USING STUDENT VOICE TO DECONSTRUCT

COOPERATIVE, MATHEMATICAL PROBLEM SOLVING

by

LISA ANN SHEEHY

(Under the Direction of James W. Wilson)

ABSTRACT

Researchers have documented positive effects of cooperative learning on students' achievement, motivation, interpersonal skills, self-esteem, higher order reasoning and race relations. Although cooperative learning has been widely researched from a variety of perspectives, there is a need to both provide students' a stronger voice in this research base and to consider the impact of cooperative learning on mathematical experiences. Framed by a poststructural orientation, this study is an account of my inquiry into how the experience of cooperative, mathematical problem solving (CMPS) effects the mathematics of individual group members. Data were collected as the participants, three female, pre-service secondary mathematics teachers, worked cooperatively to solve different mathematical problems. Each of the three problem solving session was immediately followed by a group interview in which the participants discussed their perspectives on the PSS. The following day, I conducted individual, video-based recall interviews where each participant and I paused the video often to point to and discuss ways, primarily with respect to mathematical activity, in which she engaged in and experienced CMPS. As I applied Earley's (1997) theories about levels of consciousness as a theoretical framework for data analysis, emerging self/other binary tensions were identified. In analyzing the effect of these tensions on individual mathematical activity, it was apparent that many of the tensions hindering individuals' mathematical activity were connected to traditional cooperative learning structures. Using Derrida's (1997) deconstructive analytical tools, traditional structures were rethought in such a way that individual mathematical activity was privileged. I theorized that when the primary goal of implementing CMPS is to enhance individual mathematics, rather than improve interpersonal skills and joint products, the cooperative group becomes an invaluable setting for enriching individual mathematical autonomy as well as fostering a strong sense of community among group members.

INDEX WORDS: Cooperative Learning, Problem Solving, Mathematics Education, Poststructuralism, Deconstruction

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B.S. Ed., University Of Georgia, 1989

M.Ed., University Of Georgia, 1995

A Dissertation Submitted to the Graduate Faculty of The University of Georgia
in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2004

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DEDICATION

In loving memory and honor of my grandparents

Mrs. Virginia Russell Doty

Dr. Norman H. Russell, Jr.

The late Reverend Almer & Mrs. Vivian Sheehy

and my other "grandparents"

Mrs. Joan Light & the late Honorable Robert Light

The late Mr. & Mrs. Scott Blackstock

ACKNOWLEDGEMENTS

Of all the words I searched for as I wrote this dissertation, these were the hardest to find and the least adequate. To list someone's name on this page seems trivial in comparison to what he or she has done for and meant to me during my doctoral work. You will just have to trust me when I say that without the help of the people below, there would most certainly not be a dissertation following these words. So, thank you to…

my entire family for always believing I could do this and for all the times you didn't ask, "When will it be finished?"

all six of my absolutely adorable nephews (Bo, Noah, Luke, Gus, John, and Owen) whose births marked both the beginning and the end of my doctoral program

Mary Ann for your incredible patience, for marrying my dad, for all the times spent writing on the porch, for the many ways you inspired and taught me

Grandpa Russell for the "and then they were quietly proud" advice, the bond between us and the inherited passion for teaching

Grammy for your wonderful laughter that always makes everything seem okay

Karen for being my little sister, my playmate, my confidant, and my friend

my "Little Lyn" for the honor of being a part of your life as your "mom"

all my graduate school friends (Jake, Nancy, John, Chris, Andy, Becky, Kelli, Dennis, Amy, Brian, Serkan, Kursat, Samuel (just to name a few) who walked this sometimes treacherous but always exciting journey with me

my writing partner, Signe, for your listening ear, insightful advice and beautiful friendship

Wendy for all the cards, notes, smiles, laughter, advice and fun

my committee for the hard work, guidance, helpful suggestions, encouragement, challenges and mentoring you have provided.

all of my professors at UGA who helped me become not only a better teacher and researcher, but also a better person

Bettie St. Pierre for Derrida, the freedom and confidence to wonder, the invaluable help with my research questions, advice, friendship, and, of course, for being my "idol"

John Olive for the gentle and creative encouragement

my participants for their time, commitment, and desire to be
inspiring teachers—I know you are

Mom for always coming to my rescue and for the "dose of reality" when I said, "I quit, no more." You are the best cheerleader, best mother, and truly, my best friend.

my girls' night friends and my bible study group for the refuge and the most honest friendships I have ever known. They say friends are the family you choose—thank you for choosing me. You have made the good times better and the hard time easier.

And then there's Amy, Denise, Keith

Two are better than one, because they have a good return for their work; If one falls down, his friend can help him up…a cord of three strands is not easily broken.
Ecclesiastes 4: 9-12

There have been many times you helped me up and held me up.

For all those times, I am eternally grateful.

My last words of gratitude are to two men God has provided to take care of me, protect me, worry about me, inspire me, push me, believe in me and catch me if I fall.

The first is my father who is without a doubt my hero.
You have given Karen and me lives filled with unconditional love, safety, and lots of silliness. Whether it was reading Dr. Seuss, wrestling matches that Karen and I "won," super suppers, pick-up-sticks, thirteen year old heartaches, cheerleading "moms' meetings, graduations, careers, moves, relationships, celebrations and losses…
you've always been there for me and always known just where to be,
what to say or do, and how to make any situation better.
Any good that lies in me has been modeled to me by you.
Never has a daughter felt so grateful to and proud of her dad.

The other is my major professor. I met you at age 18 when I came into your office and rather timidly said, "I want to be a math teacher, a really good one. Can you help me do that?" From then on, there has never been a moment in my life when you weren't there helping me be just that. Now at age 36, after the three degrees and all the good and bad times you have walked me through, I still come to you with dreams and ideas.
And your answer is always the same, "Sure, I'll help you do that."

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They [teachers] think that something is happening when it really isn't, because of what they learned about cooperative learning—that cooperative learning does this and this for the student—when really they don't know what the student is thinking.

(Courtney, Group Interview 3)

What *are* students thinking? What are students experiencing and perceiving? What are students feeling? I believe these are always important questions to ask, and in the present study, they were at the heart of my investigation of the experiences, perspectives and mathematical activity[1] of three college students engaged in cooperative, mathematical problem solving (CMPS). More specifically, this dissertation is a written representation of my search for and unfolding of answers to the following questions:

1. *How do students engage in and experience cooperative, mathematical problem solving?*
2. *What binary tensions are present or emerge within cooperative, mathematical problem solving?*
3. *How are these tensions related to students' individual mathematical activity?*

As these three important questions framed and guided my research, answers emerged that were significant, meaningful and unexpected. Analogous to a mathematics problem, deeper understanding of an answer lies in the process of interpreting the meaning of, developing strategies for, and finding a solution(s) to the problem. Thus, in order to provide the reader with a richer understanding of what the participants said about their experiences with CMPS and the subsequent implications, my purpose in writing this dissertation chronicles the process of my study.

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[1] Mathematical Activity is a broad term encompassing a variety of ways students engage with mathematics, such as: conversations, reflections, written representations, observations, and demonstrations.

When I reflected on how to share the journey of my doctoral research, I found myself searching for the beginning and asking, “How did I get to this study? Where did it originate? What was the point of departure?” One could posit that a research study begins with identifying the problem or stating the research question. Others might argue that the theoretical perspectives or methodological orientations of the researcher are the foundation of a study. Although the above decisions and orientations were significant in shaping aspects of the present study (and are discussed throughout this dissertation), my “wonderings” about cooperative learning emerged from my experiences long before these decisions were made. St. Pierre (1995) began her dissertation with the musings of Foucault on the impetus for engaging in research—I echo those words of Foucault as I begin my own work.

> Each time I have attempted to do theoretical work, it has always been on the basis of elements from my own experience—always in relation to processes that I saw taking place around me. It is in fact because I thought I recognized something cracked, dully jarring, or disfunctioning in things I saw, in the institutions with which I dealt, in my relations with others, that I undertook a particular piece of work. (Foucault, cited in St. Pierre, p. 9)

Through a variety of experiences, I became aware of and curious about some of the “cracks, jarrings and disfunctionings” in my own understanding of cooperative learning. What follows is a discussion of those experiences most influential and memorable in the journey that brought me to this present study.

Background

Over many years, I have learned about cooperative learning through studying pedagogical methods, learning theory, educational research, curriculum development, and cognitive psychology. By using cooperative learning extensively as a high school mathematics teacher, I have learned much from my classroom experiences and from my students. I have also learned from my colleagues by both attending and conducting staff development courses focused on cooperative learning. Thus, I have organized this background discussion into three sections: learning about cooperative learning, teaching with cooperative learning, and teaching about cooperative learning. Although these sections are somewhat chronological, there is a great deal of overlap and interplay.

Learning about Cooperative Learning

I remember participating in reading groups in elementary school, feeling a spirit of cooperation during my two years in a Montessori elementary school, working with lab partners in high school, and being assigned out-of-class group projects throughout my college career. However, I do not remember any of those experiences referred to as "cooperative learning." My first formal introduction to the term *cooperative learning*, along with its associated pedagogies and underlying philosophies, was in 1986. It was my sophomore year in college, and my introduction to educational psychology professor presented a unit on classroom cultures. Through this unit, I was first exposed to the work of two leading researchers in the field of cooperative learning, David Johnson and Roger Johnson. Johnson, Johnson, and Holubec (1991) classified the structure of general classroom instruction into three main forms: competitive, cooperative, and individualistic. They claimed:

> For the past fifty years competitive and individualistic efforts have dominated classrooms…. Cooperative learning has been relatively ignored and underutilized by teachers even though it is by far the most important and powerful *way* to structure learning situations. (emphasis added, p. 26)

As I reflected on this three-way differentiation and on my own experiences as a student, I intuitively believed Johnson et al. were right and subsequently wanted to provide my students with a classroom that was cooperative in nature. In accepting Johnson and Johnson's claim, I felt challenged to answer the following question: What is meant by this *way* they called cooperative learning? In an effort to answer this question, I enrolled in courses (throughout my undergraduate and graduate studies) that were either explicitly about cooperative learning or that incorporated the study of this pedagogy into the curriculum. Gaining further knowledge through coursework and my own independent readings not only increased my interest but also enriched my understanding of cooperative learning.

When I discuss what I believe cooperative learning is, I find it helpful to consider first what cooperative learning is not. According to Johnson and Johnson (1994a), putting students into groups does not necessarily structure cooperation among students. They summarize that cooperation is not:

1. Having students sit side by side at the same table and talk with others as they do their individual assignments.
2. Having students complete tasks individually after instruction with those finishing first helping other students with their work
3. Assigning a report to a group where one student does all the work and others put their name on it.

Many of my students' and my own previous experiences with group work were conducted in a manner reflected in one of these above statements. In order to dispel some preconceived ideas about cooperative learning, I found the statements significantly helpful as a way to introduce and begin to implement cooperative learning in my classroom.

Identifying these noncharacteristics is a helpful start, but there is still the question, What *is* cooperative learning? In many ways, the answer depends on whom you ask. Because it is true that "many different techniques are couched under the beguilingly simple umbrella term *cooperative learning*" (O'Donnell & Dansereau, 1992), the term can be difficult to define. Throughout my own studies, I have continued to read about four of the most widely researched, discussed, and implemented methods (and variations of them) for cooperative learning: Student Team Learning, Jigsaw, Group Investigations and Learning Together.

The *Student Team Learning Method* was developed by Slavin (1983a), and while many possible variations of this method have emerged, the two most popular are Student Teams-Achievement Divisions (STAD) and Teams-Games-Tournaments (TGT). In both variations, teachers divide classrooms into teams who then compete against each other. After presenting a lesson, the teacher assigns a follow-up assignment for the teams to complete. With respect to team make-up and formal assessment, the two methods diverge. STAD teams are heterogeneous with respect to achievement and use individual quizzes to assess the improvement of each team member from the previous lesson. The teams' overall grade is then determined based upon the collective improvement of the team members. TGT employs games and tournaments in which individuals in homogenous teams (based on past performance) compete at tournament tables in order to earn points for their teams. The tournaments are organized such that students compete with students from other teams who have been identified as being at the same academic level (Artz & Newman, 1990; Johnson & Johnson, 1994a; Kagan, 1985; Kluge, 1990; Owens, 1995; Slavin, 1985,1990).

The *Jigsaw Method*, developed by Aronson (1978), is primarily structured by the division of material or tasks among individual group members. Each student is responsible for completing a task and then reporting to and teaching the rest of the group members. An intermediate step may be employed in which a student from one group meets with students from other groups who had the same assignment in order to compare results. For example, students studying the Civil War would each research a different battle on their own. Group members would then come together to share information in order to prepare for an individual test or quiz or to create a joint assignment. An example from the mathematics classroom might involve students in a team each working to represent a set of data or a problem in different ways, such as, algebraically, geometrically, and numerically. The students would then come together to compare and discuss the various representations (Artz & Newman, 1990; Johnson & Johnson, 1994a; Kagan, 1985; Kluge, 1990; Owens, 1995; Slavin, 1983a, 1985, 1990).

The *Group Investigations Method*, developed by Sharan & Sharan (1976), is similar to the Jigsaw Method in that groups must allocate sub-tasks to individual group members. However, with the Group Investigations Method, each group has a different task and thus works to create a presentation in order to teach the rest of the class. Unique and central to this method is the use of open-ended problems that provide students a significant amount of control in determining subtasks. Once the teacher divides the class into groups, these groups then discuss and determine (with other groups) how to further divide up tasks in order to investigate a problem or topic. Next, further divisions occur within each group as individual subtasks are assigned according to the students' decisions. When groups have completed and discussed their subtasks, group presentations are made to the class. Students are then assessed on their presentations (Johnson & Johnson, 1994a; Kagan, 1985; Kluge, 1990; Owens, 1995; Sharan & Sharan, 1992; Sharan, 1994; Slavin, 1990).

Learning Together is an approach to cooperative learning offered by Johnson and Johnson (1975). In contrast to the three methods discussed above, Johnson and Johnson suggest that all students should work together on the completion of one assignment, thus sharing a common goal. Rather than dividing work or being assessed individually with scores contributing to a group score, the *Learning Together* approach is based on the belief that students should work simultaneously on the same task or problem. Johnson and Johnson (1994b) offer five elements essential to creating successful cooperative learning environments in classrooms: positive interdependence, both individual and group accountability, teaching students the required interpersonal and smallgroup skills, promotive interaction, and reflective group processing. They also offer methods and strategies for encouraging each element. For example, suggested methods for encouraging positive interdependence include using one set of materials or assigning each group member a specific role (e.g., recorder, presenter, calculator, or leader). Emphasis is placed on helping students learn to work in a group and on encouraging group members to reflect on and discuss the ways in which their group is functioning. Assessment occurs in three forms: group behaviors, the collective output of the group (e.g., a group presentation, test score, or problem solution), and follow-up individual assessment (Artz & Newman, 1990; Johnson & Johnson, 1990, 1994a; Kagan, 1985; Kluge, 1990; Owens, 1995; Slavin, 1990;).

A variety of pedagogical techniques and classroom management strategies are associated with cooperative learning. Although each of the above models of cooperative learning has its own unique qualities and philosophies, important commonalities have also been discussed. For example, Kluge (1990) pointed to a unifying characteristic among the models when he defined cooperative learning as “a method of classroom instruction in which students are placed in small groups and work together to achieve a common goal” (p. 1). Offering a more specific description of student actions, Artz and Newman (1990) claimed “cooperative learning involves a small group of learners, who work together as a team to solve a problem, complete a task, or accomplish a common goal” (p. 2). In examining the philosophies that teachers need to adopt to successfully implement cooperative learning, Evans, Gatewood and Green’s (1993) review of related literature led to the following observation:

> Using cooperative learning as a teaching strategy requires three basic assumptions:
>
> - Students can learn from each other.
> - Students can, with instruction, govern themselves.
> - The teacher is not the only source of information. (p. 3)

Although there are numerous similarities among these many descriptions of cooperative learning, enough differences exist to make it difficult to know what a teacher means when he or she claims to be using cooperative learning in his or her classroom. Thus, it is important that researchers communicate as clearly as possible what style of cooperative learning is used and how it is implemented. As with any pedagogy, teachers mold cooperative learning to suit their own students, classrooms and styles of teaching. In order to provide a deeper understanding of what cooperative learning means *to me*, what follows is a description of how I molded and implemented cooperative learning in my own classrooms.

Teaching with Cooperative Learning

As noted, teachers make decisions to adapt and implement pedagogical methods in a variety of ways based on many factors, including prior knowledge and experience, students and classroom culture, currently implemented pedagogies, and knowledge about cooperative learning. An individual teacher's philosophy of teaching shapes and influences many of these factors. Thus, before I discuss how I implemented cooperative learning in my own classrooms, I will share my philosophy of teaching.

Philosophy of Teaching

As a mathematics teacher, one of my primary responsibilities is to help bring my students to a place where they not only feel confident in their abilities to solve mathematical problems, but also have the desire to explore the unknown. Thus, one of the goals I have for my students is that each one develops *mathematical power* (National Council of Teachers of Mathematics (NCTM), 1989, 1991). Mathematical power is the ability and desire to explore, conjecture, and reason logically, as well as the ability to use a variety of methods for effectively solving nonroutine problems. Students exemplify mathematical power in the words I most like to hear in the classroom, "Miss Sheehy, don't give me anything more than the problem. I want to figure it out by myself or with my team." In striving for this kind of mathematical power, I believe it is important to help students acquire and learn to use tools to solve mathematical problems. This acquisition of strategies and skills, however, is only a part of the strategy for encouraging the development of mathematical power. Another part lies in fostering a sense of curiosity in students. It is essential that students ask *why*, receive opportunities to make and explain their own conjectures, and not accept something as truth until they feel it is sufficiently justified. Until desire and curiosity are integrated into a student's learning process, all the mathematical ability in the world will not give the student what is, in my opinion, true *mathematical power*—a desire to solve a problem or explore a situation and the confidence to engage in the process.

I consider another of my primary responsibilities to be the fostering of a sense of *community* in the classroom. Thus I work hard to help students get to know, appreciate, and respect each other. Einstein is frequently quoted as having said, “Do not worry about your difficulties in mathematics. I can assure you that mine are still greater.” This has been one of my students’ favorite quotes. Knowing that the esteemed Albert Einstein possessed a learning disability, they have referred to him as “Al, the learning disabled genius.” This label reflected an understanding and a connection with Einstein as a brilliant and accomplished mathematician who, at times, experienced difficulties with mathematics. Learning is a struggle, a process, and an achievement for all of us. It does not happen in the same way or at the same time for any two people. If students and teachers understand these aspects and choose to be patient with one another both as individuals and as learners of mathematics, the classroom community offers a unique and supportive environment for learning together.

The implementation of cooperative learning is an integral and necessary part of the development of both mathematical power in students and a sense of community in the classroom. Cooperative learning blends well with my philosophy about learning, students, and teaching, and reflects my belief that people are relational beings. Cooperative learning is the primary framework within which I plan instruction, organize my classroom, and support mathematical learning.

Implementation of Cooperative Learning

The theories and recommendations of Johnson and Johnson heavily influenced the decisions about and style of cooperative learning I implemented. They maintained that “in order for cooperative learning to be effective, lessons must be structured so students do in fact work cooperatively with each other…. [This structuring] requires an understanding of the components that make cooperation work” (Johnson & Johnson, 1994b). I strove to incorporate the following five basic conditions Johnson and Johnson (1990) claimed must be present in any cooperative activity or relationship into my classroom activities:

1. Teachers must clearly structure positive interdependence within each student learning group.
2. Students must engage in promotive (face-to-face) interaction while completing math assignments.
3. Teachers must ensure that all students are individually accountable to complete math assignments and promote the learning of their groupmates.
4. Students must learn and frequently use required interpersonal and small-group skills.
5. Teachers must ensure that the learning groups engage in periodic group processing. (pp. 105-106)

This list of conditions has served as a yardstick to develop and assess the effectiveness of cooperative learning activities. The following activity is a specific example of how I integrated and implemented these five elements into lessons and, as such, demonstrates the role of cooperative learning in aiding mathematical learning.

Example of a Cooperative Lesson

Believing students must learn about cooperative learning before they learn through cooperative learning, the nature of activities that took place in my classroom relied heavily on the amount of experience students had with cooperative situations. I used the following activity discussed in this section in a ninth grade Pre-Algebra course with students who had limited experience with cooperative learning. Thus, prior to engaging in cooperative learning activities, I spent a few days with students discussing the following questions: What does it mean to work with others and to be in a team? How or when do you need to assign roles? What is appropriate conversation in a team? What are the benefits of being in a team? What are some problems that may arise? How can I be encouraging to others in my team? It was only after cooperative learning was discussed and initial concerns were addressed that cooperative activities like the following were introduced.

Materials Needed: One set of at least twenty 1 x 1 inch squares, paper, pencil, graphing calculators, overhead transparencies, and overhead pens.

Activity: The task of your team today is to explore all the different dimensions of rectangles that can be constructed using specific numbers of your square manipulatives. For example, you can make only one type of rectangle (dimension 1 x 2) using exactly 2 squares. So the questions to you are: How many rectangles of different dimensions can be made with exactly three squares? Four squares? and so on. And why? Investigate this situation with your team. Please keep a record of the data your group collects and a record of any representations of your data (scatterplots, graphs, pictures, etc.).

Roles: Working with your team of four, please discuss these suggested roles and how they might function best in your team and then assign them accordingly. Feel free to create new roles and/or use these suggestions: Leader/Encourager, Technology Expert, Records Keeper, Presentation Coordinator.

Discussion Questions: Because I did not want the following discussion questions to guide their initial explorations, I gave them to the teams after they had investigated the problem for a while.

1. Make a list of all the interesting patterns that your group has found.
2. Can you find and discuss any mathematical reasons why these patterns would be occurring?
3. Can you think of ways to categorize different numbers of squares based on their ability to form one or more rectangles?
4. Can you develop any theorems or algebraic formulas to model the conclusions of your group?

Presentation: Each group will now develop a presentation of their mathematical findings and/or conjectures to the class. Each member needs to have a part in your group's presentation. Please hand me your team’s discussion questions at the end of your presentation.

Journal Entry: Describe what you and your team have been working on today. Explain how your team approached the investigation. What do you think you personally contributed to this investigation? Briefly summarize the conjectures your team made about this activity. Please bring me your journal when you have finished this entry and return to your team to discuss how your team worked together on today's investigation and ideas for about how you could work together even better tomorrow.

Teaching about Cooperative Learning

As I incorporated cooperative learning into my teaching of secondary mathematics, I observed students learning in powerful and meaningful ways—ways that I believe are unique to cooperative learning. I observed firsthand the positive results I read about in my university coursework and/or heard about through in-service workshops. For example, greater self-confidence and an increased interest in mathematics replaced students' insecurities and anxieties. As a spirit of camaraderie increased in the classroom community, racial tensions decreased. In my experience, students became more motivated, more successful, and more engaged when cooperative learning was a significant part of the mathematics classroom. Thus, students' mathematical learning was influenced in positive ways. In survey after survey, when I asked my students to reflect on cooperative learning, the majority echoed the above observations and shared my enthusiasm for cooperative learning.

Because of my enthusiasm, as well as that of my students, I was invited on a number of occasions to serve on panels, conduct workshops, or make presentations about cooperative learning to fellow classroom teachers and school administrators. I expressed (to my county curriculum director) my reasons for and expectations of leading workshops about cooperative learning like this:

> Cooperative learning is amazing! It has made such a difference in my classroom and in the ways my students engage with mathematics. I just know if other teachers knew about this method I have been learning about—they would use it,

too. Then more students would have wonderful experiences with mathematics and in the classroom in general.

The teachers I worked with in staff development courses were also enthusiastic and optimistic about the potential benefits of cooperative learning as they engaged in a variety of activities including:

- Learning about cooperative learning strategies (mostly from Johnson and Johnson)
- Listening to student testimonials (letters or actual visits from my own students)
- Studying research results
- Participating, as students, in a variety of cooperative learning activities.

I saw my role in these workshops as twofold: to provide relevant information about and experience with cooperative learning and to demonstrate to teachers through both student testimonies and personal experiences how effective cooperative learning could be.

At the end of a workshop or presentation, teachers generally communicated that they planned to incorporate cooperative learning in their classrooms and, consequently, expected their students to learn mathematics in ways that were more meaningful. It proved to be a common occurrence that most teachers left workshops believing they would experience success with cooperative learning. Most teachers, however, also experienced frustrations when trying to implement cooperative learning in their own classrooms. I repeatedly received feedback like the following:

- "I just couldn't make it work for me."
- "My students couldn't handle it."
- "I know it works for some teachers. I did it just like we did, but I didn't get similar results."
- "My students hated it."

Many of the teachers with whom I talked found it difficult to formulate specific questions about the apparent failure of cooperative learning. They were, however, able to express frustrations with the research and theory that described the learning opportunities that they were having difficulty fostering in their own classrooms. Two general questions I often heard directed to me were "Why do you believe so strongly in cooperative learning?" and "How does it work for you and others?" I wanted to understand why this scenario occurred so often and about my lack of acceptable responses to their questions. The experience of teaching about cooperative learning caused me to begin wrestling with these tough questions—questions about cooperative learning that felt seemingly unanswerable.

As I searched for answers, I began to believe this previously described scenario so often played out because, in general, these teachers did not view cooperative learning as more than just a pedagogical tool, system, or procedure to be learned. Although I believed I was effectively incorporating cooperative learning in my classroom, I did not fully understand why the method worked. I had a good idea of what I was doing and thinking, but I did not understand the process my students were going through. I realized that, for me, cooperative learning had become a *black box phenomenon*. For years, I placed students into this pedagogical "black box" called cooperative learning and out came motivated students who were much more involve in mathematics. Research results and experiences of others corroborated the effectiveness of this treatment and the positive outcomes I was observing. We must not be content with achieving success without addressing the questions of *how* and *why* it works, when it does. By opening up the "box" and revealing the processes of students, a deeper understanding and picture of the classroom experience called cooperative learning will emerge.

Pilot Study Implications

Over several years, my discontent with my own understanding of the processes and experiences of student grew. As I entered the doctoral program, I knew my understanding was limited and that there was (and still is) much to be learned about cooperative learning. In an attempt to understand cooperative learning more deeply and to get a glimpse inside that elusive black box, I designed and implemented a pilot study in the fall of 1999. In addition to testing ideas and methods for my dissertation research, my goal for this pilot study was to develop what I called a *cognitive video clip* of my students' mathematical thinking during cooperative problem solving. Theoretically framing this inquiry were social constructivist learning theories (discussed in chapter 2). More specifically, I collected data from a single cooperative learning experience of four teenage girls and analyzed it with respect to Vygotsky's general *genetic law of cultural development,* which states:

> Any function in the child's cultural development appears twice, or on two planes. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category…. Social relations or relations among people genetically underlie all higher functions and their relationships (cited in Wertsch & Toma, 1995).

I developed observation and interview protocols with the intent of identifying the movement of students' thinking from the intermental plane to the intramental plane. In particular, interview questions addressed when and how each participant used the ideas of a group member and offered her own ideas to the group. I specifically looked for instances of scaffolding (see Appleton, 1997). Therefore, I asked questions like the following:

- How did the group, the other people here, help you make sense of this problem? Either in general or at a specific moment?
- Do you remember a point when you thought, "Her idea helped me"?

The day I collected data, I heard the participants more often talking about what "blocked" their learning than sharing instances of scaffolding. As I sat with my data later that evening and transcribed the interview tape, some of the statements I heard repeatedly truly surprised me. For example, the girls explained:

- How frustrating it was to have only one set of manipulatives because they did not get to try out their own or someone else's idea. They could only try out either the ideas that the team agreed on or those that the girl with the manipulatives liked.

- How brainstorming was good but that they did not get to think through an idea before they threw it out there.
- How they really wanted some time by themselves with the data they had collected from the problem solving scenario.

I realized that these young women were speaking about the *nature* of their cooperative learning experiences. The direction and design of the present study was rapidly changing. Understanding the experiences of students engaged in CMPS should involve much more than a careful documentation and analysis of interactions with respect to the intermental and intramental planes. By challenging my previous beliefs and assumptions about group work, I began rethinking and rebuilding an understanding of this *way* called cooperative learning—its purpose and its design. I began to question other assumptions I had made about cooperative learning. For example, because students said one set of manipulatives was possibly hindering their learning, I wondered what else in the traditional discourse of cooperative learning might not be what I thought it was—or did what I thought it did.

The possibility that cooperative learning might limit individual mathematical activity was, quite honestly, not something I had considered before. I was frustrated with myself—feeling confused and guilty that I had not focused on what now appeared to be such an important aspect of CMPS. I had just seen positive results with the participants (the problem was solved and everyone participated), and they had each expressed positive feelings about cooperative learning claiming they liked using it in mathematics classrooms.

Because I had given my students end-of-course surveys that focused on what they liked and did not like about cooperative learning, I thought I was paying attention to their responses. However, these surveys consisted of questions and opinions about the outcome, the final product, and the emotional responses at the end of a course. In this pilot study, I heard about students' thought processes, not in the theoretical sense I planned for, but in a practical way connected to mathematics and learning. One significant implication of the pilot study was the decision to focus my future subsequent efforts on investigating the experience of cooperative learning from the perspective of the students involved. To do that I needed to listen to *student voice* in a way to allow me to hear and be aware of both the limits and the possibilities cooperative learning creates for students.

Another significant implication of the pilot study was the identification of aspects of my own implementation of cooperative learning that, although it may make students feel more confident or increase their motivation, might interfere with their *individual mathematical activity*. My job as a teacher of mathematics is to create the best possible environment for students to engage in, enjoy, and benefit from the study of mathematics. My job as a researcher is to ask questions and work towards answers about how best to create such an environment. In his discussion of research on cooperative learning in secondary mathematics, Owens (1995) argued the connection between how the content of mathematics is learned in the context of problem learning needs to further investigation. Recognizing "studies that integrate context and content [are] difficult to perform," he claimed these types of studies "are most likely to have a significant impact on the profession" (p. 169). The notion of integrating content and context challenged me to listen to students as they talked about their general experiences with cooperative learning and how these experiences effected their own mathematical activity.

Poststructural Orientations

The design of the present study reflects my desire to provide students opportunities to discuss their cooperative and mathematical experiences and provide the means for connecting those experiences to individual mathematical learning. Poststructuralism offered the language and theory needed to articulate the direction and purpose of the present study. To provide the reader a deeper understanding of the previously stated research questions, I offer a discussion of the two aspects of poststructural theory I consider most central to the study—the notion of self and orientation to research.

Notion of Self

Foundational to the enlightenment and the humanist movement was the centering of humans and their ability to apply reason in order to construct knowledge—to understand, define, classify, predict and even control the world they lived in. Leaders of this movement professed "by illuminating, through human reason, the prejudice, dogma, and authority of older religious and political institutions, [humanist] individuals would become individually and collectively empowered to make the world a better place" (Walshaw, 2001, p. 472). For Rene Descartes (1596-1650), this use of reason can be seen in his project of "rejecting those foundations he had accepted all his life as true and rethinking them for himself" (St. Pierre, 2000, p. 494). During this process of rejecting all knowledge based on his senses, Descartes (1993/1637) proclaimed that the "truth–*I think, therefore I am*–was so firm and so certain…[he] could accept it without scruple" (p. 19). With this claim, Descartes reflected two significant ideologies. He established the mind/body and knower/known dualities of humanism, and he essentialized the self as a reasoning subject able to know an objective world.

Poststructuralism is a theoretical orientation rooted in questioning the assumptions that produce the objectivism/realism dualities of humanism. Poststructuralists maintain that truth and knowledge are multiple, historical, temporal, bound up in power relations, and situated in cultural discourses and practices. Consequently, they reject Descartes' proposal that knowledge is separate from the known (Hacking, 1999; Griffin, 1998). St. Pierre (2000) suggested, "As the centered center of his world, the Cartesian ego founded modern philosophy, and the humanist individual [was essentialized as] the origin of truth and knowledge" (p. 494). Understanding the term *essence* to mean "the inner essential nature of a thing, the true being of a thing" (Van Manen, 1990, p. 177), poststructuralists call into question the notion of a centered and essentialized self. The *self* of humanism is now challenged by "poststructuralist theories of subjectivity [which] reject the humanist notion of a unified, fixed self that has a stable essential core" (Jackson, 2001, p. 386). Rather, poststructuralists propose that the self is multiple, shifting, constantly being constructed, and always in process.

Entering the discussion about what it means to be "constantly constructed" and about who or what is doing and/or influencing the constructing, Howe (1998) proposed, "Human beings are self-creating…. It is not as if human beings are simply pushed to and fro by existing social arrangements and cultural norms. Instead, they actively shape and reshape these constraints on behavior" (p. 16). I agree with Howe's interactive description of individuals interacting with their environment. With respect to this interaction, however, I further assert that the self is symbiotically connected to, but not essentialized by, its subjective experiences of the world. Best and Kellner (1991) characterized this symbiosis as a "transcendental [intuitive]/empirical [experiential] doublet whereby Man both constitutes and is constituted by the external world" (p. 42). For me, that statement begs the question: How does this "constitution" occur?

Throughout the present study I became increasingly aware of "a distinct sense in which [the self] is produced, and *re*-produced, in and through social interaction" (Freeman, 1999, p. 106) within a cooperative learning group. In her own words, each participant described her subjective experience as a relational process where she "sensed [herself] as both constituted by and constituting, the other [participants]" (Gergen, 1999, p. 172). For example, if a participant perceived the mathematical knowledge of a teammate as superior to her own, this participant may have a tendency to abandon her own mathematical ideas and adopt those of her teammate. Whereas, if this same participant felt superior in her own knowledge of the problem, she might work hard to align her teammates with her ideas and be less likely to listen to others.

The self on whom I focused this study (the individual in a cooperative team) was multiple, shifting, and constantly a part of a double move within her environment. Thus, in order to open-up the experiences cooperative learning both created and limited for students, I investigated the "double moves" of three students as they engaged in the discourses and practices of CMPS.

Research Process

Within a positivistic paradigm, the work of researchers is grounded in a desire to reveal objective truth by using reason and logic to produce unbiased, reliable and valid results. When these researchers recognize a problem or ask a question, their methodology for finding an answer often involves identifying and isolating variables. These variables are then analyzed with respect to regularities and/or relationships, thus, the goal of research is to find and/or create structures. When these structures become limited or inadequate, positivistic researchers ask new questions or identify new variables in order to adapt or change previous structures.

Those who work from a poststructural research paradigm believe that truth is subjective, multiple, and contextual, and thus are fueled by different, and somewhat contrasting, desires. In “a nutshell,” the work and challenge of poststructural theory is to continually question our taken-for-granted structures and/or assumptions (Derrida, 1997). Thus, through the lens of poststructural research, the problem to investigate is found in structures already in place. By questioning regularities, the goal of researchers is to deconstruct by illuminating the subjectivity and multiplicity of the objective truth inherent in positivism.

Taking up a poststructural theoretical orientation meant my work and challenge throughout this study was to orient the participants and myself toward the continual questioning of the structures and traditions of cooperative learning activities. My reading of Bové’s (1990) essay entitled “Discourse” significantly influenced the critiques and methods I employed as I examined the functions and effects of cooperative learning. In his discussion of the poststructural concept of *discourse,* Bové claimed we should no longer focus on the question “What does it mean?” but rather on another set of questions. Within his list of proposed questions, I replaced Bové’s word discourse with cooperative learning and subsequently focused on:

- how cooperative learning functions.
- how cooperative learning gets produced and regulated (p. 54).

The metaphor of the black box was no longer just to open the lid and look inside, but to open up the possibilities of questioning and rethinking the entire structure of the box. Therefore, in the present study, the participants and I questioned the functions and effects of the structure of cooperative learning as they related to CMPS.

As I developed my own research questions about mathematical, social, and personal experiences during CMPS, I emphasized the value of including student voice, recognizing the double move of subjectivity, and asking questions reflective of a poststructural orientation to the study. However, phrasing research questions, which kept the notion of the double move of subjectivity central, was linguistically problematic. Because constructivist learning theories are so pervasive in mathematics education literature, the meaning of phrases common in poststructural literature such as, "Students simultaneously constructed and were constructed by their cooperative learning experiences" was potentially ambiguous. Readers' prior understandings and uses of the word "construct" to describe the process of students creating their own mathematics increased the likelihood of hindering and/or limiting my ability to represent the goals of the present study. Acknowledging this potential to misconstrue vocabulary common in the discourse of poststructuralism, I represented the notion of the double move by replacing phrases like "simultaneously construct and be constructed by CMPS" with "simultaneously *engage in* and *experience* CMPS." Thus, the present study was framed and guided by the following research questions:

1. How do students engage in and experience cooperative, mathematical problem solving?
2. What self/other binary tensions are present or emerge within cooperative, mathematical problem solving?
3. How are these tensions related to students' individual mathematical activity?

In the following chapter, I place this study within the literature and research base of cooperative learning and offer a more thorough discussion of theoretical positions intended to illuminate the significance of these research questions.

CHAPTER TWO: SITUATING MYSELF WITHIN THE LITERATURE

The publications of academic work, research studies, and theoretical discussions provide a forum for conversation among individuals with similar or overlapping interests. This chapter offers a discussion of my position within the academic conversations most relevant to the present study. Because cooperative learning is "one of the most widespread and fruitful areas of theory, research, and practice in education" (Johnson & Johnson, 2000, 1st section), the conversations surrounding it are diverse and extensive. Thus, I have organized this review and critique of literature into two broad categories: the *empirical* literature on cooperative learning research and the related *theoretical* literature.

Empirical Literature

When Lou, Abrami, Spence, Poulsen, Chambers, and d'Apollonia (1996) performed a comprehensive search of the literature, they found over 3,000 published articles with which to perform a meta-analysis on within-class grouping studies. Johnson and Johnson (1994b), who have spent their well-respected academic careers studying cooperative learning, have proposed that we know far more about the effectiveness of cooperative learning than we know about almost any other facet of education.

Within this significant (and almost overwhelming) body of research, the "positive outcomes of cooperative learning strategies have been well documented by studies at all grade levels and in all subject areas" (Artz & Newman, 1990, p. 7). Reviews and meta-analyses of this research have been conducted within a variety of frameworks and paradigms (e.g., Johnson & Johnson, 1993; Johnson, Johnson & Stanne, 2000; Sharan, 1980; Slavin, 1983a, 1985). Similar reviews have been conducted within the field of mathematics education (e.g., Cohen, 1994; Davidson, 1985; Davidson & Kroll 1991; Springer, Stanne & Donovan, 1999; Webb, 1989, 1991). The empirical reviews and studies I have selected to discuss can be categorized using the three following questions that reflect the evolution of my own thinking regarding cooperative learning research:

- What are the *effects* of cooperative learning on students?
- What *processes* are students involved in during cooperative learning activities?

- What are the *opinions* of students involved in cooperative learning activities?

Effects

Many of the positive effects of cooperative learning mentioned previously were identified and verified by research studies. These studies were generally quantitative in nature and conducted within a positivistic research paradigm. In a significant meta-analysis in this area of research, Johnson and Johnson (1992) claimed to have “reviewed all the published and unpublished research conducted over the past 90 years comparing competitive and individualistic efforts” (p.176). They found over 520 experimental studies and 100 correlation studies focusing primarily on the effects of cooperative learning (p. 176). The most common design of these treatment and effect studies involved researchers applying a treatment (a form of cooperative learning) and then collecting measurements of a dependent variable (e.g., achievement, attitude, race relations) by using pre-designed instruments (e.g., tests, Likert scales, researcher observations, surveys). The objects of study were, generally, either a one group of students given pretest and a posttest or a control group (students in a traditional, teacher-centered classroom) and an experimental group (students in cooperative, student-centered classrooms). Either way, statistical analyses of the data were performed, results compared, and conclusions drawn about the effectiveness of cooperative learning.

Researchers have overwhelmingly concluded that cooperative learning is an effective classroom strategy *if* it is clearly understood by the teacher and used in a manner paralleling its underlying philosophies. Rottier and Ogan (1991) asserted that if this understanding is present, “cooperative learning tends to promote the following positive effects: higher achievement, the greater use of reasoning strategies, more positive relationships between students, more positive attitudes toward subject matter, and higher self-esteem” (p. 16). Likewise, in their review of the cooperative learning literature, Bellanca and Fogarty (1991) reported that research has repeatedly produced the following results:

> All students of all ability levels in cooperative learning groups enhance their short and long-term memory as well as their critical thinking skills and that because cooperative learning leads to positive interaction among students, intrinsic learning motivation and emotional involvement in learning are developed to a higher degree. (p. 242)

With respect to assessment and grading, students seem to perceive joint (group) grades as more fair than traditional grading systems (Slavin, 1985). Cooperative learning has also shown favorable results with respect to ethnicity and cultural issues in the classroom. For example, Kluge (1990) summarized that "in some cases, cooperative learning was found to be more likely to increase the academic achievement of non-white students than that of white students" and that "liking among students of different races increases when cooperative learning methods are used" (p. 12).

