THE MATHEMATICS CLASS

AS A MATHEMATIZING

COMPLEX SYSTEM

by

THOMAS EARL RICKS

(Under the Direction of Dr. Jeremy Kilpatrick)

ABSTRACT

In this study, I investigated three middle school mathematics classrooms from the complex perspective, a theoretical framework that redefines classrooms as potentially complex phenomena. A mathematics class can become a learning system when individual students form a collective mathematizing entity. Such entities parallel the type of communities mathematicians learn and work in. This perspective suggests that student learning can be augmented by attending more closely to cognizing classroom collectives. Questions abound, however, about how such complex entities emerge and are sustained. I selected and studied three classroom episodes that demonstrated complex formation through fine-grained videotape analysis. My study demonstrated the existence of mathematizing complex systems that jointly created mathematics and regulated themselves—all at the class level. I describe a variety of underlying principles for teacher action that provided for the emergence of mathematizing complex systems in these classes, although the method of each varied. Such research can help teachers and teacher educators augment individual learning in their own mathematics classrooms by occasioning similar collective behavior.

INDEX WORDS: Adaptability, Classroom discourse, Complexity theory, Emergence, Mathematics, Mathematics education, Middle school education, Philosophy of mathematics, Qualitative research, Self-regulation, Videography

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B.A., Brigham Young University, 2001

M.A., Brigham Young University, 2003

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia

in Partial Fulfillment of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2007

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August 2007

DEDICATION

To those who for our tomorrow give their todays.

…

But I must refrain from such trite clichés.

That dedication will never do!

For using such unlinguistic ways,

'Twill set my committee's wrath ablaze…

Remember, tongue, 'thy doctoral defense,'

Your destiny hangs upon their review:

With chalk and blood and sweat and tears…

…

No, no—Not that again!

Cast such language, far flung & hence.

Remember, soul! The ivory walls,

Black robes. The hood. Academia's jargon!

You've left your youth, a new world calls

Suffice it be, tho' ere I part

The wind's my beacon; stars, the chart.

I'll try again, admit my fault!

One last time, is all I dare:

To teachers, everywhere.

ACKNOWLEDGEMENTS

I would like to thank Jeremy Kilpatrick, Paola Sztajn, George Stanic, Andrew Izsák, and the other members of the Department of Mathematics and Science Education in the College of Education at the University of Georgia. I would also like to thank my supportive parents, the three unnamed teachers, and their students.

Thank you

all y’all.

TABLE OF CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

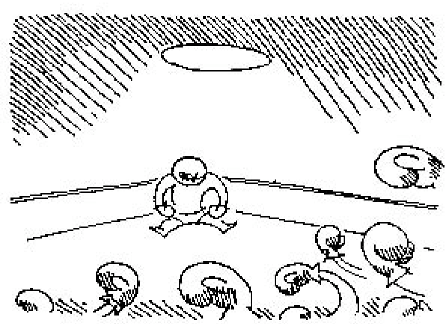
PREFACE

grook /gruk/ *n.* “A grook ("gruk" in Danish) is a form of short aphoristic poem. It was invented by the Danish poet and scientist Piet Hein. He wrote over 7,000 of them, most in Danish or English, published in 20 volumes. …His gruks first started to appear in the daily newspaper "Politiken" shortly after the Nazi Occupation in April 1940 under the signature Kumbel Kumbell. The poems were meant as a spirit-building, yet slightly coded form of passive resistance against Nazi occupation during World War II. The grook are characterized by irony, paradox, brevity, precise use of language, sophisticated rhythms and rhymes and often satiric nature.”[1]

Example:

Problems By Piet Hein[2]

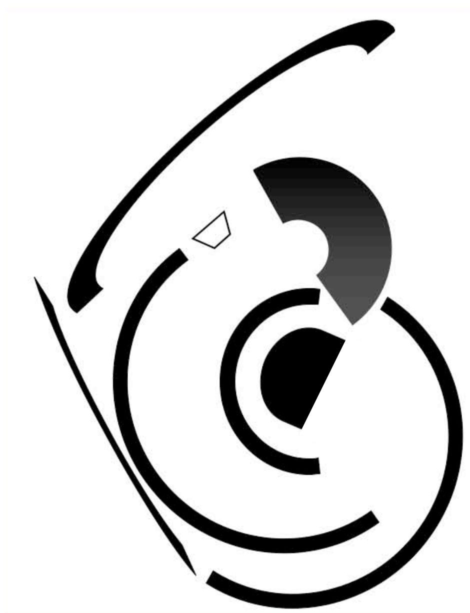
Problems worthy
  of attack
prove their worth
  by hitting back.



[1]From Wikipedia, the free encyclopedia
[2] From http://chat.carleton.ca/~tcstewar/grooks/grooks.html

My Dissertation 'Tis A Grook

My dissertation
'Tis a grook;
Just simply marks?
Perhaps, but look!
E Pluribus Unum
A sketch, half-done?
Put 'em together,
Out of Many, comes One!
Can 'ya see it now—?
What d'ya think?
I see nothing — just lines.
Or splashes of ink.
Mighty separate and lonely.
Combine'm! Perceive'm!
The Parts, if arranged,
Together should cleave'm.
Perceptions transformed!
"Now a Whole."—Claims the I*:
"United, we stand,
Divided, we die."



CHAPTER 1

INTRODUCTION

When we try to pick out anything by
itself, we find it hitched to everything
else in the universe.
—John Muir

How one looks determines what one sees. Mathematics education has used a variety of psychological perspectives on the teaching and learning process to inform its work. These outlooks, by necessity, have provided visions and perceptions of mathematics education shaped by the particular lens being used. Although the myriad viewpoints each offer vital insight unique to themselves, these perspectives have, with rare exceptions, focused on how individual students make sense of mathematics (B. Davis & Simmt, 2003). Intensive work with constructivist epistemologies, for example, involves only the individual pupil's cognitive constructions. Even social constructivist researchers, although acknowledging the social component of mathematics learning, still maintain the individual as the "locus of learning" (B. Davis & Simmt, p. 153) when they examine the social implications for learning; classes still are viewed as "groups-of-individuals" (Lave, 1996, p. 149).

When it comes to teaching mathematics, the story is much the same: Contemporary mathematics instruction is designed for augmenting the learning of individuals. Differentiated instruction is a clear example of carrying classroom mathematics learning to the individual-is-the-focus extreme, where some researchers suggest teachers should adapt their instruction to meet the diverse needs of unique individuals (Darling-Hammond, Ancess, & Ort, 2002). Other researchers have called for smaller class sizes (Bloom, 1984) to facilitate the teaching of the individual. Homogenizing the mathematics class through tracking or grade-level advancement is an attempt to “individualize the group,” perhaps glossing over the richness inherent in classroom environments. The fact that students are assembled in a classroom to learn mathematics results from efforts to make schooling efficient and affordable, and not from a plan to create and harness community action for learning. In addition, the competitive nature of classes, the assignment of individual grades, and the assessment of the isolated individual’s capacities through closed-book, written examinations confirms that although students are together in classrooms, mathematics learning has focus on the individual.

Various researchers have suggested such dominant individualistic perspectives could be balanced with a perspective on the social (Boaler, 1999; B. Davis & Simmt, 2003; B. Davis & Sumara, 2001; Schoenfeld, 1994). Such a view would be beneficial, especially as many believe mathematics, as a domain, transcends any individualistic perspective. These social supporters view mathematics in a similar way as many individualistic researchers: not as a static knowledge domain—an external thing to be internalized by a learner—but rather a socially created, culturally dependent, often-fallible domain (Ernest, 1990) linked to a community’s exploits; mathematics becomes an interconnected, dynamically fluid, ever-changing *something* that exists through nested collective action. Individuals are necessary, to be sure, for that action, but not sufficient by themselves to explain the domain of mathematics.

These people maintain that mathematics exists through the collective actions of many people over thousands of years. It belongs to no one and yet is accessible to many, if not all; it is a constant communal humanistic creation (Romberg, 1994). Great discoveries by many individuals and groups have woven the tapestry of current mathematical thought: people like the

Pythagoreans, the Arab algebraists, Cardano's band, Descartes, and Newton and Liebniz. Mathematics does not reside in any single mind, nor is it the result of any single individual's effort. Mathematics is collective. What is considered mathematical—fundamental axioms, appropriate terminology, conventional representations, mathematically valid propositions—is all socially driven, "a cultural product" (Ernest, 1990). Axioms, theorems, and proofs—they all arise out of social intercourse. Whether mathematical principles exist independent of human experience is an intriguing philosophical question, but the mathematics we know is known through us—collectively. People determine the rules, set the bounds, create the language, and argue the case.

From this perspective, mathematics emerges through communicative correspondence, socially posed questions, and group deliberations. Possibly because formal mathematics strips ideas to the simplest abstract structures of which people are capable, however, society often views the domain of mathematics as sterile and separate , forgetting that mathematics as a discipline has arisen out of and is embedded in humanity's activity. There is nothing pure or simple about the domain, and heated debates still rage over various mathematical concepts (see the *Mathematical Intelligencer*, a journal discussing such debates), as well as over the nature and boundaries of the discipline itself.

All mathematical discovery, all mathematical activity, all mathematical explanation can be seen to take place in a complex social context. Newton recognized this social dependence: "If I have seen farther than others," he said, "it is because I stood on the shoulders of giants." Mathematics is determined as much from human history as it helps to shape history's course—the two are intrinsically linked. Much more than numbers or computations, mathematics is of, by, through, and for the many.

With increasing understanding of individual student thinking, the field of mathematics education has been developing novel approaches to the study of collective classroom learning (Cobb & Yackel, 1995; B. Davis & Simmt, 2003). As Boaler (1999) stated, “The behaviors and practices of students in mathematical situations are not solely mathematical, nor individual, but are emergent as part of the relationships formed between learners and the people and systems of their environments” (p. 260). This recognition of the dependence of mathematics on the collective indicates a fruitful area for research, virtually untouched by a century of mathematics education research. A handful of researchers have begun to investigate mathematical learning in collectives using the newly-developed theory of *complexity theory* (B. Davis & Simmt, 2003), which investigates the cognition of learning systems: Mathematics is a collective action by thinking entities that engage in communal mathematizing—as a whole—and provide substantial opportunities for individual contributions. Because “complex systems transcend their components” (B. Davis & Sumara, 2001, p. 88), creating novel phenomena unpredictable from the components’ behavior, a new kind of research is needed to understand the behavior of these systems:

> At each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other.… At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry. (Anderson, 1972, p. 393)

Complexity theory allows mathematics classes functioning jointly as mathematizing superminds to be envisioned and studied from a scientific perspective. It opens up a vista heretofore unseen by previous individualistic perspectives.

These complex systems are composite entities formed from interacting, interrelated components: “For reasons that are not fully understood, under certain circumstances agents can spontaneously cohere into functional collectives—that is, they can come together into unities that have … potential realities that are not represented by the individual agents themselves" (B. Davis & Simmt, 2003, p. 141). These smaller entities interact synergistically to form a whole whose potentialities are larger than the sum of the parts. The whole becomes an object with power that did not exist previously in any of the components, and the whole often exhibits holistic learning capabilities at the system level (Delic & Dum, 2005). Examples of complex systems abound, from ant colonies to economies to nations. The human body is made up of trillions of individual organisms—cells—that are independent living creatures (they can even be separated from the larger host and kept alive, as in blood transfusions, organ transplants, skin grafts, etc.), but when brought together these tiny creatures interact in such a way as to form a larger whole that is much more than the sum of its parts. The cells exist together not only in one location, they also exist together functionally. And just as individual cells can form a larger person, so too can individuals in a classroom merge to form a larger learning entity—a mathematically functioning classroom "organism."

Complexity theory may help researchers understand classroom dynamics because students' actions are affecting the system they constitute while simultaneously being affected by that system (B. Davis & Simmt, 2003). The mathematical development occurring is an entire class phenomenon—the result of "joint productive activity" (Stein & Brown, 1997, p. 175). Knowledge becomes "stretched over" (Lave, 1988, p. 1) the entire class, not the domain or possession of any one individual; the mathematics is situated, social, and distributed (Putnam & Borko, 2000). Whereas individual and social constructivist paradigms focus on the individual as the "locus of learning," complexity theory sheds light on how the class as a whole develops mathematics. Individual knowledge in such a situation cannot be understood, complexity theorists claim, by slicing up the classroom and ignoring the larger collective entity of which the individual is an active part.

This perspective suggests that individual student mathematical learning can be augmented if teachers attend closely to such cognizing classroom collectives (B. Davis & Sumara, 2001); these entities, as “mathematizing communit[ies]” (Sfard, 2003, p. 381), parallel the type of communities mathematicians learn and work in (Romberg, 1994). The goal of the present study was to understand more about how complex classes learn mathematics, and what contributes to their formation as complex systems. I hoped to illuminate mathematics learning in these classrooms from this unorthodox perspective and also to specify strategies for creating such environments:

> Complexity research tries to identify general principles of emerging organizations common to such systems, … to understand the organizational structure of these systems in a coherent, possibly compact and rigorous way, and ultimately to simulate and optimize their behaviors. (Delic & Dum, 2005, p. 1)

I investigated three teachers’ middle school mathematics classes as they engaged in collective mathematical action. Using the work of Brent Davis and his colleagues (e.g., B. Davis & Simmt, 2003; B. Davis & Sumara, 2001), and additional complex perspectives (e.g., Jackson, 1991; Johnson, 2001; Lovelock, 1991), I analyzed how these classes developed mathematics jointly. In other words, I put down the common mathematics education “individual psychologizing” lens and picked up the complexity one to consider how the collective class itself—as an intelligent, identifiable entity in its own right—develops (or learns) mathematics. In particular, I addressed the following research questions:

(1) Is there evidence for the existence of mathematizing complex systems in mathematics classes?

(2) If so, what contributes to the development of such systems?

(3) How could complexity theory contribute to mathematics education?

In this study, I describe the theoretical perspective of complexity theory, and the manner in which I prepared for and conducted my study. I then dedicate one chapter to each research question. First, I provide evidence of joint lesson emergence and whole-class regulation as indicators of complex systems in each of these teacher's classrooms. In other words, adopting the framework of complexity theory allowed me to see complex systems operating in these classrooms. Second, I detail the common teacher actions that I observed as contributing to forming and sustaining these systems. I conclude with envisioned benefits complexity theory may bring to mathematics education.

CHAPTER 2
THEORETICAL FRAMEWORK

What is mathematics? How is it best learned?

—Blake Peterson

The two questions above have haunted me and fueled this research. The study is an attempt to answer the questions, which were posed in 2000 by my instructor in a course on methods of teaching secondary mathematics. As an undergraduate, I had rather superficial responses—my vision was definitely limited. But when I began graduate work in mathematics education at Brigham Young University 2 years later, the stimulating environment of graduate education provided a fertile ground for transforming my beliefs. After 6 years of graduate school, I believe I can provide a more thoughtful response to Peterson's two questions than I did before. In the present study I investigated learners' joint mathematical activity that gave rise to collective mathematics. This standpoint has opened up a new horizon on the nature of mathematics and how it can best be learned. Complexity theory has helped me to recognize the role mathematizing social systems play in classroom learning dynamics. Although this perspective views mathematics learning differently than common contemporary views (B. Davis & Simmt, 2003) such as constructivism and social constructivism, I will briefly discuss these two latter epistemologies as they provide a background from which to consider complexity theory.

Constructivisms

Individual Constructivism

Constructivist learning theory purports that "humans are builders, not recorders, of knowledge" (Lauren Resnick, quoted by Kilpatrick, 1986, p. 162). Individuals organize their experience by constructing knowledge, where "knowledge is whatever [a human being] holds invariant in the changing flow of experience" (von Glasersfeld, 1975, p. 10). In constructivism, unlike epistemologies of absorption or maturation, knowledge is not viewed as an external object needing to be internalized by the cognizing organism because "the world cannot enter into a cognitive organism's domain all in one piece" (von Glasersfeld, 1985, p. 91); rather, knowledge is an internal viable construction, and "cognition must be considered a process of subjective construction on the part of the experiencing organism, rather than a discovering of [mathematical] reality" (von Glasersfeld, 1975, abstract).

Therefore, constructivist epistemology might envision the mathematics teacher's responsibility as facilitating and guiding "students [to] actively construct their mathematical ways of knowing as they strive to be effective by restoring coherence to the worlds of their personal experience" (Cobb, 1994, p. 13). From the constructivist point of view, "instruction should facilitate children's construction of knowledge rather than present information and procedures to children" (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989, p. 528), for "'meaning' in mathematics is the fruit of constructive activity" (R. Thom, 1973, p. 204). The teacher forms mental models of the students' understanding (von Glasersfeld, 1985, p. 14) and adjusts instruction accordingly. If the teacher perceives that the student is creating incorrect constructions, the teacher can present tasks or situations that challenge the faulty construction of the individual, for "knowledge can…be seen as something which the organism builds up in the attempt to order the as such amorphous flow of experience by establishing repeatable experiences and relatively reliable relations between them" (von Glasersfeld, 1985, p. 17). Those teachers wishing to facilitate student understanding "should help students to construct mathematical knowledge rather than to passively absorb it" (Carpenter et al., 1989, p. 502). Lecture, presentation, or telling are not the main means of instruction. Methods of teaching based on a constructivist epistemology are "sometimes called *constructivist* teaching, sometimes *experiential* learning, and sometimes 'discovery learning'" (R. B. Davis, 1994, p. 89, emphasis in original).

Two salient principles can be gleaned from the myriad beneficial insights provided by constructivist epistemology. First, learning occurs through constructive mental action as one tries to make sense of one's experiences. Second, constructive action is a process in which incomplete understanding is reformed into a more coherent, environmentally harmonious one: Understanding does not form ready-made and complete all at once. Deeper understanding comes only through a transformation of incomplete understanding, and attempting to bypass that crucial, yet often misunderstood stage of partially faulty learning is detrimental to the stability of the learner's mental framework.

Social Constructivism

This second perspective of social constructivism sheds additional light on student learning in social settings because individual constructivism gives "priority to *individual* student's sensory-motor and conceptual activity" (Cobb, 1994, p. 14, emphasis added). Social constructivism considers the crucial social component of learning (e.g., Vygotsky, 1978); with this view, social interaction is seen as essential for personal mathematics understanding (Cobb, Wood, & Yackel, 1990). That interaction can involve the sharing of one's personal sense making, which helps to refine and hone one's private understanding, as well as the hearing of others' sense making. Hearing about another's understanding allows the individual to make sense of another's sense making, a two-layered sense-making process. Individual constructivism focuses mostly on individual knowledge organization, whereas social constructivism considers individual construction in the context of social interaction (Cobb, 1994). Social constructivism maintains that although individual students still construct mathematical knowledge for themselves, the interactions of the classroom are essential to this process, for “meaning arises through interactions. Meaning is a social product, a creation that is formed as people interact” (Yackel, 2000, p. 12).

In social constructivism, principles of the individual constructivist view are combined with an awareness of the importance that symbolic interactionism (Woods, 1992) and social norms (Cobb et al., 1990, p. 135) play in affecting the individual’s construction of meaning, for “individuals are seen to develop personal understandings as they participate in the ongoing negotiation of classroom norms” (Yackel, 2000, p. 2). In a social constructivist classroom, the teacher guides the construction of classroom norms that facilitate the construction of mathematical ideas by the students:

> Instruction … focuses on conceptual development as opposed to procedures and skills. A typical class session consists of teacher-led discussions of problems posed in a whole class standing, collaborative small-group problem solving, and follow-up whole class discussions in which children explain and justify the problem interpretations and solutions they developed during small group work. (p. 4)

Dialogic functioning, or thinking about another’s thinking (as opposed to univocal transmission), plays a critical part as students digest each other’s interpretations and understanding (Wertsch & Toma, 1995). The teacher is fundamental in creating a classroom environment “where students are expected to discuss, that is, explain their thinking and ask questions of others” (Yackel, 2000, p. 17). Such discussions are critical in developing mathematical reasoning (Schwartz & Hershkowitz, 1999). Both individual and social constructivisms are useful lens for understanding certain phenomena in mathematics education.

Theoretical Framework of Complexity Theory

Complexity theory does not displace individually or socially inclined constructivist ideas, but rather augments those views to include the possibility that by engaging in unified group action, the individual interacting agents (students) form entities (classes) whose global range of cognition and action lies beyond the scope of any single agent; and also, by participating in such an active entity, individuals increase their mathematical understanding. Mathematics educators are realizing not only the importance of healthy classroom interaction for substantive mathematics understanding but also how such interactions forge the individuals in the classroom into a larger learning system.

Such unifying action holds great potential for individual mathematics learning. I agree with Richards (1991): “Mathematics is a socially constructed human activity.…Yet each individual constructs his or her own mathematics” (p. 15). Thus, an individual learns mathematics through a sense-making process of individual action coupled with the combined interactions of others’ sense making. If taken alone, individual or social constructivist paradigms (Cobb, 1994), although helpful to understand mathematics learning, do not describe larger entities that may be acting upon the learning process of students in dynamic classrooms, nor how these entities develop or learn. Although powerful, with these perspectives “the individual tends to be seen as the locus of learning and the fundamental particle of social action.” (B. Davis, Sumara, & Simmt, 2003, p. 223). Complexity theory, or the “science of learning systems” (B. Davis & Simmt, 2003, p. 137), envisions the “classroom community [as] an adaptive, self-organizing—complex—unity” (p. 164). Complexity theory provides a perspective to examinea mathematics community as a learning entity capable of a collective mathematizing that would benefit individual learning (B. Davis & Simmt, 2003). It considers the collective creating mathematics through the collaborative mathematical action of its interconnected members. Rather than viewing a class as the mere sum of individual students, complexity theory examines individual agents cohering into larger cognizing units, enabling “researchers to regard such systems, all at once, as coherent unities” (p. 140). Complexity provides an alternative perspective for mathematics education previously unseen and unstudied.

What does a complex class look like? Imagine a mathematics teacher posing a word problem to a class. After working alone or in small groups, the students have developed various ideas and possible solutions (the individual constructivist paradigm is insightful here by detailing how students are building individual mental constructions of mathematical ideas). The teacher then begins a whole-class discussion by choosing various students to present their solution methods (such action is describable by social constructivism, being the sum of individual contributions; the students are developing meaning through their sharing and exposure to other’s sense making). As students begin to question one another’s thinking and consider the resulting classroom discourse, subsequent mathematical discussion becomes a product of previous class action. If allowed to continue, the students will get new mathematical ideas from the whole-class discussion, and those new ideas can trigger other ideas. None of the ideas existed previous to the whole-class discussion. At this point, the emerging ideas could not have been predicted; they are interconnected and adapted by the students’ participation. The class has become, at least temporarily, a complex system, a unified whole that can regulate its own mathematical behavior. The whole develops mathematical terminologies, definitions, strategies, representations—even some of the mathematical problems they raise themselves. The domain of mathematics is being carved out by the community itself.

The nascent science of complexity theory provides mathematics education with a perspective on this class that can help explain the nature of mathematics learning in such dynamic situation:

> Still very much an infant, this new science of complexity promises to describe the universe in much more accurate and appropriate terms, yielding, in consequence, deeper understanding and more reliable prediction. It also promises to much more closely ally the physical world with that of the mind, unifying what was previously thought dichotomous. (Peak & Frame, 1994, p. 5)

Complexity theory attempts to grapple with mystifying phenomena. How do researchers understand situations such as the growth of civilization, weather patterns, the flocking motions of birds, or any complex phenomenon? Broadly speaking, *complexity* is the study of phenomena that cannot be described by merely studying components of the phenomena.

Although many authors cite Weaver's (1948) seminal paper as the formal inauguration of complexity theory (e.g., B. Davis & Simmt, 2003; Delic & Dum, 2005), its roots stretch deep into history. A variety of thinkers have proposed holistic perspectives that recognized certain phenomena as more than the sum of their parts. One of the earliest was Aristotle, who in *Metaphysics* described objects whose wholes were greater than the parts. In 1932, prior to Weaver's (1948) paper, Sellars had described "organized complexity" (Ellis, 2003). However, these beginnings of holistic considerations came to be eclipsed by *scientific reductionism*—a prevalent view that understanding comes from breaking apart phenomena to study the components. The parts explain the whole:

> The reductionist seeks to reduce a description of a complex system to a simple recipe of ingredients. A reductionist views a human being as "nothing but" so many liters of water and so many grams of calcium, etc. However, the facts set out by a reductionist are of very little use if one needs to understand the workings of the system. From the list of the ingredients constituting the human body, it would be very difficult to predict the intelligence of the human being or its emotions.

> When stated in this way, reductionism obviously has severe limitations, but it is surprising how entrenched reductionism has been amongst scientists who see a human being as “nothing but” a computer or “nothing but” a biological organism engaged in the search for survival of the species. (Kaye, 1993, pp. 8–9)

Although providing much insight for science, the reductionist view neglects the combined influence of various components that give rise to new phenomena that cannot be described by summing the analyzed parts: “Although it is certainly true that all matter is made up of extremely large numbers of pieces, these pieces often behave collectively, yielding new properties” (Peak & Frame, 1994, p. 121).

The reductionist paradigm held sway in the scientific community until the middle of the 20th century. In fact, the way many mathematics classes operate reflects such a perspective for the discipline itself, as Stigler and Hiebert (1998) explained:

> If one believes that mathematics is mostly a set of procedures and the goal is to help students become proficient in executing the procedures, as many U.S. teachers seem to believe, then it would be understandable also to believe that mathematics is learned best by mastering the material incrementally, piece by piece. This view of skill-learning has a long history in the U.S. Procedures are learned by practicing them many times, with subsequent exercises being slightly more difficult than the exercises that preceded them. Practice should be relatively error-free, with high levels of success at each point. Confusion and frustration should be minimized; they are signs that the earlier material was not mastered. The more exercises, the more smoothly learning will proceed. (p. 7)

The breaking of the reductionist monopoly on science can be linked to such projects as the Manhattan Project (which developed the first atomic weapons) and Ultra (Britain's top-secret code breaking group), which provided an unprecedented occasion for disparate groups to merge and coalesce. The Manhattan Project had over 200,000 people working around the clock. Warren Weaver was head of the Applied Mathematics Panel of the U.S. Office of Scientific Research and Development, coordinating mathematicians doing operations research. In 1948 he published his groundbreaking paper delineating complexity and called for science to actively embrace phenomena describable only by considering holistic group action. In his paper, Weaver described research perspectives over the previous several hundred years, setting the stage for his argument that science needed a new branch of inquiry. He described *simple* systems, *disorganized complex systems* (later termed *complicated systems* [B. Davis & Simmt, 2003]), and *organized complexity*. Simple systems "tend to involve only a few interacting objects or variables…. For a simple system, actions and interactions of each part can be characterized in detail and the behavior of the system can be predicted with great precision" (B. Davis & Simmt, p. 139). A billiard's trajectory or a satellite's orbit are examples of simple (easily predictable) systems. Complicated systems are larger versions of simple systems, "situations such as astronomical phenomena, magnetism, and weather that might involve millions of variables or parts [but still] phenomena [that] are determined and reducible to the sum of their parts" (p. 139). Both simple and complicated systems can be described by analyzing the component pieces of the system. The parts (if properly understood) explain the whole. *Organized complex systems*, however, possess properties that are emergent and adaptive, rendering impossible a reductionist approach to understanding the systems. The parts by themselves do not explain the whole.

Since 1948, increasing numbers of scientists have left the ranks of the reductionists to consider the world from holistic perspectives. Examples are Herbert Simon, Nobel Prize winner in economics, and P. W. Anderson, winner in physics. Interest in complex phenomena gradually developed until blossoming into a full-grown science in its own right in the mid-1980s, crystallizing in the founding of the Santa Fe Institute in 1984 (Delic & Dum, 2005). Since then, there have been other centers dedicated to the study of complexity, such as the University of Michigan's Center for the Study of Complex Systems and Duke University's Center for Nonlinear and Complex Systems.

Complexity theory (the study of complex systems by *complexivists*) is not a discipline in its own right (B. Davis & Simmt, 2003). Rather, it is a perspective that stretches across a variety of disciplines. Because the concept of complexity is embraced by scholars in many fields, it has become an ambiguous term, similar to the path of such educational terms as *constructivism* (Kilpatrick, 1986), *community* (Grossman, Wineburg & Woolworth, 2001), *pedagogical content knowledge* (Shulman, 1987), and *reflection* (Rodgers, 2002), with people using the same term in many different ways. Understandably, a large number of scholars from various fields are investigating complex phenomena, and no universal definition of *complexity* exists (B. Davis et al., 2003; Delic & Dum, 2005; Gell-Mann, 1995). Some have tried to measure it quantitatively (e.g., Kolmogorov's measure as described in Li & Vitányi, 1993), but in doing so they discovered an intriguing paradox. A random pattern should be considered the least organized—after all, it is only random. In measures of complexity, simple patterns rate lower than more complicated patterns because they can be reproduced with fewer initial conditions and thus reduced in length. Amazingly, the least complex pattern of all, a random pattern, turns out to be the most complex; the only way to duplicate it is with itself—it is absolutely irreducible.

