PROFESSIONAL NOTICING: HOW DO TEACHERS MAKE SENSE OF STUDENTS' MATHEMATICAL THINKING?

by

GINGER RHODES

(Under the Direction of Denise S. Mewborn)

ABSTRACT

The purpose of this study was to understand how teachers make sense of their students' mathematical thinking. Specifically, learning trajectories and professional noticing were used as a way to examine how teachers understand and use students' mathematical thinking in their teaching practices. A qualitative research methodology was employed to address three research questions that focused on teachers' "informal" learning trajectories, what teachers notice about students' mathematical thinking during classroom interactions, and the ways that teachers respond to that thinking during classroom interactions. Two high school geometry teachers were observed and interviewed during one semester. In addition, the teachers attended biweekly working-group meetings to discuss students' mathematical thinking. I created two learning trajectories for the teachers' lessons to represent their thoughts about their students' mathematical thinking before and after lessons: the projected learning trajectory (PLT) and enacted learning trajectory (ELT). The two learning trajectories were compared to identify instances of teacher noticing. The PLT and ELT were similar in some instances but not others. The teachers described and interpreted what they noticed in terms of their uncertainties and surprises about students' mathematical thinking. Furthermore, the teachers described and interpreted what they noticed about students' mathematical thinking in terms of the mathematics tasks, their own mathematical knowledge, and their actions with students in the classroom. The teachers typically responded to students' mathematical thinking in five ways: posed a question, asked students to share, told the students something about the mathematics, posed another task, and used a pedagogical content tool. What the teachers noticed in the classroom interactions and the ways that they responded to students affected their development of PLTs and ELTs, and vice versa. When the students led the ELT, the teachers gave detailed descriptions for the learning trajectories and gave more details for what they noticed during classroom interactions. In contrast, when the teachers led the ELT the teachers struggled to describe learning trajectories and what they noticed during classroom interactions. The reporting and analysis of the data reveal implications for both research and teacher education.

INDEX WORDS: Teacher development, Learning trajectories, Professional noticing, Teacher knowledge, Teacher listening, Instructional practices

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GINGER ALAYNE RHODES

B.S., North Carolina State University, 1998

MAEd., East Carolina University, 2000

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GINGER ALAYNE RHODES

Major Professor: Denise S. Mewborn

Committee: Jeremy Kilpatrick
George Stanic

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
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TABLE OF CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

> As she [the teacher] continued to listen to their explanations and request their ideas, she continued to be surprised at second-graders' understanding of mathematics. She discovered that their thinking was far more sophisticated than she had previously assumed. (Wood, Cobb, & Yackel, 1991, p. 601)

Students come to school with sophisticated and remarkable mathematical capabilities. In Wood, et al.'s (1991) study, the teacher made significant changes in her instructional practices when she began listening to her students. She noticed the mathematical power that students possess. She noted, "I have been teaching all this time [15 years], and I never knew second-graders knew so much about math" (p. 601). I began by sharing this teacher's experience because it highlights the value of thinking about the ways that teachers make sense of students' mathematical thinking. Through this teacher's interactions with students, she began to interpret their mathematical thinking and, consequently, made instructional decisions based on that thinking. For this study, I was interested in the ways that secondary school teachers make sense of students' mathematical thinking.

It is the responsibility of the teacher to create meaningful learning experiences for all students. For teachers to create such experiences, they must have not only a deep understanding of the content they teach but also knowledge of their students (National Council Teachers of Mathematics [NCTM], 1989, 1991, 2000). That knowledge includes knowing students as learners of mathematics, and "teachers need opportunities to examine children's thinking about mathematics so that they can select or create tasks that can help children build more valid conceptions of mathematics" (NCTM, 1991, p. 144). Similarly, Steffe and Wiegel (1992) commented that the responsibility of the teacher is "to learn the mathematical knowledge of their students and how to harmonize their teaching methods with the nature of that mathematical knowledge" (p. 17).

When teachers listen to students, they have to negotiate between what they are expected to teach and their desire to attend to students' mathematical thinking. Mathematics educators often quote Ball (1993):

> But I wonder: *How* do I create experiences for my students that connect with what they now know and care about but that also transcend their present? *How* do I value their interest and also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention? (p. 375)

Many teachers have faced this dilemma in their efforts to base their instruction on students' current ways of thinking as well as cover the state-mandated curriculum. Dewey (1902/1976) addressed this dilemma of pitting what is within the child against the knowledge that society developed over time. He thought this view was problematic because the material then becomes formal and symbolic, creates no motivation for learning, and loses value. Dewey's (1902/1976) recommendation was to take a fresh look at these two ideas and reconstruct their meanings. The child presents his or her current understanding and attitudes, whereas the curriculum presents potential understanding and attitudes. The teacher's role is to interpret the child's current and growing understanding to "determine the medium in which the child should be placed in order that his or her growth may be properly directed" (Dewey, 1933/1989, p. 286).

My Personal and Professional Journey

Two specific experiences during my high school teaching led to my interest in understanding the ways that teachers make sense of students' mathematical reasoning. First, during my master's degree program, I took a course designed to investigate secondary mathematics education issues. One day in class the instructor gave all of the students a mathematics task to solve. I remember that I solved the task by creating an algebraic equation, which took me about 5 minutes. The task seemed interesting, and I thought I might give it to the high school students that I was teaching at the time. Next, the professor showed video clips of different middle school students correctly solving the same problem, without algebra and in less time. I was amazed at the capabilities of those students. This experience showed me that I typically approach problems using algebra, even in situations where arithmetic might be more appropriate. The experience brought my attention to the remarkable and clever ways that students reason in mathematics. Furthermore, it set me on the path to considering the ways that my students were thinking mathematically and how I could build on that knowledge.

As I continued in my high school teaching career, another experience challenged my thinking about the ways that teachers understand students' mathematical thinking. I tutored a colleagues' geometry student twice a week for 5 months. During that time I came to understand how the student was thinking mathematically. While I was tutoring him, I frequently had conversations with his teacher about him. I was often surprised at the teacher's interpretations of the student's mathematical thinking because they contrasted with what I knew of him. This experience led me to question the ways that teachers (including myself) learn about how students think and what impact that has on instruction. I learned a lot about the ways the student was thinking mathematically through hours of interacting with him solving mathematics tasks.

Teachers do not typically have much class time to spend with one student, but they do spend a lot of time with groups of students. So, what do teachers learn about students through their numerous interactions with them in groups?

I have shared these anecdotes to describe experiences that affected my professional growth. Much like the teacher quoted in the beginning of this report, I began to make changes in my instructional practices when I focused on students' mathematical thinking. The two experiences left an impression on me that challenged my view of students' mathematical thinking and even my own teaching practices. In addition to these experiences, over my years of teaching I have had many classroom interactions with students. Through those interactions I expanded my knowledge about how students think mathematically and the mathematical power that they possess. Now, as a teacher educator, I have begun to ask questions about the value of this kind of knowledge, how it develops, and how teacher education and professional development can support and encourage teachers to notice and build on their students' mathematical thinking. I have chosen to continue this line of inquiry by studying what geometry teachers notice about students' mathematical thinking and how they use what they notice in their teaching practices.

