Mathematics, Philosophy, and Proof Theory

by

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(Under the direction of O. Bradley Bassler)

Abstract

Our purpose shall be to introduce revisions into the foundational systematic introduced by Brouwer and Hilbert in the early part of the last century. We will apply these revisions to develop a symbolic calculus for the study of extralogical intuition (in formalism, logic at the metalevel), which we shall show to be not weaker than intuitionistic propositional calculus, and rich enough to encode all of finitary set theory. Our calculus will be efficient in its principles and based on a small, compact set of axioms, and its consistency will be shown. In the main it will be based on two departures from traditional developments: (1) the interpretation of logical conjunction as a mathematical operation of set formation, and (2) the interpretation of logical implication as the exchange (in time) of actual or intuited objects. Its rule structure, in addition, will possess two novel features: (1) generalized substitution, or what we call herein *deposition*, and (2) a formal method of assumption.

Index words: Foundations of Mathematics, Proof Theory, Formalism, Intuitionism, Intuitionistic Logic

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Dedication

To David Galewski, Richard Feynman, and the king of carrot flowers.

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Contents

Chapter 1

Introduction

1.1 Some Remarks on Recent Developments in Foundations

By the time the flashing insights of Lakatos, the great student of Polya who nearly achieved so much, were first beginning to take hold in the mathematical world, the bold critical reexamination of many old ideas in the philosophy of science had already begun to suggest, in broad, cloudy strokes later refined by Hersh, Tymoczko and others,[1] for mathematics, a new and very different kind of foundational program. From diverse quarters of the humanities and the sciences, new arguments were advanced and old ones were revived, all casting doubt on Hilbert's claim that formalistic finitary methods were capable of providing the range of concepts that would be required to ground mathematics scientifically, even countering that the notion of a formal system is intrinsically limited to but a meager fraction of the activities, both physical and mental, which we are happy to call mathematical in the highest sense: infused with brilliance, insight, and technique. A more textured coding of the tools of mathematics, as finds partial expression in the prescient and creative work of Wittgenstein, and the solid and enduring contributions of the intuitionists, seemed now in earnest to be standing up for consideration. A new synthesis of science, one which reincluded the mathematics which had distanced itself, after Euler, Lagrange, and Gauss, by proclaiming itself *pure* now seemed possible, perhaps even realistic; some even declared the arrival of a new "humanist" school, which might restore mathematics to something more like that idyllic realm to which Euler had made his luminous contribution so long ago.

[1]See Lakatos [26], [27], and Tymoczko's anthology [32].

However, the champions of this recent movement in foundations seem at times willing to underemphasize or even to ignore the sobering lessons bestowed on mathematics by the discoveries of Weierstrass, Cantor, Peano and the other analysts of the nineteenth-century dawn of mathematical modernity, for in general their doctrines tend not to present a *system* to the mind, a way to convert one clearly stated idea into another in the symbolic language that precise reasoning and communication requires. Thus the mantras of Hilbert, and even far older ones dating to classical times, today still command the loyalty of most mathematicians. Though they might listen with interest when told about progressive new ideas coming from anthropology, physics, philosophy, education, neuroscience,[2] and the intrepid outliers among them, they fail in general to see clearly what is to be gained by making drastic changes to the principles and methods of mathematics. Though they may find something to like in the cases made by constructivists and others, they find it doubtable nevertheless that anyone will ever successfully disprove that certainty is an admixture in their work, and by and large conclude that a caution route is the best way forward. Indeed, it remains unclear, in our time, precisely to what a bold departure from formalism would lead, and even whether such a venture could be safely carried out. From the relative quiet of recent decades, what the collapse of the uneasy détente in foundations might lead to is, it seems, within no one's capacity to predict.

Could the murmur of "picture theories," and "tacit knowledge," and the loyal advocacy of some for "quasi-empiricism," "mathematical humanism," "social constructivism," etc., ever be carried beyond the regular in- and outflow of debate about mathematics, and infiltrate the tall, noble hierarchy of the actual mathematical community—or appear, without a trace of code-switching, alongside the language of an actual proof? Some might wonder how the spread of these movements and ideas might effect the working mathematician's output. All mathematicians value the fabulously innumerable accomplishments that have been achieved during the century recently passed, and there is general agreement that mathematicians

[2]Anthropology: see [35], physics: see [9], education: see [6], neuroscience: see [8].

of our era have succeeded in eliminating clouds that previously hung over their science and impeded its progress. They worry that concessions to historicism, fallibilism, and speculative philosophizing—in the style of Weyl's papers on the logic of the infinite, for example, or those of Poincaré, in which he attempted, in the 1890's, to deductively derive the topological characteristics of physical space—represent a labor-intensive step away from the clear path championed by Hilbert, that great disdainer of history and the endless memorializing of tradition. They patiently await the repair and rejuvenation of formalism, and are content in the meantime to abide its sometimes imposing officialdom in the midst of daily life, rather than opening the boundaries of mathematics to all manners of philosophy, when, they feel, the result of this could very well be only that the theorems which might have been the accolades of their own generation, might be lost amidst distraction, and fall ahead to become the encumbrances to clear and direct thought which future generations alone are able to exceed.

This "aphilosophical" attitude has flourished amidst the creation of formal logic and, in recent times, the growing influx of concrete, results-oriented computer-related research, while in the meantime the traditional areas of mathematics have themselves grown to many times their size in 1900. During the past three-quarters-century since Hilbert's passing, the purity of mathematics has been maintained, and the formal foundation he envisioned for it has by and large endured. Were he alive today, he would no doubt be pleased, none the less upon seeing that the net effect of this state of affairs has been overwhelmingly positive upon science as a whole. Yet this generally glad outcome has had rather the opposite effect upon the growth and advance of foundations.

The long and strenuous training of modern mathematicians conditions them to be wary and even hostile to anything they take to be the intrusion on the rules of play in their science by rhetorically well-heeled, but inexperienced commentators. This fact has long presented a forbidding obstacle to interdisciplinary investigation of the foundations of mathematics, adding considerably to the already formidable philosophical challenges inherent in such an analysis. Foundationalists are well aware of the relative obscurity of even their most significant achievements outside of their field, and the low regard that many mathematicians have towards even their most insightful philosophical reports. Meanwhile comparable achievements of indeed great mathematical—but far less philosophical merit (such as the work of Gödel and Cohen in set theory) are the recipients of universal praise among mathematicians, and are held up as models of foundational work. The case of Brouwer is an outstanding example of a foundationalist of first rank, whose work was misrepresented or misunderstood by fellow mathematicians.[3] Wittgenstein's middle and late work is also controversial within the hard sciences, yet there is general agreement in the humanities that Wittgenstein's far-reaching influence was beneficial and progressive.

Thus, high forbidding ramparts, which today encumber both exit and entry, encircle those parts of mathematics grounded in set theory and formal systems, and leave it quarantined to the circles of debate which exist in other areas of science. Though intended to bolster and protect, they at times present a stubborn obstacle to those who are working to integrate fast-evolving new ideas from other areas of science into the matrix of leading concepts in foundations, and maintaining the pace of progress in science as a whole. Today, there is a lack of observers and participants in research in foundations who are qualified in enough disciplines to accurately evaluate all of the work that has been done by serious-minded philosophical and mathematical researchers, resulting in a deeply fragmented and somewhat dishevelled depiction in the literature of a subject which is widely felt to possess deep underlying unity. It may be wondered, then, even in spite of the immense and ever-growing legacy of our own golden age, whether it is time yet to say that, decades after the mathematician's embrace of the diminished modern faith in creative instincts and the power of abstract philosophical reflection, he has come upon an unexpected impasse.

It is noteworthy, too, that the effects of this estrangement within science are not limited to the scientific domain. In spite of a growing tide of opinion, the gulf remains wide as ever

[3] *Cf.* Brouwer 1927a [3].

between those willing to imagine mathematics to be the bare symbolic landscape generated by a formalizing mind, and those who have reasons for opposing this view. This chasm, which begins benignly inside the scientific community, extends from there well beyond it; as the matters under debate become less well understood, the divide widens and sometimes fosters treacherous misunderstandings. Whether this can ever be changed—and whether the damage already wrought can be undone with the steady progress of time—future generations may be able to tell. An individual of today, told he must choose, be he committed to coherent justification in mathematics, and to the clear symbolic ideal which is the very love of mathematics itself—who follows the call for certainty and knowledge which was at its origin the inspiration of philosophy—is bent toward those schools who offer to their adherents a symbolic system, a mathematically connected family of ideas.

1.2 The Success of Formalism

There are historical facts which shall for all time shape the character of the twentieth century. The indelible mark of Hilbert's program through its influence on the ensuing development of mathematical logic by Church and his students at Princeton, Gödel and Gentzen in Germany, Herbrand in France, and many others, as well as its influence upon the creators of the twentieth-century orthodoxy in mathematics (the synthesis of Bourbaki) impel us to reflect upon what was nearest—of all things—at its apex to becoming the *de facto* end of the search for a settled, rigorous mathematical source of truth.

It is impossible to separate the formalist school from the personality and figure of its founder. Long before formalism was a school and a method, it was the tendency and character of a certain David Hilbert. Among the many contributions Hilbert made to mathematics, one finds numerous examplars of the work that would have such a profound impact on its direction, always written in the original style that would become legendary. His *Zahlbericht* of 1897 was a celebrated textbook which significantly impacted the field of algebraic number theory. Hermann Weyl later wrote, of his own experience reading it (during his first summer off at Göttingen),

> It is as if you are on a swift walk through a sunny open landscape; you look freely around, demarcation lines and connecting roads are pointed out to you before you must brace yourself to climb the hill; then the path goes straight up, no ambling around, no detours.[4]

The *Grundlagen der Geometrie* of 1899, a mathematical bestseller, was a compact, lucid, and penetrating development of geometry from a newly clarified set of axioms. Though it drew from work by Moritz Pasch and others who were at that time already working to place geometry on a firm modern foundation, Hilbert's book, written hardly more than a year after he had entered the field from number theory, thrilled its readers and immediately overshadowed all the previous work. If Hilbert's ideas captured the loyalty of working mathematicians more successfully than any other philosophy since Platonism,[5] this was surely in part because Hilbert, when he entered the foundational debate in earnest in 1921, carried with him a considerable reputation built over the course of a 40-year-long career. When the Hilbert of 1910 or 1920 spoke, mathematicians were ready to listen. When Hilbert raised his voice in criticism, as he famously did against Weyl and Brouwer (see [18]), it could ill be ignored.

If it is impossible to separate formalism from the figure and personality of its creator, it is extremely difficult to separate Hilbert from the school where he spent most of his adult life, for the Georg August University of Göttingen was known at that time simply as "the mecca of mathematics." During those years, the prospect of education under the aegis of Klein, Hilbert, Minkowski, Courant, Noether, van der Waerden, Weyl, and still other luminaries attracted the generation of mathematicians and logicians—Curry, von Neumann,

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[4]Quoted from Reid in [30], p. 94.

[5]According to Jacob Klein, "Aristotle never tires of stressing that Plato, in opposition to the Pythagoreans, made [mathematical objects] 'separable' from objects of sense, so that they appeared 'alongside perceptible things'...as a separate realm of being." [23], p. 70.

Gentzen and MacLane among them—who would carry the foundations of mathematics into its future. Hilbert's abiding influence there must surely also be considered among the reasons for formalism's lasting impression upon the approaches and attitudes of mathematicians toward their work.

The success formalism enjoyed was also due to the long list of celebrated problems which it emphasized (among these, the continuum hypothesis, the consistency of arithmetic, and the *Entscheidungsproblem*—the first, second, and tenth problem on Hilbert's famous list of 1900—led the pack) and the novelty of the suggested method—metamathematics—for approaching them, which immediately won its share of committed partisans. To its adherents, the dispassionate technique—a radicalization of the axiomatic method—obeyed a simple, direct, and universal pattern of thought. It seemed to offer a peaceful channel into which all of one's energy could safely be invested, one which gracefully sidestepped the restrictions that were laid on mathematics by intuitionists. Through this peaceful channel, it was felt, the infinite itself could be seen and even touched, without the slightest danger of outside disturbance, or upheaval from within. Metamathematical philosophy has an unusual but (by now) undeniably powerful impact upon the human mind. Although in the rapid evolution of science effects are often difficult to separate from causes, its spirit was certainly in keeping with the powerful movement at that time towards greater abstraction. Thus it was, to many, a logical step forward.

There is, however, as I will argue, another, more weather-proof reason for the success of formalism than these. This is the philosophical principle upon which Hilbert based his formalism, the philosophical principle he was drawn to prepare and publicize after studying the work of Brouwer and Weyl. It is a principle rightly called, for it is but one and simple. Quite unlike the philosophical dilations of a Brouwer, it is easy to state and difficult to misstate. It is a common idea—yet we should not discredit Hilbert's philosophical achievement in boldly applying it with conviction. To the contrary—its commonness serves to remind us, and even alert us, to a certain remarkable endurance through the vicissitudes of physical time and space. Its resonance in human feeling is rooted in distant ages, yet in communications of the highest and lowest order, it is constantly being restated, revitalized, rediscovered, and reclaimed. It says merely this: that mathematical truth ought to depend upon and flow, with uttermost unity and directness, from the universal experience of seeing, holding, and manipulating natural objects.

> As a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain *extralogical concrete objects* that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else *nor requires reduction*. This is the basic philosophical position I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.[6]

In the heart of a man lies his philosophy, and from this follows the tone and tenor of all his work. This in turn shapes his destiny, and the particular methods he selects. One is led to suspect, therefore, that the success of formalism—as a working mathematician's answer to philosophy, as a logician's methodology, in all the ornateness of its subsequent development—is due in no small part to its unassuming, commonsensical point of departure.

Hilbert, in sum, sounded the call for the extralogical study of logic, to be used to supply the centerpiece of his foundational project, a proof to validate all proofs and ensure their existence. This was not to be achieved; there is an inner circularity in this self-referential *raison d'être* of metamathematics famously revealed by Gödel, one on which future mathematicians and philosophers ought to be able to shed a clearer light. Herein we will not

[6]Hilbert, from a well-known address [19] delivered in Münster in 1925 (quoted from [14], p. 376), emphasis mine throughout. He directly continues: "And in mathematics ... what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizeable."

speculate about the cause of formalism's deeply intertwined failure to be all that it promised that it would be. This stage in the development of formalism does not concern us, since we shall not adopt for our use the Hilbertian notions of metamathematics or ideal propositions. We shall, however, adopt the *basic principle of formalism*, in just the form that it was clearly announced by Hilbert.

1.3 Our Point of Departure

We stand now at the threshhold of our own investigation. We have adopted the *basic principle of formalism* as our first grounds for mathematics. Before we go further, however, let the distinction be drawn between *mathematics* and *philosophy*—for we do not make the same concession for the latter. We are are not content, only because a thing is sufficiently manifest, that nothing further can be learned or stated about it. We do not forbid ourselves from tampering with our principle, for the philosophical pursuit of meaning must continue as ever. We grant only that it is upon that inexorable kernel of certainty which persistently surrounds us (which Hilbert laudably characterized in the quotation above) that a mathematical proof may rest upon for its validity: not a classical channel for reception in the mind of *a priori* truth, but rather, the extralogical intuition of direct, local experience. We shall now carefully embark on an extended examination of the idea, the image, which this hypothesis throws up before us, in order to understand what it allows, and what it strictly forbids.