Unfortunately, the situation is clouded further because some readers may confuse complexity with something very complicated, with the interpretation that if all the initial conditions and all the variables affecting the system were known, and if sufficient computing power existed, then the behavior of the system could be predicted. For example, such a reader might interpret Peak and Frame (1994)'s description of complexity in this way:

> Even if all the rules are strictly deterministic and are known exactly, there is still room for the intrusion of disarray when many variables are involved. An accurate description of a system consisting of a large number of independent pieces, despite all of them behaving perfectly deterministically by simple rules, can require more information than we can process…. The complexity of the system

> overwhelms our ability to make accurate predictions; its behavior is as good as random. (p. 121)

To assume that complex systems are just complicated versions of simpler systems is a fallacy, as is the assumption that if we were not overwhelmed by the information, we could predict precisely their behavior. Such a view returns one to the reductionist perspective, that only understanding components will explain the higher order structure arising from them. B. Davis and Sumara (2001) stated:

> Unlike complicated objects, which are the sum of their parts, complex systems transcend their components. Their actions can be spontaneous, unpredictable, and volatile. As well, whereas complicated systems tend to be described in the language of classical physics, researchers draw more on biology to describe the unfolding of complex systems. Terms like *organic, ecological*, and *evolutionary* have come to figure much more prominently in studies of complex behavior. At the same time, attempts to discern or impose direct causes and simple correlations on complex systems have been largely abandoned. Complex forms do not lend themselves to simple analyses or interventions. (p. 88, emphasis in original)

Similarly, Anderson (1972) remarked:

> The main fallacy in [thinking of a complicated system as complex] is that the reductionist hypothesis does not by any means imply a “constructionist” one: the ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe. In fact, the more the elementary particle physicists tell us about the nature of the fundamental laws, the less relevance they seem to have to the very real problems of the rest of science, much less to those of society. The … hypothesis breaks down when confronted with the twin difficulties of scale and complexity. The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of the few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other …. At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry. (p. 393)

Anderson called for research to investigate such complex nonreductive behaviors. An organized complex system “is not just the sum of its parts, but the product of the parts and their interactions” (B. Davis & Simmt, 2003, p. 138). Therefore, it is essential that complex phenomena be studied holistically as complete entities, at the level of their emergence, for these systems

> emerge in the interactions of agents that are themselves dynamic and adaptive. These nontrivial systems change their own operations through operating. Such phenomena are not entirely predictable, as they have capacities to respond in different ways to the same sorts of influences. More significantly, they can learn new responses. (pp. 139–140)

Two Indicators of Complex Systems

In this study I will examine the existence of complex systems based upon two indicators of complex systems: *emergence* and *self-regulation*.

Emergence. The first indicator of a complex system is emergence. Delic and Dum (2005) described two approaches to the study of complexity, with each perspective possessing its “own language and priorities: one looks into complexity as an *emergent phenomenon* to be understood, while the other looks into complexity as an *engineering problem* to be tackled” (p. 2, emphasis in original). Because the present study related to education and involved human beings, I took the first perspective, that complex human behavior is an emergent phenomenon to be understood.

It is impossible to read articles about complexity without encountering ideas about emergence. But what exactly is emergence? It has become a nebulous, even mystical, concept about which scientists and other scholars have little agreement. In this report I define *emergence* as a macro-level phenomenon of some collective that none of its components exhibits. That is, the collective taken holistically gives rise through the combined actions of its components to new phenomena that did not exist previously in any component—in fact, they could never exist in those components. This view is in line with many other researchers’ definitions.

A principle common to all complex systems is the emergence of higher-level behavior not present in the components of the system:

> Emergence and unpredictability are hallmarks of [complex] systems. Emergence is in essence the acknowledgment that systems as diverse a economies, cells, or ant colonies cannot be characterized by the behaviour of their individual components—humans, chemicals, ants—but only by the higher level organisations that grow out of them…. Complexity research tries to identify general principles of emerging organizations common to such systems across diverse areas, to understand the organizational structure of these systems in a coherent, possibly compact and rigorous way, and ultimately to simulate and optimize their behaviors. (Delic & Dum, 2005, p. 1)

These emerging complex systems cannot be studied by just looking at the individual components but must maintain their integrity to be properly understood. The system's behavior is not the result of individual components' isolated behavior, but rather "it is composed of and arises in the co-implicated activities of individual agents" (B. Davis & Simmt, 2003, p. 138). The linked behavior of the system's components gives rise to the overall system dynamics:

> Discussions of emergence are often accompanied by such illustrative examples as the flocking of sandpipers, the spread of ideas, or the unfolding of cultural collectives. These sorts of self-maintaining phenomena transcend their parts—that is, they present collective possibilities that are not represented in any of the individual agents. (p. 140)

Feedback and rules are critical to allowing such collective behavior to develop (Johnson, 2001).

Self-regulation. A second indicator of complex systems is self-regulation. This concept means that individual agents can coalesce through self-organization to form sustaining, self-regulating complex systems through their own interdependent activities (B. Davis & Simmt, 2003), without apparent outside governing forces: "Such self-maintenance can arise and evolve without intentions, plans, or leaders" (p. 140). Johnson (2001) described the *myth of the ant queen*: Many people believe the queen somehow governs the behavior of the ant colony. Although the queen lives deep in the nest with little contact with other ants, ant colonies exhibit mystifying behavior, such as progressing collectively through infant, adolescent, and adulthood stages without any single ant controlling that behavior. Individual ants live only a few years. The queen isolated in the egg chamber does not send out commands to the hordes of worker ants. They operate the same way in queenless ant farms. The ant colony is an example of a self-regulating entity not governed by a leader. It solves problems such as finding the shortest distance to food sources and creating the colony trash mound optimally distant from the nest. Johnson also described the baffling behavior of slime molds. Single-celled backyard organisms under certain conditions coalesce to form moving collectives. For decades, mycologists believed in a *pacemaker hypothesis,* which specified that certain cells triggered and governed collective slime mold behavior (slime mold is a misnomer—slime molds are not molds, but actually colonies of bacteria). But recent research has demonstrated that there are no pacemaker cells controlling collective slime mold behavior: Rather, the collective self-organizes through the mutual interactions of the individual bacteria.

How does this phenomenon connect to complex mathematics classrooms? Certainly a mathematics teacher exercises some control in his or her class. But just as ants or bacteria interact among themselves and shape global colony behavior, so too does classroom interaction contribute to classroom-specific culture and sociomathematical norms (Cobb & Yackel, 1995) as the community, through its operating, defines itself. Sure, a teacher exerts influence in this process, more than any other agent, but the teacher is not ultimately responsible for the final collective behavior. Independent student action contributes greatly to the emergent personality and intellect of the mathematics class. Ant colony behavior develops or emerges as the ants go about their daily tasks. Bacterial colonies develop colony-wide defense tactics or sporulate (form spore stalks) through the intricate interaction of the organisms. Similarly, the way a class operates mathematically in a complex class develops in like manner as students are allowed to interact and share ideas with one another, and the mathematics is “seen to emerge as the collective practices of the classroom community evolve” (p. 31).

Such formation into an organized entity is not controlled from outside but is initiated by the interaction of the individual agents in the system (the teacher is one of those agents): “These events of self-organization might be further described as ‘bottom-up,’ as emergent macrobehaviors … arising through localized rules and behaviors of individual agents, not through the imposition of top-down instructions” (B. Davis & Simmt, 2003, p. 141). In addition, not only can complex systems form and then regulate their own behavior, but they often coalesce into even-higher-level systems, with more sophisticated self-regulation. Why would something that appears to self-regulate, such as a thermostat-controlled room, not be considered a self-regulating complex system? Because it exhibits no complexity. The mechanism of self-regulation did not organize itself into its present structure. It was prestamped by design. Nor could a thermostat repair itself if disturbances to its functioning occurred. It is not truly self-regulating. It cannot adapt, for example, if a spring comes loose, or a lever breaks.

Part of the reason complex systems are able to maintain their self-regulating organization is because they are adaptive, or possess the remarkable ability to learn, “where learning is understood in terms of the adaptive behaviors of phenomena that arise in interactions of multiple agents” (B. Davis & Simmt, 2003, p. 137).

> Examples on Earth of the operation of complex adaptive systems include … learning and thinking in animals (including people), the functioning of the immune system in mammals and other vertebrates, the operation of the human scientific enterprise, and the behavior of computers that are built or programmed to evolve strategies—for example, by means of neural nets or genetic algorithms. (Gell-Mann, 1995, p. 4)

Through the interaction of individual agents, the collective system can exhibit the capacity to adapt to changing circumstances: “Broadly speaking, complex systems consist of a large number of heterogeneous highly interacting components (parts, agents, humans, etc.). These interactions result in highly non-linear behavior, and these systems often evolve, adapt, and exhibit learning behaviors" (Delic & Dum, 2005, p. 1). An example is the genomic web response of bacterial colonies when faced with a lethal antibiotic. Bacteria share genetic information, jointly solving the problem of developing genetic antibiotic resistance in as little as 48 hours (ben Jacob, 1998). The colony has adapted to its changing environment, creating a response when individual bacteria share their genetic information. Thermostats exhibit no such behavior when landlords try to install an upgraded system.

Necessary Conditions for Complexity

Various authors have cited particular aspects or conditions necessary for the development of complex systems, sometimes referred to as the "laws of emergence" (Corning, 2002). Johnson (2001) described a mixture of negative and positive feedback, structured randomness, neighbor interactions, and decentralized control. B. Davis and Simmt (2003), adapting selectively from various authors, highlighted five necessary conditions: internal diversity, redundancy, neighbor interactions, decentralized control, and organized randomness. I used the Davis and Simmt list of criteria for the present study, hereafter referred to as *Davis's criteria*.

Davis's Criteria

Internal diversity. The first criterion needed for the system to exist is internal diversity, which highlights the varied nature of the components of a complex system. The components are not identical, and the variation in the components and their features provides the system with a rich repertoire of resources from which to draw. For example, in studies of medical pathology, microorganisms are isolated and then grown rapidly on nutrient-saturated Petri dishes. This procedure produces microbial colonies that grow rapidly in stress-free environments, but the bacteria display virtual genetic homogeneity. The layperson might consider bacteria as eating and reproducing machines—certainly not capable of intelligent action, decision making, or social cooperation. Environmental microbiologists, however, who study natural microbial colonies in their stress-filled native environments, have discovered that such colonies possess amazing internal genetic variation, which provides for highly sophisticated collective action including communal engineering, collective defense, wolf-pack hunting strategies, colony-wide communications systems, social memory, and group problem solving—all possible because of the diversity that individual microorganisms contribute to the larger colony's operation (ben Jacob, 1998).

Redundancy. Redundancy is a second criterion vital for the formation of complex systems. Whereas internal diversity provides a system with the qualities needed for creative action, too much diversity can cause the system to disintegrate. Redundancy provides the element of commonality around which varied individual agents can coalesce. For example, although bacteria in a natural colony possess varied genetic information, they also come from the same species; their methods of communication, defense, reproduction, and movement are identical. Such similarities provide the footing for cohesive action. Too much redundancy, however, stifles the growth of a complex system. The richest complex systems demonstrate a delicate equilibrium between redundancy and internal diversity.

Decentralized control. Complex systems demonstrate a democratic operation; they are not dictatorial—no one is calling the shots or dictating the directions. This third critical property was mentioned previously in the discussion of self-regulation. Decision making is dispersed among the individual interacting agents. Some agents may have a greater impact than others in determining a system's outcome (the mayor of a city exerts more influence than a postal clerk, for example), but the action is still distributed among the interacting components. There is no controlling super power.

Neighborhood interactions. The ability of the individual components, especially close neighbors, to interact with each other is the fourth essential part of a complex system. Through this interaction, information is shared, decisions are made, and complexity emerges. Restricting neighborhood interactions dampens the ability of individual components to operate systemically, which minimizes the global collective behavior. The more freedom given to intercourse among agents, the more creative, rich, dynamic, and unpredictable the system's actions.

Organized randomness. Of the five criteria needed for complex system formation, none is as important as, or more difficult to understand than, organized randomness. The term seems paradoxical; B. Davis and Simmt (2003) call it an "oxymoron" (p. 154). Yet just as with internal diversity and redundancy, complex systems display a fragile suspension between organized collective behavior and unpredictable, unplanned behavior. Complex systems are rule bound, which means that they operate according to universal, orderly principles: Not just anything can happen. At the same time, individual agents can operate with great freedom within the rule framework, which means they have great latitude for action: Just as not everything is allowed, neither can nothing happen. Rather, the restrictions placed on complex system components are enabling. The rules provide for coherence and redundancy, and the freedom to operate within the rule system creates creative spaces for individual action from which the system draws its energy. For example, teachers who prescribe everything that students should do for an assignment restrict the students' learning opportunities, yet giving students complete latitude can create chaos. A balance must exist between allowing unpredictable responses and having those responses be manageable by the system as a whole. As I describe below, I refer to this criterion as organized chaos because events in classrooms are not purely random.

Davis's criteria are necessary conditions for the development of a complex system. They should not be mistaken for indicators of complexity. For example, rocks exhibit internal diversity of minerals. A mosaic tiled floor exhibits redundant elements. Billiard balls interact with their neighbors (with help from the cue, or course). The Mandelbrot set appears as mathematically organized chaos. None of these are complex systems. In fact, B. Davis and Simmt (2003) described how all five criteria could be met by a group of agents, and even then a complex system might not form. But the criteria are still necessary for the possibility of a complex system to arise, or for it to be *occasioned* (B. Davis & Simmt). Conversely, emergence and self-regulation are indicators of the presence of a complex system. Davis's criteria produce the possibility, and the indicators demonstrate the existence, of the product of complexity. Where emergence and self-regulation are, there lies complexity as well.

Complexity Research

General Complexity Research

Complexity theory has been used in a wide array of studies. E. Thompson (1991) applied complexity to perception of color and argued that color "emerges from the mutual encounter of the (visual) brain and the universe" (p. 86). Scientific arguments that color resides in an object (e.g., an apple appears red because the skin of the apple is red) versus the more prevalent theory that color is the reflected light perceived by an observer (e.g., the apple is not red; red is the only color of light reflected to the observer) are recast with complexity theory to describe chroma as "an ecologically emergent visual domain" (p. 86) where the "experiential domain … emerges from the codetermination of perceiving animals and their environments" (p. 90). Jencks (1995) described urban architectural movements from a complex perspective in her book *The Architecture of the Jumping Universe*. Space constraints and creativity merged to create modern trends in architectural designs. Zajonc (1991) applied complexity theory to understanding cognition, arguing that “we are the inheritors not only of material monuments [by great artists] but cognitive ones as well” (p. 113). He argues that all cognition for intelligent beings is linked, and that breakthroughs in discovery are precipitated by some sort of connection.

Current trends in immunological research (Tada, 2004; Varela & Anspach, 1991) consider the immune system to be a multicellular complex organism residing within an animal’s bloodstream, and that internal superorganism overcomes disease through a process of adaptive learning capable of memory, basic cognitive mechanisms, and even recognition and self-awareness (e.g., a healthy immune system does not attack its host or itself).

Bacteriologists over the last decade (ben Jacob, 1998; Shapiro, 1998) are considering bacterial colonies from a holistic perspective. Rather than being a mighty clump of individual organisms, like a herd of single-celled wildebeests on a microscopic savanna, the colony itself is a single entity—a multicellular (though not always connected) organism. Margulis and Guerrero (1991) described the paradox of the microscopic *Mixotricha*, a tiny organism that assists certain Australian termites in digesting wood cellulose: the organism is composed of five species of microorganisms working synergistically. Two species of spirochete helically motile bacteria provide the motions on the external portion of the organism for movement. The nucleoplasm is another organism, embedded with two other species of organism—one unidentified. They stated:

> In the arithmetic of life, one is always many. Many often make one, and one, when looked at more closely, can be seen to be composed of many. Conventional arithmetic leads us astray making us think that there are eternal numbers identifying real “things”—things that we tend to think in science are only known

> when they are numbered. But in life "things" have a different way of adding up. (p. 51)

For several decades researchers have considered ecosystems from the complex perspective (Delic & Dum, 2005; Jackson, 1991; Todd & Todd, 1991), the most extreme view being that of *Gaia* (Lovelock, 1991), that the earth is a self-regulating entity that maintains a stable temperature for life on its surface despite the variable heat produced by the sun over time. This view reflects Hutton's 1785 claim when he stated before the Royal Society of Edinburgh, "I consider the earth to be a superorganism and that its proper study should be by physiology" (Lovelock, p. 31).

Wolfram (1983) used complexity theory to describe research with mathematical cellular automata. W. I. Thompson (1991) used complexity theory to describe discovery in the scientific process:

> What the mind is bringing together is precisely what the elitist culture strives to keep apart …. Discovery … is essentially a surprise. It is the result of putting things together that, ordinarily, are kept apart [so they] can be seen together. (pp. 16–17)

Similar examples of applying complexity to research problems could be drawn from economics, geography, political science, history, and so forth (Johnson, 2001).

Complexity Research in Mathematics Education

Only recently have mathematics education researchers used complexity theory in their research. The dominant group of researchers has been Brent Davis and his colleagues (B. Davis & Simmt, 2003; B. Davis & Sumura 2001; B. Davis, Sumara & Kieren, 1996). Davis et al. (1996) described a brief, two-paragraph example of elementary children's fraction solution strategies to launch a theoretical discussion about curriculum emerging in "mutally specifying relationships” (p. 151) between pupils, teachers, and context. The latter portion of the article detailed a nonmathematical example of an anti-racism discussion in a secondary school and how complexity sheds light on that situation. B. Davis and Sumara detailed professional development work with a group of teachers after school using complexity as the theoretical lens. They explained their work with inservice teachers and how the spontaneous attributes of the teachers’ collective related to complexity theory. B. Davis and Simmt described similar professional development work from a complex perspective and elaborated that activity against the backdrop of Davis’s criteria. They also analyzed a single lesson from an elementary classroom by investigating whole-class discussion after 10 minutes of partner work. Focusing on one student’s response, they described the criteria. They concluded that class action is appropriately described through a complex perspective and that attempts to fragment classroom action prevent observers from understanding how children learn mathematics in school.

Leikin (2004) studied the cooperative learning of preservice and inservice teachers. One participant described the cooperative learning group as a “collective brain” (p. 239). Leikin concluded that working cooperatively might develop a collaborative Vygotskian Zone of Proximal Development that would “attribute strongly to the development of teachers’ collective mind” (p. 246). J. Thom (2005) studied a 30-minute example of a group of 3 fifth-grade students solving a cube counting problem. Using the theoretical lens of complexity, she investigated the emergent mathematics of this collective and how their strategies affected the mathematical functioning of the coherent entity. Each of these studies used examples to illustrate attributes of complexity as well as how complexity might shed light on better understanding various dynamics apparent in the phenomena. It is not apparent whether the studies were specifically designed to investigate complexity in educational settings or to expand complexity theory for educational research, but I assume they were not.

Why Use Complexity Theory?

Complexity theory is an additional lens for understanding the learning of mathematics in classroom environments because it can "redescribe a classroom collective as a learning system" (B. Davis & Simmt, 2003, p. 144), furnishing a fresh vision to teachers and researchers that the class as a whole may be another entity in the classroom, one capable of learning, creating, and acting—in short, all the behavior that any individual student would demonstrate, but behavior at the class level. John Muir (1911/1988), the famous naturalist, claimed: "When we try to pick out anything by itself, we find it hitched to everything else in the universe" (p. 110). Such a perspective applies to individual students in mathematics classrooms: We cannot select a student without finding that he or she is hitched to everyone else in the classroom. We discover "how the whole becomes not only more than but very different from the sum of its parts" (Anderson, 1972, p. 395). There is a growing movement in mathematics education to recognize the role that community involvement plays in individuals' developing mathematical understanding, and "complexity theory informs … how collective learning practices can support personal learning" (B. Davis & Sumara, 2001, p. 85).

Mathematics teachers can teach not just individual students but a larger *something* in their classroom: "The teacher's main attentions should perhaps be focused on the establishment of a classroom collective—that is, on assuring that conditions are met for the possibility of a mathematical community" (B. Davis & Simmt, 2003, p. 164). Then the individual students can learn mathematics through the interactions they have in this larger learning system. Rather than classes merely being "groups-of-individuals" (Lave, 1996, p. 149), where the teacher can struggle to deliver personalized instruction to every student, a class can become a new individual—a group-as-individual. Complexity theory augments understanding the development of mathematics for students, for “in collaborative learning, distributed expertise and multiple perspectives enable learners to accomplish tasks and develop understandings beyond what any could achieve alone” (Edelson, Pea, & Gomez, 1996, p. 32). The union with the whole helps extend the learning possibilities of the parts. Creating complex unities in the classroom is far more intricate than merely clumping students together in groups or forcing classroom dialogue:

> Within the context of the mathematics classroom, an implication here is that group work, pod seating, and class projects may be no more effective at occasioning complex interactivity than traditional straight rows—if the focus is not on the display and interpretation of diverse, emergent ideas …. Without these … the mathematics classroom cannot become a mathematics community. (B. Davis & Simmt, p. 156)

In the next chapter, I discuss the methods employed in this study to investigate such mathematizing class systems in context, presenting evidence for their existence and delineating the developmental factors that appeared to allow such systems to thrive.

CHAPTER 3
METHOD

Children and teachers are not
disembodied intelligences,
not instructing machines
and learning machines,
but whole human beings
tied together in a complex
maze of social interconnections.
—Willard Waller

The present study concerned the mathematics classes of three middle school teachers. In this chapter I detail how I came to work with these classes, what I did while working with them, and how I managed the resulting data. Three main sections compose the chapter: I describe the participants of the study, the procedure used to collect data, and data analysis.

Participants

Selection of Classes

Without selecting classes that displayed frequent complex activity, I would have been unable to investigate the phenomenon of interest. Good sites yield the rich, thick data prized by qualitative researchers (Bogdan & Biklen, 2003), whereas poor sites provide thin, possibly useless, data. I began my search in January 2006 by talking with several public-school mathematics teachers I knew. I described my interest in finding classes in which students engaged in robust mathematical conversations. In January and early February, I visited two teachers who claimed to have such classes. One was a high school class that yielded an intriguing diverse classroom environment but that lacked the rich student-to-student discussions I needed; the students engaged in many one-on-one dialogues with the teacher. The second class, a first-grade class, engaged in the type of discussions I was looking for, but not nearly at the intensity I wanted.

These two attempts highlighted the fact that asking teachers whether their classes fit certain criteria was risky because few teachers will readily admit that they do not have student-to-student mathematical discussions. Asking teachers about their classes proved an ineffective way to identify classes with good student-to-student discussions. I decided to contact people who could recognize rich student-to-student mathematical discussions and who had observed a large number of teachers. I described the type of class I was looking for to mathematics education faculty members and graduate students at the University of Georgia (about two dozen people, in person) and at Brigham Young University (two people, by email). The two criteria I described were that the teachers' lessons should have healthy student dialogue and that student ideas should form a substantial part of the lesson. I considered these criteria to be essential for complex class behavior to emerge: If students were not able to share their ideas with one another in an open manner, and if those ideas were not valued, I assumed no mathematics class collective would materialize.

Healthy student dialogue. By *healthy student dialogue*, I meant that students would actively consider and discuss each other's ways of thinking. Students would be talking to each other about mathematics, not just responding to the teacher; and such oral action would be a natural part of their class culture. It would also be dialogic in nature (Wertsch & Toma, 1995). Such dialogue is not common in typical mathematics classes given restricting, often mandated, timetables for content coverage and looming pressure to test. Most classes I had observed in prior work during my career where student dialogue occurred had engendered only a type of "number talk" or "answer giving" (Richards, 1991), as if the students' role were to fill in the teacher’s blank by mind reading (e.g., Teacher: “What would be the next step in solving this problem?” Student: “Do such-and-such, just like you showed us in the previous examples”). In this type of unnatural discussion, a single answer needs to be “found.” Sometimes various students chime in to agree or disagree, but that is not what I term *healthy* student dialogue.

One might be concerned that my presence as an observer in the initial site screening process might have altered the dynamics of class dialogue. I do not believe, however, that a teacher whose class was not already producing healthy student dialogue could suddenly get students talking productively about mathematics on the day I visited. Students not accustomed to mathematical discussion require time to acquire this ability (as do teachers in learning to maintain such activity).

Student ideas should form a substantial part of the lesson. I wanted classes in which not only would students engage in rich mathematical discussions with each other (often mediated by the teacher), but also their developing ideas would be incorporated into the lesson, forming much of its substance. This approach would not result in a lesson formed by teacher talk, a preplanned presentation with occasional participation by students, but rather a lesson with key foundational questions posed by the teacher (usually by posing and maintaining good mathematical tasks) that formed the skeletal structure around which the bulk of the lesson would be fleshed out by the students. Students would talk actively about their mathematics, and it would get considerable classroom “air-time.” Their ideas would often be put on the boards; and the ideas would carry weight and would often be referred to later in the lesson or in subsequent lessons.

I define *lesson* to be the class’s public enacted curriculum or, alternatively stated, the class’s shared mathematical actions and objects. This includes all discussion, class action, representation, ideas, commentary, problems, definitions, etc. that are publicly presented, discussed, debated, argued, or questioned. A teacher may privately prepare for the lesson by developing a *lesson plan*, which specifies concepts, goals, examples, and problems to be raised in the lesson. But the lesson plan is not the actual lesson, because much may occur in the lesson that is not included in the lesson plan. Similarly, not all of a lesson plan may be enacted in the actual lesson. It is privately prepared, and not the product of shared action. For this reason, I do not consider homework to be part of the lesson, as students work independently; however, if public discussion occurs over certain ideas from the previous night's homework, this discussion of the homework would now be considered part of the current day's lesson.

Observations. The mathematics educators gave me the names of roughly 2 dozen teachers whose classes they thought met the criteria. I contacted these teachers identified by the mathematics education experts, explained my desire to consider their classes for the study, asked permission from them and their principals, set up appointments, and visited their classes (preferably sitting in the back of the room to give me a wide view). I needed to observe the recommended mathematics classes because I wanted to choose those that exhibited the most potential for collective mathematical activity.

I visited 10 teachers and their classes in the southern United States and 8 teachers and their classes in the western United States to select the sites. Appendix A lists the classrooms I visited and my evaluation. I made 25 classroom visits, with repeat visits of some teachers. I observed the lessons but did not participate in them. I considered classes at all levels of schooling: 2 kindergarten classes, 1 first-grade class, 1 fourth-grade class, 10 middle school classes, 3 high school classes, and 1 graduate school class. I found one teacher in the southern United States and three in the western United States whose classes exhibited regular healthy student dialogue and in whose lessons student ideas played a substantial role. One of the western teachers was pregnant, and I decided that her classes would not be suitable for the study because she might occasionally be absent during data collection. I visited the remaining three teachers (all middle school teachers) again to verify that their classes met the 2 criteria for selection. During the summer of 2006, I formally invited the three teachers to participate in the study. I also met with each of their respective principals, as well as one district research supervisor who asked to discuss the study with me. All names used in this report are pseudonyms.

The Teachers and Their Schools

Ms. Auburn at Orange Blossom Middle School. Ms. Auburn taught for 12 years at another middle school in her same district, and then 3 years at Orange Blossom. She was bilingual in Spanish and English, had lived abroad for several years, and taught Spanish classes in addition to mathematics. She said she was “dedicated to understanding how students think about and learn math as a means of creating mathematical literacy in children.” Ms. Auburn held a B.A. in mathematics education (Secondary Level 4) from a nearby university and was also Nationally Board Certified. She was the district’s mathematics specialist for 3 years while the district attempted to implement a mathematics curriculum program funded by the National Science Foundation and recognized by the National Council of Teachers of Mathematics (NCTM) as embracing a problem-solving approach. Because of backlash from parents, the district made the innovative curriculum optional, and many teachers had returned to the curriculum used previously. Ms. Auburn had served as past president for her state’s chapter of the NCTM. Her department was tolerant and open to the way she taught mathematics. I once overheard her suggest to a colleague that she (Ms. Auburn) might consider moving to a new junior high being build in a nearby city and hand-picking the mathematics teachers to form a cohesive department.

Orange Blossom Middle School. Orange Blossom Middle School was located in the small but rapidly growing city of Plumgarden (population 15,000) which bordered two larger cities. The school enrolled 1,200 students in seventh, eighth, and ninth grades. The school district, which encompassed several cities, had 46 elementary schools, 10 junior highs, and 7 high schools. Of the student body, 96 percent were Caucasian and 2 percent Hispanic, with 10 percent receiving free or reduced lunch. Plumgarden was a rapidly developing affluent suburb of a metropolitan area of over 1 million to the north, and included hundreds of plush new homes worth over half a million dollars each.