The results of "cause and effect" studies on cooperative learning have been informative and invaluable both to the field of mathematics education and to me personally. The positive results from this genre of research first inspired me to integrate cooperative learning strategies and philosophies into my own teaching. In my opinion, however, effect studies on cooperative learning have reached a point of saturation; and yet such studies continue to make up a significant proportion of the research done on cooperative learning. For example, I recently conducted a search (using ProQuest) of dissertation studies conducted from 1998 to 2003 using the keywords "cooperative learning" and "mathematics." From my reading of abstracts, I initially identified 73 studies fitting these criteria. I was surprised to find that 47 of these studies were effect studies (much like I ones have described) and, for the most part, reported the same positive results. There can be little doubt that "cooperative learning is here to stay…. [Not only is it] based on a profound and strategic theory, [but there is clearly] substantial research validating its effectiveness" (Johnson, Johnson, & Smith, 1995, p. 5). Although duplicating research can be valuable, the above research base lacks examination of a crucial component of cooperative learning—process.

Processes

As I alluded to in chapter 1, cooperative learning can feel like a *black box phenomenon*. It is as we could put students into cooperative learning groups and expect the outcome to include positive results, cooperative students, and enriched learning. Without knowledge of the group processes that occur, “it is [difficult] to understand *why* some cooperative learning studies have found positive effects on achievement” (Webb, 1991, p. 366). Because studies that employ a black box approach neglect to investigate the manner by which these results are produced, we are often left with the questions of *why* cooperative learning does (and does not) work. Shifting research attention toward a focus on the *processes* involved in cooperative learning—to ask what is going on inside that box and offer descriptions of this activity—is one way to address the question, “Why?”

- The literature is replete with calls for more *process*- and less *product*-oriented research designs. For example,
- In their call for programmatic research on small-group process, Good, Mulryan and McCaslin (1992) discussed a need for “an improved knowledge of cooperative group processes” (p. 193).
- In an overview of research of cooperative learning in mathematics education, Davidson and Kroll (1991) suggested “an important question to consider is just exactly what goes on during various types of cooperative learning” (p. 363).
- With an emphasis on the cognitive processes of the individual (rather than the impact, product, or result), Maurer (1987) posed the question: “What learning mechanisms take place in group situations that foster individual cognitive growth?” (p. 180).
- In his discussion of cooperative learning in the secondary mathematics classroom, Owens (1995) argued that most studies of cooperative learning have had cooperative learning rather than the individual as the object of study. He claimed that “research in mathematics education…is replete with studies of cognitive

> structuring, representation, and so on—but not in the context of cooperative learning" (p. 169).

In considerable number, researchers have been responding to these calls for more process-oriented research. I have selected three reports from this literature that represent a variety of research goals and methods intended to identify and describe the activity inside the box

In a review that brought together a comprehensive body of research, Webb (1991) systematically analyzed the relationship between verbal interactions and mathematical learning in small groups. Her theoretical approach was based upon the belief that the process of giving and receiving help positively influences learning outcomes. All 17 studies reviewed by Webb share similar methodological approaches. Students in heterogeneous groups "were instructed to work together on problems and to help each other" (p. 370). Groups were audiotaped or videotaped, and categories and frequency of peer interaction were coded from these data. The type and quantity of helpful interactions were compared to achievement test results. Webb concluded that "two kinds of verbal interaction significantly relate to mathematics learning in peer-directed small groups: giving elaborate explanations (positively related to achievement) and receiving non-responsive feedback (negatively related to achievement)" (p. 384). Webb's review departs from the purely outcome-centered genre of research by examining an aspect of how results are produced (in this case, giving and receiving help.)

In a more qualitative study of student interaction, Walters (2001) investigated and analyzed the questions high school students in a precalculus course posed to each other while working on mathematical problems designed around a nautilus shell. This observational study of four students used a social constructivist theoretical perspective. Using videotapes and field notes, Walters coded the students' discourse into the following broad categories: representation, demonstration, idea, and question. She subdivided each category into additional subcategories. In the report, Walters discussed findings only within the "question theme." From the data, she developed four codes with respect to the types of questions students posed. Listed below, each code is accompanied by a quote representative of the theme and the percentage of the 24 questions asked during the 6-minute episode that Walters analyzed for the report.

Attunement	(29%)	So, ninety, right? Are you listening to me?
Procedural	(4%)	How do you go from radians to centimeters?
Interrogatory	(42%)	Why do you [have] point three and then here it's point two?
Confirmation	(25%)	But wait. That seven was at that seven, right?

Walters' report, like Webb's, moves away from focusing solely on outcomes. By investigating students' questions through participant-observation, Walters also shifted from looking at effects to examining process. Although she was present while the students worked on the problem and she carefully reviewed videotapes, Walters assigned the above codes to the data based on her perceptions of student intent. Students were not asked to provide explanations of the intent of their questions.

Some process studies are designed to provide students opportunities to discuss with the researcher their own perspectives on the experience of working with other students. For example, working from a social constructivist theoretical perspective, Appleton (1997) investigated the cognitive processes of middle school students engaged in cooperative lab-based science activities. Intended to challenge students' preconceived ideas about certain scientific phenomenon, these "discrepant-event [activities] were chosen because they readily generate cognitive conflict" (p. 305) by generating outcomes contrary to what students intuitively expected. The students were videotaped as they performed the labs and discussed the outcomes within their group and in whole class discussion. Along with field notes and videotapes, primary data were collected by asking students to watch pre-selected (by Appleton) segments of the videotaped sessions. Using the data he collected, Appleton designed a detailed model (see Figure 1) of the cognitive-processing phases students move through as they encounter cognitive conflict during cooperative science investigations.

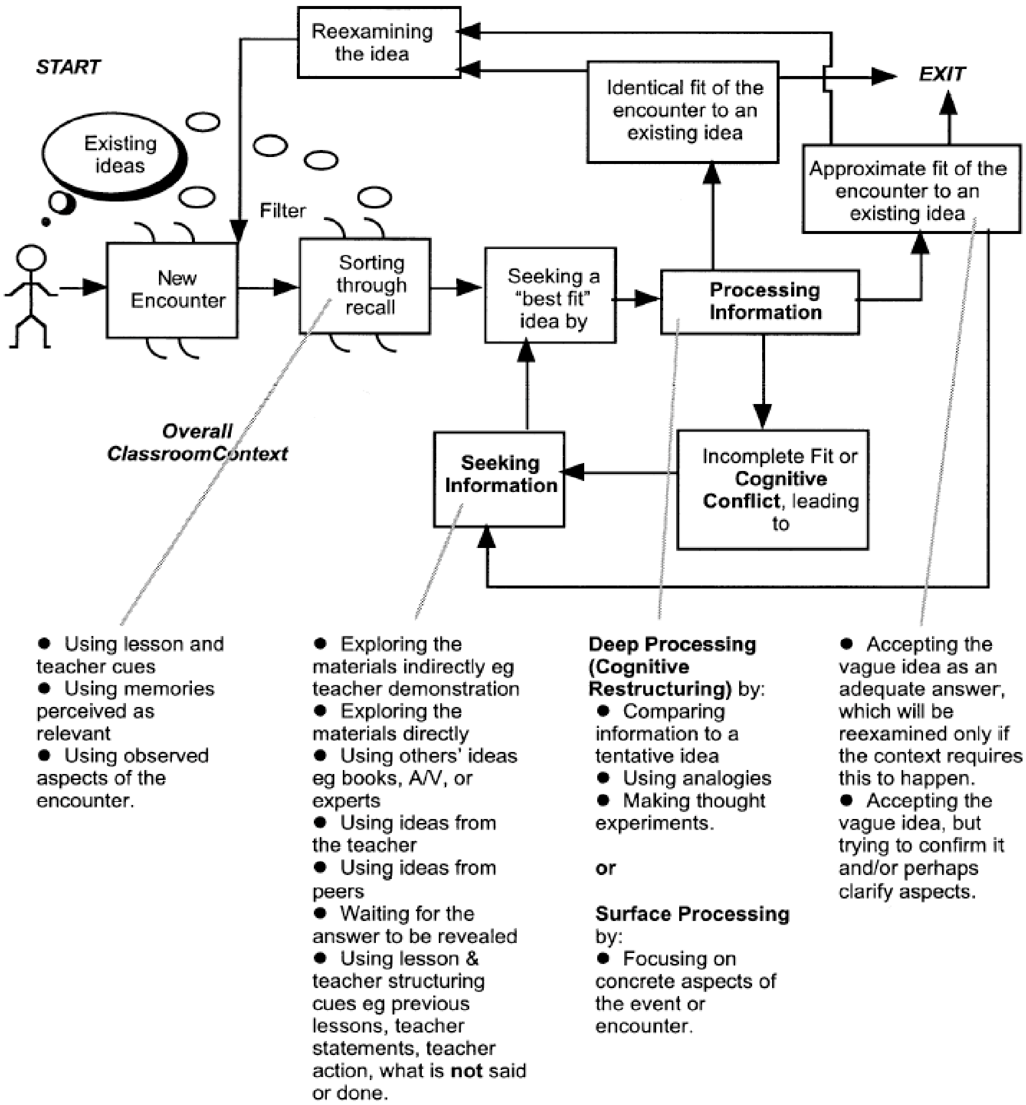

Figure 1. Appleton's Constructivist Model (p. 313)

Like Walters (2001), Appleton focused the data analysis on the discourse between students in order to describe the cognitive processes of students as they worked in groups. More specifically, however, Appleton looked for instances of scaffolding—building on each other's ideas, questions, and suggestions—between students. As an example of "a student involved in attempting to construct a new idea from the information gained [when] the existing ideas on which he attempted to build appeared to be somewhat tenuous" (p. 312), Appleton shared the following quote from a student:

> When [another student] said [“suction,”] I thought about it and I thought, “Oh, yeah.” So I started thinking about suction as well. . . . It ticked my mind off and turned it on and I started thinking . . . thinking on the line about the suction bit and about how did the balloon bring it up and down and when [the teacher] said, um, suction, um, sucking it up, I thought, “How could that suck it up?” And then I started to think even deeper. I started to think about air pushing against water and water pushing against air. (p. 312)

Through students’ discussion of their own cognitive processes as well as his theoretical knowledge and past research projects, Appleton developed a model that represented (among other results) his conclusion that students use scaffolding as one way to identify and resolve cognitive conflict.

There a solid research base providing answers to the question, “What is different about students (the effects) after we put them into the “black box” of cooperative learning?” There is also a rapidly growing area of research that, using a variety of data collection and analysis techniques, describes the activity (the process) of students engaged in cooperative learning, hence, providing answers to the question, “What is the activity of students while inside the “black box?” Although Appleton (1997) did ask students about their thought processes, there are researchers who claim that this type of process research (as well as the observational studies previously discussed) is not enough if we want to understand the experience of cooperative learning from the perspective of a student. These researchers are also interested in identifying and understanding affective issues of cooperative learning.

Opinions

The last area of research I discuss provides answers to the question—What are students’ opinions about the experience of being in the “black box?” Research studies striving to answer this question use a variety of data collection methods (e.g., open-ended surveys, journals, interviews, course evaluations, and informal conversations) as tools for students to reflect on their experiences. The studies have generated a wide range of results.

For example, as seven mathematics professors (Reynolds, Hagelgans, Schwingendorf, Vidakovic, Dubinsky, Shahin, & Wimbish, 1995) prepared to write a book that would "provide a valuable resource for any instructor who is considering cooperative learning groups in an undergraduate classroom" (p. iii), they each collected data on the reactions of students and instructors to their experiences with cooperative learning. Stating that "students have much to say about their experiences with cooperative learning groups," these researchers used comments from course surveys, course evaluations, and journals to report students' opinions about their experiences. These opinions were reported in the areas of learning and doing mathematics, changing attitudes, social skills and socializing, difficulties and responsibilities, group grades and the overall experience. Here is a sampling of the student comments the professors reported:

- I thank God that I work in a group and we can all help each other (Student journal, Calculus I)
- I don't want his laziness to destroy my grade like it almost did last semester. (Student journal, Calculus I)
- To sum up my feelings, forced group work activities aren't beneficial and in some cases extremely unfair. (Student journal, Calculus II)
- We worked in groups which helped spark one another. It seemed as if each person put in their own way of doing mathematical problems. (Student survey, Calculus II) (pp. 77-82)

Although they reported diverse comments, the authors concluded that, at least for them, cooperative learning was generally successful and they would continue to implement it in mathematics courses.

Whereas Reynolds et al. (1995) used cooperative learning within classroom activity or for out-of-class group projects, science professors Towns and Grant (1997) implemented cooperative discussion groups as a supplement to normal class activity. Then they audiotaped Friday discussion sessions, had informal conversations with students about the experience, and administered exit questionnaires to the students. Approaching the data "from a phenomenological point of view [to] get at the essence of the experience" (p. 820), Towns and Grant strove to answer the following research questions about the cooperative learning experiences provided for students:

1. What was structure of event?
2. What did these specially designed sessions mean to the students involved?

In significant detail, Towns and Grant described the structure (the processes) of the event. They also added the component of student perspective and identified two areas in which students discussed what cooperative learning meant to them: increased conceptual understanding and the development of interpersonal skills. These claims are evidenced by student comments such as the following:

- I believe I will go out of this class actually knowing something—not just walking out of here memorizing a bunch of formulas and then forgetting them.
- After the Friday sessions, I can recognize what I do know and understand those materials and I can see the relationships between the topics.
- The interactions with peers and the sharing of ideas was helpful to me. We could help each other and share the responsibilities!
- Concern—I didn't want to look like a fool.
- I felt additional pressure. I was uncomfortable, but I survived. (pp. 829-831)

They concluded their report by encouraging more research in the area of social interactions among students in cooperative groups, assessment issues, and the nature of group interactions.

Critique

All of the studies discussed in this review of empirical literature shared a common goal of understanding cooperative learning and its impact on students. Studies within each category (effects, processes, and opinions) have influenced and shaped my thinking about cooperative learning as not only a classroom teacher but also as a researcher. However, as I searched the cooperative learning literature, I found only partial answers to my research questions about the impact of cooperative learning on individual mathematical activity. Although researchers of the effects of cooperative learning reported increased mathematical achievement and attitude toward mathematics, they did not explicitly address the question of "How?" Conversely, whereas studies of process described the types of interactions and cognitive activity students engaged in during cooperative learning, these processes were not explicitly connected to individual learning. Finally, the research on students' opinions was neither explicitly tied to individual student learning nor group processes. I realized that, while appropriate for the goals of the previous studies, the research designs provided a piecewise rather than holistic understanding of students' experiences with cooperative learning. Such a holistic design would complement the literature base and add to what we know about CMPS.

Thus, my goal in designing the present study was to integrate studies of effects, processes, mathematics and experiences—to find out how these aspects of CMPS influence individual learning. Because I wanted to do more than just observe what happened during cooperative learning activities, or test/survey students after cooperative learning activities, the methodology designed for this study implemented a "participant as researcher" strategy and encouraged the structure of the CMPS to be theoretically deconstructed. Combined with my theoretical orientations and commitment to student inclusion, the methodology explained in chapter 3 built on and expanded the literature base on cooperative learning in an integrative, holistic approach to understanding the cooperative and mathematical experiences of students.

Theoretical Literature

As I considered stating a theoretical framework *before* conducting the present study, I found myself facing a difficult dilemma: Should I or should I not I? Initially the notion of an *a priori* theory was rather enticing. While reading the theoretical work of others and searching for something to claim for my own study, I was enamored by and quick to embrace several theoretical approaches. For example, as I read Barbara Jaworski (1994), it was clear I would be a “constructivist researcher” and “co-construct” my study with the participants. I later read Jacques Derrida (1997) and knew the approach I would use was “deconstructing” cooperative learning. When I met Dr. Bettie St. Pierre, studied under her guidance for a year, and found myself in agreement with much of what she said, I realized that I, too, might have a postmodern/ poststructuralist view of the world. It was when I studied social constructivism in a learning theory course in mathematics education that I became certain social constructivism was the theoretical framework for studying cooperative learning.

Perhaps I was easily influenced by the smooth talk of experienced and respected researchers and philosophers. Maybe I was too eager to align myself with and use someone else’s theory as my own *a priori*. This “idolizing” and eagerness can be a dangerous move. I feared that once I stated my theoretical framework and collected data, I would find myself trapped in a framework that was not helpful but be obligated to make it fit. One alternative was to simply avoid situating myself theoretically before the study. Such a move, however, has dangers of its own.

In his discussion of the practicality of theory, Wenger (1998) supported stating theory up front. He then described theory as a perspective, “not a recipe; it does not tell you just what to do. Rather it acts as a guide about what to pay attention to, what difficulties to expect, and how to approach problems” (p. 9). I knew when I observed my students and interviewed them about their cooperative learning experiences I would focus on certain aspects. Thus, I agreed that situating myself theoretically before data collection would provide helpful guidance in making these choices.

Nevertheless, a theoretical framework could not serve solely as a data collection guide. Denise Mewborn (a member of my doctoral committee) explained repeatedly, “A theoretical framework is what you will use to analyze your data” (personal communication). Providing a framework within which to make sense of data I had yet to collect was problematic for me, and I found myself back in the dilemma of whether or not to state theory *a priori*. My solution to this dilemma rested in making the distinction that “situating myself theoretically” included two interrelated components. One was an *a priori* theoretical *perspective*—a way of viewing. This perspective would guide my methodology. The other component was a more specific theoretical *framework*—ways of making sense of what I was viewing. This framework would inform my data analysis. Making this distinction between “what I would see” and “how I would make sense of what I saw” was essential. The remainder of this section is a discussion of the development of my theoretical perspective for studying students’ experiences with cooperative learning and of my theoretical framework for making sense of students’ experiences as they do mathematics together.

Struggling with Constructivism

As I mentioned in the discussion of my pilot study, the assumption that my theoretical framework would be or would rely heavily upon social constructivism became problematic. Questions arose in several areas:

- Theoretical—What is constructivism? Is it a framework, a philosophy, a perspective, a learning theory, or something else?
- Methodological—Would constructivism inform a study of cooperative learning?
- Epistemological—Did I really believe that constructivism adequately describes how students learn?

As I searched for answers to these questions, it became increasingly problematic to confine myself to social constructivism as a theoretical framework for investigating students’ experiences with cooperative learning.

Von Glasersfeld (1996) said that the “key idea that sets constructivism apart from other theories of cognition… [is] the idea that what we call knowledge does not and cannot have the purpose of producing representations of an independent reality, but instead has adaptive functioning” (p. 3). Although most would agree with him on this general level, agreement often stops there. Educational researchers have a difficult time answering the above questions with consistency. They have constructed their own meanings of constructivism that are both contextual and political in nature. Boudourides (1998) offered a summary of constructivism and its many forms (in a way that is reflective of my own thinking) in the following discussion:

> A first 'mild' (or 'trivial') version of constructivism originating in the work of Piaget holds that knowledge is actively constructed by the learner and not passively transmitted by the educator. In addition, there is the radical constructivism of von Glasersfeld,… in which cognition is considered adaptive in the sense that it is based on and constantly modified by a learner's experience. Beyond that, there is the social constructivist version of Vygotsky, who in an effort to challenge Piaget's ideas developed a fully cultural psychology stressing the primary role of communication and social life in meaning formation and cognition. The latter version of constructivism is accentuated by theories of sociology of scientific knowledge,… which argue that all knowledge is a social construct in the frame of science and technology studies. (Introduction section, para. 1)

Although some researchers discuss the commonalties of the types of constructivism (e.g., Ernest, 1995; Staver, 1998), it seems more popular (or at least more fun) to polarize[2] the versions, choose an extreme, and enter into debate. With the radical constructivists on one end of a continuum and the social constructivists on the other, they dispute "whether the mind is located in the head or in the individual-in-social-action, and whether learning is primarily a process of active cognitive reorganization or a process of enculturation into a community of practice" (Cobb, 1996, p. 35). I was not sure where I would place myself on this continuum, nor was I sure that I really wanted to. It was (and is) an argument I would rather not engage in, and yet I sensed that, as a theoretical perspective for my study, constructivism had potential.

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[2] In the following discussion, I use the terms polar and dichotomous to represent two opposing positions. There exists an inherent debate between the two positions.

In his criticism of constructivism, Kozloff (1998) claimed, “Constructivist ‘theory’ is a mishmash of overlapping platitudes and absurdities…. Taken separately, constructivist ‘propositions’ are merely simpleminded. Taken together, they are indistinguishable from the verbal behavior of a person suffering from chronic schizophrenia” (Offenses section, 4th point,). This “schizophrenia” is apparent in the following journal entry I made as I struggled to make sense of the constructivist literature I was reading:

> I argue that socially constructed knowledge is individual knowledge…. I do not think there is such a thing as social knowledge. How can it have knowledge? It is not possible. I like Paul Cobb’s notion of social knowledge as “taken as shared” where cultural norms, traditions, language, etc. are taken as shared meanings. But, it is always the individual with his or her own knowledge, which is dynamic and “multiple” in truths. I would argue that the knowledge I hold is, has been, and continues to be influenced by, adaptive to, and created in response to those and that which I encounter…. There is a constant exchange of information from the inter to intramental planes. I believe we have Inner Voice (Vygotsky) in our intramental plane, which allows and affords us the opportunity to hold multiple truths. (Journal entry, April 9, 2001)

I found value in both ends of the individual/social debate but clearly had a lot of difficulty deciding where to locate myself on the continuum. I ended up feeling as if I was just running back and forth.

When I was first introduced to constructivism, it sounded so simple and wonderful. We construct our own knowledge based on our experiences. Social constructivism added the idea that we do this constructing with other people. Because I believe cooperative learning is effective and meaningful, it seemed clear I had found my theory. However, things got fuzzy when I reflected on the meaning of theory, critically investigated what constructivism is, and thought about cooperative learning from a social constructivist perspective.

Not ready to totally abandon the idea of adopting a social constructivist perspective for my proposed study, but wanting a way to escape the apparent dichotomies and inconsistencies within the theory, I decided to take the following challenge offered by Kozloff (1998):

> If one were interested in cogent, well-written, intellectually rigorous and illuminating works on how human beings collaboratively produce knowledge… one would [look beyond] the watery soup served up by constructivists in education and psychology. (4th point, Consequences section)

Instead he suggested one read from an extensive list of authors from a variety of fields, such as, Durkheim (1912), Hume (1738), and Mead (1934). Although I did not read all of his specific recommendations, I took seriously the challenge to read about (social) constructivism in a variety of fields including science, sociology, philosophy, anthropology, and theology. Suchting (1992) described vividly this quest for information as he discusses constructivism and the danger of its rhetoric:

> Certain words and combinations of words are repeated like mantras, and while this procedure may well eventually produce in some what chanting is often designed to do, namely, produce a certain feeling of enlightenment without the tiresome business of intellectual effort, this feeling nearly always disappears with the immersion of the head in the cold water of critical interrogation. (p. 247)

As I read, I identified similar conversations about dichotomies, inconsistencies and polarization within the discourse of each discipline. The “cold water of critical interrogation” shocked me as I learned there was no apparent escape from the polarization of the individual and the society, the internal and the external. Wanting more than a position on a polarized theoretical continuum, it was time for me to engage in these discussions.

Joining Dichotomous Conversations

As I read about social constructivism across the disciplines, ideas seemed very disconnected. Language quickly became an issue. I found that many words (e.g., *knowledge, reality, constructivism, constructionism, postmodern, objective*) were clearly context, discipline, and author dependent. For example, consider the following three books:

The Social Construction of Reality (Berger & Luckman, 1966)

Social Constructivism as a Philosophy of Mathematics Education (Ernest, 1998b).

The Social Construction of What? (Hacking, 1999)

Hacking cited both Berger and Luckman and Ernest. Not only did Ernest cite Berger and Luckman, but the table of contents in the two books closed resembled each other. Because of these similarities, I imagined the books would have similar content or at least the language would be used consistently, but that was not the case.

Berger and Luckman are sociologists, Ernest is a professor of mathematics education, and Hacking is philosopher of science. Because these authors discussed the possibilities and implications for socially constructed knowledge within different contexts, they initially seemed to be discussing different ideas using different terminology. Upon further inspection, I realized that (at least for me) Berger and Luckman's concept of "society as subjective reality" was analogous to Hacking's notion of "social construction" which was analogous to Ernest's view of "fallibilism." Each author also presented analogous dichotomous views of social constructivism: society as objective reality, realism, and absolutism, respectively.

In my continued investigation of the long-standing debates that have developed and are visible in many disciplines, a common theme emerged. In each discipline, struggles seems to be between an emphasis on the individual (the internal) or an emphasis on society (the external). The dichotomies in their respective fields of study presented in Table 1exemplify this struggle.

Table 1

Dichotomies Across Fields of Study

Field of Study	Dichotomy
Anthropological Methodology	Objectivity / Subjectivity Heshusius (1992)
Language Development	Piaget / Vygotsky Boudourides (1998)
Mathematics	Absolutism / Fallibilism Ernest (1998b)
Mathematics Education	Radical Constructivism / Social Constructivism Cobb (1996)
Pedagogical Strategies	Competitive / Cooperative Johnson, Johnson & Holubec (1991)
Philosophy	Reductionism / Holism Hacking (1999)
Psychology	Participatory Levels of Consciousness / Reflective Levels of Consciousness Jay Earley (1999)
Sociology	Objective / Subjective Reality Berger & Luckman (1966)

Ian Hacking (1999) metaphorically described the ongoing theoretical debates as

> contemporary versions of problems that have vexed Western thinkers for millennia.… Old words tend to become hulks encrusted with barnacles. But scrape off the parasites for yourself, and you might glimpse the gleaming hull of an Aristotle or a Plato shining through. My observation is not that we ought to be

> doing the same things that they began, but that the same old things are still being done." (p. 63)

I believe there is value in continuing this polarized debate—it is important to hear and be challenged by different perspectives. Nevertheless, there is perhaps value in striving to find some common ground.

Integrating Theoretical Perspectives

Steffe (1995) claimed that "conflicts can paralyze action" (p. 489). If we consider the influences of constructivism/realism on classrooms, this "action" is linked to children, and they are much too valuable to become casualties of theoretical wars. Steffe went to give the following invitation:

> I agree with Ernest that we should give respect to those positions with which we disagree. But I go even further by asking the proponents of particular epistemologies to seriously consider how they might modify their ways of thinking in view of their differences with other epistemologies. (p. 489)

It seems he was suggesting a sort of "meet in the middle" peacekeeping strategy—an integration of perspectives.

Varieties of postmodern reactions to longstanding polar debates are embodied in integrated theories. What follows is a list of theoretical orientations that are integrated reactions to the dichotomies presented previously in Table 1. I argue that by offering integrated theoretical perspectives, these scholars have begun to open up the confining notions and language of polarity.

Mathematics Education. Paul Cobb (1996) asserted that instead of "adjudicating a dispute between opposing perspective," we would do well to "explore ways of *coordinating* [emphasis added] cognitive constructivist and sociocultural perspectives in mathematics education" (p. 35).

Philosophy of Science. Michael Polanyi (1958) took the position that the knower is not separate from the known: knowledge is personal and *integrated*. In his discussion of Polanyi's work, Rothfork (1995) emphasized "equal stress must be put on both terms—'personal' and 'knowledge'—to avoid reductionism in either direction: into fuzzy subjectivism and taste or sterile formalism" (para. 5).

Qualities of Consciousness. Jay Earley (1999) identified the "planetary crisis" as the loss of meaning, empathy, and vitality in today's world. As a solution to this crisis, he offered a philosophy of the consciousness realm "involving *integrating* [emphasis added] participatory and reflexive levels of consciousness."

Science. David Griffin (1988) proposed a unified science that he called "Postmodern Organicism" where science and the world are reenchanted. He claimed in adopting a postmodern organismic philosophy we would give heed to Toulman's call for "'a *single integrated system* [emphasis added] united by universal principles,'" (p. 30).

Sociology: Gerard Delaney (1997) proposed that "the constructivist-realist debate is a confused one" (p. 131) and that this "divide is in fact a false dichotomy…. The two sides can in fact be interpreted in reconcilable fashion" (p. 133). He offered the *integrated* [emphasis added] notion of "constructivist realism."

Cooperative learning requires individuals working as a team; thus, it is important to study both the individual and the group. As such, an integrated theoretical perspective was enticing. It was a method for "work[ing] out what Bauersfeld identified as the 'urgent research questions'—how we understand and describe in detail the relationships between observable social realities and individual development." (Steffe, 1995, p. 507). Perhaps this acknowledgement made him nervous, because Dr. Steffe used to say to me, "Please don't lose the individual in your study." I always assured him I would not and walked away wondering how that could be possible. Although confident I would never lose sight of the fact that I was working with individuals who form a group, I was aware that the language of an integrated perspective could easily shroud the individual. When I considered the possible risks in a move toward the middle of a continuum, I wondered what could be left behind/out by integrating dichotomous perspectives. Greer (1996) cautioned that "in the ferment of new ideas, liberalization of methodology, and openness to concepts from many disciplines, there are risks of losing balance through over-compensation" (p. 182). Not only could balance be compromised, but "this constant intertwining and blending of elements once seen as distinct… the traditional [dichotomies], so dear to our heart—where have they gone?" (Crotty, 1998, p. 209). Inherent in the idea of "meeting in the middle," is the assumption that a continuum is the only way to think about apparently dichotomous positions.

The polarization of theoretical perspectives is a product of language—a structural move. We classify objects, people, and ideas in polar terms, such as, positive/negative, man/woman, full/empty and legal/illegal. St. Pierre and Pillow (2000) urged us to continually recognize that dichotomies are a part of the structure of our language and

> point out that this sort of structural mistake is difficult to avoid since we are always speaking within the language of humanism, our mother tongue, a discourse that spawns structure after structure after structure—binaries, categories, and other grids of regularity. (p. 4)

Awareness of this difficulty may bring about recognition (a poststructural theoretical perspective) or action (deconstructive methodologies) against the limits of our humanistic language. In their discussion of overcoming and moving beyond dichotomous language, Heshusius and Ballard (1996) explained that "postmodern deconstructionist thought demystifies Cartesian dualisms in that it claims the impossibility of modernity's belief in representation…and that it points to the impossibility of describing reality through a transparent language" (p. 9). If we are able to point to the impossibilities of language, then we have applied "deconstructivist methodologies that aim to generate skepticism about beliefs often taken for granted within… social scientific discourses" (Collins, 2000, p. 53). Challenging the reality of the language and beliefs the aforementioned polarities can free us up to accept Steffe's invitation to modify views in ways other than integration.

For this study on the experience of cooperative learning, I wanted to focus attention on the group as a whole, the individual students, and the interactions among and between the individuals. Understanding that an integrated theoretical perspective which blended those elements to guide methodology would be limited, I sought a perspective cognizant that "each of these traditions [theories of social practice and theories of identity] has something crucial to contribute" (Wenger, 1998, p. 15). The next section offers an approach to adopting a theoretical perspective that, rather than simply integrating theories, maintains the integrity and strength of the individual theories.

Complexity Theory

Although integrated theories do present a solution to polarized debates, I agree with Hacking (1999) who

> does not want peace between the constructionist and the scientist…. It's a dilemma that doesn't need to be solved. … We analytic philosophers should be humble, and acknowledge that what is confused is sometimes more useful than what has been clarified. (p. 29)

In reaction to this type of acknowledgement, the field of complexity science has emerged over the past 30 years (see Davis & Simmt, 2003). A variety of scientific, sociological, and philosophical ideas about the study of dynamic systems are "grouped together loosely under the title 'complexity theory'" (Lissack, 2003, p. 1). The common thread, however, is a holistic "philosophical position claiming… that every apparent whole [dynamic, complex system] can be understood only in the context of the larger whole containing it" (Web dictionary of cybernetics and systems).

In the review of empirical literature surrounding cooperative learning, I presented a variety of methods and purposes for studying the effects on, processes of, and opinions of students engaged in cooperative learning activities. As discussed in the literature critique, it is important not just to examine these separate aspects of cooperative learning but to also examine the interactions of them in connection with students mathematical activity—the dynamic and complex system of CMPS. "What is exciting [about complexity theories] are the tremendous advances being made in understanding emergent phenomena. These advances are opening up the black box that had previously obscured the real process of emergence" (Goldstein, 2003, p. 68). Taking up the holistic approach of complexity theory and a subsequent theoretical perspective, the walls of duality I had previously felt surrounded by were opened up in the present study of cooperative learning.

Theoretical Perspective

The director of the Institute for the Study of Complex Systems in Palo Alto, Peter Corning, entered the dialectic discussion of theory with a significantly different perspective than those previously presented. Remarking on the holism and reductionism debate of the past century, Corning (1998) argued:

> What sets the present era apart is the fact that the scientific enterprise seems to be in the process of bridging the theoretical chasm between holism and reductionism; there seems to be a growing appreciation of the inextricable relationships between (and within) wholes and parts . . . relationships which necessitate multi-leveled, multi-disciplinary, "interactional" analyses. (Introduction)

He proposed that instead of adopting a theoretical perspective focused on a single aspect of a researchable situation or integrating two or more perspectives, one should adopt a perspective guiding the researcher to look through all of these lenses.

In reacting to Einstein's statement. "We should make things as simple as possible, but not simpler," Corning (1998, Concept of Synergy section) claimed that "theoretical simplifications, or generalizations, may serve to identify key features, common properties, or important relationships among various phenomena. Equally important, a concept which encompasses a broad range of phenomena may also serve as the anchor for a theoretical framework." He offered a "synergy paradigm" as this type of concept and claimed that this research focus gave "equal weight to both reductionist and holistic perspectives" (Toward a Synergy Paradigm section).

It seemed that Corning had presented a holistic perspective from which I could "see it all" in a cooperative learning environment. Such a broad interpretation of Corning's paradigm was both idealistic and problematic. Claiming I could design a study in which I could observe all aspects of the CMPS was idealistic and problematic. This viewpoint felt similar to the perspective I had taken for years as I incorporated cooperative learning into my classroom—the approach that left me with more questions than answers. Upon further investigation and reflection, however, I discovered the adoption of Corning's (1998) proposed "synergistic" perspective would guide a systematic approach of addressing my questions about cooperative learning. As Corning described the focus of a systematic, yet synergistic, theoretical perspective:

> A synergy perspective suggests a paradigm that explicitly focuses on both wholes and parts, and on the interactions that occur among the parts, between parts and wholes and between wholes at various levels of interaction and causation. (Toward a Synergy Paradigm section).

A significant reason I adopted this perspective was Corning never described his theory as integrated— multi-leveled and interactional, but not integrated. He encouraged researchers to pay attention to the individual components without simply meshing them together with the whole. Thus, the individual is not lost which is an aspect of synergism that I value. Corning also noted that the traditional view of synergism (the whole is greater than the sum of its parts) is "actually a caricature, a narrow and perhaps misleading definition of a multi-faceted concept." Rather, he suggests that "the effects produced by wholes are *different* (emphasis original) from what the parts can produce alone" (Concept of Synergy section).

Reflecting a tri-fold synergistic theoretical perspective towards studying students' experiences in and with CMPS, I viewed the individual students as the "parts," the cooperative environment was the "whole," and group members as the "interactions." Therefore, the refined research questions that guiding many of my methodological decisions were:

1. How do students engage in and experience cooperative, mathematical problem solving with respect to:
   a. the environment?

b. the group members?

c. the self?

2. What binary tensions are present or emerged within cooperative, mathematical problem solving?
3. How are these tensions related to students' individual mathematical activity?

No longer feeling trapped by the "intellectual shackles" (Corning, 1998) of a solitary perspective, synergy offered *simultaneous direction and freedom* for me as I studied cooperative learning. The theoretical perspective provided the direction for focusing on and collecting data with a systematic tri-fold approach composed of the part, the whole, and the part-whole aspects of cooperative learning. The dilemma of where to place myself theoretically on the individual/society continuum all but disappeared. I moved back and forth along the continuum using the aspects of a specific theory or philosophy applicable to this study—those that helped me see as well as understand what I was seeing—and left the rest behind. The lenses of this synergistic perspective were blurred enough to allow the study to emerge and clear enough to provide a place which I could look from, look toward and change directions freely.

Theoretical Framework

Although the research questions guided what I viewed, there is more to educational research (or any kind of research for that matter) than just observing a situation. The obligation to analyze always accompanies a researcher's eagerness to theorize. Thus, situating myself theoretically also involved the essential component of identifying a theoretical framework—a way of making sense of what I was viewing. Along with a synergistic theoretical perspective, the two theories framed the data analysis phase of the present study were *levels of consciousness* (Earley, 1997, 1999) and *deconstruction* (Derrida, 1967/1974, 1997).

The majority of the data collected in the present study reflected the experiences and mathematics of each participant from her perspective—how she saw herself both influencing and being influenced by CMPS experience. I learned about a participant's personal experience by listening to the aspects she was aware (conscious) of and chose to share with me during Problem Solving Sessions (PSSs) and video recall interviews. In general, each participant talked about she was doing and/or thinking as well as what she thought the other participants were doing and/or thinking. A framework for identifying and describing the experiences of the participants as both members of a learning community and individual learners was borrowed and adapted from Earley's (1997, 1999) model of the social evolution of humankind in which he identified two levels of consciousness.

Although Earley developed his model to describe the levels of consciousness of social evolution throughout history and across the planet, I adapted it as a framework for theorizing the social evolution of three students involved in the experience of CMPS. Earley identified two levels of consciousness, participatory and reflexive, based on the ground and emergent qualities of the conscious realm. Replacing the word "world" with "group" in these levels are described as follows:

> The *Participatory* Level of Consciousness is "a ground quality and is characterized by a sense of belonging to the [group] and in an immediacy of [the cooperative, mathematical problem solving] experience."
>
> The *Reflexive* Level of Consciousness is an emergent quality characterized by a sense of our ability to understand ourselves in the [group]. It is being able to step back and reflect on how we experience [cooperative, mathematical problem solving]. (Earley, 1997, p. 21)

Within the context of the present study, at a participatory level of consciousness, a group member reflected on the ways in which her participation effected the cooperative problem solving experience—what she contributed mathematically, socially, or otherwise. At a reflexive level of consciousness, a participant reflected on how the cooperative environment and group members effected her individual problem solving experience.

Earley (1997) further claimed, “The human race is at a turning point in its history, a crossroads in social evolution. We are facing a crisis of planetary proportions… [and] are confronted with immense dangers and exciting possibilities” (p. 1). He argued that this planetary crisis—manifested in a major ecological crisis, economic problems, terrorism, homelessness, and rising crime among other issues—is the result of the course of social evolution that has left us in the *trap of a dichotomy* (emphasis added) between what he calls the ground (e.g., community, holistic) and emergent (e.g., individual, rational) qualities of humankind. The imbalance in these qualities of humankind, has resulted in the dichotomy he called “the crisis of social evolution.”

As a response to such limiting dichotomies, Derrida (1997) metaphorically described deconstruction as follows:

> Whenever one runs up against a limit, deconstruction presses against it. Whenever deconstruction finds a nutshell—a secure axiom or a pithy maxim—the very idea is to crack it open and disturb this tranquility. Indeed, that is a good rule of thumb in deconstruction. That is what deconstruction is all about, its very meaning and mission, if it has any. One might even say that cracking nutshells is what deconstruction is. In a nutshell. (p. 32)

Although the term “deconstruction” can mean to tear down, Derrida used the term to mean open-up, turn upside down, view in new ways, and rethink traditions. I believe that Derrida’s (1967/1974, 1997) work on the deconstruction of dichotomous language not only provided a framework for data analysis, but it also enriches the understanding of Earley’s response to and proposed integrated solution of the crisis of social evolution.

Rather than using Earley's interpretation of the term *dichotomy,* within the framework of deconstruction, the term *binary* is used to characterize two terms placed in some sort of relation to each other where

> each term, rather than being polar opposite of its paired term, is actually part of it. Then the structure or opposition that kept them apart collapses…. Ultimately, you can't tell which is which, and the idea of… opposites loses meaning, or is put into 'play.'" (Klage, 1997, last paragraph)

To deconstruct the apparent participatory/reflective dichotomous trap, Earley asserted we must restore balance by "consciously choos[ing] to regain our wholeness and vitality in conjunction with our complexity and autonomy, as individuals and [as] communities" (p. 4). I suggest that Earley's solution is a result of his own deconstruction the dichotomy he perceived between the social and the individual. To do this, Earley rethought the dichotomous levels of consciousness as a binary where one cannot exist without the other and subsequently proposed we strive for a new level of conscious participation.

> *Conscious participation* means having the aliveness and meaningfulness of participatory consciousness while using your ability to reflect, criticize, and make conscious choices. It means being able to step back from your participation and ask critical questions about what it really means, whether anything is missing, whether you are being manipulated, and so on. It means retaining your individuality and making choices about your participation. It also means having the vision to see that reality is indeed a meaningful, participatory whole. (Earley (1997, p. 228)

Earley described conscious participation as an integrative solution and represented in the graphic shown in Figure 2.

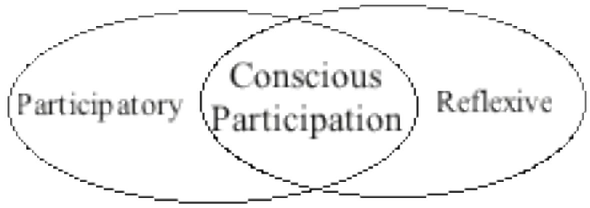

Figure 2. Earley's model of consciousness participation (1997, p. 228)

Although I agree is a valuable framework for rethinking dichotomous tensions within CMPS experiences, I suggest that Earley's graphic does not accurately reflect conscious participation as the integration of the participatory and reflexive levels. The word *integrate* means " to unite with something else; to incorporate into a larger unit" (Merriam-Webster's online dictionary). Figure 3 illustrates my adapted graphic that represents what I perceive to illustrate conscious participation as an integrated level of consciousness—in which the participatory and reflexive levels of consciousness are encompassed within the level of conscious participation.