We see before us, scattered, various fields of knowledge which it is upon us to find some kind of ground for. As we stumble through the shallows, able to look with uncomprehending awe at the vast ocean with which the waters touching us mingle, we find it is first our task to account for logic, set theory, and arithmetic. Which of these ought to come before, and which after? Should we begin, as the semi-intuitionists did, with arithmetic? Should we begin with logic alone? If we try a combined attack on two of them, how exactly should they be fitted on the plinth? We turn for guidance to the extralogical intuition. We find that by it (through some deep, hidden channel) we may see, as given, the object as it stands before us, and that objects may by immediate reckoning have parts, giving rise to elementary totalities. Less immediate, by contrast, are the elements of logic (properties, and groups of these) and arithmetic (numbers, and groups of these); insofar as they are immediate, they seem suspiciously so due to their status as objects (as the formalist would say, *signs*) of the former kind. Therefore our principle directs us to begin with the primitive concepts of finitary set theory, and try with care gradually to progress from there to the concepts that are generally taken to ground the other two domains.

We grant that we may observe sets—do we also grant that we can record these observations? Can we write down properties of sets? I suppose, for example, that I may see (extralogically, in the form of a direct experience) that one set falls in another. Have I then experienced a primitive, manifest property or relation? Perhaps I may well do so—but the difficulty does not seem to lie in their record, but rather in their transmission. The logical notions of properties and relations take on meaning and acquire potency only by means of *assertion, proposal,* or *judgment.* These notions add considerably to the purview of our project at a fundamental and untimely stage, in the form of complicated considerations which it will suffice to simply label *biological.*[7] Were we to build upon these notions, we would soon find ourselves in the midst of structures not required by the inner necessity of our problem, and might well lose sight of the contentual locus to which we assign the greatest trust. History teaches that we would run the danger, after preparing a foundational account of the inner nature of human judgment, of seeing our labor reduced to casuistry by later refinements of our common notions.

Thus, we are at a momentary impasse. It seems we must either labor with great difficulty upon the subject of the *judgment*, in pursuit of clarification of what we broadly term the *biology* of mathematics, before *valid mathematical proofs* may be written down (perhaps by way of some formidably complex definition of one human being's vocalized assertion to

[7]Nor can we leap so far ahead by admitting into our initial purview the notion of a mathematical *problem* as something well-defined or manifest. Kolmogorov's calculus of problems (see [25]), therefore, can be admired, but cannot be adopted by us.

another), or we must simply fold up our hands and wait patiently for light to be thrown on these topics by other disciplines. This perhaps grim state of affairs, however, should not be admitted until we have confirmed that it is not circumvented by the simplest and most direct route forward—an attempt to write down the laws of logic from regarding the proposition's status as *the object surveyed*, and not as the judgment asserted, denied, believed, trusted, or cursed. Since we grant ourselves the power to observe objects (to create them for the purpose of study, to manipulate them, doing all that we do freely with mathematical symbols in formalism), we might, rather than attempting to plumb and measure the mysterious confines of our minds, remain at that level where Hilbert insisted we remain, watching the proposition just as the formalist does—observing its behavior as an object beneath our awareness, manipulated extralogically by our will. We might then attempt to extract, as the intuitionist does, the further principles of calculation.

So we thus resolve to forbid the use of all the usual language of judgements, beliefs, and assertions. We shall attempt, for now in lack of these concepts, to develop what we can of logic, straight out of the constructive-formalistic kernel of certainty named above. We shall, that is, develop the system of logic as a substructure of the system of objects.

Chapter 2

Philosophical Beginnings

2.1 Implication and Time

In two brief articles of 1928 and 1929, Valerii Glivenko extended Brouwer's investigation showing that from intuitionistic principles, it follows from the triple-falsity of an expression that the expression must be false. Glivenko showed, based on a set of axioms essentially the same as those used by Kolmogorov in 1925 (though Kolmogorov did not, and Glivenko did, have use of the principle *ex falso sequitur quodlibet*), that the double-falsity of the law of excluded middle holds, that the validity of a proposition implies the validity of its double-falsity, that the classical validity of a proposition p implies its double-falsity, and that every classically valid negation is intuitionistically valid. In the passage quoted below, from the latter paper of 1929, we find Glivenko in a praxical mode, actively engaged:[1]

> Here it is a matter of verifying the schema
>
> $$\frac{\begin{array}{c}\sim\sim P \\ \sim\sim(P \supset Q)\end{array}}{\sim\sim Q.}$$
>
> To this end, we will prove the formulas
>
> $$p \supset ((p \supset q) \supset q),$$
>
> $$(p \supset (q \supset r)) \supset ((\sim\sim p \supset (\sim\sim q \supset \sim\sim r)).$$

His conclusion reads:

[1] I use the translation in [29].

> [...] This established, one has
>
> $$\frac{\begin{array}{c} p \supset ((p \supset q) \supset q) \\ (p \supset (q \supset r)) \supset (\sim\sim p \supset (\sim\sim q \supset \sim\sim r)) \end{array}}{\sim\sim p \supset (\sim\sim(p \supset q) \supset \sim\sim q).}$$
>
> from which the rule to be verified immediately follows.

On the surface, nothing out of the order has taken place. As Glivenko states explicitly, his objective is to validate a schema, or pattern of inference, in which two propositional formulas ($\sim\sim P$ and $\sim\sim(P \supset Q)$) combine in order to produce a third, $\sim\sim Q$. He obtains this result much as the student of logic would obtain it today: by applying axioms within a proof system in which there are two formal rules, modus ponens and the rule of substitution.[2]

The question, the oddity, is this: for what reason did Glivenko write

$$\frac{\begin{array}{c} \sim\sim P \\ \sim\sim(P \supset Q) \end{array}}{\sim\sim Q}$$

and then prove—?

$$\vdash \sim\sim p \supset (\sim\sim(p \supset q) \supset \sim\sim q)$$

relying upon a circumlocution through the inline domain and the rule of modus ponens?[3]

In a footnote to the same article, Glivenko notes

[2] According to the second edition of Hilbert and Ackermann [20] (see p. 29 of [21]), the use of this rule system in logic can be traced to Frege's *Begriffsschrift* of 1879.

[3] In posing this question, there are a few points I ought to make clear. First, there is a simple answer it is not my intention to discredit. Glivenko has offered us, as it would go, an exercise in inventing derived rules—"short-cuts," Troelstra and van Dalen call them ([31], p. 45); Kleene also uses the Latinized "subsidiary deduction rule" ([22], p. 51). Granted, the account could simply end there. Secondly, I do not intend to claim to be a diviner of Glivenko's true thoughts on the matter. I am ultimately only interested in using this passage from Glivenko's paper to illustrate my point, which could be made without it or by referring to any number of other places in the literature. Let us, though, keeping all this in mind, allow the question to be examined with an open mind.

> To save in writing, we will sometimes make use of the following rule:
>
> $$\frac{\begin{array}{c} P \supset Q \\ Q \supset R \end{array}}{P \supset R}$$
>
> which follows immediately from Axiom II.

Precisely what is taking place in these two quotations? All the evidence found in the brief article suggests that Glivenko respects the strict division between the notions of axiom, theorem, sentence, etc., and the carefully fenced-off notion of a rule which appears in Hilbert and Ackermann's *Grundzuge der theoretischen Logik* [20], which he cites. Why, then, does he—"informally," as we would say—break the boundary twice? The reader who has done any logic for him or herself knows that there exists a natural psychological tendency to do precisely what Glivenko has done. Is this nothing more than a human tendency to play truant to the laborious accounting that science sometimes requires? Might there perhaps be something deeper involved—some veiled mathematical principle beyond our human-born idealization? The evidence, it can be argued, tends to the conclusion that the notions of rule and axiom resist the traditional distinct and disjoint classification—that the boundary between them is not the ironclad wall Hilbert and Frege envisioned. Perhaps we ought to formalize this tendency, someone might say—we formally derive new formulae; these we call "theorems." What sort of "theorem" do we obtain when we formally derive a new rule?

Traditionally there exists, in the storehouse of logical knowledge, two distinct domains: the horizontally-oriented home of "theorems" and the vertically-oriented home of "derived rules." It might be argued that by such a state of affairs, the unity of logical intuition is corrupted. Note, especially, the discrepancy between the justification of the derived rule, and the fundamental rule adopted as purely self-justified (in this case, modus ponens). When one asks why Glivenko's derived rule is permitted, the justification flows out of the horizontal, inline level of the writable formulas and axioms (which are meant, as we take it, to "tell us" something—something meaningful, something true). Modus ponens, however, has not been so fashioned. Yet shouldn't the rule—if it is really a rule—also be said to "speak truth" to us, in the form of a kind of *possum*? To hold to the usual account of the rule as a kind of metalevel activity to which the terms *truth* and *falsehood* are not meant to be applied (being confined in order to ensure the absence of paradox within the horizontal domain of statements and axioms) seems somewhat legalistic, akin to the view that what quacks like a duck and walks like one is not a duck by nothing more than its formal definition. In mathematics, where general tendency has always led towards subsuming similar structures into new abstractions, and where more than a few distinctions have eventually given way and been forgotten, this policy appears ripe for a critique. Only the most resolute logical relativist could ignore the odd bifurcation of the source of justification (truth) in the standard form of logic. Having developed logic ostensibly for the sake of the first kind of truth (truth in the object language) our curiosity tends always, like Glivenko's, to leads us to ask questions posed to the second kind of truth: *what may I do...?* In general, from Frege on, this discrepancy has been well-hidden so as not to cause any disturbance, tucked away inside the only rule no one could possibly reject.

Further evidence of this bifurcation can be found in the symbolism. What, for example, distinguishes the semantic content of the vertical line, and the symbol $\supset$? Glivenko writes what he calls a *schema*, then proceeds, in a step whose banality belies its significance, to commit himself to obtaining a proof of the schema rewritten—since, as the structure in which he operates is designed to affirm, this will *justify*, in the manner of a guarantee from somewhere or someone outside himself, the schema given earlier, which takes the form of a personal action—a remote and innocuous, but nevertheless forcible act: the antecedent formulae are brought out, and then they are "discharged"—they are used, sold, consumed. Perhaps I go too far—but now the distinction between the semantic content of the two symbols, and the murmuring disturbances on the philosophical level that they reflect, have been underlined. There is, I think, a real problem here to be understood, perhaps even to be solved.

Let us cautiously advance further into the structure of this unsolved problem at the heart of proof theory. We are first faced with the need for some sort of elementary vocabulary. Recalling our point of departure in chapter 1, we shall limit the range of the search by ruling out, as best we can, accounts that involve a human presence in mathematics; it is not wise to introduce the anatomical features of complex living things into our justificatory premisses, until we are confident that they are absolutely necessary. Next, having thereby cleared away the field of most other possibilities, we take our choice—it seems we need only to introduce one new element, or dimension, into the analysis, namely, time. We shall reduce the rule-axiom distinction, that is, to the terms of the physicist. We say, then: we face the problem of attempting to understand an enigmatic double-role performed by the written mathematical statement, or, put slightly differently, the manner in which axioms are lifted into the temporal axis of mathematical construction, and from there, how activity is projected down into a spatial representation which rests before our eyes, the symbolic diagram. A mathematical sentence is written: this may have either a spatial, or a temporal sense. The sentence may be declared, or it may be performed. In the usual proceedings of logic, this division which we take to divide the temporal and spatial dimensions separates horizontally- and vertically-oriented computations.

The next step is a remarkable observation: the temporal dimension cannot be extricated from logic. We cannot say that logic has a simple persistent presence, like an ideal hall or museum one can enter and explore. We cannot entertain the idea that logic is a temporally frozen realm, a realm purged of change, a realm of nothing but "facts" about existence and the existent.[4] Logic *occurs*, it does not merely exist—or else logic must be trivial.

Suppose that I perceive, in a frozen, timeless sense, $a \supset b$. Let us say: this is a fact. Now I see that $b \supset c$—I see one, and I see the other. If logic only involves facts, then I am powerless to conclude a thing. I must simply wait, hoping to find $a \supset c$ somewhere

[4]In the mind's eye, it is easy to confuse the intuition of historical and mathematical facts. We want to say of historical facts that they are *true*, permanent, that they can't be undone. Do we not say the same things about mathematical facts? But historical facts are about things which have *happened*—past tense. Mathematical facts are not this way, they did not *happen*—or did they?

nearby, since I do not now see it, and cannot "deduce" without a timeless form ready upon which to base a deduction. But perhaps I have found somewhere another artifact: I see that $(a \supset b) \wedge (b \supset c) \rightarrow (a \supset c)$. I rejoice, but to my dismay, I soon find I am still plagued by paralysis. What can I do? Let me grant that I may "form" a conjunction $(a \supset b) \wedge (b \supset c)$—perhaps this may be allowed (though in most formal system, even this demands a "rule"). Now I have one, and the other—but how will I ever be rid of the antecedent? I must carry it forever, it seems; at best, I can only try to cover it up or ignore it. If I were to continue doing logic, making more inferences, my back would certainly break from carrying along with me the least effect, the smallest trace of my work. Such a logical realm would either be trivially fused into vast "theories" or pulverized into discrete atoms having no "logical" interconnections. Even were I to grant myself the (quite amazing) power to survey, without sifting and in a single timeless instant, the entire beautiful realm of truths, I would still have no way of removing the intermediate steps of an inference chain; I would have no way of cutting the baby free from its mother, of extracting from the proof the theorem which it is its purpose to derive.

This is nonsense[5]; of course I may "draw a line" as Frege did, and conclude something. I do not make sense of this act, however, unless I admit the notion that my logical calculation advances, develops, progresses. Logic *occurs*; to derive is to travel, to be in motion. Unless the heat is added which melts the ice, there can be no *new* inference; in fact, the very notion of inference utterly vanishes from view. The reduction of the list of rules to modus ponens (which is usually associated with Hilbert's name, but was also Frege's approach, was the convention used by Russell and Whitehead, and was employed without special comment by the intuitionists as well (see [16], [13], [24])) goes no further: the number of rules must be greater than or equal to one. The *conclusion* is something which—somehow, in some way—takes place, assumes a location. Carroll's paradox (as it is sometimes known) is nothing but

[5]Fitting, then, that the first to remark on this phenomenon was the famous logician and nonsense poet Lewis Carroll, in a brief note [5] published in *Mind* in 1895.

the insight that logic cannot be complete until there is posited within its constitution a point at which facts end, and activity begins.[6]

Thus our formalistic point of departure, the study of the object, has now become the study of the object in time. By applying our strategy of the preceding chapter, we have stumbled upon a principle (construction in time) which was central to the thinking of Brouwer and the later constructivists.

After this dramatic change of direction, we may feel somewhat disoriented, but now we must immediately turn to investigate whether our new ideas might contain within them the possibility of a robust system of calculation. We have seen that the occurent inference cannot be pushed out and made exterior to logic. We have identified the occurent inference—the inference made in time—with the inference justified by a rule. The axiom is not a rule—it occurs outside of time. It is a *fact*, something inert. In order to be used or employed, it must be transformed (as in the quotation from [13] above) into a rule-like construction—i.e., one which states that a form with a given property can be actively manipulated during calculation. Note that there is nothing, until that stage has been reached, which the axiom can be used for unless the objects we are interested in studying are themselves formulas. If we have the desire to learn about formulas and not about objects, then obviously the axiom's role is of crucial importance; otherwise we must certainly view its role as rather ancillary to that of the rule, which, after all, is the real source of operational power in the calculus.

