioning operations not only in the context of a sharing situation, but also in a fraction multiplicative situation such as making a bar that is 3/4 times as long as a 3-meter bar (cf. Protocol 4.16), but it was questionable whether Rosa had constructed a distributive partitioning operation that she could use in a non-sharing situation. Therefore, although the overarching goal of the teaching experiment during Phase II was to investigate emerging mathematical actions and operations in the context of fraction multiplication situations, I decided to try to pose a proportional reasoning problem [a partitive fraction division problem from a mathematician's point of view] in order to explore whether distributive partitioning operations could be evoked by Carol and, in Rosa's case, whether the problem could help her construct distributive partitioning operations.

> Protocol 4.22 on 02/23/09: Finding how much rubber is needed for one basketball if 3 pounds of rubber makes 11 basketballs.
>
> R: (Rosa reads the problem.) If one pound of rubber makes seven basketballs, and how much rubber of a pound [is required for one basketball?]
> C: (Carol interrupts Rosa's reading the problem.) One-seventh. One-seventh of a pound.
> R: Yeah, because seven represents the number of basketballs one pound makes.
> C: (With hand-gestures of distributing something from one position to the other several positions,) you just divide it and give it to the basketball.
> R: Yeah, one-seventh of a pound.
> T: How did you get it? I'm embarrassed because you were so fast. (All laugh.)
> C: The other seven basketballs and one pound equals. One-seventh, you just distributed out to each of the seven basketballs.
> R: Yeah, so you just half it by seventh.
> T: How about next, then? Good, I like it.
> R: (Rosa reads a problem on paper.) If three pounds of rubber make eleven basketballs, how much rubber of a pound is required for one basketball?
> C: Would that be three-elevenths?
> R: That's what I'm thinking.
> C: Three-elevenths?
> R: Of a pound, so it's one pound.
> C: Oh, one pound. Okay, I'm gonna write this down. Okay, so you have one, two, three into eleven basketballs. (Carol draws a column of three squares and a column of eleven circles in a vertical way. See Figure 4.23a.) So it gets one-third of three-thirds. How much is one-third of three-thirds?
> R: That's what I was gonna saying because you have...
> C: Cause you have one-third of three-thirds, couldn't you multiply? (Carol writes down '1/3 of 3/3' on her paper.) It gets three-ninths, which is just one-third.
> R: You see, you have two balls left and you have three pounds (see Figure 4.23b).
> C: (Pointing out one of three squares, which stands one pound,) they get one-third of each. (With hand-gestures of distributing a part of each pound to each basketball,) they get one-third, one-third, one-third. One-third, one-third, one-third. You know, they wouldn't get a third of each. They get an eleventh of each. So they, each ball gets three-elevenths. So one ball equals three-elevenths. How much of each they get, they get one-eleventh of each pound.
> R: No, we have to ...
> C: Yeah, (pointing out the first pound) because one-eleventh goes to each of them (eleven basketballs). (Pointing out the second pound) one-eleventh goes to each. (Pointing out the third pound) one-eleventh goes to each. So they get three-elevenths in all, but one-eleventh from each pound. Is that right or no?
> T: One-eleventh from.. Yeah, so how much rubber of pound is required for one basketball?

C: Three-elevenths?
T: Three-elevenths. Okay, do you agree, Rosa?
R: I'm thinking. Okay, you have three, three, yes, yes, cause you have three pounds and eleven basketballs...
T: So for one basketball...
R: You have to get one-eleventh from the three, so that's gonna be three-elevenths.





Figures 4.23a & 4.23b: Carol's (Left) & Rosa's (Right) drawings for 3 pounds and 11 basketballs

This protocol clearly indicated how powerful Carol's distributive reasoning was in the context of a proportional reasoning problem and that Rosa was yet to construct distributive partitioning operations for her mathematical activity with a non-sharing goal. Carol immediately assimilated the problem as a sharing situation of three pounds of rubber among eleven basketballs and easily got the answer [three-elevenths.] Such a quick solution by Carol was possible because of the availability of her distributive partitioning operation prior to actual actions of sharing. That is, she already knew that one-eleventh from each pound could be distributed to each of eleven basketballs in order to share three pounds equally among eleven basketballs as in her comments "Yeah, (pointing out the first pound) because one-eleventh goes to each of them (eleven basketballs). (Pointing out the second pound,) one-eleventh goes to each. (Pointing out the third pound,) one-eleventh goes to each." Although there was confusion to find the amount that she was to distribute from each pound to each basketball, she quickly found the error in her partitioning activities and independently corrected her wrong distributing actions.

Thus, she was able to monitor and self-regulate her distributive activities at a level above the acting subject [Carol herself] in distributing action, which indicated her distributive partitioning operation became an interiorized mathematical operation for Carol to use prior to activity.

On the other hand, Rosa did not demonstrate any indication that she independently used a distributive partitioning operation to solve this problem. She seemed to be dominated by Carol's quick solution, which might have suppressed whatever operations she might have used had she not tried to compete with Carol. She did attempt to distribute eleven basketballs over three pounds but was stuck with the leftover [two basketballs] after she distributed three basketballs on each of three pounds. Even if she was attempting to keep up with Carol's pace, I conjecture that the main reason for her error was due to the lack of distributive partitioning operations that she could use to anticipate the result of her mathematical operations prior to actual activity. Such a conjecture was corroborated by the fact that Rosa could not deal with the leftover [two basketballs] after distributing three basketballs on each of three pounds. If distributive partitioning operations were available to her, somehow she might have been able to partition each of the leftover of two basketballs into three parts and distribute one-third of each basketball to each of three pounds to get the total three and two-thirds of a basketball for a pound although the problem was intended to be done in the opposite way [to distribute three pounds to eleven basketballs]. Rosa seemed to assimilate Carol's explanation in that she finally agreed with her, but whether she constructed distributive partitioning operations as a result was still to be investigated.

Summary of Phase II: Fraction Multiplication

During Phase II, I investigated what mathematical operations would emerge in the students' mathematical activities in the context of fraction multiplication situations. First of all, I posed a problem to check their construction of recursive partitioning operations. I knew that Carol had already constructed the operations through her first year of the teaching experiment. I could possibly conjecture Rosa's construction of recursive partitioning based on the fact that she had constructed a GNS. However, I wanted to make sure of Rosa's construction because a recursive partitioning operation was fundamental in students' fraction multiplication.

Posing multiplication problems of a whole number by a fraction, after indications of both students' construction of recursive partitioning operations, revealed a crucial difference between the two students' mathematical operations for fraction multiplication, that is, the use of distributive partitioning operations. Carol's use of distributive partitioning operations consistently emerged. Her distributive reasoning was powerful enough to solve problems not only in a sharing situation but also with non-sharing goals such as fraction multiplication and proportional reasoning [unit ratio].

On the other hand, I could not attribute distributive partitioning operations to Rosa as her partitioning operations were just an imitation of Carol's partitioning activities. Rosa seemed to have no clear goal in her partitioning activities. Such lack of distributiveness in her partitioning activities kept appearing during Phase II. Nevertheless, she was able to solve fraction multiplication problems with the result of her partitioning operations (although they were imitations of Carol's) by re-assimilating the results of the partitioning operations as situations for her partitive (or iterative) fraction scheme for connected numbers. It was made possible by the flexible use of her GNS involving fractional quantities.

In addition, even with a confirmation of both students' construction of recursive partitioning operations, a crucial difference was revealed in the two students' accommodation of their GNS for construction of a three-levels-of-units structure involving two unit fractions, where the denominator of one fraction was a multiple of the denominator of the other fraction (e.g. 1/6 and 1/18). When making a 23/18-bar using a 6/6-bar without erasing the marks on the 6/6-bar, Carol successfully partitioned each part of the 6/6-bar into three parts, pulled out five parts and added them to the 18 parts to make the 23/18-bar. Rosa, on the other hand, demonstrated a lacuna in using the results of her recursive partitioning operations with the same problem. She was not able to use the three levels of units [1/18, 1/6, and 1] as a given structure and was not successful in solving this problem. Such a lacuna in her mathematical thinking turned out to make a critical difference in transformation activities between fractions during Phase III.

Phase III: Multiplicative Transformation Between Two Fractions

Although we gained a lot of important data in relation to their fraction measurement division in Phase I, we also observed their struggles whenever we tried to convert the problems in fraction measurement division situations into the situations where they needed to find a multiplicative operator to transform a fractional quantity into another quantity (cf. Protocol 4.1). As mentioned earlier, the students' explicit awareness of the multiplicative operator while engaging in the transformation can be regarded as an important step in the constructive itinerary of the students for the RNA. Therefore, to investigate the two students' mathematical schemes and operations in relation to their construction of RNA, the research team decided to explore the two students' mathematical activities related to the transformation activity between fractional quantities. The challenge to the research team at the time was to devise appropriate tasks to encourage the two students to engage in the transformation activity and further help them become aware of the multiplicative operator to be conducted in the process.

Two Student's Conceptions of an Improper Fraction

As reported earlier, we already had a hard time provoking the two students to convert their ways of assimilation of fraction measurement division situations into the situation of a multiplicative transformation. In order to encourage the students' multiplicative transformation activity, I designed a step-by-step problem: 1) Find a fraction of 37/31-meter, which amounts to 1/31-meter and, then, 2) based on the previous result, find a fraction of 37/31-meter, which amounts to 31/31-meter. My expectation was that two simple problems might help the students explicitly realize their transformation processes and find multiplicative operators for each step and then coordinate the two operators from the two multiplicative transformations in order to find a single multiplicative operator to go from 37/31-meter to one meter.

> Protocol 4.23 on 02/23/09: Finding a fraction of 37/31-meter to make 1/31-meter and finding a fraction of 37/31-meter to make 31/31-meter.
>
> (The problem is "Using JavaBars, find a fraction of 37/31-meter to make 1/31-meter and find a fraction of 37/31-meter to make 31/31-meter.)
> T: Some fraction of thirty-seven over thirty-one meters amounts to one thirty-one, one over thirty-one meter.
> R: So...
> T: Some fraction of...
> C: Like a fraction of...
> T: Yeah...
> R: Okay, fraction of thirty-seven over thirty-one equals one thirty-one.
> C: Can we make that?
> T: Sure. Basically, that's what I want.
> C: Uh~~ long bar.
> T: Rather than calculating, just construct the bar. (Both Carol and Rosa make a long bar.) So your given bar is thirty-seven over thirty-one. (Carol puts 37 parts in her long bar.)
> R: (Rosa says to herself) Oh~~ That's right. No, it's not.
> T: What fraction of this bar should be one over thirty-one (indicating the bars the students already made on their screens)?
> C: (Carol pulls out thirty-one parts from the bar to make a 31-part bar. On the other hand, Rosa makes a 37-part bar and fills the left most six parts of her 37-part bar in a different color. Then she reads the problem on the paper one more time.) Wouldn't it just be one thirty-seventh?
> R: Yeah... It have to be.
> C: Yeah. Because you just take the one...

R: Yeah, if you just take one... (Rosa rests her chin on her hand.)
C: That's the same as one thirty-seventh...
R: I always do that when I don't know what to do (meaning rests her chin on her hand). I always go like this. (Rosa rests her chin on her hand again. Carol fills one piece on both the 31-part bar and the 37-part bar with a gray color.)
R: (After five seconds looking at Carol's screen and the question on the worksheet) yeah... Of course it has to be one thirty-one cause thirty-one is the one that you started with.
C: One thirty-sevenths.
T: One thirty-one? What do you mean by one thirty-one. One thirty-one of thirty-seven over thirty-one is one thirty-one?
R: No, one thirty-seven is one thirty-one of thirty-one. Oh~ that's confusing. (Rosa smiles awkwardly.)
T: (Speaking at the same time as Rosa) what fraction of thirty-seven over thirty-one should be one thirty-one?
R: One thirty-seven (pause for five seconds).
T: One thirty-seven. So... did you get it using JavaBars or just calculated it?
R: No, I calculated it but do I have to show you on here (screen)?
T: Sure, that's what I want.
R: Okay, I think I can just pull one out (pulling out one part from her 37-part bar) there.
T: Um-hm.
R: (Rosa smiles.) One thirty-seventh. This is the one of the thirty-seven over thirty-one that you have.
T: Okay, so... What is the answer?
C & R: One thirty-seventh.
T: One thirty-seventh, right? One thirty-seventh of this one (37-part bar) is one over thirty-one meter, right?
R: Right.
T: Good. Then...what fraction of one, no, thirty-seven over thirty-one meters should be thirty-one over thirty-one?
R: Thirty-seven over thirty-seven?
C: That would be one.
R: Exactly, but that's one. Is that right?
T: No.
R: It is not.
C: Wait, thirty-seven thirty-one meters... it would be thirty-one thirty-seventh.
T: Why do you think like that?
C: Because if it's thirty one over thirty-one,
R: (At the same time,) the reciprocal of it.
C: It should be one. It just be a thirty-one over thirty-seven because you have the one inside the thirty-seven pieces.
R: But it's...
T: Can you show me using JavaBars?
C: That's basically the same thing.
R: But isn't thirty... but isn't thirty-one over thirty seven less than one?

C: That’s right here (pointing at her screen). Thirty-one over thirty-one should be one meter.
R: Right.
C: You have thirty-seven and then you have, you already have thirty-one and thirty-seven. So it’d just be thirty-one thirty-sevenths cause you have the one meter instead of thirty-seven pieces.
R: Okay, I got it.

Although I didn’t expect it, this protocol ended up revealing the two students’ current conceptions of improper fractions. A fractional connected number sequence (FCNS) is a connected number sequence in which the units of the connected numbers are unit fractions, which is a result of an iterative fractional scheme (Steffe, 2002). For 37/31-meter to be constructed as a number in the FCNS, it is necessary for a student to construct three levels of units [1/31, 1, and 37/31]. In other words, a 37/31-meter bar should be conceived as thirty seven units of 1/31-meter, each of which can be iterated not only thirty seven times to make the 37/31-meter bar, but also thirty one times to make one meter. Interestingly, Rosa could not immediately solve the problem using an iterative 1/31-meter. It was Carol that provided ‘1/37’ as an answer after construction of a 37-part bar and a 31-part bar by pulling out 31 parts (one at a time) from the 37-part bar on the screen. Although the answer was correct, it was not obvious that Carol had already been aware of 37/31-meter as thirty-seven 1/31-meter prior to actual construction using JavaBars. However, the immediacy with which she pulled out 31 individual parts from the 37-part bar to constitute the one-meter bar, suggests that she did conceive of the 37 parts as 37 one-thirty-oneths of a meter. In the end, Carol established mutual relationships among 1/31, 1 and 37/31; that is, she constructed a 37/31-meter bar constituted by a three-levels-of-units structure. Within this three-levels-of-units structure, Carol could establish the relationship of the 37/31 to one meter and also the reciprocal relationship of one meter to 37/31-meter, that is one meter was 31/37 of 37/31-meter.

On the other hand, Rosa did not seem to construct such a three-levels-of-units structure for the improper fraction [37/31] even with her actual construction of a 37/31-meter bar and a 1-meter bar on the screen using JavaBars. She seemed to recognize such a three-levels-of-units structure of 37/31-meter when Carol filled out one of the 1/31-meter parts in her 37/31-meter bar and 1-meter bar with a different color. However, it turned out that Rosa's explanation was just a recapitulation of Carol's actions and explanations. When I asked them the relationship between 1-meter and 37/31-meter, there was no indication that Rosa understood the relationship among the three units [1/31, 1, and 37/31]. That is, when I asked Rosa what fraction of the 37/31-meter bar was the 31/31-meter, her answer was "Thirty-seven over thirty-seven?" There was no indication of her establishment of a quantitative relationship among 1/31, 1 and 37/31. Thus, this protocol corroborated that one of the reasons that Rosa demonstrated a difficulty in dealing with fraction multiplication (as well as fraction division) involving an improper fraction might be due to the lack of an ability to use a unit fraction in the same way as she uses the unit of one in the case of her generalized number sequence (GNS). Therefore, devising appropriate tasks to help her construct and use a FCNS as a given structure for other mathematical activities was to remain as a challenge to me as a teacher-researcher in planning the continuing teaching episode.