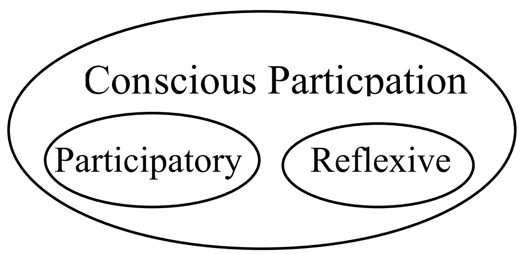

Figure 3. Consciousness Participation as an integrated solution

Rather than beginning with the assumption that participatory and reflexive levels are, by nature, dichotomous, I considered the pair a binary. A binary consists of two terms placed in relation to each other—a relation that is not necessarily polar. Thus a synergistic, holistic model exists where the integrity of the two parts of the group/individual dichotomy is maintained, an individual is simultaneously aware of both levels of participation, and is free to move between them. In the following chapters, I explain how

I applied these theories of deconstruction and levels of consciousness, what it revealed and how it helped me address the aforementioned research questions.

CHAPTER THREE: RECOUNTING THE METHODOLOGY

When you stare at an object with focused eyes, you see the same image continuously. However, if you let your eyes relax and look slightly past the object, you can see things you had not seen before—a different perspective. The challenge of research is to take us beyond (and thus, necessarily, *through*) what we already know to what we do not know. Going into this study, I knew there were some things I was not seeing about how students learn cooperatively. I wanted to design a study that would allow my eyes to relax and, as a result, make these things visible. Derrida described (1974/1967,1997) research as a way of illuminating this *other*—what we are not presently aware of—and offered the following methodological philosophy:

> But you were asking a question. I was wondering myself if I know where I am going. So I would answer you by saying, first, that I am trying, precisely, to put myself at a point so that I do not know any longer where I am going. (Derrida, cited in St. Pierre, 1995, p. 7)

Because my prior knowledge and beliefs about mathematical learning, experience, and research brought me to and were a significant part of the present study, it was not possible for me to become completely "disoriented." However, if I had mapped out a clear and exact *a priori* methodological path to take, I would have missed opportunities to go where the participants led—down the road less traveled or less researched.

Throughout the present study, in order for me to see beyond the traditional structures of cooperative learning while still keeping it in sight, I needed to both build on and question my prior knowledge. Thus my goal in designing and letting this study unfold was to *unfocus* and look, with a new perspective, at students' cooperative learning experiences as they engaged in mathematical problem solving. This chapter is a recounting of the design that, through the lens of poststructural orientations and deconstructive methodologies, emerged, evolved, and unfolded throughout the various stages of the present study.

Qualitative Paradigm of Inquiry

Rooted in postmodern thought, which rejects foundationalism, the qualitative research paradigm is "primarily concerned with human understanding, interpretation, intersubjectivity, lived experiences," and multiple truths (Ernest, 1998a, p. 33). As previously stated, the purpose of the present study was to illuminate and communicate the individual experiences within cooperative problem solving by investigating the mathematical, social, cognitive, and subjective experiences of students as interpreted by them. Thus, the methodological decisions of the present study were most appropriately framed within this qualitative paradigm of inquiry.

Qualitative research traditions come with an abundance of stances, approaches, methods, and techniques for conducting research. In Schwandt's (2000) discussion of a variety of epistemological stances for qualitative inquiry, he asserted:

> In our efforts to understand what it means to "do" qualitative inquiry, what we face is not a choice of which label—interpretivist, constructivist, hermeneuticist, or something else best suits us. Rather we are confronted with choices about how each of us wants to live the life of a social inquirer.

As I made choices about how to conduct this present inquiry, I looked to various research traditions through the lens of poststructuralism. I borrowed from them aspects most fitting in this research setting, with my research objectives, and within my theoretical perspectives. My specific methodological choices stemmed from my beliefs about the research process and were significantly influenced by Garrick's (1999) discussion of the interpretive research paradigm as well as Jaworski's (1997) and Guba's (1990) discussions of the constructivist paradigm of inquiry.

In his postmodern discussion of the philosophical assumptions of the interpretive research paradigm, Garrick (1999) stated that "interpretative research uses personal experience as its starting point…. This experience can best be understood from the standpoint of the social world of that individual" (pp. 148-149). Therefore, emphasizing the context within which participants are acting is a significant aspect of an interpretive study. Highlighting the implications of this contextual focus, he pointed to an assumption commonly shared by qualitative researchers: “The world is made up of tangible and intangible multi-faceted realities [that are] best studied as a whole rather than being fragmented into independent and dependent variables” (Candy, cited in Garrick, 1991, p. 149). Although I identified specific aspects of CMPS in my research questions, the synergistic perspective I strove to maintain promoted a holistic, interpretive method of data collection and analysis that accounted for multiple perspectives.

In addition to this interpretive paradigm, I also adapted a constructivist paradigm of inquiry to the present study. A doublet of constitution central to this paradigm is the recognition that “my presence in the research domain, as in all situations in which I live, affects the situation… I am not a neutral factor. Together, with others, we researchers both constitute the situation and are constituted by it” (Lerman, 2000, p. 224). The constructivist paradigm of inquiry is rooted in subjectivist epistemology where data depend “heavily on a reconciling of interpretations, or intersubjectivity, between participants in the research” (Jaworski, 1997, p. 114). Thus, data are representative of the words, actions, thoughts, and beliefs of all participants.

Because “the inquirer and the inquired into are fused into a single entity and findings are literally the creation process of the interaction between the two” (Guba, 1990, p. 27), I found it important to identify and keep in mind that I was also a participant, in a different role than the other three, but nevertheless, a participant. Moreover, a constructivist paradigm promotes methodologies where “individual constructions are elicited and refined hermeneutically… with the aim of generating constructions on which there is substantial consensus” (Guba, 1990, p. 27). In order to create this consensus, I employed data collection techniques such as individual, group, and stimulated-video-recall interviews and used member checking throughout the constant-comparison analysis. The remainder of this chapter illustrates the ways in which I designed and implemented a qualitative research design reflecting an interpretive, constructivist paradigm.

Data Collection

Table 2 provides an overview and timeline of the various phases of the present study. In light of the emergent nature of this study, I did not and do not view these phases as linear and discrete. As I employed a constant-comparison method of data analysis, I made methodological decisions as the study progressed in reaction to these on-going analyses. For example, the design of the mathematical activities for the problem solving sessions (PSSs) was open to (and did) change as I both collected and analyzed data. I transcribed interview sessions before each new interview in order to inform protocol by identifying new questions or adapting original questions. And, believing that writing is a method of inquiry, I journaled throughout all aspects of the study.

Table 2

Overview and Timeline of the Study

Phase	Strategy	Description	Dates
Phase I: Planning	Pre-field Work	Obtained IRB approval, selected participants, and designed mathematical activities.	Nov 2001 – Jan 2002
Phase II: Data Collection	Preliminary Interviews	Conducted individual interviews about mathematics and cooperative learning.	Jan 14, 200
	Problem Solving	Video and audio taping 3 PSSs.	Jan 15, 22,
	Group Interviews	discussed the group experience immediately following the PSS	Jan 15, 22,
	Individual Interviews	interviewed individuals after each PSS.	Jan 16, 23,
	Data Transcription	viewing and transcribing video and audiotapes after each, and before the next, PSS.	Jan 14-Feb
	Research Presentations	Twice, the participants and I gave presentations discussing initial findings.	April 25, 20 Oct 28, 200
Phase III: Interpretation	Data Analysis	Coding, identifying and exploring themes, overlaying theory	Jan 2002 – July 2003
Phase IV: Representation	Writing	Journaling, drafting and revising chapters striving to honor and represent the voices and of my participants.	Jan 2002 – April 2004

This table was adapted from (Leatham, 2002, p. 47).

Participant Selection

I selected three participants from a cohort of undergraduate, mathematics education students at the University of Georgia. There were several justifications for the purposeful selection of the three participants for this study. First, I firmly believe research is done *with* and not *on* people. Because interviews are "joint product(s)of what interviewers and interviewees talk about together and how they talk with each other" (Mischler, 1986, p. vii), I viewed the participants as co-researchers. Thus, as I reflected on the process of participant selection, I found it crucial that the selected participants should want to be a significant part of investigating and communicating the experience of CMPS.

Second, experience and research have taught me that it takes students some time and some help to learn to work together. Because the present study focused on *how* not *if* cooperative learning functions, it was important that the participants had previously experienced investigating mathematics within a cooperative group and be comfortable working with each other. It was also important that the participants were comfortable with me (and I with them) discussing not only their mathematics but also their experiences and feelings while engaged in cooperative learning.

Third, based on both research and personal experience, I determined that three or four members in a cooperative group would be ideal. Because I knew I would be videotaping the students as they worked together, the decision of selecting three students rather than four students for the present study was logistical. Four students would need to sit in a circular arrangement in order to communicate easily and have visual and tactile access to any materials being used or developed. Thus, I would be videotaping the back of one student. With three students seated at the end of an oval table, I would easily be able to videotape all the students simultaneously.

With all these factors in mind, I decided it would be beneficial to the present study for the participants to be students I had taught. Not only were they used to working cooperatively, but we had established a rapport and were comfortable interacting on a variety of levels, sharing, learning, and interacting with each other. Thus the participants were members of the Topics and Technology in Secondary School Mathematics Curriculum (EMAT 3500) I taught in Fall Semester, 2001 (the semester before this study was conducted). Because we had worked extensively in groups throughout this course, I considered all members of the class to be familiar and comfortable with investigating mathematics in cooperative groups.

The decision of which three students to select was somewhat problematic. Having shared general ideas about my proposed study with the class, I knew that of the 23 students in the class, 20 of those students would have agreed to be in the study (3 would not be on campus during the study). I was uncomfortable selecting just three of these students myself, not wanting to have any of my students feel, "Well, Miss Sheehy just liked those three people or thought they were the best students." Fortunately, Nicholas Oppong, a professor on my doctoral committee and thus aware of my research questions and proposed design, taught the same group of students the class period before I met with them. He had assigned long-term group projects in his course, whereas the group compositions in my course changed often. Therefore, I was able to elicit his help and guidance by asking him to recommend a group of three students he felt worked well together, had "gelled" as a group, and were reflective thinkers who articulated their ideas well. He sent me an e-mail recommending three young women, Annie, Courtney, and Meredith[3], for the present study. Pleased with his recommendation, I contacted those students to ask them to be a part of the study and to subsequently set-up times and dates when we could begin interviews and problem solving sessions.

I knew these participants well and not just in the classroom as my students. I knew about their families, their friends, and their lives. I worried about them, was proud

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[3] Although I identified the names of the university and courses, the participants and I decided to use pseudonyms in this written representation. We all felt it was important and illustrative for photographs to be included. There were, however, discussed reasons for a certain degree of anonymity.

of them, wanted the best for them, and so on. The present study was strengthened by the fact that the students and I knew each other well and were comfortable with each other. They trusted me and were used to talking with me about their learning, mathematics, feelings and experiences. I was, however, constantly aware that my relationship with these students had the potential to weaken the study. For example, I was concerned the participants would try hard to "say the right thing" and to give me "good data." They were excited to be involved in the study and eager to help me with my dissertation research. In the discussion below of the data collection and analysis techniques, I describe how we all worked to focus on and discuss what we saw and experienced rather than what we thought we should have seen and said. I feel I used my knowledge of and relationship with the participants to aid them in explaining and reflecting on their experiences with cooperative learning. As Ball (2000) suggested, being both teacher and researcher "offers the researcher a role in creating the phenomenon to be investigated coupled with the capacity to examine it from the inside, to learn that which is less visible" (p. 388). Although teacher and researcher roles may have been difficult to balance, the benefits of engaging in educational research with students whom I taught far outweighed the struggles for balance.

Preliminary Interviews

This first phase of formal data collection occurred on January 14, 2002 (see Table 2). Lasting approximately one hour, these preliminary interviews offered the participants and me to reflect on their personal learning styles, views of mathematics, cooperative learning behaviors, and reasons for participating in this study. I met with each participant in a small office in the Mathematics Education Department—a familiar environment for the participants and myself. Sitting close together with a tape-recorder between us, and using the interview protocol (see Appendix B) as a guide, I asked each participant questions from in three categories: nature of mathematics, mathematical learning, and prior cooperative learning experiences and perceptions. I also used this time to describe the nature of the present study to each participant and answer her questions. I use these data as a way to introduce each participant.

Participant Descriptions

Annie

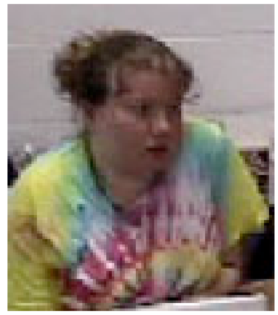

Figure 4. Annie

Annie (see Figure 4) chose a fish as the animal most representative of mathematics. She pointed to "specifically an Angel fish because of all the patterns." Annie further explained her choice by saying, "Math is nothing but a bunch of patterns put together…. You know the scales make up a pattern. Put them all together, and it's a school of fish... and everything's all connected." Annie selected the statement "*Mathematics is true because it is beautiful or because it is coherent"* as the one she felt best describes mathematics (see Appendix B for list of statements). When I asked Annie why she selected this statement, she passionately said:

> Anytime I learn something new, I just feel like I am so enlightened. I love that feeling! You know through the next day or something, I'll be told what the idea is. And if I'm trying to come up with the idea on my own, I just feel like this is so amazing, and everybody needs to know this!

In discussing the ways in which she liked to learn mathematics, Annie first explained that she did not like it "when a teacher just shows me the theorem and then expects me to understand it. I have to build up to it with examples and then present the theorem." Annie definitely likes to learn new topics by "going from concrete to abstract" ideas and representations. In fact, she felt this approach worked well with her own learning style: “Once you’ve shown me concrete to abstract, I can go home and recreate concrete on my own just fine. I’m a very good self-learner.” Although Annie enjoyed “having the personal interactions when you’re doing group work,” she maintained, “If I come up with my own ideas, I really understand it a lot better.” Not wanting her “self-learning style” to be confused with independent learning, Annie explained that she did not want a teacher to just give “me a book and let me explore. I don’t like that. I need someone to at least show me an example, and then I’ll get the idea.”

With respect to cooperative learning experiences, Annie said before college she could not “recall a specific instance,” but she did know that she “always got stuck doing all the work.” Although this aspect of cooperative learning was Annie’s “most unfavorite part of working in groups,” her favorite aspect is the “exchange of ideas.” She described this as: “The ‘Oh I didn’t think about that,’ ‘That is so cool,’ or the “What are you talking about? Explain that to me.’” Another aspect of cooperative learning that Annie really enjoyed was when “you explain it to someone else” and “you make sense.” She said that she got a lot of satisfaction from knowing she “just taught somebody something!”

Annie identified her “mathematical background” to be a personal strength she would bring to a cooperative group. She explained that because she had “tutored for so long, [she knew] a lot of stuff that most people have forgotten. It [was] only because [she] tutored.” When asked about a weakness, Annie quickly replied, “Probably taking over.” However, she pointed out that both her opportunities to work with Courtney in classes and her tutoring experiences have helped her with this weakness. “At first I kind of did all of it. And then slowly I began to understand—whoa, wait, slow down, you gotta let them come up with these things.”

When asked to describe the ideal cooperative group, Annie first described an experience from a previous course that she felt was less than ideal:

> When we worked together for the golden ratio project, we separated it into sections. I don’t think I really liked that, because it wasn’t very cohesive. But, at

> the same time, it was a huge project, and I don't think all of us could have gotten together to do all of it together.

So I asked Annie, "What if you were working on one mathematics problem instead of a big project?" She responded:

> I don't think there would be roles. I think there would be more along the lines of "What do you know about this? What do you think we should do?" Each person would say, "Oh, yeah, I agree with that." Or if we're all clueless, we would just be like, "We have no idea." Then people would bring to the table what their strengths are.

Courtney

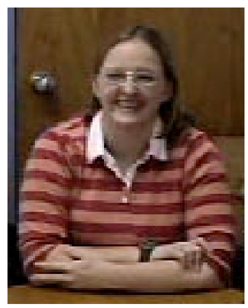

Figure 5. Courtney

As an animal she thought represented mathematics, Courtney (see Figure 5) chose a dog " because it's like man's best friend—you know, real reliable." She observed that mathematics is "something that you can use a lot" and thought of a dog as being like that: "an everyday thing that's useful and kind of dependable." From the list of statements about the nature of mathematics (see Appendix B), Courtney chose two statements with which she most agreed, the first being "*Mathematics is true because it has been elicited in a way that reflects accurately the phenomena of the real world."* Courtney explained, "All this math we have learned—now I realize that it is found in nature and in all different stuff." She further reflected, "Mathematics is … a way to express it because it was already there. But then humans found a way to express it or to write it down and work with it." Similarly, the statement "*Mathematics is true because it is an accurate* *expression of primal, intuitive knowledge"* also reflected her belief that mathematics is "just an expression of what's already here."

When I asked Courtney to talk about the ways she liked to learn mathematics, she initially claimed she was "kind of a procedural person… step by step.… I have to practice, practice over and over again." She quickly added, "But that's just like a starting off point." Courtney went on to describe how her procedural preference to mathematics had been changing:

> I've never really been exposed, that much, to exploring on your own, except for last semester. I did a lot of [exploring] in all of my classes [last semester], and I think that has helped me see math in a different way as not being so procedural and as being more discovery. So, now I think that it's a balance between those two.

When I asked her about favorite aspect of studying mathematics, Courtney remarked, "I like it because it's kind of like a puzzle that you have to figure out." Her least favorite aspect was the "ambiguity you sometimes find in it—you can do it this way for just one kind of thing, but then when you have this other type of situation, you have to do it a completely different way." Courtney added that this ambiguity "is kind of interesting and neat, but it is also kind of frustrating because you don't ever know where to start sometimes."

Similar to Annie's experiences with cooperative learning, Courtney did not "really remember doing much group stuff in high school." She recalled that her classmates would study together sometimes and "we did a few projects, I think." Courtney added, "They weren't ever really cooperative learning… in that you are learning together." She went on to talk about the experiences she had with cooperative learning the fall semester before the study. She said, "The best part about it is that you get so many different ideas." She laughed as she added, "But that's kind of the worst part about it, too." She further discussed this difficult aspect of working together:

> Sometimes you have two dominating personalities and have conflicting ideas. They're not necessarily wrong, but they are just different. It's sometimes hard to come to an agreement, but then in the end it makes your outcome better because you've explore all different aspects. Whereas if I had done something on my own, I would only see what I was thinking and only go down that one path. So I think in the end, you get a better outcome or better product, but it might be kind of frustrating getting there.

When I asked Courtney to discuss a personal strength she would bring to a mathematical cooperative group, she said, "I think I'm more of the concrete thinker than I am an abstract thinker…. I don't get as many of those 'out of the box' ideas.'' She continued to explain this "concrete" strength:

> I guess what I am trying to say is that I seem to be more of the part of mathematics that is more grounded and is more you get one answer—you do it this way, you get one answer…. I think that thinking of all those out of the box things is important when you are brainstorming and stuff, but in the end you kind of have to come back to an absolute.

Not surprisingly, Courtney said a weakness she would bring to a cooperative learning group is the fact she did not "think outsides the box as much as other people do." She explained how this weakness had the potential to make cooperative problem solving difficult for her: "I see that in other people, but I just don't think that way—so it is hard sometimes to see how they're thinking that way."

Courtney decided to talk about the ideal mathematical cooperative learning situation with respect to her anticipated work in the study with Meredith and Annie. She said that in the end "we would come to a conclusion, but I think we would do it all together. One person would think of the idea, the other two would question that idea, and then together we would come up with what really is true and what is right." She pointed to specific roles that each person would potentially play in this process:

> If it was the three of us… Annie always seems to… think of off-the-wall ideas, abstract things that I wouldn't think of. Meredith does this too. I think that I would most likely be the one to think, "Well, does that idea really make sense? Does that really follow mathematics or is it just us thinking? Does that mathematically make sense, or is it just something that we think might happen?

Courtney believed this type of questioning would play an important role in solving a problem cooperatively: "I think by me asking the questions, they would explain what they really mean and then we would start talking together."

Meredith

Figure 6. Meredith

Meredith (see Figure 6) claimed that if mathematics were an animal, it would be an *ant*. She chose ants because "they are all everywhere and into everything. Just like the existence of math and how you can find math in so many different aspects—so many different subjects and so many different areas." Although she liked a lot of the statements on the list I presented to her (see Appendix B), the one that most reflected her views on the nature of mathematics was "*Mathematics is true because it reflects accurately the phenomena of the real world*." Meredith explained that "math has been constructed and created so that we can communicate. We can establish these axioms and these theorems to try and make laws and understand—I don't know—I guess trying to establish a reality."

Meredith shared that, when learning mathematics, it was important for her to "start off with the basic information and then absorb it." Absorbing it meant "writing it down, looking back over it and then applying it to examples." She also explained that talking aloud about mathematics was extremely important to her learning. Even if

> [I was] at home doing my homework by myself, if I read out loud or… speak out loud while I am going through it, I can dig deeper into it and figure it out faster and usually more accurately, too. I'm just thinking more—I don't know if it's the auditory hearing it or hearing myself dig through it.

For Meredith, learning and doing mathematics was very empowering. She described the "feeling of conquering a problem that you've been working on… staring at it for like two hours, and then finally the light bulb comes on and your like 'Ah, I got it!' That's very empowering." Because Meredith really enjoyed understanding mathematics, she felt frustrated when she had to "just memorize it so that I can just regurgitate it on the test. I don't like that, because I really feel empowered by understanding mathematics. And so when I don't understand it, and I just memorize and regurgitate, it is like I am getting nothing from it."

With respect to her previous experiences with cooperative learning, Meredith told me that in high school, cooperative learning meant group projects in a variety of her subject area courses, but she remembered doing group work in only one mathematics class. Meredith went on to explain that in college, before she began her course work in the college of education, cooperative learning was implemented in much the same way, "for a final project that was due at the end of the term or something like that but not really small assignments or even in-class activities." Meredith reflected on the pros and cons of working cooperatively and said that the best part of working together is "pooling our ideas all together." Meredith explained that she

> might not see something that someone else does, and if we all communicate with each other—whether it's wrong or right—we can help each other all figure it out. It is even sometimes understanding why an idea is wrong that can help you get to what the right idea is.

Meredith shared two aspects of cooperative learning she disliked. One was “when the work isn’t distributed evenly” and someone got stuck doing all the work. The other aspect was when

> there is another group member who is really controlling or takes over—you know the perfectionist who has to have everything done a certain way. If they don’t do it themselves, they don't think that it will get done correctly—that’s probably the worst.

Meredith told me that because she enjoyed it so much, problem solving would be a strength she could contribute to a cooperative group. She added that because she was more “big picture” and less detail oriented, she would need the people in her group to “recall the old laws and rules.” She thought she would “recognize there is something you can do with [the problem] but I couldn’t remember the formula exactly…. Problem solving and analyzing more application type problems—I enjoy that more.” She described her ideal cooperative group for solving a mathematical problem as involving

> communication from everyone, analyzing your thoughts, ideas and the problem (or whatever we are working on), designating, making sure that everyone is working on some aspect (either together or in our own little separate ways) in order to be able to come back together and communicate those ideas to each other.

Through these individual interviews, it is apparent that Annie, Courtney and Meredith share a love of mathematics, a curiosity about mathematical learning, and similar classroom experiences within their secondary mathematics education coursework. The participants also expressed different beliefs about the nature of mathematics, different preferences for studying mathematics, and different perspectives on cooperative learning. We all looked forward to taking these varied beliefs, perspectives, and interest into a problem solving situation and engaging in some mathematics cooperatively.

Problem Solving Sessions

At the heart of the present study, the next stage of data collection concentrated on the participants' involvement in three separate cooperative investigations of mathematical problems. The data collected from the subsequent group and individual interviews following each Problem Solving Session (PSS) were the primary data sources for this study. I carefully considered both the selections of the mathematical tasks and the manner in which I conducted the PSSs.

Selecting the Problems

As criteria for selecting mathematical problem solving activities in which the participants engaged, I followed Polya's (1957) suggestion that a "problem should be well chosen, not too difficult and not too easy, natural and interesting" (p. 6). More specifically, I wanted to offer the participants mathematical activities that would:

- Place participants in problematic situations novel to them, where solutions were not readily apparent,
- Be problems participants… not only understand, but also desire to solve (Polya,1957, p. 6),
- Have multiple entry points, allow multiple solutions or strategies, and have multiple exit points (especially considering the 30-minute sessions),
- Be conducive to cooperative learning, encouraging the sharing of ideas, the collecting of data, the selecting of strategies, group products, and independent thinking,
- And, most importantly, provide opportunities for participants to engage in and learn some new and meaningful mathematics.

I also decided that a similar mathematical theme for the three investigations would add some continuity and contextual similarity to the problem solving sessions. I wanted each investigation to provide for the participants to model the problem situation with manipulatives, gather numerical data, look for patterns, create multiple representations including algebraic models, make and prove generalizations, and extend the problem.

Based on these criteria and my own classroom experience using a variety of problems as cooperative investigations with students, I selected the following three problems for the participants to investigate: the Square Game (see Appendix C), the Locker Problem (see Appendix E), and the Polygonal Numbers Investigation (see Appendix G). The moves of the Square Game produced linear and quadratic data sets, the Locker Problem dealt with factors of numbers and square root functions, and the Polygonal Numbers Investigations involved formulas for arithmetic sums[4]. Although chapter 4 provides a detailed description of how the problems were introduced to and solved by the participants, the following section discusses the general format of the PSSs.

Solving the Problems

The participants and I determined that Tuesday evening from 5:00 to 7:00 was the best time for us to meet for the PSSs. On the three Tuesday evenings we selected, Annie, Courtney, and Meredith took a mathematics education course from 3:15-4:45. This class met down the hall from the conference room Patricia Wilson (the mathematics education department chair) graciously invited us to use as a place for our research. Thus, as shown in Table 2, we met for three consecutive Tuesday evenings. Because the participants had just gotten out of class, I provided dinner before we "got to work." Sitting around a small table in the mathematics education break room and enjoying salads or pizza, we engaged in informal discussion, not only about school, work, and friends but about the study as well. Although not videotaped or recorded, notes from this time of conversation filled several pages in my journal.

Around 5:30, we would head down the hall to the conference room to begin the problem solving session. I had previously set up the video camera in the front of the room, placed the microphone in the middle of the table, and laid out any materials for the problem (e.g., manipulatives, problem statement, data sheets, scratch paper, calculators,

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[4] Solutions to the each problem are found in Appendices C, E and G.

computer, pencils). The participants came in and sat at the end of an oval table in the same order for all three sessions—the seating arrangement was their choice. I then turned on the video camera and presented the problem they would be working on as I sat at the other end of the table and observed. I was both a participant-observer and their teacher. I engaged in their problem solving process when they asked clarification questions, wanted to make sure I had just seen a great insight, got stuck. I also intervened whenever else it felt appropriate (examples of these interactions are given in chapter 4). Many times my answer to their questions was another question, but sometimes I provided more of a "hint." The sessions were intended to last from 30 to 45 minutes. Although the participants could have worked much longer on the problems, I honored their time commitments and stopped them at a point that provided some closure to the exploration.

Group Interviews

Group interviews occurred immediately following each PSS. We stayed in the same room, turned off the video camera, turned on the tape recorder, relaxed, and talked comfortably about the participants' individual, group, mathematical and emotional experiences during the past 30 to 45 minutes. The intent of these interviews (see Appendices I, K, and M) was to provide the participants an opportunity to debrief—to share thoughts and comments on the PSS while it was fresh in their minds. It also gave the participants a chance to hear, react to and reflect on each other's ideas. I took notes during this interview regarding issues to address in the individual interviews conducted the next morning. For example, in the group interview following the first PSS, the participants spoke repeatedly about the size, use, and placement of the manipulatives and materials. Realizing that this aspect of the PSS was important to them, I was careful to discuss "manipulatives" with each of them individually. This proved to be a very fruitful approach to adapting interview protocols as well as allowing time for the participants to incorporate "group processing" (see p. 9) into the cooperative learning activity.

Individual Video Recall Interviews

Poststructuralism offers “critiques and methods for examining the functions and effects of any structure or grid or regularity that we put into place” (St. Pierre, 2000, p. 6). I selected “Video Based Stimulated Recall Interviews” (see Appleton, 1997; Prawatt & Anderson, 1994) as a way for the participants and myself to examine the functions and effects of cooperative learning during this phase of data collection. Thus, an individual participant watched the videotape of each PSS with me while we discussed in detail what we viewed.

These individual interviews occurred the day after the PSS in the same conference room in which we had worked the evening before. Annie, the early bird, claimed the 9:30 to 11:00 slot for each Wednesday morning. Meredith had a break between classes at lunchtime, so she came from 11:00 to 12:30. Courtney was flexible in the afternoons, so we decided to keep the schedule fluid by having her interviews be from 12:30 to 2:00.

To prepare for the interviews, I spent Tuesday nights previewing the tape of the previous night’s PSS. Using my research questions as a filter, I watched the tape (pausing often) and made notes referring to specific episodes or moments I wanted to discuss with one or all of the participants. I then drafted an interview protocol to be used the next morning (see Appendices J, L and N). From 8:00 to 9:00 on Wednesday mornings, I viewed the video again with the drafted interview protocol in hand. I noted any additions or adaptations to the protocol I deemed necessary and noted the times of specific episodes to be discussed.

Because of the previously noted limits of Appleton’s (1997) methodology, I thought it very important that we watch the entire tape, not just episodes I had identified. I encouraged each participant to watch the videotape and recall, at liberty, what she had been thinking, doing, and feeling. Each participant sat close to the “pause” button and was encouraged to pause the video at any point in the interview. I also pressed *pause* when I had something specific to ask or when a participant began to shared an idea or observation I wanted further clarification or discussion about. For example as we watched the PSS2 tape, without pressing *pause*, Annie commented, “See there, I am much more engaged this time than the last problem.” At that point I wanted to know if she could identify why she was more engaged, so I pressed *pause* and she explained that “the manipulatives were much closer to [her] this time and that helped.” Although conversations ensued around the comments, clarifications, and questions of both the participants and me, the goal in each of these interviews was that the interview be more representative of the thought process of the participant than of the researcher. The video-based interviews promoted this goal by allowing each participant watch herself (and group members) and then point to and discuss those aspects of the PSS that were important and relevant to her.

I provided time before and after watching the videotape for us to discuss broader questions with respect to the PSS or to CMPS, in general. The participants and I were pleased with the timing and the structure of the interviews. We all thought it was “good that we did it today and didn’t wait another day. [The videotape] really does bring a lot of it back” (Meredith, PSS1). I believe that data were richer and the study was stronger because of this immediate and extensive use of the stimulated-video-recall technique.

Research Presentations

An unexpected yet welcome opportunity for further data collection occurred in April 2001, when I was asked to give a presentation on this study to preservice mathematics education teachers at the University of Georgia. Because I viewed the participants in my study as not only participants but also co-researchers, I invited them to be a part of the presentation. Prior to the presentation, I had completed a preliminary analysis of the data. As the participants and I prepared for the presentation, they discussed and reacted to my initial findings. The new data provided by their feedback were invaluable at this point in the study. In addition to the input my participants provided while preparing for the presentation (see Appendix O for the agenda), I also had access to more data during our presentation and the classroom discussion that followed.

Data Analysis

In his discussion of the "spiral" of qualitative data analysis, Creswell (1998) remarked, "Data analysis is not off-the-shelf; rather it is custom built, revised and choreographed" (p.142). This adaptive and ongoing process of data analysis provides researchers flexibility within data analysis techniques, but it also adds to the complexity of communicating such an emergent process. Doucet and Mauthner (1998) claimed "the literature on qualitative research methods has relatively little to say about the detailed and concrete processes of data analysis" (para. 2). Not only is the literature on analysis limited, but so are many of the descriptions of the analysis processes found in research reports. Lists of codes, names of computer software, references to the constant-comparative method, cross-case analysis and grounded theory are common analysis techniques referred to in recounting data analysis. Although these theories and techniques are central and important aspects of qualitative data analysis, the simple naming of them does not provide a reader enough analytic detail on which to base decisions about the reasonableness of emergent findings. I find that the difficulties to report analytical processes that "blur the lines between data collection and data transformation" (Leatham, 2002, p.56) lie in the intuitive nature of such analyses. However, since the validity of this study lies in the ability of the reader to make judgements about the integrity of the reported findings and conclusions. What follows is my account, intended to be sufficient enough to allow such judgements, of the data analysis phases specific to this study.

With the goal of systematically interpreting my data so that I could provide meaningful, reasonable, and defendable answers to my research questions, I conducted data analysis in four phases using *adaptive*, *rhizo*, *theoretical* and *deconstructive* analysis techniques. The first phase, adaptive analysis, occurred throughout data collection as I modified the protocols for future interviews based on previous interviews and problem solving sessions. I also employed the adaptive analysis tool of member checking as I discussed my interpretations of the data and emerging themes with the participants. New codes were developed, emphasized or de-emphasized. For example, the notion that silence was an act of resistance to perceived power with a group was not an aspect of the PSSs I focused on until Meredith pointed to it when I suggested that silence could signal a lack of understanding. I was then able to discuss this issue of silence with the other participants. New insights and codes (discussed in chapter 5) were developed within the theme of silence.

Botanically speaking, growth that "pops-up" in a seemingly disconnected manner (i.e., crabgrass) may actually stem from the same "creeping, horizontally-growing underground root system"— a rhizome. (Greenfiled, 2000, para. 1). The second phase of analysis employed the "rhizome" model of Deleuze & Guattari (1987/1980) as a metaphor for describing the intuitive nature of initial data sorting and subsequent emergent coding. Because "rhizomes do not have clearly identifiable beginnings and ends, a [rhizoanalysis] concentrates on the middle, rather than trying to follow" a linear path of coding and analysis (Honan, 2002, para. 3). Along with a linear read of the data, I also read interview transcripts in different orders, viewed segments of videos—started in the middle. I reflected on the data both formally (e.g., reading, re-reading interview transcripts, comparing interview and video data, writing data stories for each problem solving session) and informally (e.g., the insights that wake you up in the middle of the night, the ones you have to jot down while driving). The rhizoanalysis then consisted of making notes of instances and discussions that "popped out" and felt important to understanding the participants experiences. As I studied these notes, connections and commonalities emerged as codes. For example, the code "manners" emerged from the following list of notes:

- Sacrifice your own mathematics for the "good" of the group
- There is a need to ease tension
- Feeling that you have to make second order models
- Trying not to take over … maybe it is hardest to be the leader because now you are thinking about math, and others ideas, as well as am I too bossy

- They seem to respect other's train of thought and not interrupt but are willing to drop their own (Researcher's Journal, compiled list)

The theoretical phase of analysis began by using the pre-established synergistic perspective guiding the study. I sorted the emergent codes and placed them into the categories delineated in my first research question: environment, group members, and self. Within each category, codes were then combined. For example, within the "self" category the following codes:

- Self Perception
- Time for Reflection
- Mathematical Knowledge offered and taken-up
- Perspective shifting
- Getting Ideas Validated and Supported
- Grounded Qualities… taking care of yourself
- Helping each other

were collapsed into the themes "autonomy" and "matheamtical ownership." The theoretical phase continued as I looked for what made each of these themes identifiable. I realized what was consistently apparent was the binary tension the participants felt between the individual and the group—a tension that frequently interfered with individual mathematical activity.

The final phase of analysis was the application of Derrida's method of *deconstructing binaries* that can be summarized in four steps: find, articulate, reverse and resituate the binaries (Graves, 1998). By identifying the ground and emergent qualities in each theme, the application of Earley's levels of conciousness model as a framewok helped identify and articulate binary tensions within the three categories. In chapter 5, I discuss these binaries, how they effected individual mathematical activity, and the ways in which the particiants reversed the binaries. In chapter 6 I discuss the implications of reversing group/individual binaries and offer suggestions for resituating the binary so that cooperative problem solving enhances the mathematical activity of the individual members of a group.

Representation

At the doctoral committee meeting where I defended the proposal for the present study, we discussed a variety of ways I could have represented the data in the next chapter. Options included: write one case study or data story per individual participant or use the group activity or interactions as the medium for representation. Jeremy Kilpatrick pointed out that this decision reflected the binary tension I was striving to deconstruct. He encouraged me to consider a representation that held true to a synergistic perspective by representing cooperative learning as a complex system and not as just a sum of its parts. When faced with the desire to move beyond the dichotomous social/personal approaches to mathematics education research, Kieren (2000) suggested using the conceptual tools from both perspectives "to create conceptual binoculars to… look explicitly at mathematics knowing as embodied action" (p. 231). The goal of the present study was to understand better the complex system of the mathematical activity of individual students in the context of cooperative problem solving. Thus, I chose to use the PSSs as a medium for telling the data stories. This holistic, embodied approach to representation brought the mathematics to the forefront and strengthened the investigation of the research questions.

As I decided the context within which to situate the data stories, I felt troubled by inscriptive process of writing of qualitative research. I heard many times in my qualitative research courses:

It is never really the thing, but only a version of the thing.

Part of doing constructivist inquiry is the realization that "what we call our data are really our own constructions of other people's constructions of what they and their compatriots are up to" (Geertz, 1973, p. 9). The experiences of the participants as they engaged in CMPS (the "thing") would be written (the version of the "thing") based primarily on the observations and discussions of the participants and myself—our constructions. I became increasingly aware that the text I created would now be "divorced from the context of which it had been a part and where in it had been produced" (Farran, 1990, p. 96). It could only be a re-presentation based on interpretation. The responsibility was great and the task daunting as I worked to create a text that would embody of the context, honor the words of the participants, and be of value to its readers.

My grandfather is a well-published Native American poet. He has a gift for providing a voice to the people he feels so connected to—our ancestors. I imagined that if he did a qualitative study investigating the experiences students engaged in cooperative learning activities, he would write a beautiful representation that mixed the actual voices of the students with the voices he heard in their actions. He would find a way to give voice about personal, social, learning, and mathematical experiences to those who are not quite sure how to (or if they should) do that for themselves. My grandfather would see beyond their written words and hear more than just their spoken words. I hope that part of him was within me as I struggled to tell the important stories of three young women I feel so connected to.

CHAPTER FOUR: SOLVING THE PROBLEMS

Meredith spoke for all of us when she said, “I have learned so much from this experience. I am so glad I did it. I have definitely learned a lot of ways to have cooperative learning be a lot more effective.” All the participants agreed they would be creating, in their classrooms, similar opportunities for their students to gain insight about themselves as mathematical and cooperative learners via reflection. Annie pointed out that not only do the participants learn about themselves, but [the teacher] “can find out a lot about people and their mathematical thinking and stuff by having them reflect on an experience like this…. I’m sure [Miss Sheehy] has learned more about us.” Annie, Courtney, Meredith and I all felt that through our participation in this study, we gained new and valuable insight and now have much to share about our experiences. Courtney graciously wished all students, teachers and pre-service teachers could have been involved firsthand in our in-depth and multi-perspective investigation of CMPS. Acknowledging such involvement would have been physically impossible and very impractical, we talked about ways to provide others the opportunity to glean their own insights from our experience. One way was to tell the stories of the participants and the PSSs by writing about the problem solving sessions and subsequent interviews.

This is my dissertation, so I am, of course, the one with the responsibility (and the honor) of “telling our story.” What follows are three detailed descriptions of and reflections on the PSSs. The ever-present goal in writing these detailed descriptions was to invite you, the reader, to be in the moment with us, and thus gain insight into what the participants were experiencing. Weaving together the videotapes of the sessions with the participants’ descriptions and interpretations of the events, this chapter is a re-presentation of the PSSs from multiple perspectives.

This weaving of the data sources in order to describe the sessions posed a few stylistic challenges in terms of verb tenses and quotation styles. In order to distinguish between participants’ quotes from videotapes (during the PSSs) and quotes from the subsequent group and individual interviews, I developed and consistently used the following styles. Quotes directly from the videotapes are written in regular font and *italics* are used for interview quotes. Verb tenses were potentially confusing because as the participants viewed videotapes and talked with me during interviews, they were commenting on conversations that occurred during the PSSs. To decrease this confusion, interview quotes are in past tense, and videotape descriptions and quotes are in present tense. In addition, I used the following abbreviations and acronyms throughout the remainder of this dissertation (see Appendix A for a full list).

PSS(no.)(Initial)	Individual interview quote Example: PSS2A represents a quote from Annie's individual interview following the second problem solving session.
PSS(no.)(Initial)(G)	Group interview quote Example: PSS3AG represents a quote from Annie during the Group Interview following the third problem solving session.