Mr. Murano at Green Acres Junior High School. Mr. Murano was in his eighth year of teaching: the first year as a full-time intern at a local high school, and the next seven at Green Acres. He had a B.S. in mathematics education from a nearby university, where he said he had been first exposed to the reform agenda in mathematics teaching recommended by the National Council of Teachers of Mathematics (NCTM). His primary goal was to have students doing mathematics in his classroom, thinking, reasoning, and exploring. He focused more on the process than the result of “learning mathematics,” passing a test, and so on. He said: “[My] goal is to provide [the students] with opportunities and experiences that will challenge them to do excellent work and create many connections.” He had made five presentations at local or national mathematics education conferences in recent years. A speaker of English and Spanish, he lived for 2 years in a Latin American country. The first day I observed his classes, he tutored in Spanish a student who had come from Guatemala only 3 weeks before, while instructing the rest of class in English. Mr. Murano was a National Board Certified (Early Adolescent Mathematics) teacher, a winner of several prestigious local awards, and the winner of a sizable cash grant from a local bank. He taught mathematics only.

Green Acres Junior High School. Green Acres Junior High School enrolled 1,500 seventh, eighth, and ninth graders in the two cities that bordered Plumgarden. Their populations were 24,000 and 30,000. The school district was the same as Ms. Auburn's. Both cities were lower middle class. One-quarter of the students at Green Acres Junior High were on free or reduced lunch, with 92 percent Caucasian students and 5 percent Hispanic. Mr. Murano taught in a portable classroom. He had 36 desks crowded into the room, with 38 students (2 of whom sat not at desks but in a side aisle on chairs borrowed from the hall).

The mathematics department at Green Acres had considerable friction over how mathematics should be taught. Mr. Murano and a few other teachers supported communities of inquiry, but most did not. He commented to me once how this division created quite a bit of stress as he interacted with colleagues, especially as they soon had to jointly choose a new department textbook. He also indicated that some felt resentment among the other faculty, as he was being recognized by various organizations for the exemplary work he was doing in his classes, while other teachers, mostly senior in status, were not.

Ms. Sandy at Bridgewater Middle School. Ms. Sandy had 14 years of school-teaching experience. She taught a year in each of two public schools, the next year in a private school, and then stopped teaching for many years to raise a family. She returned to teaching and had taught at Bridgewater for 11 years. She had a B.S. in music education when she began teaching and later returned to a local university for an M.Ed in middle grades education (mathematics concentration) in 1999. She earned an Ed.S in middle grades education (ESOL concentration) in 2005 from the same university. She had worked with a regional education services agency to help local schools in the district implement the current mathematics curriculum. She taught a social studies class in addition to her three mathematics classes.

Ms. Sandy was a well-known teacher who, with her classes, had been the subject of several research projects, including a three-year National Science Foundation-funded study of students' algebraic reasoning and a dissertation study of the teaching of algebra. In addition, videotapes of her lessons were available on the Web site of the state department of education. Consequently, she had much experience with teaching while being videotaped. She said, "All of these projects helped me to reflect on my teaching and students' learning."

Bridgewater Middle School. Bridgewater Middle School enrolled about 800 students in sixth, seventh, and eighth grades. It was located in a poor section of the small city of Hamilton with a population of under 4,000 in a rural county of 15,000 that was on the verge of becoming a suburban community for a large metropolitan area. Almost 90 percent of the students received free or reduced lunch. Roughly a third of them were African American, about 2 percent were Hispanic, and the remaining 65 percent or so were Caucasian. The county school district had a primary school (K–2), an elementary school (3–5), Bridgewater Middle School (6–8), and a high school (9–12). Of the three teachers I studied, Ms. Sandy came from the most supportive department, and the school administration was strongly committed to improving mathematics education. I had a chance to observe three of Ms. Sandy's colleagues, and each had student communities forming in their classrooms. Her classroom was located in a new wing added to the school, which had previously functioned as the county high school.

Procedure

The goal of the study was to investigate whether mathematics classes could form mathematizing complex systems, and if so, how such systems developed. The goal required a research approach that allowed for clarifying insight and deepening understanding about a specific phenomenon. I chose a qualitative approach because the research questions required understanding complex phenomena. Qualitative methods are especially well suited to dealing with classroom phenomena because “qualitative researchers are concerned with process rather than simply with outcomes or products” (Bogdan & Biklen, 2003, p. 6). The research questions addressed processes: the process of forming a complex system and the process by which such a system developed mathematics.

Qualitative research is composed of a variety of submethodologies or research genres. The particular genre for my study was socio-communicative, which “explores the meaning participants make in social interactions and settings, [with] the locus of interest … communicative behavior [and in which] researchers … turn their focus onto fine-grained interactions of speech, acts, and signs” (Rossman & Rallis, 2003, p. 100). These methods were a good match for my research questions because I scrutinized both verbal and nonverbal human communicative actions in classrooms.

Data Collection

Because “complex unities must be studied at the levels of their emergence” (B. Davis & Simmt, 2003, p. 143), I had firsthand, continuous, minimally unobtrusive and documented contact with the classes during data collection. Thus, I was using an ethnographic approach for data collection (Rossman & Rallis, 2003) as I attempted to understand the emergent mathematical culture (Bogdan & Bilken, 2003) in these classes.

By *firsthand*, I mean that I stayed in the classroom, engaging in a type of “environmental research” similar to the methods employed by environmental microbiologists to study bacteria in their natural environments (ben Jacob, 1998) and identified by Stigler and Hiebert (1999) as a critical factor for understanding class dynamics: “No state that we know of regularly collects and uses data directly related to instructional processes in the classroom …. We need to know what is going on in … classrooms" (p. 8). I did not rely on secondhand accounts of class action by using interviews or questionnaires or by examining student assessments. Instead, I was in the middle of the action as it unfolded naturally.

Although the students I had observed for the site screening in early 2006 were not the same as those that participated in the study the next school year, I assumed that teachers manifesting complex systems with one set of students could reasonably be expected to have such systems with the next set, whereas if a teacher was not creating a complex system one year, chances were he or she would not the following year. By studying three teachers, I also provided an element of redundancy in case for some reason a selected teacher was inconsistent.

By *continuous*, I mean that I made uninterrupted classroom observations, without skipping a day, for 3 to 6 weeks, depending on the class. Such continuity was essential to understanding the development of the various lessons and actions I would analyze in depth. I did not, as some researchers have done, make periodic visits and then attempt to reconstruct the class's behavior by filling in the gaps. The continuity gave me a picture of what had happened prior to any particular day I was studying (except the first day, obviously), and where the lesson led (except for the last lesson). Continuity became context enabling.

By *minimally unobtrusive*, I mean that I attempted to be as an unobtrusive as possible given that I had cameras and equipment in the room. I did not interfere with the instruction; I conducted no interventions or student interviews. I tried to be a "fly on the wall." I did chat in a friendly manner when spoken to, and I handed out mechanical pencils when students expressed the need. After the first few days, the students' interest in my presence noticeably subsided as they saw that day after day, I just stood behind the cameras, monitoring the equipment and jotting notes. I knew normal class dynamics were returning when I observed students located right in front of me and my primary camera secretly pass candy and notes, and quietly poke and kick each other when the teacher turned around. During one lesson, I overheard a visiting university supervisor ask the teacher whether the cameras bothered the students. The teacher responded, “No, they’re quite used to it. The first few days they noticed them, but then they just ignored it.”

By *documented* contact, I mean that I attempted to capture the activity in reproducible form for later analysis: high-quality stereo sound audio recordings of class dialogue recorded by multiple cameras videotaping from two (in one class, five) perspectives. One videocamera remained stationary in the back of the classroom to capture as much of the whole-class interaction as possible. I directed the second camera at the individuals who were speaking or producing mathematical work at the focus of the class’s attention. From August until November 2006, I was an active observer of the activities in these selected classrooms as I videotaped, took fieldnotes, and conducted after-class interviews on the class dynamics. In addition, I took after-class fieldnotes while reviewing the videotapes, concentrating on events related to complex mathematical behavior. The videotaping and fieldnotes were an attempt to capture collective mathematical development in the classrooms. I also collected data on the less-public aspects of the classroom collectives by interviewing the teachers (audio- and videotaped). I paid the teachers $25.00 per hour of interview time. Mr. Murano declined to be interviewed because he was so busy. He had two student teachers that semester, was working with two researchers from a nearby university, and had recently been appointed vice chairman of the school-community council in addition to his other responsibilities. I conducted one interview with Ms. Auburn and two interviews with Ms. Sandy. The interviews were transcribed verbatim, and the transcriptions rechecked multiple times for accuracy. Archival copies of all videorecordings were made, and the recordings were digitized to facilitate analysis. Because the data (with a few minor exceptions) can be used in instruction and research, the complex activity of these classrooms has been captured and preserved for continued analysis and instruction. I approached the data collection task as creating a high-quality dataset I could use throughout my career, rather than just for this study.

I purchased high-quality omnidirectional microphones to hang from the ceiling of the classroom and table microphones for the tables near the cameras. Both sets of microphones had cables running through a mixer into the primary and secondary cameras. During whole-class discussion, the ceiling microphones recorded, and the table microphones were turned down using the mixer which was next to me and the primary camera. When whole-class discussion would give way to individual, partner, or group work, then I would turn down the ceiling microphones, and turn up the table microphones. When I was using more cameras in the classroom, the sound was recorded by those additional cameras through the cameras' own internal recording systems. Although the table microphones were effective in picking up sound, they were ineffective in trying to capture the dialogue of students at a distance. As the transcripts attest, I was able to capture almost all class dialogue. Sometimes students would speak so low the ceiling microphones would not pick up their murmur, but in those cases the teacher could not hear them either and would ask the students to repeat their comment more loudly.

I had previously submitted and obtained human subjects permission for the study. I sought informed consent from the teachers and students (and their parents or guardians) and indicated to the teachers and students that their participation was voluntary. Students declining to participate (or students whose parents or guardians declined to have their children participate) were not used in the study, and most were reseated out of camera range. Table 1 lists information on the eight classes used in the study. The participation response was high, with all students in one class agreeing to participate (this was the only class filmed with five cameras, giving complete classroom video coverage that eliminated camera blindspots as opposed to the usual two-camera coverage for the remaining seven classes). Table 2 catalogs the dates in 2006 when I was in teachers' classes.

Table 1

Class Enrollment and Participation in Study

Teacher	Class	Period	Grade	N(class)	N(study)
Auburn	Prealgebra	80 min., daily	7	32	28
	Prealgebra	80 min., daily	7	36	32
	Algebra	80 min., daily	7	25	24
Murano	Algebra	80 min., every other day	7, some 8	38	37
	Algebra	80 min., every other day	7, some 8	38	36
Sandy	Prealgebra	50 min., daily	8	19	17
	Prealgebra	50 min., daily	8	17	16
	Algebra	50 min., daily	8	26	26

Data Analysis

The study was designed as a systematic attempt to study classroom-based complexity and the resulting development of a mathematical collective through case studies of classes. The case study approach is particularly well-suited for classroom-based research because it is "an especially good design for practical problems—for questions, situations, or puzzling occurrences arising from everyday practice" (Merriam, 1998, p. 11). At this point, the study developed into

Table 2

Dates of Data Collection

Teacher	Dates of observation	Dates of filming	Note
Auburn	23–25 Aug. (3 days)	28 Aug.–15 Sept. (14 days)	Audiotape of teacher discussion on 24–25 Aug. No class on 26 Aug.
Murano	23–26 Aug. (4 days)	28 Aug.–15 Sept. (14 days)	Audiotape of teacher discussion on 24–26 Aug.
Sandy	None	21 Sept.–13 Oct., 23 Oct.–8 Nov. (30 days)	16–21 Oct.: Intersession (no regular school; no filming)

a combination of instrumental case studies (Stake, 1994) that would generate theory for broader application. This decision also allowed me to compare and contrast a specific teacher's classes or to compare and contrast several teachers' classes.

During the analysis stage, I departed from the usual multiple case study approach (which begins by analyzing cases individually), and I began to look across cases for comparison of puzzling, insightful, curious, or confusing events. I examined similar instances in other teachers' classes, crossing the sequential case-analysis boundary at the outset. I have subsequently viewed these classes not as isolated *cases* of a shared yet rare phenomenon but rather as isolated *classes* with common underlying properties that contribute to the emergence and sustaining of the phenomenon. So although I had designed my study to follow case study methods of analysis, I adopted a more strategic approach of blending the cases and looking instead at the underlying properties contributing to complex class activity and their subsequent effect on the lesson development. The data set for this study was therefore composed of the particular episodes I selected, set against the background of the full data set of classes collected through the ethnographic approach.

During the first stage of data analysis, I identified the subset of 3 complex episodes, one from each teacher, I would use as the core episodic segments for the main data analysis out of all the videotapes I had available. Selection of episodes occurred as I reviewed lessons identified in my fieldnotes during data collection or during perusal of the videotapes afterward. I was looking for compact units of class activity demonstrating that student ideas were forming substantial parts of the lesson through healthy student dialogue: the same two criteria I had used for selecting research sites. I chose one complex episode for each teacher to form the core set of data for analysis, followed by 6 supplementary episodes that informed my analysis. For two of the teachers, the episodes I selected were, according to my observation and later reflection on the lessons, the best examples of complex activity I saw. For the remaining teacher, because many episodes existed of roughly equal quality, I chose an episode that was characteristic of the teacher's style. Table 3 lists the main and supplementary data sources for the analysis, including the 1999 Trends in Mathematics and Science Study (National Center for Educational Statistics, 2003). U.S. lessons from that study were used for comparative purposes (described later in chapter 5).

I selected these episodes so that I could study collective mathematical development through fine-grained constant comparative analysis. I analyzed the main episodes by examining what features seemed to contribute to forming the complex system and how the system was sustained. In addition, I transcribed the three main episodes, with multiple rechecking and revising for accuracy. Using the transcripts, I was able to track how the collective participation constructed the joint mathematical ideas. Such fine-grained analysis was crucial; I took the perspective that “the qualitative research … demands that the world be examined with the assumption that nothing is trivial, that everything has the potential of being a clue that might unlock a more comprehensive understanding of what is being studied” (Bogdan & Biklen, 2003, p. 5).

Table 3

Data Sources

Teacher	Name	Episode	Length (min.)
Auburn	Soccer Problem	Main	24
Murano	Susan’s and Manuel’s Problems	Main	80
	End of Manuel’s Problem	Supplementary	80
Sandy	Perimeter Problem	Main	34
	Algebra Balance	Supplementary	5
Auburn	Interview 1	Supplementary	Approx. 60
Sandy	Interview 1	Supplementary	Approx. 60
	Interview 2	Supplementary	Approx. 60
TIMSS Teacher 1	U.S. Lesson 1	Supplementary	Approx. 50
TIMSS Teacher 2	U.S. Lesson 2	Supplementary	Approx. 50
TIMSS Teacher 3	U.S. Lesson 3	Supplementary	Approx. 50
TIMSS Teacher 4	U.S. Lesson 4	Supplementary	Approx. 50

I reconstructed the formation of the system during each of the three main episodes by documenting the individuals’ active participation in the episode. Using Spradley’s (1980) qualitative framework of *space, actors,* and *activities* (I use the word actions in this dissertation), I emphasized the actors participating and actions occurring in the classroom space. What occurred? Who did it? How did it emerge? And what was the action's apparent effect on other actors and later actions? Because the emerging ideas were not the product of any one individual but rather the synergistic interplay of multiple sources of understanding being shared in a community, the structure and evolution of the mathematical concepts in each episode were mapped with contributing individuals and events tagged and followed throughout. This approach allowed me to delineate who contributed what ideas to the growing concept and how suggestions and items of understanding were accepted, rejected, or modified by the collective. In particular, I examined how mathematical ideas emerged and evolved throughout the episode: how individuals added to the collective mathematizing and reciprocally how the collective appeared to influence individual understanding (as manifested by later comments or student work). Through this process, I was able to document the mathematical possibilities afforded students and how certain students contributed to those possibilities. In this way, I developed theory from the bottom up:

> Qualitative researchers tend to analyze their data interactively. They do not search out data or evidence to prove or disprove hypotheses they hold before entering the study; rather, the abstractions are built as the particulars that have been gathered are grouped together.
>
> Theory developed this way emerges from the bottom-up (rather than from the top-down), from many disparate pieces of collected evidence that are interconnected. The theory is grounded in the data. (Bogdan & Biklen, 2003, p. 6)

Instead of looking at individual lessons separately, I looked across the spectrum of class actions in the main episodes, comparing and contrasting.

Constant Comparative Method

The data were analyzed according to the constant comparative method as originally described by Glaser (1965). The constant comparative method "is concerned with generating and plausibly suggesting (not provisionally testing) many properties and hypotheses about a *general* phenomenon" (p. 438, emphasis in original). I chose the constant comparative method because it fit well with my desire to investigate a poorly understood phenomenon for which little theory existed. In addition, the method was well suited to the diverse data sources I was using, including videotapes, interviews, fieldnotes, student written work, literature, and my own experiences in classrooms as a teacher, supervisor, and student—all considered valid data sources under the constant comparative method. In the method, a wide data net augments theory creation. A final reason I used the constant comparative method was its inherent "flexibility which [aids] the creative generation of theory" (p. 438).

I followed the four main stages of the method: comparing, integrating, delimiting, and writing. Initially, I compared events in the main episodes, writing memos to record my developing ideas. As those developing ideas began to coalesce, I began to limit the type of comparing I was doing to refine the categories as my theoretical lens became more focused. I reached a point of saturation where the incidents became more understandable and codable, and I began the writing process by attempting to describe what I was learning, knowing that much of what I had discovered I would use in this report.

CHAPTER 4

WERE THE CLASSES

MATHEMATICALLY COMPLEX SYSTEMS?

We teach who we are.
—*J. J. Irvine*

Chapter 4 has two sections that describe evidence substantiating the claim that the three mathematics classes operated as complex systems. The indicators I present as evidence are mathematical emergence and self-regulation. All the class actions I detail were mathematical. Before discussing the two indicators, I describe each episode to provide a context.

Three Teachers' Episodes

Ms. Auburn's Episode: The Soccer Problem

Ms. Auburn's episode lasted 24 minutes. It was the third day of school and the first day of filming. She used a problem (see Figure 1) that she had given the students the night before to complete as homework. She began the period with a short opening activity and pop quiz, and then she gave the students about 2 minutes to review the work they had done on the problem the night before and to gather their thoughts. She began a whole-class discussion by asking for a volunteer, Lillian, to put her solution on the board. As Lillian was going to the board, Ms. Auburn told the remaining students to compare their own strategy and answer with Lillian's.

Lillian shared her solution strategy with the class: She had simply added the number of players together to get 22 high-fives. Asked to show her method, another student, Jasmine, said, "Mine is completely different, because I did it in a completely different way." Jasmine had double-counted the high-fives (when the high five was given, it counted as a high-five for both

> At the conclusion of a soccer game of two teams, each including eleven players, each player on the winning team gave high-fives to each player on the losing team. Then each player on the winning team also gave high-fives to each other player on the winning team. How many high-fives were given?

Figure 1. The Soccer Problem.

Persons), obtaining an answer of 242 high-fives (11 × 11 = 121; 121 × 2 = 242). At this point, many students appeared eager to share their strategies, raising their hands to volunteer. Ms. Auburn surveyed the hands and asked Bryan to go to the board and share his method. He did, presenting a restructuring of the problem in which only some of the winning soccer players gave high-fives, resulting in 66 high-fives (5 winners × 11 losers = 55; 55 + 11 winners = 66).

Now the class was full of palpable energy, manifested through loud comments and many students wiggling in their seats with hands raised. Ms. Auburn said, "Now we have three totally different answers on the board. How are we going to figure out what's correct or what's not correct?" Many students whose method differed from the three already given wanted to share theirs. The next volunteer Ms. Auburn chose was Trevor, a bright, vocal student who had been anxiously but patiently waiting with his hand up. He shared his strategy, which involved breaking the problem into two smaller problems: first counting the number of high-fives the winning team gave to the losing team (11 × 11 = 121), and then devising a pictorial representation without double counting to explain how the winning team high-fived each other. As Trevor showed his solution, affirmative murmurs rippled across the class, indicating that others had used a similar strategy. But Trevor made a slight error in adding up the number of high-fives the winning team gave each other (56 instead of 55), which several students openly questioned.

Rather than correcting the error, Ms. Auburn mentioned that some students were saying the partial sum was 55 rather than 56. As Trevor silently checked his arithmetic at the board, several students were allowed to describe the similar methods they had used. Hayden had invented a method of pairing addends that added to 10 (such as 9 + 1, or 8 + 2) to achieve a quick sum of 55, and it avoided the pitfalls of Trevor's less systematic strategy. Trevor then announced that he had made an arithmetic error, and the partial sum should be 55, making the grand total of 176 instead of 177 high-fives.

Jasmine was allowed to return to the board, revise her method, and correct her answer. She adopted Trevor's strategy and arrived at 176. Lillian, when she returned to the board and modified her strategy, was still struggling with double counting, which another student, Addison, identified. Ms. Auburn said to the class that they as a class were coming to a consensus on both a method and an answer. But another student, Danielle, commented that the last winning team member would not high-five himself or herself, so the partial addition sum should be 54, not 55. When Ms. Auburn asked the class what they thought about Danielle's suggestion, soft mumbles were heard as the students thoughtfully considered Danielle's claim.

Ms. Auburn selected six volunteers, three on a team, to enact a high-five simulation for the class. She asked the class what type of strategy they should use to keep track of the high-fives given by the students in the simulation. One student proposed that each simulator should just keep track of how many high-fives he or she gave (which would result in double counting). This proposal was a clear indication that some students were still struggling with what constituted a high-five. Ms. Auburn asked the class how many agreed with this student's proposition, and enough students concurred that she decided to proceed with his suggestion.

The students did two simulations in a row before the class recognized they were double counting when the winning team high-fived each other. They needed to develop a system to prevent double counting the high-fives. As part of the clarification during this activity, Trevor commented that the class did not yet have a clear definition of what *high-five* meant. The class finally agreed that one high-five consisted of a unique contact between two individuals. A method was proposed to avoid double counting during a third simulation.

Ms. Auburn readdressed Danielle's concern that the last person did not high-five himself or herself, so the last 1 in Trevor's method should not be used. Trevor, and many others in the class, now agreed that Danielle's point was valid and that the partial sum should be 54, not 55. Hayden then interrupted and convinced the class with a persuasive argument that although the last person on the winning team would never give a high-five to himself or herself, the last high-five counted was between the second-to-the-last and the last winning team member. He made a clear explanation for distinguishing between the cardinality of the set of the remaining winning players and the number of high-fives given between them. His argument convinced the class that the partial sum was 55, and the total number (and the solution to the Soccer Problem) was 176 high-fives.

Then another student, Wyatt, explained his method. His method was valid, but like Trevor, he had made an arithmetic error, which students quickly pointed out. Wyatt's method counted the number of high-fives each winning player gave, starting with the first who gave 11 high-fives to all the losing team members and then 10 to his other 10 winning team players, and so on. Another student, Devon, offered a corrected method based on Wyatt's method. Ms.

Auburn wrote it on the board, adding it to what she had written when Hayden offered a rebuttal to Danielle's 54. Ms. Auburn asked Hayden to repeat his partial-sum counting strategy (9 + 1; 8 + 2; 7 + 3, etc.). The class again came to a consensus that the correct answer was 176, fortified with this similar though different method.

Mr. Murano's Episode: Susan's Problem and Manuel's Problem.

Mr. Murano's episode took almost the entire 80-minute period. The episode began with the students completing a worksheet (see Appendix B) that focused on Susan's Problem shown in Figure 2. Mr. Murano reviewed with the class what a *recursive routine* was and how to do it on a calculator. He asked the students whether 25 + 2.5x was the same as 2.5x + 25, apparently to remind the students about some of the work they had done 2 days before (because they were on block scheduling, their mathematics class met every other day). He then instructed the students to work with their partner to find a good window on the graphing calculator that would adequately display the graph of $y = 2.5x + 25$. The students moved their desks together and worked quietly in pairs to find and record a window, with Mr. Murano circulating silently. After most students had finished this part of the worksheet, he began a whole-class discussion and asked for a volunteer to show an example of a "good window." Kaleb used the overhead display calculator at the front of the room to show his window; then Mr. Murano led the class in a discussion of what made a "good window choice."

Mr. Murano asked for a volunteer to explain a method for solving the problem of how many weeks would be needed for Susan to save $139.99, one of the subproblems. Nadia volunteered to show her method on the overhead, which the class discussed together. Andy, a rather quiet student, was asked by Mr. Murano to show his method, which he did. Then Mr. Murano asked for another volunteer; Boston volunteered and explained his method, which he said was similar to Nadia's but different because he used a recursive routine. Unfortunately, while Boston demonstrated, he lost count of how many times he had pushed the enter button on

> Susan's grandmother gave her $25 for her birthday. Instead of spending the money, she decided to start a savings program by depositing the $25 in the bank. Each week, Susan plans to save an additional $2.50.
>
> 1. Make a table of values for the situation.
> 2. Write a function rule for the amount of money Susan will have after *t* weeks.
> 3. Find a viewing window for the problem situation.
> 4. How much money will Susan have after 7 weeks? Write this equation. Show how you found your solution.
> 5. Susan wants to buy a school ring. When will she have enough money to buy the $139.99 ring? Write this equation. Show how you found your solution.

Figure 2. Susan's Problem.

his calculator. A student mentioned that Boston was using Andy's method. Boston started to enter the recursive routine again, but rather than continue all the way to the answer, he said, "You just keep pushing 'enter'," and started to sit down. Mr. Murano asked Boston to finish his recursive procedure at the front of the class. A lively discussion ensued because Boston started counting weeks when the initial $25 was deposited at Week 0. He had counted 47 enters. Some students argued that the first entry of $25 at the beginning did not count as Week 1 (Week 1 started once the first deposit of $2.50 had been made), so the answer should have been 46, not 47, weeks. After some discussion, Boston recognized his error. This worksheet was now completed, and the students put it away as Mr. Murano distributed the next worksheet.

Mr. Murano had the students work on Manuel's Problem (Figure 3) in pairs (see Appendix C for a copy of the worksheet for Manuel's Problem). After a few minutes, he asked for attention and stated, “Excellent! Now we’re all going to go through and make sure all have come to the same conclusion about what is happening with Manuel. We need to make sure that we all, all agree. And again, hopefully you’ll notice, that there are several different ways.” Different solution strategies were presented by different students, similar to the discussion of Susan’s Problem. One student, Bradley, eagerly asked to share his method; Mr. Murano agreed. He recommended that students write down other students’ ideas on the worksheet, and the class period ended.

Manuel worked all summer and saved $1090. He plans to spend $30 a week.

1. Make a table of values for the situation.
2. Write a function rule for the amount of money than Manuel will have after *t* weeks.
3. Find a viewing window for the problem situation.
4. How much money will Manuel have after 11 weeks? Write this equation. Show how you found your solution.
5. When will Manuel be out of money? Write this equation. Show how you found your solution.
6. [There is no number six on the worksheet. A numbering error.]
7. How will the line change if Manuel had initially earned $1300? Graph the line. What changed? What did not change?
8. How will the line change if Manuel spent $200 on school clothes and started the year with only $890? Graph the line. What changed? What did not change?
9. How will the line change if Manuel starts with the $1090, but decides he will only spend $25 a week? Graph the line. What changed? What did not change?

Figure 3. Manuel’s Problem.

Ms. Sandy’s Episode: The Perimeter Problem

Ms. Sandy’s episode occurred in the third week (the fifth week of school) of the six weeks I spent with Ms. Sandy and her students. The episode was roughly 30 minutes long. Before the students entered the classroom, Ms. Sandy wrote the problem shown in Figure 4 on the front board. She began the lesson by asking Cassidy to come to the board and be the scribe for the class. Cassidy drew a table with four columns: a ‘base’ column, a ‘height’ column, a ‘perimeter’ column, and a ‘correct solution?’ column. Ms. Sandy took a back seat in the classroom and asked the class what they needed to do to solve the problem. The students, who had not yet worked with equations but had solved similar problems with “guess-and-check tables,” recommended creating a table (which was also in the problem statement). Cassidy created a table on the board, and Ms. Sandy said, “Let’s start guessing.”

> The base of a rectangle is three centimeters more than twice the height. The perimeter is 60 centimeters. Use a guess & check table to find the base and height.

Figure 4. The Perimeter Problem.