The problem posed in Protocol 4.23 did not encourage the students to carry out multiplicative transformations and to find necessary operators for the transformation processes, which had been my goal. Rather, Carol solved the problem using her three-levels-of-units structure for an improper fraction, while Rosa was constrained by her lack of the use of such a structure. After several similar problems were posed, resulting in similar behavior on the part of both students, I decided to present a problem beginning with a proper, rather than an improper fraction. I began with the problem of what fraction of 3/4 of a meter amounts to one meter and what fraction of 3/4 of a meter amounts to 31 meters. Both students struggled for about ten minutes to solve the first problem: what fraction of 3/4 of a meter amounts to one meter? Although they constructed a 1-meter bar_4 and a 3/4-meter bar by pulling out three parts from the 1-meter_4 bar, the problem that began with “What fraction of ...” did not seem to provoke any fraction scheme available to them in order to cope with the problem situation. When the witness-researcher changed the problem into “How many times is 3/4 contained in one, 4/4?” they were able to assimilate the problem using their unit-segmenting scheme with a remainder that they had constructed during Phase I. That is, Carol was able to solve the problem through segmenting the 1-meter_4 bar with the $3/4\text{-meter}_3$ bar using her unit-segmenting scheme with a remainder as indicated by her comment “ because it’s in there, one time and one of the thirds” and Rosa immediately agreed with Carol. Before continuing with the second problem (what fraction of 3/4 of a meter is 31 meters?) I wanted to check to see if the students’ assimilation of the first problem after the witness-researcher’s rewording would enable them to solve a similar problem, stated in the same way as before.

Rosa’s Construction of a Unit-Segmenting Scheme with a Remainder and its Strategic Use

Protocol 4.24 on 02/27/09: Finding a fraction of 5/7-meter that amounts to one meter.

T: What fraction of five-sevenths should be one-meter? Five-sevenths.
(Rosa makes a 5-part bar and pulls out two parts from the bar to arrange them with the bar in a row.)
C: Sevenths. One and two-fifths?
R: (Rosa does not seem to listen to Carol’s answer.) I should’ve set those together. And you have one, two, three, four, five and you need to make one meter?
T: Yeah. What fraction of...
R: Okay. One... and two-fifths.
C: That’s what I said.
T: One and two-fifths.
R: Yeah. Because five is your whole. We were looking at four instead of three[29].

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[29] Rosa was referring to the original problem: what fraction of three-fourths meter amounts to one meter? right before this protocol.

T: Let’s listen to your, Rosa’s explanation.
R: Okay. Instead making seven, um... I took, this (5-part bar) is five-sevenths. And here is two other ones to make the whole one meter. But... so this (5-part bar) is one and you need two more to make seventh-sevenths. But two more of this (5-part bar), because this is your whole. So if there is five in total and you need two, it’s gonna be two-fifths plus one original bar that you have right there.
....
T: Okay, let’s go back to the second problem. How could we solve this problem? Some fraction of three-fourths meter amounts to thirty-one meters.
R: It’s ninety-three over three. Right?
T: Ninety-three over three?
R: Ninety-three over three because four over three times thirty-one over one since four over three is one meter.
T: So, what is the answer?
R: No, I’m sorry. I’m sorry. So a hundred and twenty-four over three.
....
T: What did you multiply?
R: Oh, what did I multiply. Four over three times thirty-one over one.
T: Why?
R: Why. Okay, I know that four over three is one meter. And you’re trying to see how many, um... I’m trying to get thirty-one meters. So you just multiply it times thirty-one.

Carol was able to immediately solve the similar problem of what fraction of 5/7 was one meter using her unit-segmenting scheme with a remainder. However, this protocol also revealed that Rosa had constructed a unit-segmenting scheme with a remainder. In the previous teaching episodes involving fraction measurement division problems (cf. Protocols 4.5, 4.6, 4.7 and 4.8) it was Carol who provided explanations first and who led communications among us. Rosa mostly assimilated Carol’s activities and explanations to find solutions, rather than by herself, although she was very quick to assimilate Carol’s explanations and find the pattern of Carol’s work. However, in this protocol, although Carol suggested her answer first, Rosa did not seem to attend to Carol’s answer and instead found her answer by herself. Her comments before Carol’s explanations, “But two more of this (5-part bar), because this is your whole. So if there is five in total and you need two, it’s gonna be two-fifths plus one original bar that you have right there” corroborated that Rosa had constructed a *unit-segmenting scheme with a remainder*. That is, she was explicitly aware of her segmenting unit [5/7-meter] and conceived the leftover [2/7-meter] in terms of the unit used in segmenting operations [two-fifths of the 5/7-meter]. Moreover, Rosa demonstrated a similar strategic use of her unit-segmenting scheme to solve a more complex problem [What fraction of 3/4-meter does amount to 31 meters?] in a way similar to Carol in Protocol 4.8 [How many times is 5/3-meters contained in 176 meters?]. Rosa knew that four-thirds of 3/4-meter was contained in one meter and used it to find a fraction of 3/4-meter to get 31 meters by multiplying four-thirds by 31/1. I consider this indicates significant mathematical progress for Rosa, when compared with her struggles in Protocol 4.8, because at that time she could not use the result of her unit-segmenting scheme with a remainder for other mathematical situations, even with Carol's very detailed explanations.

Introduction of Dilation in GSP for Multiplicative Transformation Activity

As the previous two protocols indicate, a transformative construction in JavaBars, say, the construction of a one-meter bar from a given 37/31-meter bar, tended to induce the students to establish a part-whole relationship between the given bar [37/31-meter] and the result bar [31/31-meter]. However, since I wanted to encourage the students to construct a scheme for a multiplicative transformation between two (fractional) quantities, that is, to become aware of the multiplicative operator to be used in the transformation as well as the transformation activity itself, I decided to pose such transformation problems in the milieu of Geometer's Sketchpad (GSP) using the dilation option. DILATION option in GSP enables a student to conduct a multiplicative transformation of a geometric figure such as a segment or a triangle. When a DILATION is chosen from the menu, a student can enter a scale factor [in a fractional form] in the 'Dilate' box (see Figure 4.24) by which to dilate a selected figure.

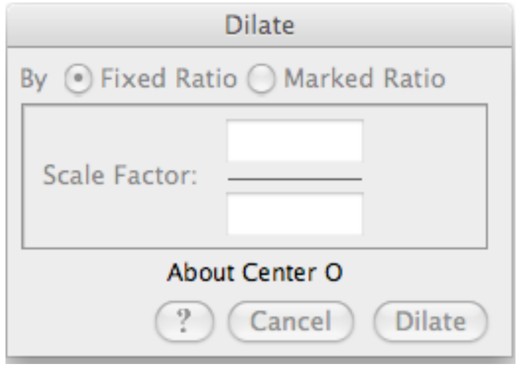

Figure 4.24: Dilation box in GSP

In contrast to the program JavaBars, where a student can produce her transformation activities step by step, the dilation option in GSP involves generating a scale factor for the transformation prior to engaging in the transformation. Therefore, I expected that a transformation activity in GSP using DILATION could open the way not only to allow the participating students to produce a multiplicative geometric transformation, but also to provide an occasion for the students to reflect on and abstract their mathematical activities involved in the transformation, which might lead to the construction of the scale factor, that is, the number to be multiplied [the multiplier].

In the teaching episode on March 11th of 2009, the GSP program was introduced for the first time to Carol and Rosa [Carol had no experience with GSP although this was her second year of the teaching experiment]. Along with my guidance, they were introduced to the term 'Dilation' and explored the effect of the 'Dilation' option in GSP by dilating a given segment by '2' and by '1/3.' Although the meaning for 'dilation' that they knew before was to "enlarge something," they realized that they could make the size of a given geometric figure decreased through dilating the figure by a fractional scale factor less than one. When the students finished transformations of two segments based on the original segment FG dilating by '2' and by '1/3' (see Figure 4.25) the witness-researcher posed a question to find a scale factor to directly transform the segment dilated by '2' into the segment dilated by '1/3'

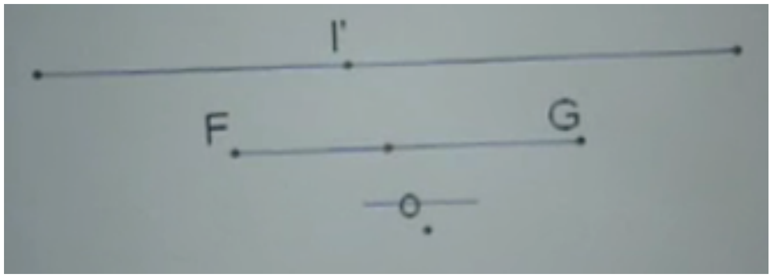

Figure 4.25: Dilation of FG by 2 and 1/3

Upon the witness-researcher's asking, although they experienced a temporary confusion between the number to be put for dilation [for multiplication] and the number for division [they argued six over one at first], they finally found one-sixth as an answer as indicated by Carol's comment, "If it's the larger one which is two times bigger than FG and the one we just made was three times... like there is three pieces in FG. Then it would be one-third. And that means there're six pieces in the larger one." This was an indication that the two students were able to carry out recursive partitioning operations in the milieu of GSP just as they did in JavaBars. They constructed a three-levels-of-units structure for the doubled segment. That is, even though there was no partition lines on the doubled segment, the doubled segment could be re-presented as two units of three-thirds so that one-third could be iterated six times to make the doubled segment and then such an ability to re-present the doubled segment enabled the students to find the scale factor [1/6] to transform the doubled segment directly into the segment dilated by one-third.

Interiorization of Recursive Partitioning Operations with One-Half in Transformation Activity

In the teaching episode held on March 19th, the transformation problem from 1/9A1/9B[30] to 2/3A2/3B was posed to the students in GSP environment. When the problem was posed to the students, they did not seem to have any clue for how to establish a quantitative relationship

[30] I (the teacher-researcher) and the two students made an agreement about naming for the constructed segments. That is, 1/9A1/9B was a name for the segment made from a given segment AB dilated by 1/9.

between a 1/9A1/9B segment and a 2/3A2/3B segment[31]. Rosa attempted several numbers for dilation by trial and error, and Carol's conjecture also seemed to be based on guessing. If JavaBars was used for this problem, it might have been solved much more easily because construction in JavaBars could give the students perceptual representations of two segments to easily compare the two fractional quantities [if one-ninth were one bar, then two-thirds would be six of those bars.] In other words, given a 1/9-bar in JavaBars, they might have been able to construct the referent unit bar by iterating the 1/9-bar nine times, and then pull out six parts from the 9-part bar of the referent unit to make a 2/3-bar since they already had constructed an iterative fraction scheme for connected numbers. However, in the GSP environment the students needed to anticipate a scale factor for transformation first. It means that they would need to have constructed a 2/3A2/3B segment consisting of six units of a 1/9A1/9B segment in re-presentation prior to actual construction. In that sense, finding a scale factor in GSP would require a higher level of mathematical thinking than the construction in JavaBars. Specifically, for this problem, interiorization of results of recursive partitioning operations seemed necessary because the three-levels-of-units structure with 1/9, 2/3, and 1 needed to be used as a given structure.

Therefore, for the next teaching episode on March 23rd, I prepared a question that might require students' interiorized use of recursive partitioning operations, but one that seemed the easiest (from my perspective) among such problems, that is, recursive partitioning operations involving '1/2.' The task was to find as many ways as possible to make 1/2A1/2B by selecting and dilating a fractional part of AB (cf. Figure 4.26).

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[31] The students were given a referent segment AB and a 1/9A1/9B segment on their computer screens.

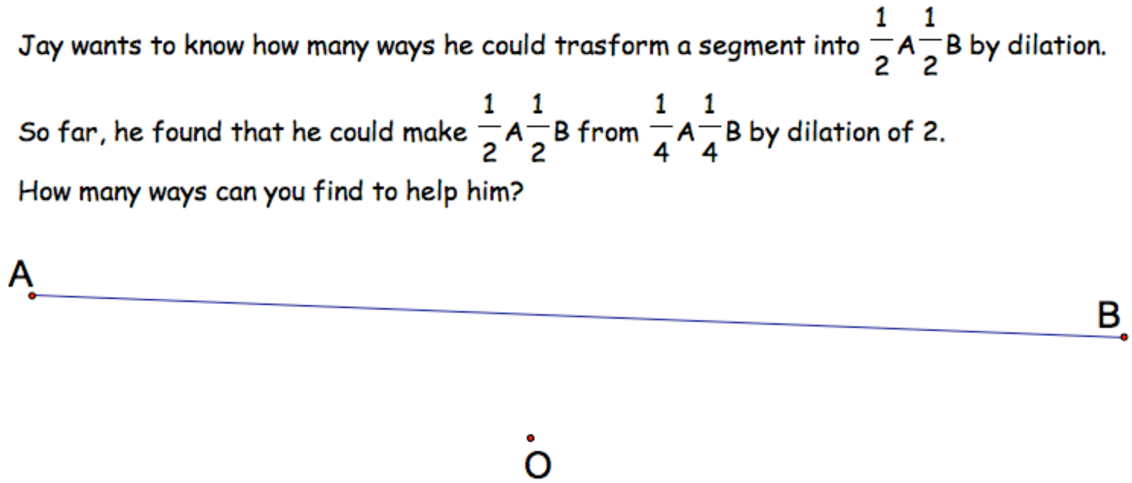


Figure 4.26: A dilation problem related to recursive partitioning operations involving 1/2

In contrast to the previous problem, both students flexibly constructed a 1/2A1/2B segment by dilation in various ways. In order to make a 1/2A1/2B segment from a 1/4A1/4B segment using dilation, Carol identified 1/2A1/2B as two units of 1/4. Further, she also knew that 1/2A1/2B was equivalent to four units of 1/8 when positing and then transforming 1/8A1/8B. The latter case was possible by her use of a three-levels-of-units structure using one whole as a given material. That is, she was able to construct a referent whole segment consisting of two units of a 1/2A1/2B segment and at the same time 1/2A1/2B as four units, each of which can be iterated eight times to make the re-presented referent whole. Further, as implied by her comments, “And 1/8 (1/8A1/8B) you have, you have to have four eighths pieces of AB, which equal to 1/2AB. And then it goes on and on.” she constructed a class of fractions equivalent to one-half. On the other hand, Rosa found a pattern for her dilation activity; “If you have even number denominator with the numerator of one, you can divide the denominator by two and take that number and multiply it by that fraction.”, which meant that she abstracted her dilation patterns for construction of 1/2A/12B from various (unit) fractions. Although Rosa’s explanation relied on manipulation of numerals in fraction multiplication calculation, I would attribute to her an interiorization of her recursive partitioning operations involving 1/2 because she also understood Carol's explanation very quickly.

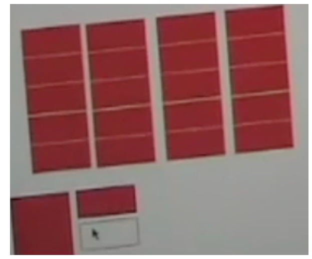
In sum, although the situation might be restricted to the specific case involving a unit fraction, '1/2', both students demonstrated that they were able to take results of recursive partitioning operations for granted for their transformation activity. Nevertheless, whether they could use such a three-levels-of-units structure based on recursive partitioning operations as given prior to activity in general was yet to be investigated.

Carol's Distributive Partitioning Operation and Rosa's Common Partitioning Operation in Transformation Between Two Unit Fractions

The last problem in the teaching episode on Mach 23rd of 2009 was to transform a unit fraction [1/14] into another fraction [1/4] in GSP. Although only one minute was allowed for the students to think about the problem because of a time constraint, they experienced difficulty when dealing with the problem situation. Rosa was still inclined to utilize numerical calculation for fraction multiplication and Carol did not seem to find a clue to cope with the problem. Finding a scale factor for a multiplicative transformation between the two unit fractions in GSP seemed to require too abstract mathematical reasoning for Carol and Rosa. They needed anticipatory schemes [whatever the schemes might be] that they could execute in re-presentation prior to activity. Therefore, I prepared the JavaBars program as well as GSP to use for the next teaching episode because I believed that the use of JavaBars could provoke their construction processes step by step and thus help the students to figure out a scale factor to be used in GSP.