Why Geometry?

Clements (2003) described geometry in the United States for grades pre–K through 12 as "the study of spatial objects, relationships, and transformations; their mathematization and formalization; and the axiomatic mathematical systems that have been constructed to represent them" (p. 151). In this study, I focused on high school geometry teachers for several reasons. First, NCTM advocated an increased emphasis on geometric instruction at all levels (1989, 2000). Clements commented, "U.S. curriculum and teaching in the domain of geometry is generally weak, leading to unacceptably low levels of achievement" (p. 152). For example, secondary students scored at or near bottom on all of the geometry tasks in the Third International Mathematics and Science Study (Beaton et al., 1996; Lappan, 1999). Next, geometry is required for entrance into many colleges and is considered a "gatekeeper" course—a course that allows students to meet minimum college entrance requirements (Stone, 1998). Last, I have had more experience teaching geometry than teaching other subjects. This experience was valuable, as it allowed me to observe classroom interactions and student thinking and be comfortable with the content, how students might solve various tasks, and how teachers might approach various topics.

Why Focus on Teachers' Understanding of Student Thinking?

A growing body of literature in mathematics education suggests that teachers make instructional changes when they focus on student thinking (Campbell & White, 1997; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Franke, Carpenter, Levi, & Fennema, 2001; Kazemi & Franke, 2004; Lin, 2006; Lubinski & Jaberg, 1997; Mewborn, 2006; Steinberg, Empson, & Carpenter, 2004; Vacc & Bright, 1999). A recent book series titled, *Teachers Engaged in Research*, (Mewborn, 2006) presents multiple articles written by practicing K–12 teachers that detail action research in which the authors investigated student thinking within particular domains of mathematics. In the series forward, Cochran-Smith (2006) acknowledged that "the teacher researchers in these books demonstrate how they came to understand their students' reasoning processes and thus learned to intervene more adeptly with the right question, the right comment, a new problem, or silent acknowledgement and support" (p. xix). For example, C. Huhn, K. Huhn, and Lamb (2006) studied students' understanding of three tasks through conversations with the students about their thinking. C. Huhn et al. made connections between classroom environments and students' conceptual understanding. The findings motivated the three teachers to become leaders and advocates for learning environments that encouraged student dialogue and promoted student mathematical authority.

In addition to action research reports, there have been numerous studies of Cognitively Guided Instruction (CGI), a professional development program that "focuses on helping teachers understand children's thinking by helping them construct models of the development of children's mathematical thinking in well-defined content domains" (Franke et al., 2001, p. 657). Fennema et al. (1996) reported on a 4-year longitudinal study that examined 21 primary teachers who participated in a CGI professional development program. Eighteen of the teachers made changes in their beliefs and their instructional practices that directly influenced student achievement. In a follow-up study of CGI professional development participants, "all 22 teachers maintained some use of children's thinking and 10 teachers continued learning in noticeable ways" (Franke et al., 2001, p. 682). The 10 teachers' learning was called *generative growth*. The teachers:

> (a) viewed children's thinking as central,
> (b) possessed detailed knowledge about children's thinking,
> (c) discussed frameworks for characterizing the development of children's mathematical thinking,
> (d) perceived themselves as creating and elaborating their own knowledge about children's thinking, and
> (e) sought colleagues who also possessed knowledge about children's thinking for support. (Franke et al., 2001, p. 653)

In a professional development program that specifically examined student work, teachers developed a deeper understanding of their students' mathematical thinking and began to develop possible instructional trajectories for their students (Kazemi & Franke, 2004). Lubinski and Jaberg (1997) reported changes in teachers' beliefs, and those changes were reflected in their instructional practices. "The focus of the intervention was on developing students' understanding of mathematics by using teachers' knowledge of students' thinking processes" (Lubinski & Jaberg, 1997, p. 223). The project Increasing the Mathematical Power of All Children and Teachers (IMPACT) "encouraged teachers to foster children's interpretation of mathematical relationships in problems based on whatever intuitive knowledge or unique experiences the children had outside of school and to build on those experiences with instructional activities that might extend or deepen existing knowledge" (Campbell & White, 1997, p. 313). Over 3 years a total of 73 teachers of kindergarten through fourth grade attended professional development sessions. "The results of project IMPACT reinforce the notion that as teachers come to think more deeply about how children understand and 'construct' mathematical meanings, teachers can make instructional decisions and organize their classrooms in ways that support and encourage more meaningful mathematics learning, resulting in significant gains in student achievement" (Campbell, 1997, p. 107).

The literature related to high school teachers that focus on students' mathematical thinking is sparse. Heid, Blume, Zbiek, and Edwards (1999) studied three secondary teachers who use task-based interviews with students to learn more about how students think. The results showed that the teachers measured students' understanding to some standard and that the interview approach influenced their actions. For example, LeAnne's (one of the participants) interview was focused on the mathematics curriculum. In her interactions with the students, she used questioning as a way to get her desired outcome – the curriculum. In another study that focused on one high school teacher, Doerr (2006) found that the teacher listened to students' alternate solutions and developed schema that represented a diverse group of students' thinking. In addition, Doerr reported that the teacher was asking students to describe and explain, which in turn led students to evaluate their own solutions.

Across these studies there is one common result—teachers make instructional changes when they focus on students' thinking (except Heid et al.'s (1999) study, which only looked at what teachers learned). In addition, instructional practices that support and build on students' thinking promote student mathematical understanding (Fennema, Franke, Carpenter, & Carey, 1993; Hiebert & Wearne, 1993). It is important for teachers to incorporate instructional practices that support and build on students' mathematical thinking, and therefore we need to learn more about how teachers' understand and use students' mathematical thinking. It is evident that what teachers do and say in classrooms shapes classroom interactions and affects the level of the task (Stein, Smith, Henningsen, & Silver, 2000), which in turn influences student learning. The literature describes the significance that teachers' understanding of students' mathematical thinking has on instructional practices, but it provides little detail about how teachers make sense of students' mathematical thinking and how teachers use that thinking during classroom interactions.

Research Questions

The literature presented above develops a strong rationale for learning more about the ways that teachers conceptualize students' mathematical thinking. Much of this literature focuses on elementary teachers and on teacher change. With few exceptions (e.g., Doerr, 2006; Heid et al., 1999), there is less known about the ways high school teachers understand students' mathematical thinking. The literature also brings attention to the significance of classroom interactions, and, specifically, teachers' actions within those interactions. This report details a study of high school teachers, their understanding of students' mathematical thinking, and their instructional practices.

Teachers do not have many opportunities to discuss with colleagues their practice or their ideas about the ways students think mathematically. With that observation in mind, I offered a professional development experience to support teachers in their endeavor to verbalize their sense making of students' mathematical thinking. The purpose of that professional development experience was to offer teachers an opportunity to reflect on and discuss students' mathematical thinking. It was my intent to examine teachers' understanding of students' mathematical thinking within their natural environment.[1] It was not my goal to study teacher change or the professional development experience.