There is no compulsion that we are aware of, in fact, such as we felt with respect to the rule under the duress of Carroll's paradox, to make use of the notion of an *axiom* at all. Readability is the only consideration preventing us from replacing every instance of $\supset$ in Glivenko's proof cited above with a vertical line. There are certainly practical considerations in play, it is true, for the logician, who finds that difficulties tend to arise when working with

[6]Confirming evidence can be noted in many quarters. We do not simply "see" mathematical proofs—rather, we read them. Nor do we read them by starting at the end, and moving backwards—proofs, like books, poems, musical compositions, etc. have a beginning, middle, and end. Even in a trivial proof these categories are not transcended, only equated. Every proof, if understood, contains the ability to go on, with additional time, proving ever more.

systems—Gentzen's natural deduction systems for example—with a large number of rules. However, for the mathematician, who is not so concerned with proofs at the metalevel of consistency or completeness, these concerns are very miniscule, and perhaps even nonexistent. It is also true that the semantic *sense* of the vertical line and the inline inference $\supset$—the *possum*—is hauntingly similar. By equating them we would obtain, in addition to the structural elegance that shrinking the basket of fundamental concepts would be sure to grant, a system that pulls more philosophical weight. If it were only possible, it would be absolutely irresistible to fuse the two notions into one. We aim our investigation accordingly: we will try to obtain precisely such a logic: one in which each inference is actively performed by the mathematical subject. This will be done by removing the semantic estrangement caused by maintaining two separate inferential constructions, eliminating the dichotomy of performance and declaration.

Now it is unmistakable: a new vision of logic has begun to emerge. A logic without axioms, a science of derivation *in the pregnant sense*, a science of motion, hence—kinematics. We no longer consider a science of logic whose domain is the linguistic marrow of true statements—rather, the matter of logic has now become a proof-theoretical *pseudocode for praxis*. The metalevel (the level of the object in time) has been embraced and proclaimed the level of mathematics itself.[7] This, it may be worth noting, was not done through an appeal to naturalness, intuitiveness, or the like, although these factors were weighed; rather, it was through deliberate application of our principles, based upon observed constraints. Our goal remains to find a way through the difficulties noted in chapter 1; our strategy has yielded us a small glimmer of light to guide the investigation forward.

[7]Crucially, our assumption system (see section 3.3) allows us to avoid the Euclideanism that this would seem to imply.

2.2 Justification

We require some sort of notation to represent temporal processes. We shall write $a \to b$ when it is meant that *we exchange, in time, the object a for the object b*. To mathematical activity (viewed from the disengaged state in which Wittgenstein encouraged us to observe ourselves) we thereby attribute no more structure than is absolutely necessary to the mathematical subject, or *possessor*: this, if nothing else, occurs during the moment of inference from a to the conclusion b. a is given over for b; a and b change hands; in other (more dispassionate) language, we might say that *a transforms* into b; thus we will call such a construction either a *process* or a *transformation*.[8] These shall be construed not as *binary* operations but as *chain-forming* operations. The basic idea is simply that $a \to b \to c$, for example, is taken to be a basic expression, a chain beginning with a and ending with c.[9]

We may infer from this interpretation that $\to$ gives rise to a universal poset $(\mathcal{W}, \to)$ of objects: reflexivity, anti-symmetry, and transitivity[10] are manifest properties of the temporal development of objects as we recover it from the mathematical intuition. For the intuitionist, they flow from the intuition of subjective time on which all mathematical construction is

[8] It is lore among mathematicians that the Indian genius Ramanujan had an unusual method: instead of writing out a careful record of his progress on paper, he would write an expression out in chalk upon a slate, erase it, and then rewrite it in a new form; he might repeat this several times. When he was finished, he would write down the result, having no concern for its proof. If we imagine this emission-acquisition event (write-erase-rewrite), we can conceive of it as an object (an "act" or process), and diagram it: $a \to b$. This idea is akin, though not identical, to the "first act of intuitionism," the generation in mental awareness of a two-ity. Brouwer calls two-ity "the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory" ([4], pp. 140-141).

[9] If I travel from Athens to Washington, D.C., and then continue north to Baltimore, then I may view two *trips*, or simply *one*—there is categorial heritability here. Perhaps the most important reason for proceeding in this way is that it smooths the proof theory considerably in section 3.2. Later we will prove that transitivity of the fundamental relation $\to$ in fact holds, on the basis of its mathematical nature as a chain-forming operation, among other things. We will, however, adopt a transitivity axiom and a binary transformation relation once we begin working with models (see p. 62).

[10] A relation $\leq$ is reflexive if $x \leq x$; anti-symmetric if $x \leq y$ and $y \leq x$ implies $x = y$; and transitive if $x \leq y$ and $y \leq z$ implies $x \leq z$.

based. For the formalist, they are the properties that it would be impossible to withdraw from linguistic symbols, because they are intrinsic in them.

Our next task is to obtain a theory of mathematical justification which respects the boundaries sketched in the previous section. We shall secure our deductions through the adoption of proof-theoretical rules, processes which we shall henceforth call *axioms, rules,* or *proof-theoretical axioms*. (We refer back to the axiom concept abandoned in the last section with the term *logical axiom*.) Collectively, we shall refer to this element of the calculus, following Herbrand, as the *theory*, or the *active theory*.[11] Since we are well aware that the philosophical investigation of the rule concept inexorably leads to deep-seated challenges related to the aforementioned *biology of mathematics*,[12] we make note of the difficulty without attempting to resolve it in the space we have here. We simply grant that a certain representation, a certain object—a unity, a diagram in space of a schematic pattern of progression through time—can be understood *as* a rule, and can also *become* one—can, in other words, be *adopted* or *assumed* by a mathematical subject, and become a part of theory. In such a representation *variables* (a, b, c, etc.) may be present; it is understood that the rule denotes a permitted transformation regardless of the identity of the objects in the positions held by variables.

In our view, then, theory (the justificatory basis of proof-theoretical theorems) exists in a purely temporal state, as tacit knowledge of *can* and *must*. It is—what we are. It is the whole invisible fund of processes delimiting our freedom of action as mathematical agents—visualizable as channels or gateways that recur indefinitely. We will agree that there is this overarching presence in all of our proofs, namely, the patterns of motion which are legally ordained, i.e., possible in the mathematical sense.

[11]The active theory contains *all* that which governs us—the fundamental axioms, definitions, and abbreviations. Putting aside the style in which they are introduced, it seems that the distinction between these concepts is that in the case of a definition or abbreviation, consistency and independence are intrinsic, though to verify this conjecture would demand that we define terms more carefully.

[12]See the *Philosophical Investigations*; among others, Wittgenstein's analysis was furthered by Millikan and Kripke.

As euphony suggests, we will sometimes refer to objects synonymously as *expressions*. The distinction between an object which is a process $a \to b$, i.e., one which is semantically a transformation, and has the top-level form $a \to b$ or $a \leftarrow b$, and an object which does not possess this property,[13] will be important throughout our work, so we shall reserve the term *object in the strict sense* (or *strict expression*[14]) for the latter, and usually *process* for the former.

We agree to write $a = b$ when we mean that a is reversibly exchanged for b. For this we *can* write $a \to b \wedge b \to a$, or, alternatively, regard the note "this process which I now do is reversible" as only a courtesy (the distinction will not matter after chapter 4). We call such a process a *reversible process*, *equality*, or *equation*. The use of the symbol $=$ and the terms equality, etc. is quite deliberate. We make no distinction between arithmetical, set-theoretical, and logical interchangeability (equivalence, equality), since there is no basis for such a distinction given our point of departure in chapter 1.

Herodotus once wrote, "Egypt is the gift of the Nile." A river flows through the still valley. The lesson we draw from proof theory is that mathematics does not begin until time breaks an abstract silence, until what becomes (τὸ μὴ ὄν) calls on what is (τὸ ὄν) to give reply. In this sense, perhaps, the infinite, like the pea beneath the mattress, has been hiding in the bedrock of formalism all along.

2.3 The Economic Myth

We conclude this section with a short intuitive discussion to help orient the reader, so that he or she can more rapidly comprehend the whole created from the various elements of the theory introduced below, which might perhaps appear strange and new at times. It also serves to disclose somewhat the design principles which have motivated the author during his work.

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[13] This distinction is made using the principle of excluded middle; intuitionistically the two-sided distinction between strict expression and process may "split" into three or more.

[14] For example, polynomials, integrals, etc. fall under the heading of objects in the strict sense.

Let us imagine that all of mathematics is taking place within a vast concrete trading economy. In this economy are owners, who go about their business, trading according to laws which govern all just and fair exchange. Among these owners there are a unique kind, who seek only to gain, to purchase, the knowledge of new laws. Such new laws, once found, are added to the ones already known, and supply owners with an understanding of the inner nature of the economy itself—the system, as a whole, governed by the laws.

How is it that a new law may be obtained? In this ideal economy, they are generated from the principles of value. A law is known which says that a thing a, exchangeable for a thing b which may itself be exchanged for c, may—by a Principle of Absolute Value holding among owners—be exchanged also directly for c. Likewise, a may be exchanged for a symmetric copy of itself, and two entities, each of which may be the price at which the other is obtained, are said to be the same in value.

Another observation. A collection of things, when it is known that individually they may be exchanged for others, may be exchanged *en masse* for the batch of items they may be exchanged for in isolation, by a certain Principle of Independent Value, proclaiming that the intrinsic value of any entity is independent of any transfer of ownership, and the locations of other objects within the economy.

Finally, this. Whatever may be deemed fair from the presumption that a set of possessions may be exchanged for a given product or commodity, becomes a law of second order, that is, a law governing the exchange of law for law.

What does the mathematical owner have in his possession at the start of his economic life? What does he need with which to purchase the first law? Since law is a public commodity—freely shared by all within the realm—what he purchases comes at no price. He need only share with other owners in the principles of value, and he is left free to pursue his aim. In the mathematical economy, one begins in possession of nothing—an equal amount for all.

In the next section, we will implement and formalize the ideas which this brief illustration motivates, in order to obtain a functional deductive system.

In presuming the concept of a rational and fair economy, have we already implicitly assumed the presence among agents of a judgement faculty, and the classical logic of assertions? The question is meaningful, perhaps, and many others like it might be raised which either reflect the spirit of the illustration, or are critical of it. The reader should be sure to understand this. However the author implores the reader to consider not only the purpose of the above discussion as a beginning point of the formalism which begins soon, but also as one of its *endpoints*. This is related, for the author, to the important and tricky matter of how philosophy and mathematics can have a successful exchange. For it is neither a steadfast law in philosophy, nor in mathematics, that with added scrutiny, comes added depth. While (therefore) the aim of this illustration is to make clear in as few symbols as possible in just what manner we have departed from the norm in logic (while striving to preserve a great deal—as much as was felt possible—of its mechanics and other particulars), it also carries a purpose of some centrality, which is to clarify the sense in which we say: to find the inner, look outwardly. This, which we have once again returned to, is the hypothesis at the seat our work, which we believe is the mathematician's confident conviction, and the keen psychological force which led Hilbert to formalism. In fact, it led Brouwer to a philosophy by no means dissimilar in its faithful optimism amidst the ever-steepening decline of apriorism in the West: the conviction was equal in both of these two great figures that the source of the laws is not beyond our knowing, but beneath our very noses, among us in the midst.

Chapter 3

Fundamental Proof Theory

In this chapter we present the proof-theoretical tools we shall employ: *deposition, condensation,* and the method of *assumption*, discussing mathematical and philosophical motivation, and familiarizing notation. Though the material here may seem basic, it forms the vital link between the contents of the preceding chapters and those that follow.

3.1 Deposition

We see that not all expressions are isotone[1] with respect to transformation. This fails immediately: suppose that a implies b, for which (from the considerations above) we may write $a \to b$. We know that a transforms (at will) into c in time. If such a transformation occurs *within* the form $a \to b$, then surely $c \to b$, a fallacy. Consider, too, expressions involving the unary operator $\neg$. It is everywhere clear that isotonicity gives way to antitonicity when the object to be transformed within a context is colored by a negation symbol. However, intuitionistically and classically, $\neg\neg a$ gives $\neg\neg b$ when $a \to b$.

These considerations lead us to ask: when, in general, is an expression isotone—or, posing the question slightly differently, *in what spatiotemporal contexts does $a \to b$ mean that a may be transformed into b*, and in what contexts is this not the case? One of the tasks ahead is to prove (for we shall be careful not to assume it) that all strict expressions which we define are isotone. We shall also be able to obtain a succinct set of laws characterizing the phenomenon

[1]Following Birkhoff (see [1]) a function ψ on a poset $(W, \to)$ shall be *isotone* if $a \to b$ implies (is or may be exchanged for) $\psi(a) \to \psi(b)$, and *antitone* if the latter condition holds whenever $b \to a$.

of antitonicity in the wider realm of processes, in part due to the definition we use for negation $\neg a$, following the intuitionists in reading an abbreviation for $a \to \bot$.[2]

These principles shall constitute a *generalized system of substitution.* In all past accounts which the author is aware of, substitution—taken in the broadest, most universal sense—has been only incompletely developed: when an object a and an object b are equivalent, there is universal exchangeability, irrespective of context. Nothing else is said, even about the possibility of a substitution which might be partial—a substitution, that is, might be performed in which only the i^{th} occurence of a in an expression ψ is exchanged for b—in spite of the fact that such substitutions are not infrequently made in mathematics.[3]

These tools will be carefully prepared for general use in our calculus. This will greatly distort the sense in which we normally refer to substitution, so in order to avoid confusion, we will only use the term *substitution* as it is already commonly understood. For the generalized notion, we shall introduce the term *deposition.* The process of deposition, in which we depose the object a, the *deposand*,[4] and in its place leave the *deposit* b, will be written by using what we refer to as an *axis*, or *deposition axis*, thus:

$$\begin{array}{r|l} \psi(a) & \to \psi(b). \\ a \to b & \\ \hline \end{array} \tag{3.1}$$

The L-shaped line is a convenient way to specify the transformation which justifies the conclusion without an obtrusion on the flow of the inference chain being followed in the top row (which might have several steps preceding, and several more following after). It should

[2] Practical consideration of generalized substitution (see *infra*) impels one to consider negation to be a process, in order to achieve simplicity, as much as one is impelled to say out loud that the value -1 is a real, concrete integer, and not a construction out of two or more different elements. Philosophically, too, we have observed that due to Carroll's paradox, it is manifestly not possible to extricate from calculation the notion of time. Therefore the "process-free" logic that might be gained by transforming all logical processes into objects in the strict sense, say, by using the form $\neg(a \wedge \neg b)$ must be unobtainable. These various considerations make it highly likely that negation is—and should in fact be—a process, regardless of whether logic is viewed Platonistically or instrumentally.

[3] To give very simple but quite typical example, x^2 might be revised to xy, when $x = y$.

[4] On analogy with the term *subtrahend*, "that which is subtracted," this is "that which is deposed."

not be taken on analogy with the “branching” of a proof tree; the process in the lower row of the axis is, *sensu stricto*, nothing but a courtesy which may be omitted at the discretion of the writer. One should consider it an exchange of $\psi(a)$, whose justification is politely noted to the reader, for $\psi(b)$, and not a combination of the two elements $\psi(a)$ and $a \rightarrow b$. Also note—in any deposition, the *subthesis*, the process in the crook of the L, cannot be chosen arbitrarily—it must be valid[5]—and in our work it is usually an axiom, or an assumption recently made. We may also write, using Sub for substitution:

$$\begin{array}{r|l} \psi(a) & \underset{\text{Sub}}{=} \psi(b). \\ & \\ a = b & \\ \hline \end{array} \tag{3.2}$$

In a substitution, every instance of a must replaced with the object b; thus in most substitutions the deposand is a variable or constant.