The first problem for the very next teaching episode on March 30th of 2009 was originally 'Given a 1/4-meter segment and a 1/10-meter segment, find a scale factor to transform the 1/4-meter segment into the 1/10-meter segment using dilation in GSP.' However, as being expected, when I posed the transformation problem, both students had a hard time solving it by using dilation. Even though Carol and Rosa were familiar with the effect of dilation through several teaching episodes, say, a transformation of a 2-meter segment into a 1/3-meter segment through dilation by one-sixth, they seemed stuck with finding one number that they could use to dilate 1/4-meter into 1/10-meter. They just seemed to guess a scale factor based on visual comparison between lengths of the two segments and put several numbers in the dilation box by trial and error. For Carol and Rosa, the GSP environment seemed too abstract to anticipate whether their activity would produce the desired result. Even though I encouraged them to dilate the given 1/4-meter segment in two steps to make a 1/10-meter segment (from 1/4-meter to one meter and from one meter to 1/10-meter) they could not coordinate their two-step dilation by finding one number (4/10) to be used in one-step dilation to get from 1/4-meter to 1/10-meter. Therefore, I had no alternative but to intervene and suggest that they turn to JavaBars and construct a 1/10-meter bar from a 1/4-meter bar. At that moment, I hoped that the construction activity using JavaBars could help them to find a scale factor for dilation in GSP by reflecting on their construction processes in JavaBars.

> Protocol 4.25 on 03/30/09: Transforming a 1/4-meter bar into a 1/10-meter bar in JavaBars.
>
> (The problem is “ Using JavaBars, make a bar and pretend that is a 1/4-meter bar. Transform the 1/4-meter bar into a 1/10-meter bar.” Carol makes a 4-part bar for one-meter from a 1/4-meter bar by copying three more pieces of the 1/4-meter bar and joining them with her original 1/4-meter bar.)
> C: So we need to make this into one-tenth?
> T: Um-hm.
> C: (Carol partitions each of the four parts of 1-meter bar into ten parts so that she has a 40-part bar.) Okay. So then I just take one of... (Carol pulls out one out of ten parts from each of four 10-part sections. See Figure 4.27a)



Figures 4.27a & 4.27b: Carol's (Left) & Rosa's (Right) constructions for transformation from a 1/4-meter bar to a 1/10-meter bar

R: (Rosa already made four separated bars for one meter from her original 1/4-meter bar by copying it four times and then divided each of four 1/4-meter bars into five parts.) Okay, it's just twenty. So I pull out... (Rosa pulls out two parts from one of four 1/4-meter bars, which amounts to 2/20-meter. See Figure 4.27b)
R: Okay, I think I got it.
C: I think I did it backward so it's long.
T: All right. So who will explain first?
R: Okay, what I did is I have this is one-fourth (her original bar), and so I copied it.
T: Is that your original?
R: Yes, this is my original. I copied four to make one meter and since four isn't divisible, ten isn't divisible by four, I went up to twenty. And I make five pieces out of one-fourth, so two of those pieces gonna equal one-tenth.
C: Right, I did mine backwards.
R: What did you do?
C: I didn't get any answer.
T: (To Rosa) I cannot follow you. Can you say that again?
R: Okay, this is my original bar and I copied it to four to equal one meter. And because ten isn't divisible by four because you will get a decimal I went up to twenty because four times five is twenty and so instead of pulling out one-twentieth, I pulled two-twentieths, which reduce down to one-tenth.
T: Oh, got it. (To Carol) what about yours?
C: I didn't do mine. I wasn't. I didn't get it.
T: You didn't get it?
C: Yeah, I understand how she did it but I wasn't sure what I did.
T: What, what's your solution? It seems to be okay.
C: I don't really have...
R: Did you do forty?
C: Yeah.
R: That's what I was thinking first.
T: What were you trying to do? Just, just explain [to] us what you did.
C: Well, I took the four pieces which is the one meter and I divided them each into ten. Then I took out one from each and it's four-tenths or four-fortieths which is same as one-tenth.
T: Why did you pull out one from each?
C: Because four-fortieths equals one-tenth.

With a 1/4-meter bar, Carol copied three more 1/4-meter bars and joined them with her original bar to make a 4-part bar of one-meter. Then to make a 1/10-meter bar [to take one-tenth of the 1-meter bar], she partitioned each part of the 1-meter bar into ten parts and pulled out a part from each of her 4-part bar. Therefore, for Carol this was another situation in which she needed to use her distributive reasoning. Although she was embarrassed by Rosa's relatively simple solution [to partition each 1/4-meter into five, not ten] Carol was explicitly aware that taking one-tenth of each 1/4-meter bar would make one-tenth of the whole 1-meter bar by combining them.

In terms of Rosa's mathematical operations, I already argued, in the situations of previous teaching episodes (cf. Protocols 4.14 and 4.16) that might require a student's distributive reasoning, that the nature of Rosa's partitioning activity had no clear goal and it was just an imitation of Carol's distributive partitioning activity in a different way. However, this protocol indicated that Rosa actually had assimilated Carol's distributive partitioning operation and modified it into her own partitioning operation. In order to transform a 1/4-meter bar into a 1/10-meter bar, Rosa copied her 1/4-meter bar three times and arranged them as a similar way as Carol did. Then she independently partitioned each of her 1/4-meter bars into five parts and pulled out two parts from one of the four 1/4-meter bars (see Figure 4.27b). Unlike Rosa's partitioning activity in the previous episodes, the partitioning activity that she demonstrated in this protocol seemed to have a clear goal. As indicated by her comments, "because ten isn't divisible by four because you will get a decimal I went up to twenty because four times five is twenty and so instead of pulling out one-twentieth, I pulled two-twentieths, which reduce down to one-tenth," Rosa was able to conduct her units-coordinating operations at three levels of units, which was a coordination of two iterable composite units. That is, she was able to find a greater number of parts [twenty] that was one of the multiples of both of the given partitions [four parts of 1/4-meter and ten parts of 1/10-meter.] Her coordination of two three-levels-of-units structures based on her GNS, that is, to see one meter as four units of five [1/4-meter] and simultaneously ten units of two [1/10-meter] enabled her to partition each of the 1/4-meter bars into five parts. By using her iterative fraction scheme for connected numbers, she could pull out two parts for one-tenth of one meter consisting of twenty parts. Since there was no evidence that Rosa's one-tenth of one meter was a collection of one-tenths from each of the four 1/4-meter bars, which could be regarded as a distributive partitioning operation, Rosa's partitioning operation should be distinguished from Carol's distributive partitioning operation. Therefore, I would attribute Rosa's partitioning operation to a *common partitioning operation for unit fractions* in the sense that she produced a comeasurement unit, 1/20 of one meter, so that both 1/4-meter and 1/10-meter are multiples of the unit fraction (Olive, 1999).

Rosa's Construction of Multiplicative Relationship Between 1/5 and 7/5

The teaching episode on April 1st of 2009 was the second meeting with Rosa alone because of Carol's unexpected absence. Therefore, the teaching episode gave us one more chance to wholly focus on Rosa's mathematical thinking in relation to fractional knowledge.

> Protocol 4.26 on 04/01/09: Transforming 1/5-meter into 7/5-meter and finding the scale factor for dilation.
>
> T: Make a bar and at this time this is labeled by one-fourth.
> R: One-fourth?
> T: No, sorry. One-fifth. (Rosa puts '1/5' on the bar using LABEL tool.) And based on that bar, you should construct seven-fifths meter of a bar. (Rosa copies seven pieces of her one-fifth bar and joins them together.) Okay. You got it. In terms of dilation, can you explain to me?
> R: Um... Because you're starting a number smaller, you would have to dilate, well I guess you just dilate by seven-fifths. Because if you do the reciprocal as five-fifths and anything more is seven-fifths. So I'm guessing that you just... or seven over one.
> T: Seven over one. Or seven over one? What's your final answer?
> R: Um...

T: What do you think you have to dilate by to make one-fifth into seven-fifths?
R: I'm thinking seven over one.
T: Seven over one? Okay, we can check in GSP. (Given one meter segment on the screen in GSP, Rosa makes a 1/5-meter segment dilating by one-fifth from one meter.) What number did you say you dilate by to make one-fifth into seven-fifths?
R: I'm thinking seven over one because seven... times one, no, one-fifth times seven over one is gonna be seven-fifths.
T: Seven-fifths? So...Let's try it.
R: (Rosa DILATES one-fifth by '7/1' and a little longer segment than 1-meter segment appears a little above the 1-meter segment.) Oh, it's bigger than one. That's right. Isn't it?
T: Yeah, did you say seven, seven over one?
R: Um-hm.
T: I heard seven-fifths, right?
R: No. Seven over one. Seven over one times one-fifth.

Although Rosa' demonstrated a lack of a reversible iterative fraction scheme (cf. Protocol 4.20), it was surprising that Rosa experienced a minor confusion when finding a scale factor to transform a 1/5-meter bar into a 7/5-meter bar after completion of construction of the two bars [1/5-meter and 7/5-meter] on the screen using JavaBars. At the time, the overarching goal of our teaching experiment was to investigate what mathematical operations a student might need or produce in the process of transformation of a fraction into another fraction. The problem to transform 1/5-meter into 7/5-meter in this protocol was posed as sort of a warm-up question that was supposed to be used as an example to explain what the problems were about in the meeting. Unlike the problem in the Protocol 4.20 where Rosa had to use an interiorized multiplicative relationship of a unit fraction [1/11] with [19/11] in a fraction multiplicative situation, the present problem clearly demanded Rosa to construct a 7/5-meter bar based on a 1/5-meter bar. We also knew that Rosa was familiar with the effect of DILATION option in GSP and identified the function of DILATION with that of multiplication operation.

However, surprisingly, Rosa could not immediately find a scale factor to transform a 1/5-meter bar into a 7/5-meter bar. Although she made seven copies of her original 1/5-meter bar and joined them to construct a 7/5-meter bar right before, her first answer was seven-fifths, not seven. My conjecture is that such a temporary confusion means that her first conception of seven-fifths did not stand in for seven units of one-fifth, each of which can be iterated seven times to make seven-fifths. Even though she corrected her answer right after her confusion as seven over one, the initial error indicated that to interiorize a multiplicative relationship between a unit fraction and one whole was not trivial for Rosa. Her comments, "if you do the reciprocal as five-fifths and anything more is seven-fifths." implied that she was still inclined to assimilate seven-fifths as one and two-fifths based on the additive relationship with one whole. Moreover, when a 7/5-meter segment appeared in GSP as a result of her dilation of a 1/5-segment by seven, Rosa's comments, "Oh, it's bigger than one" indicated that the result was an unexpected construction, which meant that her conception of 7/5 was yet to be interiorized in relation with 1 and 1/5.

Nevertheless, Rosa demonstrated that she has constructed a multiplicative relationship between one-fifth and seven-fifths in action by herself. However, it was to be investigated whether this simple activity would help Rosa construct a multiplicative relationship of unit fractions in relation to other (especially improper) fractions in general and, further, interiorize such an iterability of unit fractions as given material for further mathematical activities like fraction measurement division.

Rosa's Lack of Interiorization of Recursive Partitioning Operations Involving an Improper Fraction

> Protocol 4.27 on 04/01/09: Transforming 7/5-meter into 1/15-meter and finding a scale factor for the transformation.
>
> T: Let's go to the next problem.
> R: Okay, seven-fifths to one-fifteenth? (Rosa draws a bar, puts '7/5' on the bar using LABEL and copies one more bar under her original bar.) Okay, you want one-fifteenth? You divide that by seven and then I'm going to fill... (Rosa partitions her copied bar into seven parts and fills five parts with a blue color from the left.) Okay, one-fifteenth? Then,

twenty one...over fifteen. Okay this is twenty one over fifteen. (Rosa labels the 7-part bar as '21/15') Okay, and one-fifteenth. Since fifteen over fifteen... (Rosa partitions each of the five blue parts into three, and then pulls out one smallest piece from the leftmost part of her 7/5-meter bar. See Figure 4.28) Okay.

Figure 4.28: Rosa's construction for transformation of a 7/5-meter bar into a 1/15-meter bar

T: Okay? Is that... What is that, is that one-fifteenth? Okay. Yeah, can you explain to me?
R: Okay. Um... I pulled out that, I made the denominators same. And so seven-fifths is twenty-one over fifteen, and here (pointing out blue parts) fifteen, this is one and I just pulled out one-fifteenth out of these.
T: Okay. All right. Good. Then what number do you need to dilate by to make?
R: You need to dilate by...twenty over fifteen... Um... Hm... Twenty-one over one?
T: Twenty-one over one. So, if you dilate by twenty-one over one, dilate this one (Rosa's original bar) twenty-one over one, make this one (one blue part)? Is that what you mean?
R: I think so. Yes.
T: Twenty-one over one. So if you dilate this one (Rosa's original bar) by twenty-one over one, does it increase or decrease?
R: Oh, I'm sorry. Then it's gonna have to be one over twenty... No. Hm... Because I know if you dilate that by... Hm...What number... One-fifteenth. Hm... it seems like I'm doing it in a complicated way. Um... it's gonna have to be... one over thirty-five?
T: Thirty-five? Why, why do you think?
R: Seven times one is seven and... five times seven is thirty-five. If you divide this by seven, it's gonna equal one-fifth, that's three-fifteenths. Okay, forty-two thirtieths... Hm... One-fifteenth. (Twenty seconds passes.) Um... one-third?
T: So if you take one-third of this (Rosa's original bar), what will be, can you guess the bar?
R: No, that's not it. It's gonna be seven over fifteen. (Twenty seconds later) wow!

Rosa knew that 7/5 was same as 21/15 through numeric calculation as she usually did in previous teaching episodes. Also, she demonstrated that she was able to use the result of her recursive partitioning operations as given material to establish a quantitative relationship among 1/15, 1/5, and 1 as indicated by her partitioning operation of each of the blue 1/5-meter parts into three in order to construct a 1/15-meter bar (cf. Figure 4.28). That is, she had already constructed a three-levels-of-units structure with 1/15, 1/5, and 1 with an awareness that 1/5 consists of three units of 1/15, each unit of which can be iterated fifteen times to make one meter. This was possible by the modification of her GNS with one, three, and fifteen into a fractional context involving two unit fractions [1/5 and 1/15].

However, when I asked her to find a scale factor to dilate 7/5-meter into 1/15-meter, she did not seem to realize that one twenty-first of 7/5-meter was 1/15-meter, even though she visually constructed a bar for 1/15-meter from her original 7/5-meter bar using JavaBars. My conjecture about her unawareness of the relationship was corroborated by her hesitation indicated by the comments "Then it's gonna have to be one over twenty...no. Hm... Because I know if you dilate that by... Hm... What number... One-fifteenth. Hm... it seems like I'm doing it in a complicated way. Um... it's gonna have to be... one over thirty five?"

I conjecture her difficulty in finding a scale factor for such a transformation activity was possibly due to her 1/15-meter bar being actually constructed only in relation to a one-meter bar [15/15-bar], rather than to her original 7/5-meter bar [21/15-bar]. At first, Rosa made a bar for 7/5-meter and partitioned it into seven parts. However, such partitioning operations were not carried out based on the multiplicative relationship between 1/5-meter and 7/5-meter, that is, 7/5-meter consists of seven units of 1/5-meter, each of which can be iterated seven times to make 7/5-meter. Rather, the intention of her initial partitioning operations was to indicate one-meter part by filling out five parts [one meter] out of seven partitioned parts with a different color. Her second partitioning activity of only five blue parts for one meter into three parts corroborated that the meaning of one-fifteenth was initially limited as one out of fifteen parts of one whole [15/15], not a part from seven-fifths consisting of twenty one parts of one-fifteenth. I conjecture that in order for her to provide an answer immediately, she should have expanded her three-levels-of-units structure with 1/15, 1/5 and 1 to the structure with 1/15, 1/5, and 7/5 with an awareness 1/5 consists of three units of 1/15, each of which can be iterated twenty one times to make 7/5.