As I analyzed the following detailed descriptions of the PSSs with respect to my research questions, I identified emerging themes within the environment, group members, and self categories outlined in the first research question. It was evident to me the participants were both actively experiencing, effecting and being effected by the CMPS experiences in response to self/other tensions they experienced throughout the PSSs and in the follow-up interviews. The common tension for the three participants across the three PSSs can be loosely generalized by the following question, "Who is more central, the group or me?" As you read the following data stories, look for these types of tensions in the following areas: using manipulatives, adopting roles, being polite, sharing ideas, resisting power, and seeking autonomy. In chapter 5, I discuss these tensions within the context of their effect on individual mathematical activity.

Problem Solving Session 1: The Square Game

The Problem

This problem involved paying a game. I instructed the participants to set up colored squares on game board as shown below (see Figure 7).

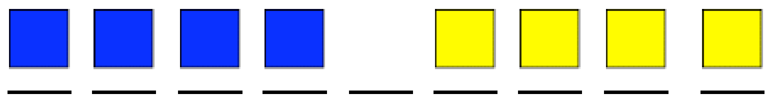

Figure 7. Initial set-up for the Square Game

They were then told that winning this game meant moving the squares so that the yellow and blue squares exchanged positions. The initial question posed was "What is the minimum number of moves it takes to win the game?" The following conditions under which each move could be made were read and the participants began working.

- The blue squares may only move to the right. You may move them one space to the right to fill an empty space or you may jump one square that is immediately to its right (providing that there is an empty space to jump into).
- The yellow squares may only move to the left. You may move them one space to the left to fill an empty space or you may jump one square that is immediately to its left (providing that there is an empty space to jump into).

Once the students were familiar with the game, my plan was to present them with the following types of questions:

- What is the minimum number of moves it takes to win this game?
- What is the minimum number of moves to win if you start with only 2 pieces on either side?
- Can you predict how many moves it would take to win if you started with seven pieces on either side?
- Do you notice any patterns in the colors you are moving or in the type of moves that you are making?
- Can you classify and count the moves you use in other ways?

Participants' Descriptions of the Problem

Courtney

We were given a piece of paper with nine dash marks on them, nine places I guess. And we had four blue squares and four yellow squares. And we started with blue squares on the right and four yellow on the left, and there was a space in the middle. And we had to figure out, well, the question was the minimum number of moves by sliding and jumping the squares that you could get the yellow squares on the right and the blue squares on the left… no, the yellow squares on the left and the blue squares on the right. And the blue squares could only move towards the right and the yellow squares could only move towards the left but you could slide or jump—you could jump one square at a time.

Annie

You gave us a game board, and we had four yellow pieces and four blue pieces, and we had to get one color on each side.

Meredith

[We had] four blue squares and four yellow squares on nine spaces with a space in the middle with all the squares on spaces leaving one in the middle with each color moving in one direction…. [We] rearranged the color so that the yellow sits where the blue did and the blue sits where the yellow did…just doing slides and jumps. And then try and do that with a minimum amount of moves and find a formula for how to figure out how many moves you have to make based on the number of squares that you have.

Description of the Environment

Figure 8. At the start of PSS1

Around the end of an oval table (see Figure 8), the girls are sitting (by their choice) from left to right: Annie, Courtney, and then Meredith (I am at the other end of the table). The laptop computer is unopened and on the side of the table with Annie along with a graphing calculator. The game board and square pieces are in front of Courtney and Meredith has a stack of blank paper and a pencil in front of her. (Note: I place these materials at the various seats before we entered the room together.) The girls seem to be a little uncomfortable—maybe it is just that they are unsure of what is about to happen, as this is our first problem solving session. They are relatively quiet compared with their usually talkative selves, and what little conversation they share is sprinkled with nervous laughter. They glance several times at the video camera and then look to me for direction before handling any of the materials in front of them. Everyone is leaning forward, and although they are not yet aware of the activity they will be working on, I have a sense they are ready to engage in a mathematics problem.

Learning the Moves

As I begin setting up the problem by saying, “You start with four blue squares and four yellow squares,” Courtney is counting out the appropriate number of squares from the materials in front of her and places the game pieces on the board according to my instructions. I then explain what it means to "win" the game and offer the conditions for moving the pieces. As I say, "The blue pieces can only move to the right," Courtney turns the game board once so that it is facing Meredith and then again so that it faces Annie. I then continue with the directions (defining the two legal moves – jumps and slides) and pose the problem of finding the minimum number of moves to win the game. As I explain the rules of the game and pose the problem, Meredith looks at me almost the whole time, while Courtney looks back and forth from the board to me and Annie never takes her eyes off the game board.

As they begin working on the problem, Courtney moves the square pieces first as Annie and Meredith stare at the board. Within 15 seconds, Meredith reaches over and is moving the pieces. She remains in control of the pieces for about 30 more seconds and offers some generalizations about the possible moves. Meredith later discussed this time with me:

L: *Now I'm curious, you are staring at the game board. Do you remember whether you were trying to think of a way you could do it or if you were trying to follow what Courtney was doing?*

M: *I think I was trying to figure out what would work…. And then I jumped the blue and then I realized that if I jumped the blue again, I would have two right beside each other and an empty space. But then if I did the same thing on the opposite side, I was going to be in the same predicament, either way, like whichever direction. So I was trying to think, "Well, what other options do you have after that move?" Because I knew the first move was going to have to be a slide, I was thinking what other options I had.*

Meredith voices these generalizations aloud, which prompts Annie to test them on the game board so she reaches in and begins moving some pieces. Annie ends up disagreeing with one of Meredith's generalizations but does not immediately tell her that.

A: *I didn't completely agree with that [generalization].*

L: *Did you tell her that you didn't agree?*

A: *No. I was trying to understand why she thought that instead of saying, "No, I don't think that's right."*

When I asked Courtney (who had been silently watching Meredith and Annie try out moves on the board) what she was thinking about during this time she responded, "*I think I was more thinking about what I would do next. As she [Annie] would move, I would try and think of options that would be better, but it didn't really matter because she would just move it.*" Courtney now takes back control of the game pieces, points out that the yellow squares are getting stuck and grouped together on one end, and offers the suggestion "We need to go one this way and one that way to have it even." Meredith quickly supports this suggestion by agreeing "Yeah, to have it minimized" and then suggests "We need to keep that space in the middle." Courtney and Meredith (both moving pieces) test this new approach, and Courtney laughs as she notes that they are once again stuck. At this point Annie (who has been sitting quietly and watching the board) reaches over and without saying anything begins to move pieces. The three girls alternate moving game pieces for short time – they say very little to each other except for statements like "Move this one here." And then, almost simultaneously, they all stop moving pieces, lean back in their chairs, and look at me with a look I know well. I take the cue and offer the suggestion, "Would it help to make the problem smaller?"

Making the Problem Smaller

As Courtney talks about George Polya (she had studied his problem solving strategies in another course) and agrees with the strategy of "making it smaller," Meredith immediately picks up two yellow and two blue squares and draws her own game board on the paper in front of her. She later explained,

> "*I was like, make the problem smaller, put it right in front of me, it will be easier to see if it is facing my direction—basically, get it closer to me where I can relate to it a little better*."

At this same time Annie, noticing that Meredith has chosen two squares of each color, pulls the game board toward her and methodically rearranges it by removing four of the squares and placing the graphing calculator on one end of the board to cover the last four spaces. As if she had prepared the board for Courtney to use, she then slides it back neatly in front of Courtney.

Initially, Meredith is the only one moving game pieces (on the board she just constructed) as Courtney and Annie intently and silently observe. After about five seconds Courtney begins to mimic Meredith's moves on the game board in front of her by watching Meredith and then repeating the same moves. When I asked Courtney, "*What makes you switch from watching [Meredith] to moving the pieces in front of you*?" she offered, "*Because I think I had to do it myself*." For the next minute Meredith and Courtney are moving the pieces in an attempt to win the game, glancing at each other's board and starting over each time they feel "stuck." Annie occasionally reaches in and moves the pieces in front of Courtney or points to the ones that Courtney should move but never verbally states any move she is making or that she wants Courtney to make. They all three continue looking back and forth from one board to the other. When asked about why and when they looked to what the others were doing Meredith explained:

M: *I wasn't really paying attention [to the other board until] I got frustrated. (laughs).*

L: *So, it was when you got frustrated that you looked over at what they were doing?*

M: *Yes. When I looked over, it looked like they were getting stuck, too. So, I … (pause)*

L: *You went back to your own?*

M: *Uh huh.*

And Courtney commented: "*Yeah, I think…. I might have been looking if I was stuck to look and see what she [Meredith] was doing and get a help from her*." Because Annie does not have her own game board and pieces, she is able to watch both Courtney's and Meredith's moves more closely. She later told me that this proved to be difficult for her and got in the way of solving the problem: *"Now I'm just like, 'Oh, gosh!'" And I need to look at what Meredith is doing and also look at what we need to do and I am like, 'No, I* *can't do this' [keep both things straight]*." Shortly the pieces again stop moving. Meredith and Courtney look to me for help as Annie stares at Courtney's board. I probe: "What are you finding that doesn't work? Are you both getting stuck in the same place?"

Getting "Un-stuck"

Courtney notes, "It seems like we keep on getting stuck with the end ones still there. They are trapped behind two squares, and they can't jump over two squares. So maybe if we got those out to begin with?" I interject briefly, with the intent of helping them communicate more easily, and offer the suggestion of naming the moves as they are performed. They use the suggestion, as they are all three calling out "slide" and "jump" while Courtney performs the moves. Courtney explains to the group why she believes certain moves do not work and why they end in getting "stuck" or "trapped." Meredith offers an idea about how the pieces are getting trapped and then begins to record the moves as they are made on her paper. She recounted:

> *Well, I knew that from the beginning there was some obvious moves that you had to make and so I wanted to write those down – the ones I knew were safe and were okay before you get trapped. So that once we were getting trapped, you had to think of, okay, what else could we have done here and what else maybe would have worked to get us up to this point if we should have set it differently, you know?*

Everyone expresses agreement by saying "uh huh" and nodding when Courtney states, "Jumping the end one doesn't work." I respond by asking the clarifying question, "So you know your first move has to be what?" and Courtney answers, "A slide" with Annie and Meredith echoing "slide." Now Courtney sits up straighter and becomes more dominant by increasing the volume of her voice as she says to the group, "Okay, what are our options now?" I asked Courtney about this question and her intentions in asking it:

L: *When you were asking "What are our options?" were you asking for your group's benefit or were you asking because you didn't know?*

C: *I don't think I was asking because I didn't know, but I was asking all of us…so I guess to stir up things*

L: *And do you know why you felt like you needed to facilitate the conversation that way? Or why you wanted to?*

C: *I think because Meredith was trying to do it at one row and whenever Annie moved the pieces, she doesn't say anything, she just moves them because she is just thinking to herself so nobody else…. Meredith was talking too, but she was doing it over there on the piece of paper and this is the board they were supposed to do it ... and nobody else is really talking. I guess Annie is the only other one who was working on the board and every time she started moving, she wasn't ever saying what she was doing, she was just doing it because she was just thinking about it while she was doing it. I didn't really know what she was thinking. I guess I was just thinking out loud.*

L: *Modeling that, too?*

C: *Yeah, definitely*

With Courtney talking through the possible moves and Meredith verifying Courtney's decisions by repeating her words or offering words of agreement, the game is played several more times. Meredith takes care to write down the moves made in each game thus allowing them to back up when they get stuck rather than having to start all over. At the same time, Meredith and Courtney see the series of moves that will give them a win. They are both quite animated while Meredith is pointing and calling out the moves and Courtney is quickly moving pieces and saying, "I see it." As she was so excited about finally winning, Meredith has not recorded these moves so Courtney begins attempting to repeat the series of moves. She calls them out by color and type of move ("slid yellow, jumped blue") as Meredith diligently records. Annie is leaned forward and slumped down with her elbow on the table and her head resting on her hand as she watches Courtney and Meredith try unsuccessfully several times to repeat the win. Not uncharacteristically for this PSS, Annie continues to be very quiet during this entire investigation with two and two pieces. However, she has been focused on the game board both visually and by occasionally pointing to the squares she felt should move. Meredith later offered the following explanation for Annie's silence:

> *Maybe it had a lot to do that she was so much farther away from [the board]. You see, I moved my chair closer to Courtney and Annie is still farther away. I'm still trying to look at it horizontally and Annie is looking at it vertically almost. So maybe it was harder for her to see it—because she didn't even seem excited [about winning the game], like reacting, "Yeah, that's it."*

On the fifth attempt with Courtney verbalizing the moves, Annie now pointing to the correct move when they got stuck and Meredith using her records as a guide, they successfully reconstruct and create a record of the winning moves.

The girls exchange smiles after these moves are recorded and then immediately they all look to Meredith's paper while searching for patterns. Each one is leaning forward, engaged both visually and verbally, and offering hypotheses about possible patterns. Courtney claims, "You just alternate between slides and jumps;" Annie describes the pattern in the colors "blue-yellow-yellow-blue-yellow-yellow;" and Meredith points out that Courtney's observation is incorrect because there is a "jump-jump" and goes on to explain, "But the jump-jump is in the middle. There's a slide-jump-slide (pointing to the paper) here [the first three moves] and a slide-jump-slide here [the last three moves] and the jump-jump in the middle." Courtney offers a suggestion for why this might be true in all cases (all numbers of squares) and Meredith points out another pattern that she sees on the paper. Although Meredith rarely turns the paper so that others can more easily view it, she did observe later that she might have had an advantage in pattern identification: "*It might have been hard for them to see or understand what I was saying because I was the one that was writing it down and looking at it.*"

Meredith now suggests, "Let's try it with 3 and 3." Knowing that it would be easier if they had already identified patterns in the moves, I interject, "How about one and one?" Annie sets up the board for one and one by removing pieces and covering blanks as Meredith labels her paper and prepares to record and organize moves. Courtney has already visualized the winning sequence of moves so as soon as Meredith finishes setting up to record, she quickly moves the pieces calling out "you just have to slide and jump and slide." Meredith quietly probes "so we just slid what?" Courtney takes the cue and calls out the moves again, this time slowly and including color. Meredith notes that there is still a jump in the middle and Courtney proposes that "the more things you get, the more jumps you have in the middle." "Well, let's try it with three and see if there are three jumps in the middle," suggests Meredith.

Looking for Three Jumps in the Middle

As Courtney adds more squares to the board so that there are three yellows and three blues, Annie again assists her by covering the unnecessary spaces on the board and clearing the excess squares out of the way. Recalling the hypothesis that the first move must be a slide, Courtney begins the game with a "slide blue." On the third move she is interrupted and corrected by both Annie and Meredith (who are referring to the previous pattern for two squares) – Annie is mentally reconstructing the patterns and Meredith is looking at the data sheet she has developed. As Courtney continues to move the pieces, all three girls offer suggestions and predictions about what moves will get them stuck and what they should do next. Suggestions are diminishing and everyone is leaning back (Meredith even pushes her chair away from the table). Sensing frustration brewing, Courtney quickly chimes in with a quiet but assuring, "We can do this." Meredith smiles, leans back in with a very positive "Okay" and begins offering possible moves again. Annie returns to the "head in the hand, staring at the board" position.

As Courtney and Meredith brainstorm about possible ideas, Annie reaches in and moves pieces. In unison Courtney and Meredith express their frustration – Courtney blurts out "Whoa!" and Meredith with her hand up in the halt position says, "Hang on, slow down, we are writing this down." Meredith later explained that she did not mind Annie moving the pieces she just did not want to "*go ahead too fast. We got here, it's* *looking good, let's get this information down before we jump any more ahead…. I was nervous about forgetting about what we had done."* Meredith and Courtney patiently backtrack by un-doing the moves that Annie made. Annie and Courtney begin discussing possible future moves and identify that they keep getting two yellows stuck. So Courtney clears the board and suggests they start over. Meredith, in a very calm and almost questioning tone, interjects: "I think the key is that instead of starting over, we need to go back to…" Courtney interrupts with, "That's what I'm doing" and moves the pieces based on the first moves recorded on the data sheet. Courtney continues trying the moves that all three are suggesting while being careful to call out her own moves and also name the moves that Annie, almost always silently, reaches in and performs. Meredith is repeating each move as she writes it down. At one point, Annie reaches in and makes a move that elicits "yeahs" and "oohs" from Meredith and Courtney. Meredith remembered,

M: *Then it worked and I was like, "ooohhhh." I think I was kind of more in awe—whatever she just did, just worked." But it was so fast that I didn't really have time to grasp what it was that she saw.*

L: *Where do you think she got that idea?*

M: *I guess she's moving the squares in her own head, trying to see what would work, visualizing it, because she wasn't doing it.*

Almost reluctantly, I call to their attention that, in fact, they just moved the yellow pieces in two different directions. Wondering if I observed correctly, Courtney repeats the moves that Meredith reads and all three girls rather quickly identify the incorrect move. With no one appearing disheartened, Annie reorganizes the game board while asserting that the yellow squares should start on the right. In a slightly abrupt tone Courtney reminds her that "it doesn't matter which ones go where." Annie replies with "Yeah, but yellow has always been going this way (pointing to the left)." As Courtney draws arrows on the game board in the directions that Annie suggested, Meredith is busy erasing the sequence of moves they had previously constructed for the three squares.

They start over and work for about two minutes trying to win the game. During this time the following approaches occur: Courtney offers two choices of moves and then does what Meredith chooses, Courtney and Meredith turn toward each other and discuss each move, Annie shifts her position several times from leaning back in her chair to reaching in and moving pieces, and Courtney does a lot of thinking out loud as she tests various moves. Annie said that during this time, "*I'm not paying attention to what they are saying; I'm trying to look at the pattern*." Meredith later decided that Annie's strategy might be a good one:

> *Should I let them just go with it and just sit back and just watch? Put the ball in their court since they are already talking…. I didn't see [the solution]. I think that might have been a lot of the reasons why… maybe I didn't see things. Like Annie was sitting back, she was just observing, she wasn't communicating that much. Maybe that's why she saw things a little bit more.*

Courtney echoed this belief about Annie's role in the PSS:

> *She just naturally—she kind of brainstorms and then she gets the idea and she says it and it's like, "Whoa, she's right….. Secretly she's sitting over there in her chair thinking about it the whole time where we are more concerned with moving the pieces and writing it down and we don't have as much time to think.*

Things shift as Meredith takes control of the activity for awhile. She moves closer to Courtney and the board. She both moves the pieces and poses questions about possible moves, and she offers summaries of strategies and ways in which they end up stuck. It is during this time, when Meredith is more dominant, that Annie becomes much more verbally engaged. She and Meredith now have lots of eye contact across the table and are discussing possible moves. After getting stuck one more time, this conversation quickly fades, as does any movement of the pieces. Courtney (who has been observing) looks at me and shrugs her shoulders while Annie and Meredith lean in to rest their chins on their hands and stare at the board. Wanting to keep them encouraged and knowing that they were close to a solution I remark, "That was a really good try, though."

So they regroup—Courtney puts the squares back in the starting position as Meredith gets a new, blank sheet of paper. They confidently begin with "slide blue, jump yellow" as Meredith writes these two moves down. Then Meredith hesitantly proposes, "We need to keep it blue-yellow-space, blue-yellow-space so there's always room for an empty space." Courtney quietly answers her with, "No, because remember if we have two blues or two yellows next to each other, we can't get out of that." "That's what I'm saying," asserts Meredith, "we don't want two colors next to each other when we are jumping." Courtney moves back a few steps with what she has been trying and reiterates "So we have to keep consecutive colors away from each other." Meredith responds with an agreeing "Sure" and recalls the idea "Let's try to keep that space in the middle." Courtney then moves into her familiar mode of operation (thinking out loud to encourage discussion) as she poses the question "What are our options?" She then states and shows every possible move from this position, concludes that no move will work, and looks at me with a suspicious smile as she posits "Well, maybe you can't do it with three." I tell her, "You can do it with three" and that they "almost had it a second ago." She laughs and says, "Don't tell us that!" as she mentally recalls the previous moves (by tapping on the table below the board) and proposes, "Okay, that was when we were sliding."

As Courtney leans back slightly and pulls her hands away from the board, Annie reaches in and completes three moves. In a slightly frustrated tone, not really with Annie but with the game, Courtney sighs and says "But, you can't do that" and then undoes Annie's moves. Annie remembered feeling her own sense of frustration here: "*Courtney was basically in control of the board and whenever I tried to do something else, she was like, 'you can't do that because of this.'*" Still sure that the first two moves are "slide-blue, jump-yellow," Courtney returns the board to this position to discuss the possible third moves. She resumes her familiar role (talking out loud to encourage discussion) and once again asks, "What are our options?" Without pausing, Courtney shows and states two possible options. Annie points to the yellow square on the end dash closest to her and acts out on the table above jumping it – an option not suggested by Courtney. She recalled: "*I saw it because I knew we had to open up that other last yellow and jump*." But Meredith reminds Annie, "We want to keep [the blank space] as much in the middle as possible." Courtney makes and calls out a series of three moves (slide-yellow, jump-blue, jump-blue), pauses as she looks up, and says "Yeah, cause we need those two jumps like before." Meredith writes these moves down immediately. Courtney slides a yellow and Annie now points to the blue square on the opposite end of the board and motions that it should be slide. Meredith again says, "No, we want to keep it in the middle" and Courtney says, "You can't do that." Annie retorts, "Well, you can't do that either" as she points to the last move Courtney made. Annie then removes her hands from the board and pushes herself back from the table. Annie later shared:

A: *I knew that you had to get that end piece down when we were doing the three-one and I just kept thinking "I want to do that, let me try this".*

L: *You didn't ask to try it out.*

A: *I think because I didn't want to take control—I didn't want that.*

L: *Why not?*

A: *I don't know (laughs). I think in tutoring I just really try hard to get other people to understand and make their own connections than just me. And …I think I was just trying to…I sort of saw what was happening, I knew you had to move it. Once I did that, then I could see what the rest was…. I think the main part was that I wanted them to see it, too, and if I could have just moved that piece and shown them it would have worked.*

At this point Meredith says, "We're stuck so I think we shouldn't have slid the yellow" as Courtney backs up one move. Annie now, rather than just motioning, verbally says, "slide-blue" (the move she had pointed to a moment ago). Looking at Meredith, Courtney states, "Yeah, slide the blue and then jump the yellow." "Mm, hmm," Meredith says excitedly as she writes down these two moves. Now Annie points to the yellow square on the end closest to her and says, "jump the yellow." Meredith quickly adds on "and slide the blue" as she is recording these moves. Annie and Courtney now call out the next sequence of moves. At last they have won the game!

Immediately after the last move and with her eyes on the data sheet, Meredith proposes, "Let's go back and test it – I think I might have written it down wrong." As Courtney resets the board, Meredith leans back in her chair and sings a celebratory "heya, hey" and then summarizes for herself and her group, "So it's a process of jumping it, creating an empty space, and sliding it—deep, huh?" They all three smile and laugh about her "stating the obvious" conclusion. You can feel the tension from moments ago just melting away. They test the series of moves that Meredith has recorded and find that they were correctly recorded. Meredith brings them back to the question of "Are there three jumps in the middle?" by saying "Okay, now let's look at the patterns and compare."

Comparing the Patterns

Meredith places the two data sheets between herself and the middle of the table so that Courtney sees the paper vertically while Meredith and Annie look at it from the side. Meredith also points out the different data sets, "This is three, this is two, this is one." "So let's figure out the middle," Meredith proposes, as she counts to determine that with three squares of each color it took fifteen moves to win the game. She marks the center at the seventh move and then observes, "Look, they are all jump yellows. This [pointing at the data for two squares] was two jump blues." Courtney says "There are three jump yellows in the middle here and two jump blues for the two." Now Courtney and Meredith, both pointing at the paper, alternate describing the patterns they see in the list of fifteen moves for the three squares and decide that the pattern "goes out the same" from the middle. They state this pattern together and then Courtney points to the data for two squares. She explains how the patterns are similar, gets no response from either Meredith or Annie, asks "Do you see what I am saying?" and still gets no response. Courtney then slides her chair over to Meredith and takes a piece of blank paper *"because I [had] to write it down, I can't just talk."* Meredith later said that during this time she was *trying to compare the list, basically to when we had two squares and see if there is a similarity between the colors—or…but, see, when I was looking at it, initially, I was trying to see the relationships between the color and the slide or the jump.*

Courtney writes down a section from the sequence of moves for the three squares and explains to Meredith how they are similar to the moves for the two squares. Annie leans way in and points to the paper she can reach, as Meredith has now pulled the paper with the three squares data directly in front of her, but says nothing. While Annie silently watches Meredith write her ideas on the paper, Meredith is beginning to generate ideas about the four squares. She says, "so, for the next one…" and Courtney chimes in with, "You'll probably have four blues in the middle" with Meredith simultaneously stating an equivalent hypothesis.

Figure 9. Courtney and Meredith sitting together.

Courtney and Meredith are now sitting almost directly across from Annie with shoulders almost touching (see Figure 9). They are comparing the lists of moves with each other and hypothesizing about the patterns they see. They are very focused as they speak confidently and quickly. Annie recalled that while Courtney and Meredith were sharing ideas about patterns, she *"had no idea what they were talking about. I was like "What are you talking about?"* Meredith reflected on Annie's silence as she asked me:

M: *Do you think she can really stay with what we are saying? She is looking at lists from left to right and we are looking at top to bottom, you know?*

L: *While you were looking together did you have any sense that Annie was lost?*

M: *Not necessarily like…Courtney was saying, Annie will sit back and observe and then she will pop out something… really good and you'll go, "Oh, yeah!"*

When I asked Courtney if everyone understood the hypotheses she was making, she replied:

C: *I don't know about Annie.*

L: *Is she just watching?*

C: *Yeah, and just thinking about it. But I don't know if she is objecting because she's not saying anything.*

Courtney and Meredith continue to try to predict and construct the sequence of moves for the four squares on paper while Annie puts the game board in front of her and sets it up with four squares of each color. Meredith looks at the board and says, "Well, do you want to try to find the pattern? There still might be stuff in between there [the pattern she and Courtney have developed so far]." Meredith later expanded on her suggestion to move to the board:

> *The two center values were jump yellow, jump yellow for two and then for three it was jump blue, jump blue so the color was opposite. So I was wondering, "Okay, does that mean as you go up, like even and odd, that the color will change but there will still be jumps in the middle because there were jumps in the middle…. I could tell by the patterns that we had there was no way we knew what that stuff was in the middle so we had to figure it out on the board. That part we couldn't figure out without doing it on the board.*

Now that Annie has finished setting up the game board, she turns it back to Courtney. Meredith reads the pattern and Courtney moves the pieces – they have constructed about eight moves. They reset the pieces and repeat the same moves. Unsure about what to do next, Courtney asks, "What could that in the middle stuff be?" and then poses, "Well, what are our options here?" Annie is looking at the paper in the middle of the table and now asks Meredith to hand her one of the other data sheets. Meredith does not respond so

Annie goes back to watching Courtney go through the options. As Courtney and Meredith decide in what direction to go with the pattern of moves, Annie reaches in and moves two squares. Courtney immediately reverses those two moves and says, “Whoa, whoa!” as they all three laugh. The pieces continue to move but now Annie is saying, “Uh-huh,” “Mm-hmm,” and nodding as Meredith calls out moves. Courtney is now looking at Annie between each move. At one point Annie reaches in again and points to a square and says, “Jump the blue.” Courtney immediately does this move and then she and Meredith move the pieces about five more times. Courtney described some frustration and confusion:

C: *See there, she [Annie] just wants to do it and I’m like, “Whoa, I don’t know what you are doing?” Or just don’t do it so fast…. See, she won’t even, she doesn’t even say what she is thinking. All she is doing is shaking her head and nodding her head.*

L: *Were you trying to figure out what she was thinking?*

C: *It was hard because she said, “No” or whatever. Then she said, “Jump blue” and I had to like pull it out of her, what she was thinking. She wasn’t volunteering it at all.*

L: *And why were you trying to pull it out of her?*

C: *Because I know she’s smart, I know she can think about stuff, I know she probably knows how to do it, she’s just not telling us (smiling).*

L: *(laughs). Why wouldn’t she tell you?*

C: *I don’t know.*

As I watch the girls try different moves and back-up and try new moves, I have been sitting on my hands and biting my tongue trying not to point to the end square or blurt out “Just pull the end in!”

Pulling the Ends In

I am aware of the time (they have been working for thirty minutes and I told them these sessions would last approximately 30 minute). I want them to feel successful in finding this pattern for the four squares so I say, “Can I give you a hint?” Without taking her eyes off the board and still moving pieces Meredith says, “Yeah, that’d be great” while Courtney looks up at me and smiles a smile that tells me “You don’t think we can get it, do you?” I continue with, “The key to this is pulling the end in. You almost got there.”

Courtney backs up a few moves and reminds Meredith and Annie to think about pulling the ends in. With Annie watching, Courtney and Meredith make some moves until they realize that they have just put three blues in a row and they move back. Annie reaches in and points at a yellow square. Courtney looks up and asks her, “Do you want to jump this yellow?” With a nod from Annie, Courtney makes this move. Meredith says, “Okay, now jump the yellow on the end” and you can see the “I see it!” in Courtney’s grin as she says, “Yeah then slide the yellow in!” They now do four “jump-blues” that they hypothesized would be in the middle and continue with Annie pointing, Courtney completing and naming the moves, and Meredith recording. Meredith looks at me when the last square has been moved to successfully win the game and says, “There’s a pattern here.”

There’s a Pattern Here

Courtney described the pattern as “kind of obvious when you get to the end… you do the same things on one side and then you do it to the other side.” “Yeah, it’s like 1, 2, 3, 4, 5, and then 4, 3, 2, 1,” offers Meredith as she draws an inverted V in the air. As Meredith is writing out the pattern again – she wanted it done neatly—I remind them of the original question by asking, “So what was the minimum number of moves it took to win the game?” Annie summarizes, “one is three, two is eight, three is…” Meredith lets them know she’ll have the number of moves for four squares in a minute as Courtney is thinking out loud saying, “2^3 is 8.” Courtney takes a sheet of paper and creates a t-table organizing the minimum number of moves as shown in *Figure 10*.

# of squares on one side	Minimum # of moves
1	3
2	8
3	15
4	
5	

Figure 10. Table of numerical data

While Courtney creates this table, Meredith asks Annie to help her check the sequence of moves she has recorded for the four squares; she feels there is a mistake because the patterns were not consistent. Annie sets up the board as Meredith describes the inconsistency:

> If you just look at the colors coming in from the top and the bottom, it goes one blue from the top and the bottom and then two yellows, and then three blues. It looks like it goes four yellows but I have three yellows on the bottom. I might have written it wrong. And then it has the four blues in the middle. So it might be one, two, three and then all the fours.

They have yet to test the moves on the game board as Meredith goes on to describe the patterns she sees in the slides and jumps. Based on these patterns, Courtney and Meredith decide that they have left out a jump yellow and write it in. Annie is pushed back a little from the table with her hands on her chair, swiveling back and forth. Courtney says "So, how many moves is that?" Meredith counts and reports "24." Annie asks, "Is that with the inserted one?" Twenty-four is verified and Courtney poses the question, "So what is that in comparison to four? Meredith states that there are different ways to compare as there are moves by color and moves be action (jumps and slides). I ask which type they will use and Courtney proposes that they use jumps and slides because "those are actual moves and that's what we are trying to figure out – the number of moves." Meredith later described this advice from Courtney as helpful:

Initially, I was trying to see the relationships between the color and the slide or the jump. And I wasn't trying necessarily to look at just the color or just the move. I was trying to look at both of them combined to see a pattern. And so, I don't know if that might have been hurting me when I was looking for that pattern. But I think Courtney suggested, "Well, just look at the color" and I was like, "Oh, okay."

Courtney reads the results as she records the number of slides and jumps for each game and compares it with the number of squares as shown in Figure 11.

# of squares on one side	Minimum # of moves	# slides	# jumps
1	3	2	1
2	8	4	4
3	15	6	9
4	24	8	16
5			

Figure 11. Slides and jumps data

She says, "We had eight slides [for the four squares] so we had sixteen jumps. Over here [the data for one square] we had two slides and one jump. Annie and Meredith both note that "it's half." Courtney writes as she says near the one square data, "So maybe you do $2n$ slides?" She then checks (after I suggested verifying it) this proposed formula when n=2, 3, and 4 and reports that $2n$ is correct. Courtney immediately goes on to investigate jumps – but does this silently. With the paper in front of her, it is not surprising that when Courtney says, "Oh! It's n^2," Meredith responds with "What is?" Courtney turns the paper to Meredith and explains that n^2 was the number of jumps. Meredith pretty unconvincingly replies, "Oh yeah" as Courtney is busy summarizing, "So we have $2n$ slides and n^2 jumps." Meredith provided some insight into her mood at this point:

I might have been taking a break. I remember when they were like trying to figure out the formula for it that I was just kind of trying to take a mind-break. When I

> *was writing was when I started trying to figure it out for myself.... [That] was when I was really concentrating on what was really going on. But I think I was just distracted there, I wasn't necessarily thinking about anything else, I was just kind of taking a little mind break, you know? Like "maybe they can figure this part out."*

Courtney writes the formula "no. of Moves = $n^2 + 2n$" on her paper, reads it, and says, "that's the formula for number of moves." Annie reaches across the desk to get the data sheet with the sequence of moves for four squares and tests the formula. It works.

At this point the conversation lags. I feel that this is a good stopping point so I tell them what "wonderful little problem solvers they were" and compliment them on a "nice job." As I get up to turn the video camera off Annie, Courtney, and Meredith are all relaxed, smiling and talking about the formula.

Problem Solving Session 2: The Locker Problem

The Problem

For this PSS, I gave the students one sheet of paper on which the following problem and illustration shown in Figure 12 was printed.

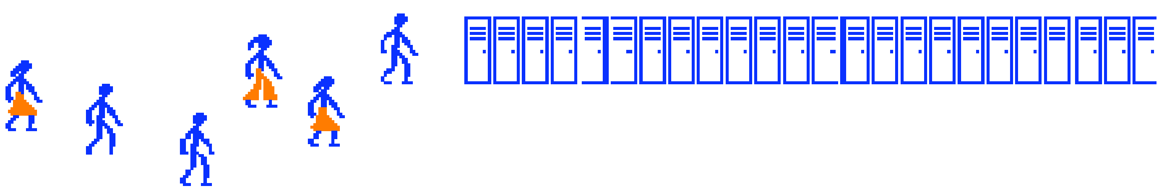

Figure 12. Illustration of the Locker Problem

Imagine you are at a school that still has student lockers. There are 1000 lockers, all shut and unlocked, and 1000 students.

- Suppose the first student goes along the row and opens every locker.
- The second student then goes along and shuts every other locker beginning with number 2.

- The third student changes the state of every third locker beginning with number 3. (If the locker is open the student shuts it, and if the locker is closed the student opens it.)
- The fourth student changes the state of every fourth locker beginning with number 4.
- Imagine that this continues until the thousand students have followed the pattern with the thousand lockers. At the end, which lockers will be open and which will be closed? Why?

Participants' Descriptions of the Problem

Annie

The problem was that we started with so many lockers and the same amount of students and you have to do these certain operations to them and turn the locker over…. The second person turns over 2, 4, 6, 8. Whatever person that is, they turn over their multiples. Basically, all the way through until when you finish for the students. I didn't understand that at first, but once they kept doing five and six, I realized, "Oh this goes on until the students are done." And then we had to figure out which ones were open and why. Not only which ones, but also how many and then why. And then we tried to manipulate that and change the conditions on it and found out some interesting stuff…. I thought we worked very well together last night.

Courtney

The problem was set up that if you have a thousand lockers in a hallway and you have a thousand students. The first student came along and all the lockers were closed but unlocked. The first student came along and opened every single locker. The second student came along starting with the second locker and opened every other one. The third student… changed if it was open they closed it and if it was closed, they opened it. So the second student came along and did that to every *other one starting with two. Third student came along starting with three and did every third one. Etc, etc until all one thousand students had done their job. That was the problem and then we started off with…. I just remember thinking that it went—we seemed to work as a team more.*

Meredith

Well we had a thousand lockers and there were students that walked past the lockers. And the first student opened every locker and the second student started with lock two and opened every other—I mean did the opposite like if it was open then they'd shut it. And then the third person started on three and did every third and then the fourth did every fourth. And then we um, we found a pattern. And then we had to try and recognize why it was the pattern that it was which was the squares were open, were the ones left open…. I think it was a little bit more shared this time than last time.

Description of the Environment

Figure 13. At the start of PSS2

We are sitting around the end of the same oval table (see Figure 13). The girls sit in the same order that they sat for PSS1 – from left to right–Annie, Courtney, and

Meredith (I am at the other end of the table). They are, however, sitting much closer together this time. Before the students arrived, I had intentionally placed two choices (different sizes) of materials on the table. I also intentionally place all the materials in the center of the table so the choice of seat would not as strongly influence the role each girl adopted (as it did in PSS1). Consequently, in the middle of the table is a bag of plastic discs that are red on one side and yellow on the other, a stack of square dot paper, a half sheet of paper with the problem written on it (turned face down), two decks of cards, and three pencils. Not only are the girls sitting closer, but they are also smiling and talking to each other. Everyone (including me) seems relaxed and ready to get started on tonight's problem.

Deciding on a Strategy

I tell the girls that "the problem for you to solve is on that sheet of paper turned faced down." Meredith flips the paper over, reads out loud, "A thousand lockers," and places the paper so that all three of them could see it. All three lean in with one elbow on the table and chin in hand and read the problem silently. After about fifteen seconds, Courtney lowers her arm flat on the table and leans back. Annie then folds her arms, leans back and says, "Hmm." Annie and Courtney watch Meredith as she continues to read; she also folds her arms on the table and sits up when she is finished reading.

Annie speaks first and suggests, "I am thinking mod 2, mod 3, or whatever." Meredith supports this idea with, "Yeah, use mod arithmetic." Courtney later told me what she thought about Annie's suggestion: "*What did I think about that? I wasn't too excited (laughs)…. I'm just kind of like, 'Uh.'"* Meredith then takes a piece of paper and a pencil but writes nothing on the paper at this point. At this same time, Annie has opened the bag of discs and has spread about 12 discs out in front of her. She begins to organize them in a horizontal line in front of her and flips the red ones over so that the yellow side is showing. As Annie arranges discs, Courtney, with her hand on the problem statement, proposes, "Let's think about if we use Polya." Annie laughs and says, "I thought you were saying polio the other day [PSS1]" while Meredith supports Courtney's idea with, "Yeah, yeah, make it smaller."

Making it Smaller

Annie takes a deck of cards from the middle of the table and suggests, "Make it ten." She later described this decision to change manipulatives from the discs to cards:

> *Yeah, I started looking at these chips and I started thinking, "Oh, this is not going to be good because it was going to be the same thing we ran into last time [PSS1]—what color was doing what—without thinking, "Okay, what was red again or what was yellow again."*

Courtney agrees that they should use cards. She and Annie decide that face down will represent a closed locker, and Annie lays out ten cards across the end of the table (see Figure 14). Annie said later that choosing the cards instead of the discs was a good decision because "*it was a bigger board. We could all see it, we could all touch it, and we could all be a part of it.*"

Figure 14. Setting up ten lockers

Meredith has been and is still intently working on her sheet of paper, looking up just once as Annie reached past her to take the deck of cards. When I asked Courtney if she had been aware of what Meredith was working on at that time she remembered:

> *It is like last time [the Square Game], she is writing out the spaces of the lockers and then she started with opening then all. She put like an O, O, O [for open] and*

then she went back and put a C for closed, which would work if you didn't have manipulatives. But it seems easier to use the manipulatives.

Meredith later explained what she had been working on when she told me that she was

drawing out little squares on my sheet and I was doing one: open, open, open, open; and then two; and then put another C underneath it so that whatever the letter was on the bottom was the status of that locker currently. I was going to try and do it on paper and then I was like, that is going to take way too long, we might as well just flip them…. It is going to be better to do this with manipulatives.

So Meredith leans back, pushes her chair back a little, focuses on the ten cards and listens to Courtney ask, "Okay, so what do we do first?"

Meredith now directs the activity. She states, "First you open every locker," as Annie and Courtney flip all the cards face up to represent *open*. Meredith reads the directions from the problem statement and tells them that now the second student opens every other locker. She then corrects herself by saying, "Oh, sorry – changes the state of it." Courtney clarifies "shuts every other locker," as she and Annie flip every other card to a face down position (starting with the second card). Meredith now looks at the problem statement and offers the directions: "The next student changes the position of every third locker beginning with locker three. With her left hand holding up three fingers (to indicate where they are in the problem), Courtney helps Annie flip the appropriate cards. The same process occurs when modeling the "fourth student:" Meredith directs, Annie flips one card, and Courtney flips the other card as she holds up another finger. There is a slight pause because the directions state "continue in this manner until all the students have gone." Annie stares at the problem statement and then Meredith says, "We need to do five." She points to the cards and counts them out loud emphasizing "five" and "the last card." Annie does not flip over this last card (which is directly in front of her) so Meredith repeats herself and points more emphatically at the last card until Annie flips it. Meredith counts to six as she gestures (by flipping her hand over) and tells Courtney to “switch it over.” Meredith later explained why she was directing the activity at this point: “*Well, Annie hadn’t read all of the directions so she was kind of lost…. I think I understood the problem at little bit more so I was kind of telling everybody what to do.*”

As Meredith continues to count on from seven, Annie remarks, “Oh we keep on going past four?” Courtney fails to change the position of the ninth card so Meredith reaches over to the other side of the table and flips it herself. Now that all the flips have been done correctly, they begin to discuss what they see. Annie points to the cards and says “2, 4, 6” – she is rapidly pointing to and counting the number of “closed lockers” that are between the “open lockers.” Meredith said*, “I had no idea what she was talking about when she said that*.” Courtney, also unclear about Annie’s observation, then points to each one of the cards and observes, “So the first one, the fourth one, and the ninth one are going to stay open.” There is a brief pause as they all look at the cards and then Annie says, “Squares!” Later, Annie reflected on what made her think “squares:”

> *I think at first actually I was like, “There’s two here and there are four there and next would be six.” I was actually concentrating on that and then I heard Courtney—subconscious kind of thing—I heard her say, “One, four, nine” and I was like “Oh, I’m back in the world now, these are squares!”*

Courtney points at the cards and says, “One squared, two squared, three squared” as she nods and her head in agreement as clicks her tongue several times—the “yeah, we got it” posture. Meredith says nothing and continues to stare at the board. As Annie reaches for more cards, she now asks, “Well, do you think it works with twenty?”