We now make observation of what we do not deem an “axiom”; it is simply manifest from the kinematic semantics we have already discussed:

Observation.

$$\begin{array}{r|l} a & \underset{\text{Dep}}{\rightarrow} b. \\ & \\ a \rightarrow b & \\ \hline \end{array} \tag{3.3}$$

From this tiny mustard seed, if you like, results which strengthen deposition will grow gradually (beginning in this chapter and continuing in chapters 4 and 5) until we have obtained a comprehensive and robust tool.

The reader has probably noticed that the arrow on the main row in process 3.3 has below it the three letters Dep, standing for deposition. When we desire to be perfectly explicit in our reliance upon the justification supplied by the performed transformation above, we might write this symbol below the arrow. We call the markers we place below transformation arrows *referentials*. They are never required, but are a very convenient and legible system for

[5]As we will see when we come to section 3.3, it is more correct to say that it is *locally permissible*, or *locally valid*; this distinction is illuminating, though it has no formal effect.

annotating proofs.[6] Most axioms and theorems will include referentials in their statements from now on, so that they may be referred back to in later proofs. For those that do not contain referentials, we may refer to them by the number under which they are listed, e.g., T. 11 for Theorem 11, etc.

3.2 Condensation

Recall that, since it symbolizes sudden or gradual change, we understand $\rightarrow$ to be a chain-forming operation. One can conceptualize the transformation or process as graduating along three possible levels. First, relative to its predecessor transformation it can proceed as though walking forward: this we denote by $a \rightarrow b \rightarrow c$ and call a *chain*; it is an object, a regular expression. It can also proceed as though walking backwards, $a \leftarrow b \leftarrow c$, and we call this a *reverse chain*, and do not allow any other kind of chain; processes that are reversible, however, can always be written using the symbol $=$. Second, a transformation can proceed "upward" to a higher order process: this we denote using parentheses: $a \rightarrow (b \rightarrow c)$. Third, it can proceed "downward" to transform a process to (possibly) a strict expression: this we denote $(a \rightarrow b) \rightarrow c$. This forms what we informally call a *stack* of processes, a much more stubborn construction than the other two—for example, $\neg\neg a$ is a stack.

For the reader's convenience, we now extend this discussion, laying out the complete set of order of operations and parenthesis conventions we will use throughout our work. Among our basic *operations* or *connectives*, conjunction $\wedge$ and intersection $\cap$, intersection binds most strongly. Following these in strength are the transformation symbols $\rightarrow$, $\leftarrow$, and $=$. Following the usual convention, $\neg$ for negation binds most strongly of all, and we allow the strength of transformation symbols to be greater than that of disjunction $\vee$. The argument of the symbol $\underline{a}$ is always explicit, so it needs no parenthesis convention. We shall write long transformation arrows in place of parentheses in expressions in which there is a transformation of one or

[6] We would also like to comment, on the pedagogical side, that they are an excellent aid to the learner. They are also convenient when one is examining the interdependencies of theorems upon one another.

more processes (a process in which processes are exchanged, or a process *of second order*). For example, instead of $((a \to b) \to (c \to d))$, we may write $a \to b \;\longrightarrow\; c \to d$. We allow disjunction to bind more strongly than long arrows. Therefore, the binding sequence in full reads: $\neg, \cap, \wedge, \{\to, \leftarrow, =\}, \vee, \{\longrightarrow, \longleftarrow, =\!=\}$.

We shall now, temporarily, elaborate our notion of transformation slightly. Let us agree that axioms, i.e., assumed processes, are atomic, *one-step* processes. For the sake of concreteness (though we will never bother with this) let us agree that should we desire to write a one-step process explicitly, we can use the symbol $\Rightarrow$, hence $a \Rightarrow b$, etc. Now: we say that the standard process $a \to b$ is in fact, in general, a condensed string of one-step processes, or *condensed chain*. When we write it, that is, we are essentially running rapidly through all the individual steps, without pausing to write any of them down.

We shall thus call a *proof by serial condensation* any proof of a process $a \to b$ which has the form $a = c_1 \to c_2 \to \ldots \to c_n = b$, or an analogous proof of a reverse chain. If desired, we can write in the "condensation" step—*though this is only a heuristic*, which is often useful for the sake of clarity. It is not a "formal" transformation; we thus write it using the symbol $\rightsquigarrow$. We will announce this method of proof as a principle.

Principle 1 (Proof by Serial Condensation)**.** *From already known proofs*

$$a_1 \to a_2 \quad a_2 \to a_3 \quad \cdots \quad a_{n-1} \to a_n, \tag{3.4}$$

to conclude

$$\rightsquigarrow \; a_1 \to a_n, \tag{3.5}$$

by Serial Condensation.

As an illustration, we can now strengthen Deposition somewhat with the following observation (antitonicity in the antecedent):

$$\left.\begin{array}{r} a \to b \\ \\ a \leftarrow c \end{array}\right| \underset{\text{Dep}}{\longrightarrow} c \to b. \tag{3.6}$$

This follows immediately from transitivity:

$$c \to a \to b \quad \rightsquigarrow \quad c \to b. \tag{3.7}$$

The important observation here is that it is often useful to treat the process as a deposition (rather than a law about chains) since the expression in the antecedent behaves, with respect to $\leftarrow$, precisely as normal expressions do with respect to $\to$.

We now define: $a \wedge b$ shall be the *conjunction* of a and b. *The conjunction $a \wedge b$ is an object; it is the set of the object a and the object b.* Because the conjunction is understood by us to denote a *finite set*, we may adopt the following axiom as a constructive principle, which shall imply the isotonicity of conjunctions.

Axiom 1 (Conjunction Formation and Decoupling)**.** (1) *From a pair of objects a and b, to obtain their conjunction,*

$$\begin{matrix} a \\ b \end{matrix} \;\underset{\text{Form}}{\longrightarrow}\; a \wedge b, \tag{3.8}$$

and (2) *from a conjunction of a pair of objects a and b, to obtain their abstract grouping,*

$$a \wedge b \;\underset{\text{Dec}}{\longrightarrow}\; \begin{matrix} a \\ b. \end{matrix} \tag{3.9}$$

The style of this axiom (the only proof-theoretical axiom we shall assume which strains the use of the term "axiom") may appear more familiar to the reader if processes 3.8 and 3.9 are written in the vertical format of the "and-introduction" rule commonly employed in proof theory:

$$\frac{a \quad b}{a \wedge b.} \qquad\qquad \frac{a \wedge b}{a \quad b.} \tag{3.10}$$

Their justificatory basis is simultaneously constructive and practical. They offer the advantage of removing a proof-theoretical burden left in place by Heyting's axiom

$$a \to b \;\longrightarrow\; (a \wedge c) \to (b \wedge c) \tag{3.11}$$

given our stated goal of employing the Deposition construction as a basic tool.[7] Axiom 1 provides a new way in which processes can be "condensed" together; we present it in the full generality easily granted once associativity of conjunction is proven in the next chapter, on the basis of the case when $n = 2$ (which immediately follows from axiom 1).

Principle 2 (Proof by Parallel Condensation)**.** *From already known proofs*

$$a_1 \to b_1$$
$$a_2 \to b_2$$
$$\vdots$$
$$a_{n-1} \to b_{n-1}$$
$$a_n \to b_n,$$

to conclude

$$\rightsquigarrow (a_1 \wedge a_2 \wedge \ldots \wedge a_n) \to (b_1 \wedge b_2 \wedge \ldots \wedge b_n), \tag{3.13}$$

by Parallel Condensation.

Note that this principle may be written in the succinct form

$$\bigwedge (a_i \to b_i) \rightsquigarrow (\bigwedge a_i) \to (\bigwedge b_i). \tag{3.14}$$

We will now combine the principles of Serial and Parallel Condensation into a more general reasoning pattern which conveniently replaces them both. The idea of such a demonstration is simple and quite intuitive: some organized mathematico-logical information is displayed. Then it is "condensed" down in an ordinary way into a synoptic statement.

[7] The difficulty, briefly, is the following: our desire is to obtain

$$\left.\begin{array}{r} a \wedge c \\ \\ a \to b \end{array}\right| \underset{\text{Dep}}{\to} b \wedge c. \tag{3.12}$$

whenever $a \to b$ is valid. Consider what happens if we (globally) assume process 3.11. Then difficulty would erupt since $a \wedge c \to b \wedge c$ still needs to be (formally) proven based on the validity of 3.11 and $a \to b$—this requires additional principles and additional work. Even axiom 6 below stubbornly refuses to yield the straightforward exchange of $b \wedge c$ for $a \wedge c$!

Principle 3 (Proof by Condensation)**.** *From a collection or display of demonstrations*

$$\begin{aligned} &a_{1,1} \to a_{1,2} \to \ldots \to a_{1,n_1} \\ &a_{2,1} \to a_{2,2} \to \ldots \to a_{2,n_2} \\ &\vdots \\ &a_{m,1} \to a_{m,2} \to \ldots \to a_{m,n_m} \end{aligned} \tag{3.15}$$

to conclude

$$\underset{\text{Cond}}{\rightsquigarrow} \quad \bigwedge_{i=1}^{m} a_{i,1} \longrightarrow \bigwedge_{i=1}^{m} a_{i,n_i} \tag{3.16}$$

by Condensation.

In such a proof, we refer to the chains in the array (3.15) (somewhat improperly) as a *condensation matrix*. The process 3.16 is called the *condensate*. The components of the condensate have names. The objects $a_{i,1}$ are the *factors* of the condensate, and the a_{i,n_i} are the *products*. This language is occasionally useful to an involved mind. The method of condensation provides very legible proofs, whose complexity is often markedly decreased compared to other formal methods. Note that once a metalevel is admitted, it is easy to verify principle 3. One simply verifies that whenever $a \to b, b \to c$, and $c \to d$ are justified, the processes $a \to c$ and $a \wedge c \to b \wedge d$ are justified as well.

3.3 Simple Assumption

Our last method of proof involves the introduction of our most valuable tool: the *simple assumption system*. This system—and it does make sense to call it by that name, for it is like a wonderful machine for proving almost anything with the greatest of ease—involves the inverse of the concept of *assumption* introduced above in section 2.2. It says by *assuming* temporarily that a process is permitted, we may, having performed some derivation, return to the previous order by *unassuming* the process, and thereby obtain a new law. So far, we have obtained ways to confirm the validity of processes which involve only the strict observation of axioms; these, then, are direct methods—the justification of repeated use of processes proven in this way is that the demonstration given in the proof may simply be repeated (gone through "very fast, without writing anything down"). The method of assumption is more subtle than this, and more intertwined with the philosophy matters discussed above. However, since we characterized the axiom as a rule governing creative activity, the power of assumption is not taken over arbitrarily—we have already introduced the notion of assuming or adopting a rule, and now only with to augment the concept with the principle that it provides sufficient justification for certain kinds of transformations, certain kinds of trades. If we had proceeded in some other way, we might not, having reached this stage, be so ready to introduce the notion that a hypothesis can be exchanged for processes taken from the world of possible actions put into play by its validity. This, we think, bodes well for the decisions made in coming thus far.

We write £ $a \to b$, "let $a \to b$," when we wish to assume[8] a given process. We call this the *assumption clause.* Since initially we wish to assume only processes—we will see later on how to assume conjunctions and other constructions—we define a *simple process* to be an object writable in the form $a \to b$, for arbitrary a and b. For example, $a \leftarrow b$ is also a simple process, but not $a = b$, that is, $a \to b \wedge b \to a$ (see p. 22). We say that when a process is extracted from active theory during the method assumption, it is *unassumed,* or *abolished.* This we call the *abolishment clause.* We can now state our principle succinctly:

Principle 4 (Proof by Simple Assumption)**.** *From assuming a simple process $a \to b$, and thereby find a demonstration of a process $c \to d$, to conclude*

$$\underset{\text{Ab}}{\rightsquigarrow} \quad a \to b \longrightarrow c \to d \tag{3.17}$$

by Simple Assumption, provided that $a \to b$ is the most recently adopted assumption that has not yet been abolished.

[8]Applying the assumption system, subjectively, is rather like employing a local hypothesis, rather than a global hypothesis (an axiom or definition). Of course, as the complexity of the existing assumption structure increases, these categories naturally become less meaningful, but *local assumptions* will generally be assumptions which are made for the express purpose of applying the method of assumption.

The adjective "simple" is meant to distinguish this principle from a more general assumption system which we will define in section 4.3. The referential Ab stands for "abolish": as the proof comes to an end, we remove the transient freedom granted by the assumption by abolishing it as a rule, attaching it to the conclusion and discharging it from the active theory, producing what amounts to a report, in one environment, from another environment, which has been converted into an exchange of objects.

"It must be admitted," said Frege in his famous work of 1879, "that the way followed here is not the only one in which the reduction can be done... There is perhaps another set of judgments from which, when those contained in the [proof-theoretical] rules are added, all laws of thought could likewise be deduced" ([10], in [14], p. 29). We find, from our own long experience of the overwhelming variation in logical axiomatics, which is more multitudinous than the ways in which, in chess, the opening can give way by line of play to a middle game, that such intellectual honesty on Frege's part would be unanimously expected, were it not so extraordinary. In practice, however—when one is thinking broadly about how to condense or uncondense, and whether something should be treated as an assumption or as a factor for condensation—the objects under consideration behave in such a way that little else could be more natural than the notions introduced in this chapter.

Finally, concerning consistency. Should the constructive-formalistic kernel of certainty be accepted—i.e., should we countenance the notion that the extralogical foundation of proof theory and finite, surveyable construction in time and space will not lead to a contradiction $\emptyset \rightarrow \perp$ due to its own primordial compulsion upon the inner workings of our minds—then neither will the same body of thought once the *assumption system* has been added to it, for assumption proceeds only from hypothesis to conclusion. With an assumption open, there is always an emergency cord to pull to save mathematics from the universal calamity of a conditionless antilogy,[9] therefore the use of assumption can never result in any inconsistency.

[9]That is, inconsistency, the total collapse of value, Value Equivalence, or Unity ($\emptyset = \perp$). It is worth noting the affinity here to the Plotinian concept of the One, as well as the obvious comparison to be drawn with the primordial consciousness of two-ity so important in Brouwerian philosophy.

If we proceed forward by building only on a sound basis by introducing axioms which are verified together in surveyable, computable models, this consistency shall be maintained. Since we already have all the supple power of assumption free at hand, we can now rapidly develop the consequences of these axioms, frequently with only a small investment of effort.

Chapter 4

Conjunction

In this chapter, we introduce several axioms, and begin applying them to construct the lattice which is fundamental to our system, staying throughout within what in logical literature is sometimes called the *and-implication fragment*.

4.1 Selection, Generation, Order

Given a set whose elements are distinguishable and objectively surveyed, the act of *discarding* whatever part of them one has no present desire for is observed, in intuition, to be among the most essential constructive freedoms. The following axiom codifies and makes definite this principle.[1]

Axiom 2 (Selection)**.** $a \wedge b \underset{\text{Sel}}{\rightarrow} a.$

In order to prevent conjunctions from behaving as sums,[2] we introduce the next axiom.