Actually, Rosa made a bar for 7/5-meter and partitioned it into seven parts to find one whole. Such mathematical behaviors of students are usually interpreted as the construction of a reversible iterative fraction scheme for seven-fifths, which means that the students constructed a three-levels-of-units structure with 1/5, 1 and 7/5. In other words, establishment of a multiplicative relationship between 1/5 and 7/5 is attributed to the students, so that any 1/5 can be iterated seven times to make 7/5, with an awareness that five units of 1/5 make one whole. This protocol, however, implies that a student's construction of three levels of units does not necessarily mean that the student is able to use the three-levels-of-units structure as given material prior to further mathematical activity. The structure of Rosa's three-levels-of-units with 1/15, 1/5 and 1 failed to expand to include 7/5 because of her additive assimilation of 7/5. Rosa kept assimilating seven-fifths as one and two-fifths, not multiplicatively as seven units of one-fifth, each of which consists of three units of one-fifteenth. Therefore, being able to coordinate and use two different kinds of three-levels-of-units structures [one from interiorization of recursive partitioning operations based on GNS and the other from construction of an iterability of a unit fraction based on FCNS] was key to opening the path to construct a scheme for a multiplicative transformation from 7/5 to 1/15.

Students' Conception about Equivalent Fractions

In the teaching episode held on April 13th of 2009, I posed similar problems to the problems used in the previous teaching episode because Carol missed the previous teaching episode. When the students were asked to find a scale factor to transform a given 7/8-meter segment into another given 7/4-meter segment using dilation in GSP, they easily found the answer, '2' by manipulation of numerals for fraction multiplication. That is, they argued that they could get seven-fourths by multiplying seven-eighths by two because seven-fourths equals fourteen-eighths. Although they were very confident in their answers, I wanted them to confirm their answer by constructing a 7/4-meter bar from a 7/8-meter bar in order to investigate whether their numeric calculation actually symbolized their construction of the equivalent relationship between 7/4 and 14/8 based on interiorization of the result of recursive partitioning operations; that is, by using the three-levels-of-units structure with 1/8, 1/4, and 1 as a given material to conceive 7/4-meter as seven units of 1/4-meter, each unit of which consists of two units of 1/8-meter that can be iterated fourteen times to make 7/4-meter.

Rosa made a 7-part bar for 7/8-meter and pulled out a part from the 7-part bar to arrange it beside the 7-part bar. Her aim seemed to be to visualize a one-meter part. Then she copied her original 7-part bar twice and joined them together as an answer. At that time, Rosa seemed to be aware of three levels of units [1/8, 1/4, and 1] in her construction (see Figure 4.29b[32]) in that she conceived of 7/4 as twice 7/8. On the other hand, Carol made an 8-part bar, partitioned each part of the 8-part bar into two and then pulled out one of the two parts from each part of the 8-part bar to join them together as an answer (see Figure 4.29a). She explained her construction upon my request; she exclaimed that she should have made a 7-part bar and then doubled her original 7/8-meter bar rather than halving it. On her second attempt, she constructed a 7-part bar for 7/8-meter and made two copies for construction of 7/4-meter. The Protocol 4.28 began when Carol was labeling her result bar as '7/4-meter.'

[32] Note that Rosa labeled her result bar as '14/8', not as '7/4'

Figures 4.29a & 4.29b: Carol's (Left) and Rosa's (Right) first constructions of a 7/4-meter bar (below) from a 7/8-meter bar (above)

Protocol 4.28 on 04/13/09: Labeling a 7/4-meter$_{14}$ bar.

(Carol labels her 14-part bar as '7/4 meter')
R: Seven...
C: Fourths? No...
R: Yes.
C: Yes.
R: No. No.
C: No. I have to half that. No, not half it.
R: No, just put fourteen over eight cause you took these (pointing out Carol's original 7/8-meter bar), all each one-eights. So there is no way you can make it one-fourth unless you...
C: Wait, but then they (7/4-meter$_{14}$ bar) wouldn't be seven-fourths, would it?
R: It's eighteen over, I mean fourteen over eight.
C: Fourteen over eight, but... It's basically the same thing.
R: No, it's not because you have to half it (pointing out Carol's original 7/8-meter bar.)
C: You can get fourths out of that, so...
R: You have to half it and pull these out.
C: Right, but if you look at the bar, not the lines.
R: Um, no. Because there is no way one-fourth can be the same size. That's what I did on the last one.
C: I know, I know. But if you pretend those bars aren't there, aren't there then... You are just looking at the bar, not the lines. You see? (Rosa answers in the negative.) I was labeling in the bar. I was ignoring the lines.
R: (inaudible) You put four-sevenths?
C: Yes, cause I'm talking about the bar.
R: I don't know.
C: Not the bar with the line.
T: So, you seem to be disagreeing with something right?
C: Huh?
T: Disagreeing with something.
R: No, it's the same fraction. It's just... The thing I'm confused about is that the seven-fourths pieces are same as the one-eighth pieces.
C: I'm not talking about the pieces though.
R: I know, I know the pieces, it's just the... But, you're right. You're right.
C: Talking about the bars whole.
R: Um-hm. If you just reduce it. Um-hm. I mean she's right.

T: (To Carol) what was the final labeling, one... What was that? One and what?
C: No, that's fourteen.
T: Oh, fourteen-eighths. At first, you labeled that as seven-fourths, right?
C: Right.
T: Okay. Actually based on eight-sevenths[33] you copied one more and joined each other right?
C & R: Um-hm.
T: Basically you did the same. But your naming is, at first your (Carol's) naming was seven-fourths. And your naming (Rosa's) is fourteen-eighths. Is that different?
R: I mean the fraction is the same.
C: It's the same thing.
R: The fraction is same. It's just the size of the... Because this (Rosa's 7/8-meter bar) is in eighths, not in fourths.
T: Right, that's what I'm asking. In terms of size is that different [from] each other?
R: No. I mean how she divided it. She has to combine like these two (Carol's two 1/8-meter bars) would have to equal one. Like these (Rosa's two 1/8-meter bars) equal one. So it would be one, two, three, four, five, six, seven. (Rosa counts the number of parts of the 7/4-meter$_{14}$ bar by two from the left.)
T: So, in your construction, okay, in your construction, let's assume that your construction (Carol's) is seven-fourths meter and your answer (Rosa's) is fourteen-eighths meter. And in your construction, can you fill the color for one meter bar?
R: One meter bar? I see what's you're saying, Carol. (Rosa fills eight parts out of her 7/4-meter$_{14}$ bar.) What you were saying... if that the fractions are the same, it's just bar size gonna be different. Instead of being like um... Cause this (one part of her 7/4-meter$_{14}$ bar) is one-eighth, one-eighth, one-eighth, one-eighth. So if you take these two and combine them together, then that way she could have seven-fourths. I mean the fraction is the same but the size of the pieces would be different because one-eight is smaller, and one-fourth would be bigger. But the fractions are the same. It's just how you divide the bars.

It was surprising but very interesting to observe their struggles when they were naming their resulting 7/4-meter bar. Although the problem explicitly denoted that the final result should be 7/4-meter and they were actually aiming for construction of a 7/4-meter bar from a 7/8-meter bar, they hesitated to give the final result of a 14-part bar the name '7/4-meter.' Their confusion seemed to come from manipulation of numerals in whole number multiplication. That is, Carol's comment, "No. I have to half that" and Rosa's comment "No, it's not because you have to half it (pointing out Carol's original 7/8-meter bar.)" indicated that they somehow felt a necessity to

[33] The teacher-researcher (I) accidently said "eight-sevenths" but it should have been "seven-eighths."

double a quantity, which seemed to lead them to divide each part of the 7/4-meter$_{14}$ bar into two parts in order to increase the number of parts in the 7/4-meter$_{14}$ bar. However, Carol realized that she actually had a correct construction for 7/4-meter. Her comments, “I was labeling in the bar. I was ignoring the lines.” implied that she correctly conceived two units of 1/8-meter as one unit of 1/4-meter so that she could consider fourteen parts of 1/8-meter as 7/4-meter. Her consistent confidence for her argument demonstrated that she constructed a structure of 7/4-meter on the basis of three levels of units [1/8, 1/4, and 1]. Further, it also means that her construction was framed on another three-levels-of-units structure for improper fractions [1/4, 1, and 7/4 or 1/8, 1, and 14/8] because she might not be able to see 7/4-meter as seven units of 1/4-meter, each of which consists of two units of 1/8-meter that can be iterated fourteen times to make 7/4-meter without construction of such three levels of units for conception of the improper fractions.

Rosa’s struggle lasted longer than Carol’s. Even with Carol’s explanation, she argued that the 7/4-meter$_{14}$ bar should be named as ‘14/8’ because it had fourteen parts. For Rosa, the most important factor was the number of parts in the bar and she did not seem to realize that the size of the bars should be constructed based on the referent bar [actually she constructed a one-meter bar for her reference] regardless of the number of partitions on the bar. When I asked them to fill out a one-meter part in their own 7/4-meter$_{14}$ bar, she finally seemed to understand Carol’s explanation and rephrased it in her own word, “I mean the fraction is the same but the size of the pieces would be different because one-eighth is smaller, and one-fourth would be bigger. But the fractions are the same. It’s just how you divide the bars.” In sum, although it should be claimed that their production of 14/8 as commensurate to 7/4 was the result of their mathematical actions and operations, it revealed that the ability to take for granted the results of recursive partitioning operations (especially in the context of a situation involving an improper fraction) would be one of the critical factors (or an obstacle) in their transformation of a fraction into a commensurate fraction.

Test for Construction of Three Levels of Units for Commensurate Fractions Involving an Improper Fraction

Since the ability to use three levels of units based on recursive partitioning operations as given emerged as one of the important factors in the students' transformation activity, I posed a problem to test whether their construction of three levels of units in the previous protocol was stable and permanent. When I asked them to construct an 8/8-meter bar, they easily made an 8-part bar. Then when I also asked them to construct a 4/4-meter bar, with no hesitation, both students claimed that a 4/4-meter bar would be same as the 8/8-meter bar. Encouraged by the students' quick responses, I posed one more comparison problem between a 3/2-meter bar and a 6/4-meter bar.

Protocol 4.29 on 04/13/09: Comparing a 3/2-meter bar and a 6/4-meter bar.

(Both students have a one-meter bar with no partition on their own screen.)
T: Using one meter bar, can you make a... three-halves bar?
C: Three...
T: Just leave original bar. Three-halves meter bar.
C: Three over two?
R: Three over two? Okay.
(Both students make a copy of their original bar, partition it into two and pull out a part from the bar. Then they join it with their copied one-meter bar.)
T: Okay, I think you constructed in the same way. So, you made, based on one meter bar, you made [a] three-halves--three over two meter bar--right? So, my question is, if you make, based on this meter bar, your one-meter bar (pointing out one meter bar on each screen) if you make [a] six-fourths meter bar...
R: Um-hm. You just double this.
T: Would it be larger or smaller than your three over two meter bar?
C: It would be the same.
R: It would be the same.
T: Would be the same or larger?
C: Same.
T: Same? (To Rosa) what's your opinion?
R: The fraction is the same, but you're going to have to double this (3/2-meter bar.)

T: I mean in terms of size.
R: Bigger.
T: Bigger? What's your opinion?
C: Well, couldn't it... if you started from that...
T: Start from one-meter bar.
C: It would be the same as that because you're just dividing the same, the original bar is getting bigger. You just divide into four pieces and then you have the six pieces of one.
R: It really depends on how you divide this bar (3/2-meter bar.) Because see, how she divide it into three pieces, she is gonna have to divide that (1/2-meter part) by half.
C: And then you get the same thing.
T: Okay, then let's construct it, let's construct it. Based on your one meter bar,
C: Do you want us to erase this one?
T: No. Just leave it and based on your one-meter bar, would you construct six-fourths meter bar? Using your one-meter bar.
(Carol divides her one-meter into four parts, pulls out two parts from it and then joins them together. Rosa joins a one-meter bar, which has no partition, with two 1/4-meter bars.)
T: What was your conclusion? Is it larger or smaller in terms of size?
C: The same.
R: It's the same size, but how I was saying, like you see, you have three here, like I was just thinking it double.
C: No, because you are not doubling. Now you start from there. You pretend that was not in there. You just start (inaudible).
R: Oh, okay. Mine is the same size, just the same fraction.
T: So do you agree as [they are the] same? Larger or smaller? What's the final?
C & R: Same.
T: Same. Good.

Carol, with no hesitation, exclaimed that a 6/4-meter bar would be the same size as the constructed 3/2-meter bar. That was a confirmation that her construction of three levels of units for two equivalent fractions [in this protocol, 6/4 and 3/2] on the basis of the result of recursive partitioning operations became a permanent mathematical conception. On the other hand, Rosa's first answer was bigger, which means that the 3/2-meter bar should be doubled for construction of a 6/4-meter bar. As expected, her conception of such three levels of units was unstable. Rosa had yet to interiorize the result of recursive partitioning operations to see the equivalent relationship between 3/2 and 6/4. That is, she could not see the three-levels-of-units structure with fractions: 1/4, 1/2 [consisting of two 1/4s] and 3/2 [consisting of six 1/4s, each of which can be iterated six times to make 3/2]. However, unlike the previous protocol where Rosa did not establish 14/8 as commensurate with 7/4, she easily assimilated Carol's explanation while still insisting that each of 1/2-meter parts of the 3/2-meter bar was to be divided into half. So, I judged that Rosa demonstrated progress in her use of the three levels of units. It is possible that she will be able to take three levels of units as given prior to activity in further tasks like the ones in the last two protocols because her assimilation was relatively faster in the latter than in the former and she correctly constructed a 6/4-meter bar based on the given one-meter bar by herself. However, it turned out that my conjecture was not true when I investigated whether the students constructed reversible recursive partitioning operations in the teaching episode on April 24th, which will be described later.

Rosa's Evoking a Unit-Segmenting Scheme in Transformation between Two Unit Fractions

When I posed a similar problem [finding a scale factor to transform 1/4-meter into 1/10-meter] in the teaching episode held on March 30th, the GSP environment seemed too abstract[34] for the students. Thus, I encouraged them to carry out their transformation activity in JavaBars. As a result, we were able to observe their transformation of 1/4-meter into 1/10-meter in their own ways [Carol's distributive partitioning operation and Rosa's common partitioning operation]. Therefore, the problem in the present protocol held on April 15th [finding a scale factor to transform 1/7-meter into 1/3-meter] was posed in the GSP environment again with the anticipation that such mathematical reasoning, indicated in JavaBars in the previous teaching episode (cf. Protocol 4.25), could emerge also when the students used GSP.

[34] What I mean by 'abstract' is that the students' mathematical operations need to be conducted in their re-presentation prior to actual action.

Protocol 4.30 on 04/15/09: Finding a scale factor for transformation of 1/7-meter into 1/3-meter.

T: We will do, we'll continue on the same type of problem. So, we'll start from the GSP. The first problem would be, transform a 1/7-meter bar into a 1/3-meter bar. So, at first just think about it in your mind rather than doing something. What you can do is just, click to show one meter and check the location and the length of one meter (see Figure 4.30).