One student said 25, and Ms. Sandy asked whether that was the base or the height. The student said it was the base. Cassidy wrote 25 on the board in the base column and then wrote 22 in the height column, doing the height computation herself. Ms. Sandy now asked Cassidy where she got the 22, and Cassidy, assuming she must have done something wrong, erased her 22. Cassidy then asked the student who had proposed the original 25 what the height should be. This question triggered discussion in which students debated whether they should divide 25 by 2 and then subtract 3 or should subtract the 3 first and then divide by 2. Cassidy wrote 9.5 on the board, saying that you divide the base by 2 and then subtract 3. Ms. Sandy asked how to find the perimeter (perhaps not noticing the 9.5 already on the board or perhaps just ignoring it temporarily). One student said 25 + 9.5; others mentioned that you should multiply 25 and 9.5. The class noisily went back and forth on how to compute perimeter. To restore order, Ms. Sandy sharply asked to see some hands if people wanted to talk. She chose another student, Macky, who said that to find the perimeter you should double the base and double the height and then add them together. Murmurs of agreement were expressed, which abruptly ended the argument. Ms. Sandy, noticing Cassidy's 9.5, digressed momentarily from the perimeter discussion to ask the class how they knew that 9.5 was a correct value for the height. She said that doubling 9.5 gives 19 and adding 3 yields 22, which is not 25. One student argued that the height was 9.5 because you added 3 to get 12.5 and then doubled the height to get a base of 25. It took a few minutes of discussion for the class to agree that only one of the methods of computing the height was valid. They realized that the height divided by 2 minus 3 was different than the height minus 3 divided by 2. The problem called for the latter computation. One student said the height should be 11, and Ms. Sandy asked for justification, which the student correctly gave. Cassidy wrote 11 on the board. Returning to the perimeter discussion, the class found that the original guess of 25 for the base did not yield a perimeter of 60. Cassidy put an × in the 'correct solution?' column, signifying that the row was not the solution.

Braxton guessed another base value of 20.5, gave the height as 9.5, and then revised it to be 6.5. Both times, Ms. Sandy doubled the proposed height and added 3, which did not yield 20.5. Another student said they should choose a base value that was easier (a whole number), and other students concurred, so they decided to choose 21. The students eventually decided that the height for a base of 21 was 9 and that that combination gave the desired perimeter of 60.

Ms. Sandy asked about a general rule. The class discussed $2(b + h) = 60$, and Cassidy put that in the perimeter column. They tried to list b in terms of h, which was now seen as easier than starting with b and working backwards to derive h. They decided on $b = 2(h) + 3$. Ms. Sandy asked the students to work individually to come up with an equation for the perimeter in the perimeter column that used the base expression and the height expression. After about three minutes the students were brought back together for a whole class discussion, Braxton proposed that the perimeter was equal to twice the height plus 3 plus twice the height again: $p = 2\ (h) + 3 + 2\ (h)$. Opinions were split between whether this expression yielded 60 when h was 9 (the students were not following the conventional order of operations) or whether it was 39. The class computed it together, with Cassidy writing on the board, and together they agreed that it was 39. Cassidy wrote $(2\ h + 3 \times 2) + (h \times 2)$. Ms. Sandy encouraged the class to substitute 9 for the height and see whether 60 was the result. After more discussion (and trial-and-error computations), the class realized they needed to adjust the parentheses slightly: Cassidy wrote $p = (2\ h + 3) \times 2 + (h \times 2)$. They cheered once they had written a formula that gave 60. During the remainder of the period, the students worked in groups of four to create equations for similar word problems from their textbook.

Mathematical Emergence

One of the indicators of the presence of a complex system is emergence: Something arises through multiple individual interaction that supersedes any individual's characteristics. It is a property that belongs to the whole system and not to any single member. In all three teachers' classes I witnessed *mathematical emergence,* or the joint creation of mathematics. Not exclusively the product of teacher action, mathematics developed—or emerged—in these episodes, through collective action as the class jointly created the lesson. No single individual was responsible for the lesson, or the shared public enacted curriculum. Different strategies, different terminologies, different representations were all woven together to form a larger fabric superseding any individual's lone labor. The final tapestry of class action was a holistic phenomenon, understandable only through considering the collective's actions.

Joint Lesson Construction

The classes that I studied demonstrated consistent, reliable joint lesson construction. I define *joint lesson construction* to be formative class action in which the lesson becomes an embodiment of the class's collective mathematical activity. Such a phenomenon, where the emergent accomplishments supersede that of any individual, is a hallmark of complex systemic action. In each episode, the lesson was an active construction by the entire class rather than the result of any single individual's actions or even the sum of multiple individuals' actions. Students' ideas built on previous student ideas, and influenced later class actions.

Some general examples follow that differentiate lessons arising from a single individual's action, the sum of individuals' actions, and complex action. The purest form of a single individual creating a lesson is the *lecture*—a lone production. Univocal transmission, as opposed to dialogic functioning, dominates. An example of a lesson formed through the sum of individuals' actions is when the teacher shows, tells, presents, or demonstrates the mathematics to be learned, with occasional questions to students that have only one correct response (as if the job of students is to fill in the blank). Such a lesson is a construction of more than one individual, but it is a preplanned presentation with slight adjustments to accommodate students' remarks. A second example of a lesson formed through the sum of different individuals' actions is slightly more dynamic: The teacher has the students work on a worksheet and then asks several students to come to the board and present their solutions to different problems. If a student makes a computational error, other students may point it out and give correcting help or redo the problem correctly. This lesson is the result of multiple people's actions, but again it is just the sum of various individuals' actions—additively—like laying a hardwood floor with each piece of wood fitting with others to make the floor.

A lesson arising from complex action involves student participation in such a way that the lesson grows out of the individuals' contributions. For example, the teacher poses a challenging question to the students, who work individually. Then they have a whole-class discussion about different solution methods. As different students present their methods, and discussion about the accuracy or efficiency of different methods occurs, the lesson becomes a dynamic entity. It can be likened not to a hardwood floor but to a living tree. Later parts of the lesson grow out of and receive nourishment from earlier parts. Dialogic functioning is essential for this growth to occur; students consider previous ideas and use them for further action. Another way to express this is that students' strategies often were shaped by contributions by others. Students question one another and use each other's ideas to further the discussion. The lesson becomes a jointly constructed object, meaning that the actions of individuals develop and shape subsequent action. Sections of the lesson are not simply presented or demonstrated, plugged into place like precut blocks; rather, the lesson grows organically. It is dynamic, unpredictable, and unique. A teacher who might try to involve a different set of students in the "same lesson" discovers that the second lesson is far different from the first.

Although the three teachers contributed to the formation of these jointly created lessons, the lessons belonged neither to the teachers nor to any student. Instead the class as a whole owned the lesson, for it was the creator. To illustrate joint lesson construction, I describe how the emergent mathematical lesson was the result of collaborative unified activity by many individuals—an indicator that a complex system was present.

Ms. Auburn's Joint Lesson Construction

Ms. Auburn's episode demonstrated the emergence of joint lesson construction. No single individual—not Ms. Auburn, not Trevor, not Hayden, no one—was responsible for the creation of the lesson. The lesson was joint in the sense that individuals participated by giving their ideas to the whole class to contribute to whole-class action; it was joint in the sense that such action influenced later action. Dialogic functioning, not univocal transmission, dominated. Sometimes the whole class appeared confused (e.g., when Trevor's arithmetic error produced competing ideas); sometimes the class appeared stumped (e.g., by Danielle's insightful remarks, which halted all discussion temporarily); sometimes the class understood (e.g., when Hayden convinced the class that Danielle's concern was not valid). Unlike lessons that are largely preplanned and then enacted by the teacher, the lesson in Ms. Auburn's class included emergent whole-class action. The class as a whole developed the lesson through the synergetic actions of individuals. The students were dialogically involved, creating their ideas in response to other students' ideas. Several students, such as Jasmine, Lillian, Caitlyn, and Trevor, showed visible signs of cognitive change. Students cited each other's means of operation (e.g., Danielle said, "I kind of did it his way"). Vocabulary was developed, different strategies considered, terminology refined, and definitions created (a definition of *high-five*, for example). There was student justification, reasoning, questioning, and debate. Ms. Auburn's class was operating not as a mere sum of individuals' actions but as a complex system—action critically influenced subsequent action.

The class was fragmented in its understanding as it considered various solution methods. With some students' validation of Trevor's corrected method, the class began to accept his strategy as valid but was still confused about whether to count the last high-five. By doing some simulations and defining a high-five, the class was able to rectify Danielle's concern, and further insight was gained into the problem's solution by the consideration of Wyatt's method. Hayden's creative partial summation added mathematical creativity to the end of the lesson—helping to tie everything together. The halting stumblings of the class gave way when Trevor's strategy was presented. It drove the remainder of the lesson. The act of jointly solving this mathematical task became a "togethering activity," transforming the class into a complex system.

Mr. Murano's Joint Lesson Construction

In Mr. Murano's episode, the class as a whole jointly completed a previously assigned worksheet and began another, but in an unusual manner strikingly different than worksheet work involving dozens of similar problems. The activity was different because of teacher-guided student sharing and thinking about each other's ways of operating. In other words, unlike classrooms in which the worksheet is often a practice instrument that teachers give students to complete alone (or possibly to work on in small groups) to reinforce the day's lesson or to review previous concepts, the worksheet in Mr. Murano's class was completed a section at a time, with partner work followed by whole-class discussions after each section. It did not reinforce the lesson; it's completion constituted the lesson. Previous class action influenced (and was sometimes cited) in later class action. The worksheet acted both as a repository for the problems the class would work on together and as a location for student mathematical work and recording. Mr. Murano's class jointly wrestled with and solved problems that had not been previously presented in class. The worksheet was not a collection of repetitious exercises but rather an instrument for innovative whole-class instruction.

This episode was different from Ms. Auburn's episode: It was more structured, and the work was more tightly guided by the teacher. Certainly the worksheet contributed to the structure. As with Ms. Auburn's class, however, Mr. Murano's class as a whole was jointly constructing the lesson. It emerged out of the discussions by the class that occurred in work with a partner. Mr. Murano engaged his algebra class in an oscillation between whole-class discussion and partner work. When the whole-class discussion stopped, the students were working in pairs on the same problems. It was not like some lessons in which during the 20 to 30 minutes of worksheet time allocated at the end of the lesson, some students race ahead while others move more slowly. The synchrony of the work in pairs was critical so that whenever Mr. Murano brought the students back together, they could have a coherent discussion because they had all worked on the same problem.

The questions on Mr. Murano's worksheet were not numerous problems very similar to previously demonstrated examples. Rather, the worksheet was a guide to the problems the students would work on jointly, almost as if Mr. Murano had given the students copies of his lesson plan with the solution paths left blank. Also, the problems were original in the sense that the students had not done such problems before. They were problems that took thought, time, struggle, effort, and discussion to solve. In the 80-minute episode, the class did 11 worksheet problems, the last 3 of which were very similar (only the *y*-intercept of the equation varied), which meant that roughly 5 to 10 minutes of class time were dedicated to each problem, quite unlike typical worksheet problems, which can normally be solved rather quickly. The episode took place during the third week of the course, and already the students were engaged with complex algebraic concepts that are usually not studied until much later: writing general linear equations, identifying slope, graphing and interpreting linear functions, working with *x*- and *y*-intercepts, and even writing equations in point-slope form. The students had begun the algebra course not having been taught about linear relationships.

As the students worked on Susan's Problem and Manuel's Problem, there was no clear solution path and no mimicking of previous steps Mr. Murano had demonstrated. In the whole-class discussions, various strategies and methods were shared that the students began to incorporate into their own work. When students referred to each other's methods by name (e.g., Nadia's way or Andy's method), they showed that they were thinking through each others' work and comparing it with their own or that of others.

Thus, Mr. Murano's lesson, like Ms. Auburn's, was a group construction as the class worked to solve the problems on the worksheet. Mr. Murano broke the class into pairs to work on the problem before bringing them together to talk about their work. This approach could have had several advantages for the students' mathematical learning. Allowing the students to work on the problem prior to the whole-class discussion might have yielded a healthier, more robust discussion. In addition, it might have allowed those students who might not willingly speak out in whole-class discussions the option of thinking through the mathematics by themselves prior to thinking about how others dealt with it. Mr. Murano also took advantage of the work in pairs to ask a pair pointed questions or provide individualized assistance. But after letting the pairs work independently, Mr. Murano always brought the class back together to come to consensus about the tasks worked on and to explore jointly the finer details of the problem. This discussion prepared for the next worksheet problem.

Not all of the lesson activity might appear to have been mathematical. For example, how is finding a good window on the graphing calculator related to mathematics? Is there even a right answer? Yet the students were learning how to evaluate choices, make wise decisions, and communicate them to other students in a convincing manner. And they were forming the sociomathematical norms so vital to mathematics learning (Cobb & Yackel, 1995). They were learning proper ways to interpret a visual representation of a linear relation, the parts of a graph (like the *x*- and *y*-intercepts) that were important to include in the window, and the language of technology so the calculator could be a beneficial tool. The work in Mr. Murano's class, even when using calculators, was mathematical.

Ms. Sandy's Joint Lesson Construction

Ms. Sandy's episode is a third example of joint lesson construction. It was also an embodiment of class action. No students solved the problem on their own, and Ms. Sandy, although guiding the class in certain directions, let most of the ideas emerge through the joint student talk. As the students were talking about possibilities, especially in the formation of the general equation, they would incorporate each other's ideas (or debate them as invalid by offering counterexamples).

An occurrence in Ms. Sandy's episode should be illustrative of this characteristic of complex joint behavior. During the emergence of the lesson, Braxton wanted to choose 20.5 as the new base, with 9.5 and then 6.5 as the height. Ms. Sandy doggedly showed that these heights were not valid for a base of 20.5. Multiple students murmured that this was too tricky of a number to use, and wanted a whole number, which other students agreed with. Hearing the shift in students' attitude toward a new base, Ms. Sandy agreed to investigate a new base. She accepted Mckenna's base of 21. Easier to compute with, this new value proved to yield both a rapid height by the students and the valid solution to the problem. The point is that the holistic behavior of the agents—the actions of the collective—were dynamic and adaptive to the circumstances. Braxton was eagerly trying to find a valid base value after the first attempt of 25 proved too large. Other students grew frustrated with his unusual decimal guess. The resulting conversation involved a group movement toward consensus (wanting to use a whole number value) that began almost unheard, but ballooned rapidly to become the dominant direction of the group. No single individual was responsible for this behavior—it was complex emergence.

Other examples for these teachers could be cited. But suffice it to say that the lesson in each episode was an unpredictable chain of events through mutual action by varied participants that relied on previous class action and utilized other's ideas. The resulting emergent mathematics was macrobehavior none possessed by himself or herself. The entire class's action would be needed to understand the resulting lesson. For example, mathematical terminology evolved during class lessons. This emergence and crystallization of terminology was a dynamic process involving many individuals. Conventional terminology was introduced as the lessons progressed: Words like *steepness* or *slant* were replaced over time by *slope*. Ms Auburn described how she would highlight and model appropriate terminology as it began to emerge in class discussions:

> I don't know that when I first started teaching I ever consciously made a decision about that, but over the last few years, I have realized that, yes, I do have to model [appropriate language]. And the language that I use is the language that they will pick up. So when we're talking about *slope*, it is easy to start out talking about the *steepness* of the line, but then I do want the correct mathematical language out there, so that I will start using that. So it's not just the *steepness*, but it's the *slope* of the line. What does the "slope" tell us? What does the "*y*-intercept" tell us? So that those are words that they hear, and become familiar with as well. (Interview 1)

Often, however, correct mathematical terminology was introduced by a student who had heard it used in that context before, and the teachers highlighted the correct terminology when it arose.

All of these episodes can be viewed from the standpoint that the class as a unified whole was constructing the mathematics, which was embodied in the lesson. Just watching the actions of the teacher or of some particularly vocal student would not explain the dynamic emergence of the mathematics in these episodes. Such mathematical emergence is evidence of a mathematically complex system.

Self-Regulation

The second indicator of the presence of a complex system in these classes was self-regulation. Each of the classes exhibited self-regulation during the selected episode. Members of the system approved or disapproved of other members' mathematical actions, and they made such approval or disapproval known to the others. Although some regulation came from the teacher, a substantial amount came from the students. The longer they jointly operated with a specific concept, the more sophisticated their self-regulation became, probably because of their comfort with the topic. Some of the self-regulation was modeled by the teacher, and in time the students began to incorporate the teacher's actions into their own methods. The more some students operated in those ways, the more other students followed suit. But the students would also spontaneously react to each other's mathematical discussions, often providing further evidence or counterexamples for another student's claim. Such action led to rules and ways of operating that were understood by the entire class. I take spontaneous student mathematical discussion not initiated by the teacher, especially the approving or disapproving of others' mathematical ideas, as evidence that the class as a whole was self-regulating its mathematical behavior.

Two examples from Ms. Auburn's class illustrate the self-regulation. First, when Jasmine was correcting her first attempt to solve the Soccer Problem, she forgot to add the last 10, giving 45 instead of 55 as the partial sum. Trevor interrupted and commented that she needed to include the other 10, which she did. This intervention was not initiated by Ms. Auburn. Second, toward the end of the lesson when the class was about to accept Danielle's incorrect notion that the partial sum was 54 (because the last person would not high-five himself or herself), Hayden interrupted with a comment, contrary to the direction the teacher was headed, and gave a clear description of why 54 was not correct. This intervention, too, was not initiated by Ms. Auburn. Hayden demonstrated that the last 1 in the partial sum was how many high-fives were given by not the last person, but the second to last. This comment led to a careful diagramming of the problem on the board by Ms. Auburn, which was used in the final summary.

An example from Mr. Murano's class also illustrates self-regulation: Boston incorrectly described how many weeks passed when he was counting in front of the class (by how many times he pressed the enter key during his recursive routine). Several students instantly pointed out that the number should be 46 instead of 47. This intervention was not initiated by Mr. Murano. The students' uproar seemed to have initiated Mr. Murano's question as to why it would have been 46 weeks instead of 47 weeks. His question led to an insightful discussion about the *y*-intercept.

Similarly, near the end of Ms. Sandy's episode, many students erroneously computed the perimeter to be 60 when checking one of the heights. In fact, many students reported that it should work. Without Ms. Sandy saying anything, Raul jumped in to disagree. He said that the height was 39, which he then revised to 40—not 60. As the class rechecked their arithmetic, they realized that they were not following the order of operations and that Raul was correct.

Nonmathematical Systems

This study dealt only with complex systems containing mathematically based phenomena. Complex nonmathematical phenomena also arose that were legitimate and deserve brief mention even though they were not the focus of the study. I had hypothesized that any classroom could be the scene of various complex systems because of the nature of human beings' social interactions when a group is brought together for an extended time. People often begin to exhibit unified intelligent behavior as they interact. I observed a variety of complex systems at the research sites that I would not consider mathematical complex systems. Examples include clandestine distribution networks (to distribute candy or other forbidden objects), clandestine communication networks (e.g., small-group or class-wide note passing), classroom group movements (such as rapidly escalating disruptions that elicited an untypically harsh teacher intervention, or sudden student mobbing of an area), secret student-student helping (coordinated efforts to help other students appear to understand teacher questions or mathematical concepts without the teacher's knowledge), and nonmathematical small-group or whole-class humor. These actions all met the criteria for a complex system because they included some form of global behavior that no single participating individual demonstrated.

I recorded examples of each of the complex nonmathematical systems mentioned above. Some were so subtle they could be easily overlooked: Students would rather craftily form "illegal" partial-class systems that excluded the teacher. Often the hidden systems I observed involved fewer than a dozen actively participating students, but occasionally the systems grew quickly.

The teacher was usually not part of the observed nonmathematical systems, aside from those dealing with whole-class nonmathematical humor. Many of these systems exhibited considerable sophistication and coordination—evidence that even so-called slow learners, problem students, or lazy pupils could exhibit intelligence, motivation, and concentration when functioning as interested parties in a purposeful social unit. Many students who might be considered "mathematically uninclined" exhibited behavior that demonstrated their hidden abilities. Unfortunately, some of those students viewed school as a boring game that needed some social spicing-up, classroom learning as an amusing obstacle they enjoyed collectively dodging, and school mathematics as ritualized hoop jumping. They channeled their energies into mathematically unproductive systems. Although some of those systems (such as a system manifesting class-specific humor) were natural and probably valuable for maintaining a healthy ecological classroom balance, others were a distraction from learning mathematics. Nevertheless, such intriguing multi-agent behavior possessed the complex qualities of emergence and self-regulation. The behavior deserves mention because, as discussed in chapter 5, when the teachers successfully combated disruptive nonmathematical systems and rechanneled the students' energies into the developing mathematical system, the system profited thereby.

The perspective of complexity theory helped me to see two phenomena in these classes, emergence and self-regulation, which indicate the presence of complex systems. I describe in the next chapter the factors I observed influencing the formation and sustaining of thes mathematizing complex systems according to Davis's criteria (B. Davis & Simmt, 2003).

CHAPTER 5
HOW WERE THE COMPLEX SYSTEMS FORMED AND MAINTAINED?

In the fields of observation, chance
favors the prepared mind.
—Louis Pasteur

This chapter addresses the second research question of what contributes to the development of complex systems in mathematics classes. More particularly, I investigate how complex systems were formed and maintained in these mathematics classes. I identified a variety of common actions shared by the three teachers' classes that I believe provided for complex system formation in those classes. These actions can be organized under the framework of Davis's criteria. I conclude that creating a complex system in one's class depends on the actions taken by the teacher.

Because the norms and the ways of operating in the system had been established early in the year, as the classes continued over the next several weeks, each teacher spent less time training the class how to act, which left more time for the lessons. Although all three teachers took similar actions, subtle differences existed in the methods they employed. I first highlight the shared principles by which these teachers operated and then describe variations. In other words, the overall strategies for conducting their classes appeared similar, whereas the teachers' tactics had both similarities and differences. This chapter has five sections, one for each of the criteria B. Davis and Simmt (2003) identified as essential for complex system formation and function:

internal diversity, redundancy, neighborhood interactions, decentralized control, and organized chaos.

I began my study with the belief that I would find additional criteria needed for complex class formation beyond Davis's criteria. I anticipated augmenting the list of essential criteria, but during the analysis I discovered that their criteria are robust. Therefore, I discuss each criterion below, with the various manifestations of each criterion.

Internal Diversity

Internal diversity refers to the differences among the individual agents operating in a complex system. Such diversity provides a fertile reserve of abilities that can be used by the system as a whole. All three teachers highlighted for their students, and subsequently capitalized on, the diversity among the students. For example, Mr. Murano and Ms. Auburn used similar tasks the first week of school to emphasize the importance of diversity in the class. The diversity was already present, but they used activities that highlighted it. Through the discussions they made the diversity apparent to the students and helped them recognize its importance for the successful operation of the class.

For example, the first day Mr. Murano repeatedly emphasized the importance of working together while reviewing his disclosure document (Appendix D). It stated:

> Success in class is directly affected by student participation. *Understanding mathematics requires discussion, reasoning, and presentation of concepts by students…. Participation in class is essential for students to learn for understanding*. (emphasis in original)

The same day, he partitioned the class into groups of four and asked each group to take various polygonal cardboard pieces and form four squares. The catch was that he gave each student in the group a resealable bag with several pieces inside, but no bag contained the pieces needed to form a complete square. The two rules the groups were to follow were as follows: (1) There could be no talking, and (2) all pieces had to be used in making the four squares. Mr. Murano had mixed the contents of each bag with pieces from the other bags, so that each group had the pieces needed to complete four squares, but no bag alone could make a square. Over time, the students recognized what he had done, and began to work together.

After the students completed the activity, Mr. Murano led a class discussion of the importance of students using each other's ideas in learning mathematics and in joint problem solving. He commented to me later that there was much variation across groups in the time it took to realize that they needed to share pieces.

Ms. Auburn also highlighted student diversity. The first day of class, she played a Name Game, in which by starting in one corner of the room and working their way around, the class could learn everyone's names: The first student said his or her name, and then the second student repeated the first student's name and said his or her name. This process continued with each student repeating the names of all the preceeding students before saying his or her own name. At the very end, Ms. Auburn attempted, with some help, to say every student's name. The activity would prove crucial over the next few days as the students began to talk about each other's ideas in whole-class discussions. Because they knew each other's names, their ability to hold a discussion and talk about each other's ideas was greatly enhanced. Although learning names is not a mathematical activity, Ms. Auburn saw it as important for strengthening the mathematical potential of her class:

> I always make a big deal about kids getting to know each other's names, not only me, but also other kids in the class, because if they know somebody's name, they are more likely to talk to them than just [to say], "Oh, the kid in the black shirt." So we spend time getting to know each other's names. (Interview 1)

Ms. Auburn saw learning names as contributing positively to mathematical whole-class discussions. She then described the discussion she has with her students after the Name Game, where the team nature of basketball was compared to learning and doing mathematics:

> Then I also do "On the Court." I talk to kids about: What is it like at the basketball game? If you're in the stands, what does it feel like? What are you doing to participate in the game? Do you have an outcome? Like, do you have any effect on the outcome of the game [in the stands]? Whereas if you are a player in the game, you definitely have an outcome.… You actually get to participate and get to be a part of what is going on. So we get to talk about "On the Court." What does it mean if you're "on the court?" What are you participating [in]? How are you involved in mathematics? Are you just cheering on the smartest kid in the class, or are you actually doing your math and thinking about it? … I want the kids to realize that they have to be involved. They can't just leave it for someone else. (Interview 1)

Ms. Auburn also implemented a Mathematician of the Week program. Each week she would pick one student from each of her classes and put his or her name on the Mathematician of the Week poster at the front of the room. She would describe to the class what mathematical actions the student had performed the previous week to become the Mathematician of the Week. She also engaged her students in a mathematical task with partners, to help facilitate their enculturation into the class:

> We start out at the very beginning working in … partners, and they have to talk to each other. I always do an activity at the beginning of the year that has them working with someone else. And I want them to do that, because otherwise there are always kids who would say, "Well, I would rather work on my own. I don't want to talk to someone else." …. So I always start out the beginning of the year with at least one activity where they have to think [together]…. It is that whole idea of working in a group, starting out…. We start at the very beginning and have them go to the board sharing ideas, and start out from the very beginning modeling the behavior that I want them to do, asking questions, thinking out loud, and working in groups. (Interview 1)

On the second day of class, Ms. Auburn introduced her disclosure document (see Appendix E), which also highlighted the importance of students' diverse contributions. It read in part as follows:

> Students will receive 10 or 15 participation points each week if they participate 2 or 3 times a week in class. Participation means sharing ideas with the class, reading out loud, asking appropriate questions…. Being in class and participating in the class discussions is a significant part of the student's learning and understanding.

These were some of the activities Mr. Murano and Ms. Auburn conducted for their students to highlight the importance of recognizing and appreciating the diversity in their classes. Ms. Auburn said, "So, they have to start thinking about 'Oh, our classroom is a group, we are a team, we are a team together'" (Interview 1). These teachers wanted their students to understand that the talents and capacities of every individual in the classroom were important, and that respecting others' ideas, as well as participating oneself, was essential for healthy class discourse.

Although I did not observe the first few weeks of Ms. Sandy's class, she did say similar things to her class: "I like [my students] to work in groups, and I like them to get in groups right off the bat [at the beginning of the school year]" (Interview 1).

The three teachers also capitalized on the diversity in their classrooms; they turned it into an advantage for mathematics instruction. By *internal diversity*, I refer not to demographic differences but instead to differences among students in how they think about and operate with mathematics. In fact, I believe that mathematical diversity is the most important form of diversity in classrooms for mathematics learning. Each teacher created an environment that brought mathematical diversity out into the open for the students to consider in the development of their own mathematical understanding. I do not contend that differences in race, culture, social economic status, or other qualities are unimportant to mathematics learning, but in considering the impact on the class as a system, I want to emphasize the diversity of students' mathematical thinking. The great differences in how the students thought about, talked about, represented, and understood mathematical ideas were essential to the collective mathematical action in these classes.

Mr. Murano and Ms. Auburn, whose classes were relatively homogeneous racially and economically, and Ms. Sandy, whose classes were very diverse racially and economically, highlighted for students and capitalized upon the inherent diversity of students' ways and means of operating mathematically. These teachers (1) presented and maintained cognitive demand of difficult mathematical problems with no clear solution path, and (2) created a respectful social space for different ideas to be presented and discussed.