Axiom 3 (Generation)**.** $a \underset{\text{Gen}}{\rightarrow} a \wedge a.$

Per stands for "Persistence."

Theorem 1. $a \underset{\text{Per}}{\rightarrow} a.$

Proof. $a \underset{\text{Gen}}{\rightarrow} (a \wedge a) \underset{\text{Sel}}{\rightarrow} a \quad \underset{\text{Cond}}{\rightsquigarrow} \quad a \rightarrow a.$ □

[1]This axiom was proven by Heyting (see [16]), but his proof is not valid for objects in general. We will return to Heyting's system in section 6.1.

[2]The reader is invited to use the following dictionary on all the work done so far: for $\wedge$, $\rightarrow$, replace $+$, $\geq$.

Theorem 2. $a \wedge (b \wedge c) \underset{\text{Soc}}{=} (a \wedge b) \wedge c$.

Proof. $a \wedge (b \wedge c) \underset{\text{Sel}}{\rightarrow} a$

$$a \wedge (b \wedge c) \underset{\text{Sel}}{\rightarrow} b \wedge c \underset{\text{Sel}}{\rightarrow} b$$

$$\cfrac{\underset{\text{Cond}}{\rightsquigarrow} \quad (a \wedge (b \wedge c)) \wedge (a \wedge (b \wedge c)) \rightarrow a \wedge b \qquad (a \wedge (b \wedge c)) \wedge (a \wedge (b \wedge c)) \underset{\text{Gen}}{\leftarrow} (a \wedge (b \wedge c))}{} \;\underset{\text{Dep}}{\longrightarrow}\; (a \wedge (b \wedge c)) \rightarrow a \wedge b.$$

Beginning again, we have

$$a \wedge (b \wedge c) \underset{\text{Sel}}{\rightarrow} b \wedge c \underset{\text{Sel}}{\rightarrow} c$$

$$\underset{\text{Cond}}{\rightsquigarrow} \quad a \wedge (b \wedge c) \rightarrow c.$$

Which together give

$$\underset{\text{Cond}}{\rightsquigarrow} \underset{\text{Gen}}{\rightarrow} \quad a \wedge (b \wedge c) \rightarrow (a \wedge b) \wedge c.$$

The converse is derived similarly. □

In this proof, in which some of the formal steps are only sketched, we observe (twice) a Condensation step, followed by a step in which the doubled assumptions are fixed using Gen and Dep. This step is common and will be suppressed from now on.

Theorem 3. $a \wedge b \underset{\text{Cmu}}{=} b \wedge a$.

Proof. $a \wedge b \underset{\text{Gen}}{\rightarrow} (a \wedge b) \wedge (a \wedge b) \underset{\text{Soc}}{\rightarrow} a \wedge (b \wedge (a \wedge b)) \underset{\text{Sel}}{\rightarrow} b \wedge (a \wedge b) \underset{\text{Soc}}{\rightarrow} (b \wedge a) \wedge b \underset{\text{Sel}}{\rightarrow} b \wedge a$. □

Suppose that you hold or possess something—let's say a book. Now imagine that you *give* it away. (You can imagine giving it to someone else, or you can imagine discarding it—it makes no difference.) Think of what you now hold in your hand. Consider the inner conceptual unity of that which you now momentarily possess. We denote this possession, commodity, object, with the symbol Ø.

Now pick an object—let's say your desk. Now imagine an object which it forms a part of—your room or office. Now consider that this object is a part of another object. And another object, and another... Imagine reaching an object at the limit of this indefinitely continuing pattern of thought: one that encompasses or contains every object you encounter or will ever encounter. We denote this indefinite object with the symbol $\perp$.

$\varnothing$, then, we can think of as the familiar empty set, while $\perp$ is a maximal element given to the poset of objects $(\mathcal{W}, \to)$. To the Greeks these were known as *ho kenon* and *ho plethos*; Boole knew them as the classes 0 and 1. We pronounce them *null* and *down*, respectively, and define them formally through the following axioms.

Axiom 4 (Order)**.** $\perp \underset{\text{Ord}}{\to} a.\ \ a \underset{\text{Ord}}{\to} \varnothing.$

The referential Ord shall also refer to the next two theorems.

Theorem 4. $a \underset{\text{Ord}}{=} a \wedge \varnothing.$

Proof. As in Theorem 2, there are two processes to verify.

$$a \wedge \varnothing \underset{\text{Sel}}{\to} a.$$

$$a \underset{\text{Per}}{\to} a$$

$$a \underset{\text{Ord}}{\to} \varnothing$$

$$\underset{\text{Cond}}{\rightsquigarrow} \quad a \to a \wedge \varnothing. \quad \square$$

Theorem 5. $a \wedge \perp \underset{\text{Ord}}{=} \perp.$

Proof. $a \wedge \perp \underset{\text{Sel}}{\to} \perp;\ \perp \underset{\text{Ord}}{\to} a \wedge \perp.$ $\square$

4.2 Deposition Revisited

Next, we return to our development of Deposition, begun in chapter 3. The proof of the following theorem (our first to employ the method of proof by assumption) is remarkably intuitive. When an active assumption is employed, it will usually, as a courtesy, be noted, thus: $\underset{\text{£}}{\to}$. This is to say, "by assumption..." A more precise referencing system is conceivable, but we shall not provide one here.

Theorem 6. $a \to b \underset{\text{Dep}}{\longrightarrow} a \wedge c \to b \wedge c.$

Proof. $\pounds\, a \rightarrow b$. (Let $a \rightarrow b$.)

$$a \wedge c \underset{\text{Sel}}{\rightarrow} a \underset{\pounds}{\rightarrow} b$$

$$a \wedge c \underset{\text{Sel}}{\rightarrow} c$$

$$\underset{\text{Cond}}{\rightsquigarrow} \quad a \wedge c \rightarrow b \wedge c$$

$$\underset{\text{Ab}}{\rightsquigarrow} \quad a \rightarrow b \longrightarrow a \wedge c \rightarrow b \wedge c. \quad \square$$

Corollary 1. $a \rightarrow b \underset{\text{Dep}}{\longrightarrow} (c \rightarrow a \wedge d) \rightarrow (c \rightarrow b \wedge d)$.

Theorem 7. $a \leftarrow b \underset{\text{Dep}}{\longrightarrow} (a \wedge c \rightarrow d) \rightarrow (b \wedge c \rightarrow d)$.

Proof. This may be shown by repeating the argument given in chapter 3 (p. 30). After assuming $b \rightarrow a$, use Theorem 6. $\square$

Thus antecedents are antitone, consequents are isotone. From now on we write

$$\begin{array}{r|l} a \wedge c & \underset{\text{Dep}}{\rightarrow} b \wedge c, \\ a \rightarrow b & \\ \hline \end{array} \qquad \begin{array}{r|l} c \rightarrow a \wedge d & \underset{\text{Dep}}{\longrightarrow} c \rightarrow b \wedge d, \\ a \rightarrow b & \\ \hline \end{array} \qquad \begin{array}{r|l} a \wedge c \rightarrow d & \underset{\text{Dep}}{\longrightarrow} b \wedge c \rightarrow d, \\ a \leftarrow b & \\ \hline \end{array}$$

to incorporate valid processes into a running inference chain. It can be shown using Theorem 9 below that the antecedent of an antecedent process is again isotone, etc. In Frege's concept calculus these principles require several pages to prove.

4.3 General Assumption

Where do adopted assumptions go? The answer, according to our proof-theoretical orientation, is that they become the attributes of the calculator or mathematical subject—intuitively, they don't go anywhere, they simply slip into the context of the calculation itself. Of course, with practice, it is quite natural to think otherwise, and to conceive of active theory as an abstract group of laws, akin to the Constitution or the U.S. Code. In fact, once the notion of an abstract group of assumptions has arisen and been associated with theory itself, it is once again natural, for the sake of universality, to think of theory as a *conjunction* of adopted processes. This is because, so to speak, this *moves back the formal* *envelope*, i.e., just as at the surveyable surface of our derivations we write $a \wedge b \underset{\text{Cmu}}{=} b \wedge a$, we can think of the active theory as a conjunction like any other, which is being actively exchanged in the most primitive assumptive environment, i.e., the environment in which there are no assumptions at all. This would amount to the adoption of the following principle.

Principle 5 (General Principle of Assumption)**.** *The active theory takes the form of a conjunction θ of all the processes which a mathematical subject has adopted at a given time. Therefore* (1) *conjunctions of processes may be adopted and abolished, and* (2) *adopted processes may be abolished in any order.*

We do not adopt principle 5; for various reasons it is best to adopt an equivalent set of axioms. However, because principle 5 is so readily believed—and it is—a question arises. If the active theory takes the form of a conjunction of adopted processes, then what should be done about the object null, the algebraic identity of the conjunction operation? Given that $\theta = a \wedge b$, can I freely adopt or abolish a and b? If we but admit as much, then it follows that calculation

$$\begin{array}{ll} \pounds\, \emptyset. & \\ p & \\ \underset{\text{Ab}}{\rightsquigarrow} & \emptyset \to p. \end{array} \tag{4.1}$$

is valid for every process p. In other words, $\emptyset$ can not only be thought of as the "empty" set—but also as a formal tautology. While it is not entirely implausible that calculation 4.1 is invalid—intuitively it seems to merely tell us that valid processes are "free" and may be obtained at no cost—it is possible to argue that the calculation 4.1 is invalid, since $\emptyset$ is not a *process*—in that case, the tautology and the empty set must remain separate in standard fashion. However, consider that the proof theorist will recognize the validity of the statement

$$\emptyset \vdash \mathrm{T} \tag{4.2}$$

where T symbolizes the tautology of propositional calculus, since in normal practice, to the left side of the symbol $\vdash$ is placed a set S of axioms and propositions, and on the right is placed some formula for which there exists a formal proof using only the elements of S. Thus our vision of an axiom-free, proof-theoretical logical foundation encourages the former view. Moreover, no inconsistency arises from the assumption that the calculation 4.1 is valid (see section 6.2). Since by it a compelling structural simplification is gained (*cf.* to Theorem 11 below), and since its consequences have never yet to our knowledge been investigated, we shall assume that it is indeed valid. A direct and simple way to do this is to simply write $\varnothing = \varnothing \to \varnothing$, that is, to adopt the following as an axiom.

Axiom 5 (Null)**.** $\varnothing \underset{\text{Null}}{\longrightarrow} (\varnothing \to \varnothing)$.

As noted previously, we write $\neg a$, "not a" for $a \to \bot$. Similarly, from now on we shall write $\underline{a}$, "vis a", in place of $\varnothing \to a$.

Lemma 1 (Tacit Null)**.** $a \to b \longrightarrow \underline{a \to b}$.

Proof. $£\, a \to b$.

$$£\, \varnothing \to \varnothing.$$

$$a \underset{£}{\to} b$$

$$\underset{\text{Ab}}{\rightsquigarrow}\quad \left.\begin{array}{c} \varnothing \to \varnothing \longrightarrow a \to b \\ \varnothing \to \varnothing \underset{\text{Null}}{\longleftarrow} \varnothing \end{array}\right| \underset{\text{Dep}}{\longrightarrow} \varnothing \to (a \to b).$$

□

In the following proof, we shall readopt the assumption $a \to b$ assumed in the proof of Lemma 1. Therefore within the environment of the proof, the exchange $\underline{a \to b}$ is permitted. We shall denote this with a "knowledge clause": we write $\mathfrak{K}p$ or $\underset{\text{R}}{\mathfrak{K}}p$ to mean that within the assumptive environment in which the clause appears, a proof of p can be given, where an optional referential R refers to previous work. We think of these processes as "known," and may write $a \underset{\mathfrak{K}}{\to} b$ when using them. We shall also make first use of the symbol $\divideontimes$, introduced for the sake of readability as proofs grow lengthier. It will be used as a kind of local pronoun, either to assign or to denote the objective nearest at hand, and can be thought of as representing "that which is sought."

Theorem 8. $((a \to b) \wedge (c \to d)) \to e \;\longrightarrow\; (a \to b) \to ((c \to d) \to e)$.

Proof. $\pounds(a \to b) \wedge (c \to d) \to e$.

$$\pounds\, a \to b.$$

$$\underset{\text{L. 1}}{\mathfrak{K}}\; \underline{a \to b}.$$

$$\begin{array}{r|l} c \to d \underset{\text{Ord}}{=} \varnothing \wedge (c \to d) & \underset{\text{Dep}}{\to} (a \to b) \wedge (c \to d) \underset{\pounds}{\to} e \quad \underset{\text{Ab}}{\rightsquigarrow} \quad \underset{\text{Ab}}{\rightsquigarrow} \quad ※\,. \\ \underline{\varnothing \underset{\mathfrak{K}}{\to} (a \to b)} & \end{array}$$

□

Theorem 8 brings us close to the full principle of general assumption, only on the basis of the Null axiom. If we assume that $(a \to b) \to ((c \to d) \to e)$, we have

$$\begin{array}{r|l} (a \to b) \wedge (c \to d) & \underset{\text{Dep}}{\to} ((c \to d) \to e) \wedge (c \to d). \\ \underline{(a \to b) \underset{\pounds}{\to} ((c \to d) \to e)} & \end{array}$$

Thus the only gap in the proof is the principle of Modus Ponens in the weak form

$$(a \to b) \wedge ((a \to b) \to c) \;\longrightarrow\; c. \tag{4.3}$$

Obviously this would be provable if the principle that conjunctions of processes can be assumed and abolished together were adopted. This we grant in the form of an axiom.

Axiom 6 (Weak Modus Ponens)**.** $(a \to b) \wedge ((a \to b) \to c) \;\longrightarrow\; c$.

As for Modus Ponens in the *strong* form

$$a \wedge (a \to b) \;\longrightarrow\; b, \tag{4.4}$$

one is better off avoiding this principle, since it is not intuitionistically valid. Because the transformation relation $\to$ plays the role of containment among strict expressions, a weak counterexample can be produced in intuitionistic set theory: if a is a strict expression and the most that can ever be said of whether b is contained in a is $\neg\neg(a \to b)$ (that is, $\neg\neg(a \supseteq b)$), then process 4.4 does not hold—rather its status, too, is that of absurdity of absurdity (for a simple model to illustrate this, see p. 65).

We therefore have the valuable "collective assumption" theorem.

Theorem 9 (Collective Assumption)**.** *For all processes p, q, and objects a,*

$$p \to (q \to a) \underset{\text{CA}}{==} q \to (p \to a) \underset{\text{CA}}{==} (p \wedge q) \to a. \tag{4.5}$$

Proof. We have that $p \to (q \to a) == (p \wedge q) \to a == (q \wedge p) \to a == q \to (p \to a)$, from Theorem 8 and the considerations of the discussion following it. □

Corollary 2 (Transitivity)**.** $(a \to b) \wedge (b \to c) \longrightarrow a \to c.$

From this point on we shall use the general assumption system, as is justified based upon Theorem 9. This is a great convenience—for example, we obtain a short proof of the following theorem.

Theorem 10. $\underline{a \wedge b} = \underline{a} \wedge \underline{b}.$

Proof. $£\, \varnothing \to a \wedge b.$

$⨳\, \underline{a}, \underline{b}.$

$\varnothing \underset{£}{\to} a \wedge b \underset{\text{Sel}}{\to} a, b.$

$£\, \underline{a} \wedge \underline{b}.$

$\varnothing \to a$

$\varnothing \to b$

$\underset{\text{Cond}}{\rightsquigarrow} \quad \varnothing \to a \wedge b \quad \underset{\text{Ab}}{\rightsquigarrow} \quad ⨳.$ □

Lemma 2 (Birthday)**.** $\varnothing \to a \underset{\text{Bir}}{\longrightarrow} a.$

This lemma says that, in general, objects behave rather like half-processes, with respect to null.