Simon's (1995a, 1995b) work on the Mathematics Teaching Cycle provided a lens for studying teachers' conceptualizations of students' mathematical thinking. An important component of the Mathematics Teaching Cycle was what he called "hypothetical learning trajectories." He used the word *hypothetical* to draw attention to the fact that teachers cannot know in advance what the learning trajectory truly looks like until after students engage with the mathematical activity. A learning trajectory is model that represents children's starting points, the changes that occur because of mathematical activity, and the interactions that were involved in those changes. These learning trajectories represent students' mathematical thinking.

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[1] I recognize that my mere presence in the classrooms, the multiple teacher interviews, and the professional development experiences influenced the teachers and the classroom environment. I used the words *natural environment* to signify my attempt to understand these teachers at this point in time in their professional lives. Furthermore, I was not studying changes in instructional practices.

Researchers in mathematics education have spent considerable time developing learning trajectories for children (e.g., Battista, 2004; Clements & Wilson, 2004; Gravemeijer, 2004; Simon & Tzur, 2004; Steffe, 2004). They hope that learning trajectories will support teachers by providing them with ways that students develop concepts and tasks that might guide that development. Learning trajectories are not designed to take away the teachers' autonomy in classrooms; teachers are still responsible for making instructional decisions before, during, and after lessons. Steffe (2004) notes, "The construction of learning trajectories of children is one of the most daunting but urgent problems facing mathematics education today" (p. 130).

Two of the immediate difficulties with learning trajectories are the limited availability and the complexity of these models. Teachers do not have access to the learning trajectories that are created by researchers, and therefore they cannot use them. Also, the structure of teaching does not provide teachers with professional time to read long technical reports detailing learning trajectories created by researchers. Descriptions of learning trajectories need to be concise and easily accessible to teachers in order for them to reap the benefits. I claim that teachers with experience have developed "informal" learning trajectories for their students through their interactions with students. These informal learning trajectories inform their instructional practices and guide their interactions with students. Learning trajectories are developed and modified through interactions with students doing mathematics. Classroom interactions present opportunities for teachers to learn more about how students think mathematically and are a vital component in considering how teachers are making sense of students' mathematical thinking.

A significant contribution to the field would be to understand more about how teachers understand students' mathematical thinking. I examined this focus from the perspective of two components of the Mathematics Teaching Cycle—learning trajectories and classroom interactions (specifically, teacher noticing in classroom interactions). The research questions for this study were as follows:

1. What learning trajectories do teachers describe of students' mathematical thinking when they participate in a professional development experience that focuses their attention on their own geometric thinking and their students' geometric thinking?
2. What do teachers notice about students' mathematical thinking during their teaching practices?
3. In what ways do teachers respond to students' mathematical thinking in mathematics classrooms?

These research questions are addressed in the following chapters. In the next chapter, I discuss relevant literature (chapter 2). The subsequent two chapters address the research methodology (chapter 3) and the results (chapter 4). In chapter 5, I outline implications of my study for teacher education and future research.

CHAPTER 2

LITERATURE

The broad purpose for the present study was to consider the ways teachers understand students' mathematical thinking. Simon's (1995a, 1995b) work on the Mathematics Teaching Cycle was significant in how I conceptualized the ways teachers understand students' mathematical thinking. The Mathematics Teaching Cycle described the relationship between teachers' knowledge, learning trajectories, and classroom interactions, all of which related to the ways teachers understand students' mathematical thinking. The Mathematics Teaching Cycle led me to two bodies of literature that framed this study – learning trajectories and professional noticing. In addition, this was a study about teachers' understanding of students, so I reviewed literature on students' mathematical thinking and teachers' knowledge.

This chapter reviews relevant literature for the study. I begin with a brief overview of students' mathematical thinking. Following that description, I elaborate on the Mathematics Teaching Cycle and the components applicable to this study.

Students' Mathematical Thinking

Using Wood, Williams, and McNeal's (2006) definition, I define mathematical thinking "as the mental activity involved in the abstraction and generalization of mathematical ideas" (p. 226). This definition was built on Williams' (2002) work in which he connected specific types of mathematical thinking to observable actions. Through this work, he developed three cognitive activities: r*ecognizing* (comprehending and applying); *building-with* (analyzing, synthetic-analyzing, and evaluative-analyzing); and c*onstructing* (constructing and synthesizing). It was not my goal to categorize what teachers noticed about students' mathematical thinking into one of these three categories, but the categories were helpful in identifying the ways that students' mathematical thinking appears in classrooms.

In the United States, K-12 geometry is "the study of spatial objects, relationships, and transformations that have been formalized (or mathematized) and the axiomatic mathematical systems that have been constructed to represent them" (Clements & Battista, 1992). Spatial reasoning underlies most geometric thought, including the "ability to 'see,' inspect, and reflect on spatial objects, images, relationships, and transformations" (Battista, 2007, p. 843). Geometric thinking involves three cognitive processes: visualization, construction, and reasoning (Duval, 1998). Visualization refers to illustrating a statement by a space representation. Constructions are models of mathematical objects created through the use of tools. Reasoning involves pulling out new information from given information. These cognitive processes are closely connected to one another and may influence each other.

Geometric Thought

Two pairs of researchers laid the foundation for research on students' geometric thought: Jean Piaget and Barbel Inhelder, and Pierre and Dina van Hiele. Piaget and Inhelder (1967) noted two major themes in geometric thinking. First, children develop their representation of space from manipulation of the spatial environment. Second, the organization of geometrical ideas follows a definite order. Children develop topological relationships and then progress to projective and Euclidean relationships. The van Hieles created a model of geometric thinking (van Hiele, 1986) that students progress through five levels of thought (see Appendix A for levels). The van Hieles reported that the levels of thinking were qualitatively different, sequential, and hierarchical and that each had its own language and way of thinking (Clements, 2003). In recent years, researchers have reported new developments pertaining to the van Hiele levels. First, the level descriptors have expanded to three-dimensional shapes. In this expansion, most of the work followed the original van Hiele levels. The extension to visualization was more difficult than the other areas of geometry for researchers to grasp, because visualization was not necessarily connected to knowledge of properties (Battista, 2007). The second development involved researchers examining the notion that the levels are discontinuous. For instance, Clements, Battista, and Sarama (2001) concluded that the different van Hiele levels were types of reasoning that developed simultaneously. Teachers were viewed as a key factor in helping children progress through the five levels (Fuys, Geddes, & Tischler, 1988). In addition to the five levels, there are five phases that teachers can use to help students attain a higher level of geometric thought. In these phases the teacher makes a transition from teacher involvement and instruction to more student independence. The phases are as follows:

Phase 1: *inquiry* – the teacher and students discuss the object of study

Phase 2: *directed orientation* – the teacher selects a one-step task for the students to explore

Phase 3: *expliciting* – the students refine their vocabulary with little prompting from the teacher

Phase 4: *free orientation* – the students engage in multi-step tasks

Phase 5: *integration* – the students internalize objects and relations to create a new domain of thought. (Clements & Battista, 1992)

There were some similarities between Piaget and Inhelder's and the van Hieles' views about geometric thought. Clements and Battista (1992) noted three similarities. First, both pairs of researchers believed that students must actively construct their own knowledge. Second, neither believed good telling was good teaching. Third, neither believed that the goal of education was to accelerate development. Nor did either they believe that one should devalue thinking at a lower level once higher levels were achieved. There was both support and skepticism in the educational community pertaining to the two views of geometric thought. For example, researchers questioned the mathematical accuracy of Piaget and Inhelder's use of terms, such as topological and Euclidean figures. Pertaining to the van Hiele levels, some researchers questioned whether five levels were appropriate and what the relationships were between the levels.