Working with More Cards

As Courtney says, “Let’s try it, just to make sure,” she is pushing all of the materials in the middle of the table forward to make room for more cards. Annie asks, “Or should we go to twenty-six?” and mumbles something about twenty-five. No one answers so Annie counts out sixteen more cards, hands them to Courtney who has now removed and stacked up the previous ten cards. Annie described her reasons for changing from twenty to twenty-six cards: "*I said twenty but I actually put out twenty-six cards. I wanted to test more as they kept working. And the reason I put out twenty-six, I was like, "Okay, let's try twenty-five as well."*

At this point Meredith re-enters the conversation by asking, "So it's the first one, the fourth one, and the ninth one that stay open?" Courtney repeats this statement to Meredith as she begins to arrange the cards that Annie is dealing. Meredith confidently says (in that voice that relates the "now I get it" feeling), "Oh, one squared, two squared, three squared…" I asked her later, *"At what point did you know it was squares?"* She responded,

> *When they first said it, I was… trying the logic out in my own head—what they were talking about. Then when I said it back out loud later, I was like, "Oh, okay, I get it."*

Before all twenty-six cards are arranged on the table (directly in front of Annie), Meredith reaches in and says, "Well, we already know it's the first and the fourth and the ninth" as she flips those card face up. She suggests, "Now we should start with the eleventh – oh wait we forgot we have to…" "Do it all," Annie jumps in. "Yeah, let's do it all over again," says Courtney as she flips the three cards face down. Courtney instructs, "Okay, first they are all open" as she turns over the first few cards. They all laugh and say together, "we should just call this open," as each girl helps turn over all the cards. Each girl helps in the process as they now go on to count out (by pointing to each card) and flip over every second, third, fourth and fifth card. Annie takes a pencil and places it between the fifteenth and sixteenth cards because "*it was a little easier to find the numbers."* Now, on the sixth move, Meredith points to the cards as Annie turns over the cards and directs with "Six, twelve, eighteen, twenty-four." They quietly alternate calling out numbers as Annie turns over cards for the seventh through the thirteenth moves, with Meredith and Courtney occasionally pointing to the next card that needs to be flipped. When they get to the fourteenth card, they seem to all realize that the rest of the moves will just require one flip of the card with the corresponding number. So, Annie quickly flips the rest of the cards.

As Annie and Courtney are redoing the last two flips, Meredith examines the board and observes, “Okay, so first, fourth, ninth, sixteenth, and twenty fifth.” Courtney offers “Yeah, five times five” and Annie nods in agreement. Meredith leans back, smiles, and cheers, “Woo-hoo, it worked!” Meredith explained to me why this was such a positive moment for her:

> *I was glad that we were able to use the same... theory that we had come up with and use it on an even bigger set and it still works. It was like, “Okay, good, we are getting somewhere. We are right with our thinking and our idea.”*

Meredith then looks at the paper and laughs as she says, “Now we got a thousand.” Courtney responds, “So from one squared to…” and she pauses. Meredith begins to write something down as Annie quietly asks, “What would be a thousand squared?” She later commented, *“I had it in my mind, I just wasn’t saying it right. I think I said ‘the least squares,’ a term from statistics, and it should have been the greatest square less than a thousand.”* Annie then sighs, leans way back in her chair, and laughs as she says, “I don’t know that high.” I offer a calculator and Courtney says, “Well, let’s just think about it, three zeros.” Meredith asks for clarification, “Are you talking about one thousand squared?” Courtney says, “No, what squared will give you the closest to one thousand” and Annie adds, “So we’ll know what the highest…” “Oh,” says Meredith, “square root of a thousand – that’s one hundred times one hundred.” Courtney and Annie both chime in with “No” and Courtney adds, “one hundred times one hundred is…” Meredith finishes her sentence with “ten thousand.” Courtney then poses the question, “So what’s the closest square to one thousand – like what squared is closest to one thousand?” Courtney and I later discussed her question posing:

L: *When you are asking that, are you saying that for you? Are you saying that for Meredith, are you thinking out loud…?*

C: *Out loud, yeah. But I think I'm also explaining to her because I don't think she really got it at first, what we were saying. I think it's for my benefit, too, because I always think out loud (laughs).*

Meredith calculates 50^2 on her paper and reports that "It's lower than 50." Annie suggests 20 and Meredith writes this down. As they wait for Meredith to do the calculation, Annie laughs at herself as she says, "Boy, take us away from those calculators…" Meredith quickly reports, "20 is too few, Courtney." As Meredith points me to where the calculator is in her book bag, Annie remarks that "30 is 900." Courtney suggests 32 as Meredith says, "What about 35?" Meredith takes the calculator from me, types 32^2, reports that it is too high, and that 31^2 is 961. As Meredith takes care to write this value down, Courtney states, "that's going to be the closest" and Annie concludes "So it's 31."

Meredith looks at Annie and asks, "So, what was the purpose in us finding out this square? 'Cause after that all the other ones will be turned over?" Annie answers her with "Yeah, they will be turned down—closed." Courtney interjects, "Yeah, like 1^2, 2^2…all the way up to 31^2 – all of those squares will be open." Since Meredith still appears to be confused, Courtney points to Meredith's paper and says, "like the 961…" Meredith finishes her sentence with "will be open." Annie now points to the board and explains, "And that's the 31st one that's open. She points at the cards that are face up and illustrates, "This is the first one that's open, this is the second one that's open." "So 31 lockers will be open," adds Courtney. "Oh, okay," says Meredith. Meredith reflected on this conversation:

> *See that is part of me—having to talk out what we just did to understand the relationship of it to the problem. I have to re-say it back out and then I understand it better than with it all just jumbled up in my mind. Because there is so much going on in my head, I had to talk it out. And it helped me a lot because I had them there to be like, "Yeah, that's right." And so if there is somebody else, like a group, to verify my thoughts that I just did, like vocal about it, try and think out the problem, it helps me because they are like "Yeah, that's right. Now, let's*

> *move on to a new step, a new process because we got that one down." So, it kind of reassures me.*

Annie and Courtney are sitting back and Meredith is focusing on her paper. After a brief pause, I ask "So, why does that happen?"

Investigating Why That Happens

Courtney answers first with, "Well, let's think about it. At the very beginning we turned over that one to begin with. Since we started with that second one, that first one is never going to be turned over." She maintains eye contact with Annie (who is leaning back and listening) as she tries to further this explanation but gets confused about which cards are face-up and which are face down. Annie leans in and uses Courtney's idea as she offers,

> That one (pointing to the fourth card that Courtney is holding) gets flipped twice so it goes back to its original. This one [the ninth card] gets flipped twice. This one gets flipped (pause) on the two, four, and eight, and one and sixteen.

Courtney picks up on this idea and pointing to the fourth card says, "Yeah this is a product of one, four, and two" as she counts to three on her fingers. She then acts out the flips with the fourth card as she states, "So it starts open, gets flipped over for the one, gets flipped back over for the two, and gets flipped back over for the four." She then does the same process for the ninth card and concludes "It has to be flipped for, hold on (as she flips the fourth card over)." Annie finishes her sentence with, "An odd number of times." Taking off her glasses and rubbing her eyes, Courtney agrees and says, "Yeah it has to be an odd number of times to return back to being open." I asked Courtney if she had thought about an odd number of flips before Annie offered the idea and she explained:

> *No, because I said four. I think I was trying to think of too many different things and I didn't see that it was obvious that it was an odd number of times. But Annie wasn't really—she was just kind of observing what I was saying so she automatically picked up that it was an odd number.*

There is now a pause in conversation as Annie e leans forward and begins to write on Meredith's paper. Annie explained that she was exploring factor trees because:

> *I remember her [Courtney] saying the three and I was like, "Yeah" and I went to the eighth [card] because it had obviously more. "The one, two, four, the eight" and I was like "Okay, let's try the factor tree." And I was like, "yeah, this is what we need to concentrate on."*

Glancing up only twice, Meredith has been diligently working on her paper during the exchanging of ideas about the number of flips between Courtney and Annie. There is a pause in their conversation and Meredith takes the opportunity to offer, "I've been working on some mod stuff over here – if that's going to help us." Annie continues writing on the top of Meredith's paper while Courtney stares blankly at what Meredith has been writing down. Wanting to have Meredith's ideas included in the discussion, I ask Meredith about what she has been working on and she informs us that she "got stuck." She explained further:

> *I knew that 31 squared was 961. So I said that one thousand is equal to 961 plus 39, which is equal to 39 mod 961. But I don't know enough about mod arithmetic to be able to change it around.*

Tentatively Courtney says, "I don't think we need…" Meredith interjects, "Might not help us?" "I don't think it's gonna, yeah," replies Courtney, of whom I later asked:

L: *Did you try to make sense out of what [Meredith] was saying?*

C: *I didn't really try to. I just didn't think that mattered at all. I thought the direction we were going was going to get our answer and I didn't think we needed to worry about something too complicated… that we might not really know a lot about.*

L: *Okay. But you listened?*

C: *I listened to her but it didn't make much sense.*

L: *Why did you listen?*

C: *To be polite.*

Annie shared with me that she was "*not listening to Meredith*" at that point (although she did know that Meredith was doing "*something with modular arithmetic*.") When I asked her if Courtney was listening to Meredith, Annie laughed as she told me, "*I think she is trying to look at what I'm writing and also pay attention to Meredith, that is what I would have done if I was sitting right there.*"

Annie waits until Meredith and Courtney have paused and then explains what she has been working on by pointing to the factor trees that she has constructed on the paper that Meredith was also working on. Courtney is nodding in agreement with Annie as Meredith now focuses on what Annie has written. Meredith talked about her shift in focus to Annie's work:

> *I think that's when I was starting to abandon my own thoughts, I'm starting to look over at them, and I was listening to what Annie was saying. I was like, "I don't think it is going to work but I'll still tell y'all what I was thinking so maybe if y'all think something will work, y'all can build on it where I couldn't." Which was kind of trying to offer it to them, but they had their own thing that was going for a little while, too. So I'm sure they didn't want to abandon that and start something new that in their mindset wasn't on so I guess they are still going with what they were going with. I think once you have your mind set for a type of way to problem solve…you kind of try and think of all logical ways that you can work it out. But I think if somebody else in the group introduces something, like a new way to look at it or whatever, it is kind of hard to totally abandon your idea—You kind of don't want to totally abandon it. But maybe need some help from somebody else to help explore it.*

As she puts the pencil down and leans back in her chair, Annie offers the hypothesis, "Do you think it has to do with the number of factors? You know the factor tree (as she makes and inverted V with her arms). Annie explained that

I started doing a factor tree, but… I didn't really understand. I knew that it had something to do with it but I didn't understand how and Courtney pointed out the number of factors it has, not the number of times it appears or that kind of thing.

Courtney takes a pencil and turns the paper towards her as Meredith rubs her forehead and asks for clarification, “Like the least common multiple – is that what you mean?” Meredith later commented that:

I wasn’t following [Annie]…. I was like, “What would odd and even numbers of turns?” You saw me kind of wipe my face, I was kind of like, “What is she…talking about? Where is she going with that?” I didn’t understand—I think I was still trying—in the back of my head, I was still trying to think, “Is there a way we can do this with mod arithmetic?” I was given in a little bit more to listening to new ideas, too.

Courtney directs attention to the paper and suggests, “Think about a number that isn’t open like ten. The only thing with ten is five and two. That’s like the whole thing of why they are squares.” She looks at Annie and asks, “You know?” Annie nods as Courtney continues, “Because there is some number that when you multiply it by itself you are going to get this number (as she taps the pencil on the fourth card).” Annie takes the pencil out of Courtney’s hand, describes the factor trees that she had drawn, and then drops the pencil on the paper. Meredith reads the numbers quietly while Courtney says, “Un-uh, all you have is one, two, five, and ten—that’s four.” Annie replies, “You don’t have ten.” “Yes you do, because it gets flipped over on the tenth time,” answers Courtney and then points out, “See here you have one, sixteen, two, eight, four…” Courtney stares at Annie for a second and then asks, “Is that what you were thinking about?” No answer, from Annie so she then offers, “That’s what I was thinking about—the numbers of factors. Like one and itself and then what else?” Courtney is met with silence and blank stares from Meredith and Annie. Meredith explained that she was feeling

kind of lost, like I was trying to figure out what they were talking about. But I was on a different train of thought …I don’t know what all was going on in my head *but I wasn't totally with them. But I was trying to just sit back and trying to understand what they were talking about. "I don't know, maybe I'll get it soon."*

Meredith is now leaning back with arms crossed— but listening and watching. Annie is leaning in as Courtney takes the pencil to the paper and says,

> Because of that itself—it's gonna get flipped back over whenever the sixteenth student comes it's gonna get flipped back over. When the first student comes it's gonna get flipped, when the second, the eighth, and the fourth comes it will get flipped. So it gets flipped an odd number of times total, so that's why it will be open. All the other ones will get flipped an even number of times, so that's why they will stay closed.

Courtney glances at me, Meredith says, "hmm" and Annie swivels back and forth in her chair as they are all now staring at the table (see Figure 15). At this point I ask, "Does that make sense to you, Meredith?"

Figure 15. Everyone staring at the table

Meredith Making Sense

Looking at Courtney, Meredith replies, "Yeah. – What you just said did." Her voice sounds confident but Meredith's face still expresses confusion. Courtney further explains her thinking, "That's why I was thinking 'Let's look at some of the ones that are still closed and make sure that it seems like it is true.'" Annie then points to a card and suggests, "Let's do twelve." As Courtney begins writing, Meredith asks, "Well, why is it that the squares are…" She doesn't finish her sentence because Annie is now laughing as she notes that she is pointing to the eighteenth card instead of the twelfth. Courtney is busy drawing a factor tree for twelve and showing Meredith, "See it's got 1, 2, 3, 4, 5" (counting each factor) as she points to it. Annie tells Courtney that she has forgotten the factor of four and Courtney says, "Oh, right… see you need to think about all the different ways it can be written." Annie now counts that there are six factors for the number twelve and Courtney claims, "Right, that's why it stays closed." Courtney goes on to summarize, "So it has to do with the number…" "Of divisors," Annie interjects, "including one…" "And itself," Courtney finishes.

Meredith rubs her forehead and says, "hmm" as Courtney proceeds, "That's why these stay open. Because they are only flipped an odd number of times. And since they start out closed, an odd number of flips is going to leave it open." Meredith says, "Yeah" as Courtney adds, "All the rest start out closed and an even number of flips is going to leave it closed again." Courtney told me that she felt her explanation was helpful to Meredith:

> *She kind of caught on… I think she wasn't really sure what the factors had to do with it being opened and closed, but then I said the thing about, " If it gets flipped five times it is going to end up open and if it gets flipped four times, it is going to end up closed." Because just a second ago she was like, "Oh, yeah."*

Meredith commented:

> *Well, I was understanding what [Courtney] was talking about with the odd and the even, you know, the odd number of flips and the even number of flips—how its last state is going to be. [But] making the connection between the amount of factors it has is the number of flips—I hadn't made that connection.*

Meredith also told me that at this point she had decided to stop agreeing and get some questions answered about that connection. She explained:

M: *Oh yeah. I was kind of lost on how the factors, the number of factors, related to the open-closed. And so I kind of let them go with it for a little bit while I was lost* *and then eventually I had to ask and Courtney explained to me, you know, what the relationship was.*

L: *And why did you let it go for a little while before you asked?*

M: *'Cause they were thinking hard. They were like on a choo-choo train and I did not want to stop it.*

This is why she now asks, "What does that have to do with its multiples, or its divisors, or factors that y'all were talking about?"

Annie mumbles, "thirteen has one and thirteen" and then looks down to draw on the table with her finger while Courtney directs Meredith's attention to the paper and tries to restate her previous explanation, "That's what I was saying – when the first student comes it gets flipped. When the second student comes it gets flipped 'cause it's a multiple of two." "Oh," says Meredith hesitantly as she watches Courtney pointing at the factor tree for twelve. Courtney, talking very quickly, continues to explain, "When the third student comes, it's gonna get flipped, too, 'cause it's a multiple of three. Remember at the end (as she motions to the cards) we were thinking about multiples – you know we were going 'seven, fourteen, twenty-one…" "Right," agrees Meredith with a higher degree of confidence. Courtney continues, "Because that's all the times [for the seventh student]." "Right," Meredith repeats. Courtney points the pencil back at the factor tree for twelve and concludes, "Same with the fourth student, the sixth student and the twelfth student – they are all going to flip that one card." She puts the pencil down, leans back in the chair and looks at Meredith who is still sitting with crossed arms and a puzzled expression. Courtney adds, "so it gets flipped a total number of six times." "Oooh…so if it's even, it's shut," Meredith claims. Courtney nods and says, "Right!" Meredith does not move from her leaning back, arms crossed, serious expression position as they all pause and look down at the cards or the paper.

Then Annie looks at the sheet of paper with the problem statement on it and poses the question, "So, what happens if you change the number of lockers?" Annie answers her own question by making the claim that "if it's greater it would be the square root of the least square." With nods of agreement but no comment from either Meredith or Courtney, Annie then goes on to read the next part of the problem statement, "What about if you change the initial conditions? Well, if they all started out…" "Open," Courtney says. Then Courtney and Annie, talking at once and finishing each other's sentences, reach the conclusion that Annie later summarized: "*If you start with all the student lockers open and a student comes and closes it, it would just be the opposite. Everything would be open except for the closed squares*." Meredith agrees with them saying, "Yeah, it's just the opposite." And Courtney concludes that, "Changing the number of lockers would change the number of lockers left open but it wouldn't change the pattern." Meredith and Courtney nod in agreement and then Courtney adds, "Well, as long as you have as many students as you do lockers…" Annie offers the question, "So what if you have less students than lockers?"

Less Students than Lockers

Courtney points to the cards and suggests, "Well, you wouldn't reach the end." Because I thought this was an interesting direction to take the problem, I offered the suggestion, "What if you had half the number?" After a brief pause, Annie says, "You could only go… the least square, the least number of students." As Courtney started to say, "Because you could only get," Meredith interrupts with, "There would be more open though." Annie says, "Well, up to that point the pattern works for fifty, forty-nine, 'cause seven squared is forty-nine so you would have seven lockers open. But then after that the pattern wouldn't work any more." Courtney repeats, "Yeah, it wouldn't work any more," as Meredith rubs her eyes.

Aware of the time (only ten minutes left in this problem solving session) and curious myself about this question of half the number of students, I remind them of the "make it smaller strategy" by asking, "So if you had ten lockers but only five students, what would happen?" As Courtney begins flipping all the cards over, Meredith hypothesizes, "One and four would be." Annie agrees, "Yeah, one and four" and adds, "But there would be other ones that are open, too." Courtney jumps in with, "At the end," as she points to the cards. “Because the first student opens all of them,” Meredith begins to reason. At the same time Courtney is noting, “All the multiples aren’t going to get…” and gestures a flipping motion with her hand over the cards.

As Annie now clears the cards off the table of all except for one row of ten, Courtney directs Meredith’s attention to the paper they had been writing the factor trees on (see Figure 16) and explains, “All these things we were talking about [the factors] – all those flips aren’t going to happen because we don’t have enough students.”

Figure 16. Courtney explaining to Meredith

Annie places a pencil between the fifth and sixth card signifying that she is set-up and ready to model the situation with ten students. Meredith verbally directs the moves as Annie and Courtney flip the cards each time for the first, second, third, fourth and fifth student. Courtney looks at Meredith after the “fifth student” flips and says, “That’s it – you only have five students.” Meredith observes, “So it [the square numbers being open] works up to five. Courtney reiterates, “So it works up to how ever many students you have, but after that it’s not going to follow a pattern.” Meredith agrees as she and Courtney join Annie in studying the row of ten cards. I observe, “It’s everything but the square.” Courtney immediately responds with, “Yeah, that makes sense,” but then pauses as Meredith and Annie both lean back in their chairs and excitedly say, “Wow” and “It sure is.” Meredith later told me that during this time she was feeling much more involved in the activity: *"I'm sooo much more engaged now. Before I was just sitting back and watching and listening and staying lost and hoping that I would catch on eventually."*

After a few seconds of silence as each girl thinks about this new observation, Courtney offers the following reasoning (speaking mostly to Meredith):

> Because only half of the flips that needed to be done were done so it's halfway through the process…. It's halfway there. The rest of the flips will take it opposite of what it's supposed to be there (pointing at the sixth through tenth cards).

Meredith slowly nods in agreement even though she later told me that:

> *I didn't agree with her when she was saying that because I knew that depending on the number—[that] would make like how many flips you were going to have for that set. You know, you start with one, you flip all of them, you start with five, you are only flipping two, three, you are flipping three of them. And so, when she was saying that, ...I wasn't agreeing with her, but I didn't say anything because I didn't want to step on anybody's toes either, you know? But I didn't agree with what she just said.*

Annie then suggests, "However many students you have is proportional to how many of that least square you'll have." Courtney and Meredith verbally agree as I ask, "So, for twenty students would everything be turned…?" Annie jumps in to ask for clarification, "For how many students?" I suggest that they stay with half as many students as lockers. Annie thinks for a moment and then says, "Well, then the least square is sixteen so you would only have four open and then everything but…" As Annie pauses, Courtney picks up on Annie's suggestions and claims, "So on twenty, four and nine would be open and then…" "After that," Meredith chimes in as she looks up, marks a "halfway point" in the air, and motions to the "cards" that are to the right of this point. Courtney watches Meredith and then adds, "After that, eleven, twelve, thirteen, fourteen…" In unison all three girls continue with, "fifteen, seventeen, eighteen, nineteen and twenty open and sixteen would stay closed." They all look at me and then Courtney suggests that they "try it." As Annie sets up two rows of ten cards, they all discuss that they liked the cards because they look like lockers and that they thought they might be playing poker. After some giggles, they are ready to test their theory on the twenty cards.

This time it is Courtney who directs the moves as all three girls are involved in the flipping of cards as they complete the appropriate flips for each of the ten "students." As they examine the board, Courtney points to the twelfth and thirteenth cards, which are face-up, and suggests, "Maybe we messed up."

Maybe We Messed Up

After a brief pause, Meredith restates, "So twelve and thirteen were closed, hmm…" They all three stare at the board trying to figure out what was happening with these two numbers. Meredith told me that this quiet time was a time when she was able to make some much-needed connections:

> *I don't know if I was just trying to figure out why they were closed based on the flips that we had done so far. Or maybe trying to think of a different relationship…. I think at this point I was starting to make more of a connection between the multiples and whether there is even or odd multiples…which is a long time after they tried to explain it to me.*

After about ten seconds of looking at the board and looking at each other, Annie is the first to offer an idea claiming, "You can't go past ten. The least square here is nine so the pattern doesn't work past nine." Courtney reminds Annie, "Remember that before they were all open except (Meredith joins in on this word) for the squares? We were trying to see if that would be true here." Courtney and Meredith look at Annie as Annie looks at the board. Hoping to encourage them to continue discussing this apparent dilemma, I ask, "So what is it about those two numbers?" After a few seconds, Annie offers, with a laugh, "They are consecutive." Meredith observes, "Yeah, and twelve and thirteen add to twenty-five." "And that's a square number," Courtney quietly laughs. A few more seconds of silence and the looks at me that say "I'm waiting for you to help us more," I decide to be a little more directive with my prompt this time. I had seen them "mis-flip" when they were all three flipping the cards so I ask them, “When would thirteen get flipped?”

Talking by building off each other’s ideas and finishing each other’s sentences, they go through the factors and respective flips for both twelve and thirteen. Originally they consider all the factors of each number (instead of just those factors less than ten) until Meredith reminds them that for thirteen the card would only flip one time (to the open position) because there are only ten students. Annie and Courtney both reach for the thirteenth card to correct its position and immediately begin naming the factors for twelve –omitting the factor of twelve this time. They quickly realize that twelve should also be open and Courtney tells me, “I think we messed up.” I am excited about this pattern that they have found and offer, “That’s really cool if it works.” Courtney, looking a little skeptical, suggests to her group, “Should we do it again?” Annie quickly responds, “No, I think we skipped it.” “Skipped what?” Courtney asks. Annie does not answer but does suggest that they think about seventeen. All three girls now spend time naming the factors and the moves for fourteen through seventeen. Annie and Meredith are now certain that all the cards except for the sixteenth one should be open (face-up). Although Courtney is excited to have shown this to be true for twenty, she told me that she felt uncomfortable with the strategy:

C: *I didn’t like that. If I had done it myself, I would have started all over…just because I am always afraid something is not going to happen right. Like we did that, and we missed two of the cards. I think we should have started over from the beginning and just made sure we did it right. I mean, I guess with this pattern it is obvious what is going to happen because we’d already figured it out, but if we really were still sure about the pattern and why those cards were opened and closed—I definitely would have probably insisted that we start from the beginning to make sure.*

L: *So, why didn’t you insist?*

C: *Because I was already sure of the pattern and I already knew why the cards should be opened or closed. So we could go back and figure it out whether it was right or wrong.*

Annie, Meredith and Courtney now agree their hypothesis that "all the numbers but the square, beyond the halfway point, will be open for any number of cards." They all lean back and smile at me as I tell them that I have never seen anyone do this extension before. Meredith told me that during this time she was thinking, *"Well, maybe it does work for half but maybe it doesn't work for all proportions".*

Testing Other Proportions

Before Meredith is able to voice this thought, Annie offers, "If you have half—if it's proportional—if we have thirty cards and we only chose ten students?" Courtney proposes that the pattern will still work as Meredith suggests that "you could just lay down ten more cards." Meredith goes on to explain that all the flips do not need to be repeated and talks through the flips for the twenty-first and twenty-second card. Annie lays out a third row of ten cards (see Figure 17) as Meredith and Courtney discuss a strategy for catching up the third row.

Figure 17. Working with thirty cards

Meredith directs as Annie and Courtney take turns flipping the cards. They get confused for a moment, knowing something is not right, but then realize that they had considered the first card in the row to be the twentieth rather than the twenty-first card. Back on track now, they continue completing the appropriate flips on this bottom row of cards for the third through tenth student. Courtney and Meredith are ready to move on to the eleventh student when Annie reminds them that “No, we are only working with ten students.” “Oh, okay,” Meredith replies as she and Courtney join Annie in looking at the cards. Annie explains that she is “trying to look for a proportion here.” Meredith asks, “How many squares range from twenty-one to thirty?” Courtney tells her, “Just one—twenty-five.” Meredith ponders this as she slowly says, “One square and four open cards.” Annie and Courtney both told me later that they were hoping that there would be just three cards open so they went back to the twenty-fourth card and recounted the factors. To their disappointment, they find that they have done the flips correctly. Annie leans way back in her chair in a frustrated gesture as Courtney and Meredith lean in with puzzled looks.

When I ask them what they were looking for and hoping to find, Courtney explains, “An equal proportion, somewhere.” Meredith offers two rather detailed hypotheses about what she is looking for that involve the number of squares in certain intervals compared with the number of students and lockers. Annie, sparked by Meredith’s ideas, now leans back in and says, “Ooh, ooh, here’s what we are looking for.” She goes on to point out a pattern that she sees in the twenty-first to twenty-fifth cards. Courtney points to cards at the end of the row and asks, “What about those?” Annie puts her hands over those cards and laughs, “They’re not there.” Courtney summarizes, “So we have five cards that by our theory shouldn’t be the way they are.” Annie and Meredith remind her that the theory worked for half but that now they were investigating the situation when there are one-third the number of students as lockers. Leaving the “third as many students” problem unsolved, Meredith, goes back to the “half as many students” hypothesis, and suggest that they “check up to fifteen to see if their original theory for half still holds.” Soon afterward, with Courtney calling out the numbers and Annie flipping cards they complete the respective flips for the eleventh through fifteenth students. Looking satisfied, they all say, “Yes, it still works.” Meredith takes two plastic chips and begins playing with them—Courtney and Annie are now interested in how Meredith is making the chips jump on the table—and time is up for this PSS. This is a clear signal to me that they are at a place in the problem where they feel "finished" and time for this PSS is up. As I get up to turn off the video camera, Courtney says, "That was cool."

Problem Solving Session 3: Polygonal Numbers

The Problem

The goal of the problem presented in PSS3 was the investigation of both the numeric, algebraic and geometric representations of a variety of polygonal numbers. Note: I defined polygonal numbers for this activity as follows:

> Given a number of equal circular counters, then the number of counters which can be placed on a equilateral polygon so that the tangents to the outer rows form the equilateral, but necessarily equiangular, polygon and all the internal counters are in contact with its neighbors, is a "polygonal number" of the order of the polygon.

I posed aspects of this problem throughout the PSS, beginning with square numbers and including triangular and pentagonal numbers. What follows is a description of the types of questions I asked as we began the session.

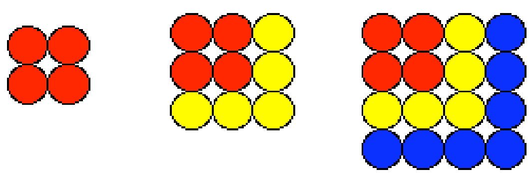

Figure 18. Geometric representations of square numbers

The first image (see Figure 18) to the left represents the second square number, the next one represents the third square number and the last represents the fourth square number. What would the *n*th square number look like? What is the relationship between the *n*th and the $(n + 1)$th square number? What would be an algebraic representation for the *n*th square number? Let's extend our investigation of these types of questions to triangular numbers and then other polygonal numbers.

Participants' Descriptions of the Problem

Annie

> *The problem was dealing with polygonal numbers. So with a square you made a square with the same number of dots on each side. With a triangle you made—and so on. So we had to figure out what was the pattern if you kept adding from one to the next—what was the nth triangular number and nth square number and the nth pentagonal number?*

Meredith

> *Well we started out with square numbers and we were looking at them with the dots and trying to figure out for each square number how many more dots we had. And we were trying to recognize a pattern and we were using some algebra to try and figure out how to go from the nth term to the $n + 1$ term and then did the same thing with triangular numbers.*

Courtney

> *Last night we started talking about square numbers and a general formula for the nth square number. And then we talked about triangular numbers and a general formula for the nth triangular number. And then we just kept on going to pentagonal numbers and then hexagonal numbers. And we were just trying to see a pattern. We looked at it algebraically and geometrically—how these numbers are represented. We were trying to come up with general formulas for the nth pentagonal number or whatever.*

Description of the Environment

Figure 19. At the start of PSS3

We are sitting around the end of an oval table (see Figure 19). From left to right the order is (as in the last two PSSs) Annie, Courtney, Meredith, and me. Annie has just opened the laptop computer. In the middle of the table are two graphing calculators, a box of colored markers, a bag of two-sided red and yellow plastic discs, two pencils, a stack of plain paper, a stack of dot paper, and a table microphone. Again, I have provided multiple choices I have taken one sheet of paper and am ready to introduce the problem.

Starting with Square Numbers

I begin by stating that we are going to be investigating polygonal numbers. I remind the girls that they are familiar with square numbers and ask them to tell me the first thing that came to mind when I say "square numbers." Courtney says 2 and we all look at her strangely. I ask, "Why 2?" and she says, "Because you raise to a 2." Then I ask Annie the same question and she responds with a 4. Meredith echoes with 4 and 9, 16, etc. I conclude, "So you are thinking digits" and then ask, "Can you think about square numbers geometrically?" Annie draws a square with her finger on the desk and says, "2x2". Courtney chimes in with, "Like the area of a square?" and Meredith says, "Oh the side lengths." So I draw a square with side length two and we briefly discuss the connection between the numerical representation ($2^2 = 4$) and the geometric representation of the area 2x2 with area 4. Then I show them both the dot paper for this type of geometric representation and the two-sided discs for 4-square (see Figure 20).

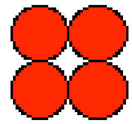

Figure 20. The "4 Square" with discs

I now ask, "So, what would nine look like?" Courtney responds with a "square of side length 3" while Meredith takes the discs and begins to build the 9-square on the table in front of her. When I asked Meredith*, "How did you get to be in charge of the manipulatives here?"* she responded by saying, "*They were just right there in front of me so I just grabbed them*." As Meredith lays out the discs, Courtney tells Meredith that she needs to use nine discs and reminds her to put one disc in the middle. Meredith later reflected:

> *Courtney helped me there when I was setting it up for the 3x3 because I forgot to put a dot in the middle and she's like, "no 3x3, you need one in the middle." At first I was thinking why and she was like, "Nine. You know three squared is nine." I was like, " Oh yeah, I forgot that." So that helped me, just setting up the problem and seeing what was going on.*

Annie is working on trying to open the spreadsheet software while this building of the "9-square" is going on and "*at this point I was like, 'Okay, what's the question?' I was thinking where is this leading*?" I then pose the question, "Can you see the four square in [the 3X3]?" Annie now turns her attention to the model that Meredith and Courtney have just built, reaches across the table, and points out an arrangement of a 2x2 square within the 3x3 square (see Figure 21):

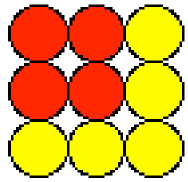

Figure 21. 2 x 2 Square within the 3 x 3 square

Now that I feel they understand what I mean by a geometric representation of a square number, I ask them to think about connections between the representations and then numbers. I begin by asking, “In going from four to nine, what did you do?” Courtney responds by saying, “You add a row and a column” as she points to the model. Meredith offers “three and three” as Annie agrees, “Yeah.” I ask, “So you add three and three?” “No,” all three girls say in unison. Annie reports, “three and two” as Meredith and Courtney assert that you only add five. I then ask, “So, how many do you think you’d add to get to the next square?” Courtney begins mentally constructing a 4x4 square by pointing on the table where the dots should. Meredith follows up by placing discs on the model to expand it to a 4x4 as she and Courtney exchange words like “4 and 3” and “4 across and then a column of three down.” Meredith summarizes their process by saying “first you added five and now you’re adding seven… on the next one would you add nine?” and as she builds the 5x5 square with the discs, she observes, “Yeah the five across.” Courtney finishes her sentence with, “And four down.” Annie is watching this process of model building with the discs, but she remains silent. With the intent of encouraging them to generalize, I inquire, “So how do you get from the nth square number to the $(n + 1)$th square number?”

Getting to the “n + 1” Square Number

Talking all at once, Annie says, “Multiply,” Courtney suggests, “You add $n + 1$,” and Meredith claims, “You add the next odd integer.” I respond by asking “Does that have any relationship to the n?” Courtney asks for clarification from me about whether n is 4 or 2, and I tell her, “Let’s have it be 2.” She then immediately offers the hypothesis, “You would add $n + 1$ circles” as she points to the model. Meredith quickly agrees with

Courtney as she observes, "Yeah, with 4 you add 5." However, in retrospect, Meredith said she thought they should have worked with her "odd integer" observation at that point:

M: *I wish I would have gone with what I said right there—'Cause I mean I recognized that if it's a 3 x 3 you add 7 and I knew that it was an odd integer that you were going to be adding—I recognized that it was 7 and 9. So I wish I would have gone with that idea that I had…*

L: *Why do you think you didn't'?*

M: *Well, there were a lot of other ideas that we were throwing out on the table and so I don't know maybe that's one of those things where if I had been working independently maybe that would have helped me get to the solution faster. But with other people helping you think, you want to examine what their thoughts are too, because you don't necessarily know if yours was the best one.*

Annie has not been involved in this interchange between Courtney and Meredith because immediately after I posed the question about the relationship to *n*, Annie took a sheet of paper and began to develop a multiplication she felt was representative of this situation. Annie "didn't realize that Courtney and Meredith were working together" because she busy was working on

> *trying to think about how to explain that you add, you know, the n,* $n + 1$. *I didn't quite have that idea yet but I knew it was one more here and one less here or something like that… the multiplication table helped me figure out how to do this…. This is where I was like 'yeah I see it' and I started making dots so I could explain it to them… so I could show them.*

Courtney and Meredith continue to discuss how many discs you add each time and are trying to make a connection to the question I had posed. Annie directs the attention to the table she was developing and begins explaining. At this point, Courtney reaches for her own sheet of paper because she was

C: *trying to think how it's not working—cause I'm trying to say formulas in my head—so that's why I just grabbed a sheet of paper to write it all down.*

L: *Do you remember if you were paying attention to what [Annie] was doing?*

C: *I don't think so. Because I was trying to answer your question in my head but I couldn't so I got a piece of paper.*

L: *She was making a multiplication table.*

C: *Yeah, but I didn't know what that had to do with [what we were working on].*

As Courtney works at recording the data she and Meredith have developed so far, Annie and Meredith are discussing the table that Annie was developing. Meredith appeared confused and Annie spoke more softly. After a few seconds of silence Courtney offers, "Well, when you have four, when n is four, you add five." After a little confusion from Courtney about defining n, I help them by saying, "If nine is the third square number, let's let n be 3."

After a few more seconds of silence, Meredith inquires, "So, what's the question? What are we figuring out?" I remind them that I had asked, "How do you move from one square number to another?" "Oh, okay." Meredith responds to me as Courtney restates, "So how do you go from n to $n + 1$ number of dots?" Meredith's attention is now on Courtney's paper as they discuss possible patterns they are seeing while Annie continues to work on her representation. Meredith and I talked about her shift in focus:

L: *So I watched you switch. Now you are not looking at Annie's paper and now you are looking at*

M: *Courtney's [paper], yeah.*

L: *Was it because she spoke up?*

M: *Yeah, just because she was thinking up ideas and she was talking with you and I thought y'all were about to go somewhere. So I was just checking out what she had because I wasn't really getting anything out of what Annie had over there…. I understood what Courtney was doing more so I was, you know, contributing to what she was doing over there and kind of interacting with her more than I think I* *did with Annie. I understood what she was doing and how it was related to the problem.*

Courtney now asks Meredith, “Can we get a general formula from these?” as she points to the data on her paper. Meredith offers a few ideas when Annie chimes in and reports (by talking to me) that, “If this is the nth number, you add on a 1 x n column and then you add on a $n + 1$ row.” Although it appears Courtney had not been listening to Annie’s idea (as she was working on her own paper), she now looks up, points to Annie’s paper (see Figure 22) and says, “But that 1 is on both of them.”

Figure 22. Courtney pointing to Annie’s paper

Annie tersely responds, “No,” and goes on to explain her thinking to Courtney while Meredith now reaches for her own sheet of paper. Courtney decided to listen to Annie’s explanation and try them out on her own paper because she wanted to

see what Annie was doing. I think what she does then makes me understand exactly what the formula is. ‘Cause she is saying that we are adding a row (an n-row) and an $n + 1$ column or vice versa. If you multiply, add those together, you get $2n + 1$.

Meredith was copying onto her paper what Courtney and Annie had been recording. I also discussed Annie’s ideas and her decision to begin working on her own paper with Meredith:

M: *It looked like [Annie] was trying to make a table and I was trying to figure out how she was making that table and how she was setting it up…. I couldn't really understand why she was doing a multiplication table….*

Actually, at this point, I was kind of confused about the problem. So, I was just checking out what they were doing…. I'm looking like back and forth and back and forth…. [I'm thinking], "What are you doing? What are you doing?"

L: *I can see your eyes shifting from Annie to Courtney, Annie to Courtney*

M: *Like who do I want to go with? They were representing the problem in totally different ways…. [Annie is] talking about n x n column and n + 1 row—I was like "Whoa, your terminology is just way too intense for this problem!" She was saying, " an n x n row," and I was thinking what is an n x n row? I wasn't understanding it so I abandoned her approach totally. I was like whatever…. Maybe that's just because she recognizes that terminology better and maybe I don't. But it made me* ***<u>even more</u>*** *want to abandon… her approach…. I had to get my own paper and just do it for my self.*

About five seconds pass with the girls each working on her own paper while glancing at each others' work and muttering things about "$2n + 1$"—it's the talk to yourself, but out loud kind of muttering.