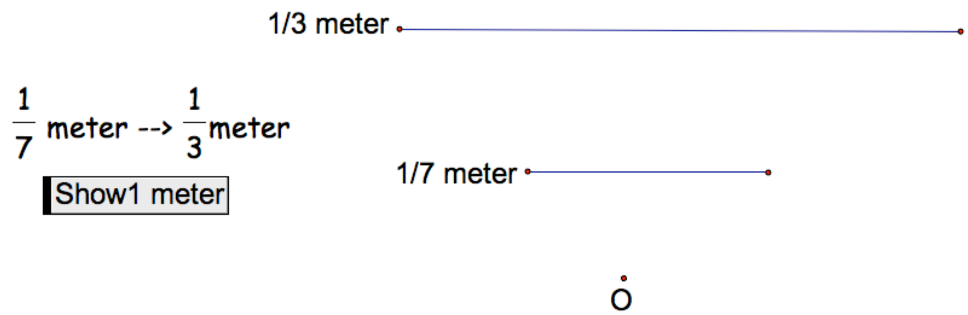
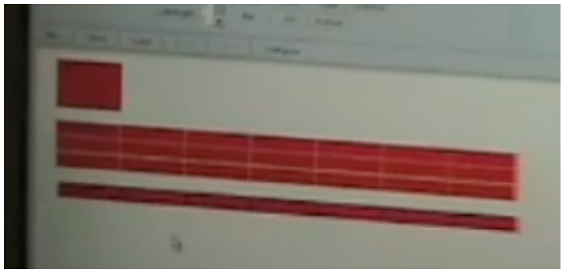
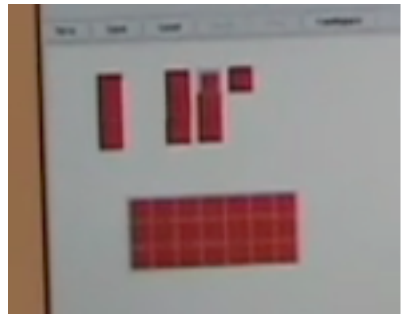


Figure 4.30: Transformation problem from a 1/7-meter segment into a 1/3-meter segment in GSP

C: Three-twenty one and Seven-twenty one...
R: Do I show it (1-meter bar, which is hidden now)?
T: Yeah, you can do that, but just for a minute, just think about [it] in your mind. What would be the... would be the answer for dilation?
R: Is it five-twenty one?
T: Five twenty-one?
R: No, no. Because this is three-twenty one and that is seven-twenty one.
C: Yeah, I wrote over that down. That's what I was thinking.
R: So how do you get to it? Two and one-third?
C: I think two and one-third.
R: Yeah, six times two and then one, and then half.
C: Because there is six, yeah, three and three is two and then there is one-third I thought.
T: Two and one-third.
R: Um-hm. That's what I got.
T: (To both) do you agree?
R: Yeah, that's what I said.

The GSP environment still seemed too abstract for them to directly find a scale factor based on their partitioning activity observed in the Protocol 4.25. Nevertheless, their ways of operating were quite intriguing. Rosa's first answer was 5/21 after she converted 1/7 into 3/21 and 1/3 into 7/21. She seemed to look for the middle value additively [the arithmetic mean of the two fractions from my point of view]. When I encouraged them to reflect their transformation processes without actual action, Rosa suddenly suggested two and one-third as an answer and Carol immediately agreed with Rosa's answer. As indicated by Rosa's comment, "Yeah, six times two and then one, and then half" and Carol's comment, "Because there is six, yeah, three and three is two and then there is one-third I thought" it was clear that their answers were the results of provoking their fractional unit-segmenting schemes. Although Carol did not independently initiate her fractional unit-segmenting scheme, her comment described above was an indication of her assimilation of the problem as a situation for her fractional unit-segmenting scheme. Regardless of both students' quick solutions, I encouraged them to use JavaBars for the transformation activity because their evoking of fractional unit-segmenting schemes was totally unexpected.

In JavaBars, their mathematical behaviors were as I expected. Carol made seven copies of her original 1/7-meter bar and joined them with a label of '1 meter bar.' Then she immediately partitioned each of seven parts of 1-meter bar into three pieces and pulled out one piece from each 1/7-meter part (see Figure 4.31a). Such partitioning activity was a confirmation of her distributive partitioning operation that she consistently demonstrated during the teaching experiment. Rosa also went through the same processes. She copied her original 1/7-meter bar seven times, joined them together and then partitioned each of seven parts of 1-meter bar into three pieces to pull out a piece from each 1/7-meter part (see Figure 4.31b). However, her comments, "One whole bar is one-third of the one-seventh because I divided it into twenty one pieces and I pulled out seven and that's three..." implied that her partitioning activity was a common partitioning operation rather than a distributive partitioning operation.

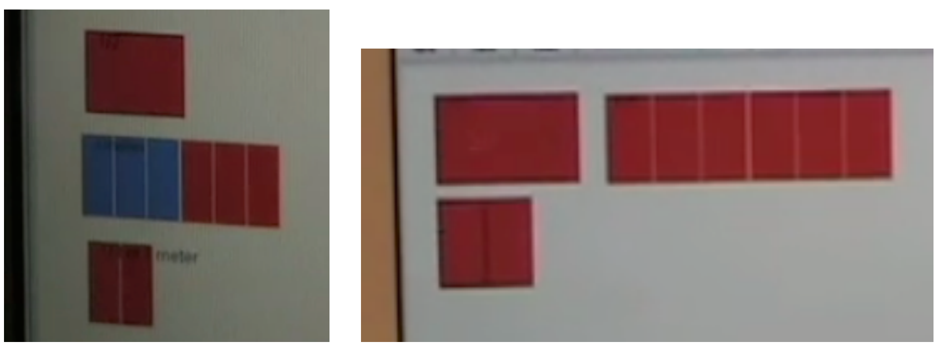
 

Figures 4.31a & 4.31b: Carol's (Left) & Rosa's (Right) transformations of a 1/7-meter bar into a 1/3-meter bar

Carol's Construction of a Multiplicative Scheme for a Transformation Between Two Unit Fractions

As reported in the previous protocol, both students' evocation of their fractional unit-segmenting schemes was not expected. Since the overall goal of the teaching experiment at the time was to help the students become explicitly aware of a multiplicative operator to be used in their transformation activity--a two-step transformation via a whole referent or a comeasurement unit fraction--I had to find another way to provoke their engagement in such transformation activities, in which there was an awareness of the multiplicative operator. Therefore, although the next problem was supposed to be a transformation problem from 1/3-meter to 1/2-meter, I changed the direction of the transformation to transform 1/2-meter into 1/3-meter on the spot. The aim of such a change was to constrain the evocation of their unit-segmenting schemes in the context of transformation activity because they demonstrated difficulty in assimilating a problem as a situation of their unit-segmenting schemes in a decreasing situation, that is, a situation of measuring a smaller quantity with a larger quantity (cf. Protocol 4.6). Due to such an aspect, if the students assimilated the transformation problem of 1/2-meter into 1/3-meter as a situation of their fractional unit-segmenting scheme again like in the Protocol 4.30, they might encounter a perturbation similar to the one they had before, which might lead them to evoke other available mathematical operations to cope with the situation.

When the problem was posed to the two students in GSP, they could not find an answer easily like in the Protocol 4.30. Although they began with converting 1/2 into 3/6 and 1/3 into 2/6 similar to the previous problem for 1/7 and 1/3, the students' conjectures did not seem to be anticipatory mathematical thinking. When I intervened and encouraged them to construct the transformation processes using JavaBars, as expected, Carol copied her original 1/2-meter bar twice for 1-meter, partitioned each of two 1/2-meter bars into three and pulled out a part from each 1/2-meter bar to construct a 1/3-meter bar on the basis of her distributive reasoning (see Figure 4.32a). Rosa also made two copies of her original 1/2-meter bar and joined them to construct a 1-meter bar. Then she partitioned each 1/2-meter part into three parts to make a 6-part bar of 1-meter and pulled out two parts of the 1-meter$_6$ bar for 1/3-meter (see Figure 4.32b).



Figures 4.32a & 4.32b: Carol's (Left) & Rosa's (Right) transformations of a 1/2-meter bar into a 1/3-meter bar

After both students' construction for transformation in JavaBars, Carol found a scale factor for dilation in GSP based on construction of two bars in JavaBars as two-thirds. Her answer seemed to be derived from establishment of a part-whole relationship between the 1/2-meter$_3$ bar and the 1/3-meter$_2$ bar based on perceptual information on the computer screen. On the other hand, Rosa had a hard time figuring out a scale factor for dilation in GSP even with her constructions for the two bars right in front of her and Carol's direct explanation [it's two-thirds of one half.] Rosa seemed to be lost when she went through the construction processes from 1/2-

meter to 1-meter and then from 1-meter to 1/3-meter. For Rosa, going by way of 1-meter seemed to come up as an obstacle in her constructive itinerary for 1/3-meter from 1/2-meter.

At that moment, the witness-researcher intervened and asked the students to dilate a 1/2-meter segment in a couple of steps to make into a 1/3-meter segment. Upon the witness-researcher's question, Rosa argued that we could make 1/2-meter by starting from 1/6-meter and doubling it and Carol mentioned a use of 1-meter in her comment, "if you think about one meter and then you finally you should have the one meter again" although she could not exactly explain the whole dilation processes. Then Protocol 4.31 followed:

Protocol 4.31 on 04/15/10: Construction of two-step dilation between two unit fractions.

W: How would you get, if you, if we didn't have the 'show one meter' button, how would you get the one meter through dilation?
R: Two over one.
C: Wait, from one-third or one-half?
W: Let's see, you guys are going from one...
T: One-half.
C: Then you just dilate by two.
R: Yeah.
T: By two.
W: And then what would you do to get the one-third meter?
C: Then you...
R: Three.
C: Divide by three over one.
W: You dilate by three over one? So, can you guys try it real quick?
C: So you gonna start from one-half?
W: Start from one-half and do the dilation in two steps.
(Carol dilates her 1/2-meter segment by two to make a 1-meter segment and then dilates the 1-meter segment into a 1/3-meter segment by one-third.)
T: (To Rosa) did you start from one-half?
R: Um-hm.
T: And make one meter.
R: One meter, yeah. (Rosa dilates her 1/2-meter segment by two to make a 1-meter segment.) Right there.
T: And then based on your dilation, can you construct, transform into one-third?
R: One-third... Two-thirds? Is that what we just did?
C: No, from one.
T: From one meter. Can you transform one meter into the 1/3-meter? What number do you need to dilate by?

R: ...three.
C: Think about what is one-third right there. How did they get that?
T: Three. Is [it] three?
R: No, one-third.
T: One-third, right. (Rosa dilates the 1-meter segment into a 1/3-meter segment by one-third.) Okay, for the first dilation, I mean, for transform one-half into one meter, what number did you dilate by?
R: The reciprocal.
T: Yeah, what number?
R: Two over one.
C: Two.
T: Two, right? And from one meter to one-third what number did you dilate by?
C & R: One-third.
T: Can you combine?
C: And then, oh~ because we have the two... is that... and then one-third. That would be two-thirds.
T: Yeah, can you combine those numbers for the transformation from one-half into one-third? (The teacher couldn't hear Carol's argument.)
C: That would be two-thirds.
T: (To Rosa) two-thirds?
R: (Rosa nods her head.) Yeah, I see.
C: Is there another problem like this? Cause I wanna try it on GSP.

Unlike Rosa, Carol seemed to be able to use her construction processes in JavaBars to interpret the two-step dilation processes in GSP. At that moment, it was not clear why Carol used her operations in JavaBars to interpret the two-step dilation processes in GSP when Rosa had much difficulty in combining two dilation processes guided by the witness-researcher. Therefore, I decided to pose one more similar question to investigate their mathematical operations emerging in their transformation activities between unit fractions.

Protocol 4.31: (Cont.)

T: All right, let's try this problem. One... Um...let me give you a large number[35], one-eleventh into one-thirteenth (The teacher writes down '1/11→ 1/13' on the problem sheet.)
C: Let's see. You do it by eleven over one... Can we write our steps down?
T: Yeah, sure you can.
(The teacher asks the students to open a new sketch in GSP and to make a 1-meter segment, and then to make a 1/11-meter segment and a 1/13-meter segment based on

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[35] I was referring to the denominator part of the following fractions.

their 1-meter segment by dilation. They construct a 1/11-meter segment and a 1/13-meter segment dilating by one-eleventh and one-thirteenth for each. Although the students have technical problems in construction of the two segments, the teacher helps them construct their segments. After construction, the teacher hides their 1-meter segments. So the 1/11-meter segment and the 1/13-meter segment are left on each student's screen.)
R: And you want to know what we need to dilate by?
T: Yeah, the problem is, what number do we need to dilate by to transform one-eleventh into one-thirteenth meter, of a meter.
C: Eleven over... Would it be eleven-thirteenths? No.
T: Eleven-thirteenths.
C: Yeah, is that right?
T: Think about it.
C: No. That wouldn't be right cause it ends up being smaller.
T: So we kind of, using this problem, you kind of want to do step by step processes right?
C: Wait, wouldn't it be eleven-thirteenths of one-eleventh?
T: Why do you think like that?
C: Because you had to multiply or dilate by this (1/11-meter segment) by eleven over one to get to one meter and then you dilated one meter by one-thirteen to get one-thirteenth. So that will be eleven over thirteen of one-eleventh.
T: Right!
(Rosa converts '1/11' to '13/134' and '1/13' to '11/134' but she seems stuck at this step. Carol dilates her 1/11-meter segment into a 1/13-meter segment by eleven-thirteenths and confirms that her answer is right.)
C: Yeah, that's it.
R: One-eleventh, one-thirteenth.
T: You can take time if you want.
R: I feel like the answer... I did it before and all the other one.
C: Should we do on JavaBars?
T: (To Carol) you can try it. (To Rosa) and you can take time to think about it, take it easy.
R: (Twenty seconds later) I have no idea what I did before.
(Although the teacher goes back to the previous problem involving 1/2-meter and 1/3-meter with Rosa and explains two step dilation processes, she does not seem to keep track of such dilation processes with the numbers to be used in dilation. On the other hand, the teacher asks Carol to explain her answer due to time constraint although she does not finish her construction in JavaBars yet.)
C: If you start with one-eleventh and you want to get one meter right? So, you go eleven over one. It's way up there, one meter (on the screen). It's way up there because I did that. And then what you wanna do is to take the one meter and you wanna get one-thirteenth. So you do one-thirteenth from one meter, right? So then you have one-thirteenth. And then you just think, um... you did eleven over one or eleven to get the one meter so that's gonna be a numerator and then thirteen is your denominator cause you did it second and that's the fraction that you're trying to get. So it'd be eleven-thirteenths.
R: I don't know. I mean I see what you are saying. I just took like the one as bigger is a numerator and the second one is the denominator because it's smaller.

When I asked the students to find a scale factor using GSP for transforming a 1/11-meter segment into a 1/13-meter segment, Carol immediately said “11/13” with an explanation that she had to multiply the 1/11-meter by 11 to get to one meter and then dilated one meter by 1/13 to get 1/13-meter, which would be 11/13. Carol’s behavior indicated construction of a scheme to multiplicatively transform one unit fraction into another. My conjecture is that such ability of Carol’s to produce the two-step dilation processes in re-presentation prior to actual activity might be because of her distributive reasoning. That is, the ability to take one-thirteenth of her re-presented 11/11-meter segment using her distributive partitioning operation seemed to enable her to find the fractional operator, 11/13, for the transformation by combining 11 with 1/13. On the other hand, Rosa could not independently establish the number to be used in dilation prior to activity throughout the teaching experiment even though she was able to enact transforming a unit fraction into any other unit fraction without an explicit awareness of how she operated. Considering that the nature of Rosa’s partitioning operation was a “common partitioning operation” rather than a “distributive partitioning operation,” lack of her distributive partitioning operations seemed to constrain her transformative activity between two unit fractions via a referent whole.

Test for Reversible Recursive Partitioning Schemes

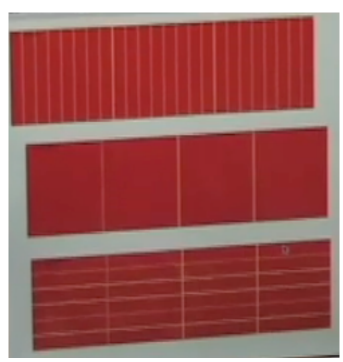
As revealed in the teaching episode on April 13th, Rosa had yet to interiorize her recursive partitioning operations to establish an equivalent relationship between two unit fractions (e.g. 1/6 and 3/18). Therefore, I decided to pose one more problem related to recursive partitioning operations in the context of transformation activity in GSP as well as JavaBars. The problem was to transform a whole referent unit [one meter] into another unit fraction in two-step dilation, say, to transform one-meter by one-fourth first and then transform the result by one-sixth to make into 1/24-meter. I anticipated that such a problem might require a reversible use of their recursive partitioning operations, which might be an indication of interiorized recursive partitioning because the problem asked the students to use the result of their recursive partitioning schemes[36] [1/24-meter] as material for the construction of a possible situation of the recursive partitioning scheme [1/4 and 1/6 for dilation].