Geometric Concepts

I end this section with a brief review of literature related to the concepts of length, area, and volume, as these concepts are important in the secondary geometry curriculum. This literature is relevant for the present study as it identifies ways that students might reason about the concepts length, area, and volume. I used this literature as a way to hypothesize and identify what teachers might notice about students' mathematical thinking in a geometry course.

Measurement is important to understanding various aspects of geometry (e.g., structure of shapes). Barrett, Clements, Klanderman, Pennisi, and Polaki (2006) described three levels (with sublevels) of reasoning about length measurement:

1) visual guessing to assign length;

2a) inconsistent ways of identifying or iterating units; uses of salient markers as a counting set for measuring;

2b) consistent identification or iteration of units;

3a) coordinating iterated-unit items, side lengths, and collection of side lengths to obtain perimeter; and

3b) coordinating length attributes, yet with further tendency and ability to relate multiple cases.

In the Barrett et al. study, students were asked to find all the rectangles with a perimeter of 24. Students in grades 8 through 10 quickly generalized without maintaining the physical representation of the rectangle. Many students struggled at times “with untenable cases based on arithmetic computation that had become disconnected from spatial operations” (p. 215). When the high school students did use drawings they tended to draw them proportionally, or they used drawings that functioned as labels for sets of numbers. Several studies described the difficulties students have with differentiating length, area, and volume. An example is a study by Tierney, Boyd, and Davis (1990), who found that teachers and students often believed that: 1) as perimeter of a figure increases, so does the area, and 2) when the side lengths of the faces of a cube doubles, so does the volume.

A core idea in developing competence with area and volume “is understanding how to *meaningfully* enumerate rectangular 2d and 3d arrays of squares and cubes, respectively” (Battista, 2007, p. 897). Battista identified five basic cognitive processes as essential for that development: abstracting, forming and using mental models, spatial structuring, units-locating, and organizing-by-composites. Furthermore, he identified seven levels of sophistication in students’ structuring and enumeration of arrays.

Level 1: Absence of units-locating and organizing-by-composites processes

Level 2: Beginning use of the units-locating and the organizing-by composites processes

Level 3: Units-locating process sufficiently coordinated to eliminate double-counting

Level 4: Use of maximal composites, but insufficient coordination for iteration

Level 5: Use of units-locating process sufficient to correctly locate all units, but less-than-maximal composites employed

Level 6: Complete development and coordination of both the units-locating and the organizing-by-composites processes

Level 7: Numerical procedures connected to spatial structurings, generalization. (pp. 897–898)

In the beginning level, students were unable to “see” the units in the arrays and often double counted. As they developed more sophistication, they were able to group and iterate the units. At the final level, students were able to reflect on and analyze the enumeration of arrays.

Most of the literature on length, area, and volume has treated the concepts separately. Researchers have identified levels for understanding length and separate levels for understanding area and volume. Recently, Battista (2007) developed a general theory of geometric measurement that connected the disparate literature. Battista thought that a common theme in measurement was students’ initial difficulty in “seeing” the unit. Therefore they struggle to correctly iterate the unit. Then, they progress to using an iteration process with the unit to measure the shape. Subsequently, students numerically operate on measurement without iteration. At the final level, students inferentially operate on measurements without iteration. Battista’s work on connecting the literature on length, area, and volume was relevant for the present study because I wondered if teachers would use their own language to describe whether their students were operating at one of Battista’s levels.

The Mathematics Teaching Cycle

“The Mathematics Teaching Cycle (Simon, 1995a) refers to a conceptual framework that describes the relationships among teacher’s knowledge, goals for students, anticipation of student learning, planning and interactions with students" (Simon, 1995b, p. 76). Simon's efforts to describe teaching based on a constructivist learning theory resulted in the framework. He argued that new models of teaching were necessary and that the teacher's own models of teaching influenced her or his instructional practices.

The interrelated components of the cycle were the teacher's knowledge, the hypothetical learning trajectory, and interactions with students. Figure 1 is an abbreviated diagram for the Mathematics Teaching Cycle. The figure is abbreviated because it does not elaborate on each of the three components. In the Mathematics Teaching Cycle, the teacher's knowledge influences the development of hypothetical learning trajectories for her or his students. These developments influence what happens in the classroom. Simon designed his diagram with arrows going in one direction. He also stated that the arrows do not represent all the connections and directions of influence. A Mathematics Teaching Cycle may last from a short period of time of a day or two to several weeks, and the teacher may be at different points in different cycles simultaneously. Teachers are a vital component of the cycle, because the instructional decisions they make before, during, and after classroom interactions underlie the cycle. Within these decisions there is an "inherent tension between responding to the students' mathematics and creating purposeful pedagogy based on the teacher's goals for students' learning" (Simon, 1995b, p. 76). Teachers must decide when and how to act on students' mathematical thinking during classroom interactions.

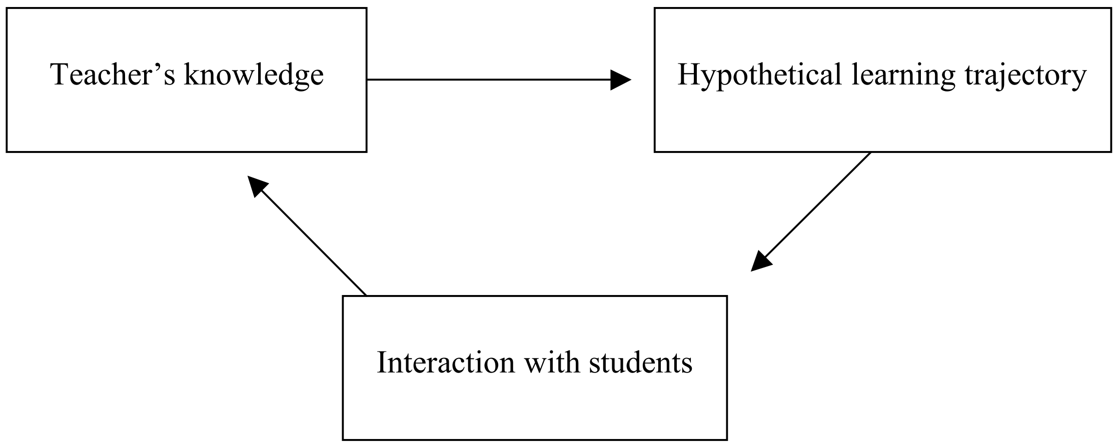

Figure 1. The abbreviated Mathematics Teaching Cycle (Simon, 1995b).