Pose Challenging Problems

Stein, Smith, Henningsen, & Silver (2000) have identified the posing of challenging tasks, and then maintaining the cognitive demand of the tasks as critical to mathematics learning. Each teacher posed such tasks and strove to maintain the cognitive demand. The main mathematical problems in which the three teachers engaged their students provided seedbeds for diverse mathematical thinking. Each teacher used his or her own type of problem. Mr. Murano used original worksheet problems that had a variety of subcomponents, Ms. Sandy gave problems from the curriculum textbooks and materials, and Ms. Auburn used an amalgamation of problems either borrowed from a variety of curriculum resources or originals. They all gave challenging problems that allowed for a variety of solution methods even though each main problem had a single right answer, and thus the students had space for exercising individual creativity and mathematical insight. In the episodes, one can see how the Soccer Problem, Susan’s Problem, Manuel’s Problem, and the Perimeter Problem were all challenging open-ended problems, especially as they involved new mathematics with which the students were not familiar. The problems allowed for a variety of different mathematical viewpoints to be used in their solution. In addition, no teacher lessened the cognitive demand of the problems later in the lesson. For example, Ms. Sandy commented on maintaining the cognitive demand of a problem in her first interview:

> If they don’t do it, they don’t get it. I think that it is very good to struggle through a problem. I know that I do not necessarily like to struggle through a problem, if I have to, but I know I understand it when I have…. You can always tell when the lightbulb comes on. If they have struggled through something, and they have had to do it themselves—…. Telling them something is the last resort for me. I am going to think of all the questions I can think of, to get them going where I think they should go, without being too leading. But if they’re not doing it, it is not helping. They have to really do it…. I think they are going to learn it by thinking about it. (Interview 1)

Create Respectful Spaces

In addition to posing challenging problems, the respectful social space that the teachers created was essential for bringing forth the mathematical diversity allowed by the main problem. This attribute of the community was highlighted by the National Council of Teachers of Mathematics (NCTM, 2000) as critical to student learning:

> The … teacher should strive to establish a communication-rich classroom in which students are encouraged to share their ideas and to seek clarification until they understand…. To achieve this kind of classroom, teachers need to establish an atmosphere of mutual trust and respect…. When teachers build such an environment, students understand that it is acceptable to struggle with ideas, to make mistakes, and to be unsure. This attitude encourages them to participate actively in trying to understand what they are asked to learn because they know that they will not be criticized personally, even if their mathematical thinking is critiqued. (p. 270)

The teachers worked hard to mold a classroom climate that would be conducive to the sharing of tentative mathematical ideas. For example, Ms. Auburn solicited volunteers to share their solutions to the Soccer Problem. Later in the lesson, she allowed those students who wanted to revise their method (e.g., Lillian, Jasmine) in response to other students' comments (e.g., Trevor's, Hayden's) the opportunity to return to the board and do so. The message was clear: Revision of prior work was allowed in her classroom. Tentative ideas could be shared, competing viewpoints debated, and opinions raised—all within a safe social climate. Mr. Murano made encouraging comments whenever his students volunteered ideas. He constantly used such terms as *excellent*, *good*, and *very good* to describe students' ideas (69 such affirmations in his episode). With such affirmations, he communicated to the students that sharing ideas in his classroom was to be commended. Ms. Sandy would refuse to move forward in any aspect of the lesson when the students were quiet. She would simply wait for a response, or repeat the question and wait some more. In her class, thoughtful student comments were expected, and once they were expressed, she would ask the class what they thought about the student's ideas rather than pass judgment herself.

Ms. Sandy in an interview talked about the attitude she tried to instill in students:

> Who do you think is right? Let's vote. It's okay, if that is what you believe, then you should stick to it. If somebody proves you wrong, and you agree that you were wrong, it's okay. It is better to make a mistake and figure out how to fix it, than not to figure out how to fix it…. It is all just a matter of when people have their own opinion, and everybody being comfortable with their own opinion. But when you find that your opinion may not be the way things should really work, you have the guts to change your mind. And I think that they do. (Interview 1)

She noted that the students in her algebra class in particular, would be willing to stand up for their opinion until evidence convinced them otherwise:

> They are not afraid to change their mind, and they are not afraid to be wrong. And in a lot of cases they are willing to admit that they were wrong, and [say], "Oh, yeah, I see it now." And I think that comes with having math confidence. (Interview 1)

Ms. Auburn expressed the same opinion about her class atmosphere:

> I work really hard to get kids to respect each other, listen to each other's ideas even. I know that I have a lot of kids that say, "Oh, he has got to be right because he is the smartest kid in the class." And my comment to that is, "Well, okay, I am the teacher, does that mean I'm always right? Does that mean I never make mistakes? Does that mean they never can make mistakes, or think about a problem in a different way than you thought about it? So it is always good to ask questions and wonder: "Well, how did you get that? Does it make sense in this context? Does it carry on?" So I always want [my] kids to also think about their mathematics, and question where did it come from. (Interview 1)

Part of creating a respectful space was the movement of students' seating. This movement allowed students to work with and get to know their classmates. The teachers moved students around to new seats several times a semester. Ms. Sandy commented, "I like to change them around, like every book we do, for every unit, so they do not have to work with the same people all of the time" (Interview 1). Ms. Auburn indicated such repositioning helped unify the class:

> Ricks: You said a phrase [earlier in the interview]: "The class as a group."
>
> Ms. Auburn: Oh.
>
> Interviewer: Could you say a little more about that?
>
> Ms. Auburn: I usually do a new seating chart about once a month, so they get to know other kids. They always, they always have a different squish partner with whom they have to work. So when you talk about the class as a whole group, they really are a group as a whole. And they have to learn to respect and listen to each other's ideas as the kids are at the board presenting and talking. (Interview 1; see p. 97 for a definition of *squish partner*)

Redundancy

Internal diversity provides complex systems the resources needed for creative movement. Yet too much diversity can spell disaster. A nascent system can be torn apart by differences. B. Davis and colleagues (B. Davis & Simmt, 2003; B. Davis & Sumara, 2001) claimed that the characteristic of redundancy provides a counterforce against internal diversity’s impulse toward separation. Redundancies are various shared qualities among the system’s smaller agents that allow for cohesive joint operation—the adhesives for complex systems. I noted several redundant categories shared by these complex classes: (1) common norms, (2) common tasks, and (3) common mathematical orientations.

Common Norms

Mr. Murano and Ms. Auburn developed class norms jointly with their classes during the first few days of school. These teachers made joint norm formation an explicit activity culminating in a tangible product. They spent considerable time on it, first discussing how important it was for the class to develop shared expectations for all to learn equally, and then engaging each class in creating a unique written list of the norms the class would strive to follow (within the broader climate the teacher created through his or her disclosure document). Ms. Auburn said:

> We talk about: Norms are normal behavior. What we would expect from other people, when we’re working in a group of 2, or 4, or as a whole class? And I post them up on the bulletin board. And I have to keep reminding kids all year long. It’s not just like, “Oh, we have our norms, and we are set to go!” It’s like every day, every week, every month, we have to remind them what the norms are, and be polite and let others [have] a chance to talk, and listen while somebody else is talking. And not just to keep your mouth shut, but actually listen to what they are saying, and go, “Oh, does that concept make sense to me? If not, what do I disagree with? What do I agree with in their presentation?” (Interview 1)

These norms were not imposed by the teacher but arose through student discussion about good ways of acting in groups and together as a whole class. Because the norms were posted, the teachers and students could cite them easily. Any disciplinary action by the teacher had an added effect because the students had previously agreed on the norms. The consistent reference to the norms by the teacher reinforced their effectiveness.

Ms. Sandy did not have posted norms, but she did comment in an interview that the way she acted helped students learn proper modes of conduct:

> Maybe I should post [norms] on the wall. If you are … doing what you are supposed to be doing, everything goes smoothly; respect each other; opinions are good no matter what they are…. I expect them to know how to follow rules, and to know what decent classroom behavior should be. And I hope, without having to spell it all out, that I treat them the way they feel comfortable, they feel valued, they feel like they can say anything and not be laughed at. (Interview 1)

Thus, for all three teachers, the implicit or explicit norms provided for the maintenance of respectful social spaces for student discussions.

Common Tasks

The second redundant element I observed these teachers use was the concept of jointly solving the same task. Each teacher engaged the students in a shared class-wide effort on the same mathematical task, which yielded rich mathematical discussions. Whether the teacher asked students to work individually or in groups prior to a whole-class discussion was less important than asking all students to work on the same problem together at the same time. Once I watched as Ms. Sandy deviated from this practice, asking different groups to solve different problems and to create individual posters to present to the class. The next day, as each group presented its results, she had noticeably more management problems. Some of the students who were not presenting and were not familiar with the other groups' problems, quickly tuned out of the discussion. But for the most part, all three teachers involved students together in common tasks so the subsequent whole-class discussions could be coherent, meaningful, and communal.

Common Mathematical Orientations.

A third form of redundancy that I witnessed was the development of a consistent *mathematical orientation*: similar educational experiences, whole-class formation of shared terminology, mathematical language and its structure (mathematical grammar), mathematical ways of operation (strategies and methods), and representation. Cobb (Cobb & Yackel, 1995) has used the term *sociomathematical norms* to describe “normative aspects of whole class discussions that are specific to students’ mathematical activity” (p. 8). I use *common mathematical orientations* because when it comes to describing redundant mathematically related elements in the classroom, I believe *sociomathematical norms* is too narrow a term. I observed the shared ways students operated mathematically, including their dispositions, confidence levels, tenacity, and even being mathematically skeptical. That behavior permeated the class action and was not confined to aspects of whole-class discussion. I see sociomathematical norms as a subset of common mathematical orientations.

An additional distinction between mathematical orientations and sociomathematical norms is that sociomathematical norms are negotiated by the community—like an individual’s beliefs and values, but at the class level. Some aspects of common mathematical orientations, however, may be inherent in a class because the students have had similar curriculums or textbooks in previous year, and may even have had the same teachers. This experience is an obvious source of redundancy, as the members of the community know enough collectively about the subject to begin a conversation. The students have had previous mathematics courses together and have used textbooks, calculators, and group work to do mathematics. To some extent they all understand basic arithmetic and elementary mathematics terminology. They share some overlapping “knowledge in the head” (Andrew Izsák, personal communication) from which they can build common understanding. Other aspects of a common orientation may not be beliefs or values at all—as Cobb and Yackel (1995) claim all sociomathematical norms are—such as mathematical grammar (how mathematical symbols are ordered or arranged to properly communicate ideas) or common mathematical strategies understood by the class as a whole. In addition, assisted by community construction of sociomathematical norms, the way in which students engage with mathematics, whether privately or interactively, also has a redundant element.

In Mr. Murano's and Ms. Auburn's classes, the teachers from the beginning of the year developed further the common mathematical orientations by modeling appropriate forms of mathematical discourse and behavior. They consciously made explicit certain ways of mathematical operation so that the students could more easily communicate and understand each others' work. Through the class's joint actions, initially modeled in appropriate ways by the teacher, the students continued to develop and refine class-specific language for describing mathematical ideas, objects, and procedures. The class as a whole developed a unique language system. The emergence of classroom-specific mathematical humor, with inside jokes no outsider would understand, showed that some of the class's language was unique. In addition, the teachers began to use conventional mathematical language, which was then integrated into the class's discourse. An example of this language was in Ms. Auburn's class when students first used an arm or a hand motion to describe how one line differed from another, then adopted the word *steepness*, and finally began using *slope* after Ms. Auburn introduced the term.

The approaches to solving problems, the ways of representing mathematical ideas, the mathematical strategies and solution paths, and the manner of appropriate mathematical debate—all were developed through joint class action from disparate initial elements into a cohesive, shared, class-specific orientation toward mathematical action. I give two clear examples of this development from the episodes. In Mr. Murano's class, the various student ideas about solving Susan's Problem and Manuel's Problem began to converge toward a common understanding.

Mr. Murano tried to make this convergence explicit when he began the discussion of the linear equation for Manuel's Problem:

> Excellent! Now we're all going to go through and make sure all have come to the same conclusion about what is happening with Manuel. We need to make sure that we all, all agree. And again, hopefully, you'll notice that there are several different ways. Okay. So, some of you probably came up with things slightly differently than others.

Ms. Auburn, in her episode, emphasized several times to the students that the class was coming to a common understanding:

Ms. Auburn:	Okay. So now, does [her] answer match Trevor's answer?
Trevor:	Yes.
Students:	Yes! Yes!
Ms. Auburn:	Oh, so we are starting to come to more of a consensus?

And at the end of the episode:

Ms. Auburn:	So, 176 [high-fives].
Students:	Yeah. Yes!
Ms. Auburn:	Do we have a consensus?
Students:	Yes. Yes.

Additionally, from the various solution methods presented, each class would gravitate toward simple, concise, efficient, and correct strategies through the communal discussions.

In the beginning, the students often struggled to express their ideas orally. In Mr. Murano's class, a week before the main episode, Boston was trying to explain how he found a trend line (approximation of a line of best fit) for some data the class was working on. He graphed his trend line on top of his data points using the overhead display calculator.

Mr. Murano:	So, tell us about it. What did it do?

Boston: That. [Looks sheepishly at Mr. Murano.]

Students: [Laughter.]

Boston: Like that. It looks kind of okay.

A little while later, after Boston showed his equation, his difficulty in describing his mathematical thought-processes was again evident through his cryptic language:

Mr. Murano: What is the deal with the 3 there [referring to Boston's *y*-intercept]? Negative 3? Why did you do that?

Boston: 'Cuz.

Students: [Laughter.]

Mr. Murano: 'Cuz why?

Boston: [Showing with his hand.] I had to move it, like [waves his hand] this way. See? [More waving.] Yeah. I don't know. It would fit it better. 'Cuz, um, I can't explain it very well.

Mr. Murano: Okay.

Boston later became much more proficient at explaining his thinking. Many times the students would attempt to explain their mathematical thinking and stop by saying, in effect: "I know what I am trying to say, but I don't know how to say it," or "I know how to do it myself, but I can't explain it."

The careful efforts by these teachers to model precise explanation and to push for precision when students explained their thinking to others laid the groundwork for developing the class-specific mathematical orientations that eventually involved conventional terminology. For example, Mr. Murano relied heavily on technology to introduce the concept of linearity; his students learned about doing the recursive routine either by hand or on the calculator to model a constant rate of change. Ms. Auburn's students, when learning about linear relationships, did not learn about recursive routines on the calculator (at least while I was there). So although both groups of students were learning the same formal mathematics, the context was different.

Decentralized Control

Decentralized control is the third criterion B. Davis and Simmt (2003) considered essential for complex system formation. I observed three features of decentralized control that were shared by the three teachers: (1) They relinquished mathematical authority, (2) their language was dominated by plural (*we*, *us*, *our)* language rather than singular (*I*, *me*, *my*) language, and (3) they engaged their classes in mathematical discussions that veered between chaotic and controlled. This is consistent with what the NCTM (2000) has described as reminding "students that they share responsibility with the teacher for the learning that occurs in the lesson" (p. 60).

Relinquishing Mathematical Authority

The teachers took pains to remove themselves as the mathematical authority in the class. They regarded the social unit—the collective—as the mathematical authority (except in rare instances when clarification was needed). The common plea by students to a teacher hovering over the desk—"Is this answer correct?"—illustrates the position a teacher often holds in the power dynamics of a mathematics class. In all three of these teachers' classes, however, the teachers took explicit action to demonstrate to the class that they were distancing themselves from the position of a mathematical authority. These actions were verbal or physical, and they left mathematical issues unresolved.

The teachers used explicit language to communicate that they were not going to tell whether a student was right or wrong. For example, Julio asked Ms. Auburn to just tell the class which method was correct after three students had presented different answers to the Soccer Problem. She responded simply, “I don’t have the answer.” When a student asked Ms. Sandy to tell the class the answer to an equation in another episode, a student, Raul, said that she would never tell the answer, because “she never does.” Although some students continued to try to get answers from her, Ms. Sandy was adamant about responding with another question for the questioner or the whole class.

Ms. Auburn said: “I like for them to think of themselves as a team, the whole class as a team, so they can say, ‘Oh, we have to work together. It is not just me against other people.” Or, “I am competing against them. But we really *are* a team’” (Interview 1). The teachers still maintained an adult presence, ultimately responsible for management, student participation, assessment, homework assignment, and other activities needing adult authority in a middle school classroom. They also set the tone for the expectations for mathematical expression, mathematical description, precision, and questioning. But when it came time for passing judgment on the correctness of a method or answer, or for choosing sides on a mathematical debate, the teachers deftly and deliberately deflected any questions back to the students. Thus the mathematical authority was passed back to the class itself.

This authority reduction was highlighted by the NCTM (2000) as helping students develop “productive habits of reasoning” (p. 345):

> In order to evaluate the validity of proposed explanations, students must develop enough confidence in their reasoning abilities to question others’ mathematical arguments as well as their own. In this way, they rely more on logic than on external authority to determine the soundness of a mathematical argument. (p. 345)

Additionally, all three teachers would sometimes leave issues unresolved when students were unable to come to a consensus. Mr. Murano was especially prone to let ideas remain unresolved until a later time or lesson when the students had learned more—a characteristic of the teaching of the mathematician R. L. Moore (Cohen, 1982), who commonly left unresolved issues in his teaching.

All three teachers also displayed physical manifestations of their removal from a position of judgment. For example, all three would leave the front of the room and let students take their place: The student who held the marker became the focus of class attention. The teachers often went to the back of the classroom during discussions, as if removing themselves from the students' range of vision allowed the students to focus more on who was talking and not to be preoccupied with the subtle physical cues that teachers often give during mathematical discussions. Mr. Murano, in his cramped portable classroom, would sit in a student chair or stand against the back wall between rows. Ms. Auburn, in a larger room, would sit at her desk or on an empty desktop at the back of the room. Ms. Sandy would sit in a chair at the back of her room.

Singular–Plural Ratio

During analysis, I noticed teachers' constant use of *we*, *us*, and *our* in their dialogue. I compared the use of the singular forms *me*, *I*, and *my* with the use of *we*, *us*, and *our*. For each teacher, I examined the entire lesson from which the sample episode came and obtained data that showed how singular or plural forms were used by the teacher. As can be seen in Figure 5, for all three teachers plural usages were more frequent than singular ones and increasingly so as the lesson progressed. The horizontal axis represents the time as the lesson progressed. The vertical axis represents the accumulated number of singular or plural forms used by a teacher during his or her lesson. For each page of transcript, I counted all the singular language (*I*, *me*, *my*) and all the plural language (*we*, *us*, *our*).

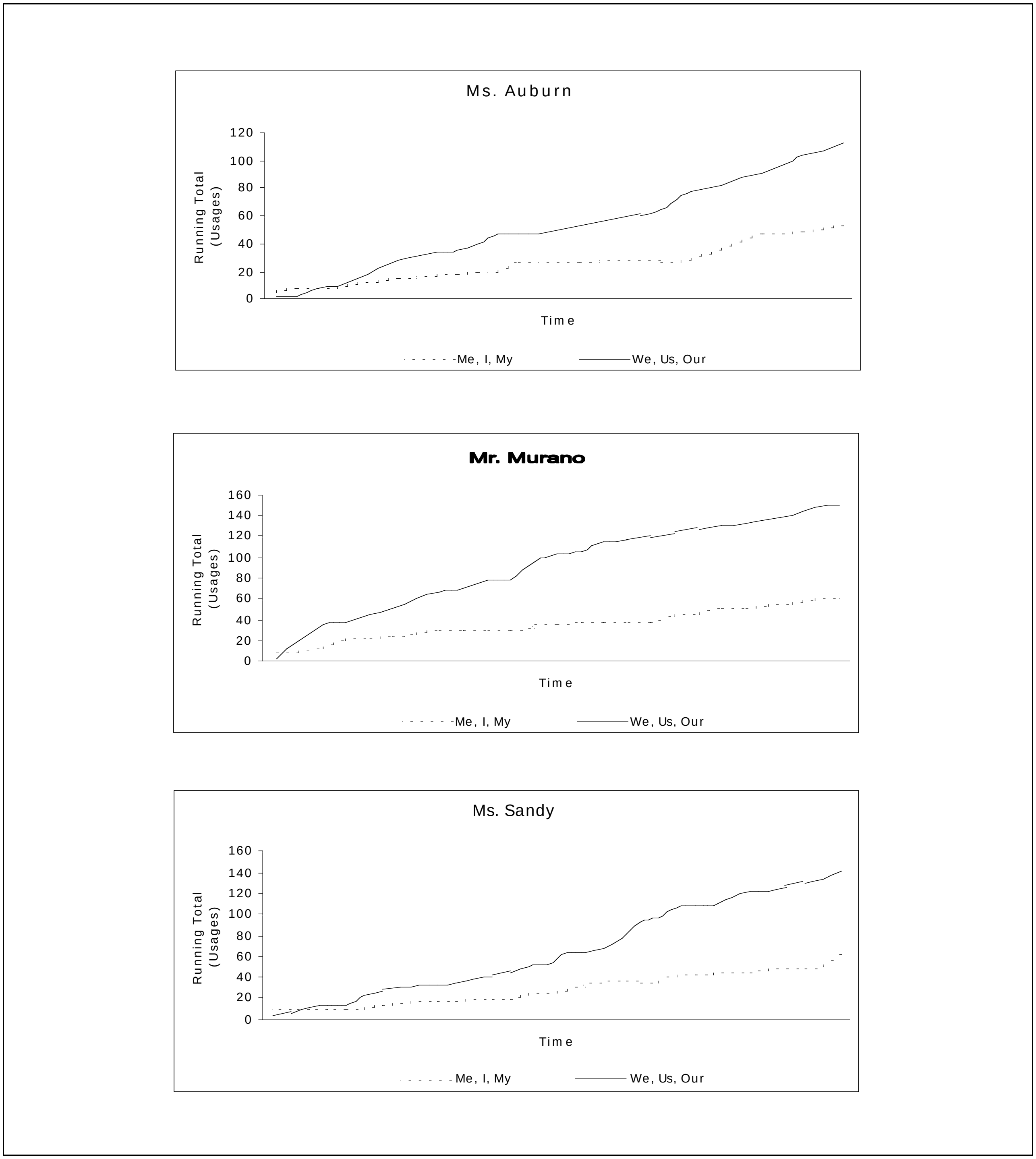


Figure 5. Three teachers' singular-plural language.

For comparison, it proved helpful to transpose my data against other American classrooms. I chose the four lessons highlighted in the 1999 Trends in International Mathematics and Science Study (TIMSS) video study as typical classes (National Center for Educational Statistics, 2003). The TIMSS study investigated several dozen American lessons, and researchers chose four to highlight what the researchers considered “typical American lessons.” Sophisticated sampling techniques were used, as well as rigorous analysis by the researchers, which are described by the National Center for Educational Statistics (NCES, 2003).

As can be seen in Figure 6, the ratio of singular–plural usage is quite different for the U.S. teachers in TIMSS. As a group, the TIMSS teachers did not follow a consistent pattern of usage. In Lessons 1 and 2, singular uses were more frequent and tended to be increasingly so as the lesson progressed. In Lessons 3 and 4, plural uses were more frequent but not by much, and not increasingly so. Only Lesson 4 showed any similarity to that of the three teachers in the present study, but toward the end of that lesson, the *me* usage rebounded sharply. Compare the absence of a pattern in the TIMSS data with the dramatic consistency among Ms. Auburn, Mr. Murano, and Ms. Sandy. They have significant plural growth in plural usage and similar singular-plural ratios. These graphs provide evidence that, compared with typical U.S. teachers, the teachers in the present study were more likely to use language that signaled their identification as part of a classroom community and less as the authority.

Table 4 shows these totals by teacher. Figures 5 and 6 provide additional information not visible in Table 4 because the progress of language usage throughout the lesson can be seen.

Ms. Auburn mentioned that this approach to teaching, by involving the students jointly as a mathematics community, enabled her to understand what the class as a whole was understanding. In other words, it facilitated assessment—almost as if the teacher, by listening to whole-class discussion, was “reading the system’s mind.” Here is a portion of the interview:

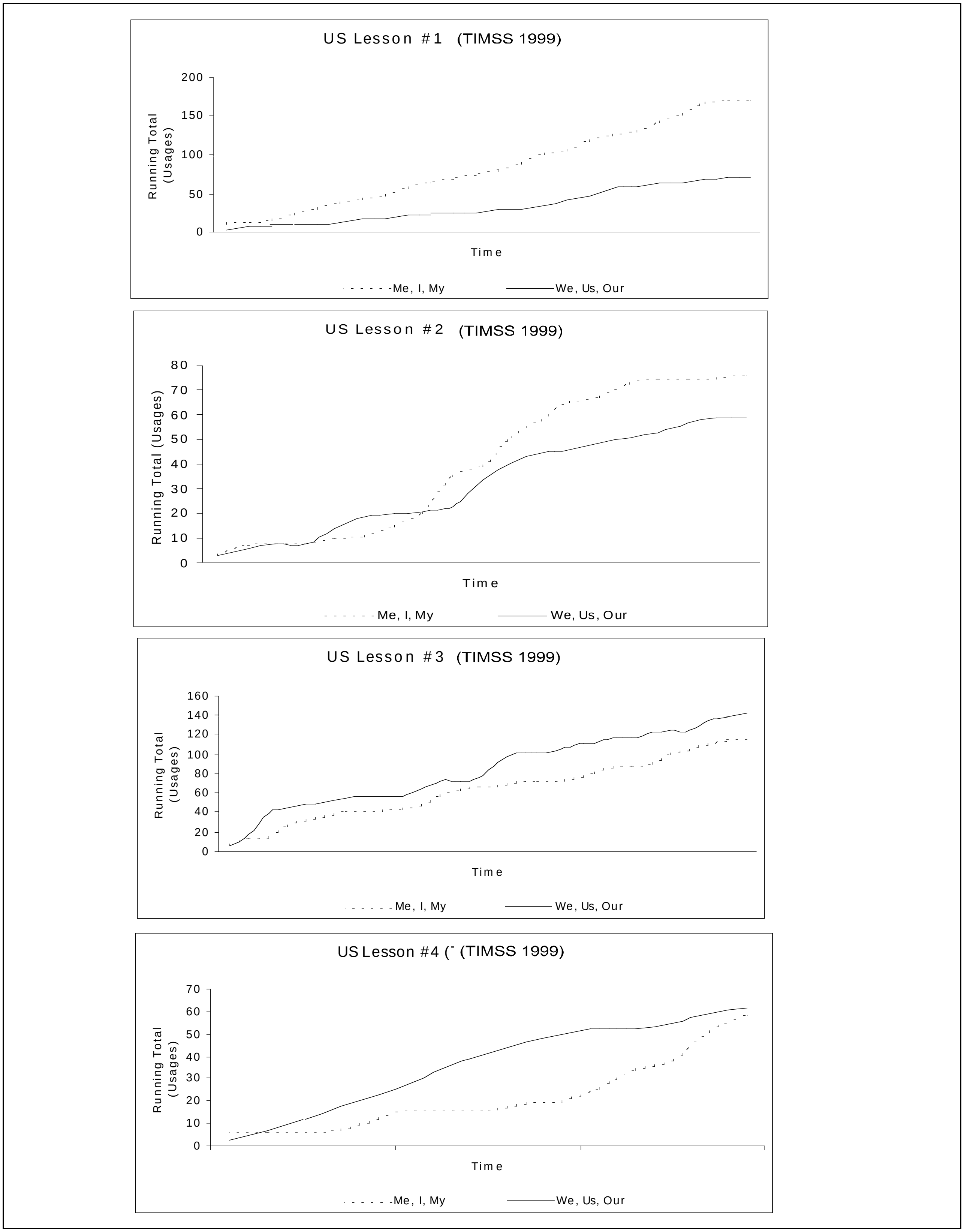


Figure 6. TIMSS 1999 U.S.teachers' singular-plural language.

Table 4

Number and Percent of Singular and Plural Forms in Lessons

Teacher	Singular N	Singular %	Plural N	Plural %
Ms. Auburn	53	32	113	68
Mr. Murano	62	29	151	71
Ms. Sandy	62	30	142	70
Total	171	30	406	70
TIMSS Teacher 1	172	70	72	30
TIMSS Teacher 2	76	56	59	44
TIMSS Teacher 3	114	45	142	55
TIMSS Teacher 4	59	49	62	51
Total	421	56	335	44

Ricks: As a teacher, do you do things, or have kind of a sense of, how the class as a whole is doing?

Ms. Auburn: Yeah. Like mathematical-wise? Why, how they are doing?

Ricks: Yes.

Ms. Auburn: Or just working together?

Ricks: I'm thinking more about the mathematics.

Ms. Auburn: Yeah.

Ricks: Like are they getting a concept? Are they not getting it? But now I am talking about the class as a whole.

Ms. Auburn: As a whole.

Ricks: I'm kind of interested about this process. If like, if you have ever thought about it, or—?

Ms. Auburn: Actually, I do, and it is very interesting, because, well, especially when we have new teachers, and I mentor them and stuff, it is really interesting to talk with them about before you give a quiz or before a test. "How do you think your class is going to do? What do they know, what do they don't know?" For me, teaching this way [through my method] allows me to know a lot more about what my kids as a class [motions with her hand in a circle] as a whole [know], because I get to walk around the room, and see their work, and as a whole, I would say, I could determine whether they're working together. They're strong. Their mathematical ideas are good, and there might be one or two kids who are struggling, and those are the ones that I have to help, and give extra help with. Or, as a class as a whole, they are not doing so well. So you need to go back and to figure out where their holes are. What are they missing? What do we need to do to bring them up? It really is, it is really that personality thing, where a class as a whole—they have a good feeling, and they're working mathematically, or they struggle…. I am not always exactly correct in my thinking, but I usually have a good feel about what the kids know, and when to move on, and when to approach the next concept. (Interview 1)

Navigating Between Chaos and Control

I observed an interesting pattern that all three teachers manifested. The teachers relinquished control of the lesson until the class action reached a critical point, at which time they regained lesson control. By *lesson control*, I refer to the regulation of the way the mathematics was enacted in the whole-class discussions—that is, who spoke, who went to the front, who responded to questions, and so forth (which is different from being a mathematical authority who controls what is considered correct or incorrect mathematically). As a discussion progressed, the teacher would turn over to the students more and more of the creative act of the lesson formation—the enactment of the mathematics. The students would then become more engaged and animated as the mathematical discussion accelerated. For example, they would become increasingly less likely to raise their hands to speak before commenting. The dialogue became rapid-fire. Sometimes students would stand up at their desk they became so involved. The volume of discussion would increase as well. At such a juncture, the teacher would step in and retake control of the lesson. Like a parent teaching a toddler to walk, the teacher would move back and forth between letting go and holding on. This navigation between chaotic and controlled lesson enactment required constant give-and-take maneuvers by the teachers. There was a liberating action when students were given more control (providing for rich dialogue about mathematics) that no doubt would not have occurred had teachers not let go of the lesson's reins. As the discussion became more excited and louder, however, becoming increasingly difficult to understand as more students began participating, the teacher almost always reined in the class before control was lost. When the class came under more control, the joint actions appeared to slow, but then the teacher would give more control back to the class. Such navigation was apparent when I transcribed the lesson. I would reach points of difficulty because so many students were talking over each other that I had a hard time figuring out who was saying what. But then I would observe that the teacher would calm things down, and the dialogue would become easier to transcribe.