When the problem [to transform one meter into 1/24-meter in two steps] was posed to the students in GSP, Rosa suggested one-twelfths and one-half and Carol did one-third and one-eighth. However, their answers seemed to be based on the result of numeric calculations of fraction multiplication. Further, Rosa was abstracting her way of calculating to find a pattern to transform one meter into 1/24-meter as indicated by her comments, "I think if you find a half of the number, and then like if you take that number like half of the denominator, and then, um...use that as yours one of whatever that is... I guess we can find the... not multiples, factors?" Carol did not show any indication that she saw one meter as twenty four units of 1/24-meter as well as three units of 1/3-meter, each of which consists of eight units of 1/24-meter, either. Therefore, I encouraged them to confirm their answers by two-step construction in JavaBars as I did in the teaching episode on April 13th.

> Protocol 4.32 on 04/24/09: Transformation of 1 meter into 1/24-meter using two steps in JavaBars.
>
> T: (To Carol) you said that you can use one-third and one-eighth and you (Rosa) one-fourth and one-sixth, right? Then using JavaBars, can you kind of figure out, kind of your process? Construct your process.
> R: Using JavaBars?
> T: Yeah, using JavaBars.
> C: Do you want us to...
> R: So you want me to use one-fourth and one-sixth?

[36] I intentionally use recursive partitioning *schemes*, not recursive partitioning *operations* because I need to mention a reversibility of the recursive partitioning scheme where a result of the scheme [1/24] is taken as a situation of the scheme.

T: Yes, you said one-fourth and one-sixth, right?
R: Yes, or one-half and one-twelfth.
T: So, let's say, make a bar. Let's assume that this one is, let's label this one is one-meter bar right? We should start from [a] one-meter bar.
(Rosa makes a bar and partitions it into twenty four parts.)
R: Okay, one meter, and you just want me to go from there? (Rosa makes a copy of her 24-part bar.)
C: And we want to do two steps right?
T: Yeah, two steps like in GSP. You can use two steps.
R: I should've made it a whole a lot bigger. I'm gonna make it bigger.
(Carol divides her 1-meter bar into two parts and pulls out one part from it. Rosa makes a new bigger 24-part bar and makes another copy of it. She starts to partition each part of her copy bar into four parts.)
R: (After partitioning eleven parts among twenty four parts into four) I really, um...
(Carol partitions her pulled-out 1/2-meter part into twelve, pulls out one piece from the 1/2-meter$_{12}$ bar and labels it as '1/12 of 1/2 meter or 1/24 of 1 meter'. See Figure 4.33a.)
W: What do you make, Rosa?
R: I don't know. I'm trying to think of how to do this.
T: My question is, can you confirm your answer in GSP?
R: I mean I can. I'm just trying to figure out the easiest way to do it.
T: All right.
W: Can you do in two steps? (Rosa erases all her work except her original bar with no partition.) Can you do in two steps? You did one step, right? You just did, you just made a bar twenty-four parts, right?
R: Um-hm. (Rosa copies two more bars, divides her original bar into twenty four parts vertically, the first copied bar into four parts vertically and the second copied bar into twenty four parts, but using four vertical parts crossed with six horizontal parts. See Figure 4.33b.)

Figures 4.33a & 4.33b: Carol's (Left) & Rosa's (Right) transformations of a 1-meter bar into a 1/24-meter bar using two steps

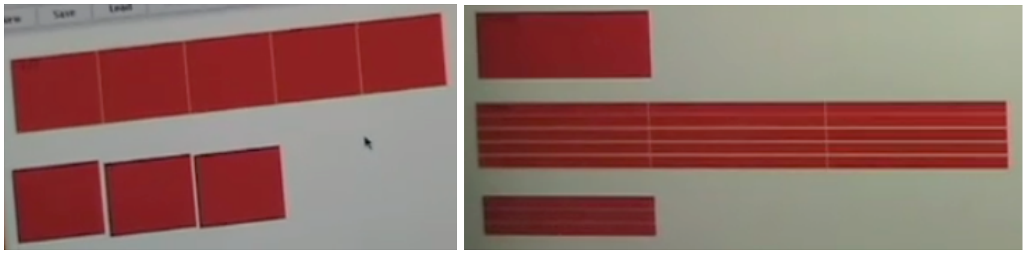
Carol's construction processes were clear. She partitioned her 1-meter bar into two and took a twelfth of the 1/2-meter part by partitioning the 1/2-meter part into twelve. Although she made a different two-step construction from her initial answer [her first answer was one-third and one-eighth], her two-step construction clearly indicated that she had constructed a reversible recursive partitioning scheme. That is, she was able to partition a one-meter into twenty four parts in re-presentation prior to her actual activity and see one-half meter as a unit of twelve parts of 1/24-meter. Therefore, her explanation based on numerals of fraction multiplication was actually symbolizing her reversible recursive partitioning scheme.

Unfortunately, Rosa's two-step construction processes in JavaBars revealed that her construction of a quantitative relationship based on recursive partitioning operations was not completed in the sense that her recursive partitioning scheme was a one-way scheme, not a reversible scheme. First of all, she partitioned her original 1-meter bar into twenty-four and then began to divide each part of the 24-part bar into four. Considering that her answer was one-fourth and one-sixth, the goal of her partitioning activity of each part of 1-meter$_{24}$ into four was possibly to take one-fourth of the whole 1-meter$_{24}$ bar. However, she seemed to feel that something was going wrong when she was partitioning the eleventh part of the 1-meter$_{24}$ bar; she might have realized that the completion of the partitioning activity produced ninety six parts in the 1-meter bar, which was an unexpected number for her. In addition, 1) her struggle to make a two-step construction in her second attempt to confirm her answer and 2) the fact that her pulling-out operations from the previous steps in her constructions were not explicitly conducted, corroborated that Rosa did not use a three-levels-of-units structure involving 1/24, 1/4 and 1 as given to construct a two-step dilation for transformation of 1-meter into 1/24-meter. It means that she was yet to interiorize the three-levels-of-units structure based on the results of recursive partitioning operations. Her cross partitioning of the second copy of the one-meter bar by four and six could have symbolized her solution based on the two co-factors of 24 that she used for her dilations in GSP, rather than on her recursive partitioning operations.

Confirmation of Rosa' Common Partitioning Operation in Transformation Activity Between Two Unit Fractions

Protocol 4.33 on 04/24/09: Transformation of a 1/3-meter bar into a 1/5-meter bar.

W: Make a bar. (Both students make a bar on the screen.) Pretend that bar is one-third of another meter bar. Let's pretend that's one-third of the bar. Using that, I want you to make one-fifth of the bar.
R: Use this bar to make one-fifth? Fifteen... (While staring on empty space, Rosa seems to figure out something in her head.) And you want one-fifth?
(Carol makes three copies of her original 1/3-meter bar and joins them together.)
R: Okay, that's one-fifth... Yes, that's one-fifth. (The camera was not capturing Rosa's construction processes because the witness-researcher was following Carol's work, but Rosa has one 5-part bar, and three parts pulled out from it. See Figure 4.34a) So, I dilated one-third by five and multiply it times three? No, I think it's just one step thing.
T: How did you get it?
R: Um... I took the one-third piece and divided it into five sections. And then so I mean the whole bar then would then equal fifteen over fifteen and so fifteen is divisible by three and five. So I pulled out three of the pieces because three times five is fifteen.
T: Then what number do you think dilate to transform from one-third to one-fifth? I think, I mean you got it right. So can you guess that?
R: Three-fifths?
T: Three-fifths? Why, why do you think like that?
R: Um... Because three, if you multiply together, one-third times three-fifths is three over fifteen, which is one-fifth.
....
T: How did you get that?
C: I did, I did basically same way as Rosa except I did in two steps. I copied it. I copied it three times. So I get one bar and I divided each of them into five pieces and I took one out of each piece to get (see Figure 4.34b).
R: How many pieces do you have in total?
C: I had fifteen.
R: Yeah, fifteen, fifteen. I just made it smaller.
T: Then what number do you need to dilate by for your transformation?
C: Three.. fifths? Or three, three-fifths from the one-third meter. Right? (Teacher nods his head.)
R: Because if you divide by three-fifths then the one-third is five so the whole bar is gonna be fifteen.



Figures 4.34a & 4.34b: Rosa's (Left) & Carol's (Right) transformations from a 1/3-meter bar to a 1/5-meter bar

Given a bar with no partition considered as a 1/3-meter bar, the aim of this problem was to transform the 1/3-meter bar into a 1/5-meter bar using JavaBars. Interestingly, Rosa's first word was 'fifteen' as in her comment, "Use this bar to make one-fifth? Fifteen..." Then she quickly partitioned her 1/3-meter bar into five parts and pulled out three parts from the 1/3-meter$_5$ bar for construction of a 1/5-meter bar. On the other hand, Carol copied her 1/3-meter bar three times and joined them together to make a 1-meter bar. Then she partitioned each of three parts of the 1-meter bar into five parts and pulled out a part from each 1/3-meter part to construct a 1/5-meter bar by joining those parts together. Carol's mathematical behavior was another confirmation of her construction of distributive partitioning operations, which was expected by the teacher-researcher [me] because Carol already showed indications of her distributive reasoning several times before (cf. Protocols 4.14, 4.16 and 4.22).

In contrast to Carol's consistent use of her distributive partitioning operations, I already argued that Rosa assimilated Carol's partitioning activity and modified it to a common partitioning operation (cf. Protocol 4.25). Now I claim that Rosa's mathematical behavior in this protocol confirmed that she interiorized her common partitioning operation in the context of transformation activity between unit fractions as an available operation prior to actual action. That is, she was able to find a co-measurement unit [one-fifteenth] for one-third and one-fifth through construction of a 1-meter$_{15}$ bar in her re-presentation prior to action. This was made possible by the coordination of two three-levels-of-units structures as indicated by her comments, "I took the one-third piece and divided it into five sections. And then so I mean the whole bar then would then equal fifteen over fifteen and so fifteen is divisible by three and five. So I pulled out three of the pieces because three times five is fifteen." In other words, in the context of transforming a 1/3-meter bar into a 1/5-meter bar, she knew that both could be simultaneously obtained from a 1-meter bar consisting of fifteen parts by coordination of two iterable composite units, 3 and 5. Her ability to coordinate the two iterable composite units enabled her to divide the 1/3-meter bar into five parts as one-third of fifteen parts in her re-presentation and further pull out three parts for one-fifth of the re-presented 1-meter bar_{15}. She was not only able to construct a three-levels-of-units structure with 1/15, 1/3 and 1 but also to coordinate it with another three-levels-of-units structure with 1/15, 1/5, and 1. This was a typical behavior of a student who had constructed common partitioning operations (see Nathan's similar solution in Olive, 1999). Also, her common partitioning operation seemed to help her to find out the number to be dilated in two steps in this problem although her explanation was based on her numeric calculation. The critical difference from Carol's transformation activity was that Rosa's transformation was based on a co-measurement unit fraction for two given other unit fractions through a common partitioning operation, say, construction of a co-measurement unit fraction one-fifteenth for one-third and one-fifth. On the other hand, Carol consistently relied on the use of one whole in her mathematical activity. Her partitioning operation was distributive based on a sequential association of two three-levels-of-units structures. It was also noticeable that establishment of a visual part-whole comparison between her constructions of two fractional quantities seemed to play an important role in finding an answer.

Summary of Phase III: Multiplicative Transformation Between Two Fractions

In transformation activities during Phase III, Rosa demonstrated that she was able to strategically use the result of her unit-segmenting scheme for further mathematical activity as Carol did in Phase I. That is, when Rosa was asked to find what fraction of 3/4-meter was contained in 31 meters, she knew that four-thirds of 3/4-meter was contained in one meter and could use it to find a fraction of 3/4-meter to get 31 meters by multiplying four-thirds by 31/1. Although the evocation of her unit-segmenting scheme was not what I intended to encourage for the students during Phase III, obviously it constituted mathematical progress for Rosa, when compared with her struggles in Phase I.

In order to facilitate the students' transformation activities with an awareness of the multiplicative operator to be used in transformations, I, as a teacher-researcher, decided to introduce dilation activity in GSP. I expected that transformation activity in GSP using DILATION could provide an occasion for the students to reflect on and abstract their mathematical activities involved in the transformation, which might lead to the construction of the multiplicative operator.

However, it turned out that the GSP environment was too demanding for the students to execute transformation activities, in that they needed to anticipate a scale factor for transformation prior to actual transformation. Even though Rosa had constructed recursive partitioning operations, she was able to flexibly conduct transformations only in simple situations, for example, transformation of 1/6 into 1/2. When the situation became complex, say, transformation of 1/9 into 2/3, she experienced difficulty in using the results of recursive partitioning operations [a three-levels-of-unit structure involving 1/9, 2/3, and 1] as given for the transformation.

Nevertheless, I could help them conduct transformation activities by coordinating two computer tools: JavaBars and GSP. Carol was able to transform a unit fraction [1/4] into the other unit fraction [1/10] using her distributive partitioning operations in JavaBars. On the other hand, Rosa's transformation activity for the problem indicated that she constructed common partitioning operations while engaging in the transformation activity. Common partitioning operations were made possible by her ability to coordinate two iterable composite units based on her GNS. This was significant mathematical progress for Rosa when compared with her partitioning operations during Phase II that had no clear goal and were just an imitation of Carol's partitioning operations.

Carol, based on her distributive reasoning, constructed a multiplicative scheme for transformation between two unit fractions. This was another indication of how powerful Carol's distributive reasoning was in non-sharing situations. Rosa's common partitioning operation was powerful enough to transform a unit fraction into another unit fraction using JavaBars and entailed more potential in that she had constructed a comeasurement unit for both unit fractions. However, I could not investigate more about her powerful use of common partitioning operations because the academic semester was over.

Rosa, during the teaching experiment in Phase III, revealed a lacuna in her recursive partitioning scheme. She could conduct recursive partitioning operations but was not able to use the results of her recursive partitioning operations a priori for further mathematical activity, such as for generating commensurate fractions to a proper fraction. She also experienced difficulty in conceiving improper fractions based on a FCNS as a given structure. It needs to be noted that Rosa's outstanding ability to calculate fraction multiplication and division (even without paper and pencil) seemed to inhibit her from developing necessary mathematical operations in a given situation. For example, even though Rosa had constructed an iterative fraction scheme, she did not conceive 1/31-meter as one-thirty-seventh of 37/31-meter on the basis of three levels of units [1/31, 1 and 37/31]. She knew that the answer should be 1/37 by numeric calculation, but there was no indication of her establishment of a quantitative relationship among 1/31, 1, and 37/31 even after constructing perceptual materials [bars] on the computer screen using JavaBars.

Table 4.4 (provided below) is a brief summary of the two students' schemes and operations that I hypothesized based on their activities during the teaching experiment.