In defining the Mathematics Teaching Cycle, Simon (1995b) had a particular kind of teaching in mind. He detailed it as teaching in which

> the teacher is involved in model building, and hypothesis generation (e.g., models of students' mathematical understanding, predictions of how learning might progress). The teacher's learning is a key aspect of teaching. Teaching is goal directed, yet the goals are constantly being modified. The teacher's relationship with the students is one of both direction setting and following. (p. 80)

The descriptions of mathematics teaching that he provided were the aspects of teaching that my study explored, and therefore this Mathematics Teaching Cycle was a fundamental construct to consider. It was not my intent to identify a teacher who teaches as Simon described. However, his descriptions connect aspects of teaching that are relevant to considering the ways teachers understand students' mathematical thinking. Furthermore, the Mathematical Teaching Cycle provides a way to examine learning trajectories teachers create and what teachers notice about students' mathematical thinking within classroom interactions.

Because his previous figure did not represent the complex nature of teaching, Simon offered a second figure for the Mathematical Teaching Cycle (see Figure 2). This extended diagram connects the learning trajectories that teachers create and what happens in the classroom (specifically, what teachers notice and do pertaining to students' mathematical thinking). Within the components of the Mathematics Teaching Cycle are two ideas that were important in my study: hypothetical learning trajectories and what teachers noticed about students' mathematical

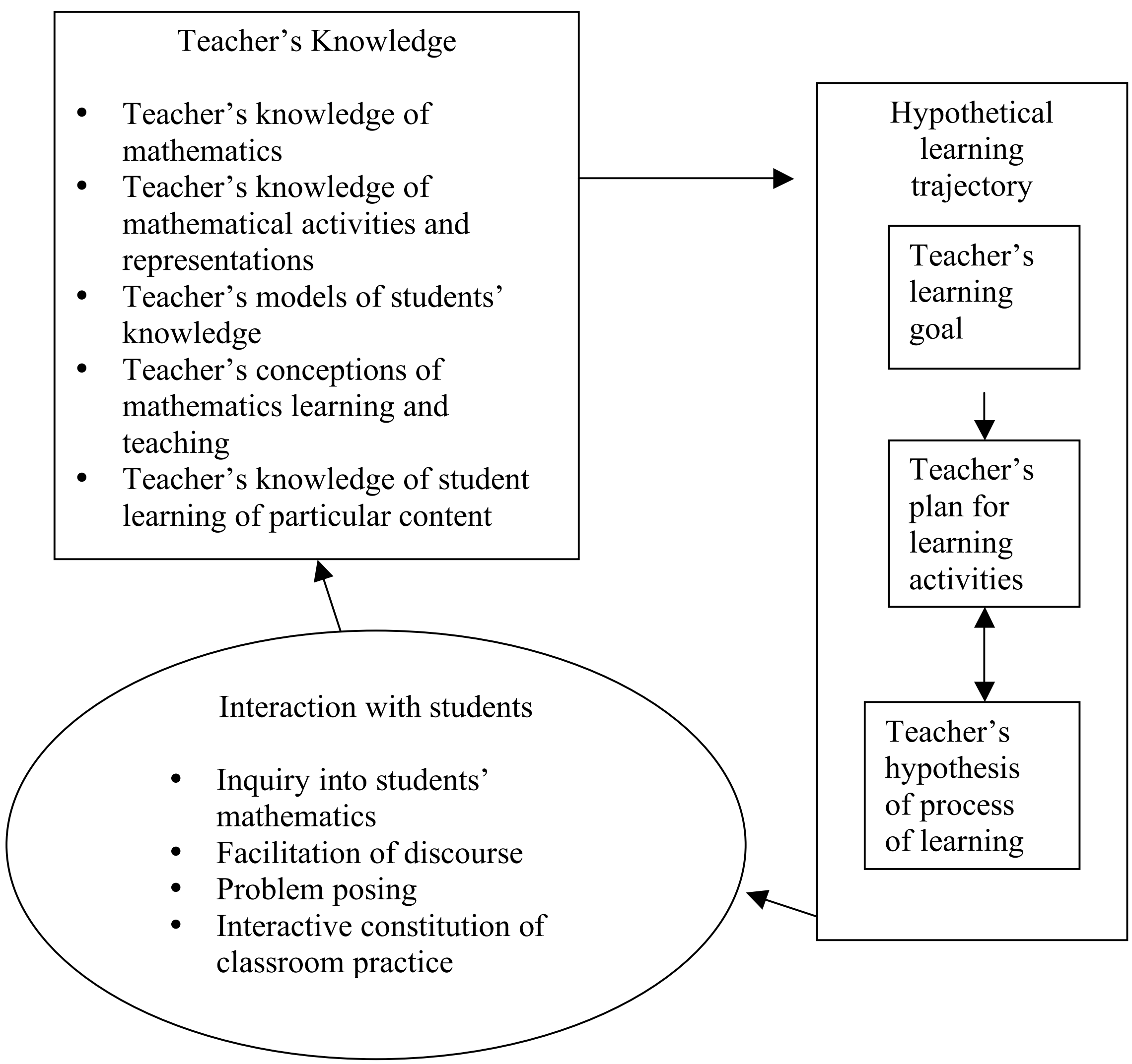

Figure 2. The Mathematics Teaching Cycle (Simon, 1995b).

thinking. Teachers' mathematical knowledge for teaching was a component that I considered to situate learning trajectories and what teachers notice during classroom interactions.

The Mathematics Teaching Cycle: Learning Trajectories

Learning trajectories are used to describe models of students' mathematical thinking. Steffe (2004) stated:

> A learning trajectory of children includes a model of their initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situation of learning, and an account of the mathematical interactions that were involved in the changes. (p. 131)

Learning trajectories represent children's starting points, the changes that are due to mathematical activity, and the interactions that were involved in those changes. The mathematics of children is considered to be "legitimate mathematics" in the sense that children have rational reasons for what they say and do (Steffe & Thompson, 2000). To create learning trajectories, the teacher must understand children's mathematics, which requires an adult to interact extensively with children doing mathematical activities.

Simon (1995b) defined a hypothetical learning trajectory as "the path by which learning might proceed. It was hypothetical because the actual learning trajectory was not knowable in advance. It characterizes an expected tendency" (p. 135). Hypothetical learning trajectories are different than the learning trajectories defined by Steffe (2004) because they are predictions about how learning might proceed. Steffe's learning trajectories are models that describe how learning proceeded.

> The hypothetical learning trajectory has three components:
> the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p. 136)

The teacher's creation and modification of hypothetical learning trajectories are the central component of the Mathematics Teaching Cycle (Simon, 1995b). Through interactions with students, teachers generate new or modified existing trajectories. Simon highlighted the importance of having a goal and rationale for instructional decisions. Teachers' thinking behind instructional decisions is hypothetical because they do not know in advance the goals that students set for themselves or how instructional decisions will influence student learning. The learning goals that teachers have for their students are not the same learning goals that students set for themselves. Teachers' understanding of students' thinking and learning is necessary for the hypothetical learning trajectory.