For example, at a certain point in solving the Perimeter Problem, Ms. Sandy appeared overwhelmed by the students' discordant dialogue and asked to see some hands (which was not a common occurrence in her normal whole-class discussions, unless things started to get rowdy):

Ms. Sandy: Well, how you find perimeter then?

Student: By adding!

Ms. Sandy: Well, it said "add."

Braxton: No, you find the perimeter by *multiplying*!

Raul: It just said find the perimeter by adding.

Student: And then add 3.

Ms. Sandy: Excuse me, I need to see some hands. We are going nowhere fast. We have a lot of problems to do today.

[Hands raise.]

Student: You add up—you multiply the—uh, the—.

Student: Hey, wait?

Student: It's, it's—.

Kody: Oh, I know! I know!

Ms. Sandy: I still hear people talking, and I have hands up ready to go.

Ms. Sandy: Macky, how you find the perimeter of the rectangle?

Macky: Add double the base and the height, and add them together.

Ms. Sandy: Double the base *and* the height?

Macky: Yeah.

Another example from Ms. Auburn's class demonstrates essentially the same thing:

Tevor: So from 10 to 1, that would end up being 56, so I just added 121 and 56, and that was 177.

Ms. Auburn: 'Kay, I saw some answers over here that were very close to those.

Students: It is 55. I thought it was 55? 55!

Ms. Auburn: Oh!

Students: I thought it was 55! It is 55! 55!

Ms. Auburn: Oh, wait, you guys. Hayden and—tell me [to Devon] your first name again—?

Devon: Devon.

Ms. Auburn: —Devon both think it's 55 instead of 56.

Each teacher seemed to have a different point at which he or she became uncomfortable and regained control of the lesson: Mr. Murano had short cycles of lesson control, Ms. Auburn had longer cycles, and Ms. Sandy exerted the least control—so much so that occasionally her class became so boisterous that she had to exert tremendous effort to get the students back on task. I make no judgment about which approach would be most beneficial.

Neighborhood Interactions

The fourth characteristic of Davis's criteria (B. Davis & Simmt, 2003) is neighborhood interactions between components of a complex system. The most noticeable characteristic of the three teachers' classrooms was the dynamic interaction of the students. All three classes had dialogic (Wertsch & Toma, 1995) student discussions about mathematics. The three teachers used similar strategies to enhance the way students conversed about mathematical ideas. As a side note, I use *neighborhood interactions* in a way consistent with typical literature on complexity (e.g., Johnson, 2001), which describes individual agents interacting. B. Davis and Simmt (2003) described neighborhood interactions as the ideas of students that come in contact with one another. Presumably, if students interact, so will their ideas. Ms. Auburn said:

> That is probably my biggest philosophy: The more kids are involved sharing and talking about the mathematics, the more likely they are to do it than if they have to sit back and be a notetaker, or just plugging in problems. And because of that, I spend a lot of time at the beginning of the year setting up my classroom so that kids know that I want them to participate. They work with partners. They work on their own. So it's not always a partner group. They have to think on their own as well. But also getting kids to come to the board, be willing to share, and for some kids just been able to explain they're thinking is a whole new thought process. (Interview 1)

Partner Work

All three teachers engaged their students in considerable work with a partner. Because Ms. Sandy's room had two-person tables with two chairs at each table, she would have the students at each table collaborate on mathematical problems. She would specify only when discussion was not allowed. Most of the time, however, talking to one's neighbor about mathematics was not considered cheating or otherwise inappropriate. Unless discussion was prohibited, the students engaged with each other about mathematics. Certainly, the two person tables facilitated partner interactions.

Mr. Murano and Ms. Auburn, in classrooms with separate student desks, taught their students a movement strategy to promote partner interaction: They created *squish partners*. The first few days of school, they taught their students to move their desks together for partner work. Ms. Auburn arranged six parallel rows of desks in three sets of pairs and taught the students that when she said "Squish with your Squish Partner," the students were to bring their desks together so that their desk was touching that of their partner from the other row. She taught the students how to join desks quickly: They used several minutes the first day to practice squishing and unsquishing.

Mr. Murano had his room laid out in six rows, too, and he taught the students to squish together in a similar arrangement as Ms. Auburn. He also used the words *squish* and *squish partner*. In both classes, it was obvious when the teacher wanted the students to work as partners because they were specifically asked to squish. Students would also talk quietly to one another even when their desks were not together. Such quiet discussion was allowed unless explicitly forbidden or disruptive.

Group Work

All three teachers regularly engaged their students in group work, typically a group of four. In Ms. Auburn's and Mr. Murano's classes, the students were taught how to rotate their desks so they were facing each other in a group of four. In Ms. Sandy's classes, the students in front would usually rotate their chairs to the table behind them to form a group of four. The students became efficient at forming a group quietly and disengaging to come back to the whole-class discussion with minimal disturbance. Mr. Murano and Ms. Auburn also practiced this four-person group movement the first day, forming and unforming groups.

Whole-Class Discussion

Most important, all three teachers would involve students in daily whole-class discussions (Yackel, 2000). Ms. Sandy was most likely of the three to begin such a discussion without prior independent, partner, or group work. Dialogic functioning was a key component of the whole-class discussions. The students were able to talk to each other and question each other's mathematical activity: They interacted mathematically. Like mathematicians, they engaged in justification, explanation, interpretation, contradiction, reasoning, and the like. The whole-class discussions allowed the classes to become mathematical communities. As Ms. Auburn claimed:

> Because, even as a mathematician, if you are working on something, very few people do isolated mathematics on their own. And even if they do, when it is published, it becomes the work of everybody, and [there are] lots of peers who review it. So in everyday life, when people solve mathematical problems, they talk to someone else. They think about: "Oh, what is the problem that actually has to be solved? What are we doing? What's our approach? How are we going to take this on? Is this idea going to work? Is it not going to work?" So I think, very few things in this world are isolated ideas. People work in groups, and they collaborate, and they talk back and forth. So I think as far as mathematics, that is part of what they do as well. (Interview 1)

During the whole-class discussion the complex system was most clearly present as joint lesson construction emerged. I observed the teachers using four techniques to maintain healthy whole-class discussions. They (1) modeled transparent thinking, (2) modeled critical colleague commentary, and (3) thoughtfully selected student strategies for demonstration.

Transparent thinking. The teachers modeled for their students appropriate interaction that required clearly described sense making, something the NCTM (2000) has stated as necessary for effective mathematics instruction (p. 123). That clarity allows others a window into one's thoughts. Ms. Auburn described the modeling she did:

> I do realize that my behavior effects my classroom a lot, so the language I use is the language my kids will pick up. How I respect them and behave towards them is a lot of how they respect each other. And then I also encourage my kids, like, if I see behavior that I think is a problem, I will ask them "What's wrong?" so that they can see, and model their behavior. (Interview 1)

For example, when a student was at the board describing something he or she had done, each of the teachers would often stop the student and ask for a clearer, more precise explanation—an explicit description of what the student was thinking when he or she used the strategy. The teachers rejected opaque communication; they required an ungarbled, understandable rendering of the student's thinking. Not surprisingly, most students were unaccustomed to producing precise descriptions of their mathematical actions. The students struggled to express their actions appropriately, as they developed a suitable production vocabulary to meet the teacher's expectations.

Critical colleague commentary. In addition, to further model clear interaction with others, the teachers also modeled appropriate questioning of others' thinking. While one student was describing his or her thinking in a transparent manner, the other students were encouraged to critically assess the mathematical description. In other words, it was the responsibility of those receiving the communication to critically examine its mathematical validity. As Ms. Auburn said:

> What's their mathematical reasoning behind it? If someone else can talk them into changing their mind, is it a valid reason? Can they be convinced to change their mind just because someone else says: "Oh, my way is better. Or is it really

> mathematically valid?" So I do push towards explaining why. Where did it come from? So it's not just the two equations are equivalent. "Okay, yeah, we all agree. *Why* are they equivalent ones?" (Interview 1)

The receiving party was to accept, question, or reject the first party's discussion. Such action fostered mathematical interactions, as the two parties jointly developed a clearer understanding of the phenomenon. Respectful critique was an appropriate form of commentary; respectful rebuttal or appropriate acceptance was expected. This approach reflects a perspective sanctioned by the NCTM (2000):

> In mathematically productive classroom environments, students should expect to explain and justify their conclusions. When questions such as, What are you doing? Or Why does that make sense? Are the norm in a mathematics classroom, students are able to clarify their thinking, to learn new ways to look at and think about situations, and to develop standards for high-quality mathematical reasoning. (p. 341)

Thoughtful Selection. I observed two aspects of teacher actions that contributed to the dynamics of class interaction: equalizing participation and careful selection. Critics of the study of complex systems in education sometimes claim that only the aggressive, vocal students benefit from the actions of the system because only they are participating. In this study, each teacher used his or her own technique to encourage student participation and to help all students become involved in the discussion. The NCTM (2000) said that "all students should have the opportunity to learn" mathematics (p. 29), which requires such equalizing.

Mr. Murano, for example, gave each student a form on which the student was to record whether he or she had been tardy, the main idea of the day's lesson, and any homework and quiz scores. As part of the form, he had a space for each day to indicate when the student had participated orally. He would use a cylindrical stamp with a little "smiley face." When a student went to the board to present or discuss his or her ideas, Mr. Murano would sit in the vacated seats. While the student was up front, he would pull out the stamp and stamp the student's form. He would tell the students that they needed to keep the form on the corner of their desks so that he could come by and give them a stamp for participation. In that way he encouraged participation, as the stamps contributed to students' grades. Sometimes Mr. Murano would select a quiet student to share his or her thoughts. For example, he asked Andy, an especially quiet student, to share his strategy during the discussion of Susan's problem. Mr. Murano could quickly glance at the students' sheets as he walked around the room to tell how much a particular student had participated that week.

With a marker Ms. Auburn wrote the names of all the students in her class on a side board, and she would put check marks after the students' names when they participated. She gradually delegated this responsibility to students who sat near the board.

> As I look around my kids, some of them are very quiet. They'll sit there and not say anything unless they are specifically asked to participate, or I ask them a question about what's going on, and those are the kids I want to become involved as well. And so, I love having all my names on the board because then I can look over and go though. Trevor has participated six times; Damien has not participated at all! "Damien, what you think? How is it going? Where does it come from? What are you thinking as you go along?" Because otherwise it is very easy to always let the kids who are always outgoing to drive the conversation. Because the other kids have great thoughts as well. It's just they are not as willing to stand up and go, "Wait, pick me! Pick me!" So, I love having that as an involvement so I can track what kids are saying and where they are going and how they are thinking about mathematics. For even the quiet kids. And having them share and see what's going on. (Interview 1)

Because Ms. Auburn frequently rotated the seating arrangements, all the students in a class had an opportunity to record the participation on the side board. If a student declined to be a recorder, another one was asked to do it. With this system, Ms. Auburn could easily glance over during a whole-class discussion and see which students were lacking in participation points. Each week the students were required to have a certain number of participation points. Because the system was public, some students would start encouraging the quieter ones to participate so they could earn participation points. Ms. Auburn remarked about this practice in an interview:

> It is interesting to see kids that haven't participated, then they are like: "Oh, Damien hasn't had a chance! Let us let Damien have a chance to come to the board!" And it is fun to see different personalities of kids as they get to … become the cheerleaders and they get to stand back and cheer someone else on as they go along. And I had a class last year that every time that someone came to the board, they wanted to applaud. To give them the hand [demonstrating by clapping]. And I am like, "Okay! It is fun to cheer kids on and stuff." (Interview 1)

Ms. Sandy had no tangible way of keeping track of which students participated, but she would often walk around the room, sit next to quiet students, and engage them as part of the classroom discussion. She made a concerted effort to ask the quieter students their thoughts during whole-class discussion. For example, she mentioned to me in an interview that although Addison was a good thinker, she would not normally participate voluntarily unless Ms. Sandy called on her. And so she called on Addison frequently.

Careful selection is another technique that I observed the teachers using to facilitate whole-class discussion. In addition to selecting less vocal students, the three teachers attempted to select students based upon the observed mathematical substance of their strategies. For example, while the students worked individually or in pairs, Mr. Murano would circulate around his room with a clipboard, making notes about the different ways the students were approaching a problem. This practice is similar to the way Japanese teachers have been observed to teach (Lewis & Tsuchida, 1998). During the later whole-class discussion, Mr. Murano would refer to his clipboard, if needed, to ask a student to share a particular strategy. Similarly, Ms. Auburn and Ms. Sandy would selectively choose students based upon the mathematics they had done rather than purely nonmathematical criteria. Ms. Auburn commented about this tactic:

Ricks: When you go around, like when you select examples of student work to take pictures of, or when you ask specific students to go to the board, do you, do you choose those kids for a reason?

Ms. Auburn: I do…. Like, if as I am walking around and everybody is doing the problem in the same way, and so there might be one or two ways, then I will choose kids who have both ways…. If I'm walking around the room, and I see several different strategies, then I will pick—let's say, I see five different strategies—I will pick three of them that I think are good to focus on. And sometimes I will pick work because I want kids to discuss "Oh, wow! That was an interesting way that they did it." Sometimes I will take work because this is definitely more efficient than I saw from these kids, and I want them to go: "Oh, I hadn't thought about it [like that]. I could do it that way." And so, it depends on the lesson, and it depends on how many different strategies. Or if it's something that I see kids are going to struggle with later. But usually it is pretty purposeful; most of the time it is. (Interview 1)

There was often foresight and mathematical justification for the choice of the students that were chosen. It was often premeditated selection—on mathematical grounds—rather than arbitrary or based on equalizing participation. The teachers had mathematical purpose behind the decision making they engaged in. True, at certain times it did not matter whom they chose to present; someone was needed to start the discussion. For example, in the beginning of the Soccer Problem discussion, Ms. Auburn chose Lillian seemingly randomly. Yet at other times in her teaching, it was clear she was carefully selecting a certain student's work to highlight it for the class. She would sometimes take a digital picture of the student's work to later project on the front of the board for the entire class to see. These thoughtful techniques of equalizing participation and carefully selecting students because of the mathematical contributions they could make to the lesson greatly facilitated the whole-class discussion.

Organized Chaos

Organized chaos is the final key feature of Davis's criteria (B. Davis & Simmt, 2003; B. Davis & Sumara, 2001) and was a key element of the teachers' actions. I will use the term *organized chaos* instead of *organized randomness* which B. Davis used. With regards to organized chaos, I have identified two common techniques I noticed these teachers use: (1) actuated luck and (2) mathematical management. These were qualities that each of the teachers used to organize seemingly random occurrences in the classroom.

I changed B. Davis's term *organized randomness* to *organized chaos* because *chaos* seems more appropriate for describing classroom environments. I do not believe the events in a mathematics class occur at random. They may appear to an observer to be random, but I contend that all actions have some motive and reasoning behind their occurrence, and so *random* is not the best way to describe even the most jumbled events in a classroom. Mathematically chaotic behavior contains some element of randomness or spontaneity, but it is structured against some underlying organization. Over time, patterns emerging out of turbulence indicate hidden constraints. Complex systems require some background organization but with the freedom for individual creative expression, which organized chaos implies.

Actuated Luck

An observer might consider that these teachers had great luck during their lessons. The students appeared say the right things to help the discussions along, or a student's mistake early on would prove to be a significant asset in understanding a later concept. After observing these complex classes, other teachers might consider themselves unlucky, saying, "Why don't my students talk in my class like that!" or "Why don't my students bring up these points?" The three teachers, however, displayed a principle I call *actuated luck*: Their so-called luck was the phenomenon of preparing for, recognizing, and using effectively the spontaneous occurrences in a discussion to further the mathematical goals of the lesson. These teachers took opportunities to highlight comments that would benefit the lesson. For example, during an interview, Ms. Auburn noted how she took student comments and integrated them into the lesson:

> In seventh-grade algebra today when [that student] said, "Oh, I would take the slope, times it by *x*, and add it to the *y*-intercept," I had not anticipated anybody verbalizing the equation in that form, but it was perfect! Oh, we can take that [student's comment], and we can go to $y = mx + b$. So they know slope-intercept form. (Interview 1)

The teachers appeared lucky because they prepared their classroom environments to promote beneficial events, highlighted such events as they unfolded, and capitalized on them. These classrooms had rich, sustained mathematical discussions because the teachers had the expertise and experience to pose substantial problems that would engage their students in mathematical challenges. For example, Ms. Auburn described how she had consciously practiced asking good questions:

> When I first started teaching this way, my biggest focus was on questioning. Knowing how to ask the right question, that gets kids to think mathematically. Right now, I'm a little better at questioning. I have worked on that for a lot of years, so it is getting better…. And it is a lot easier to do for me to do now than it ever would've been 5 or 10 years ago. Part of that is because I feel very comfortable teaching the mathematics that I'm teaching, and I feel very comfortable in the classroom setting that I've created. So even if I make a mistake, it's okay, it's okay to go: "Oh, you know what! I was wrong. I didn't think of it [from] that point of view." (Interview 1)

Surprising things happened in their classrooms of which they could take full advantage because they had created a dialogic space for student ideas to be elicited. The teachers appeared to recognize when a critical juncture occurred in the lesson to move the class into deeper mathematical activity. They apparently knew the right thing to say, the right point to make explicit, or how to elicit the right question because when the special, unpredictable events occurred, they seized the moment. Ms. Auburn described how in previous years she might have overlooked an aspect of student thinking that would have proved valuable:

> I used to teach … kids how to solve a proportion by cross multiplying…. I was always pushing them to do cross multiplication even though they had some great mathematical ideas, and they could see that reasoning. Now it is, like, "Okay, you're right. That works. How would you use that idea to solve this type of problem?" So it gives, it gives me more flexibility to say, "Oh, take that idea. And where can you go from this idea and still be able to solve a proportion? But solve it a different way." (Interview 1)

What started out as a straightforward mathematical task in these classes yielded intriguing discussions through the orchestrating actions of the teachers. They were able to elicit student ideas, would continue to ask the right questions, and made appropriate choices so that rich, mathematical class discussions would ensue. NCTM (2000) supported such actions:

> To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks. They need to understand and be committed to their students as learners of mathematics and as human beings and be skillful in choosing from and using a variety of pedagogical and assessment strategies. (p. 16)

Ms. Auburn offered some advice about how other teachers could use student activity for the lesson's advantage:

> You have to be able to say, "Okay, if I'm looking at these ideas and I give kids a chance, can I take it farther than I thought?" Or, maybe, I might have to take a little longer than I thought because their ideas, they're not understanding as in-depth as what I wanted them to go. So you have to be flexible time-wise, and either advance your lesson plan and go farther, or be able to back up and say, "Maybe I need a little more time on that idea." (Interview 1)

Part of the principle of actuating luck was appropriate teacher action during critical moments in the lesson. The three teachers possessed the ability to ask the right question at the right time. They seemed to recognize when the classroom action was at the point where their intervention in a specific way could propel the class into more productive mathematical experiences and acted before the moment passed. Their lessons were like a "swiftly flowing river" (Lewis & Tsuchida, 1998). An example was when Boston was explaining his recursive method in solving one of Susan's problems in Mr. Murano's episode. He was at the board and had lost count of how many times he had done the recursive routine. Rather than redoing the entire 40 or so recursions on his calculator, he simply said, "I just kept doing that [pushing the calculator's enter button] until I got down, to, like 40." When Boston began to move toward his seat, Mr. Murano seized the moment and explicitly asked him to finish the routine up front so the class could see what would happen. Here is the dialogue in context:

Boston:	That is 2, right there, so because she started out with 25 [dollars], right there. And then she, the second week, she put in 2.5 [dollars]—so that is the second week. And that is [pushing the calculator's enter button] third, fourth, fifth, sixth, seventh. I just kept doing that [pushing the calculator's enter button] until I got down, to, like 40. [moves to sit down.]
Mr. Murano:	Keep going. Let us see what you got.
Boston:	Okay, it is…
Student:	[Forty-]seven.
Student:	Six.
Boston:	Seven.
Mr. Murano:	Six or seven.?
Student:	Six.
Student:	That's six.
Student:	Six.
Student:	Seven.
Boston:	That is [forty-]seven because you started out with 25 [dollars].
Mr. Murano:	Okay. Let us see what he comes up with. What did we start out with?

A less-experienced teacher might have simply let Boston sit down and resumed the lesson. When Boston redid the routine, the class began a heated debate about whether the final value was 46 or 47. The students recognized that the initial value $25.00 was not the first week, a critical idea in understanding the *y*-intercept. This recognition led into a deeper discussion of the *y*-intercept.

All three teachers recognized that their classroom actions would have considerable impact on the direction of the lesson and tended to act in ways that would provide for rich mathematical discussions. They engaged the students in an organized though unpredictable discussion through actions that affected the entire lesson. They seemed to be asking, “How will my action affect the big ideas in this mathematical lesson?” They showed the sort of knowledge that the NCTM (2000) has called for:

> Teachers need several different kinds of mathematical knowledge— … the whole domain … curriculum goals … important ideas [for the] grade level … challenges [for] students … how the ideas can be represented … [and] how students’ understanding can be assessed. This knowledge helps teachers make curricular judgments, respond to students’ questions, and look ahead to where concepts are leading and plan accordingly. Pedagogical knowledge, much of which is acquired and shaped through the practice of teaching, helps teachers understand how students learn mathematics, become facile with a range of different teaching techniques and instructional materials, and organize and manage the classroom…. Their decisions and their actions in the classroom—all of which affect how well their students learn mathematics—should be based on this knowledge. (p. 16)

Mathematical Management

The management style of the three teachers was related to the mathematics. Books on management talk about *consistency* in management. If one student acts in a certain way, and the teacher provides a certain type of reward or punishment, the teacher should to act similarly with other students. Often, however, these teachers would allow multiple students to talk at the same time without doing anything. At first, I was baffled by their lack of intervention. At other times a student might make the slightest comment, and the teacher would provide instant discipline.

Their actions seemed contrary to contemporary theories of equitable classroom management. Were these teachers poor classroom managers? I do not think so. I found that the criterion they used for imposing discipline was not the volume of talk but whether the activity had stopped contributing to the actions of the mathematical class system.

If students were talking, even quietly, and the teacher recognized that the talk did not relate to the mathematics of the lesson, the teacher was quick to impose discipline. If students were talking loudly about appropriate mathematics, however, as long as it was part of the system's actions, then the teacher was likely to let such activity continue. In all three classes there was a clear pattern of positively reinforcing mathematical actions and dampening nonmathematical actions. This pattern is directly connected to complexity theory's idea of positive and negative feedback (Johnson, 2001).

As can be seen in this chapter, the teachers' actions played a critical role in the development of a complex system in their classes. The level at which internal diversity, redundancy, decentralized control, teacher interaction, and organized chaos were present in the class was directly correlated with the teachers' action. The three teachers appeared to operate in a similar manner following common principles that led to fertile ground for creating complex mathematizing systems in their classrooms. These actions are listed with the Davis framework, in Table 5.

Table 5.

Davis's Criteria and Common Teacher Actions.

Davis's Necessary Criteria	Action	Subaction
Internal diversity	1. Highlighting for student the importance of diversity.	
	2. Capitalizing on diversity	a) Posing challenging problems
		b) Creating respectful spaces
Redundancy	1. Common norms	
	2. Common tasks	
	3. Common mathematical orientations	
Decentralized control	1. Relinquishing mathematical authority	a) Verbal
		b) Physical manifestation
		c) Leaving issues unresolved
	2. Singular-plural ratio	
	3. Navigating between chaos and control	
Neighbor interactions	1. Varied size of neighborhood	a) Partner work
		b) Group work
		c) Whole-class discussions
	2. Ways of interacting	a) Transparent thinking
		b) Critical colleague commentary
	3. Thoughtful selection	a) Equalizing participation
		b) Careful selection
Organized chaos	1. Actuating luck	
	2. Mathematical management	

CHAPTER 6
CONTRIBUTIONS OF COMPLEXITY THEORY FOR EDUCATION

Each of us is a multitude.
—*Walt Whitman*

In chapter 4, I provided evidence that the three teachers' classes demonstrated the presence of complex systems. Each class manifested mathematical emergence and self-regulation. In chapter 5, I detailed the role the teacher played in elaborating the five criteria identified by B. Davis (B. Davis & Simmt, 2003) as essential for complex system formation and function: internal diversity, redundancy, decentralized control, neighborhood interactions, and organized chaos. Chapters 4 and 5 affirm and contribute to the complex theory espoused by complexivists (B. Davis & Simmt; B. Davis & Sumara, 2001). In this chapter, I discuss contributions of complexity theory for education. I first describe some similar perspectives to complexity theory and how complexity theory differs from these perspectives. Next, I detail the phenomenon of shifting class arrangements, which delineates, through the perspective of complexity, a phenomenon I observed in the three teachers' classes. Finally, I conclude with some general contributions I see complexity theory making to mathematics education.

Differences from Similar Perspectives

The reader might ask how complexity theory is different from other perspectives such as the emergent approach (Cobb & Yackel, 1995) and the situated perspective (Boaler, 1999). Researchers use many theoretical lenses, perspectives, sub-perspectives, and bridging perspectives. Take *constructivism*, for example. Doolittle (2007) described "a constructivist continuum" detailing exogenous, cognitive, information processing, psychological, dialectical, social, sociocultural, symbolic interactionist, endogenous, schema-based, and radical constructivisms (not to mention social construct*ionism*) to try to make sense of the bewildering array of theories and subtheories.

The emergent perspective. Fortunately for the constructivists, Cobb and his colleagues (Cobb & Yackel, 1995; Cobb & Yackel, 1995) illuminated the emergent perspective as a way to harmonize individual psychological constructivism with interactionist constructivism. They claimed that their approach coordinates (Cobb & Yackel, pp. 11–24) those two constructivisms, where "learning is a constructive process that occurs while participating in and contributing to the practices of the local community" (p. 19). The individual contributes to the community, and vice versa. Whether a researcher takes an individualistic or interactionist perspective depends on the question at hand, always being mindful that the other perspective is intrinsically linked and always in the background.

Such a blending suggests a complex perspective, and indeed, the emergent perspective shares many similarities with complexity theory. For example, the emergent perspective considers classrooms as "ecosocial systems," as does complexity theory. The emergent perspective considers domains nested within domains, just as complexity theory considers systems to be nested within systems. The emergent perspective also draws parallels between the larger social actions and the individual actions within the social. Cobb and Yackel (1995) stated: "Students are seen to always perceive, act, and learn by participating in the self organization of a system which is larger than themselves—the community of practice established in the classroom" (p. 14). That statement might have been written by a complexivist. It sounds like emergence.

What is the difference between complexity theory and the emergent perspective? Complexity theory is about higher order behavior developing through the interaction of agents. It highlights the new entity formed by such action and the holistic nature of action. As such, complexity theory emphasizes that reducing the emergent system to its components would destroy the macrobehavior of the larger system. The emergent perspective, although identifying the classroom community as an organized social system larger than the student, is about harmonizing individual cognition with the cognition arising from interactions. It still maintains the view of the individual as the “locus of learning” (B. Davis & Simmt, 2003). The emergent perspective remains silent about the larger ecosocial system as a cognitive, adaptive, self-regulating entity in its own right. So although acknowledging an extra-student system, the emergent perspective does not investigate that larger system holistically (as complexity theory considers necessary if the behavior of the whole is to be regarded). As B. Davis and Simmt (2003) wrote: “Such analyses … stop short on the matter of the actual identifications of classroom collectives…. Social norms ... emerge and evolve [and are] described on the level of interacting agents, not as properties of an emergent unity” (p. 144). But if one wished to coordinate analyses of class cognition with individual student learning, the emergent perspective would prove helpful. I see complexity theory as a friend of the emergent perspective, and not a twin, because of its different focus.