Somewhere in this muttering, Annie thought she heard Courtney say "$n + 1$" which conflicted with the geometric approach she had developed. "So I started doing the algebra with the geometry. [Using] $n + 1$ quantity squared and figured it should actually be $2n + 1$ and she said $n + 1$." While Annie was working on this, Meredith was taking advantage of the quiet time because she really valued:

> *having that time to sit back and think and look over the problem. And just let yourself have your own reasoning about what's going on and really find where you're lost, where you don't understand, what you understand up to then, where you need more help from someone else to build on the ideas that you already have, or whatever. Just [a time] to kind of organize everything and get caught up.*

Meredith realized the $2n + 1$ they were now discussing was just what she "was talking about at the very beginning when I said "Oh, it's just the next odd number" and "wished she had gone with that in the beginning more."

The 2n + 3 Confusion

Courtney summarizes where they were at this point by saying, "Miss Sheehy's asking what would you add to get from the nth to the $(n + 1)$th square number? You would add $2n + 3$ dots." She points to her paper and says, "This 3 is $2n + 1$, this 5 is $2n + 1$." Silence. Hence, Courtney reminds them they just figured this out. She reaches over and points to Annie's geometric representations while explaining "this is n [the row] and this is $n + 1$ [the column] so we are adding n plus $(n + 1)$ number of dots." Meredith, who has been writing on her paper while Courtney was speaking, now offers, "So $n + 1$ plus $2n$ plus 1 would be $3n + 2$." Annie and Courtney stare at Meredith and her paper with bewildered looks. Courtney leans in and tilts her head to better see Meredith's paper (see Figure 23).

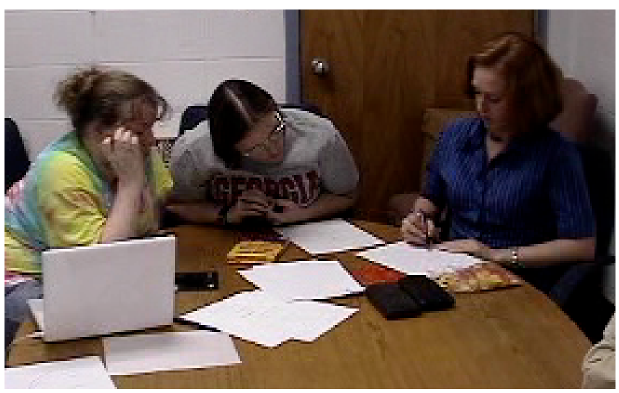

Figure 23. Courtney leaning in

Noticing this apparent confusion, Meredith points to her work as she asks, "So we're wanting to see how you get from n to $n + 1$, right?" As Courtney answers, "Right, but what would you add?" Meredith continues her explanation with "we added $2n$ plus 1." Meredith rubs her eyes, looks back at her paper and says, "But we want to know how many dots? I'm getting confused between if we're trying to figure out how many we're adding or if we're dealing with just the dots." Courtney quickly clarifies that "it's how many dots we're adding." Meredith glances at her paper and says (with a "now I get it" tone in her voice), "Oh! We're adding $2n + 1$ dots to get to the next term." As she is finishing her sentence, Courtney jumps in to further explain, "Yea, and here (pointing at Meredith's paper) your n is $n + 1$ so it's 2 times $n + 1$ plus 1. As Courtney is leaning back as if the confusion is cleared up, Meredith pauses and asks, "Why?" Quickly, Courtney points to her own paper and explains, "Because our formula is $2n + 1$. Here our n is $n + 1$ so you have to plug that in where n is in the $2n + 1$. "Oh," says Meredith, but not too convincingly. Courtney offers more explanation about the algebraic substitution $2(n + 1) + 1$ and its equivalence to $2n + 3$. Meredith now says, "Okay, I see what you are saying," and verbally repeats it while working the substitution on her own paper. Courtney watches and checks Meredith's work

Throughout this conversation between Courtney and Meredith, Annie has been sitting still with her head resting on her chin and paying attention. However, at this point she raises up out of her chair, adjusts her position, and turns away from the center of the table to work on the computer. Meredith suspected that Annie was either "*not following what we were doing*" or "*maybe she [was] taking a brain break*." Meredith elaborated, "*Sometimes when you work on a problem for a long time and you know you aren't quite getting there yet, [you] have to take a little bit of a brain break and just stop thinking for a few minutes*." I was feeling the tension, as I knew Annie was clear about this aspect of the problem and was ready to move on. I also knew that Meredith needed to make sense of the $2n + 3$ formula before moving on in the investigation. Hoping to get everyone focused on the original questions, I ask, "What's the nth square number?" Courtney hesitantly answers, "n^2." Yes, I assure her that is meant to be an obvious answer. Now, when I ask, "And the $n +1$ square number is?" Courtney replies, "$n + 1$ squared." Annie clarifies that this means $n + 1$ quantity squared as she cups her hands to form parenthesis. Looking at Meredith, I ask, "What is $n + 1$ quantity squared?" All three girls answer in unison saying, "$n^2 + 2n + 1$." Hoping that the next question I pose will provide a common place for discussion, I ask, "So, how do you get from $n + 1$ to $n + 1$ quantity squared?"

All three girls look to their own papers. Annie answers first with, “you add one row and one column… which is $n^2 + 2n + 1$.” Courtney clarifies that “it’s $2n + 1$ because previously we had n^2 and now we’re adding.” “Right, right,” Annie interrupts. Meredith has been working on her own paper and now states,

> It would be 3n + 2 squared, right? Because we would just do the same things as when we moved from n to n + 1. Now we’re moving from n^2 to n + 1 squared and the change is always 2n + 1. So all we have to do is put n + 1 into the formula, which we already know is 3n + 2, and put that quantity squared.

Courtney pauses and then re-explains her own solution. She was frustrated with Meredith because,

> *[Meredith] was still talking about this 3n + 2 and I have no idea where that was coming from…. I thought that we had already established that it was 2n + 3 not 3n + 2…. We had already talked about it. She had already said, “3n + 2” and I was like, “No it’s 2n + 3.” I showed her why and I thought she got it. And now she’s bringing up 3n + 2 again and I’m like, “What are you talking about?”*

At this time of awkward silence, Annie speaks up and calmly summarizes the derivation of the formulas incorporating both Meredith and Courtney’s work. As she explains, both girls slowly realize how one’s work was just a different representation or interpretation of the problem. Within 10 seconds, Meredith realizes the error she made with $3n + 2$ and Courtney sees the connection between her work, Meredith’s formulas, and Annie’s geometric representations.

Although this discussion ends with everyone feeling satisfied with her own understanding of the problem, the scenario also frustrated Courtney. She believed, “*Annie got it but she didn’t tell us…. I felt like Annie knew what [we were] doing, but she wasn’t telling [us].”* Meredith’s reaction to Annie’s help was quite different. She was impressed that Annie “*can just sit there and be totally quiet. Then she’ll just lay it all out on the table and it’ll just like the whole shebang*.” Wrapping up the square number investigation, I say, “See how much fun you can have with this type of investigation. Let’s have even more fun and investigate triangular numbers.”

Moving on to Triangular Numbers

“What do you mean ‘triangular numbers?’” asks Meredith. Courtney and Annie both begin drawing geometric representations in order to show Meredith. Courtney lets Meredith know that she and Annie had investigated triangular numbers last week in another class. Meredith draws her own geometric representations of the first 3 triangular numbers and then looks to Courtney’s paper who explains, “See it’s just 1, 1 + 2, 1 + 2 + 3. Meredith noted, “That helped me…. Hearing [Courtney] say that made me recognize each triangular number is the sum of the integers…. That made light bulbs go off in my head—it triggered it. I don’t know if I would have necessarily seen that.” I remind them of the formula for the *n*th square number (n^2), and then pose the question, “Can you generate a formula for the *n*th triangular number?” No one answers but everyone begins writing on her own paper. After a few seconds of silence, Meredith suggests, “That’s just like the formula for the sum of the integers which is n x (n + 1) over 2. When I ask, “Why is that the formula for triangular numbers?” Courtney explains that “If you start off with 1, to make a triangle you have to add 2. To make another triangle you need to add 3 to make a triangle.” Meredith wonders, “Why do you divide it by 2? I know we looked at this the other day.” Annie has been quiet so far because she “wanted to see what they remembered… and let them see it was the sum of the integers. Satisfied that both Courtney and Meredith have made that connection, she directs Meredith to her paper to explain why you divide by 2. “What if I rearranged the dots so they were directly on top of each other?” asks Annie as she reaches for a purple marker. With the full attention of Courtney and Meredith, Annie draws an adjusted representation of triangular numbers as shown in *Figure 24*. Then Annie suggests thinking about this representation as half of a rectangle and asks Meredith to fill in the rest with a different color (orange).

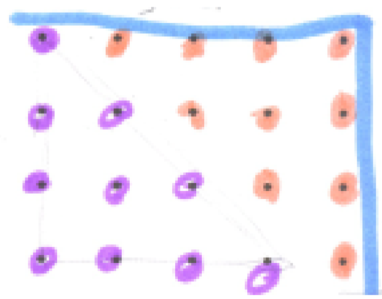

Figure 24. Annie's geometric representation

Annie then explains that the dimensions of this rectangle are in fact n x (n + 1). In unison, Courtney and Meredith “get it”—why you divide by 2. While Annie and Courtney look to the diagram and restate the formula, Meredith writes down the formula and says, “Y’all look at this. If you have $n^2 + 2n + 1$, can you factor out the n? Let’s see.” They each begin working individually to make a connection with the previous investigation of square numbers.

Looking for a Connection

Courtney asks Meredith for clarification, “What are you starting with?” “You know how we went from n to n^2,” answers Meredith. Later she told me she “was trying to see if there was some way to relate the two, but it wasn’t really getting anywhere. It was just like a random thought.” Influence by her job as a tutor where she “is not supposed to just directly tell them. You are supposed to ask them leading questions,” Annie asks Meredith, “Don’t you need another row here in order to make it a square?” Looking at the algebraic formulas she had written down, Meredith continues her thoughts aloud about connecting triangular numbers to square numbers. Again, Annie points to the geometric representation she has developed and suggests adding another row in order to form a square from the rectangle. When Meredith and Courtney agree, Annie changes her diagram by adding a pink row as shown in Figure

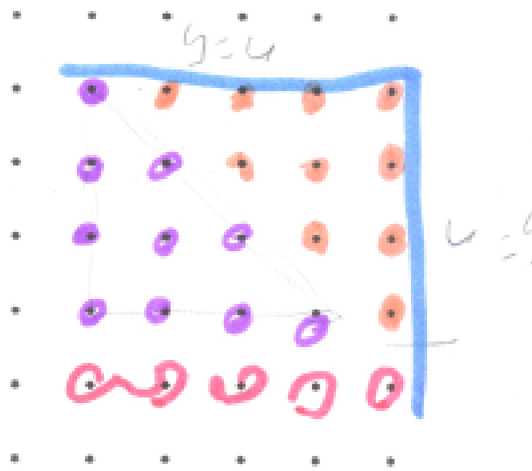

Figure 25. Adding a new row to form a square

Courtney is very animated as she explains that you should now have an *n* x (*n* – 1) rectangle and thus the formula for the nth triangular number is *n* (*n* – 1)/2. There is much debate for the next 2 minutes over the dimension being *n* – 1 or *n* + 1. Meredith explained that the confusion occurred because they were defining n differently. She noted, “See they were talking about making it a square by adding one more row, and I was talking about already having a square and subtracting *n*-1.” Annie also realized, “We were all saying the same thing, but we were just saying it differently.” As they agreed to define *n* the same way, they settled on the formula *n*(*n* + 1)/2 to represent the *n*th triangular number. Annie asks me, “I bet we are going to look at pentagonal numbers next, aren’t we?”

Modeling Pentagonal Numbers

Immediately everyone picks up a pencil or marker and then stares at her paper trying to figure out how to model pentagonal numbers. Marissa speaks first asking, “Should with start with 1 or with 5?” They all were sketching ideas on dot or blank paper as well as looking at and discussing each other’s ideas (see Figure 26).

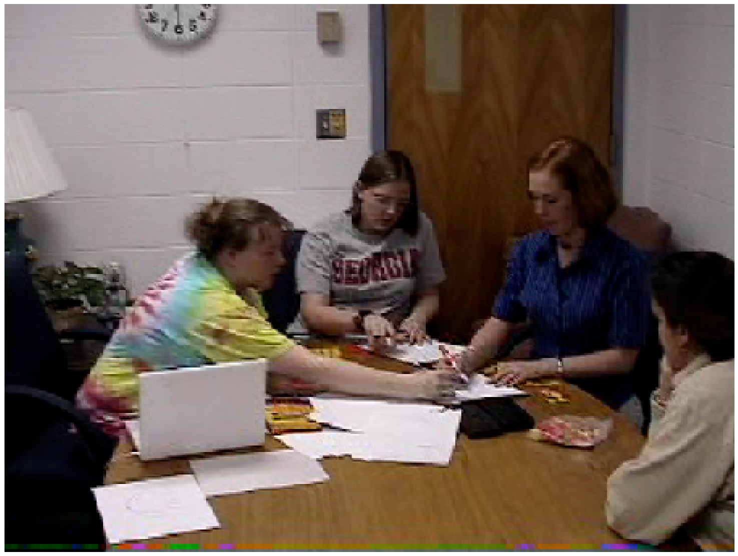

Figure 26. Testing and sharing different models

Annie claimed one reason they paid so much attention to what the other one was doing was

> *We knew what the triangular numbers were and we knew what the square numbers were so we were all worked on different pages. But, once we got to something new we had to combine all of our thoughts together, we had to pay attention to how the person was seeing it.*

Courtney realizes that "drawing them on the dot paper doesn't work… [because] this diagonal is not equal to one so when I am drawing the diagonals to make… the pentagon they don't have equal sides." At this point, Annie decides to try using the two-colored discs to model pentagonal numbers. Realizing that they are trying to model a regular polygon (and that we are running out of time), I take five discs and model the second pentagonal number with an equilateral, but not equiangular, pentagon as shown in.

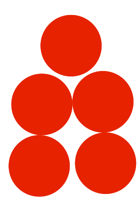

Figure 27. Geometric model of the second pentagonal number

Courtney and Meredith each take a handful of discs and begin modeling the third and fourth pentagonal numbers. After a minute, Courtney offers her observation, "It's just like a square with a triangle on top." As Meredith and Courtney continue building and talking about their models, Annie has begun to investigate her own ideas about the square and triangular number observation. As Courtney finishes the model of the fourth pentagonal number and Meredith finishes the third, Annie claims, "Okay, I think I got it." Annie turns their attention to Courtney's fourth pentagonal number and asks what square and what triangular number is represented. Courtney notes there is a square of 16 discs and the third triangular number comprised of 6 discs. Annie then points to her paper and says, "See you have an $n \text{ x } n + (n-1)\, n / 2$." Hmm and ooh are heard from Courtney and Meredith as they reach for new paper and find the scratch paper on which they previously noted formulas for the square and triangular numbers. This time to work individually was important to Courtney because "we had to add the fourth square number plus the third triangular number and make sure that we get the same thing as Annie did—to believe that what she wrote was right." Meredith said that working on your own paper is important because, "Now it's formula time."

Formula Time

As shown in her work (see Figure 28), Meredith tries to write a formula for the pentagonal numbers based on her previous representations.

Figure 28. Meredith's initial work on the pentagonal formula

When she examines Meredith's work Annie notes that although Meredith "said square with a triangle on top, I don't think she recognized that it was the *n*th square and the one less triangular number." So, together, Annie and Meredith work on Meredith's paper to develop the algebraic representation shown in Figure 29.

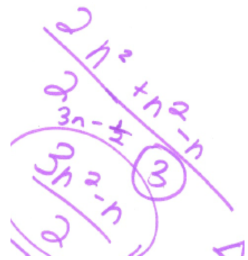

Figure 29. Annie and Meredith's algebraic representation

After a few seconds of discussion and restating the rationale for this formula to each other, Courtney suggests they compare all the formulas. "Yes, there might be a pattern, says Meredith. Annie looks at me and says, "I'm sure there is." Courtney quickly summarizes the algebraic formulas as shown in Figure 30. She leaves the formula for hexagonal blank until Meredith suggests that the pattern of the leading coefficients should be 1/2, 3/2 and then 5/2.

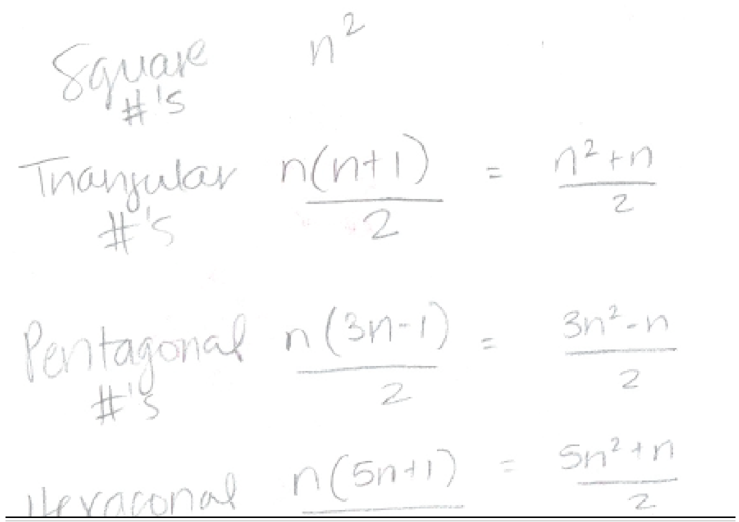

Figure 30. Courtney's summary of algebraic representations

Although I know rich mathematical conversations could continue, I let the girls know that we have gone five minutes over the allotted session time. Annie wraps up the activity with, "What about heptagonal numbers and n-polygonal numbers? We could go crazy with this."

CHAPTER FIVE: SHARING FINDINGS

The site where their stories and voices become transformed into theory

(Doucet & Mauthner, 1998, conclusions)

Review and initial analysis of the three PSSs presented in chapter 4 consisted of an intentional search for the ways students engage in and experience cooperative, mathematical problem solving. As I analyzed at a deeper level, looking for characteristics that made each of these emerging themes identifiable, it occurred to me that a common thread among the themes was the individual/group binary tension felt by the participants. As evidenced in the previous chapter, Annie, Courtney and Meredith did not make comments such as, "I felt a self/other binary tension there." However, what they did do was talk about what they were consciously aware of and thinking about as they both solved and watched themselves solve problems. Thus, using Earley's (1997, 1999) theoretical levels of consciousness models, I identified and described these binary tensions. Reflecting the synergistic nature of my study, I reported the findings of this study within three broad categories—environment, group members and self. Within each category, I presented a discussion of the tensions, how they effected (and sometimes interfered with) the individual mathematical activity of the participants. Then I demonstrate how, by deconstructing these tensions and moving towards a level of conscious participation, the participants were able to orchestrate cooperative learning in ways more instrumental to their own learning.

Problem Solving Environment

This category of themes includes the use of manipulatives, adoption of roles, and the related proximity to materials and other group members. Because these aspects of the cooperative environment were connected and thus continuously effected each other, talking about them separately is difficult. I begin with the issue of manipulatives as that appeared to impact the environment most significantly.

Accessing Manipulatives

PSS1 was the "Square Game" in which the participants were provided with one game board, one set of game pieces, and one pad of paper with one pencil. This design intentionally mirrored my own previous experiences using this activity with my own students as well suggestions from the cooperative learning literature (see chapter 3). During the interviews following PSS1, it quickly became evident that this utilization of just one set of manipulatives was a significant issue for all three participants. They felt that the size of the game pieces and board was problematic because it was too small. Courtney said we should:

> Think about manipulatives—they are usually very small. And if you think about when you usually use manipulatives, it is in groups. That kind of doesn't make sense. Maybe they are small because they are easy to store, but do they really help in the groups? Or would it be better for them to be big? (PSS2G)

Meredith noted that the number of manipulatives is also important to consider. In PSS1 she actually made her own game board in front of her in order to "get it closer where [she] could relate to it a little better" (PSS1M). While everyone agreed it was harder to think about the problem if you were not the one moving the pieces and feeling the pattern, Courtney had the most control of the manipulatives in PSS1, as the game board remained in front of her throughout the entire session. However, Annie occasionally reached in and moved the pieces without verbalizing her moves. Although Courtney felt she should let Annie have access to and move the game pieces, she worried that "if that board had been in front of Annie, she would having been doing the slides. But would she have talked while she was moving the pieces?" Courtney wondered if she could have felt as mathematically engaged.

In PSS2, the participants chose to use bigger manipulatives (playing cards) to represent the lockers and in retrospect felt this was a good decision because they "could all see it, we could all touch it and we could all be a part of [solving the problem]" (PSS2A). "The space we were using the manipulatives on was bigger" (PSS2M), thus there was less of a division between communal and individual space on the table. Annie observed, “All of us could reach in and flip it or say, ‘No, that’s not the right number’ by pointing to it…. It was a much bigger board to play with, expand our ideas, or correct each other” (PSS2G). By allowing the participants to feel more involved in the activity, this increase in communal space from PSS1 to PSS2 facilitated the mathematical thinking of the individual participants and the interactive nature of the group.

Because of the previous interviews, the participants and I were aware of the “manipulative issue.” So, in PSS3, a larger number and variety of manipulatives were provided and placed in the middle of the table. While the session started with Meredith modeling the square numbers using the discs, Courtney quickly got some blank paper to record what she was seeing and Annie took some dot paper to make her own geometric representations.

The access to only one game board almost forced the participants to work together throughout the entire PSS1. Johnson and Johnson called this strategy positive interdependence and argued that it encourages students to work cooperatively. Smith (n.d.) suggests interdependence can be fostered when “you see a student working alone and independently of the team. You can… limit the resources of the group, for example, provide only one pencil and piece of paper…. In this way, the student will be forced to work with the group” (section 2). The limiting of resources in PSS1 was the primary issue that created binary tension within the participants. At both the participatory and the reflexive levels of consciousness, these tensions inhibited each individual participant’s mathematical thinking. For example, in the first PSS, Meredith and Annie both felt frustrated at times because their only access to the materials was through Courtney. They could make suggestions to Courtney as to what moves to make but ultimately Courtney made the decisions. Acting on their own idea about how to move the squares required remembering it until there was a time to offer it. Thus, their ability to participate reflexively was hindered.

During PSS1, after Annie quietly observed for some time, she reached in and "un-did" a move she felt Courtney had done incorrectly and then made the moves she felt were correct (pulling in the last square). For Annie, this was a way to be more reflexive—to engage in her own mathematical thoughts. She just "kept thinking, 'I want to do that, let me try this!'" (PSS1A). Frustrated, Annie sensed that this movement on her part was not well received by either Meredith or Courtney, so she abandoned this approach and temporarily disengaged from the group. For Courtney, this meant having her own train-of-thought disrupted and having to now try to figure out what Annie was doing (since Annie was not verbalizing her thoughts, this made it even more difficult). Courtney felt a responsibility to be listening to her group members and moving the pieces based on what both of them were thinking. If Annie had just said what moves to make—had just spoken up—Courtney would have been more than happy to pursue Annie's ideas.

For Meredith, having one set of manipulatives caused her to move between participatory and reflexive levels of consciousness throughout the session. Wanting to be a good group member, she carefully recorded the ideas of Annie and Courtney (this role is discussed further in the next section). But she felt this was keeping her from her own thoughts about a solution so she created her own game board on the data sheet and began trying out her own ideas. This lasted less than 30 seconds as she then felt she should either offer her ideas to the group or re-engage in what they were doing. When Annie began reaching in to move the pieces, Meredith was troubled because now she had to write down both Courtney's ideas and Annie's unarticulated moves. Meredith also found this situation problematic because she was nervous about the tension between Annie and Courtney. She felt the tension and tried to keep it from escalating.

This movement from sharing the manipulatives and carefully listening to each other to trying their own ideas (either off to the side on paper or mentally) caused significant tension for each individual participant in PSS1. The majority of the time the participants were in a participatory level of consciousness and any departure from there caused them to feel uncomfortable or even uncooperative. This discomfort has serious implications for individual mathematical thinking. The participants forgot ideas because they could not immediate act on them; they were not able to formulate tactile ideas completely because they we not able to access the materials; they were not able to process others' mathematical ideas as well because the pieces were small; and it was sometimes difficult to see the game board. Because they focused on the same materials, each contributed ideas, and produced one correct solution they all understood and appeared proud of, an observer might have viewed the participants as an example of an excellent cooperative group. Although the participants would agree that those aspects of cooperative learning were indeed present, they left the session feeling unsure about their personal ownership of the Square Game solution. It was as they progressed through PSS2 and PSS3 that they became more aware of the lack of and need for reflexive consciousness in PSS1.

From the group interview and individual interview after PSS1, the participants were aware that the size of the manipulatives was problematic for their own mathematical thinking. Consequently, in PSS2 the participants arranged and used manipulatives in a slightly different way. Even though Annie and Courtney were the primary "card flippers," all three participants felt they each had access to the cards used to represent the lockers. In contrast to the PSS1, when each participant either stayed and worked primarily in the individual space directly in front of her or observed the space in front of Courtney, the introduction of larger manipulatives created a much larger communal space on the table. Although this space was primarily in front of Annie and Courtney, as Meredith took the responsibility of writing down ideas and patterns, Meredith felt that she could reach in to help and offer ideas that would be tested. The ability to be in reflexive levels of consciousness was supported by the larger manipulatives as the participants felt they had more involvement in solving the problem—their mathematics was a part of the solution.

The participatory level of consciousness was still primary in PSS2 but some of the tension felt by the participants was eased. They found that tensions occurred more around conflicting mathematical ideas and less on who should be in charge of using the manipulatives to model the ideas of the group members. The participants felt freer to think about their own mathematical ideas reflexively and then share them with their group members. Because they could more easily see and touch the cards, developing a hypothesis, testing it and drawing a conclusion before sharing was more practical and perceived as less of "backing away from the group." While this use of larger manipulatives eased tension by making group activity easier, it was in PSS3 that individual mathematical thinking was most significantly fostered during CMPS.

As the participants began PSS3, Meredith was the only one creating a model for the square numbers with discs. Courtney and Annie were both sharing ideas and since Courtney could easily reach, she pointed to the discs as she talked about them. Within approximately two minutes of being asked to generalize the problem for the n^{th} and $(n + 1)$st square number, each participant reached to the middle of the table and took her own materials with which to represent the problem. The placement of the manipulatives and resources was a important factor that promoted this choice. Courtney said it is important to "have is enough paper [so that] everyone has paper and feels free to get the paper—not like it's on one side or the other side [of the table] because there is also the influence of where things were on whether you would reach out and get it or not" (PSS3GC). Each participant was now able to represent the problem in the way that made the most sense to her. Annie created pictorial representations with dot paper, Courtney worked on algebraic formulas, and Meredith focused on numerical patterns. While each participant created her own representations of the polygonal numbers investigation, each became increasingly aware of the others' representations and how those might be related to their own work. As they shared ideas and struggled to make connections among the geometric, algebraic and numerical representations, each participant used these new representations to build and deepen her own knowledge of the problem and its solution.

The participants stated that the decision for everyone to use their own materials was primarily responsible for the feeling of being, not only more involved in the problem in terms of interaction, but also more involved at a conceptual, mathematical level. As we watched the videotape we all noticed that the demarcations between the communal and individual space were blurred this time as individuals put their work in the middle of the table for discussion, brought it back in front of them, reached over or across to write on the papers of others, and so on. The following montage of quotes from all three participants exemplifies why they felt this blurring of space and roles was helpful to their own individual problem solving process.

> This time we all had the manipulatives, we all had paper…. We all had our own dots, and we were all trying to figure it out at the same time rather than watch someone else figure it out…. It really made me grasp the concept a lot better… because having our own manipulatives helped us to be able to think independently and not have to depend on someone else to help us get the next idea…. Even though we were all kind of thinking independently, we were still contributing ideas to the group and to where we were going with the problem as a group…. We were all contributing and that was important. (PSS3G; PSS3A, C, M)

In comparison to PSS1 and PSS2, the participants moved more freely from the participatory/reflexive binary toward a level of conscious participation when they had equal access to manipulatives. What we learned is that, in a cooperative learning situation, decisions about the use of manipulatives are, as Annie put it, "Huge!" The possible tension created by too small or too few manipulatives is one that significantly affects students' individual mathematical thinking. A student should not have to sacrifice his or her own ideas or put them on hold because he or she thinks it is rude to break in and use the supplied manipulatives or recreate some on your own as Meredith did. We learned it is not that we all have our own work or we that we have to share everything, but instead that there are ways to take advantage of both of those aspects.

Adopting Roles

Just as the provision of one set of materials/manipulatives is a common strategy for encouraging students to work cooperatively, so is the adoption of roles. Roles represent tasks or jobs that need to be accomplished during a cooperative activity. In Chapter 1, I provided an example of an activity where roles were suggested: Leader, Encourager, Technology Expert, Records Keeper, Presentation Coordinator. Traditionally the assignment of roles is a strategy used to help individual students feel that they have a part to play in solving the problem or completing the activity. Roles can foster a sense of belonging as well as encourage students to plan a problem solving strategy. I have this method of assignment/adoption of roles consistently in my classrooms. When students are new to cooperative learning, I create the roles and ask the groups to assign those roles. As students become more comfortable with cooperative learning, I ask them to reflect on what roles will be helpful and assign them accordingly. When I no longer needed to remind students to plan and assign tasks, I felt they were "advanced cooperative learners." I still believe in using roles to encourage sharing the work, including everyone, using individual talents to benefit the group, fostering a sense of team, and so on. However, the participants in the present study talked about the tension inherent in having role and how the role consistently interfered with mathematical learning of the individual. As I listened to them identify and begin to resolve the binary tensions created by roles, I began to rethink the purpose and place of roles in CMPS.

Although I did not assign roles in PSS1, the participants chose and maintained distinct, task-oriented roles. These roles emerged primarily in relationship to the proximity of each participant to the provided materials. Recall I placed the materials on the table at the various seats before the participants entered the room. Courtney sat in the middle seat in front of the game board; Annie sat to Courtney's right in front of the laptop and a graphing calculator; and on Courtney's left was Meredith with a stack of blank paper and a pencil in front of her. Thus the roles were matched with the materials and were described by the participants as follows: Annie, the observer and thinker;

Courtney, the manipulator and leader; and Meredith, the recorder and organizer. In general, the participants remained within the boundaries of these roles. As seen in the previous section on manipulatives, tensions increased when boundaries were crossed. In this particular PSS, the participants felt that roles created too much pressure to operate within primarily the participatory level of consciousness—the obligation to fulfill a responsibility to the group members took more precedence than individual, mathematical problem solving.

For example, in the Square Game, Meredith was the "recorder." She described the difficulty of this role:

> I was trying to record [the moves]. Then I don't know if it is going to be correct or not. But if it is right, I want to make sure I've written it down so that we can redo it. And so that was kind of confusing to try and be the recorder, look at the pattern, and figure out how to do the moves all at the same time. (PSS1M)

She said she was "nervous about forgetting to write something down." This responsibility led her to not "see what was happening…. Even though she wrote it down, [she] didn't realize how important it was" (PSS1M). Meredith said she knew this was happening with her during PSS1 but felt that someone else would just have to so the thinking, and Annie was in the role to do that. Meredith explained, "Annie's moving the squares in her own head, trying to see what would work and visualizing it, because she wasn't doing it…. Annie will sit back and observe and then she will pop out something [and it will be right]" (PSS1M). Aware enough to know that her mathematics was being limited, (reflexive level), the need to fulfill her obligation to the group (participatory level) was Meredith's primary focus.

Courtney claimed that each participant could she herself in each of the roles from PSS1. She explained that:

> None of the three of us are incredibly shy. We are all fairly outgoing and dominant. [The way we function as a group] just depends on the role that we choose. I don't think we even choose it—it just happens." [PSS1C]

Recognizing the pros and cons of the roles, Courtney empathized with the role of the recorder. She said, “Sometimes you are so busy writing down what other people are thinking that you can’t think for yourself. I think that happened with Meredith maybe with the squares.” She also talked about the limitations of her own role. Although her role of moving the pieces allowed her to “feel” the pattern, Courtney is someone who likes to write down her mathematics. She needed time to reflect and yet, because she was considered the leader, she felt pressured to keep the activity moving as. In a way, she believed that both Meredith and Annie were in a position more conducive to the mathematical investigation of the Square Game.

At a reflexive level, Annie knew she had an opportunity to watch and reflect on the investigation, which she did for a while. “And then Meredith started writing things down; and I was trying to understand why Courtney [thought] that you keep it in the middle; and then I just kind of threw the towel in” (PSS1A). Although Annie was laughing when she talked to me about throwing in the towel, she did realize, at a participatory level, that disengaging from the problem was easy for her because her role had less obvious responsibilities to the group. This led to Annie feeling slightly left out and definitely less focused on finding a winning strategy and identifying patterns. Meredith agreed with Annie’s frustration in the group interview saying, “I think it is very easy to be someone sitting back listening to get lost in slides, jumps, yellow, and blues and just throw in the towel” (PSS1GM). Although Annie felt mathematically hindered by her role of observer, she was very sure she would not like to have been the leader. Her reason was, “Because then I would have been thinking, ‘What are they doing wrong, what are they doing right?’” (PSS1GA) instead of thinking about my own mathematical ideas.

Based on our discussion about PSS1 and the ways in which both roles and manipulatives effected individual mathematical activity, I provided a variety of choices and sizes of manipulatives (e.g., coins, playing cards, two colored discs). Also reacting to our discussions from PSS1, the participants intentionally selected the playing cards to model lockers in PSS2 because of their size and subsequent availability and proximity to everyone. This choice of manipulatives and subsequent lack of formal roles eased the expectation to perform certain tasks in order to support the group. Each participant had a chance to flip the cards (work at a participatory level) and got time to observe and reflect on the mathematics of the problem (at a reflexive level). Annie explained why the freedom to do both was important to her:

> There [are times] when if you are not participating [flipping cards], you don't understand what it going on. But, there are parts of it where you are like, "Whoa, hold on. I know that's not right." Then, you can sit back and see why. (PSS2A)

Courtney said in PSS2, they worked together, but differently, "more evenly," than in PSS1. Since there were not "distinctive roles that we were playing, it was more like we were all kind of just working together" (PSS2 C).

This type of working together enhances individual mathematical activity. Because everyone had a chance to reflect on the flipping cards and emerging patterns, once they stopped using the manipulatives and focused on the question of "Why perfect squares?", they were "all putting [their] two cents in. Meredith with the mod 4 thing, I had done factor trees and Courtney was using that idea" (PSS2A). The importance of flexible or group developed roles lies in the fact that the lack of roles allowed the participants to focus more on the mathematics. Just as procedural learning has the potential to inhibit conceptual understanding, so does procedural cooperative learning.

The participants worked primarily at a level of conscious participation throughout PSS3. They said there was rarely a time when they felt the group/individual tension and attributed this to the absence of distinct roles. Meredith described this session and why the participants all felt that PSS3 represented the type of cooperative learning they hoped to foster in their own classes.

> This time we all had paper and we were all thinking independently but still contributing ideas to the group and where we were going with it as a group. I think [not having distinct roles] is good cause you don't feel like you have to a

> role model for what you are expected to do. Whether it's recorder or leader or whatever…. Having our own manipulatives [and paper] helped us also to be able to think independently and not have to depend on someone else to help us get to the next idea. (PSS3M)

Annie suggested, "I think the group is so much more productive if you don't try to delegate roles. There is not very much learning going on there between from person to person" (PSS3A). However, the participants explained that during PSS3, "We didn't work completely individually. It was like somebody would suggest something and we would be like, "Oh yeah, let's try that. We were talking to each other, but I think we all had our own thinking going on" (PSS3C).

Because an investigation of polygonal numbers can be approached with a variety of mathematical representations, the individual thinking at the reflexive level focused on the mathematical strengths and tendencies of each participant. Annie told me in her initial interview that she likes to build up to an abstract theorem or generalization with concrete examples and representations. Thus, it is not surprising that when given the freedom to explore individually, she used manipulatives and dot paper to model geometric representations of the polygonal numbers and made generalizations based on her drawings. In her initial interview, Meredith stated that it is important for her to "start off with the basic information and then absorb it." She prefers to write down all the data and then look back over it to make observations and generalizations. Not only did Meredith's algebraic approach of writing formulas to represent the numerical data the group collected fit her mathematical preferences, but she also explained, "I didn't feel left out. I felt on task. I felt a part of it" (PSS3M). Courtney said "I like [studying mathematics] because it's kind of like a puzzle that you have to figure out" (initial interview). She explained how the different mathematical representations of Annie and Meredith created a type of puzzle for her:

> I think we all kind of figured it out on our own and then just checked with each other.… I mean Annie and Meredith were looking at it in very different ways but

> they [were] both right. But maybe Annie was just in a more of a geometric mindset and Meredith was a more algebra person. So it is fine to do it on your own and then check. I don't think that was bad at all. And also, we were all writing it down. That [problem] was really something that you had to write down yourself—at least for me to actually see what was really going on, the formulas and stuff. So we all had sheets of paper and we were all writing down things. I think we kind of worked more individually and then grouped together at the end. (PSS3C).

All three participants described the sense of working

Group Members

A significant amount of research on cooperative learning has focused on the interactions among group members (see chapter 2). Cooperative learning unmistakably brings to classrooms the social aspects of doing mathematics. Thus, it is very important that we understand how the presence of group members affects students. I am, of course, concerned with how students develop friendships, deal with peer pressure, and react in social situations—cooperative learning can help students learn about all these social components. In the present study, however, I am more concerned with how the social/individual tensions found in and created by cooperative learning experiences influence the mathematics of the individual participants. As I listened to the participants talk about the social experiences of CMPS, I identified the two themes as areas in which tension was prevalent and individual mathematics was potentially hindered or limited: being polite and resisting power.

Being Polite

Perhaps the most difficult binary tension to allay was evidenced in the theme regarding being *polite*. I was surprised at how quickly this theme emerged from the data and how significantly the need to be polite affected and changed the mathematical thinking of the individual participants. As I continued to listen to and analyze what the participants were telling me, it became apparent that using cooperative learning as a tool for developing social skills has the potential to limit the mathematical experience of the individuals in a group. At least for these three participants, being a good group member included being nice, sharing both materials and time to talk, thinking about the ideas of others, doing your part, easing tension between other group members and pretending to understand someone else even if you did not. In the present study, being this type of "good group member" was often more important to a participant than her own mathematical thinking. Evidenced throughout the PSSs were binary tensions surrounding the question, "Which is more important, my teammates' feelings or my mathematics?" While the "politeness tension" has been visible in previous sections (e.g., sharing manipulatives and maintaining roles), in the following discussion I will explicitly identify and explore a variety of tensions within the context of an episode from PSS2.

Because the participants felt this particular episode provided significant insight into their struggles with the social pressures of cooperative learning, they used video footage of the episode, both in the presentation discussed in chapter 3 and in a later presentation at a conference, to discuss the ways in which the need to be polite affected them. It happened during PSS2 (about halfway through) when Meredith suggested a modular arithmetic approach to the Locker Problem. Courtney and Annie had been working on "factor trees" while Meredith had been quietly working on her "mod idea" and waiting for an appropriate time to share her ideas with Courtney and Annie. Meredith told me it was okay to interject a new idea when there is "maybe a little break of silence. Then [she] had a chance to be like, 'Okay, now that you are through, a little bit, with your thought, I should offer my new idea'" (PSS2M). Throughout the interviews the participants repeatedly explained their views on the "rudeness of interrupting" and how this social norm often made it difficult to bring up or even respond to new mathematical ideas in a timely manner. Meredith, who was particularly concerned with being polite, reminded us in a group interview:

> You don't want to interrupt someone because you know how society says it's rude. You don't want to step on someone's toes and tell them, "No you're

> wrong." Even if in your mind you are thinking, "No you're wrong," you might still… let them finish their thought… then be like, "Oh okay" like you understand or agree. (PSS3GM)

Courtney differentiated between times when interruptions were and were not appropriate. She explained if "someone is on a roll with their ideas," you wait until they are done to share your thoughts. She said you could tell in their voice and their enthusiasm if they were on a roll of if they were just thinking out loud.

Even though she wasn't sure her idea "was going to work," Meredith decided to "still tell [them] what [she] was thinking." She thought perhaps she could offer her idea to the group and maybe they could "build on it where [she] could not" (PSS2M). When Meredith did share her ideas and thus "became more a part of what was going on" (PSS2M), Annie recalled that Meredith was "trying to figure out what one-thousand was in terms of congruence mod 31, and [I thought] she was going to try and explain it [to us]" (PSS2A). This is where the consideration of the modular arithmetic strategy ended for Annie—she tuned Meredith out and continued to work on factor trees. Annie said she felt okay (not rude) doing this because she was aware that Courtney was listening to Meredith. Annie also felt that Courtney was in a difficult position at this point because she really wanted to "look at what I'm writing" but she also felt obligated to "pay attention to Meredith" (PSS2A). Annie recognized that if she had been sitting in Courtney's place (next to Meredith), she would have been in the same position and done the same thing. Annie was able to stay primarily in the reflexive level of consciousness and focus on her own mathematical ideas because she knew someone else was "taking care of Meredith" (PSS2A).