Table 4.4

Summary of the Students' Constructions of Mathematical Schemes and Operations

	Protocol	Carol	Rosa
Phase I: Fraction Measurement Division	4.1	Unit-segmenting scheme with a remainder	Sequential association of a unit-segmenting scheme and a partitive fraction scheme
	4.3	Generalizing assimilation of a unit-segmenting scheme	
	4.5		Unit-segmenting scheme with a remainder
	4.8	Student-generated algorithm for fraction measurement division	
	4.10	Fractional unit-segmenting scheme	
Phase II: Fraction Multiplication	4.12	Recursive partitioning operations	
	4.14	Distributive partitioning operations	Imitation of Carol's partitioning activity
	4.15	Distributive partitioning operations in a sharing situation	
	4.16	Distributive reasoning Lack of use of a FCNS as a given structure in fraction multiplication	Lack of distributive reasoning
	4.17	Iterative fraction scheme for composite units	
	4.18	Use of results of recursive partitioning operations as given	Lack of use of results of recursive partitioning operations as given
	4.20	Reversible iterative fraction scheme	Lack of a reversible iterative fraction scheme
	4.22	Distributive reasoning in a proportional problem	
Phase III: Multiplicative Transformation Between Two Fractions	4.24		Strategic use of a unit-segmenting scheme with a remainder
	4.25	Distributive partitioning operations in transformation	Common partitioning operations in transformation
	4.28 4.29	Interiorized use of recursive partitioning operations to establish a quantitative relationship between two commensurate fractions	
	4.31	Multiplicative scheme for transformation between two unit fractions	
	4.32	Reversible recursive partitioning scheme	Lack of a reversible recursive partitioning scheme

CHAPTER V

DISCUSSION AND IMPLICATIONS

The present study investigated two eighth-grade middle school students' construction of fractional knowledge centered around multiplication, measurement division, and multiplicative transformations between fractional quantities. The final chapter is presented in two sections. The first section presents what I learned from my teaching experiment in relation to my research questions. In the second section, I begin to connect this research to a broader point of reference. I make this connection by answering my second research question (provided below) by reconnecting with the *Fraction Project* and by suggesting a future direction for research in students' construction of RNA. The research questions that I investigated were:

- How do two eighth-grade students who were credited with the construction of a generalized number sequence construct necessary schemes and operations for fraction multiplication and division?
    1. What sort of actions and operations are necessary for the construction of a general fraction composition scheme? Does the distributive partitioning operation play an important role in fraction multiplication? If so, how do they develop such distributive thinking? If not, which other operations are critical for fraction multiplication?
    2. What sort of actions and operations are used or newly emerge in the processes of the two students' solving fraction division problems? Specifically, how do the

students' distributive partitioning operations and common partitioning operations contribute to the construction of a fraction division scheme?

- Are there any implications for students' construction of rational numbers of arithmetic?

Discussion

This section is composed of three parts. In the first part, I compare and contrast the two student's constructions of partitioning operations. In the second part, I provide a plausible constructive path for students' fractional knowledge of measurement division based on the schemes and operations that my participating students constructed in the processes of solving fraction measurement division problems. The last part is allotted to explain how I, as a teacher-researcher, struggled to help the students construct a scheme for multiplicative transformations between two fractions by combination of the two computer tools [JavaBars and DILATION in GSP].

Nature of Partitioning Operations of the Two Participating Students

The nature of partitioning operations observed in both students' mathematical activities emerged as one of the most important issues in the teaching experiment. The availability of certain kinds of partitioning operations or a scheme embedding such operations in the context of particular problem situations [especially, fraction multiplication] seemed to often account for a key difference between the two students' problem-solving activity. The partitioning operations finally determined their successful solutions.

As reported in the research literature, recursive partitioning operations were fundamental mathematical operations for students' fraction multiplication through modification of their units-coordinating operations (Steffe & Tzur, 1994). The participating students in the present study demonstrated that they had constructed recursive partitioning operations (cf. Protocol 4.12), which was not surprising on account of their construction of a GNS as determined in the pre-interviews (cf. Protocol 3.2). However, the difference in their ability to take for granted the results of recursive partitioning operations was revealed in Protocol 4.18 when they were asked to make a 23/18-bar without erasing the marks on the 6/6-bar. Carol was able to use a three-levels-of-units structure with 1/18, 1/6, and 1 as a given for construction of a 23/18-bar by partitioning each part of the 6/6-bar into three parts and pulling out five small parts to join with the whole 6/6-bar_{18}. It was the result of interiorization of her recursive partitioning operations so that she could conceive of 1/6 as consisting of three units of 1/18, each unit of which can be iterated twenty three times to make a 23/18-bar.

On the other hand, Rosa's conceptions of 1/6 and 1/18 were separated from each other. Rosa was not able to establish a quantitative relationship between the two fractions with a goal of constructing a 23/18-bar from a 6/6-bar although she knew that 1/6 is the same as 3/18 using the result of her numeric calculation with fractions. Even though she had constructed a recursive partitioning operation in Protocol 4.12, she was yet to interiorize the three-levels-of-units structure involving two unit fractions [1/6 and 1/18]. This difference in the level of interiorization of their recursive partitioning operations presaged the difference in their multiplicative transformation activities between two fractions. Moreover, there was a lacuna in Rosa's conception of an improper fraction (cf. Protocol 4.20: Rosa's lack of a reversible iterative fraction scheme) and she was yet to construct a splitting operation to establish a multiplicative relationship of a unit fraction with a whole referent unit.

Carol demonstrated that she had constructed distributive partitioning operations when she took one-third of a 2-part bar without erasing a partitioning line (cf. Protocol 4.14). She immediately divided each part of her 2-part bar into three parts and pulled out one part from each of the 2-part$_6$ bar with an explicit awareness that taking one part from each of the 2-part$_6$ bar equals to one-third of the total 2-part bar. On the other hand, Rosa assimilated Carol's partitioning activity and was able to take one-third of her 2-part bar by association of her unit fraction scheme for connected numbers with her partitioning result. Nevertheless, Rosa's partitioning operation was just an imitation of Carol's partitioning activity in the sense that Carol always initiated their partitioning operations and consequently the goal of Rosa's partitioning activity was not clear.

Thereafter, Carol's distributive partitioning operations consistently emerged in the contexts of a sharing situation (cf. Protocol 4.15), multiplication by a proper fraction (Protocol 4.16) and even in the context of a proportional reasoning problem (cf. Protocol 4.22). Sometimes Rosa independently conducted her partitioning operations (cf. Protocol 4.19) with a non-sharing goal. However, her partitioning operations were still an imitation of Carol's partitioning activities and the results of abstraction of Carol's partitioning pattern. When Rosa was asked to take a half of a 17/15-meter bar, Rosa struggled to find the size of the smallest part of her construction [a half of 1/15-meter]. Rosa's lack of distributive partitioning operations seemed to prohibit her from retaining the result of her construction as available information for further use. Also, the fact that Rosa experienced difficulty in identifying the size of the smallest part in terms of a given referent whole [one-meter] indicated that she was yet to interiorize recursive partitioning operations at three levels of units (cf. Protocol 4.21).

Carol's distributive reasoning was also apparent in her transformation activities between two unit fractions during Phase III. Given two unit fractions of 1/4 and 1/10, she was able to transform a 1/4-meter bar into a 1/10-meter bar by making three more copies of 1/4-meter, joining them with the original 1/4-meter bar to make a 1-meter bar, partitioning each part of the

1-meter$_4$ bar into ten parts and pulling out one of the 10 parts from each of the original 4 parts of the 1-meter$_4$ bar and joining them to make a 1/10-meter bar (cf. Protocol 4.25). Obviously it was a manifestation of the use of her distributive partitioning operations in a non-sharing situation, which indicated how powerful her distributive reasoning was in her mathematical activities.

On the other hand, Rosa partitioned each of her four 1/4-meter bars into five and pulled out two parts from the collection of 20 parts for construction of a 1/10-meter bar. Her justification for the construction indicated that her partitioning operations had a clear goal of finding a multiple divisible by both four and ten. Her ability to coordinate two iterable composite units [four and ten] in re-presentation enabled her to choose 'five' for the number to partition each of 1/4-meter bars prior to activity. She then converted the problem into a situation for her iterative fraction scheme for connected numbers [pulling out 2 parts from 20 parts to construct a 1/10-meter bar]. Therefore, Rosa's partitioning operations went beyond an imitating assimilation of Carol's partitioning activities and were finally reorganized into common partitioning operations for unit fractions. Further, Rosa was able to use her common partitioning operations in a transformation activity between two unit fractions during Phase III (cf. Protocol 4.33). When she was asked to transform a 1/3-meter bar into a 1/5-meter bar using JavaBars, she immediately partitioned her 1/3-meter bar into five parts and pulled out three parts from the 1/3-meter$_5$ bar for construction of a 1/5-meter bar. It was made possible by finding a co-measurement unit [1/15] for 1/3 and 1/5 through coordination of two iterable composite units [3 and 5]. Her ability to do so was an indication of interiorized use of her common partitioning operations (in transformations between two unit fractions).

Modifications of Unit-Segmenting Schemes for Fraction Measurement Division

Students' construction of unit-segmenting schemes has been studied in whole number measurement division situations, where one composite unit to be used in measuring, evenly divided the other composite unit to be measured, and the goal of which was to find how many times the measuring unit was used in segmenting the other unit to be measured (Steffe, 1992b). However, relatively little research has been carried out for studying how the unit-segmenting schemes could be modified in measurement divisional situations involving fractional quantities.

When a whole number division problem with a remainder [finding how many times 3 meters is contained in 5 meters] was posed in Protocol 4.1, Carol and Rosa assimilated the problem as a division situation, which led them to use a conventional division calculation method. However, Rosa could not convert her decimal answer to a fraction form that I had requested. She, later, re-assimilated the problem as a situation for her unit-segmenting scheme, indicated by her comment "It's gonna be one and then something fraction," but the division situation, where a composite unit to be segmented was not completely measured out by the other composite unit used in segmenting, was a novel situation for Rosa, which entailed an unexpected quantity to measure.

It was Carol who eliminated the perturbation in using her unit-segmenting scheme. With perceptual materials [a 3-part bar and a 5-part bar on paper], she was able to construct three levels of units [1, 3, and 5], which enabled her to regard 5 as one and two-thirds units of 3 as well as five units of 1 and one unit of 5. I already argued that this was made possible by her association of the result of her unit-segmenting scheme [the leftover 2-part bar] as a situation for her fraction scheme. Therefore, if a student constructed a new unit-segmenting scheme through a modification whereby her fraction scheme was embedded as a subscheme in the assimilating part of her unit-segmenting scheme, I would attribute to the student construction of a *unit-segmenting scheme with a remainder*.

In the division situations, when a fraction divisor evenly divided a whole number dividend, Carol demonstrated her generalizing assimilation of her unit-segmenting scheme, which resulted in the inclusion of fractional quantities as segmenting units in the assimilating part of the scheme. Similarly Rosa's numeric calculation of division was connected to her unit-segmenting scheme, which means that her division algorithm stood in for her unit-segmenting scheme (cf. Protocol 4.3). However, when the fractional divisor did not evenly divide the whole number dividend (e.g. finding how many times 3/4 gallons of water is contained in 4 gallons of water), Rosa seemed to fail to associate her division calculation result with her unit-segmenting scheme in order to deal with the entailed remainder (cf. Protocol 4.4). This was an indication that she was yet to construct a unit-segmenting scheme with a remainder. In the teaching episode held on December 5 of 2008 (cf. Protocol 4.5), Rosa finally constructed a unit-segmenting scheme with a remainder when measuring 1-meter with 3/5-meter. Rosa's construction was a *retrospective accommodation* in the sense that the construction was made possible through communication with Carol, rather than independently by herself.

Note, however, that construction of the three levels of units for both students' unit-segmenting scheme with a remainder was inherited from the iterability of 'one' rather than of a 'unit fraction.' Even in Protocol 4.4 [Measuring 4 gallons of water with 3/4 gallon of water] there was no evidence that Carol and Rosa were explicitly conceiving those two quantities based on the iterability of a unit fraction [1/4] in using their unit-segmenting scheme with a remainder. Such lack of an iterability of a unit fraction was indicated in Protocol 4.7 when they were measuring 2-meter with 7/5-meter. Rosa immediately converted the 2-meter into 10/5-meter to compare with the 7/5-meter numerically. However, her numeric conversion did not stand for her conception that 7/5-meter consists of seven units of 1/5-meter and 2-meter [10/5-meter] consists of ten units of 1/5-meter. Although she was able to establish a multiplicative relationship between the whole [1-meter] and a unit fraction [1/5-meter] later, it was a retrospective accommodation using the conceptual elements of her unit-segmenting scheme and fraction scheme, not derived from her numeric calculation. Carol demonstrated a conflation of units with a given referent unit [1-meter] and the unit to be used in segmenting [7/5-meter], which indicated that her conception of an (improper) fractional quantity in the structure of a FCNS was not used prior to actual construction. The fact that both students had a hard time establishing a multiplicative relationship between 2-meter and 7/5-meter when using their unit-segmenting scheme with a remainder (even with perceptual drawings for two fractional quantities on the paper), revealed that interiorized use of a FCNS as a given structure for measuring-out activity was not a trivial task for both Carol and Rosa.

Five months later, I posed several fraction measurement division problems, with the aim of determining 1) whether their FCNS could be used as a given structure in the assimilating part of their unit-segmenting scheme with a remainder and 2) whether their partitioning operations [distributive partitioning operations of Carol and common partitioning operations of Rosa] could emerge in the use of their unit-segmenting scheme. When the problem of measuring 16/17-meter with 7/17-meter was posed to the students (cf. Protocol 4.10), they still conflated the unit to be used in segmenting [7/17-meter] with the referent unit [1-meter] given in the problem, while measuring the leftover [2/17]. Both students demonstrated a certain level of progress in the sense that they were able to immediately correct their answers by themselves through reflection on their unit-segmenting operations with the constructed perceptual materials for the two fractional quantities in JavaBars. It was, however, not enough to argue that the FCNS was embedded in the first part of their unit-segmenting schemes with a remainder. Rather, it confirmed that construction activities with perceptual materials on which they could operate was still one of the critical factors for Carol and Rosa to conduct their mathematical [unit-segmenting] operations. If the iterability of a unit fraction [1/17] had been interiorized and embedded in the assimilating part of their unit-segmenting scheme with a remainder, I could have attributed to the students construction of *fractional unit-segmenting schemes,* in the sense that measurement division situations, where fractional numbers (on the basis of a FCNS) were used as a divisor [a unit to be used in segmenting operations] as well as a dividend [a unit to be segmented], were included in the assimilating situations of their unit-segmenting schemes.

The last fraction measurement division problem of the teaching experiment [measuring 1/3-meter with 1/7-meter] produced the teaching episode that indicated how the two students' partitioning operations, established during the teaching experiment, were possibly utilized in fraction measurement situations. Carol demonstrated an ability to transform 1/7-meter into 1/3-meter (or vice versa) using her distributive partitioning operations (when she was directly asked for the transformation). Further, she was able to derive a multiplicative relationship by establishment of a part-whole relationship between her constructions of the two fractions using JavaBars. Nevertheless, Carol's distributive partitioning operation was not provoked by the measurement division situation (cf. Protocol 4.11).

Rosa easily got an answer by numerical calculations in converting 1/3 into 7/21 and 1/7 into 3/21. On account of the fact that Rosa had constructed common partitioning operations, the evocation of her common partitioning operations seemed to enable her to assimilate the problem as a situation for her fractional unit-segmenting scheme. That is, in order for the problem to become a situation for her fractional unit-segmenting scheme, coordination of 1/3 and 1/7 in her re-presentation with an awareness of a co-measurement unit [1/21] would be necessary so that 1/3 could be projected on seven units of 1/21 and 1/7 could be projected on three units of 1/21 in the re-presentation. Therefore, I expected that she could solve the measurement division problem using her (fractional) unit-segmenting scheme together with an associated common partitioning scheme. Although my anticipation ended up with partial failure due to her difficulty in monitoring her common partitioning operations, Rosa's mathematical actions and operations in Protocol 4.11 suggested that a student's common partitioning operations could open a path to construction of a more generalized fractional unit-segmenting scheme to comprehend two unit fractions, which are not multiples of each other, within the range of assimilating situations of the fractional unit-segmenting scheme.

In conclusion, the following diagram (Figure 5.1) illustrates a possible progression of students' unit-segmenting operations and schemes, on the basis of their generalized number sequences (GNS) and fractional connected number sequences (FCNS). First, I suggested that by embedding the participating students' fraction schemes into the first part of their unit-segmenting schemes, they were able to deal with a remainder when measuring a whole number quantity with another whole number quantity that did not evenly divide the dividend, producing a unit-segmenting scheme with a remainder (cf. Protocol 4.1). A scheme embedded in another scheme needs to be distinguished from just an association of two schemes. When a scheme embeds another scheme in the first part of the former scheme, it means the latter scheme is ready at hand to be used as the former scheme is activated. On the other hand, when two schemes are associated, those two schemes are executed sequentially, rather than simultaneously. It requires students to re-assimilate the result of the first scheme as a situation of the second scheme.