> Note that the development of a hypothetical learning process and the development of the learning activities have a symbiotic relationship; the generation of ideas for learning activities is dependent on the teacher's hypotheses about the development of students' thinking and learning; further generation of hypotheses of student conceptual development depends on the nature of anticipated activities. (p. 136)

Teachers choose particular activities based on the hypothetical learning trajectories of their students, and these activities affect the development of students' mathematics.

To explain the hypothetical learning trajectory, Simon made an analogy to taking a sailing trip around the world. As you plan for a sailing trip, you use as much information as you can to make tentative plans for the trip. You may plan the entire trip or just the first few days. On the journey, you may modify your plans according to your experiences, such as weather conditions or interesting places. These experiences may cause you to stay longer than you initially planned. The “trajectory” represents where you actually traveled. The path that you anticipate taking at any given moment is the “hypothetical trajectory.”

The teacher begins by taking all the information he or she knows about how children learn a particular concept. Then the teacher plans learning activities. Through observations and interactions, the teacher and students produce an experience. This experience is different from what the teacher predicted. As the teacher makes sense of the experience, the teacher may adapt his or her knowledge and modify or create a new hypothetical learning trajectory. Modifications may occur in any of the three components: the teacher’s learning goal, the teacher’s plan, or the teacher’s hypothesis about the learning process. Simon (1995b) summarizes the modification process:

> The generation of a hypothetical learning trajectory prior to classroom instruction is the process by which the teacher develops a plan for classroom activity. However, as the teacher interacts with students, the teacher and students collectively constitute an experience. This experience, by the nature of its social constitution, is different from the one anticipated by the teacher. The interaction with students leads to a modification in the teacher’s ideas and knowledge as he or she makes sense of what is happening and what has happened in the classroom. (p. 78)

All three components of the hypothetical learning trajectory are influenced by the teacher’s knowledge of mathematics, the teacher’s hypotheses about students’ knowledge, the teacher’s theories about mathematics learning and teaching, and the teacher’s knowledge of student learning of particular content. In addition, the second component, the teacher's plan for learning activities, is influenced by the teacher's knowledge of mathematical activities and representations. The hypothetical learning trajectory influences how students' knowledge is assessed, which in turn influences the teacher's knowledge, hypotheses, and theories about mathematics and student learning. In the Mathematics Teaching Cycle, the teacher must go beyond listening and assessing students; she or he must also make predictions about student's learning. The hypothetical learning trajectory is an informed prediction about how students learning might progress based on the learning goal and the learning activities. In order to make these predictions, the teacher must have a deep understanding of student thinking and learning.

As a teacher is engaged in teaching, he or she is focused on sociological phenomena, and therefore is limited to making descriptions of the language and actions of students. Using support from Maturana's (1978) notion of two nonintersecting phenomenal domains of interaction, Steffe and D'Ambrosio (1995) stated, "Simon's emphasis on the social processes involved in teaching mathematics makes it quite difficult to focus on the mathematics of his students" (p. 153). Steffe and D'Ambrosio believed these explanations were "couched in terms of the mathematics concepts and operations of the teacher" (p. 153). To explain the mathematics of students in terms of their schemes and operations, Steffe and D'Ambrosio recommend a different kind of teaching experiment. They recommend a teacher interact with extensively with one to two children engaged in mathematics, which does typically happen in schools. The concern that Steffe and D'Ambrosio identified was a valuable insight for the present study as it highlights the constraints that teachers face every day. Their concern reinforced my thoughts that teachers would struggle to focus on students' mathematical thinking because of the complexities of teaching.

Gravemeijer (2004) believed that it was beneficial to "offer the teacher some framework of reference, and a set of exemplary instructional activities that can be used as a source of inspiration" (p. 107). He used the phrase *local instruction theories* to describe and rationalize the envisioned learning path as it correlates to instructional activities. He noted that these local instruction theories could be used to support hypothetical learning trajectories, and therefore add to the quality of the trajectories. Gravemeijer argued that without local instruction theories "the chances to reconcile openness towards student's own contributions and aiming for given end goals are very slim" (p. 108).

Gravemeijer (2004) commented that the previous travel analogy could be used to distinguish between local instruction theories and hypothetical learning trajectories. The local instruction theory was the travel plan, which the teacher uses to develop the day-to-day journey (the hypothetical learning trajectory). The teacher uses a local instruction theory to select instructional activities and to generate hypothetical learning trajectories for his or her particular students.

Learning Trajectories

The goal of generating learning trajectories is to provide teachers with models of students' mathematics learning in order to assist in instructional planning. Learning trajectories also support teachers in their efforts to notice evidence of students' mathematical thinking, and consequently, provide opportunities for teachers to use students' mathematical thinking during their lessons. It is important to reiterate that each student may progress differently, and therefore as teachers are guiding students along their hypothesized trajectories they must be open and responsive to students' actions. In other words, teachers can have trajectories for groups of students, but they must be open and responsive to individual student's actions.

The learning trajectories discussed above are complex and created by researchers who have interacted intensely with students doing mathematics. As I alluded to earlier, teachers are constrained by the complexities of the classroom (e.g., responsible for many students, classroom interruptions), which limits the teacher's ability to focus on the mathematics of his or her students. Therefore, one would expect teachers to have a difficult time articulating sophisticated models of students' mathematical thinking. Yet, teachers are expected to create learning opportunities for students that build on students' current ways of thinking. Teachers are faced with the complex task of "fostering the development of conceptual knowledge among her students and of facilitating the constitution of shared knowledge in the classroom community" (Simon, 1995a, p. 119). I agree with Clements and Sarama (2004) when they state, "The notion of hypothetical learning trajectories was a unique and substantive contribution to the field" (p. 11). The development of learning trajectories could provide teachers with a framework for thinking about instructional planning.

There have been criticisms regarding learning trajectories. Gravemeijer (2004) raised a concern that it was unfair to for teachers to invent learning trajectories without any means of support. He suggested that instructional theories developed by researchers were valuable for the field of mathematics education as those theories support teachers in their efforts to create learning trajectories. Baroody et al. (2004) stated that some learning trajectories are "so highly technical and complicated it is unlikely that practitioners would find them helpful" and some learning trajectories "seem overly prescriptive and inconsistent with an inquiry-based investigative approach" (p. 253). Furthermore, Doerr (2006) said, "Hypothetical learning trajectories posited on the learning processes of individual learners would create an unmanageably large set of trajectories for a classroom teacher" (p. 5). She made an argument similar to that of Gravemeijer that teachers need to know the landscape of students' conceptual development, but that is not the same as understanding one way of thinking or one way of developing an idea or following a particular learning trajectory. Baroody et al.'s criticisms do not relate to the present study, as I was interested in the "informal" learning trajectories that teachers create for their students. I was not interested in researchers' learning trajectories. I considered the teachers' learning trajectories "informal" because they were not technical and complicated like the ones researchers create through their interactions with students. The teachers' learning trajectories have not been made public for mathematics educators to examine like the ones created by researchers. In addition, teachers do not explicitly discuss them with their colleagues. Doerr's (2006) concern for the practical aspects of learning trajectories was an issue that I considered. For the purpose of this study, her concern was part of what I wanted to know more about. I claim that teachers develop less sophisticated learning trajectories for students through their teaching practices. These develop as teachers interact with students doing mathematics over time. Teachers have learned how students' learning progresses, and learning trajectories provide a way to investigate that knowledge. It is my assumption that many teachers create learning trajectories implicitly through their interactions with students. Furthermore, I hypothesized that teachers who were more aligned with Simon's idea of teaching would have more elaborated learning trajectories than teachers who used lectures. I was interested in what the learning trajectories looked like before and after classroom interactions. For instance, do teachers have different learning trajectories for different students? If so, what are those trajectories based on? Do teachers describe learning trajectories of students by singling out one or two students, or do they refer to groups of students? These are all important questions to examine pertaining to teachers' learning trajectories for students.