For Courtney, the tension created in this moment was not so easily resolved. What she really wanted was to continue working with Annie on the factor ideas they had developed together, but Courtney acted on her belief that

> it is not nice or appropriate for me to say, "Meredith I don't think that has anything to do with what we are doing. I mean it might, but I think we can figure

> it out this way." That's what I would have liked to have said because I felt like that was distracting us…. We weren't really working together there because [Meredith] was trying to do this other thing which wasn't really what Annie and I were thinking—it kind of threw us off. But, I think that's because you are in a social environment you're not going to say no." (PSS3C)

Even as Courtney discussed this tension with me (and in presentations), she felt conflicted. At a participatory level of consciousness, she did not want to imply that Meredith was not smart or did not have a good idea, so she complimented Meredith by pointing to the fact that Meredith was currently taking an Algebra course and knew way more about modular arithmetic than she or Annie. At a reflexive level of consciousness, however, Courtney was protective of her own mathematical experience and was not willing to put her mathematical ideas away in order to focus on a new idea just to validate Meredith. So she listened, but did not really try to make sense of Meredith's ideas. She "thought the direction we were going was going to get our answer and I didn't think we needed to worry about something too complicated" (PSS2C). What Courtney did want was for an appropriate amount of time to pass so she could go back to her work with Annie, without appearing rude. Courtney pointed out it was hard to listen to Meredith and keep thinking about her own ideas about the problem, but she did. For the participants, this strategy seemed to be a viable way of being aware of individual and group needs. They all agreed that in order to pay attention to someone else (especially when you don't really want to), you need to learn to "put your math on hold" (Presentation Preparation). This means trying to keep your ideas in your mind—don't let them go—while you listen to someone else's ideas. If their idea is off the wall or one you don't understand then you quickly take your idea "off of hold" and try not to forget it while you let the other person talk.

The above approach to resolving some of the self/other tensions embedded in being polite has limitations. Although it can momentarily make the one speaking feel valued and the one listening feel polite, listening to someone else (or pretending to) just to be nice can actually limit the mathematical experiences of everyone involved. Recall that once it was clear her idea would not be taken up by her teammates, Meredith reluctantly began to abandon her own thoughts in order to focus on Annie and Courtney's conversation. Trying to be gracious to her teammates, Meredith explained, "While I was kind of trying to offer [an idea] to them, they had their own thing going, too. So I'm sure they didn't want to abandon that and start something new that their mindset wasn't on" (PSS2M). Not only did Courtney feel that time listening to Meredith took her away from the problem, Meredith felt that she "kind of hindered [herself] by not being quite as a part of the group" (PSS2M). She explained that because she made this decision to "do [her] own thing" she was lost when she came back. Meredith did not feel it would be okay to interrupt and ask Courtney and Annie to "catch her up" at this point so she tried to just listen and figure out what they were talking about. She pretended to understand Annie and Courtney's solution with respect to odd numbers of factors, but told me later that it was not until she viewed the video that she realized how much she had not understood up to that moment.

This episode is representative of the self/other binary tensions the participants experienced as they strove to solve a mathematics problem together without appearing outwardly frustrated with each other. The participants came to the study with preconceived ideas about what cooperative learning should look like (see participant descriptions, chapter 3) and about how they should act in a group situation. Consequently Annie, Courtney and Meredith used their "manners" to keep them aware of what others were saying, to focus on including everyone and to be responsive to others' ideas. Furthermore, this apparent level of participatory consciousness maintained by the participants was often grounded at a more reflexive level. If this were the case, a participant tended to be polite based on a belief that she should be because of a social expectation. When this happened, the mathematical thinking of the participants was often hindered as they tried to balance a need to be nice with a need to solve the problem. If a participant operated from a primarily reflexive level of consciousness, she seemed to be aware of the potential "rudeness" of working alone or of ignoring the ideas of others. She either justified this by allowing another group member to be attentive and/or by saying she was just too into her own mathematical idea to stop. Although this reflexive level of consciousness better facilitated individual mathematical thinking, the participants said that listening to and learning from others is very valuable. In fact, Annie claimed hearing other students' ideas is "the crux of learning—you gotta hear from other people. If you only think your own ideas in your own little world… I mean math *is other peoples' ideas*" (PSS2A).

The politeness tension is not one that is easily solved. For a group member, politeness at the level of conscious participation would involve being aware of her own ideas and needs in order to be mathematically engaged while simultaneously taking advantage of the ideas of others' and group discussions. The participants in the present study believed talking about this tension with the other group members is a valuable strategy. Realizing their own tensions and reactions helped Annie, Courtney and Meredith be more aware of their own struggle for balance in this area. Perhaps "politeness" with respect to cooperative learning needs to be re-conceptualized so that being polite does not result in individual mathematical activity taking a back seat to being a good teammate. Ways in which the participants tried to take better care of themselves—to be more reflexive on a mathematical level—while not feeling guilty or rude are identified and discussed later in the section entitled "Self."

Resisting Power

A theme with less evidence in the present study, but nonetheless a significant phenomena within CMPS, was the existence and resistance of perceived power among group members. I understand power to be "a technique or action which individuals can engage in. Power is not possessed; it is exercised" (Gauntlett, n.d., introduction). Foucault (1980, 1983) asserted that power is exercised through power relations and further, "there are no relations of power without resistance" (1980, p. 142). When power got "stuck" with one group member, the participants identified their desire to resist that perceived power position and return to interactions that were bound in shifting power relations.

With respect to resistance, Foucault (1983) suggested a strategy for investigating power in a given context.

> It consists of taking the forms of resistance against different forms of power as a starting point. To use another metaphor, it consists of using this resistance as a chemical catalyst so as to bring to light power relations, locate their position, find out their point of application and the methods used. (pp. 210-211)

As I analyzed the data, I identified instances where it appeared the participants felt constituted by imposed, perceived, given or taken power. For Annie, Courtney and Meredith, power was an issue considered at both the participatory and reflexive levels of consciousness and the binary tension was difficult to manage. It seemed that no one really wanted to be in a position of power—not wanting to be perceived as bossy or in control. Nor did she want to feel oppressed by it—not wanting to feel out of control or not valued. When a participant felt that either of these conditions was present, she resisted. Thus, in order to illuminate possible power relations within the group, I discuss two forms of such resistance.

The first significant appearance of resistance to power was visible in PSS1, where there were distinct roles and limited access to materials. All three participants perceived Courtney to be the group "leader" because of her access to the game board and directive tone. Courtney said she was uncomfortable with this role because she did not want to be the boss; however when Annie reached in to have access to the materials Courtney was not willing to relinquish her role. Annie was frustrated because, "Courtney had control of the board. When I was trying to move the pieces, Courtney wouldn't let me so that was a hindrance [to my mathematical thinking]" (PSS1A). Courtney expressed her own frustration with Annie for not expressing her ideas, verbally. She hoped that by "undoing" Annie's moves, she could encourage Annie to say the moves aloud next time. Courtney believed that Annie was "sitting over there in her chair thinking about [the

Square Game] the whole time. Where we [were] more concerned with moving the pieces and writing it down. We didn't have as much time to think" (PSS1C). Courtney said this bothered her, "Because I know she's smart; I know she can think about stuff; I know she probably knows how to do it; she's just not telling us" (PSS1C). When I asked Courtney why Annie would do this, she just laughed and said, "I don't know… she probably wouldn't do it on purpose." Both Annie and Courtney (who are good friends) enjoyed talking about this instance. They felt it was an example of how misinterpreting the intentions or actions of group member can foster resistance in the form of disengagement from the whole group, a group member, or the problem itself. All three of these forms of resistance and thus tension affected the mathematics of not just the one resisting but the ones being resisted. Meredith and I talked briefly about the issue of resistance. She shared that she felt "rebellious" when one person is in charge, "Because it almost makes you despise the other person—'Why are you thinking that you are all in charge?' It makes you question their intentions" (PSS3M).

Another common thread that ran through the power theme was confidence. At the reflexive level of consciousness, participants felt less empowered as a mathematician due to a lack of confidence in their own abilities to solve the problem, a belief that someone else in the group knew more mathematics, or had a better problem solving strategy. For example, in PSS3 Annie looked away from her approach because she heard Meredith talking about sequences. She said she was

> looking at what Meredith was doing because… she is taking a MATH 4000 (Algebra) class so I knew that had influences on her with the sequences and stuff like that. So I was like, "okay let's see what she's doing." (PSS3A)

It was not only knowledge of coursework that caused a participant to cede power to another group member, it was also tone of voice. If a group member sounded confident, Meredith was apt to accept their ideas more readily. Referring to the "$2n + 3$ thing" in PSS3, she explained:

> I was just kind of going with [Courtney] on it. It sounded like she knew what she was talking about. I think another thing in cooperative learning is if someone sounds very confident about their idea, then you are more apt to be like, "Yeah, okay' than to question it and explore it for yourself—to see if that really is correct or not…. It is a lot about the power and it can really set you back if you are going in the wrong direction. (PSS3M)

Throughout the PSSs, it appeared power got "stuck" in roles, in preconceived ideas, in tone of voice and in lack of confidence. Power is everywhere and is inescapable; however, healthy power relations exist when power is shifting and multiple. In a cooperative learning environment, the balance is difficult and the tensions are real as students struggle to feel empowered and confident in their own mathematics as well as feel empowered by the ideas and presence of other students.

Self

Reynolds, et al (1995) encouraged professors to employ the common cooperative learning strategy of having students create team names. The goal of this strategy is to promote a sense of team spirit and unity. An "all for one and one for all" motto can definitely be used in a positive way to encourage students to help each other, work together, be committed to the team effort, and take pride in shared work and solutions. A sense of team or community raises a student's participatory level of consciousness and helps him or her have "a sense of belonging to the [group] and an immediacy of [the CMPS] experience" (Earley, 1997, p.21). We should promote interaction and a sense of community in mathematics classrooms, however, when I hear students describe their problem solving process as follows, I am aware of the binary tension of identity—a sense of self. Notice the emphasized pervasive and conflated "I" and "we" language present in these post PSS1 conversations:

> L: What strategies did *you* use to solve the problem?
>
> A: Well, once *we* recognized what *we* should do, *we* were just trying to (pause)…. *We* kind of jumped into it. (PSS1A)

L: How did *you* solve [the problem]?

C: Well, first *we* started by just kind of moving it around trying to get a feel for what *we* were actually doing. Then *we* started to think about starting on a smaller scale…. *We* wrote down a pattern for the two squares. Then *we* did three squares and *we* saw there was a similar pattern. (PSS1C)

L: So how did *you* find the formula?

M: *We* had one square of each color and that meant there were three moves. So *we* were trying to find a relationship between one and the three and there was a two and that took so many moves and then there was three and then there was four and that took so many moves. *We* were trying to find the relationships, a formula that will work for all the data. (PSS1M)

The sense of team between Annie, Courtney and Meredith is obvious in the "we" language and ownership of the solution process. What concerned me as I looked more carefully at these conversations is the "we" response to "you" questions. I was specifically asking each participant to talk about her own problem solving process, and yet she relayed the group process. "Are they the same?" I wondered. Courtney explained, right after it's [over], I really can't remember who brought up what" (PSS2C). Therein lies my own tension. I want students to feel a part of a group at a participatory level of consciousness but not at the expense of a loss of their individuality and individual mathematical processes.

In the following sections, I discuss binary tensions that stemmed from a desire for autonomous mathematical thinking and ownership of a solution within a cooperative group. I then offer two strategies the participants employed in order to take care of themselves and their mathematics, while still feeling connected to and benefiting from the cooperative environment.

Owning the Mathematics

The primary area in which the participants discussed autonomy was with respect to ownership of a solution. For example, in PSS1 Annie was less verbally engaged than Courtney (who was in control of the game board) and Meredith (who assumed the responsibility of keeping a record of emerging patterns). As discussed in the section on roles, Annie was able to observe and analyze the implemented strategies as well as being able to disengage from the problem. Because Annie was so reserved, it was difficult for Courtney and Meredith to determine which of these actions she was doing. When Annie verbally offered an idea, it was usually acted on (see power section), but remember when Annie silently reached in and actually moved the pieces to demonstrate her ideas Courtney and Meredith were visibly frustrated. Annie remembered feeling exasperated when Courtney reversed her moves. Annie knew she saw the correct strategy and consequently felt her idea about moving the blue square was not valued. When the game was finally "won," Annie "felt like they got it by chance…. I realize that they did do it but at the same time… I felt like they didn't understand why it worked" (PSS1A). This led to Annie feeling unsatisfied with the eventual solution. Although she did solve the problem for herself (while the others were stuck), Annie explained that "because [she] kept trying to input something and they would take it back, [she] felt like [she] didn't contribute anything. They did all that, but they didn't see part of what they were doing" (PSS1A). It was not enough for Annie to have solved the problem through her own individual mathematical processes, she also needed the group members to recognize her input and value her solution.

During PSS3, Annie felt differently about her individual ideas. Annie developed a geometric representation of the nth and $(n + 1)$th polygonal numbers—a different approach than that of Courtney or Meredith. While Annie's idea was not taken up or even completely understood by her teammates, they were interested in learning about and comparing their own representations to hers. Annie felt that the freedom to work separately, as they worked together, allowed her to feel better about her own mathematical contributions to the group. She compared this experience to that of PSS1:

> Here [PSS3] we were all involved with the problem. We all had access to the materials. We all did our own thing and contributed. Where as with the problem we did the first time, it was very limited on who could access what was happening and I think that's where you lose the autonomy. (PSS3GA)

In her preliminary interview, Annie talked about the tension around the way she preferred to do mathematics. She said,

> I really like having the personal interactions when you're doing group work… but when I come up with my own ideas, I really understand it a lot better. Then I will go to a group and see what they think or if they can make my ideas better. But I really need that personal time, too.

For Annie, the structures of PSS2 (where she had full access to the cards) and of PSS3 (where she had access to her own materials) were conducive to her desire to both create a solution herself and feel an integral part of a group.

Meredith was the participant least likely to be worried about what her teammates thought about her mathematical ideas. It was important to her mathematical learning that she was able to talk freely about her ideas. At a reflexive level of consciousness, Meredith explained previously explained the significance of this aspect to her learning:

> I like discussion a lot and… learn a lot better when I talk out loud. Even if I'm at home doing my math homework by myself, I read out loud or I think my random thoughts that make absolutely no sense to anybody else who is right around me. But if I speak out loud while I am going through it, I can dig deeper into it and figure it out faster and usually more accurately, too. Because I'm just thinking more. I don't know if it's the auditory hearing it or hearing myself dig through it, but it helps. (Preliminary Interview)

In PSS1, when Meredith was the recorder, she felt a serious obligation to her teammates to get every idea written down correctly. This inhibited her from talking aloud about the strategies. In PSS2, Meredith spent time investigating her own idea (modular arithmetic) and later realized that Annie and Courtney were not interested. She felt their disinterest might have been partly because she presented the idea (talking through it) while she was still unsure about the approach herself. After this point in PSS2, most of the conversation Meredith engaged in was about getting caught up, and she presented no new ideas of her own for the rest of the session. As she talked about the difference in solving a problem by yourself and solving it in a group, it is evident how important verbal interaction is to her ownership of a group solution.

> It really is just different, you know. When you conquer a problem by yourself, you really can pat yourself on the back because you take full credit when you get it correct. But in a group setting, you are building on other peoples' ideas as well. Well usually you are—if there is could communication of course. But it's kind of like you are empowered by a different way, different means. I don't know it's like you are becoming, all your brains, putting them together, becoming one big head and trying to solve it that way. So, I don't know what's a better feeling. (PSS3M)

Meredith claimed she felt the most ownership of the mathematical solution in PSS3 when *everyone* was talking and sharing ideas. In balancing the reflexive and participatory levels of consciousness, autonomy was strongest for Meredith at a level of conscious participation.

Being an autonomous mathematical problem solver was also important to Courtney, however, Courtney felt she was the kind of person who would be least likely to feel a loss of autonomy in a cooperative group. At a reflexive level of consciousness, Courtney explained, "I don't think there is [that loss] with myself because I think… if [someone] brings up something that I don't get, I just dismiss it. You know, because I'm trying to concentrate on what I'm thinking about" (PSS2C). She was, however, aware that other people might not have the confidence in their own ideas and thereby

> might get distracted easier. It would definitely be true that the self-thinking they are having would be disrupted by other people's thoughts and ideas…. Another

> person might be easily persuaded by someone else. And when they hear another idea, they want to go that way. And then they'll hear another idea and they'll want to go that way. (PSS3GC)

Courtney acknowledged that because she sat in the middle seat for all three PSSs and had direct access to materials and the representations of her teammates, she never felt disconnected from the problem solving processes. If Courtney thought Annie or Meredith perceived her as bossy or at an unfair advantage due to her proximity to the materials, she felt uncomfortable and was less willing to be in a leadership position.

At a participatory level of consciousness, Courtney's confidence often led her to feel a responsibility to her group members. Her awareness that others were not as mathematically involved or did not understand an idea, prompted her to help them. For example in PSS2, Meredith had a difficult time understanding that the solution to the problem rested on the fact that perfect square numbers have an odd number of integer factors—remember she had worked on a modular arithmetic approach while this idea was initially developed by Annie and Courtney. Aware that she had not given much credence to Meredith's ideas, Courtney felt she needed to take the time to catch Meredith up. Courtney explained, "I think she wasn't really sure what the factors had to do with it being opened and closed but then I said the thing about, 'It's going to get flipped…. If it gets flipped five times it is going to end up opening and if it gets flipped four times, it is going to end up closed' and then she said, 'Oh, okay'" (PSS2C). Meredith told me later that she did not really understand, but that she "might have been saying, 'okay' just to agree with her, to appease her" (PSS2M). It is interesting that what looked like helping someone else on the participatory level was at times, for Courtney, a way of making herself feel connected and needed. Perhaps this is why Courtney was so ready to accept that Meredith "got it" even when, in fact, she was still confused.

When I asked Courtney, "What's more satisfying, to work a problem solve a problem correctly by yourself or with a group?" (PSS3L), Courtney noted that there is a give and take. Although you might not be able to give yourself all the credit for a solution if you work in a group, the mathematical solution you do create could be much richer. She explained:

> I think I was satisfied doing it this way [PSS3] because we saw it a lot of different ways [to represent the problem]. If I had done it by myself, I might not have seen it in that many different ways. So in that way, I would be more satisfied in a group. *But* it is also very satisfying to figure things out on your own and to have the satisfaction that you did it on your own. It is our ego. It's an 'I figured it out' kind of thing. And in a group situation it wasn't really only you that figured it out it was a joint effort. *But* at the same time you usually get better answers when it is a joint effort. (PSS3C)

Courtney went on to explain that, for her, the sense of ownership in the "better answer" of the group solution is stronger when she felt that she was a definite contributor to the mathematical process. Conversely, Courtney also felt there was the potential for her confidence in her own mathematical thinking to hinder her from seeing those "better answers." She explained:

> I think I don't let people's ideas stray my way of thinking. And last night [PSS3] it was kind of bad because I was still stuck on this $2n+3$, $2n+3$. When really it wasn't the answer we were looking for and it took awhile for me to maybe loosen my autonomy and really see what Annie was saying about how I was wrong.

Courtney's struggle with the ownership of her mathematics was complicated. She believed it is important to be confident while, at the same time, not appearing too confident to her group members.

What troubled me about Courtney's sense of mathematical ownership and how that should be fostered in a cooperative group, was somehow she has come to believe "the actual satisfaction of, 'I did it,' might not be as high in a group. But that's not really something you should see as that important, anyway" (PSS3C). Individual satisfaction from solving a problem in a group should be high and participants should be encouraged to desire this type of autonomy. Staying primarily in a participatory level of consciousness led the participants to sacrifice their own mathematical ideas in order to be polite. Similarly, the participants struggled with the notion of maintaining mathematical individuality in CMPS. The search for autonomy in a situation where "you were supposed to be working together," (PSS1A) was both tension filled and rewarding.

Searching for Autonomy

Foucault gave a great deal of attention in his historical writings to the ways in which people constitute and cultivate themselves within a variety of situations, cultures, traditions, and power structures. Foucault (1986) argued "cultivation of the self" is rooted in the fact that one must "take care of oneself" and characterized the ways in which one does this as "*technologies of the self*." Inherent in finding ways to do mathematics with others, while still maintaining a sense of self, is a binary tension each participant readily identified as she watched and reflected on herself operating in a small group. As the interviews progressed, Annie, Meredith, and Courtney each became increasingly aware (at a reflexive level of consciousness) of the importance of cultivating herself during CMPS. As this awareness increased, the participants developed and exercised a variety of technologies of the self. In the following section, I discuss two of the strategies the participants employed in order to take care of themselves and their mathematics, while still feeling connected to and benefiting from the cooperative environment.

Validation

As we reflected on the PSSs, the participants discovered cooperative learning lends itself to the development of a unique strategy they used on the quest for the gratification that accompanies solving a problem. They found that seeking (and receiving) validation from group members could encourage confidence in individual mathematics not just at the end but more importantly, during a PSS. Meredith, who liked to talk things out in order to make sense of her own ideas, found feedback from her teammates invaluable. She explained:

> And so if there is somebody else, like a group, to verify my thoughts that I just [said to] try and think out the problem, it helps me because they are like, "Yeah,

> that's right." Now, let's move on to a new step, a new process because we got that one down. So, it kind of reassures me and I can move on in the problem…. They were there to help me. They were there verifying my thoughts, my conclusions.

Annie said that even when you are in your "own little world" solving the problem, she found it helpful to compare ideas "to see if you are right…. You still want confirmation." When Annie reached in to demonstrate her approach to the game in PSS1, remember her moves were "un-done" by Courtney. Annie explained how this affected her confidence:

> [Validation] is definitely a part of cooperative learning. It's a big part of why, for the first activity that we did, I wasn't a part of the group—basically because they felt that my idea was wrong. They were so stuck on what they were doing so I was like, "Well I'm not going to say it if it's wrong." But if you feel like you're right, you are going to say something and you want confirmation from your peers. That's really an important part of it I think. (PSS3A)

Courtney also expressed that this type of validation builds confidence. As pointed to in the previous section, however, validation for Courtney was bi-directional. Of course, it felt reassuring when her group members liked or used her idea, but Courtney also felt a deeper ownership of her mathematics when she was able to share it with or help her group. Courtney felt personally validated when she used her mathematical knowledge to take care of her group.

Although it might seem that autonomy is found primarily in the reflexive level of consciousness, the participants used validation from the group as a way to increase a sense of individual mathematical confidence. Providing validation to another group member allowed a participant to feel confident in her ability to both recognize an appropriate solution and feel she helped that group member. This strategy is a clear example how a holistic level of conscious participation—the simultaneous awareness of both individuality and belonging—eased the tension found in the question, "Whose solution is it?"

Time for Reflection

When we watched the tapes of all three PSSs, the participants and I identified several times when it appeared that a group member had "tuned-out." These were instances when someone leaned back in her chair and looked around the room, doodled on her paper, looked at me for guidance, was not a part of a discussion, and so on. Annie identified several reasons a group member might do such things. She said, "Either they get tired, they get frustrated, they are just listening, or they understand it" (PSS2A). All three participants said this tuning out is a normal part of cooperative learning and should actually be encouraged. They created the concept and the term "brain break" as this time to move from a participatory level of consciousness when you are "taking other peoples' ideas and thinking about what they are thinking" (PSS2GM) to a reflexive level of consciousness when you are "trying to think about it for yourself" ((PSS2GM). Meredith summed it up for all of us as she described and emphasized the importance of a "brain break."

> When we just had silence for a little bit. We are all kind of working by ourselves, independently. Having that time to sit back and think and look over the problem; to just let yourself have your own reasoning about what's going on; to really find where you're lost, what you don't understand and what you're up to; and then where you need more help from someone else—to build on the ideas that you already have. It is a time to kind of organize everything and get everything caught up. When you are trying to do all these things and get everybody else's ideas, I could definitely see where it could be overwhelming–too much information, brain overload. I think it's great if you can try and say, "Okay y'all, everybody just stop talking… long enough to just look at what you have and think about where you are at. I think it would be very helpful to do that in the classroom setting. It was helpful for me.

As resolutely and clearly as the participants said, "No, assigned roles!" they also determined that time for reflection was a crucial element in effective CMPS. They wondered why the concept of a “brain break” is rarely implemented in classrooms during cooperative learning. After all, applying this strategy helped the participants work at a level of conscious participation. Achieving the balance and the freedom to move from group to individual and back to group enhanced the sense of autonomy and mathematical ownership for each participant. The enriched mathematical experience and solutions in PSS3 convinced the participants that such a simple method—taking short breaks—can make such a difference in a student’s sense of self.

There were many questions throughout our conversations with respect to *why* cooperative learning is implemented in certain ways. The participants asked questions such as, “Who decided that one set of manipulatives, assigned roles, and a shared answer are the way to go with cooperative learning?” They agreed that instead of “telling [students] exactly what they should be doing” (PSS3C), teachers should “let students analyze what things helped [and hindered] them” (PSS3M). I was left with the question of, “How does a teacher structure or de-structure cooperative learning in the mathematics classroom to foster enhanced mathematical activity?” The following chapter is intended to, based on this study, address such a question and theorize possible answers.

CHAPTER SIX: OFFERING CONCLUSIONS AND IMPLICATIONS

Summary

With all of its history and traditions, cooperative learning is a firmly established pedagogy present in both classrooms and curricula. A significant body of research and literature claims the implementation of cooperative learning in mathematics classrooms results in increased achievement, motivation, and social skills among students. Influenced by this tradition and research, I incorporated cooperative learning into my secondary mathematics classrooms for a decade. Closely following the structure and methods proposed by Johnson and Johnson (e.g., 1990), I observed the positive results among my students reported in research studies and discussed in literature. With my students, the most striking result was the increased enthusiasm for learning mathematics with their peers. Students often and excitedly spoke of this phenomenon to each other and to me.

It was not until I led workshops for fellow teachers that I began to question assumptions I had made about cooperative learning. As teachers implemented procedures discussed in our workshops into their own classrooms, they reported a variety of reactions and results. Frustrated with their perceived lack of success with cooperative learning, these teachers began asking difficult questions—What exactly is cooperative learning? Why does it work? How does it work? Why doesn't it work for me? Disconcerted by my lack of answers to questions so fundamental to cooperative learning, I began my search for a different perspective and new understanding of the pedagogy I so zealously advocated.

Wrestling to find answers, I returned to the research studies that touted the benefits of cooperative learning. In general, the research on cooperative learning in mathematics classrooms fell into one of three categories: outcome based research on the effects of cooperative learning on students, interviews and surveys on students' opinions of cooperative learning experiences, or observational studies about cognitive processes of students during cooperative learning activities. A poststructural critique framed my review of this familiar literature with questions that challenged my beliefs and assumptions about the results, methodology, and implications of previous cooperative learning research. Questions such as: Was I so attached to positive results of research that I previously failed to notice the absence of a common definition for cooperative learning, thus making it difficult to synthesize results? Was I so sure that my students' enthusiastic emotional responses to cooperative learning reflected increased mathematical thinking and learning? Was there an "unwillingness [on my part] to read and think about the theories that … critiqued [my] fondest attachments and… the effects on real people of whatever system of meaning [my] attachments produce"(St. Pierre, 2000, p. 500)? propelled the direction of this study.

As a result of this poststructural critique, I became aware that although researchers carefully defined cooperative learning by a set of procedures or provide detailed descriptions of observed cognitive processes, students themselves were rarely asked to define or explain the cooperative learning experience in more than affective terms. Students have been tested, observed, videotaped, and analyzed, yet their voices seem somehow missing in the literature. It is important to understand students' social, emotional and mathematical experiences of cooperative problem solving team from students' perspectives. Providing students an opportunity to engage in their own poststructural critique of cooperative learning promotes a more synergistic view of their experience. Thus, this inquiry into how the experience of cooperative, mathematical problem solving effects the mathematical activity of individuals within a group was framed by the following research questions:

1. How do students engage in and experience cooperative, mathematical problem solving?
2. What binary tensions are present or emerge within cooperative, mathematical problem solving?
3. How are these tensions related to students' individual mathematical activity?

The research design of this qualitative study with three female college students reflected an interpretive, constructivist paradigm. Data collection centered around three videotaped problem solving sessions. As a group, the participants met once a week in order to investigate a mathematical problem. Each problem solving session was immediately followed by a group interview in which the participants discussed their mathematics, the roles and effects of group members, ways in which each participant felt helped or hindered in her mathematical thinking, group problem solving strategies, etc. The following day each student participated in an individual, ninety-minute interview in which she and I viewed the videotape together pausing often to discuss specific instances pointed to by each participant and by me. Because the research design was emergent and analysis was ongoing, I transcribed and initially analyzed data between sessions. The participants also helped to analyze data as they discussed previous interviews both with me and with each other. Our combined observations and perspectives directed discussions and influenced interview protocols.

The next phase of data analysis occurred as I addressed my first research question. I used the data from the interview and videotape transcripts to represent the experiences of three PSSs in detail from the perspective of both the participants and myself. Reflexive analysis continued as I used the next two research questions and theoretical frameworks as tools to explore further both the raw data and the newly written data stories. Using Earley's (1997) theory of Levels of Consciousness, I first identified binary tensions (a perceived choice between the good of the group or the good of the self) identified and felt by the participants during cooperative, mathematical problem solving experiences. Examples of factors contributing to these tensions occurred within three general components of cooperative learning: the environment, the group members, and the individual. The size, number and access to manipulatives along with roles that group members took on were the aspects of the environment promoting or creating self/other tensions. Issues of being polite (or being selfish), sharing (or not sharing) ideas, resisting (or submitting to) perceived power were raised by the participants as potentially problematic aspects of working with group members and were often difficult to reconcile. As the participants reflected on and discussed the notion of individuality (the place of the self) within a cooperative group, they agreed that both a desire for individual ownership of the mathematics and a subsequent search for autonomy led to yet other binary tensions.

Once binary tensions were identified, I investigated ways in which these tensions affected the individual mathematical activity of the participants. As the participants articulated how specific tensions were effecting their mathematics, they individually and collectively found ways to change their cooperative learning practices to protect or enhance their individual mathematical thinking. By deconstructing (Derrida, 1997) the individual experience of cooperative learning, the participants began to rethink self/other binaries tensions at a level of conscious participation. As they became aware of both levels of participation (the individual and group activity), the participants expressed and demonstrated a sense of freedom to move between these while simultaneously being aware of the other. It was this recognition and freedom that provided a stronger sense of self along with a stronger sense of community as the participants investigated mathematics together.

Conclusions

Traditional cooperative learning literature and suggested pedagogical strategies therein contribute to, and in some cases create, self/other binary tensions that limit individual mathematical activity. Thus, the need to continually question and deconstruct the traditions and structures of cooperative learning is vital. Deconstruction, however, is "not about tearing down but about rebuilding; it is not about pointing out an error but about looking at how a structure has been constructed… and what it produces" (St. Pierre, 2000, p. 480). In order to illustrate both the limits and the possibilities cooperative learning creates, I organized and presented the conclusions of this study within the well-known, researched, implemented and commonly accepted structures developed by Johnson and Johnson. This traditional conceptualization of cooperative learning is reflected in the five previously noted conditions Johnson and Johnson (1990) claimed must be present for effective cooperative learning in the mathematics classroom:

- Teachers must clearly structure positive interdependence within each student learning group.
- Students must engage in promotive (face-to-face) interaction while completing math assignments.
- Teachers must ensure that all students are individually accountable to complete math assignments and promote the learning of their groupmates.
- Students must learn and frequently use required interpersonal and small-group skills.
- Teachers must ensure that the learning groups engage in periodic group processing. (pp. 105-106)

I based the ensuing discussion on one of the central principles of Derrida's deconstructive methodology. In order to rethink traditional structures, identify and "examine a hierarchical binary opposition… in which one term is privileged over the other… and [then] reverse the binary opposition by re-privileging the other" (Graves, 1998, 2nd para). As illustrated in this study, I believe the group has become the privileged term within cooperative learning traditions. Thus, rather than refuting Johnson and Johnson's work, I discussed each condition with respect to its tendencies to promote self/other binary tensions and to the possibilities of re-privileging the individual within a group.

Positive Interdependence

Johnson and Johnson claimed positive interdependence "exists when students perceive that they are linked with group mates in such a way that they cannot succeed unless their group mates do (and vice versa) and/or that they must coordinate their efforts with the efforts of their group mates to complete a task" (1994b, 2nd section). Popular strategies used by mathematics teachers in order to promote this type of interdependence are assigning and/or creating specific roles for students, providing one set of manipulatives or materials, and creating the goal of one joint, agreed on product. Although these strategies can be helpful tools to encourage students to work together, as my participants voiced, they can also be tools that diminish a student's autonomy within a group. I suggest that the "sink/swim" motto for groups can foster an individual/group tension that is difficult for students to resolve. When an environment is clearly structured to promote group results and individual activity is performed for supporting the group goals, there simultaneously exists the danger of fostering unhealthy, dependent relationships. When a student perceives her responsibility to the group as more important than her responsibility to herself, individual mathematical activity is compromised and devalued.

The philosophy of Johnson and Johnson's, "We sink, we swim together " motto can be rethought to mean the desired product/goal is meaningful individual mathematical activity fostered by group membership rather than the reverse. Positive interdependence does not necessarily need to be facilitated by external environmental structures but by a desire for group members to value the mathematics of others. As students discuss multiple representations or ways of thinking about a problem, the mathematical experiences of all the group members are enhanced. The mathematics of individuals is now valued not for what it can contribute to a single group goal/output, but what it can contribute to the learning of others. For example, when participants developed and shared different representations of a problem they made connections among the representations. This type of discussion fostered an appreciation for the mathematical activity of others as well as deeper understanding of their own solution.

Promotive (face-to-face) Interaction

"Promotive interaction may be defined as individuals encouraging and facilitating each other's efforts to achieve, complete tasks, and produce in order to reach the group's goals" (Johnson & Johnson, 1994b, 3rd section). Teachers and students generally interpret social interaction that promotes learning to mean that conversation is a required characteristic of effective cooperative groups. A student who does not engage in conversation is often referred to as a "free-rider" or antisocial or is accused of not supporting and contributing to the group. The expectation of interaction creates a social pressure on group members to their share own ideas and suggestions as well as listen to those of others. Hence, an obvious self/other binary tension emerges when individuals believe that their individual mathematical activity is valued only when it is shared and possibly only when it is taken up by the group. Because students do not feel free to reflect quietly on their own mathematics or to dismiss the ideas of others without polite consideration, the expectation of interaction can significantly hinder individual mathematical activity.

This philosophy of promotive interaction parallels Vygotsky's (1978) theory that knowledge formation occurs when it is first experienced on the social (intermental) plane and then on the psychological (intramental) plane. The results of this study suggest Johnson and Johnson's required increased promotive social interaction could promote decreased activity on the intramental plane. For example, a student with an initial idea about an investigation listens to the ideas of others (intermental plane) while simultaneously trying to mentally retain her initial idea (intramental plane). It is difficult for students to balance the verbal exchange of ideas with their individual cognitive processes when the explicit or perceived purpose of interaction is the achievement of a group goal. In the attempt to balance the two, neither mathematical idea was given an opportunity to develop.

Reverse the "promotive interaction" binary by considering individual activity as privileged, intramental activity as a necessary precursor to learning, and social interaction as a promoter of individual mathematical activity. Now the primary purpose of interaction is not to come to a group consensus or develop group mathematics, but to foster and enhance individual mathematical thinking. New questions about interaction arise, such as, "When and why do students talk? When and what ideas are shared? What do individuals do with others' ideas?" In the present study, the participants were encouraged to privilege their own hindered mathematical activity as they reflected on the group PSSs. In reversing this binary the participants felt hindered by, they realized the value of the mathematical activity on both planes and the importance of creating a cooperative environment in which promotes free movement between the planes. "Face-to-face interaction" that promotes meaningful mathematical activity lies in the freedom to value both individual autonomy, as well as the sense of belonging to a group—promotive interaction rethought at the level of conscious participation.

Individual Accountability

Johnson and Johnson maintained teachers must "assess how much effort each member contributes to the group's work" in order to "ensure that each student is individually accountable to do his or her fair share of the group's work" (1994b, 4th section). Claiming when teachers observe groups and ask students to evaluate each other's contributions, this system of individual accountability encourages positive interdependence by discouraging group members to seek what they call a "free ride." Within the individual/group binary promoted by this condition, what students contribute to a group is privileged over what a group contributes to an individual. Regardless of how meaningful this contribution is to their own mathematical activity, students feel pressured to contribute an idea or the completion of a task to their group's activity. The assessment of individual accountability sends a message to students that mathematical activity supporting the group is valued more than activity enhancing individual learning. We must also find ways to determine and to help students determine the ways in which being a part of a group contributes to the work of the individual. Asking group members to reflect on the mathematics of themselves and of each other throughout a cooperative learning activity sends a more integrated message about the value of both individual and group mathematics, and how the two support one another.

In order to balance their message of individual accountability, Johnson and Johnson also explained, "Students must [first] learn it together and then perform it alone" (1994b, 4th section). That bluntly stated tension is reflected in the common practice of following-up group assignments with individual homework, quizzes, or tests. Students know that somewhere in the "doing their fair share" and "making sure the group meets its goal" in the future they will be held accountable for demonstrating a learned a mathematical skill or concept. This condition often promotes tensions when students perceive group membership as hindering their ability or speed to learn content. Students say things like, "I had to slow down to teach her and I didn't learn as a much" or "He wouldn't help me learn it so I'm going to fail the quiz." Both perspectives on individual learning are problematic when students perceive their learning or success to be conditional upon someone else's. Thinking simultaneously about these two phases of cooperative learning can ease this tension. Rather than just reversing the binary (individual work and then group work), students should be given the freedom to move within these phases in ways that support individual mathematics. As shown in this study, students can negotiate the activity of cooperative, mathematical problem solving in ways that enhance the individual mathematics of each group member. Any condition about accountability should communicate a student the importance of being accountable for to herself as she simultaneously benefits from and gives to a cooperative environment.

Interpersonal and Small-group Skills.

Johnson and Johnson (1994b) claimed that the "more socially skillful students are and the more attention teachers pay to teaching and rewarding the use of social skills," the more achievement teachers can expect within groups (5th section). The social skills needed to achieve mutual goals include trust, clear communication, peaceful conflict resolution, sharing, and accepting one another. In order for these skills to be present and functioning in the mathematics classroom, Johnson and Johnson said teachers must not only hold individuals accountable for *what* they contribute to their group, but also for *how* they contribute and interact. They pointed to the following model from research as a method for teaching students social skills and encouraging the frequent use of these skills:

> Students were trained weekly in four social skills…. Each member of a cooperative group was given two bonus points toward [a] quiz grade if all group members were observed by the teacher to demonstrate three out of four cooperative skills. (1994b, 5th section)

Cooperative learning provides a structured environment for teaching students to be nice, include and value everyone, be polite, and take care of each other. Although the ability to work with other people is an important characteristic to develop with students, the self/other binary this structure creates not only leads to a decreased focus on individual learning but also on mathematics altogether. There is an important distinction between using cooperation to foster mathematics and using mathematics as a context to foster cooperation. Too often, the latter is the actualization of Johnson and Johnson's interpersonal condition in classrooms. The findings of this study reveal that if a student perceives manners rather than mathematics as privileged in a cooperative learning environment, she tends to do what is socially expected rather than what furthers her own mathematics. This privileging leads to a number of problematic outcomes; such as, students abandoning their own ideas in order to be respectful of others, forfeiting their own reflection time in order to actively participate in group activity, or accepting one solution instead of many in order to arrive at one agreed on group solution.

In re-privileging the mathematics over manners, we need to redefine what it means to be respectful of others mathematical thinking. The participants in this study said that asking a group member to wait before sharing her idea while you finish your own thought should be seen as "polite." Students need time to process their own ideas as well as those of others. Multiple representations of the problem or the solutions support the mathematical activity of each member when they are given the time and space to reflect on and make sense of them—without feeling as if they are being rude. When students feel free to not listen or not agree or not share, while simultaneously appreciating the value of discourse within mathematical problem solving, the mathematical activity of all participants can be enhanced. Students feel more autonomous as mathematicians and more respectful of each other when "interpersonal skills" support their own mathematical thinking and their sense of community within a group.

Group Processing

Engaged in by all group members after a cooperative learning activity, “the purpose of group processing is to clarify and improve the effectiveness of the members in contributing to the collaborative efforts to achieve the group’s goals (Johnson & Johnson, 1994b, 6th section). To encourage effective group processing, teachers should provide students with an organized format in which to discuss “what member actions were helpful and not helpful in completing the group’s work and make decisions about what behaviors to continue or change.” The privileged side of the binary promoted within this condition is behavior that promotes the group goal rather than behavior that promotes mathematics. If the group goal was to solve a specific problem (for example, to create an algebraic representation of a set of data in order to predict future values), then group processing would center on questions like: “What did each group member contribute to finding this formula?” How could your group work together tomorrow to solve problems more effectively? What is one thing each individual could improve on?

If individual mathematical activity was privileged in group processing, questions would center on questions like “What was the first idea each participant had? How many different representations/ideas were discussed? How were the ideas connected? How did the group agree upon one representation? Why? How would each member proceed with or extend the problem?” As students reflect on their own mathematics and that of others, they will simultaneously, but not explicitly, reflect on the social and cooperative aspects of the problem solving activity. Because learning mathematics is a social, cognitive, emotional and physical experience, we must focus on all these aspects and encourage students to do the same as they engage in group processing. Privileging the mathematical content rather than the group processes provides students the opportunity to talk about the cooperative, mathematical problem solving experience holistically.