Proof. $£\, \varnothing \to a$

$\varnothing \to a \underset{\text{Ord}}{\longrightarrow} \varnothing \underset{£}{\longrightarrow} a$

$\underset{\text{Cond}}{\rightsquigarrow} \quad \varnothing \to a \longrightarrow a$

$\underset{\text{Ab}}{\rightsquigarrow} \quad (\varnothing \to a) \longrightarrow (\varnothing \to a) \to a.$

$\underset{\text{CA}}{==} \varnothing \to a \longrightarrow a.$ □

Corollary 3. $\emptyset \to \bot \longrightarrow \bot$.

Theorem 11 (Null-Process)**.** $a \to b == \underline{a \to b}$.

Proof. Lemmas 1 and 2. □

Theorem 11 will play a significant role in section 5.5.

Thus, using axioms 5 and 6, we have obtained the full power, conceptual as well as mathematical, of the principle of general assumption, by building upon a basis whose philosophical context has been studied.

Chapter 5

Intersection and Disjunction

In this chapter we will introduce a new operation called *intersection*, $a \cap b$, and derive its basic properties. Then we will prove some basic theorems of intuitionistic logic.

5.1 Intersection

Recall that, following our commitment to the extralogical interpretation of assertions, we have extended the notion that Ernst von Glasersfeld has called "unition" and David Finkelstein has called "bracing" into the realm of logic, by marrying the notions of conjunction and union, together along with the set-theoretical pairing operation $\{a, b\}$. From the nineteenth century to the present, the opposing marriage has played on the imaginations of thinkers who were compelled by the relationship of disjunction $\vee$ and union $\cup$ in expressions using the set-theoretical comprehension scheme,

$$\{x \mid P(x)\} \cup \{x \mid Q(x)\} = \{x \mid P(x) \vee Q(x)\},$$

to the view that the two operations were to be viewed as correlates, each confined to a distinct and disjoint domain. The nature of a *condition* governing the formation of a set, however, does not have any meaningful direct connection to the *product* which collects the set together from its constituent parts, and therefore the connection ought to rest upon some foundational proof. Closer study in foundations, however, leads one to reconsider the intuition that guides one to refer to both the conjunction $a \wedge b$ and the two-member set $\{a, b\}$ with the words *a and b*. The bare intuition's sense of $a \vee b$ is in fact quite different, by comparison—there is no chance that anyone could mistakenly say *a and b* here. If one analyzes this distinction, one is led once again to that enigmatic role which time plays in logic, and to the suspicion that *the disjunction is a process*. While the sense of $a \wedge b$ immediately appears to intuition as an expression coupling objects in space, independent (at least, for the present) of the development of time in either a real or intuitive sense, one strains to remove the presence of time (or mathematical correlates like conditions or possibilities) from any analysis of $a \vee b$. The reality of the distinction is seconded if one notes the remarkable skew in the logic of mathematics—there is uniformly common use of both the logical conjunction and the forward implication, while the use of disjunction and the reverse implication is far more restrained, and disjunctions (usually finite and small where they appear) almost always entail special proving techniques. Any reduction of the fundamental connectives to a Boolean algebraic system fails to explain why these skewed frequency levels and discrepancies in technique should exist, since the classical logical connectives have a close symmetry akin to the Boolean operations of set theory, which are used with close to equal frequency in mathematics (although the union is probably slightly more common) and with highly similar techniques.[1]

The study of proof theory reinforces this suspicion. In the parts of logic that mathematicians and laypeople are most familiar with (predicate calculus, Aristotelian logic, Zermelo-Frankel set theory, model theory) the boundary separating set theory and logic is firmly drawn. In proof theory, however, the boundary which separates set theory and logic is blurred under the relentless pressure of the highest standards of rigor. In considering many normally overlooked issues (such as how many times an axiom is used, how different regions of a proof combine to form an epistemic unity, etc.) one is quickly led down that path forged by Hilbert, to consider the proposition as an object, to form and unform sets of these, and to rely on these principles to secure one's analysis. Maintaining the separation between logic and set theory under these conditions becomes increasingly tenuous, and the argument for

[1]Moreover, such a foundational approach simply begs the question. Once it is transformed into a subject of algebraic analysis, a system gives rise to a new level at which the same logical instruments and devices as before are back in play, and the need for a foundational account for these remains, in order for the calculator existing in time to understand what he is doing and why.

their separation becomes increasingly threadbare. A fundamental metatheorem which holds in most systems, known as the deduction theorem, states that

$$A \cup \{b\} \vdash c \quad \text{iff} \quad A \vdash b \to c, \tag{5.1}$$

which gives the cluttered, but still suggestive relation

$$\{a\} \cup \{b\} \vdash c \quad \text{iff} \quad \emptyset \vdash a \wedge b \to c. \tag{5.2}$$

In order to establish the link between tautology and the empty set suggested by general assumption and equation 4.2, one must embrace the correlation of $\wedge$ and $\cup$, and vice versa. Instrumental necessity thus, as it happens, reinforces the more intuitive of the two schemes weighed above.

Tradition has played a role in preventing, to the knowledge of the author, any previous attempted unification of this kind. At play at least in part is a certain perfectly reasonable expectation that truth (tautology) ought to be *maximal* and topmost in a propositional lattice. Indeed, such language entirely fills the literature as well as the vernacular. This may be due in part to the historical fact that the high seat of truth in our cognitive schemes has a classical cognate in Plato and all later Platonic thinkers.

The two possibilities, truth up and truth down, have contrasting characters, and play in different ways upon the imagination; the latter is the "gravitational" model, in which implications tend downwards toward a central or innermost tautology, and the former is the "hegemonic" model placing truth at the utmost peak, towards which implications "ascend."

Attracted by the prospect of gaining an unknown measure of benefit from recasting proof theory in a practicable new form, we have been forced to adjust the symbolism in order to symbolically integrate two opposing orientations in our tradition. By using the two symbols $\wedge$ and $\cap$ as the fundamental symbols of meet and join, union and cross-cut, in our lattice, we believe we have preserved what we could of the on-board intuition packed into these characters, while not lending any obscurity to an important and difficult philosophical matter, one which we have been carefully sidestepping until now. One might characterize this difficulty in the following way: the genuine thought, it seems, is not due to the logical intuition alone, for it is always *viewed*. However, the genuine object is not due to the set-theoretical (or arithmetical) intuition of visual and haptic experience, for it is always thought *of*—its existence is always deemed, judged to be, or (in other words) asserted. Our project, then, is not so much to replace logic with set theory, or to replace set theory with logic. Rather, it is to view them both from a new philosophical vantage from which each (as it is traditionally understood) repairs the incomplete character of the other. The two irreconcilable visions of nature seem to circle one another without end, though under further investigation this perspective might well have to be modified.

Axiom 7 (Intersection)**.** $(a \to c) \wedge (b \to c) \underset{\text{Int}}{=\!=} a \cap b \to c.$

With respect to Selection, Deposition, and the other basic properties, intersections are much like conjunctions.

Theorem 12. $a \underset{\text{Sel}}{\to} a \cap b. \quad b \underset{\text{Sel}}{\to} a \cap b.$

Proof. $\emptyset \longrightarrow a \cap b \to a \cap b \underset{\text{Int}}{\longrightarrow} (a \to a \cap b) \wedge (b \to a \cap b) \underset{\text{Sel}}{\longrightarrow} a, b \to a \cap b \underset{\text{Cond}}{\rightsquigarrow} \divideontimes.$ □

Theorem 12 proceeds by explicit exchange of $\emptyset$ for the desired process. This is our first use of Theorem 11: a demonstration of $\emptyset \to p$ is a demonstration of p. We make little distinction, from now on, between a process p, and $\underline{p}$, the same process making claim to itself.

Theorem 13. $a \cap a = a.$

Proof. First, let $b = a$ in the statement of Theorem 12. Next, note that $(a \to a) \wedge (a \to a) \underset{\text{Int}}{=\!=} a \cap a \to a$. Because the relations $a \to a$ are valid by Theorem 1, we may replace them with $\emptyset$, thus proving $\emptyset \longrightarrow a \cap a \to a$, that is, $a \cap a \to a$. □

Theorem 14. $a \cap b = b \cap a.$

Proof. $(a \underset{\text{Sel}}{\to} b \cap a) \wedge (b \underset{\text{Sel}}{\to} b \cap a) \underset{\text{Int}}{=\!=} a \cap b \to b \cap a.$ □

Theorem 15. $a \cap \varnothing = \varnothing. \quad a \cap \bot = \bot.$

Proof. Int, Sel. ☐

Theorem 16. $a \to b \underset{\text{Dep}}{\longrightarrow} a \cap c \to b \cap c.$

Proof. £ $a \to b.$

※ $a \cap c \to b \cap c.$

$$a \underset{£}{\to} b \underset{\text{Sel}}{\to} b \cap c \quad \underset{\text{Cond}}{\rightsquigarrow} \quad a \to b \cap c.$$

$$a \cap c \to b \cap c$$

$$\underset{\text{Int}}{=\!=} \begin{array}{r|} (a \to \;\; b \cap c) \wedge (b \to b \cap c) \\ \varnothing = (b \underset{\text{Sel}}{\to} b \cap c) \\ \hline \end{array} \underset{\text{Dep}}{=\!=} \begin{array}{r|} (a \to b \cap c) \wedge \varnothing \\ \varnothing = (a \to b \cap c) \\ \hline \end{array} \underset{\text{Dep}}{=\!=} \varnothing \wedge \varnothing =\!= \varnothing.$$

☐

In this proof, we have made use of the proving technique of "reducing by equivalence to a tautology." Each step must be reversible for the proof to go through (nothing is established by showing that $p \to \varnothing$ except what is already known). We have also used another standard technique, namely, (as we would say in our usual manner) to obtain an assertion p, we progress (by transforms from what is known) to the transform $\varnothing \to p$. This having been done, a serial condensation proof can always be written of the form $\varnothing \to \ldots \to p$.

We may from now on, by Theorem 16, employ the deposition axis construction in the form

$$\begin{array}{r|l} a \cap b & \underset{\text{Dep}}{\to} c \cap b. \\ a \to c & \\ \hline \end{array}$$

Theorem 17. $(a \cap b) \cap c \underset{\text{Soc}}{=} a \cap (b \cap c).$

Proof. $(a \cap b) \cap c \to a \cap (b \cap c) \underset{\text{Int}}{=\!=} a \cap b \to a \cap (b \cap c) \;\wedge\; c \to a \cap (b \cap c)$. This, however, follows by selection:

$$\begin{array}{r|l} a \cap b & \to a \cap (b \cap c), \text{ and } c \to b \cap c \longrightarrow a \cap (b \cap c). \\ b \to b \cap c & \\ \hline \end{array}$$

The converse is similar. ☐

Table 5.1: Logic in a set-theoretical lattice.

Signature	Formula	Common Name	Set-Theoretical Interpretation
TTTT	$\emptyset$	TRUE	empty set
FTTT	$\neg(a \wedge b)$	NAND	$\{a, b\}$ is a covering of $\bot$
TFTT	$a \rightarrow b$		containment
TTFT	$b \rightarrow a$		containment
TTTF	$\underline{\underline{a} \cap \underline{b}}$	OR	at least one set is empty
FFTT	$\neg a$		a is too large
FTFT	$\neg b$		b is too large
FTTF	$\underline{a \cap b}$	XOR	disjointness
TFTF	b		b
TTFF	a		a
FFFT	$\neg\underline{\underline{a} \cap \underline{b}}$	NOR	all sets are nonempty
FFTF	$b \rightarrow a \wedge \neg(a \rightarrow b)$		strict containment
FTFF	$a \rightarrow b \wedge \neg(b \rightarrow a)$		strict containment
TFFF	$a \wedge b$	AND	union
FFFF	$\bot$	FALSE	paradoxical (ultimate class)

5.2 Set-Theoretical Semantics in Object-Based Logic

Using the symbols we have so far introduced, we are now able to define the standard logical connectives using formulas that perform as the standard family of binary logical connectives when the lattice is reduced to only two values, true and false ($\emptyset$ and $\bot$, or T and F). Some of these have interesting geometric interpretations, as shown in Table 5.1. For one of these, disjunction, we introduce the standard notation $a \vee b = \underline{\underline{a} \cap \underline{b}}$.

Some of the formulas in Table 5.1, especially those using the symbol $\cap$, may be difficult to parse for someone used to standard set theory. Such a reader should note that according to our development, an object a is treated as a set with one element (itself); the intersection $a \cap b$ is thus taken between two singletons.

Theorem 18. $p = \underline{p} \;\wedge\; q = \underline{q} \;\longrightarrow\; p \cap q = \underline{p \cap q}$.

Proof. $\pounds\, p, q = \underline{p}, \underline{q}$.

For the reverse direction $\underline{p \cap q} \to p \cap q$, apply Lemma 2. Next,

$$p \to \underline{p} \to \underline{p \cap q}.$$

$$q \to \underline{q} \to \underline{p \cap q}.$$

$$\xrightarrow[\text{Int}]{} p \cap q \to \underline{p \cap q}. \quad \square$$

Theorem 18 says that intersections of processes behave like processes. We immediately have

$$p \vee q = \underline{\underline{p} \cap \underline{q}} = \underline{p \cap q} = p \cap q. \tag{5.3}$$

The intersection of processes thus expresses their disjunction. This, we believe, is an elegant reduction of concepts which may well prove useful in the future.

The author's experience has shown him that it is more natural to maintain the use of the usual symbol $\vee$ for disjunction. This is perhaps because the symbol allows one to detect at a glance that processes are intrinsically involved in an expression, but it may well also be because of the convenient place of $\vee$ in the order of operations.[2] However, below, we will use the symbol for intersection whenever possible.

5.3 Propositions Defined

In chapter 1, we set out to recover the logical from the manifest and objective experience of intuitive space and time. Now we have at last undergone enough preparation to take an important culminating step towards achieving this goal. We now make the following definition:

Definition. An object is *propositional* (is a *proposition, assertion, judgment, relation,* etc.) if it is a conjunction or intersection of processes, or of objects which are themselves propositional.

[2] Perhaps it is true that logic and set theory evolved into separate domains because of nothing but the persistent nuisance of managing parentheses!

Hence propositions can be thought of as all those objects generated recursively out of simple processes transforming strict expressions: all simple processes p are propositional, and if ϕ and ψ are propositional, so are $\phi \rightarrow \psi$, $\phi \wedge \psi$, and $\phi \cap \psi$. By Theorems 11, 10, and 18, propositions all satisfy the relation $\phi = \underline{\phi}$. The character of propositional objects will become more clear as we continue.