The situated perspective. The situated perspective is another related theory. How is situativity related to complexity? The situated perspective examines the activity of individuals in communities of practice. Boaler (1999) explained:

> Situated perspectives differ from many [other learning theories] that have gone before them, in their focus upon broad activity systems (Greeno and MMAP, 1998) or communities of practice (Lave, 1988). Most distinctively, situativity locates learning as a social and cultural activity and success is not a focus upon

> the cognitive attributes that individuals possess, but upon the ways in which those attributes play out in interaction with the world. (p. 260)

In the situated perspective, learning is situated, social, and distributed (Putnam & Borko, 2000). Knowledge lies between individuals rather than in their heads: It “locates learning in coparticipation in cultural practices” (Cobb, 1994, p. 14). Situated cognition goes beyond constructivist views, claiming that:

> to the extent that being human is a relational matter, generated in social living, historically, in social formations whose participants engage with each other as a condition and precondition for their existence, theories that conceive of learning as a special universal mental process impoverish and misrecognize it. (Lave, 1996, p. 149)

Individual learning cannot be understood without also recognizing the social and contextual factors in which it is embedded; in fact, learning cannot be understood if looking only at an individual’s mental cognition (whether individually or socially constructed), for as Lave recognized, “learning is part of [people’s] changing participation in changing practices” (p. 150).

Unlike constructivist theories, “situated perspectives turn the focus away from individual attributes and towards broader communities” (Boaler, 1999, p. 261). Stein and Brown (1997) commented:

> Learning is seen to result from the fact that individuals bring various perspectives and levels of expertise to the work before them. As individuals work toward shared goals, they together create new forms of meaning and understanding. These new meanings and understandings do not exist as abstract structures in the individual participants’ minds; rather they derive from and create the situated practice (or context) in which individuals are coparticipants. (p. 159)

Learning is not viewed as a knowledge construction but rather as a “social practice” (Lave, p. 150); it is not “a process of intra-individual change [but is] primarily social in nature, [for] ‘learning and development occur as people participate in the sociocultural activities of their community’” (Stein & Brown, p. 158).

> Teaching … is a cross-context, facilitative effort to make high-quality educational resources truly available for communities of learners. Great teaching in schools is a process of facilitating the circulation of school knowledgeable skill into the changing identities of students. Teachers are probably recognized as "great" when they are intensely involved in communities of practice in which their identities are changing with respect to (other) learners through their interdependent activities. (Lave, p. 158)

For mathematics learning, the situated perspective "emphasizes the socially and culturally situated nature of mathematical activity" (Cobb, 1994, p. 13). It sees learning mathematics as "a process of enculturation into a community of practice" (p. 13). This claim does not mean the individual student is not important, but to facilitate individual learning requires helping students move from the edges of participation in the classroom mathematics community toward the core and helping them "from assisted to unassisted performance" (Stein & Brown, 1997, p. 172), thus changing their mathematical identities as they change their mathematical practice. Like the emergent perspective, situated learning does not explicitly focus on holistic macrobehavior through component interaction. It is another friend of complexity theory, but again not a twin. Situativity may help researchers understand processes occurring in complex systems, but it is not a theory designed to investigate or explain the spontaneous cohesion of agents into larger, more organized collectives.

Shifting Arrangements

Complex systems are often nested within larger complex systems, and boundaries are often vague (B. Davis & Simmt, 2003). This nesting and vagueness occur not because the framework for complex systems is inadequate to describe the nested phenomena, but rather because the transient, dynamic, and stacked nature of complex systems makes exact delineation difficult. For example, when does one ecosystem (a complex system) end and another begin? Also, different types of ecosystems (e.g., arboreal, river, cave, and leaf-litter ecosystems) can exist inside a larger ecosystem (tropical rain forest); and they can even overlap one another, and transform into another system. One can think of a class in such a way: a constantly shifting conglomeration of complex systems, some mathematical, some not. In the classes I observed, both the teacher and the students recognized different entities in the classroom (e.g., those detailed in Figure 7) and moved fluidly back and forth between these different systems during classroom action (as manifested by their dialogue in the examples below).

I make a distinction between complex systems and groups of individuals in classrooms. A complex *system* is composed of interacting individuals that manifest behavior that supersedes any individual. A *group* is a collection of individuals that does not produce higher-order behavior. A group is only the summation of individuals and therefore not a complex system. A complex system produces a greater whole that supersedes the sum of its components. Exactly when a group morphs into a system is not clear but is related to the emergent moment. One moment it is not a system: the next it is. This mysterious transformation has been described by researchers as "magic" (Corning, 2002). For example, Mr. Murano's square activity in which he asked his students to work in groups of four the first day of class is a good example of the difference between group and system. Imagine a group of students starting that activity. It begins with these students grouped together by four's, and making four squares from four bags of pieces. The students will begin to work independently. So although they are grouped together, there is no holistic behavior superseding the individuals. They are a *group* of students—the four minds are working independently of each other on four separate problems. Some individuals will eventually realize that completing the squares can not be done if an individual works in isolation from his or her other group members. Their actions can indicate to the other group members that they need to work together, sharing pieces to complete the four squares. As the students begin grabbing pieces from other group member's partially constructed squares or from other

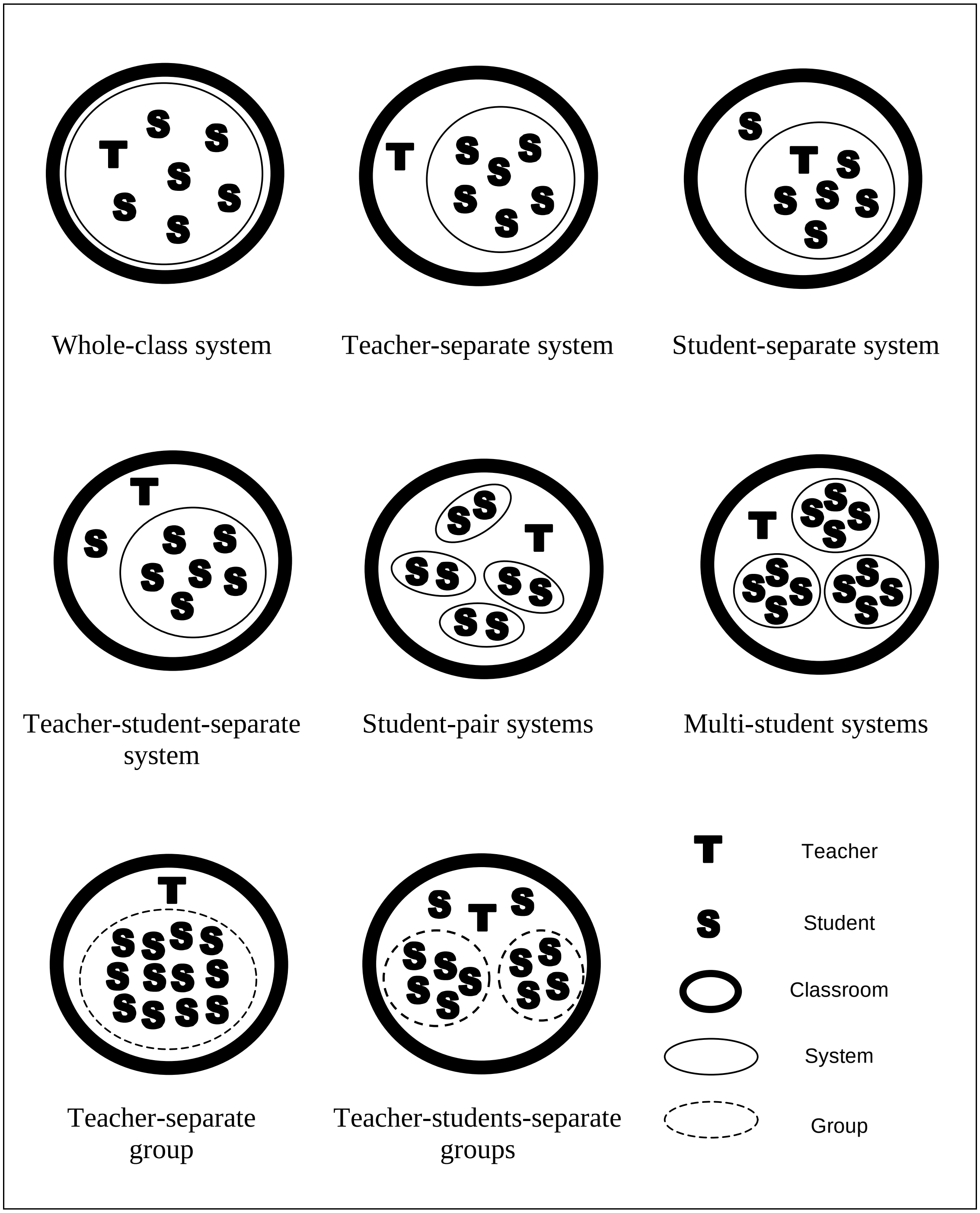


Figure 7. Arrangements of systems and groups.

members' piles and begin to work together to form one square and then another, the group is transforming into a system. They are now operating jointly, with behavior that supersedes any individual. The four problems become the property of the system. No single person is responsible for solving the four problems—they have become a system effort. Mental action of agents becomes combined.

I identified shifts from one class arrangement to another based on changes in dialogue. I discuss eight of those arrangements or shifts between arrangements, some illuminated with excerpts from transcripts. They are obviously just a sample of possible arrangements to illustrate the main idea of shifting class arrangements of systems and groups.

Whole-class system. The first arrangement is the whole-class system. This arrangement is identifiable by the plural language used by system participants: *we*, *us*, and *our*. An example is Ms. Sandy's Perimeter Problem. She had called on a quiet student (Mckenna) to explain how she had suggested the correct base that the class had been working on for some time:

Ms. Sandy: Now how did you find it so quick?

Mckenna: Um,… I just did it by "guessing and checking." I just put in a lot of numbers and just sort of….

Ms. Sandy: And the random number worked?

Mckenna: Yeah.

Ms. Sandy: That's cool! That's what "guessing and checking" is, though. We are just guessing a number, going through the process of the problem, to see if it works. But what if there was a simpler way to do this?

Student: I wish there was.

Kody: There probably is.

Ms. Sandy: There is!

Morgan: Really, I want to know.

Ms. Sandy: And that is what we are going to learn how to do.

Students: Yeah! Oh, boy. Yes!

Ms. Sandy's language suggests that she considered herself at this time to be an active participant in the whole-class system. The transcript continues, and not only does Ms. Sandy use plurals to demonstrate she is part of the system, but so does Braxton:

Ms. Sandy: So another color, like black or something. Don't erase anything…. How can we write—. How can we write a general rule, a general rule that works every time to find the perimeter of a rectangle?

Braxton: First we have to, um,… we have to divide … [mumbles].

Ms. Sandy: Multiply what?

Braxton: Like, um… [mumbles].

Ms. Sandy: You multiply *height* to get *bas*e?

Braxton: Yes.

Ms. Sandy: What if we know the *base* and *height*?

Braxton: Then I guess we….

An example of dialogue indicating shifting class arrangements arose in Ms. Auburn's episode right after the three disparate solutions were shown by three students. Ms. Auburn's dialogue indicated that she was part of the whole-class system: "We have three totally different answers on the board. So how are we going to figure out what is correct?" Julio then took a typical student action: He tried to separate her from the rest of the system and put her back into the role of teacher-as-mathematical-authority. Because it was the fourth day of school, and Julio had had little experience with Ms. Auburn's ways of operating, his remark was understandable. He might have been thinking, "Our teacher will now show us which of these three different students solutions is correct!” But Ms. Auburn responded that she was not the authority; the class was:

Ms. Auburn:	Okay, so now we have three totally different answers on the board. So how are we going to figure out what is correct or what is not correct?
Julio:	You are going to tell us the answer.
Ms. Auburn:	I don’t have the answer. We are going to see and try to figure out a way to tell what is correct. Because if you are really a mathematician, do you have someone to go to for an answer book?
Students:	No!
Ms. Auburn:	So we have to figure out a way to know which answer is correct, right?

This example shows a whole-class system that temporarily halted as Julio tried to force the teacher out of the system (to form a teacher-separate system or teacher-separate group with the teacher acting as mathematical authority). But Ms. Auburn refused, and the class continued functioning as a whole-class system.

Teacher-separate system. In a teacher-separate system, the teacher is not a functioning part of the class system; the system consists of the students only. The students are the ones who are cognitively joined, but the teacher is not. One of the supplementary episodes I analyzed was a 5-minute segment in which Ms. Sandy introduced solving one-variable equations with the variable on both sides of the equation, such as $2x + 1 = x + 6$. Although short, this episode was a remarkable example of a teacher-separate system because Ms. Sandy initiated the activity in complete silence and spoke hardly a word throughout it, with only an occasional question or comment to the struggling students here or there. Her mind was not joined with the students. Despite her silence, however, the students began vocalizing their thoughts as they jointly tried to make sense of what was going on. At first, they did not know whether it was even a mathematics problem, but by listening to other students' dialogue, the class began to form a "collective brain" as they focused on solving the problem. They jointly wrestled with Ms. Sandy's actions, and began—as a student system, without her help—to make sense of the equations she was modeling. Students who appear confused in the beginning are heard to adopt other's sense-making, and the collective discussion demonstrated a universal movement toward consensus about what Ms. Sandy was doing with the manipulatives.

Student-separate system. At times, the students' dialogue showed that they recognized they were part of separate classroom groups. For example, in Ms. Auburn's episode when Trevor was at the board speaking to the rest of the class, he referred to himself as *I*, not *we*. This usage indicates that Trevor saw himself as an entity separate from the class, so it was a student-separate system. Later, while the class was preparing to do a simulation and considering strategies to keep track of the high-fives the simulators gave, Trevor mentioned that maybe the class as a whole, of which he was now a part, should come up with a recognizable definition of *high-five*: "Maybe we need to define what we mean by—. Maybe we need to define what we mean *high-five* by?" The class was now operating as a whole-class system, and Trevor's language matched this classroom system arrangement. Such dialogue indicates that the students were becoming aware of where they were situated in the different groups that made up the class.

Teacher-student separate system. In this class arrangement, the teacher and a student are both removed from the class system's cognitive effort. An example would be when a student is presenting a solution to the class to consider, and the teacher asks the student to explain a little more to the class about what he or she is doing.

Student-pair systems. In this arrangement the larger class system is fragmented into many pairs, each pair potentially forming a small system. An example from the main episodes was when Mr. Murano's students stopped the whole-class discussion and began working in pairs to solve worksheet problems in preparation for later whole-class discussion. Two levels of complex systems existed in the classroom: the whole-class system (which became temporarily inactive) and the two dozen or so two-person systems that were working independently. The pairs formed complex minisystems because the students worked jointly to solve a novel problem. The action of each two-person system was a unified effort. As Mr. Murano walked around and observed what the students were doing (and gave suggestions to pairs that were stuck), he was part of the whole-class system but outside all of the two-person systems. I would not consider an outsider observing or asking questions about two students' joint work to be a formation of a three-person system, because no holistic behavior arises from such an encounter. As B. Davis and Simmt (2003) admit, however, the boundaries between systems are foggy. If the students began an intellectual dialogue with the teacher, the teacher might temporarily join a three-person minisystem.

When pairs of students work on meaningful mathematics, many of them will work together jointly, producing behavior that evidences complexity. Should a pair of students work side-by-side yet remain intellectually autonomous, they would function only as a student-pair group or collection. There would be no interaction of ideas, and so no mixture of ideas. Conceivably, a class arrangement of student-pair systems may also have some student-pair groups if some pairs are not operating cognitively together.

When Mr. Murano brought the class back to a whole-class discussion, most of the two-person minisystems dissolved (or became inactive), melting back to form the larger class system of which Mr. Murano was also a part. A few groups continued to work, whispering back and forth during the whole-class discussion, but those continuing minisystems were still part of the larger classroom system. I would classify the shifted systems that occurred above as an example of a *whole-class system* transforming into *student-pair systems*, and then back again to a *whole-class system*.

Multi-student systems. This class arrangement was just like student-pair systems, except that more students, usually four, were participating in each minisystem. I also observed differentiation between participants, both vocally and intellectually, and I believe the students sensed that as well.

Teacher-separate group. I noticed that the three teachers would often move out of the whole-class system when dealing with management issues or instructing the students to do certain assignments. Their language shifted from *us* and *we* to *I* or *me*. The shift was understandable because the teacher, although giving authority to students for operating mathematically, was the adult and ultimately responsible for management. Although these teachers were not the mathematical authority in the classroom, they maintained other teacher roles. Although the curriculum was jointly developed by the students and the teacher, the teacher decided what the class would do next, even if that decision was influenced by student action. None of the teachers asked students questions like the following: “Should we have a homework assignment tonight? If so, what problems would you like to work on? Should we have a test at the end of this chapter?” When it came to issues of mathematical judgment, however, the teachers did delegate great responsibilities to students, as indicated by their dialogue (e.g., “What do we think about this?”).

Teacher-students separate groups. I observed this class arrangement only once. It occurred in Ms. Sandy’s class, when she wrote $(-3)^2 = 9$ on one side of the board and $^{-}3^{2} = 9$ on the other. She told students to physically move on the side of the room that they thought was the correct equation. Soon students had formed two separate groups, one supporting the first equation as correct, and the other the second. I say group because no mental action had combined in either group of students. She then asked each group to select a spokesperson who would argue their position. She called this “playing court.” She then had the selected two students come to the board and present their arguments or proofs for why they thought their equation was correct and the other was incorrect. The two groups of students were to listen to the arguments and decide which side they now supported. As the two spokespersons presented evidence for which equation they considered to be accurate, students were constantly moving back and forth between groups. Sometimes the first equation was supported by a clear majority, with the second equation rapidly losing student support. But the support seesawed back in favor of the second, then back to the first again. Erika, the second equation’s spokesperson, eventually presented a convincing argument that because the negative sign was written up high, it was connected to the number 3, so regardless of parentheses placement, negative 3 times negative 3 was positive 9. Students seesawed back to her side. Cassidy, the first equation’s spokesperson, even began to doubt her own arguments at this point. The class ended without a clear victor. I present this example as a unique arrangement in which the teacher sat isolated from the class monitoring the debate, while the students argued their position and two groups of students listened intently.

Complexity theory helps explain these different systems and groups. It predicts that when agents are allowed to interact in certain ways, a system can form. Complexity theory also illuminates the phenomena of nested levels of systems. If the interaction of the agents is restricted, complexity theory foretells why the collection will maintain a group status without ever bumping up a level to demonstrating higher-order—or systemic—behavior (Anderson, 1972).

In addition, both Ms. Auburn and Ms. Sandy agreed in their interviews that their classes had personalities, something B. Davis and Simmt (2003) had observed and which complexity theory explains (no interview data available for Mr. Murano). A class would form a perceptible personality when it became a system because the system demonstrated holistic behavior through the interacting components that could be perceived holistically. Here is a portion of a transcript from Ms. Auburn's interview:

Ricks: Do you consider that your classes have personalities?

Ms. Auburn: [Nodding]. Very, very, very much so. Every class every year has a personality. And it is really interesting. Like my seventh graders' algebra class has a very different personality than my seventh grade pre-algebra classes. I think they are, they are much more willing to challenge each other on mathematical ideas, and—but they are also very fun-loving. They like to have a good time, and to be excited, and to have fun with math. So just from day-to-day, my, like, my B-1 class might be—part of that might be "time of the day," as A-1 is more willing to get involved and more willing to talk to each other. I am sure it is part of the day, but they are more willing to sit back and just—. So it is more of a challenge to get them, they are not as interactive with each other. And so, yeah, classes from time to time have different personalities, and it is just the makeup of the kids, how they interact with each other. Are they willing to work with each other? To come to the board? …. So, yeah, every class, every year has a different personality. (Interview 1)

One limitation of current complexity theory is the lack of any concrete description of how nested systems may transit between levels or even morph into a different system. For example, in Mr. Murano's classroom, the participants were aware of which systemic entities they were a part, and their dialogue revealed the movement between these systems. For instance, when Mr. Murano was walking between two-person minisystems and offering them advice, he would sometimes stop and talk to the entire class. He would say such things as "I want you to remember to…." Later, in whole-class discussion, his dialogue would change, indicating that he recognized he was now part of a different class arrangement. He would say something like, "Now what do we think of that strategy?" The shift from *I* to *we* indicated that he considered himself in the same operational system as the students. In whole-group discussion, he would refer mainly to the class as a whole—*us*—and later in the same discussion refer back to when the students were working in pairs and mention himself instead of the class—*I*. The following example is taken from a whole-class discussion shortly after the students had worked quietly as partners to illustrate this shifting dialogue:

> Mr. Murano: Excellent. Did everybody hear that? The whole goal of this [making a good graphing window] is that we can better see what is happening. And so if we see those [*x*- and *y*-axes], it makes it easier to read; it makes it easier to see what's happening. Excellent. All right. Let us go on to Number 4. How much money will Susan have after seven weeks? I saw several different people trying it and doing it different ways, so I wanted, I wanted some of them to come up. Anabelle, will you come up and show us what how you got it, please?

A little later, after Anabelle had shown her method:

> Mr. Murano: Questions for her [Anabelle]? Are we okay? I saw several people do it that way. Excellent. Andy did it differently. Andy, would you go up and show us how you did it?

I observed agents fluidly move into and out of systems, as well as the systems fluidly moving within a nested system framework; I recommend complexity theory be augmented to more clearly describe these phenomena.

In addition to developing a robust theory of systemic transformations, I believe that complexity theory also needs to include the possibility that students, because of their intellectual prowess (as compared to ants or bacteria), often stratify within a developing system. They develop recognizable roles. When students interact, there are certain social behaviors that may cause one to dominate over the other, and vice versa. For example, a vocal student may appear dominant in partner work, and the quieter student may let his or her partner fulfill that role. But at the same time, the quieter student may be more mathematically competent and so may dominate the mathematical direction the partnership takes. This idea of differentiation among participation of agents in a complex system is especially apparent with a teacher. The teacher is not on equal grounds with the students most of the time, and the class members recognize this. Although complexity theory recognizes decentralization, or a movement to share power, as vital for complex formation, however, there does not yet exist a coherent theory of stratifying agents and their different roles. Emergent macrobehavior can still arise with interacting individuals even if there is not equality among them. In an ant colony there are workers and guards, caretakers and a queen. Complexity theory currently lacks a robust theoretical frame to describe how systems diversify while they form, and maintain or destroy such inequality while they operate.

For example, one pair of students in Ms. Auburn's class, Quinn and Danielle, were a prime example of diversified agents. The first week I observed this pair, Quinn was the dominant partner. I considered him at the time to be the leader of the pair. Not only was he more aggressive vocally, but Danielle had recently moved into the area and so was adjusting to the move. The first week she was rather disoriented because she had come from a different mathematics background than the rest of the students in Ms. Auburn's class, having not attended the same elementary school. Danielle was timid and shy. One day she burst into quiet tears over a small mathematical misunderstanding, which Ms. Auburn soothingly resolved in a way that did not bring attention to the situation. Danielle was struggling with her own perception of her mathematically abilities, especially when paired with Quinn, who appeared to be superior mathematically because of his frequent participation. And Quinn was indeed bright.

As the days passed, however, Quinn's mathematical aggressiveness, demonstrated by his rapid assumptions while solving problems (which often yielded errors), was tamed by Danielle's plodding attempt to understand. Quinn was often ahead in partner work, working jointly with Danielle only when she asked questions—in more a partner group than a partner system. But as time passed, Danielle would often catch Quinn's errors and ask him about them. Soon, I noticed that Quinn was asking questions of Danielle, and Danielle often showed mathematical dominance as they worked together to solve problems. She would often explain to Quinn a difficult part or point out errors on his paper. I believe that Quinn was recognizing that Danielle was a peer mathematically, and they began to function more smoothly together as a student-pair system. Somehow complexity theory needs expand as to accommodate such fluid roles to be a more helpful framework for mathematics education.

General Contributions

Mathematics is to a great degree an intellectual domain enacted by a Social (Ernest, 1990). By *Social*, I refer not to a "group-of-individuals" (Lave, 1996) but to the group-*as*-individual, the collective from the perspective of complexity theory. Whether at the classroom, department, national, or international level, mathematicians of all ages interact with one another. Complexity could be used to re-envision mathematics as an emergent phenomenon of the system's actions and products. This activity may result in collective-approved definitions, terms, strategies, methods, algorithms, and so forth. These are embodied because they arise through the action of the system—as mathematical actions and objects. They are subsequently used by the system for further action, which in turn may produce more sophisticated mathematical objects.

Although in the classes I studied, the students learned conventional terminology, symbols, and algorithms, such mathematical objects were just a small portion of the mathematics that emerged. Much more lay underneath. The submerged mathematics—both actions and objects—was not readily apparent unless one watched the co-construction of mathematical ideas in those classes. That *formative* mathematics was as much mathematics as was the finished, refined, abstracted—*formal*—mathematics encapsulated in theorems, equations, terminology, algorithms, symbols, and definitions.

The rudimentary symbolization, obscure representations, halting beginnings, side scratch work hastily erased, dead-end tries, and scribbled notes—all were mathematics. From kindergarten to high school, academic lunchrooms to conference halls, mathematicians of all ages scribble and attempt and backtrack and reattempt and refine and share. They share questions and solution attempts; they communally debate; they question, raise counterexamples, reason, argue, and collectively justify (Polya, 1945/2004). On top of, and growing out of, the social forming comes the collective formalization that many consider mathematics. But I consider forming just as much mathematics as the end result.

Although different classes may engage in related activity and develop similar if not identical mathematical ideas, the mathematical activity that led to those ideas is unique to that class and its actions. In other words, each complex class develops, independently of other systems, its own mathematical domain. The mathematics is emergent. The mathematics taken as a whole includes the actions and the mental products of those actions and is unique to the systems that engaged in those actions and developed those products. If one had not been privy to its formation, one would be unlikely to understand the entire mathematics that the system developed.

Examples of this uniqueness include student ideas that are referred to by name, such as Osvaldo’s Observation, Trevor’s Way, or Danielle’s Question. Specific methods and questions to a large extent shape the resulting complex mathematical formation, and observations and questions are context specific. The existence of classroom-based mathematical humor, which only that class would understand because of the mathematical activity the class had engaged in, is more evidence that the emergent mathematics is unique to the system and has developed a distinctive microculture. The total mathematics exists as an independent domain reserved for the participants in that classroom. Although general principles may be shared with others, there is no way to recreate the actions in which all participated, so the developed mathematics is owned by those participants only. As the Greek philosopher Heraclitus claimed, one can never step into the same river twice.

The present study has identified the existence of mathematizing complex systems, as well as delineating specific factors within B. Davis's criteria that seem to support and possibly promote complexity. In particular, the critical and central role the teacher plays in this process has been highlighted. This study was the first study to take the five criteria and explore them in multiple classrooms. This study was the first to search for evidence of complexity in middle school mathematics classrooms, environments known to be generally isolating (Lewis & Tsuchida, 1998; Shulman, 1987; Stepanek, 2000; Valli, 1997). One might say that I have taken a "collective conceptual orientation" (Bowers & Nickerson, 2001, quoted in B. Davis & Simmt, 2003, p. 143) to these criteria for complex formation. In particular, I investigated how the teachers' actions contributed to each criterion, as well as affecting others at the same time. All three teachers regularly employed methods that met the criteria, and I observed mathematizing complex systems in each of their classes.

Although perhaps unaware of the term *collective conceptual orientation*, each teacher nonetheless believed strongly in the importance of establishing and maintaining working mathematical communities in their classrooms. They were acting in ways commensurate with B. Davis and Simmt's (2003) recommendation that "a teacher's main attentions should perhaps be focused on the establishment of a classroom collective—that is, on insuring that conditions are met for the possibility of a mathematical community" (p. 164). Each teacher had his or her own style and utilized unique methods, of course, but their actions related to each of the criteria contributed synergistically to the formation of complex systems in each class. The lessons I observed were animated and intriguing—in two of the episodes, the students actually cheered after solving a problem. Genuine mathematical celebration in a middle school mathematics classroom? How unusual (NCTM, 2000).

Much of what I have found as important to create and sustain complex systems is not new for mathematics educators. Many of the ideas have been practiced since the 19^{th}-century (Parshall, 2003), and elaborated or added to by later research. For example, posing a challenging question and maintaining its cognitive difficulty is recognized as important for student learning (Stein et. al., 2000). Likewise for creating a respectful space for discussion, modeling transparent mathematical thinking, developing a "community of inquiry" (NCTM, 2000, p. 345), and many other principles. And the field already recognizes the importance of the teacher for teaching (Stigler & Hiebert, 1999). So a logical question is: How does this study of complexity contribute to mathematics education? If most of these principles are already recognized by the mathematics education community as important for substantial student learning, how does complexity theory provide for an increased understanding of mathematics teaching and learning? I see two principal ways, related, and both involving vision.