The methodology of this study modeled group processing focused on the mathematical experiences of each group member. As one participant explained, “By analyzing the tape and watching ourselves and figuring out what we were thinking, how [cooperative learning] was helping us and how [cooperative learning] might have been hindering us… [we] can take the good aspects and the bad and try and make it more affective" (PSS3M). The participants acted on their observations by restructuring the cooperative learning environment to fit their own learning styles and preferences. "Think about how you learn best and do that and don't worry about if someone else isn't doing it. [Work how] you learn best, but still work together (PSS3C)," was the philosophy of cooperative learning the participants arrived at through group processing. Group processing should provide students an opportunity to investigate the ways in which cooperative problem solving effects the experiences, processes, and mathematics of both the individual and the group.

Implications

Historically, a major argument for the use of cooperative learning is that it facilitates certain aspects of learning more effectively than individual learning. The results of this study reinforced the benefits of cooperative, mathematical problem solving while simultaneously challenging us to rethink the traditional structures of cooperative learning from the perspective of students' individual mathematical activities. As I answered my research questions about how students experience and engage in cooperative, mathematical problem solving, I developed a theoretical, as well as pedagogical, orientation to cooperative, mathematical problem solving. Within this orientation, I call for a sense of community in classrooms, but not at the expense of individual mathematical activity. I have illustrated the need for students to have opportunities to figure out for themselves how best to use a group to foster their own mathematical activity. Through theory and practice, teachers and researchers strive to improve the mathematical learning of all students. With the intent of adding to and participating in this important effort, I present implications for classroom practice, teacher education and educational research.

Implications for Classroom Practice

Although the participants in this study adapted and/or adopted new cooperative learning strategies to enhance their mathematics, they also reflected, as pre-service teachers, on how they might implement cooperative learning in their future classrooms. Each of these young women is committed to providing opportunities for the students they will one day teach to "get excited about mathematics" (Meredith, Initial Interview). The participants and I all agreed that cooperative learning enriches students' mathematical learning and the classroom community. Annie, however, reminded us in our final group interview that cooperative learning is "just like life—it's what you make it." When I followed her insightful remark with the question, "So how can teachers can make it better? How can we help our students?" the participants quickly responded:

A: Give them big things to play with!

M: Keep the groups small!

C: Don't tell them specific roles!

M: Let kids reflect on their group activity!

C: Students should feel free to question each other's ideas!

M: Make sure there is enough paper to where everyone has paper and feels free to get the paper! (PSS3G)

Chapters Four and Five are the written representation of how their past experiences and participation in this study brought them to these beliefs about cooperative learning. A common thread running through the implications of this study is the philosophy that feeling free makes you more a part of a group than feeling forced to behave in certain ways. In this section I offer suggestions for how to re-structure the traditions of cooperative learning in ways that promote students' freedom to move from individual to group activity and thus enhance the mathematics of the individuals in a cooperative group.

"…that this nation, under God, shall have a new birth of freedom and that government of the people, by the people, for the people, shall not perish from the earth."
(Abraham Lincoln, 1863, Gettysburg Address)

Freedom means choices. In all aspects of cooperative learning students[5] should be given choices and encouraged to reflect on and discuss those choices by considering the effect those choices might have on themselves and on their group. Encouraging students to "govern" themselves also means trusting their choices along with their right and ability to rethink those choices. Just as Foucault (1980) argued rebellion is inevitable when power gets stuck, I believe students rebel when choices are limited. So rather than using structure to limit the choices of students, we should create an environment that opens-up new possibilities and benefits for cooperative, mathematical problem solving.

With respect to the logistical aspects of CMPS, students need to have free access both visually and tactually to all provided materials and resources. Manipulatives should be big and/or available to all students in the group. Providing a variety of materials gives students choices about how to represent their mathematical ideas and investigations. Although some students in the group may choose to work with tangible manipulative such as colored discs, others may prefer drawing on paper, and others may elect to visualize mentally. Courtney said we should tell mathematics teachers that:

> it's okay for everyone to working on their own idea—this can still be cooperative learning. Because you can all talk about it, defend your position, and everyone can learn…. I think that is the key… you can write things down separately and still be working together. (PSS3C)

She went on to suggest, "I think [teachers] should say [to students], 'feel free for everybody to write things down if you want to, if that's the way you feel are learning it the best'" (PSS3C).

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[5] As with kids, the number and kinds of choices should vary by age. Because my participants were studying to be secondary teachers, the following suggested classroom practices focus on secondary classrooms.

Although roles can be helpful in delineating tasks, roles can also significantly hinder the mathematical activity of all students in the group. The recorder may not have the opportunity to think about the investigation as she works to keep track of the thoughts of other group members. A student with the manipulatives may have difficulty listening to a number of suggestions and investigating their own ideas. A leader may feel so obligated to make sure the group runs smoothly and everyone has a chance to contribute that her mathematics becomes secondary. Again, students should have choices about when and what roles to have in a specific mathematical investigation. A teacher should discuss the use roles as well as the pros and cons of assigning them to group members. Students should feel free to rotate roles, share roles, use them for only a short amount of time, or not use them at all. The point is, members of a group should be the ones who make those decisions based on what is happening with their investigation at any given time. Although one group may divide tasks among themselves, another group may decide that one person will perform a task while the others observe, and another group may opt for all members to actively do each part of an investigation. Whether it is manipulatives, roles, or other logistical considerations, students come to class with preconceived ideas about the traditional structure of cooperative learning and how they are expected to behave within that structure. Thus, it is important for mathematics teachers to discuss a variety options, such as these, with students and encourage them to make choices to promote and enhance the ways they engage in mathematical activity.

Ask students what it means to work cooperatively and you will hear answers steeped in the traditional language and structures: “Wait your turn.” “Listen to everyone’s’ ideas.” “Don’t make fun of anyone’s suggestions.” “Share.” “Reach an agreement everyone feels good about.” “Be nice.” “Compliment each other.” “Don’t leave anyone out.” The expectation to behave in these socially accepted ways within groups can tear down, rather than build up, cooperative relationships among students. Instead of walking around with a clipboard, checking off the number of positive behaviors (like the ones noted above), teachers should focus time and energy on conversations with students about the issue of "manners." What would it mean for students to have choices about how to be respectful of each other? Does freedom from the socially accepted rules of polite behavior mean we are now rude? Framed in the context of mathematical thinking, students may decide that not listening right away is not necessarily rude behavior. It is actually more polite to listen to someone else's idea when you can really hear them—when you are ready—than when you are just pretending to listen and in the meantime forgetting your own idea. Students can begin to resent each other, and group work, when they feel forced to relate to each other in ways that take away from their learning rather than add to it. So, instead of talking about what is polite and/or rude, give students to opportunity to rethink the concept of respectful behavior in terms of supporting mathematics.

When my participants gave themselves the freedom to say to each other, "That really messed up my mathematics when you jumped in and grabbed the manipulatives," they learned that having your own materials may not be selfish and that sharing could mean exchanging ideas rather than stuff. Along with the value of discourse, quiet time should also be an integral part of cooperative, mathematical problem solving. When participants expressed to each other the need to be able to say, "Hush! I'm on a roll with my idea and your talking is going to break my concentration! I'll want to listen to your idea but later," they developed the concept of a "brain break"—a time when everyone in the group gets a chance to gather their thoughts and process the mathematics of others. This should occur periodically throughout a problem solving activity. Groups should decide amongst themselves if they want to have scheduled times for brain breaks (like every five minutes) or if an individual students should be able to call a brain break when they feel one is needed.

Individuals should also feel free to take their own break from the group activity without feeling rude or snobbish. Often times when a student presents a strategy or direction for an investigation that is not taken-up by the other group members, the student no longer pursues that line of thinking. Teachers need to communicate to students that just because a group does not use an individual's idea does not mean she cannot pursue it or that her idea was not valuable. Brain breaks are a powerful way to provide students the time and space to explore and develop these ideas so that they may benefit both the individual and the group's mathematics. When mathematics is the motive for making choices during classroom activity, students view cooperative learning as a way to enrich mathematical activity rather than viewing mathematics as a tool to enrich cooperative skills. The distinction is important and students need to know their freedom to act "socially" is grounded in the importance of creating an environment conducive to meaningful mathematical activity.

In conclusion, students should always feel *free* in a group to have their own paper, own ideas, own manipulatives, own time to be quiet, own authority to make decisions within a group. As teachers, we need to offer students choices within the cooperative learning environment and trust them to work out those choices within their group. If there is one aspect of cooperative learning that all students should engage in it is group processing—talking about options, individual mathematical experiences, ways to support the mathematical activity of everyone in the group, interactions and feelings, etc. and it promotes a level of participatory consciousness where both the individual and the group are valued. This group processing empowers students as mathematicians and as citizens of a cooperative community. And when this happens, to borrow from the words of Lincoln, students' mathematics shall have a new birth of freedom when cooperative learning is of the students, by the students, and for the students.

Implications for Teacher Education

Teachers (both pre-service and in-service) gravitate toward the implementation of certain traditions and methods of instruction based on previous experience, university coursework, mentors, personal beliefs about learning and teaching, external professional influences, teaching styles, and philosophies of mathematics, to name a few. Cooperative learning was one of the primary pedagogical tools, classroom management approaches, and educational philosophies I spent years implementing and advocating. However, nothing in my formal teacher education challenged me to analyze critically the implementation of cooperative learning in my classroom. It was only when other teachers began to question me, as I led professional development sessions on cooperative learning, that I began to critically examine what I believed about cooperative learning, it's associated pedagogies, and subsequent student learning. I now realize that my growth as a mathematics teacher rests in an assertion that, because truth is subjective, multiple, and contextual, I must continually identify and question my taken-for-granted structures in place in my classrooms.

My experiences point to an important implication of this study for preservice teacher education—the effective use of cooperative learning is not just a pedagogical problem. The conditions for effective cooperative learning and supporting classroom management strategies have been presented to many undergraduate, prospective mathematics teachers. There are some who integrate cooperative learning into a lesson plan and abandon it because, "It didn't work" in a field experience. Because it sounds reform oriented and student-centered, other undergraduates leave universities as enthusiastic advocates, with an uncritical acceptance, of cooperative learning.

As teacher educators who deliver professional development workshops and graduate courses, we must be mindful to help teachers develop a disposition to question. We often hear statements regarding the reasons why, "This won't work in my classroom, with my administration, or in a real school." In response, we need to ask teachers questions that go beyond the above reasons, such as, "Why won't it work? Why do you think this? What are your experiences with this approach? Where, when and about what do you feel at odds with what I am saying?" Then we need to be willing to think critically about our own assumptions and about what we are tied to as teacher educators. For teacher educators, this means that we must view ourselves less as experts who are trying to show teachers a better way but more as co-investigators of pedagogies. We may bring knowledge or ideas to inservice and preservice teachers, but our job is also to lead a multi-perspective, critical inquiry.

Teacher education at any level should provide teachers with opportunities to not only learn about a variety of pedagogies but to also experience and critique them. During her last interview with me, as Courtney reflected on the lessons she learned by being a part of this study, she remarked, “If you knew what was going on then you could change something.” All the participants stated that once they became aware of how self/other tensions were impacting and even hindering their own mathematical activity, they changed the way they engaged with the group. This opportunity to tailor cooperative learning to enhance her own mathematical activity left each participant feeling more autonomous as both a learner and future teacher of mathematics. The type of questioning we should seek to foster in the teachers we work with and ourselves could reveal binary tensions such as those identified in this study. The opportunity to describe these binary tensions in detail and identify the possible effects they have on mathematical teaching and learning is powerful. However, it is the activity of reversing the binaries that encourages teachers to hypothesize ways in which rethink the binary so that what used feel separate and competitive is now viewed as simultaneous and complementary to mathematical learning. As we work to deconstruct binaries that may be promoted within a variety of educational theories and methods, we will also work to deconstruct the teacher/student binary tensions that hinder the professional growth of both the preservice and inservice teachers.

There is no one way that teacher education students should experience cooperative learning or the deconstruction of cooperative learning. Participating in cooperative learning in a variety of formats and debriefing the experiences, reading research and theoretical literature about cooperative learning, watching videotapes of cooperative learning episodes, reflecting on previous experiences with cooperative learning in mathematics classes, and other such strategies are suggested and available to teacher educators. Because teaching at any level begins with knowing the students, it is not possible to prescribe the manner in which teacher educators should help their students deconstruct cooperative learning. As with any educational tradition, we would do well to remember that “unless teacher [educators] are knowledgeable enough about their students to use that information when deciding on classroom actions, even the best-intended moves may not result in desired reactions” (Brown & Baird, 1993, p. 252).

Implications for Research Traditions

In the first issue of *Emergence: A Journal of Complexity Issues in Organizations and Management*, the editor, Michael Lissack (2003), claimed:

> “Leaders’ effectiveness lies in their ability to make activity meaningful for those they lead. They do this not by changing behavior, but by giving others a sense of understanding about what they are doing. In this sense lie the potential strengths of complexity as a management tool.” (p. 122)

As leaders in the field of mathematics education, researchers have the opportunity to help others make sense of the teaching and learning of mathematics and thereby make it more meaningful. Our research is about the interactions and actions of teachers, students, administrators, curricula, government, reform movements, school systems, and university programs. Our work is complex and at times difficult, but it is vital to the continual growth of all those involved in the teaching of mathematics. As we continue in our endeavor to conduct meaningful inquiries, we can look to complexity theories for new perspectives on and methods for research.

Complexity theories share the common belief that systems, whether in science, business, education or elsewhere, should be viewed holistically in relationship to the whole encompassing them and the parts of the system effecting them. By adopting a complex view of an individual and thus rejecting the humanist notion of a unified self, poststructural theorists in education see the self as multiple, shifting, and constantly being constructed by their current environment. The goal of some researchers is to analyze or describe single aspects of an individual or classroom by finding or creating structure around it (organizing or defining parts of a system), poststructural theories challenge researchers to continually examine how these complex structures and traditions operate and what they produce.

As an example of what it means to challenge research traditions, consider the participant/researcher binary. "If we are looking through post-structuralist eyes, the once clear cut lines of demarcation appear blurred" (Crotty, 1998, p.209). Blurring the line between the researcher/participant binary challenges us to re-interpret the role of participants in research. Deconstructing this binary and thinking about what it would mean to privilege the participant. The participant now directed parts of data collection and co-analyzed data at certain stages of the study. In the present study, like so many in educational research, the goal was understand experiences from the perspective of the individual engaged in it an. In providing this opportunity to students, I developed an emergent design around the use of video-recall sessions for reflections and interviews rather than at the end or just me watching the tape. As I saw things in the tape, I was able to ask why, follow up, or clarify. This ongoing participant feedback made the participants feel valued, reflect at a deeper level, understand my research questions which helped guide our conversations. By asking them to reflect on theory (e.g., power relations, second order models) and connect it with their own experiences helped me put theory in a context and helped them provide language and I listened to how they used it. All of these activities made the term participant take on new meaning.

Carrying a poststructural approach into data analysis by again using Bove's questions to define experience, I identified binary tensions of my participants' individual mathematical experiences. Using Derrida's methodology to deconstruct their experiences, the participants were able to change the cooperative learning experience and feel more autonomous, mathematically and personally. As we work to deconstruct binaries that exist within a variety of educational theories and methods, it is my belief that the methodology I developed would help us, as researchers, gain perspectives not found in existing research. By applying poststructural theories, I collected and analyzed a new kind of data about students' mathematical, social, and individual experiences, and gained a new understanding of cooperative learning—one far beyond what I expected.

Suggestions for Future Research

A good question is never answered. It is not a bolt to be tightened into place but a seed to be planted and to bear more seed toward the hope of greening the landscape of ideas.

John Anthony Ciardi (1916-86) American poet, critic

Researchers in the field of mathematics education have and continue make significant contributions to the body of cooperative learning research. We are just beginning to explore the direction of poststructural research as an alternative orientation for not only facilitating the asking of new questions, but also for opening-up and gaining new perspectives on previously researched issues. The results and implications of this study not only inform current understandings of the experiences of students engage in cooperative, mathematical problem solving, but also they also serve as a catalyst for future research. With a focus on the mathematics of the individual, suggested areas for continued include the design of cooperative learning for classroom practice, teacher education, and research orientations.

Classroom Practice

The participants noted that the nature of the mathematics in the problem solving sessions effected several aspects of their experiences. As Annie compared the problems she commented that the while the Locker Problem "was just follow the directions and see what kind of pattern you can find," the Square Game involved developing a strategy for winning the games. She claimed that it was easier to work together when there was more direction in a problem because, "Strategy is a very personal thing. It is very hard to explain strategy to someone when you are still formulating your own ideas (PSS2A). Thus, we need to consider the ways in which the *context* (e.g., discovery, practice, technology lab, problem solving, group project, jigsaw) and the *content* (e.g., numerical calculations, proof, visualization, algebra, geometry) of mathematical investigations effect the experiences of group members.

Future questions to guide research on mathematical cooperative learning could stem from adjusting or changing a variety of the group composition variables in this study. The participants in this study were relatively homogeneous with respect to background, success, and interest in mathematics. We need to further consider how the individual mathematical activity students is supported or hindered in heterogeneous versus homogeneous groups. For example, how would students with different genders, age levels, mathematical knowledge (e.g., different vocabulary, content knowledge, interest), dispositions to mathematics, and previous cooperative learning experiences work together? Because young children are still developing social norms, whereas high school and college students have some notion of social expectations, we need to consider how the individual mathematical activity of young children is effected by issues of power, politeness, and autonomy within cooperative groups? The complexity of relationships and variables within groups also needs to be considered as we research the ways in the members in an initial group formation versus an established group are effected by cooperative mathematical activities.

Another fruitful area of research should stem from the studying the effects of implementing the cooperative learning strategies recommended in this study. For example, the participants suggested that each group member should have her own manipulatives, be allowed and encouraged to take a "brain break," and not be assigned specific roles. How does adding or removing such conditions for cooperative problem solving impact individual mathematical activity? What are the implications if we rethink the binaries that emerge from traditional cooperative learning structures, such as group processing and joint products? What do classrooms look like in that context and what is the role of the teacher?

Teacher Education

Inservice teachers, in particular, often bring more current knowledge and experiences with the milieu in which they operate to professional development opportunities. Their perspectives and daily experiences can add depth to critical reflections on pedagogies such as cooperative learning. Thus, by listening to what teachers have to say about the experience of cooperative learning, teacher educators have the benefit of many "miniature action research studies" conducted with different sets of students in different contexts. Action research for teachers in the form of asking students to reflect on their cooperative experiences and connecting this information to the development of their own mathematical thinking could add more depth to this study. We need to investigate teachers' beliefs about and definitions of cooperative learning as well as their reasons for classroom implementation—mathematical and social.

Research on the ways in which cooperative learning is presented to, understood by and implemented by preservice teachers could occur through interviews, lesson planning, reflections on videos and readings, or field experiences observations.

> A lot of times teachers think that these [cooperative learning] experiences their students are having are great and these group works are just awesome and they're learning so much from each other. When really if you ask the students they [might say], 'I don't get anything out of it.'" (Courtney, PSS3G)

Preservice teachers need the opportunities to engage in action research by talking with students and peers about personal experiences with cooperative learning. Researchers should investigate the impact of such investigations on preservice teachers' understandings of cooperative learning with respect to mathematical learning.

Research Traditions

Awash with alternative pedagogical strategies, the mathematics education reform movement has influenced classroom practices in numerous ways. Influential groups such as researchers, university professors, NCTM, textbook authors and software developers promote approaches to mathematics teaching and learning touted to be holistic and conceptual in nature. For example, assessment should be formative by employing such strategies as open-ended test items, student portfolios, and journaling. Curricula should be problem-based so mathematical concepts emerge from carefully planned and selected activities. Technology, such as software and hand-held calculators, should be implemented across curricula in order to provide students a more dynamic and visual understanding. Manipulatives and hands-on approaches should encourage a more discovery-based approach to mathematical learning. Researchers should reinvestigate common classroom traditions from new perspectives and continually rethink their implementation with respect to mathematical learning. We need to be careful that such commonly accepted and encouraged pedagogical tools do not become taken-for granted traditions. Thus, researchers in the field of mathematics education should apply deconstructive methodologies to each of these pedagogical approaches and techniques as a way to understand the experience of this pedagogy from the students' perspective. This student perspective should then be connected to individual mathematical learning. Although replicated studies provide valuable information, research should also open-up new ways of thinking rather than marry us to tradition. Just as students should not feel trapped in classroom structures that hinder mathematical thinking, researchers should also be free to adapt or adopt new structures, methodologies, and orientations.

On a Personal Note

Once a journey is designed, equipped, and put into process; a new factor enters and takes over…. We find after years of struggle that we do not take a trip; a trip takes us. (John Steinbeck, from Travels with Charlie)

At the onset of the journey of the present study, I claimed "my goal in designing and letting this study unfold was to *unfocus* and look, with a new perspective, at students' cooperative learning experiences as they engaged in mathematical problem solving" (p.54). With this *unfocusing* came a perspective that both surprised and puzzled me. As a researcher, I was surprised to find that traditional aspects of cooperative problem solving have the potential to, and often do, hinder individual mathematical activity. As a mathematics teacher I was puzzled by the lack of attention I paid to this issue in my own classrooms and by the limited discussion of this issue in cooperative learning literature.

In light of my findings and theorizing, an obvious question to ask is, "In the future, how will I implement cooperative learning in my own classrooms?" The work of scholars such as Johnson and Johnson, Sharan and Sharan, Slavin, Webb, Davidson, Slavin and others has offered valuable cooperative learning techniques and approaches implemented by many teachers in a variety of classrooms. Thus, speaking metaphorically, the structure of cooperative learning should not be torn down. Rather, the walls containing group members need more doors and windows providing freedom for individuals to come and go, as well as gaze in and out. Small changes to a standing structure can significantly increase its efficacy.

While completing this dissertation, I taught mathematics courses to pre-service teachers. Students sat primarily in teams of four as they investigated and discussed a variety of problems and activities. On the surface, my classroom environment did not look much different than it had in previous years, but if you looked closer you would have noticed students ask for "brain breaks;" seen individuals reach for more materials or manipulatives when they wanted to tryout an idea; observed team members work silently before and after sharing ideas; and watched teams present a variety of solutions. As I encouraged the recognition and value of individual mathematical activity, while benefiting from group membership, I viewed students negotiate and recreate the traditions of cooperative learning. I believe this freedom enhanced individual mathematical activity, but I also still believe:

> If we, as teachers AND students, had more of a sense of how this thing called "learning together" happens, then we, as teachers AND students, would better plan with and use cooperative learning strategies in classrooms. (Sheehy, 2000, unpublished paper).

As a researcher, I know that research is never complete and findings are never irrefutable. Thus, "throughout [our] careers we must participate in an ongoing, collaborative process of reevaluation of, and liberation from, our taken-for-granted views" (Berlak & Berlak, 1987, p.170).

Personally, I have learned that nothing is ever entirely as it seems. There is always much to be learned by viewing a situation or individual with a different perspective or by listening to a different voice. The freedom to question is the foundation for new possibilities.

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APPENDIX A: ABBREVIATIONS USED

PSS	Problem solving session
CMPS	Cooperative, mathematical problem solving
A	Annie
C	Courtney
M	Meredith
L	Lisa (researcher)
G	Group Interview
PSS(no.)(Participant Initial)	Individual interview quote Example: PSS2A represents a quote from Annie's individual interview following the second problem solving session
PSS(no.)(Participant Initial)(G)	Group interview quote Example: PSS3G represents a quote from the Group Interview following the third problem solving session

APPENDIX B: PRELIMINARY INTERVIEW GUIDE

Part One: The Nature of Mathematics

- Remember the “what animal might be representative of mathematics” question that we answered in class last fall? Well, let’s talk about that one again
- From the following list, please choose the statement that you feel best describes mathematics…
  1. Mathematics is true because humans have carefully constructed it; its fabric is knit from its axioms as a sweater is knit from a length of yarn.
  2. Mathematics is true because it is nothing but logic and what is logical must be true.
  3. Mathematics is true because it is proved.
  4. Mathematics is true because it is beautiful or because it is coherent.
  5. Mathematics is true because it has been elicited in a way that reflects accurately the phenomena of the real world.
  6. Mathematics is true by agreement. It is true because we want it to be true, and whenever an offending instance is found, the mathematical community rises up, extirpates that instance and rearranges its thinking.
  7. Mathematics is true because it is an accurate expression of primal, intuitive knowledge.
  8. Truth is an idle notion to mathematics as to all else. (Howell & Bradley, 2001, p. 12)

Part Two: Learning Mathematics

- Tell me how you learn like to learn mathematics. For example, do you like procedures, concepts, theorems? Are you a visual learner? Do you like to read the texts? Talk about it? Practice? Etc…
- What is your least liked aspect of studying mathematics? Your most liked?

Part Three: Cooperative Learning Experiences and Perceptions

- What experiences have you had working in a group? In high school mathematics? In college?
- What is your least liked aspect of working in a group? Your most liked?
- In a mathematics classroom, what is a personal strength that you bring to a cooperative group? A weakness?
- Describe the ideal mathematical cooperative learning situation. What is your role? What role do the members of your group have? What is happening?

APPENDIX C: THE SQUARE GAME

Problem Solving Session One

Set up your colored squares as shown below. Be sure to leave one space between the two colors.

Your goal is to move the squares so that the yellow and blue squares have swapped positions.

The conditions under which you may move are

- The blue squares may only move to the right. You may move them one space to the right to fill an empty space or you may jump one square that is immediately to its right (providing that there is an empty space to jump into).
- The yellow squares may only move to the left. You may move them one space to the left to fill an empty space or you may jump one square that is immediately to its left (providing that there is an empty space to jump into).

Once the students are familiar with the game, present them with the following questions:

- What is the minimum # of moves it takes to win this game?
- What is the minimum # of moves to win if you start with only 2 pieces on either side?
- Can you predict how many moves it would take to win if you started with seven pieces on either side?
- Do you notice any patterns in the colors you are moving or in the type of moves that you are making?
- Can you classify and count the moves you use in other ways?

APPENDIX D: SQUARE GAME SOLUTIONS[6]

Note: The Square Game is also known as the Toads and Frogs Puzzle, Leap Frog, and Traffic Jam, to name a few.

When *N* blue squares are placed on *N* successive positions on the left of a string of positions delineated by collinear line segments, *N* yellow squares occupy *N* rightmost positions. Overall there are $N + N + 1$ positions, so that just one position remains unoccupied. Following specified rules about possible moves, the goal then is to move yellow squares into *N* leftmost positions and the blue squares into *N* rightmost positions.

Every move is either a Slide (*S*) to the nearby square or a Jump (*J*) over one position. Thus, at every stage of solution, when it comes to selecting the next move, you may have to select between either two jumps, or two slides or a jump and a slide. For every value of *N*, the puzzle has two solutions: one can start either towards right or towards left. For every sequence of moves, a string of letters *S* and *J* can be formed to record and represent the moves. Because of the symmetry, there is no wonder that the two strings of moves corresponding to the two solutions are identical. In addition, the string of moves is always palindromic.

There is another interesting fact. In both cases, there are N^2 number of Jumps and $2N$ number of Slides. Thus the function, where *M* represents the total number of moves, is the sum of the linear function $M(N) = 2N$ and the quadratic function $M(N) = N^2$ or $M(N) = N^2 + 2N$. Students may also represent these functions in tabular and graphical forms.

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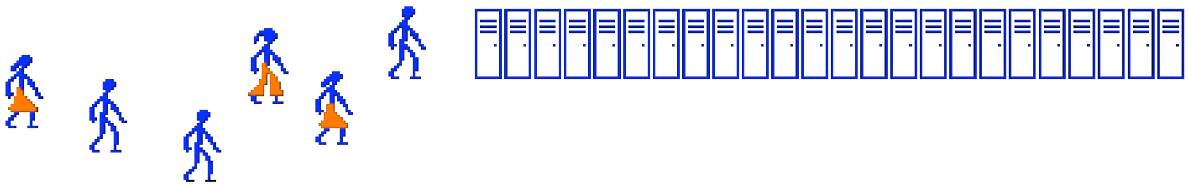
[6] Adapted from http://www.cut-the-knot.org/SimpleGames/FrogsAndToads.shtml

APPENDIX E: THE LOCKER PROBLEM[7]

Problem Solving Session Two



Imagine you are at a school that still has student lockers. There are 1000 lockers, all shut and unlocked, and 1000 students.

- Suppose the first student goes along the row and opens every locker.
- The second student then goes along and shuts every other locker beginning with number 2.
- The third student changes the state of every third locker beginning with number 3. (If the locker is open the student shuts it, and if the locker is closed the student opens it.)
- The fourth student changes the state of every fourth locker beginning with number 4.
- Imagine that this continues until the thousand students have followed the pattern with the thousand lockers. At the end, which lockers will be open and which will be closed? Why?

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[7] Adapted from http://mathforum.org/alejandre/frisbie/student.locker.html

APPENDIX F: LOCKER PROBLEM SOLUTIONS[8]

Solution: In order to begin the problem, a smaller version of the problem could be considered. A student could try flipping index cards or coins to represent opening and closing lockers. The student could begin with 25 lockers to see if a pattern develops. They would soon see (as long as they are careful not to mess up the procedure described in the problem) that the perfect squares are left open, namely 1, 4, 9, 16 and 25. This pattern will hold with all lockers 1-1,000.

Discussion: The perfect squares are the only numbers that have an odd number of factors. If a locker is touched an odd number of times, it will be open. If a locker touched an even number of times, it will be closed. Therefore, only the lockers with an odd number of factors will be left open- leaving the perfect squares. There are 31 perfect squares from 1-1,000. Thus, the 31 locker doors that will be open at the end of the process have the following numbers:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900 and 961.

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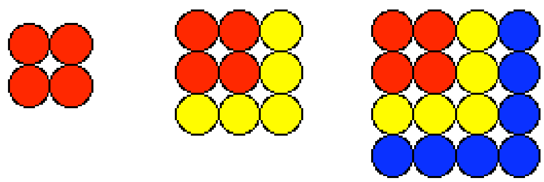
[8] Adapted from Doug Callahan's (Fall 2003) Problem Solving Portfolio found on-line at http://www.intermath-uga.gatech.edu/tweb/gwin1-01/callahan/emat6600/Locker_Counting_Solution.html

APPENDIX G: THE POLYGONAL NUMBERS ACTIVITY

Problem Solving Session Three

The goal of this problem is to investigate both the algebraic and geometric representations of a variety of polygonal numbers. We'll start with an example using square numbers:



The first image to the left that you see represents the 2nd square number, the next one represents the 3rd square number and the last represents the 4th square number. What would the *n*th square number look like? What is the relationship between the *n* th and the ($n + 1$)th square number? What would be an algebraic representation for the *n*th square number?

Let's then extend our investigation of these types of questions to triangular numbers and then other polygonal numbers (pentagonal, hexagonal…)

Polygonal numbers were defined as follows: Given a number of equal circular counters, then the number of counters which can be placed on a regular polygon so that the tangents to the outer rows form the regular polygon and all the internal counters are in contact with its neighbors, is a "polygonal number" of the order of the polygon.

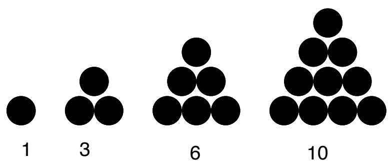
APPENDIX H: POLYGONAL NUMBERS SOLUTIONS[9]

Triangular Numbers

Numerical Representation:

The first four numbers that can be represented by a triangular array of dots are 1, 3, 6, and 10

Geometric Representation



Algebraic Representation of the *n*th triangular number

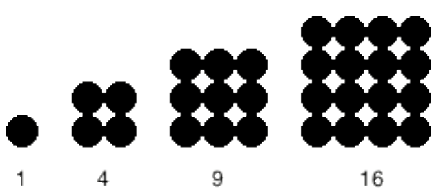
$$\mathrm{T}_n = \frac{n(n+1)}{2} \quad \text{or} \quad \frac{1}{2}n^2 + n$$

Square Numbers

Numerical Representation:

The first four numbers that can be represented by a square array of dots are 1, 4, 9, and 16.

Geometric Representation



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[9] Adapted from http://44.1911encyclopedia.org/P/PO/POLYGONAL_NUMBERS.htm and http://thesaurus.maths.org/dictionary/map/word/1580

Algebraic Representation of the *n*th square number

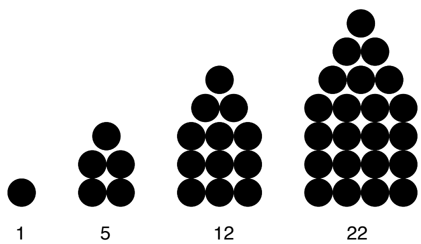
$$S_n = \frac{n(2n+0)}{2} \quad \text{or} \quad n^2 + 0n \quad \text{or} \quad n^2$$

Pentagonal Numbers

Numerical Representation:

The first four numbers that can be represented by a pentagonal array of dots are 5, 12, 22, and 35.

Geometric Representation



Algebraic Representation of the *n*th pentagonal number

$$P_n = \frac{n(3n-1)}{2} \quad \text{or} \quad \frac{3}{2}n^2 - n$$

General Form of Polygonal Numbers

The general form for the *n*th polygonal number, where *r* is the order of the polynomial, can be written in the form

$$P_n^r = \frac{n\left[(r-2)n-(r-4)\right]}{2} \quad \text{or} \quad P_n^r = \frac{n}{2}\left[(r-2)n-(r-4)\right]$$

APPENDIX I: PSS1 GROUP INTERVIEW GUIDE

1. How did your group begin working towards a solution?

2. Can you identify any roles that group members adopted?

3. Can you pinpoint specific things that a group member did or said or didn't say that helped you in this problem solving process?

4. How about specific things that a group member did or said or didn't say that hindered you in this problem solving process?

5. Was there anything, in general, about being in a group that either helped or hindered you in this problem solving process?

APPENDIX J: PSS1 INDIVIDUAL INTERVIEW GUIDE

Part One: Prior to Watching the Tape

- Can you explain the problem you solved? How did you solve it? Strategies?
- What was easy/difficult about the problem?
- What role did you play in solving the problem?
- What were your teammates roles?
- How did you teammates help/hinder your mathematical problem solving process?
- Is this how your group normally functions? Or is this how you think it should function?

Part Two: Watching the Tape

Let's watch the tape together…. You tell me to pause at anytime that you want to talk about what is doing on (and this list of questions….)

- What were you thinking here?
- What was "Susan" thinking here?
- What did you do (mathematically and/or cognitively) with Susan's idea?
- You were quiet during this time… why? What were you thinking about?
- This was a suggestion that you made…. How did you come up with this idea? Why did you opt to share it with your team?

APPENDIX K: PSS2 GROUP INTERVIEW GUIDE

1. How did your group begin working towards a solution? What solution did you use?

2. Can you identify any roles that group members adopted? How were these the same and/or different than last time (the square game)?

3. Describe the types of interactions that you and your group members had?

4. Was there anything, in general, about being in a group that either helped or hindered you in this problem solving process? How were these the same and/or different than last time (the square game)?

5. Can you pinpoint specific things that a group member said or did not say that helped you in this problem solving process? How were these the same and/or different than last time (the square game)?

6. How about specific things that a group member said or did not say that hindered you in this problem solving process? How were these the same and/or different than last time (the square game)?

APPENDIX L: PSS2 INDIVIDUAL INTERVIEW GUIDE

1. Can you explain the problem you solved? How did you solve it? Strategies?
2. What was easy/difficult about the problem?
3. What role did you play in solving the problem?
4. What were your teammates' roles?
5. How did you teammates help/hinder your mathematical problem solving process?
6. Is this how your group normally functions? Or is this how you think it should function?
7. Let's watch the tape together…. You tell me to pause at anytime that you want to talk about what is doing on (and this list of questions….)
   - What were you thinking here?
   - What was "Name of one of the other two participants" thinking here?
   - What did you do (mathematically and/or cognitively) with Susan's idea?
   - You were quiet during this time… why? What were you thinking about?
   - This was a suggestion that you made…. How did you come up with this idea? Why did you opt to share it with your team?
8. In general, how would you describe the interactions that you and your teammates had?
9. How would you work together differently next time to solve a similar problem?
10. In general, what would you do the same or different if you were the person who set up this activity?

APPENDIX M: PSS3 GROUP INTERVIEW GUIDE

Part One: Begin the interview with the same basic questions as before

1. How did your group begin working towards a solution? What solution did you use?
2. Can you identify any roles that group members adopted? How were these the same and/or different than last time (the square game)?
3. Describe the types of interactions that you and your group members had?
4. Was there anything, in general, about being in a group that either helped or hindered you in this problem solving process? How were these the same and/or different than last time (the square game)?
5. Can you pinpoint specific things that a group member did or said or didn't say that helped you in this problem solving process? How were these the same and/or different than last time (the square game)?
6. How about specific things that a group member did or said or didn't say that hindered you in this problem solving process? How were these the same and/or different than last time (the square game)?

Part Two: Share the research questions with the participants. Answer any questions they may have about what I am looking for, what the study is about, etc. Then ask for the thoughts on the research questions

Part Three: Have the participants take the statement of the problem home to think about it in preparation for the individual interviews tomorrow. Hand out the statement (see below) and answer any questions about the research questions. Tell the girls that we will focus on these types of questions tomorrow.

The following statement was printed on a half sheet of paper, handed out, and discussed:

Statement of the Problem

I propose to conduct an interpretative study, in which I will observe and interview three college students (who I taught in a secondary mathematics content course) in order to investigate the ways in which they both simultaneously construct and are constructed by cooperative learning experiences as they engage in mathematical activity. Specifically, I plan to investigate the following questions:

1. During cooperative learning experiences, in what ways are individual students both constructed and constructed by
   a) the cooperative environment? e.g., social norms, manners, expectations
   b) the group members? e.g., interactions, second order models, scaffolding
   c) the self? e.g., autonomy, metacognition, first order models

2. How are these various constructions related to the students' mathematical activity?

APPENDIX N: PSS3 INDIVIDUAL INTERVIEW GUIDE

Part One: Reflecting on the Problem

1. Can you explain the problem you solved? How did you solve it? Strategies?
2. What role did you play in solving the problem? What were your teammates" roles?
3. How did you teammates help/hinder your mathematical problem solving process?
4. How did your group function? How should it function?

Part Two: Watching the Videotape

Let's watch the tape together…. You tell me to pause at anytime you want to talk about what is doing on (and this list of questions….)

- What were you thinking here?
- What was "name of one of the other participants" thinking here?
- What did you do (mathematically and/or cognitively) with "name of one of the other participants" idea?
- You were quiet during this time. Why? What were you thinking about?
- This was a suggestion that you made…. How did you come up with this idea? Why did you opt to share it with your team?

Part Three: Reflecting on the Cooperative Learning

1. In general, how would you describe the interactions that you and your teammates had?
2. In general, what would you do the same or different if you were the person who set up this activity?
3. Reflect on research questions that we talked about yesterday using any of the three sessions as examples…

   I propose to conduct an interpretative study, in which I will observe and interview three college students (who I taught in a secondary mathematics content course) in order to investigate the ways in which they both simultaneously construct and are constructed by cooperative learning experiences as they engage in mathematical activity. Specifically, I plan to investigate the following questions:

1. During cooperative learning experiences, in what ways do individual students both construct and constructed by
   a) the cooperative environment?   e.g., social norms, manners, expectations
   b) the group members?   e.g., interactions, second order models, scaffolding
   c) the self? e.g., autonomy, metacognition, first order models

2. How are these various constructions related to the students' mathematical activity?

APPENDIX O: EMAT 4500 COLLOQUIUM AGENDA

April 25, 2002

3:35 – 3:37	Whitney	**Read Quote and State Purpose of Study** (Emphasizing the importance of understanding how certain pedagogies affect students)
3:37 – 3:47	Sheehy	**Background, theory and question** • Personal Experience Supported by Research Results • Also read calls for future research that I couldn't answer • Lead me to a pilot study in which students talked about their experiences in ways I didn't expect – began to question cl • Thus a Poststructural Orientation • So what do you look at? Synergistic Hypothesis • Research Question
3:47 – 3:55	Sheehy	**Overview of Methodology**
	Meredith	Present the Square Game
	Annie	Present the Polygonal Numbers Activity
	Courtney	Present the Locker Problem Activity AND set-up the video clip
3:55 – 4:05	All	**View and Discuss Video Tape** (each participant will take a moment to describe the experience from her perspective)
4:05 – 4:20	All	**Preliminary Findings**
		<u>Cooperative Environment</u>
	Meredith	Roles
	Annie	Manipulatives

	Courtney	Politeness
		<u>Group Members</u>
	M & A	Power
	A & S	2nd order models and Scaffolding
		<u>Self</u>
	All & S	Autonomy, Reflection time, Confirmation
4:20 – 4:25	All	**Implications**
	Meredith	Brain breaks, no roles
	Courtney	manipulatives, size of group, seating
	Annie	ways she learns, teachers role, type of problem
	Sheehy	Importance of allowing students these types of opportunities
4:25 – 4:30	All	**Discussion**

Our goal is to have more than five minutes at the end for discussion – so let's try to be very aware of time and be brief in our presentations of our parts.