5.4 Proof by Case

In chapter 4 we established that the assumption system admits conjunctions of processes as assumptions. Now we will show how, in a very natural way, intersections can be accepted as assumptions as well, and therefore that all propositions may be assumed. Consider the following argument scheme:

$\pounds\, p.$

$\quad \ldots r$

$\rightsquigarrow p \rightarrow r.$

$\pounds\, q.$

$\quad \ldots r$

$\rightsquigarrow q \rightarrow r.$

$$\underset{\text{Int}}{=\!=}\ p \cap q \rightarrow r.$$

In words, I have proven that if p or q is true, then r holds of necessity; to the mathematical reader, the style of the argument may have a familiar ring to it. Evidently, we have discovered the method of proof which proceeds by considering a sequence of what are often called *cases*. We shall now augment our assumption system in order to better accommodate the new technique, since we will likely need it again. In place of an above-styled argument, we shall instead write, using a new symbol $\mathfrak{C}$ to represent the delving into a case:

$\pounds\, p \cap q.$

$\mathfrak{C}\, p.$

$\ldots r$

$\mathfrak{C}\, q.$

$\ldots r$

$\underset{\text{Ab}}{\rightsquigarrow} \quad p \cap q \to r. \quad \square$

A proof by case, in the above format, is therefore sufficient justification for absolute validity of x under the active theory, when it is known that $p \cap q$ is valid (for example, if p says that a certain positive integer is greater than 1, and q says that it is 1).

Henceforth, any proposition ϕ may go in an assumption clause £ ϕ, and be abolished, provided correct measures are taken. Note that this machinery is justified only by the small number of principles and axioms we have assumed. Once again, we witness how the elements of logic seem to be rising up from simple extralogical principles of intuition, upon close and methodical analysis.

We close this section with an example to illustrate the method of confirming cases. Of course, many other formulas can be proven by similar means.

Theorem 19. $(\neg a) \vee b \longrightarrow a \to b.$

Proof. We need to show that $\neg a \cap \underline{b} \longrightarrow a \to b.$

£ $\neg a \cap \underline{b}.$

$\mathfrak{C}\, \neg a.$

$a \to \bot \to b.$ (Case confirmed)

$\mathfrak{C}\, \underline{b}.$

$a \to \emptyset \to b.$ (Second case confirmed)

$\underset{\text{Ab}}{\rightsquigarrow} \quad ⋇. \quad \square$

5.5 Other Theorems

In this section we prove some additional formulas. Mainly at issue here is whether a relation is satisfied by objects in general, only by propositions, or only by propositions obeying the principle of excluded middle.

From now on, Greek characters shall always denote propositions.

Theorem 20. *$\emptyset$ and $\bot$ are propositions.*

Proof. $\emptyset \underset{\text{Null, Ord}}{=\!=} \emptyset \to \emptyset$, and $\bot \underset{\text{Ord, L. 2}}{=\!=} \emptyset \to \bot$. □

Theorem 21. *With the exception of $\bot$, propositions are exchangeable only for other propositions, since*

$$\phi \to a \longrightarrow \phi \to \underline{a}, \tag{5.4}$$

for all propositions ϕ and objects a.

Proof. Write $F(\phi)$ for process 5.4. Letting $a \to b = p$ and $c \to d = \emptyset$ in Theorem 8 shows that $F(p)$ holds for simple processes p. If p and q are simple processes, then $F(p \wedge q)$ and $F(p \cap q)$ are immediate from Theorems 10 and 18. □

The following is a sharper version of Theorem 9.

Theorem 22. *For all objects a, b, and c,*

$$a \wedge b \to c \longrightarrow \underline{a} \to (b \to c), \tag{5.5}$$

but this process is not in general reversible. The relation

$$a \wedge b \to c \longrightarrow a \to (b \to c) \tag{5.6}$$

does not hold in general; nor does its converse.

Proof. Process 5.5 is trivial. In order to show

$$\underline{a} \to (b \to c) \not\twoheadrightarrow (\underline{a} \wedge b) \to c, \tag{5.7}$$

let 1 be a strict expression, and let $a = 1, b = \emptyset, c = \perp$. In order to show

$$a \wedge b \to c \;\not\twoheadrightarrow\; a \to (b \to c), \tag{5.8}$$

we use the model given by tables 6.9, letting $a = 1, c = 2, b = 1$. In order to show

$$a \to (b \to c) \;\not\twoheadrightarrow\; a \wedge b \to c, \tag{5.9}$$

use 6.9 again; let $a = \dashv, b = 1, c = 2$. □

Lemma 3. $\underline{a} \to \neg\neg a$.

Proof. $\divideontimes\, \emptyset \to a \;\longrightarrow\; (a \to \perp \;\longrightarrow\; \perp)$.

$\underset{\text{CA}}{=\!=}\; \underline{a} \wedge (a \to \perp) \;\longrightarrow\; \perp$.

This holds by transitivity. □

If $\neg a$ is substituted for a in the preceding Lemma, we obtain $\neg a \to \neg\neg\neg a$.

Lemma 4. $\neg\neg\neg a \to \neg\underline{a}$.

Proof. We must show that $(((a \to \perp) \to \perp) \to \perp) \;\longrightarrow\; (\emptyset \to a) \to \perp$. Use Collective Assumption and (since double antecedents are isotone) deposition. □

Theorem 23 (Brouwer 1923)**.** $\neg\neg\neg\phi = \neg\phi$.

Proof. Lemmas 3 and 4. □

Theorem 24. $\underline{a} \cap \neg a \;\longrightarrow\; \neg\neg a \to a$.

Proof. $\divideontimes\, \underline{a} \cap \neg a \;\longrightarrow\; \neg\neg a \to a$.

£ $\underline{a} \cap \neg a$.

$\mathfrak{C}\, \underline{a}$.

$\neg\neg a = \neg\neg\emptyset = \emptyset = a$.

$\rightsquigarrow \underline{a} \;\longrightarrow\; \neg\neg a \to a$.

$\mathfrak{C}\, \neg a$.

$\neg\neg a = \neg\neg\bot = \bot = a.$

$\rightsquigarrow \neg a \longrightarrow \neg\neg a \to a.$

$\underset{\text{Ab}}{\rightsquigarrow}$ ⋇ .

□

Theorem 25. $\neg\neg a \wedge \neg\neg(a \to b) \longrightarrow \neg\neg b.$

Proof. By Collective Assumption, one must show that if $a \to \bot \longrightarrow \bot$, if $((a \to b) \to \bot) \to \bot$, and if $b \to \bot$, there arises a contradiction. This is easily shown by using deposition twice and Corollary 3. □

Theorem 26 (Modus Tollens). $a \to b \underset{\text{Tol}}{\longrightarrow} \neg b \to \neg a.$

Proof. One needs only show that $(a \to b) \wedge (b \to \bot)$ gives $a \to \bot$. □

Theorem 27. $\neg\neg(\underline{a} \cap \neg\underline{a}).$

Proof. Note that

$$\underline{a} \underset{\text{Sel}}{\to} \underline{a} \cap \neg\underline{a} \underset{\text{Tol}}{\longrightarrow} \neg(\underline{a} \cap \neg\underline{a}) \to \neg\underline{a}.$$

and that

$$\neg\underline{a} \underset{\text{Sel}}{\to} \underline{a} \cap \neg\underline{a} \underset{\text{Tol}}{\longrightarrow} \neg(\underline{a} \cap \neg\underline{a}) \to \neg\neg\underline{a}.$$

Therefore $\neg(\underline{a} \cap \neg\underline{a}) \to \bot.$ □

Theorem 28. $\underline{a} \cap \neg\underline{a} \to \neg b \longrightarrow \neg\underline{b}.$

Proof. First, note that

$$\underline{a} \cap \neg\underline{a} \to c \underset{\text{Tol}}{\longrightarrow} \underset{\text{Tol}}{\longrightarrow} \neg\neg(\underline{a} \cap \neg\underline{a}) \to \neg\neg c.$$

The desired expression follows by using Theorem 27, letting $c = \neg b$, and finally using Lemma 4. □

Theorem 29 (Glivenko 1928). $\phi \cap \neg\phi \to \psi \longrightarrow \neg\neg\psi$, *and* $\phi \cap \neg\phi \to \neg\psi \longrightarrow \neg\psi.$

We now turn to the distributive laws.

Theorem 30 (First Distributive Law)**.** $(a \wedge c) \cap (b \wedge c) \;=\!=\; (a \cap b) \wedge c.$

Proof.

$$\varnothing \;\longrightarrow\; \begin{array}{c|l} (a \wedge c & \underset{\text{Dep}}{\to} (a \cap b) \wedge c) \\ a \to a \cap b & \\ \hline \end{array} \;\wedge\; \begin{array}{c|l} (b \wedge c & \underset{\text{Dep}}{\to} (a \cap b) \wedge c) \\ b \to a \cap b & \\ \hline \end{array}$$

$$\underset{\text{Int}}{=\!=}\; (a \wedge c) \cap (b \wedge c) \to (a \cap b) \wedge c.$$

$\divideontimes (a \cap b) \wedge c \to (a \wedge c) \cap (b \wedge c).$

$a \wedge c \to (a \cap b) \wedge c \quad \wedge \quad b \wedge c \to (a \cap b) \wedge c$

$\underset{\text{Int}}{\longrightarrow} \divideontimes \quad \underset{\text{Cond}}{\rightsquigarrow} \quad \underset{\text{Dep}}{\longrightarrow} \underline{\divideontimes}.$ □

Theorem 31 (Second Distributive Law)**.** $(a \wedge b) \cap c \;\longrightarrow\; (a \cap c) \wedge (b \cap c)$, *and conversely when a, b, and c are propositions.*

Proof.

$$\begin{array}{c|l} (a \wedge b) \cap c & \underset{\text{Dep}}{\to} a \cap c \\ a \wedge b \to a & \\ \hline \end{array}$$

$$\begin{array}{c|l} (a \wedge b) \cap c & \underset{\text{Dep}}{\to} b \cap c \\ a \wedge b \to b & \\ \hline \end{array}$$

$$\underset{\text{Cond}}{\rightsquigarrow} \quad (a \wedge b) \cap c \to (a \cap c) \wedge (b \cap c).$$

Now suppose that $\underline{a} = a$, $\underline{b} = b$, and $\underline{c} = c$. Then the same holds of $a \cap c$ and $b \cap c$, and hence

$(a \cap c) \wedge (b \cap c) \;\longrightarrow\; (a \wedge b) \cap c \quad =\!= \quad \underline{a \cap c} \;\longrightarrow\; b \cap c \to (a \wedge b) \cap c.$

£ $\underline{a \cap c}$.

$$b = a \cap c \wedge b \underset{\text{T. 30}}{=} \begin{array}{c|l} (a \wedge b) \cap (c \wedge b) & \underset{\text{Dep}}{\to} (a \wedge b) \cap c, \text{ and} \\ c \wedge b \to c & \\ \hline \end{array}$$

$c \underset{\text{Dep}}{\to} (a \wedge b) \cap c.$

$\underset{\text{Int}}{=\!=} \quad b \cap c \to (a \wedge b) \cap c.$

□

It is not likely that the second distributive law (as an equality) holds in general without some added constraint (the law of excluded middle would certainly be sufficient).

For further theorems the reader is referred to Heyting's article [16], or Troelstra and van Dalen [31].

Chapter 6

Logical Considerations

We pause to review our work thus far. We began by adopting a chain-forming connective $\rightarrow$, transformation, to which we added the binary operation $\wedge$, conjunction. The simple assumption system, a number of axioms reflecting the intuition of the set or totality, the principle that null is a process, weak modus ponens, and finally, an axiom defining the intersection $\cap$ of any two objects were then adopted. This permitted the derivation of a number of logical theorems. In section 6.1 below, we show that in our system one is able to prove anything provable in intuitionistic propositional calculus.

6.1 Comparison to Heyting's IPC

Theorem 32. *The class P—the system $(P, \rightarrow, \wedge, \cap, \emptyset, \bot)$—of propositional objects obeys the axioms of* **IPC***, Heyting's intuitionistic propositional calculus.*

Proof. The logical axioms Heyting employed in his 1930 paper [16] are the eleven given in table 6.2, along with the following rules:[1]

$$1.2 \quad \frac{a \quad b}{a \wedge b.} \qquad\qquad 1.3 \quad \frac{a \quad a \rightarrow b}{b.}$$

Our proof-theoretical axioms, which simultaneously encode finitary set theory together with the proof-theoretical rule system, are listed in table 6.1. First, note that rule [1.2] is provided

[1]There are four other rules concerning his notation for variable substitutions, definitions of abbreviations and constants, and axiom introduction. We may disregard them.

Table 6.1: Proof-theoretical axioms employed.

Formation.	$\begin{matrix} a \\ b \end{matrix} \;\longrightarrow\; a \wedge b.$
Decoupling.	$a \wedge b \;\longrightarrow\; \begin{matrix} a \\ b \end{matrix}$
Selection.	$a \wedge b \rightarrow a.$ $a \wedge b \rightarrow b.$
Generation.	$a \rightarrow a \wedge a.$
Order.	$\bot \rightarrow a.$ $a \rightarrow \emptyset.$
Null.	$\emptyset \;\longrightarrow\; \emptyset \rightarrow \emptyset.$
Weak Modus Ponens.	$(a \rightarrow b) \wedge ((a \rightarrow b) \rightarrow c) \;\longrightarrow\; c.$
Intersection.	$(a \rightarrow c) \wedge (b \rightarrow c) \;==\; a \cap b \rightarrow c.$

Table 6.2: Logical axioms of IPC.

2.1.	$a \rightarrow a \wedge a.$
2.11	$a \wedge b \rightarrow b \wedge a.$
2.12	$a \rightarrow b \;\longrightarrow\; a \wedge c \rightarrow b \wedge c.$
2.13	$(a \rightarrow b) \wedge (b \rightarrow c) \;\longrightarrow\; a \rightarrow c.$
2.14	$b \;\longrightarrow\; a \rightarrow b.$
2.15	$a \wedge (a \rightarrow b) \;\longrightarrow\; b.$
3.1	$a \rightarrow a \vee b.$
3.11	$a \vee b \rightarrow b \vee a.$
3.12	$(a \rightarrow c) \wedge (b \rightarrow c) \;\longrightarrow\; a \vee b \rightarrow c.$
4.1	$\neg a \;\longrightarrow\; a \rightarrow b.$
4.11	$(a \rightarrow b) \wedge (a \rightarrow \neg b) \;\longrightarrow\; \neg a.$

in both systems, and by weak modus ponens,

$$\frac{\dfrac{a \qquad a \to b}{a \wedge (a \to b)}}{b} \tag{6.1}$$

is valid for all propositions a and b. Thus we only need to show that for every conclusion

$$\frac{a \quad \alpha}{b} \tag{6.2}$$

where α is a logical axiom and a and b are propositions, we may derive, in our system, a proof of $a \to b$, given our own definitions of the symbols $\vee$ and $\neg$.

Axiom [2.1] is the axiom of Generation, which may be written in the vertical form

$$\frac{a}{a \wedge a.} \tag{6.3}$$

Axioms [2.11] and [2.12] were explicitly proven in chapter 4 (to hold for all objects a, b, c). Axioms [2.13], [3.11], and [4.1] are trivial by assumption. Axiom [2.14] (formula (1) of the *Begriffsschrift*) does not hold for objects in general, but since

$$\underline{b} \longrightarrow a \to b, \tag{6.4}$$

the axiom does hold for propositions, as does axiom [2.15]. Axiom [4.11] is not valid for all objects without strong modus ponens, but certainly holds when b is a proposition. The axioms of disjunction [3.1] and [3.12], too, are invalid for objects in general, but for propositions a, b, and c, they immediately reduce to Theorems 12 and 16. □

It is very likely that our system of propositions is in fact equivalent to that of Heyting, if one maps $\emptyset$ to $a \to a$, and $\perp$ is mapped to $\neg(a \to a)$. Showing this would involve showing that in Heyting's calculus one is ensured the full power of the simple assumption system.