Continuing upward in Figure 5.1, as students' fractional connected number sequences become interiorized, that is, available as a given structure prior to actual actions of their unit-segmenting operations, the students' unit-segmenting schemes with a remainder would be modified into fractional unit-segmenting schemes so that they could cope with measurement division problems involving fractional quantities (cf. Protocol 4.10). Finally, in order for the fractional unit-segmenting scheme to be generalized to solve a question like 'measuring a 1/3-meter bar with a 1/7-meter bar' in Protocol 4.11, it would be necessary for students' common partitioning operations to be associated with the fractional unit-segmenting scheme. That is, the common partitioning operations enable the students to convert the division problem situation between the two unit fractions above [1/3 ÷ 1/7] into a situation for their fractional unit-segmenting scheme [7/21 ÷ 3/21].

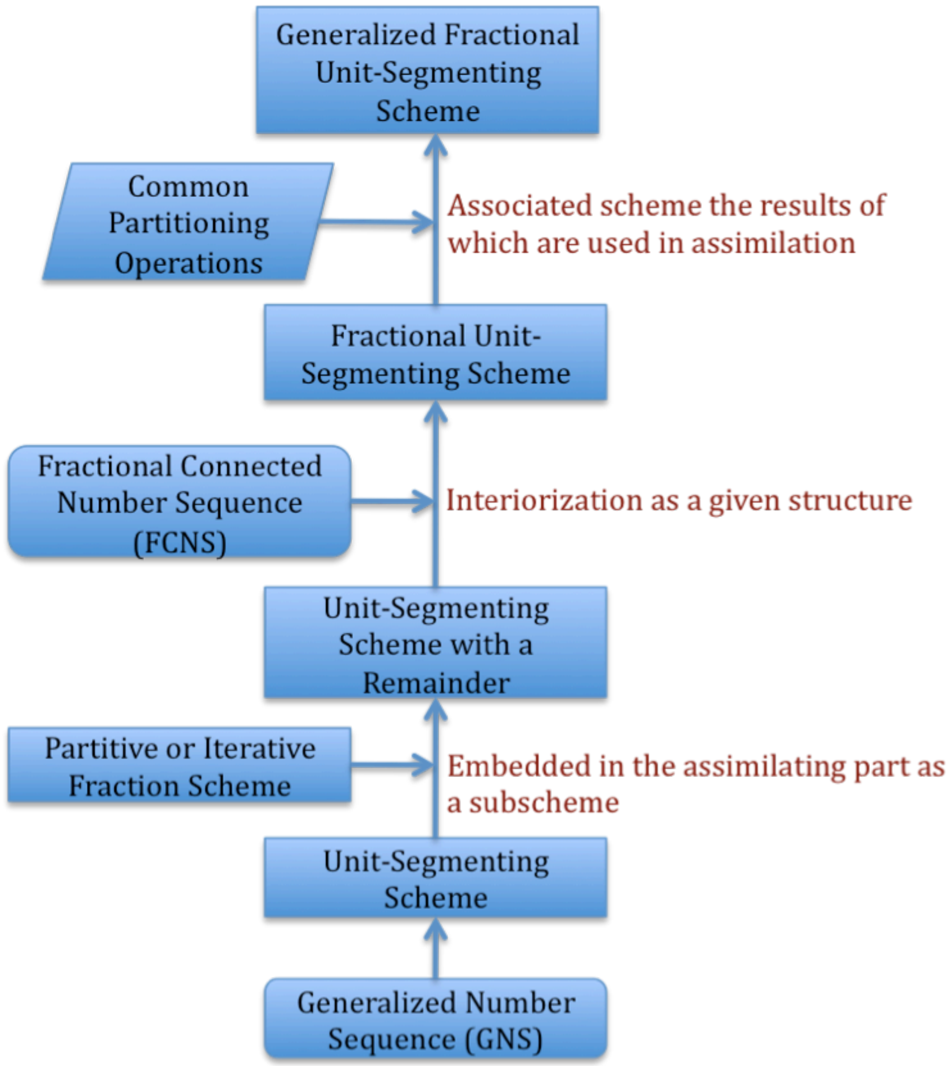


Figure 5.1 A possible constructive path of modifications of a unit-segmenting scheme for fraction measurement division problems

Multiplicative Transformation Activities in GSP and JavaBars

One of the overarching goals of the present teaching experiment was to construct a model for the two students' mathematical actions and operations in relation to their construction of RNA. I assumed that the students' explicit awareness of a multiplicative operator while engaging in transformation activities between two fractions could be considered as an important intermediate step in the constructive itinerary of the students to the RNA. However, my initial attempts to encourage them to carry out such multiplicative transformations and, further, to construct necessary fraction operators for the transformation processes did not meet with success. The reason might be due to inappropriateness of the posed tasks for the students, but another reason might stem from an affordance of the JavaBars environment as a too efficient facilitator for their mathematical activities. That is, convenient tools for creating perceptual materials of two given quantities in JavaBars helped the students easily derive a part-whole relationship between the two quantities, which might lead them to ignore their transformation processes in figuring out the multiplicative operator. Therefore, I decided to introduce a DILATION option in GSP for their transformation activities because, in GSP, the students need to anticipate the result of the dilation in order to enter a number in the DILATION box prior to the actual transformations. I expected that the DILATION option could encourage Carol and Rosa not only to construct multiplicatively transformed geometric figures, but also to become explicitly aware of a necessary scale factor for the transformations.

At the beginning of their use of the GSP program for transformation activities, my plan seemed to progress well. The students were able to find a scale factor [1/6] for transformation from a segment of length 2 units into a segment of length 1/3-unit, based on their recursive partitioning operations, on the first day that GSP was introduced. They also constructed a class of equivalent fractions for one-half in the process of finding as many ways as possible to make a 1/2A1/2B segment from other segments by dilation (cf. Figure 4.26). The construction of fractions equivalent to one half was made possible by their use of a three-levels-of-units structure as given, say, with 1, 1/2, and 1/8 with an awareness that 1 consists of two units of 1/2, each of which consists of four units of 1/8 that can be iterated eight times to make a whole [1].

However, finding a scale factor for transformation between any two fractions in a single transformation in GSP seemed to need more abstract mathematical reasoning than the activities in JavaBars, possibly because their mathematical operations needed to be anticipatory, and performed at the level of re-presentation prior to actual actions and operations. For example, when the problem to find a scale factor to transform a 1/9A1/9B segment into a 2/3A2/3B segment was posed in GSP, the students did not seem to have any clue for how to quantitatively relate the two segments, which could have been easily identified if the construction activities were conducted in JavaBars. Therefore, I decided to use both computer tools depending on the students' necessary actions and operations for their potential mathematical constructions (from my point of view). The combination of the two computer tools [JavaBars and GSP], used in order to make up for the weak points of both tools, resulted in opening new possibilities in the students' available mathematical activities and ensuing constructive itineraries. For example, using JavaBars allowed the students to easily realize their transformation processes step by step; they were then able to re-assimilate the transformation problem in GSP based on reflection of their construction processes in JavaBars, and this re-assimilation helped them find a scale factor to be used for the multiplicative transformation.

The effect, however, of the combination of JavaBars and GSP was not equally adapted by the two students. Only Carol constructed a scheme for multiplicative transformation between two unit fractions (cf. Protocol 4.31). When I asked the students to find a scale factor for transforming a 1/11-meter segment into a 1/13-meter segment in GSP, Carol provided the answer, '11/13' without construction processes in JavaBars. Carol's distributive reasoning seemed to be crucial for her to produce two-step dilation processes in re-presentation prior to activity by taking one-thirteenth of her re-presented 11/11-meter segment just as she could have done in JavaBars. On the other hand, Rosa's partitioning operations, which were not distributive, did not seem to contribute to her transformation activity between the two unit fractions via one referent whole. Even though Rosa's common partitioning operations finally became available in her transformation activity between two unit fractions in JavaBars (cf. Protocol 4.33), my encouragement to associate the transformation activities in the two computer environments to find one multiplicative operator for the transformation did not seem to work in Rosa's case due to the lack of her distributive reasoning.

Implications for Future Research

Importance of Interiorization of a FCNS for Advanced Fractional Knowledge

First of all, the present study confirms the grand assumption of the *Fraction Project's Reorganization Hypothesis*, which argued that children would construct their fraction schemes through modifications of whole number operations based on their abstract number sequences. Further, this study supports that students' construction of the operations that produced a GNS opened possibilities for their constructive activity that could not be observed in the students to whom the construction of only an ENS was attributed. The two participating students with a GNS in the present study demonstrated constructions of more advanced fractional schemes and operations, especially in the context of multiplicative and divisional situations, which were not reported in the previous literature.

In addition, the current research results imply that students' interiorization of an iterability of unit fractions, i.e. to take their FCNS as given prior to activity, needs to be considered as a critical factor in establishing viable second-order models for students' construction of more advanced fractional knowledge. In spite of the great potential for mathematical developments that the participating GNS students indicated in this study, the interiorized use of their FCNS was not evident. Actually, in the previous research literature, to establish a multiplicative relationship between a unit fraction and a referent whole in the process of modification of a partitive fraction scheme into an iterative fraction scheme, has been considered a big leap in the development of students' fractional knowledge. This leap enables the students to expand a fraction concept beyond the whole to include improper fractions (Tzur, 1999; Steffe, 2002; Steffe & Olive, 2010). However, this study additionally implies that 1) students' interiorization of such a multiplicative relationship of unit fractions to a referent whole and further 2) their being able to utilize the iterability of unit fractions as given for other mathematical activities, are not spontaneous transitions from the construction of a FCNS in action. These two developments require another level of vertical learning on the part of the students.

Of course, it was a critical part of the participating students' fractional knowledge to be aware of a referent whole and retain it throughout their mathematical activities so that the students could use it when necessary. For example, when Carol was asked to transform a 1/11-meter segment into a 1/13-meter segment using DILATION in GSP, she was able to immediately provide an answer [11/13] with an explanation that she needed to dilate the 1/11-meter segment by 11 to get a one-meter segment and then dilate the result [one-meter] by 1/13 to get the 1/13-meter segment (cf. Protocol 4.31). Her construction of a multiplicative scheme for the transformation between the two unit fractions [1/11 and 1/13] was made possible by using a referent whole [1-meter] as a mediator for the transformation and her distributive reasoning that enabled her to take one-thirteenth of her re-presented 11/11-meter segment. In the case of transformations between two unit fractions, it seemed efficient to conduct the transformations through a referent unit.

Nevertheless, sticking to the use of a referent whole for every mathematical activity, rather than a flexible combination with the use of a unit fraction, might prevent a student from developing more advanced mathematical thinking with fractions. For example, in the case of transformation from a 9/15-meter segment into a 10/15-meter segment, it might be more efficient to use a unit fraction [1/15] as a mediator for the transformation. That is, a student could dilate a 9/15-meter segment by 1/9 for the first step to make a 1/15-meter segment and then transform the resultant 1/15-meter segment into the 10/15-meter segment dilating by 10/1 in the second step. In that sense, too much reliance on the use of a referent unit might inhibit the student from developing a more general scheme for multiplicative transformations between two fractions.

Rosa experienced difficulty in constructing a unit fractional quantity [a 1/31-meter bar] from an improper fractional quantity [a 37/31-meter bar]. That is, she was not able to partition a given (unpartitioned) 37/31-meter bar into thirty-seven parts to make a 1/31-meter bar, which indicated the lack of reversibility of her iterative fraction scheme involving an improper fraction (Protocol 4.23). Also, when she was asked to transform a 7/5-meter bar into a 1/15-meter bar, even with her success of the transformation activity, she struggled to find a multiplicative operator for the transformation. Her struggle indicated that the structure of Rosa's three levels of units with 1/15, 1/5 and 1 failed to expand to include 7/5. A part of the reason for her struggle was possibly due to her additive assimilation of 7/5 as one and two-fifths, not multiplicative assimilation as seven units of one-fifth, each of which consists of three units of one-fifteenth (Protocol 4.26).

Considering the facts that 1) fractions are usually taught at upper elementary levels in schools, and 2) two participating eighth grade students were recommended by their mathematics teacher in the middle school as exemplary students in the teacher's mathematics class, it was very surprising to the teacher-researcher (me) to witness Rosa's struggles that were revealed in her dealing with improper fractions. Therefore, the struggles of Rosa concerning fraction division and transformation reported in this study, which could be attributed to her lack of use of an interiorized FCNS, suggests that the curriculum in school mathematics for students' fraction learning needs to be revisited and reorganized to take into account the importance of students' construction of a multiplicative relationship of unit fractions to the referent whole.

It is worthy of note that Rosa's outstanding calculation ability for fraction multiplication and division, which was based on procedural algorithms learned in school, seemed to play as an *obstacle* for her to develop necessary mathematical schemes and operations for advanced fractional knowledge during the teaching experiment. Rosa's struggles, due to her inclination to rely on her procedural algorithms, implies that the aims and methods for teaching fractions in school mathematics need to be seriously reconsidered. Mathematical competence cannot be reduced to proficiency in calculation. That is, students' mathematical competence is not indicated solely by computational results or performances. Rather, results of students' calculating performances become meaningful only when the results are symbolizing the students' mental mathematical schemes and operations involved in their problem-solving processes. Often, Rosa could not use her results of calculation for fraction multiplication and division in establishing a quantitative relationship in the context of problem situations (e.g. her use of an invert-and-multiply algorithm in Protocol 4.4). Her experiencing such difficulty casts a question about building a curriculum for school mathematics on the assumption that students' training procedures and skills constitute essential steps in their further mathematical learning. The present study, therefore, implies that we, as mathematics teachers, should be able to provide our students opportunities to construct meaningful mathematical structures, processes and symbols for those processes, based on their own mathematical operations, rather than convey simple operational rules as pre-packaged products for the students.

Rational Numbers of Arithmetic (RNA) and Construction of Reciprocal Thinking

Rational Numbers of Arithmetic (RNA) can be regarded as the highest level of students' fractional knowledge because the RNA is a result of "the construction of abstracted fractions as an ensemble of operations of which children are explicitly aware" (Steffe & Ulrich, 2010, p. 266). Thus, in order to construct the RNA, a student needs to not only be able to *enact* the multiplicative transformation between two fractions but also be *aware* of the operations necessary to produce any fraction of the unit whole from any other fraction. The present study implies that such an explicit awareness in transformation activities requires a more abstract level of students' mathematical thinking in fractions, because the awareness can not be entailed until the students are able to conduct their available schemes and operations for transformation in re-presentation prior to activity and monitor the transformation activities at a level above the acting subject, themselves.

For future research in students' transformation activities between two fractions, I would suggest that we need to deeply look into students' construction processes of recursive multiplicative thinking. For students' multiplicative operations to be recursive, the students are able to externalize the results of their first multiplicative operations and operate on these results without losing the resulting structure. Olive (1999) already claimed that reinteriorization of a GNS inwards produced recursive divisions of the number sequences [recursive partitioning operations] and the outward direction of a GNS could lead to construction of exponentials. Reflecting on the two participants' transformation activities in the present study, the necessary two-step dilation processes for multiplicative transformation between two fractions in GSP was actually a recursive use of multiplicative operations both outward and inward rather than in just one direction. For example, in the transformation of 1/11-meter into 1/13-meter, Carol transformed the 1/11-meter into 1-meter by a dilation of 11 [outward] and then transformed the result into 1/13-meter by a dilation of 1/13 [inward]. More generally, to transform Fraction A into Fraction B, the student would need to first multiply by the reciprocal 1/A followed by B/1, as Carol did in her transformation of 1/11 into 1/13. The construction of such reciprocal relationships can be considered as a form of a modification of recursive multiplicative thinking through reinteriorization that would enable students to form the transformation quotient (B/A) in one step (which was not achieved by the students in this study). The study of such a reinteriorization would be appropriate for future research.

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