Learning Progressions

In my analysis of the literature I also examined the idea of learning progressions. Popham (2007) identified learning progressions as a popular construct that can help to teachers design and monitor their instruction. “A learning progression is a carefully sequenced set of building blocks that students must master en route to mastering a more distant curriculum aim. These building blocks consist of subskills and bodies of enabling knowledge” (p. 83). The idea behind the learning progression is there are less sophisticated ideas that must be developed before progressing to more advanced ideas. “Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time (e.g., 6 to 8 years)” (Committee on Science Learning, 2007, p. 219). Learning progressions are sequenced based on appropriateness for students, instead of being based on the ways the field is structured. “Learning progressions describe not only how knowledge and understanding develops, but also predict how the knowledge builds over time” (Stevens, Shin, Delgado, Krajcik, & Pellegrino, p. 2). A lot of the literature on learning progressions is in the field of science education (e.g., learning progressions for the nature of matter and the molecular basis of heredity). Learning progressions are seen as a way that teachers can assess gaps in students’ knowledge in order to inform instructional practices, which denotes a difference learning progressions and learning trajectories.

Learning trajectories and learning progressions have a lot of commonalities, such as an emphasis placed on the students’ learning processes. In this study I focused on learning trajectories because the interaction of the learning activities and learning processes were valuable when considering the ways teachers understand and use students’ mathematical thinking. Learning trajectories consider how students might engage with particular tasks, whereas learning progressions specifically focuses on the developmental process. Furthermore, it is explicit in learning trajectories that classroom interactions are a vital component.

The development of learning trajectories has been a significant contribution to the field of mathematics education, but as many have explicitly stated, or alluded to, what teachers do with learning trajectories in their instructional practices is crucial. Teachers influence the mathematical activity of the classroom and influence students' mathematical learning opportunities. Many mathematics education researchers have devoted their efforts to studying and developing learning trajectories (or some construct closely related to learning trajectories) of students' mathematics (Battista, 2004; Clements & Wilson, 2004; Gravemeijer, 2004; Lesh & Yoon, 2004; Simon & Tzur, 2004; Steffe, 2004). In the present study, I took a different path in studying learning trajectories. The previous studies discuss learning trajectories created by researchers. I was interested in examining high school teachers' informal learning trajectories for the purpose of identifying what teachers noticed about students' mathematical thinking during classroom interactions.

The Mathematics Teaching Cycle: Classroom Interactions

Another component of the Mathematical Teaching Cycle (1995a, 1995b) is interactions. Interactions in classrooms occur around some mathematical activity and take place within the classroom community. Students can interact with teachers and mathematics, they can interact with fellow students and mathematics, and they can interact with mathematics. During classroom interactions the teacher observes, questions, and makes moment-to-moment instructional decisions. As mentioned earlier, teachers modify their learning trajectories for their students based on their knowledge and their interactions with students.

In this study, I was interested in what teachers noticed about students' mathematical thinking before, during, and after classroom interactions, and the ways they acted on that thinking. I used learning trajectories as a way to identify what teachers notice about students' mathematical thinking during classroom interactions. In the following two sections I review the literature on teacher "noticing" and teacher "actions."

Professional Noticing

Mason (2002) stated, "Every act of teaching depends on noticing: noticing what children were doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be or done next" (p. 7). In order to act on students' mathematical thinking teachers must notice that thinking. Studying Teachers' Evolving Perspectives (STEP) was a five-year project designed to investigate three groups of elementary teachers who were engaged in sustained professional development. Through their work, Jacobs, Lamb, Philipp, Schappelle, and Burke (2007) defined the construct *professional noticing of children's mathematical thinking*. Professional noticing was described as "how professionals view and make sense of complex situations" (p. 5). Furthermore, expertise in professional noticing consisted of four interrelated skills: (a) identifying noteworthy aspects of instructional situations, (b) describing instructional situations, (c) interpreting instructional situations, and (d) responding in instructional situations (Jacobs et al., 2007).

Identifying noteworthy aspects of instructional situations. Professional noticing requires that the teacher select what she or he will attend to within the lesson. Mason (2002) commented that as humans we are constantly inundated with sense impressions and that we are only consciously aware of some of them. He stated, "To notice is to make a distinction, to create foreground and background, to distinguish some 'thing' from its surroundings" (p. 33). Teachers cannot possibly notice everything that happens in a classroom. They must identify important aspects of teaching situations to place in the foreground and let less important aspects move to the background. Jacobs et al. brought up terms such as *highlighting* (Goodwin, 1994) and *making call-outs* (Frederiksen, 1992; Frederiksen, Sipusic, Sherin, & Wolfe, 1998) to describe noteworthy aspects of complex situations. Call-outs were used to identify noteworthy moments in a video that matched particular criteria. In making call-outs, the participants watched a video for the purpose of noting classroom moments that were important and fit particular criteria (Frederiksen, 1992; Frederiksen et al., 1998). In classroom situations, the teachers identified significant interactions in order to make decisions about what to do next. Van Es (2004) found that elementary teachers began to notice more instances of important classroom practices related to reform mathematics teaching through participating in a professional development project aimed at helping teachers notice. In the present study, I was interested in the call-outs that teachers identify that pertain specifically to students' mathematical thinking. I am using the term *call-out* differently here. The literature refers to a call-out as making notes while watching videos of classrooms. I use it to mean a teacher making a comment about his or her actual teaching.

Describing instructional situations. Professional noticing includes describing, which is different than interpreting. Describing is telling what happened without placing meaning or value on the description. Developing the skill of describing requires teachers to be as objective as possible. When teachers describe an instructional situation they identify what they specifically attended to during the situation. A teacher's subjectivities influence what she or he attends to during classroom interactions, but becoming knowledgeable about those subjectivities enhances the teacher's ability to describe those interactions. Jacobs et al. (2007) found that experienced teachers had more details in their descriptions of students' mathematical thinking than novices.