First, complexity theory helps us see the *why*. It broadens our understanding of the factors influencing mathematics learning. For example, internal diversity is recognized as important for learning mathematics. But why is student diversity beneficial? Is it only so students can consider others' sense making and learn mathematics? Rather than connecting the diversity to mathematical learning, I see complexity theory illuminating a different possibility: Diversity contributes to mathematics learning in a roundabout way by first enhancing the richness of mathematical actions in a classroom. To understand why diversity affects mathematics learning requires broadening the tunnel vision of reductionism, recognizing that diversity positively affects much more than learning, and recognizing that those other things also affect learning. There is an interconnected web whose global strength supports mathematics learning. Diversity contributes to strengthening the student interactions and interacting leads to possible redundancies in a class as students share and borrow common ideas.

Also with a complexivist understanding, student interaction helps develop common mathematical orientations. These orientations emerge through systemic behavior as the class develops a joint supermind to tackle the challenging mathematical problems it faces. Common mathematical orientations could not develop without this neighborhood interaction. Students would not be able to consider and critique each other's thinking and come to consensus as a class on common procedures, language, and representations.

Also, common mathematical orientations contribute to decentralizing classroom mathematical control. Students play active roles in developing mathematical procedures and terminology. Some of these are unfamiliar and even pleasantly surprising to teachers. Students are learning as they jointly develop a class orientation which constitutes genuine mathematical activity: the nitty-gritty of doing mathematics. Rather than the teacher acting as the mathematical authority and dictating definitions and procedures, the emergence of common mathematical orientations strengthens the sharing of mathematical truth.

And so these attributes contribute synergistically to mathematics learning. Students' mathematical diversity, when combined with appropriate redundant class understanding (taken-as-shared), under liberating constraints provided by a teacher who shares the burden of mathematical proof with dialogically functioning (Wertsch & Toma, 1995) students, develops a mathematical community. This unified approach parallels how “teachers need to … be able to represent mathematics as a coherent and connected enterprise” (NCTM, 2000, p. 17). If mathematics is together, then so could be its teaching and learning.

The present study contributes to this broadening vision by telling a story—a cohesive commentary—about how three teachers in their own ways met B. Davis’s criteria to provide apparently meaningful mathematical environments for student learning. And the presence of each of these criteria contributed to the emergence of something that was not there before—a mathematizing class entity. I have documented the presence of such a system in each teacher’s classroom. Why is that significant? Davis’s criteria have been described by mathematics education researchers, however, these have been described, for the most part, in isolation from each other. This is, comparably, rather like taking a reductionist approach attempting to understand the wholes in mathematics education.

This observation points to a simple yet tantalizing idea, the second way complexity affects our vision: It helps us see *beyond* (the parts). Much mathematics education research has focused on one topic or another. Many have investigated the diversity inherent in mathematics classrooms. Others have investigated common social mathematical norms and the importance of working together in communities. Power relationships and teacher authority have been studied, as well as enabling constraints (B. Davis & Simmt, 2003). Perhaps the field has been in a reductionist rut—each discovering intriguing tiles, but none the mosiac. In attempting to understand how students learn mathematics, and how mathematics might effectively be taught, perhaps we have focused as a field too finely on the contributing components in isolation, too intently on specific features and qualities, and have lost sight of the larger wholes that might be possible.

The history of mathematics is replete with examples of such dawning awareness of larger things. The ideals of shape and number, of geometry and analysis, long separated, were brought together by Descartes. Further work by Riemann would pave the way for Einstein's connection of mechanics with electromagnetism. Calculus is a mighty whole formed from the initial separate parts of differentiation and integration. Galois transformed our view of the structure of algebra by describing the larger system. Euler linked analysis and prime distribution. And on and on. These were moments of vision, of stepping back and seeing beyond the parts. Of forms nested in forms, of system linked to system, all seen together from a grander view. And thus by seeing, they saw more. As Johann von Goethe (1988) wrote: "Every new object, clearly seen, opens up a new organ of perception in us" (p. 39).

Perhaps we mathematics educators have been missing the forest for the separated trees, the whole for the curious parts, the thing for this or that, effective instruction for this teacher attribute or that class feature. Perhaps we have missed mathematics learning for mathematical reasoning or computational efficiency or student diversity. Perhaps we are slicing up some organized whole, some *something* that must be fully seen to be fully known. Layers of carbon, too closely seen, are mistaken for graphite, not diamond-sheen. In our zeal for knowledge, our commitment to improvement, our desire to serve, maybe we have neglected some larger entity vital to substantial mathematics learning.

I do not mean to disparage the previous work. It has been productive. It has led to increases in the nation's mathematical abilities. But maybe mathematics learning (and teaching) involves this *and* that, together. And maybe there is much more, which lies beyond the periphery, beyond the parts, if only we could see it. Interconnected, interrelated, adaptive, self-organizing features, that combined, make a larger something—invisible to even the best mathematics education researchers who lack a global perspective. Complexity helps us begin to have such an outlook.

Complexity theory illuminates the idea that each of the components for effective instruction is important, not in and of itself, but because of the effect it has on other factors in the complex process of learning. It is the togetherness that is important. Mathematical learning is a whole that emerges out of the interacting parts. "Whereas I was blind, now I see" (John 9:25). And thus we see by seeing more. At first they were just parts, now they are whole to me. The whole is more, if seen beyond the parts.

CHAPTER 7

SUMMARY, IMPLICATIONS, AND CONCLUSIONS

When the learner is ready, the teacher appears.
—Chinese proverb

Summary

Complexity theory provides a space in which to examine how classroom collectives create mathematics—mathematics not as the sum of individual contributions but as the embodiment of actions by the collective taken as a single acting organism. Such a perspective is recent in mathematics education. This study had three primary research questions. First, I wanted to find evidence for the existence of mathematizing complex systems in mathematics classes. Second, if such systems existed, I wanted to better understand the factors that contributed to the development of such systems. Finally, I investigated how complexity theory could aid mathematics education.

My study complimented previous mathematics education complexity studies by looking more closely at how such systems develop mathematics in a class environment. Leiken (2004) and B. Davis and Sumara (2001) investigated only teacher collectives. Thom (2004) studied actual students, but only one group of 3 fifth-graders isolated from their class. B. Davis and Simmt (2003) described an example of actual classroom action, which study would be most analogous to my study. Like them, I also investigated classroom action. But my study also investigated multiple-teacher whole-class environments—the place students normally learn mathematics. In addition, my study selected teachers proficient at creating complex class environments to obtain a sample of “rich” data in detail (Merriam, 1998). My study added to the previous complex studies in mathematics education research by including and integrating five dimensions of (1) mathematics students (2) in their mathematical classes (3) taught by their mathematics teacher (4) over extended periods of time (5) from specially selected classes.

I did extensive groundwork to find vivid examples of class complexity.
I began by talking to several university mathematics educators who recognized the importance of student-to-student discussions and who had extensive experience observing classes. They nominated teachers whose classes might exhibit healthy student dialogue and whose lessons might contain substantial student contributions. After observing 18 of those teachers in two states, I selected 3 middle school teachers—2 western and 1 southern—and their mathematics classes for my research. I filmed eight of their classes with multiple cameras for 3 to 6 weeks in the fall of 2006. In the words of Stigler and Hebert (1999), my investigation provides “a penetrating … look into classrooms” (p. 9) of a complex nature. Drawing from over 340 hours of videotaped observations in these classes, I selected three exemplary episodes, one from each teacher, to provide the core data for the study. As sources of supporting data, I used several additional video episodes, teacher interviews, fieldnotes taken in and after class, and memos.

I discovered substantial evidence of mathematizing complex systems in these three teachers’ classes. The three episodes exhibited joint lesson construction, where the lesson was a communal development that emerged through the class’s action. I documented the shifting arrangements in which the individuals in these classes participated, especially the teachers, who through their language indicated that they were functioning parts of the larger system of which the students were also an active part. The three classes also exhibited mathematical self-regulation as members of the class would spontaneously acknowledge or critique others' actions or ideas.

I observed several actions each teacher used to develop and sustain the complex systems in his or her classes. All three teachers recognized and capitalized on students' mathematical diversity. All three recognized the importance of class norms, with two teachers explicitly posting a list of student-produced norms in their classrooms. And all three posed challenging questions that allowed for whole-class discussion. During mathematical discussions, the teachers orchestrated the expansion or contraction of freedom to interact, which provided a space for student creativity without inducing system-halting chaotic behavior. In addition, the teachers distanced themselves from the mathematical-authority role, and the class as a whole developed the mathematics.

The teachers integrated individual, partner, group, and whole-class work, engendering an environment where individual student ideas could be respectfully debated and critically considered. Although the methods employed by the teachers varied, their teaching had the five criteria identified by Brent Davis and his colleagues (B. Davis & Simmt, 2003; B. Davis & Sumara, 2001) as needed for complex system formation: internal diversity, redundancy, decentralized control, neighborhood interactions, and organized chaos. I found evidence for the critical role the teacher plays in developing a mathematizing complex system, specifically in creating an environment conducive to the criteria listed above. By modeling appropriate behavior and removing themselves as the mathematical authority, these teachers empowered the students to begin regulating their own mathematical actions. The system of the class as a whole adapted to increasingly challenging problems, demonstrating that the system was an adaptable learning entity. Out of this joint classroom action arose a communally built lesson that embodied the social mathematizing—the social mathematizing with its products was the mathematics—a collectively sculpted version of ideas, language, representations, and strategies.

In addition, I conclude that Davis's criteria form a robust set needed to occasion complex system formation. I have detailed several common teacher actions that illustrate how the Davis criteria were enacted by the teachers, including but not limited to posing challenging problems, creating respectful spaces, relinquishing mathematical authority, varying the size of the neighborhood, equalizing participation, and mathematical management.

Implications

Implications for Practice

The holistic perspective of complexity theory holds implications for the teaching and learning of mathematics. Complexity theory allows teachers to consider instruction in a new light. Teachers who attempt to transmit to their students another Social's mathematics—embodied action from some other system—may reduce the dynamic social and intellectual domain of mathematics to a collection of external mathematical objects (definitions, terms, algorithms, etc.) needing to be remembered and memorized (R. B. Davis, 1994). Embodied action from an unknown system may become nothing more than an alien thing the students (many desperately) try to make sense of.

This observation suggests part of the teacher's role in classrooms as developing and sustaining a mathematizing class system (B. Davis & Simmt, 2003). Once created, that system then creates its own mathematics. Students in such a class could learn mathematics as they experience it firsthand by participating in a mathematizing complex system. They would experience the process by which mathematics is born, the action of the mathematizing complex system, and the collective embodiment of their collective action. Mathematics would not be an external something to be absorbed, a curiosity from the past that would help them learn about stranger things to come (in the next mathematics course, for example). Instead, it would be intrinsically connected to themselves because they produced it. Their experience would produce the action that would subsequently be embodied by the collective as a mental object. Similarly, only participants in that systemic creation could have a deep understanding of the creation. The one who knows creation best is the creator—it could be no other way.

Classrooms are good locations for learning mathematics because students are already brought together physically. Simply bringing students together, however, does not imply that they will be functioning together mentally. The power of classrooms for mathematics education, I believe, lies in the opportunity classes provide for bringing minds together through joint mathematizing—from a group of individuals to a system. Such mathematizing capitalizes on the intellectual and social possibilities already inherent in classrooms, possibilities that are necessary for the development of healthy mathematical microworlds within which students can operate to occasion substantial mathematics learning (B. Davis & Simmt, 2003).

The Social was the focus in the classes I studied. The Social is the whole class (or subsets of the whole class) taken from a complex perspective. The Social is an entity that regulates, emerges, and adapts to its environment. The Social does mathematics, creates mathematics, and is shaped by mathematics. And the Social is greatly affected—yes, even initially formed—through teacher action.

This study demonstrates that the teacher plays a critical role in determining the environment of the classroom. The teacher creates the space and the circumstances. The three teachers I studied activated the social potential of their classes in the direction of mathematical activity. They treated mathematics as a social endeavor rather than an external object to be learned. They created mathematical communities in their classrooms, with students becoming neophyte mathematicians. How to achieve such an environment is critical for augmenting student learning, for “by attending to [such complex] matters, a teacher can greatly increase the likelihood of complex transcended possibilities within the classroom” (B. Davis & Simmt, 2003, p. 145). Teachers and teacher educators can learn to appreciate the complex perspective for “when discrete agencies interact, in synergism, the total effect is greater than the sum of individual effects” (Sztajn, 2001, p. 3). And such an effect holds promise for individuals.

Similarly, the practice of creating a complex system could be scaled up to the professional development level. The isolation of American teachers is well-documented (Hart, Schultz, Najee-ullah, & Nash, 1992; Stepanek, 2000). Such isolation, often reinforced by entrenched school cultures and norms, is damaging to the profession of teaching (Ball, Lubienski, & Mewborn, 2001) and prevents teachers from developing reflective practices (Schön, 1983). Stigler and Hiebert (1999) have commented that:

> For whatever reason, teaching in the United States is considered a private, not a public, activity. The consequences of this isolation are severe…. U.S. teachers work alone, for the most part, and when they retire, all that they have learned is lost to the profession. Each new generation of teachers must start from scratch, finding its own way. (pp. 123, 136)

Recent research demonstrates the importance of breaking the isolation barrier and beginning to work together to improve one's teaching practice (Grossman et al. 2000). In addition, implementing rigorous reform proposals will prove to be impossible if teachers remain secluded in their work. Complexity theory suggests why teachers would benefit by working jointly, just as students benefit by working jointly in mathematics classrooms.

Implications for Research

The study was designed to investigate the existence and maintenance of mathematizing complex systems in public-school classrooms. My assumption was correct that a search for classes with healthy student dialogue in which student ideas formed substantial parts of the lesson would identify teachers who operated in ways conducive to forming a complex system. With all three teachers, I observed the formation of complex systems. I do not believe the composition of the class had much to do with the presence of complex systems. I observed students in an upper-middle-class affluent suburban middle school, a lower-middle-class suburban middle school, and small city middle school. Complex systems manifested themselves in each of these environments. However, investigating mathematics teaching and learning further from a complex perspective seems important, especially in other contexts and at different grade levels (as I only investigated middle school classes).

I believe researchers can flesh out complexity theory more substantially in educational settings: students with developing cognitive capacities, especially in the presence of a teacher, may show complex behavior far different than an ant or bacterial colony. In particular, mathematics education could benefit by a more detailed framework of complexity theory. Complexity needs a more defined theory about how intellectual agents that recognize a hierarchy of power—whether intellectual, political, economic, or social—organize themselves and regulate their behavior. Various leaders emerge in intellectual groups. Dewey (1933/1960) noted that phenomenon when he wrote, “In reality the teacher is the intellectual leader of a social group … not in virtue of official position, but because of wider and deeper knowledge and matured experience” (p. 273).

The teacher is the one who creates and maintains the environment where students can dialogically function and where the students' ideas are validated as substantial parts of the lesson. To investigate further the teacher's role in a mathematizing community, especially as a social leader with different powers than those of the other community members, is an intriguing research possibility. I hypothesize that a teacher using specific complex strategies might take any group of students and create the necessary conditions to occasion a complex system. This hypothesis would be an intriguing possibility for future research.

The study has several limitations with respect to understanding mathematizing complex systems, each of which opens up more research options. First, the study was not designed to investigate student learning in such systems. I have anecdotal evidence of student learning from records of classroom dialogue; videotapes of individual, partner, and group work; and artifacts. But I did not attempt to measure students' mathematical understanding. It is conceivable, though unlikely, that some students were not learning mathematics with meaning. There may have been students, particularly the quieter ones, who were not learning as much as their more active peers. I studied the whole system, so I cannot make claims about individuals. Future research could investigate individual learning during collective action. I anticipate examining student artifacts more closely in later studies to describe the learning of those students for which I have some data. I suspect that a learner operating actively in such a system would achieve better mathematical understanding than one trying to learn mathematics in a group of individuals not functioning complexly. This suspicion remains to be examined and possibly validated.

A second limitation to my study was that I studied teachers who were competent in creating mathematizing complex systems. I did not investigate teachers attempting for the first time to create a complex system in their classrooms. Although I believe teachers can learn to create such complex systems, the study did not address that issue, which would be another area for fruitful research. For example, researchers might wish to help preservice teachers emphasize the five criteria in their student-teaching experience as a way to dampen negative aspects of the “apprenticeship of observation” (Lortie, 1975), and then observe the outcomes. Additionally, the report could be shared with inservice teachers in professional development activities.

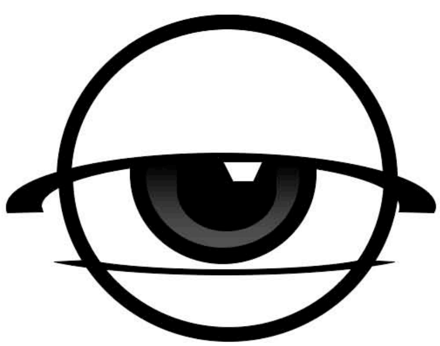
A final limitation to my study and another area of additional research could be investigating mathematics lessons where the teacher is learning mathematics along with the students, where the class as a whole takes a step into the dark into uncharted, unknown mathematical terrains. The teacher would have no previous experience and so becomes a fellow explorer instead of guide with the students.

Conclusions

The many perspectives on mathematics education provide intriguing windows on the mathematics teaching and learning process. Each contributes an alternative viewpoint that enriches understanding and enlightens action. Different perspectives often share commonalities, and at the same time, each endows new insight through its distinctive lens. Complexity theory envisions mathematics classes as possible collectives—mathematical microcommunities—which jointly construct mathematics. This perspective considers a class as a single, self-regulating entity that learns. This viewpoint predicts how substantial—individual—student learning can occur through active participation in the larger mathematizing classroom organism. Additionally, complexity theory can illuminate why certain phenomena in mathematics classrooms occur, as well as how best to deal with, and sometimes control, these phenomena for positive student gain. Finally, complexity theory explains how vital attributes of teacher action and class environments can interact synergistically to create a learning environment with potential far beyond environments with certain positive attributes in isolation.

My study observed mathematical complex systems in three teachers' classes. I also detailed the manifestations of teacher actions, paralleling Davis's criteria for complex system formation, that seemed to help develop the complex systems in these classes. And my work with complexity theory has opened up new perspectives on the social side of mathematics learning in dynamic classrooms. Complexity theory holds great promise to illuminate new avenues of understanding in mathematics education.

Maybe students' mathematical diversity, when allowed to interact in fluid, nonrepressed ways with other students' ideas around intriguing topics produces holistic *emergent* dimensions of the teaching and learning process that have yet to be fathomed by the mathematics education community. We may not even have the language to describe these constructs. Perhaps the B. Davis's criteria only indicate a wider domain of expertise needed for enhanced mathematical behavior—a this *and* that *and* those—all together. Emerson gave us great advice when he penned, "Never did any science originate but by a poetic perception" (Emerson, 1903–1904, p. 365). Complexity theory helps us step back and by seeing the whole together, perceive.



* "Now a Whole."—Claims the Eye. (Referring to the grook on page xii of the Preface)

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APPENDIX A

Classes Observed for Site Selection

Month	Teacher	Grade	Teacher-student discussion	Student-student discussion	Student ideas forming substantial parts of the lesson	Use?
			South			
January	Arnold Mendez	9	Much	None	Little	No
February	Rosa Martinez	1	Much	Some	Little	No
	Lennard Stewart	6 (Class 1)	Much	Some	Some	Maybe
		6 (Class 2)	Some	Some	Some	Maybe
	Deborah Sanchez	K	Much	None	Little	No
	Braxton Mendelli	K	Much	None	Little	No
	Richard Trolley	4	Much	None	Some	Maybe
	Jennifer Sandy	7 & 8 (Class 1)	Much	Much	Much	Use
		7 & 8 (Class 2)	Much	Much	Much	Use
			West			
March	Timothy Vereen	9, 10, & 11	Much	None	Little	No
	Sheila Auburn	7 (Class 1)	Much	Much	Much	Use
		7 (Class 1)	Much	Much	Much	Use
	Dr. Ron Tormic	Graduate University	Much	Much	Much	Maybe
	Alex Oppenheim	10	Much	Much	Little	No
	Jimmi Bremen	7	Some	Some	Little	No
	Theodore (Ted) Murano	7 & 8 (Class 1)	None (because of group work for next day’s lesson but good activity)	Much	None (because of group work)	Maybe

Month	Teacher	Grade	Teacher-student discussion	Student-student discussion	Student ideas forming substantial parts of the lesson	Use?
March	Theodore (Ted) Murano	7 & 8 (Class 2)	None (because of group work for next day's lesson but good activity)	Much	None (because of group work)	Maybe
		7 & 8 (Class 1)	Much	Much	Much	Use
		7 & 8 (Class 2)	Much	Much	Much	Use
	Misty Dickens	9	Much	Some	Some	Maybe
	Aubrie Rosen	7	Some	Some	Little	No
				South		
May	Tasha Fergusson	6	Much	Some	Some	No
	Daniel Chaucer	7	Some	Some	Little	No
	Will Wright	7 & 8	Some	Some	Little	No
	Jennifer Sandy	7 & 8 (Class 1)	Much	Much	Much	Use

APPENDIX B

Susan's Problem Worksheet

Birthday Gift Worksheet 3.1

Name____Period________

Ace____

Susan's grandmother gave her $25 for her birthday. Instead of spending the money, she decided to start a savings program by depositing the $25 in the bank. Each week, Susan plans to save an additional $2.50.

1. Make a table of values for the situation.

7

Time (weeks)	Process	Amount Saved

2. Write a function rule for the amount of money Susan will have after *t* weeks.

3. Find a viewing window for the problem situation.

Sketch your graph: Note your window: Justify your window choices:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

Use your graph and table to find the following:

4. How much money will Susan have after 7 weeks? Write this equation. Show how you found your solution.

5. Susan wants to buy a school ring. When will she have enough money to buy the $139.99 ring? Write this equation. Show how you found your solution.

[name of the school district] Algebra

APPENDIX C

Manuel’s ProblemWorksheet

Spending Money Worksheet 3.2

Name____Period__________

Ace____

Manuel worked all summer and saved $1090. He plans to spend $30 a week.

1. Make a table of values for the situation.

Time (weeks)	Process	Amount of Money

2. Write a function rule for the amount of money Manuel will have after *t* weeks.

3. Find a viewing window for the problem situation.

Sketch your graph: Note your window: Justify your window choices:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

Use your graph and table to find the following:

4. How much money will Manuel have after 11 weeks? Write this equation. Show how you found your solution.

5. When will Manuel be out of money? Write this equation. Show how you found your solution.

[name of the school district] Algebra

Spending Money Worksheet 3.2

Name____Period__________

Ace____

7. How will the line change if Manuel had initially earned $1300? Graph the line. What changed? What did not change?

8. How will the line change if Manuel spent $200 on school clothes and started the year with only $890? Graph the line. What changed? What did not change?

9. How will the line change if Manuel starts with the $1090, but decides he will only spend $25 a week? Graph the line. What changed? What did not change?

APPENDIX D

Disclosure Document for 2006-2007

[Mr. Murano]

WELCOME! New this year the Math Department is excited to unveil a webpage that is full of online help and resources for the Big Ideas associated with Pre Algebra. Watch for Algebra to be coming in future months.

[address of the webpage listed here]

PARTICIPATION:

In-Class

Success in class is directly affected by student participation. **Understanding mathematics requires discussion, reasoning, and presentation of concepts by students.** It is important that students be in class. **Students are responsible for learning and turning in material whether or not in class.**

Notebooks/Notes

Notebooks are graded for completeness, organization, and neatness. Students should keep all class work and assignments in their notebook.

Practices

Work is assigned **daily** to give students a chance to practice and learn. Absent work is due according to the school make-up policy. Late work will be accepted with approval of the teacher and **only** during the current unit. **Once a unit is finished, assignments for that unit will NOT be accepted for credit. NO extra credit**

ASSESSMENT:

Mastery is expected on all assessment items. Assessments are directly correlated to the state core and department benchmarks. We believe students must know the core concepts in order to be successful in future years. If an assessment is not mastered the opportunity to reach mastery will be given. Students that do not have mastery on all assessments will have an "I" for their grade, meaning that mastery is "IN PROGRESS".

CITIZENSHIP:

Citizenship grades will be affected by participation, attitude, and willingness to cooperate. Exceptional behavior and effort will be rewarded with outstanding citizenship.

GENERAL:

Calculators will be used! Students will benefit by having graphing technology available when at home. We recommend at least a TI-83+ graphing calculator. Be sure to scratch your name on the calculator if it is to be brought to school.

CLASSROOM BEHAVIOR:

The **[name of the school district]** and school rules, as listed in the Student Handbook, will be followed in the classroom. I expect students to discipline themselves in order to allow everyone a chance to succeed. There will be consequences for students that are unable to follow classroom rules and procedures. **Participation in class is essential for students to learn for understanding.**

GRADING:

Feel free to check your students' grade as often as you like on the Internet. Power School can be linked to through the school webpage:

[address of the webpage listed here]

Grading Scale

93-100	A	80-82	B-	
90-92	A-	77-79	C+	
87-89	B+	70-76	C	
83-86	B	Below 70	I	"In Progress"

Students with "I" grades must make up any necessary assessments by the end of the current term to change the "I" to an "A,B or C". If they allow the "I" to go past the end of the current term, the "I" can only be raised to a "C" if they complete all necessary assessments by the end of the following term. **If they do not choose to make up the "I" it will change to an "F".** Fourth term assessments must be made up by May 25th.
Cheating on a quiz/test/homework or talking during a test/quiz is not tolerated. Cheating will result in a score of zero and will be given on the quiz/test/homework and a "U" for citizenship.

PLEASE RETURN TO [Mr. Murano]

Class Period ________________

Student Name ____

Fathers Name (First and Last) ______________________________________

Father's Home Phone __________________
Work Phone_____________________

E-mail Address ____

Mother's Name (First and Last) ______________________________________

Mother's Home Phone __________________
Work Phone _____________________

E-mail Address ____

Parent and Student: Please sign below. **Your signature indicates that you have reviewed and understand the expectations of** [Mr. Murano's] classroom.

Student Signature ____

Parent Signature ____

APPENDIX E

[Ms. Auburn's] Algebra Disclosure

PHILOSOPHY: Algebra attempts to answer three essential questions: "What is a linear relationship and how do we represent that using tables, graphs, and equations?"; "What kind of relationships can we use to make predictions in real world situations?"; and "How do we know if expressions are equivalent?". These questions lead to the study of linear relationships (including slope, y-intercept, and x-intercepts), the comparison of liner relationships to exponential and quadratic relationships, and solving equations/inequalities and systems of equations/inequalities.

BOOK: *Connected Mathematics Project* and *Interactive Mathematics Project*

REQUIRED MATERIALS: 3-ring binder, notebook paper, graph paper, 5 tabbed dividers, and a pencil/pen. It is suggested that students have a calculator that they can use at home.

HOMEWORK: Assignments will take from fifteen to thirty minutes outside of class to complete. Each student will be given three "late homework" passes each term. These passes will allow them to turn in three assignments late for full credit. All other late work will receive 50% credit.

QUIZZES: Quizzes are given periodically. Quizzes are usually returned within two to three days, but no later than one week. Quizzes can be retaken, but they must be retaken within a week of when they are returned to the student.

EXAMS: An exam will be given at the end of each unit. Exams are usually returned within two to three days, but no later than one week. Students will be allowed to retake exams, but they must be retaken within a week of when they are returned to the student.

PROJECTS: At the end of some units the students will complete a project that will represent or extend the mathematical ideas of the unit. The purpose of the project is to solidify and evaluate the students understanding of mathematical skill and concept.

PARTICIPATION: Students will receive 10 or 15 participation points each week if they participate 2 or 3 times a week in class. Participation means sharing ideas with the class, and reading out loud, asking appropriate questions.

NOTEBOOKS: Students will keep a notebook of their in-class work, assignments, quizzes and tests, openers, and vocabulary words. These notebooks will be grade approximately every 3 weeks.

Grading: Grades will be determined by the following percents: Assessments (Quizzes, Exams, and Projects) 50%, Homework 20%, and Participation (In class participation, Notebooks, and openers) 30%

100 - 94	A	86 - 84	B	76 - 74	C	66 - 64	D
93 - 90	A -	83 - 80	B -	73 - 70	C -	63 - 60	D -
89 - 87	B +	79 - 77	C +	69 - 67	D +	59 - 0	F

PROGRESS REPORTS: Current grades are available at http:// 205.118.4.5 and are posted by student number in the class room.

ABSENCES AND TARDIES: Being in class and participating in the class discussions is a significant part of the students learning and understanding. Because of this we encourage students to be in class and on time. The term "on time" will be defined as being in your seat and ready to start class when the tardy bell rings. If a student is absent, it is his/her responsibility to find out what was missed and make up the work. They will have the same amount of days to make up an assignment as they are absent.

For Student:
I have signed below to show that I have read **[Ms. Auburn's]** Disclosure document and am aware that I will be held responsible for all things contained in it.

____
Student Signature

____
Print students name

Period_______________

For Parent/Guardian:
I have signed below to show that I have read **[Ms. Auburn's]** Disclosure document and am aware that I will be held responsible for all things contained in it.

____
Parent/Guardian Signature

If you would like to be part of our class email group for which I will send out up coming deadlines or announcements, please write your email below:

____