6.2 Consistency of the Axioms and Independence of the Axiom Null

It is not at all clear, given the philosophical observations we have made, what it means for our set of axioms to be "satisfied in a model." It will take more mathematics, more science, and more philosophy to understand these closed systems of objects better.

We say that an assumption gives rise (intuitively, in the space between the assumption clause and the abolishment clause) to an *assumptive environment*, or *environment*. The environment, then, is a kind of window into a domain in which the axioms under assumption locally as well as globally are universally obeyed by any object encountered.

Suppose that there is a computable model for a given environment, i.e., a finite and surveyable set of objects whose exchange (economic relationships) are fixed or algorithmically computable, and for which all the constraints on objects imposed by all adopted assumptions are satisfied. If an inconsistency (a conditionless antilogy) can be produced within the environment, it will already be present and thus manifest in the computable model. A model in which consistency can be verified—i.e., one in which there is more than one object—therefore provides a proof that within the environment a conditionless antilogy could not possibly be found.

In order to validate the axioms of Formation and Decoupling in a model in which the transformation $a \to b$ is assigned a determinate truth value for every a and b in the model, we must verify the transitivity of $\to$, and the validity of the constructions

$$a \wedge c \to \begin{array}{c} a \to b \\ \\ c \to d \end{array} \to b \wedge d, \tag{6.5}$$

that is, we must verify the following:

Transitivity. $\quad \vdash (a \to b)$ and $\vdash (b \to c)$ implies $\vdash (a \to c)$.

Condensation. $\quad \vdash (a \to b)$ and $\vdash (c \to d)$ implies $\vdash (a \wedge c) \to (b \wedge d)$.

We can view our progressive accumulation of axioms through the preceding work as being the work of the assumption system, thus:

$£\, a \wedge b \rightarrow a.$

$£\, a \wedge b \rightarrow b.$

$£\, a \rightarrow a \wedge a.$

$£\, a \rightarrow \emptyset.$

$£\, \bot \rightarrow a.$

$£\, \emptyset \;\longrightarrow\; \emptyset \rightarrow \emptyset.$

$£(a \rightarrow b) \wedge ((a \rightarrow b) \rightarrow c) \;\longrightarrow\; c.$

$£(a \rightarrow c) \wedge (b \rightarrow c) \;\longrightarrow\; a \cap b \rightarrow c.$

$£\, a \cap b \rightarrow c \;\longrightarrow\; (a \rightarrow c) \wedge (b \rightarrow c).$

[the environment.]

We wish to show that "the environment contains disunity", in other words, that the axioms we have assumed are all mutually consistent. We can, for this purpose, lay aside the assumption system, since all further use of it will only involve sojourns in other environments which will never produce a conditionless antilogy, provided all involved assumptions are valueless (true) or unattainable (false) conditions (see p. 34).

Any lattice of objects will do; it might contain five, ten, or more objects. The following binary lattice will suffice.

$$\begin{array}{c|cc} \rightarrow & \emptyset & \bot \\ \hline \emptyset & \emptyset & \bot \\ \bot & \emptyset & \emptyset \end{array} \qquad \begin{array}{c|cc} \wedge & \emptyset & \bot \\ \hline \emptyset & \emptyset & \bot \\ \bot & \bot & \bot \end{array} \qquad \begin{array}{c|cc} \cap & \emptyset & \bot \\ \hline \emptyset & \emptyset & \emptyset \\ \bot & \emptyset & \bot \end{array} \tag{6.6}$$

We can now confirm each axiom in turn, substituting $\emptyset$ and $\bot$ for instances of an arbitrary object a, b, c, etc. in them wherever they occur. It will thus be seen that they cohere together in the structure of the model. For example, we may confirm the Intersection axiom by setting

a to Ø, ⊥, b to Ø, ⊥, and c to Ø, ⊥, confirming that every possible use of Intersection will produce Ø.[2]

Note that in the model 6.6 there are no objects which are not propositions. A consistent model which contains strict expressions can be obtained from this one by adding a set S of constants $1, 2, 3, \ldots$, taking the closure over $\wedge$, and completing the tables for $\rightarrow$ (using classical truth values) and $\cap$ in the natural way.

A standard independence proof involves demonstrating that in a surveyable, computable system that contains disunity, all the axioms can be assumed but one which *cannot* be assumed. This axiom is therefore *independent* of the other axioms since a proof of the axiom from the other axioms could only result in the unity of the model, and therefore, such a proof cannot be obtained. We will not provide an exhaustive list of independence proofs for our axioms, but we will show that the nonstandard "Null" axiom, $\emptyset \rightarrow (\emptyset \rightarrow \emptyset)$, is independent of our other axioms. To prove this, consider a system of three distinct elements $\emptyset, \bot$, and T and the following values:

→	Ø	T	⊥
Ø	T	⊥	⊥
T	T	T	⊥
⊥	T	T	T

∧	Ø	T	⊥
Ø	Ø	T	⊥
T	T	T	⊥
⊥	⊥	⊥	⊥

∩	Ø	T	⊥
Ø	Ø	Ø	Ø
T	Ø	T	T
⊥	Ø	T	⊥

(6.7)

The remaining axioms of our system can all be verified. T here, as can be seen, is playing the role of the source of justification, and the system of propositions takes the form of a sublattice in the lattice of objects with a distinct minimal element.

[2]Note that to do this requires that a metalevel at which we perform these substitutions be formed.

6.3 Independence of the Principle of Excluded Middle and Strong Modus Ponens

It can be shown that the law of excluded middle cannot be proven from our axioms. In the following system, in which all nine of our axioms are verified, not all propositions obey the law of excluded middle, since $\underline{Ⅎ} = Ⅎ$, while $(\underline{Ⅎ} \vee \neg Ⅎ) = Ⅎ$, not $\emptyset$ as required.[3]

$\to$	$\emptyset$	Ⅎ	$\bot$
$\emptyset$	$\emptyset$	Ⅎ	$\bot$
Ⅎ	$\emptyset$	$\emptyset$	$\bot$
$\bot$	$\emptyset$	$\emptyset$	$\emptyset$

$\wedge$	$\emptyset$	Ⅎ	$\bot$
$\emptyset$	$\emptyset$	Ⅎ	$\bot$
Ⅎ	Ⅎ	Ⅎ	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\cap$	$\emptyset$	Ⅎ	$\bot$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Ⅎ	$\emptyset$	Ⅎ	Ⅎ
$\bot$	$\emptyset$	Ⅎ	$\bot$

(6.8)

Finally, the principle of Strong Modus Ponens, process 4.4, cannot be proven, since all of our axioms are satisfied in the following model, but $(1 \wedge (1 \to 2)) \to 2 == \bot$. (Intuitively, all that may be known is that it is not impossible that 2 is strictly larger than 1.)

[3] This set of truth values is due to Heyting [16]. The existence of Kolmogorov's negative translation which leads from intuitionistic logic to classical logic can be detected here: if one restricts one's attention to the values of the second and third rows and columns, one regains the classical set of truth values given in table 6.6 above, if one replaces $\emptyset$, truth, with Ⅎ, what Kolmogorov called "pseudotruth," in the table of $\to$.

$\rightarrow$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
$\emptyset$	$\emptyset$	$\bot$	$\bot$	$\Finv$	$\bot$	$\bot$	$\bot$
1	$\emptyset$	$\emptyset$	$\Finv$	$\Finv$	$\Finv$	$\Finv$	$\bot$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\Finv$	$\Finv$	$\Finv$	$\bot$
$\Finv$	$\emptyset$	$\Finv$	$\Finv$	$\emptyset$	$\Finv$	$\Finv$	$\bot$
$\Finv \wedge 1$	$\emptyset$	$\emptyset$	$\Finv$	$\emptyset$	$\emptyset$	$\Finv$	$\bot$
$\Finv \wedge 2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\bot$
$\bot$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$\wedge$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
$\emptyset$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
1	1	1	2	$\Finv \wedge 1$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
2	2	2	2	$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\bot$
$\Finv$	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
$\Finv \wedge 1$	$\Finv \wedge 1$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\Finv \wedge 1$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\Finv \wedge 2$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

(6.9)

$\cap$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	1	1	$\emptyset$	1	1	1
2	$\emptyset$	1	2	$\emptyset$	1	2	2
$\Finv$	$\emptyset$	$\emptyset$	$\emptyset$	$\Finv$	$\Finv$	$\Finv$	$\Finv$
$\Finv \wedge 1$	$\emptyset$	1	1	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 1$	$\Finv \wedge 1$
$\Finv \wedge 2$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\Finv \wedge 2$
$\bot$	$\emptyset$	1	2	$\Finv$	$\Finv \wedge 1$	$\Finv \wedge 2$	$\bot$

6.4 Conclusion

The term *proposition* has been constrained to refer only to certain activities of the mathematical subject under well-defined, self-imposed behavioral constraints, and not to characterizations of the world, or statements without voice, determining that which is the case. Among other things, we have seen, perhaps surprisingly, that these kinds of intuitional acts behave rather *as though* they were normal sentences in a language. Indeed, a mind distracted by some distant goal, at a far remove from what is immediately taking place, is unconsciously attracted to the fixed, transcendental sense in which propositions are oftentimes regarded. We would like to emphasize that this way of regarding propositions is not discouraged or forbidden by our considerations here. In our view, propositions are, in practice, what they have always been—mathematical propositions—and mathematics can continue as it was, without any strict dogma about the meaning of mathematical symbols being enforced. Nevertheless, by careful study of our work above, we believe that mathematicians and philosophers might be convinced that still more fruitful interaction is possible between them, if only the barriers that separate them today are worn down and overcome. Philosophies, like mathematical systems, are not all equally fecund. Like mathematical systems, too, their fecundity depends upon the dimensions of the problem immediately faced. In all of life, however, there is a persistent problem—the confrontation of a forgetful, lapsing mind with an equally forgetful, equally lapsing world. The way to a solution seems obvious—seek out whatever is, and has a way of remaining most constant and eternal. There are among us those at work on the problem, to the exclusion of other pursuits. Hence the author—for himself—finds that what distinguishes mathematics and philosophy by no means sets them at odds.

Bibliography

[1] Birkhoff, Garrett. *Lattice Theory,* 3rd ed. Providence, RI: American Mathematical Society, 1995. (1st ed., 1940.)

[2] Brouwer, L.E.J. "Intuitionistische splitsing van mathematische grondbegrippen." *Verslagen der KNAW* 32 (1923): 877-80. English translation: [29].

[3] ———. "Intuitionistische Betrachtungen über den Formalismus." *Koninklijke Akademie van wetenschappen te Amsterdam, Proceedings of the section of sciences* 31 (1928): pp. 374-379. English translation: [14].

[4] ———. "Historical background, principles and methods of intuitionism." *South African Journal of Science* 49 (1952): pp. 139-146.

[5] Carroll, Lewis. "What the Tortoise Said to Achilles." *Mind*, New Series, Vol. 104, No. 416. (Oct., 1995), pp. 691-693.

[6] Cobb, Paul and Steffe, Leslie P. "The Constructivist Researcher as Teacher and Model Builder." *Journal for Research in Mathematics Education* 14, No. 2 (1983): pp. 83-94.

[7] Davis P.J. and Hersh R. *The Mathematical Experience.* Cambridge: Birkhauser, 1981.

[8] Dehaene, Stanislas. *The Number Sense: How the Mind Creates Mathematics.* Oxford: Oxford University Press, 1996.

[9] Finkelstein, David R. *Quantum Relativity: A Synthesis of the Ideas of Heisenberg and Einstein.* Berlin: Springer, 1996.

[10] Frege, Gottlob. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* Halle, 1879. English translation in [14], pp. 1-82.

[11] Gentzen, Gerhard. "Untersuchungen über das logische Schliessen." *Mathematische Zeitschrift* 39 (1935): pp. 176-210. English translation in [12].

[12] ———. *The Collected Papers of Gerhard Gentzen.* Edited by M.E. Szabo. Amsterdam: North-Holland, 1969.

[13] Glivenko, Valerii. "Sur quelques points de la logique de M. Brouwer." *Académie Royale de Belgique* 15 (1929): 183-88. English translation: [29].

[14] van Heijenoort, Jean, ed. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931.* Cambridge, MA: Harvard University Press, 1967.

[15] Hersh, Reuben. *What is Mathematics, Really?* New York: Oxford University Press, 1999.

[16] Heyting, Arend. "Die formalen Regeln der intuitionistischen Logik." *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1930): 42-56. English translation: [29].

[17] ———. *Intuitionism: An Introduction.* Amsterdam: North-Holland Publishing, 1966.

[18] Hilbert, David. "Neubegründung der Mathematik. Erste Mitteilung." *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* 1 (1922), pp. 157-77. English translation: [29].

[19] ———. "Uber das Unendliche." *Mathematische Annalen* 95 (1926), pp. 161-90. English translation: [14].

[20] Hilbert, D. and Ackermann, W. *Grundzüge der theoretischen Logik.* Berlin: Springer, 1928 (second edition 1938).

[21] ———. L.M. Hammond, G.G. Leckie, F. Steinhardt, trans. *Principles of Mathematical Logic.* Providence: AMS Chelsea, [1958] 1999. (En. trans. of [20], second edition.)

[22] Kleene, S.C. *Introduction to Mathematical Logic.* Mineola, NY: Dover, 2002. (orig. pub.: New York: John Wiley & Sons, 1967.)

[23] Klein, Jacob. *Greek Mathematical Thought and the Origin of Algebra.* Translated by E. Brann. New York: Dover, 1992. (orig. pub.: Cambridge, MA: M.I.T. Press, 1968.)

[24] Kolmogorov, Andrei N. "O principe tertium non datur." Matematiceskij Sbornik 32 (1925), pp. 646-67. English translation: [14].

[25] ———. "Zur Deutung der intuitionistischen Logik." *Mathematische Zeitschrift 35* (1932): pp. 58-65. English translation: [29].

[26] Lakatos, Imre. "Infinite Regress and Foundations of Mathematics." *Aristotelian Society Supplementary Volume* 36 (1962), pp. 155-84. Reprinted in [28].

[27] ———. *Proofs and Refutations: The Logic of Mathematical Discovery.* Edited by J. Worrall and E.G. Zahar. New York: Cambridge University Press, 1976.

[28] ———. *Mathematics, Science and Epistemology: Philosophical Papers, Volume 2.* New York: Cambridge University Press, 1978.

[29] Mancosu, Paolo, ed. *From Brouwer to Hilbert.* New York: Oxford University Press, 1998.

[30] Reid, Constance. *Hilbert.* New York: Springer-Verlag, 1996.

[31] Troelstra, A.S., and van Dalen, D. *Constructivism in Mathematics: An Introduction,* Vol. 1. Amsterdam: North-Holland, 1990.

[32] Tymoczko, Thomas, ed. *New Directions in the Foundations of Mathematics,* 2nd ed. Princeton: Princeton University Press, 1998.

[33] Weyl, Hermann. "Über die neue Grundlagenkrise der Mathematik." *Mathematische Zeitschrift* 10 (1921): 37-79. English translation: [29].

[34] ———. “Die heutige Erkenntnislage in der Mathematik.” *Symposion* 1 (1925-1927): 1-32. English translation: [29].

[35] White, Leslie A. “The Locus of Mathematical Reality.” *Philosophy of Science* 14 (1949): pp. 289-303.