Interpreting instructional situations. Making interpretations of classroom interactions is an important skill in professional noticing. Teachers need to go beyond the describing classroom interactions to make meaning of the situation for the purpose of learning about and informing their teaching practices. Van Es and Sherin (2002) stated, "Teachers must use their knowledge of the subject matter, knowledge of how students think of the subject matter, as well as knowledge of their local context to reason about events as they unfold" (pp. 574–575). Teachers with expertise in professional noticing can provide a detailed analysis of students' understanding. There are three types of interpreting: elaborated analysis of students' understanding, analysis of students' understanding, and analysis with alternative foci (Jacobs et al., 2007). Jacobs et al. found that when experienced teachers watched a video of a classroom they were less likely to provide an analysis with an alternative focus, while beginning teachers overwhelming analyzed with an alternative focus. In other words, the beginning teachers made comments not related to students' understandings (e.g., one beginning teacher identified a pair of students who worked well together).

An aspect to bridging the classroom context to what happened in the classroom includes taking note of its culture. Cobb (2000) acknowledged, "Viewed against the background of classroom social and sociomathematical norms, the mathematical practices established by a classroom community can be seen to constitute the immediate, local situations of the students' development" (p. 73). The community that exists in a classroom is part of the context in which classroom interactions take place. Thus, an aspect of interpreting includes understanding the context.

Responding in instructional situations. Mason (2002) acknowledged that noticing and acting were intrinsically related as he defined the words *reflecting-through-action* as "becoming aware of one’s practice through the act of engaging in that practice” (p. 35). Teachers’ actions depend on what they notice. Mason and Spence (1999) made the argument that “knowing-to” requires a person to act creatively rather than reacting to stimuli based on habit. In order to act creatively a person needs to be aware of the moment. Mason and Spence acknowledged that “no-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform” (p. 135). Teachers use what they notice about students’ mathematical thinking to make decisions about how to act. The complex nature of the classroom requires teachers to make moment-to-moment decisions. “Responding is also a complex—and not wholly generic—task. It requires being able to hear and interpret what the students saying” (Ball, Lubienski, & Mewborn, 2001). I agreed with Jacobs et al. (2007) that, “responding should be considered a skill for professional noticing of children’s mathematical thinking because identifying, describing, interpreting, and responding are all interrelated” (p. 8).

Other considerations for professional noticing. Underlying what teachers notice is what they listen to during classroom interactions. Furthermore, what teachers notice is related to the ways they listen to students. Listening affects all four of the skills for professional noticing. Teacher listening is more than hearing noise; teacher listening includes noticing any product produced by students (e.g., student talk, student work, and student actions) (D'Ambrosio, 2004). Davis (1997) argued “that an attentiveness to how mathematics teachers listen may be a worthwhile route to pursue as we seek to understand and, consequently, to help teachers better understand their practice” (p. 356).

D’Ambrosio (2004) described her vision of a constructivist teacher as one who incorporated multiple voices during classroom interactions. These multiple voices include the voice of the discipline, voices of students, and the teacher’s inner voice. The voice of the discipline is the teacher's understanding of the content, and the voices of students are the students' understanding of the content. The inner voice of the teacher develops during interactions as the teacher integrates the voice of the discipline and the voices of students. The inner voice incorporates the teacher's pedagogical content knowledge. "The inner voice of the teacher includes the teacher's ability to 'unpack' formal mathematics in order to understand children's mathematics and to build a working model of the children's understanding" (p. 137).

Through Davis's (1997) 2-year collaborative work with a third-year teacher, Wendy, he identified three types of listening: evaluative, interpretive, and hermeneutic. Davis analyzed the dialogue in Wendy's class to classify instances of these three types of listening. In his study, Wendy transitioned through these three types of listening while collaboratively working with Davis. Early in the study, Wendy was listening *for* particular responses, rather than listening *to* students. This was an example of evaluative listening. In evaluative listening the teacher is not interested specifically in what students are saying, but she is listening for something in particular. In other words, the teacher knows the answer and is listening for that answer. If the answer is not given, the teacher interjects the correct answer. In evaluative listening, the teacher uses only the voice of the discipline to examine the students' mathematical thinking. In other words, the teacher uses her understanding of the discipline to interpret students' mathematical thinking. D'Ambrosio (2004) and Crespo (2000) studied written correspondences (through email and letter writing, respectively) between preservice teachers and sixth- and fourth-grade students, respectively. They both found that at the beginning of their studies preservice teachers were evaluative listening – focusing on correctness of answers. D'Ambrosio cautions that evaluative listening is not sufficient for understanding students' mathematical thinking.

Davis (1997) reported that as his study progressed, Wendy, the participant, began listening *to* students as well as listening *for* particular responses. Similarly, in both D'Ambrosio's (2004) and Crespo's (2000) studies, some of their participants transitioned to a focus on making meaning of students' mathematical thinking. Interpretive listening occurs when the teacher has a particular response in mind, but is also actively making sense of student responses. The types of questions teachers ask are for the purpose of gaining information and require students to provide more elaborate answers. Teachers use the voice of the student to examine his or her mathematical thinking. The teacher makes an effort to consider the ways the student is engaging in the mathematics and "tries to put herself in the child's place" (D'Ambrosio, p. 139).

Towards the end of Davis's (1997) study, Wendy was actively exploring the mathematics with the students, and a collective authority developed (versus a teacher authority). This type of listening was the final type of listening: hermeneutic. Davis (1997) described hermeneutic listening as "an imaginative participation in the formation and transformation of experience" (p. 369). "The notion of hermeneutic listening was intended to imply an attentiveness to the historical and contextual situations of one's actions and interactions" (p. 370). The teacher integrated the voice of the discipline, the voices of the students, and her inner voice to develop an understanding of students' mathematical thinking. The distinguishing factor was that the teacher actively made sense of the students' mathematical thinking while also actively making sense of the discipline for herself. Hermeneutic listening provided opportunities for the teacher to learn mathematics through interpreting the ways students' think mathematically. D'Ambrosio (2004) and Crespo (2000) did not find evidence that the preservice teachers they studied were at the point of hermeneutic listening.

In summary, in evaluative listening the teacher is not making any effort to understand students' mathematical thinking, whereas in interpretive and hermeneutic listening the teacher is actively trying to make sense of that thinking. The way that teachers listen in classrooms determines what they notice. It defines what aspect of classroom interactions that teachers mark as noteworthy, what they are able to describe, and thus how they interpret and, consequently, how they act. I expect teachers who listen hermeneutically to have more elaborated details of what they notice during class, whereas teachers who listen evaluatively have limited insights into the ways students think mathematically.

Summary of professional noticing. Researchers report teachers' professional growth when professional development experiences focus teachers on noticing students' mathematical thinking (Crespo, 2000; Sherin & van Es, 2005; van Es & Sherin, 2002). In those studies the teachers examined student thinking detached from the complexities of the classroom through the use of teaching artifacts (i.e., watched videos or analyzed student work). Furthermore, a lot of the literature surveyed related to expert and novice teachers' noticing. The second research question in the present study explicitly asked what teachers notice about students' mathematical thinking. Specifically, I wanted to know what teachers notice during the act of teaching, not while watching a classroom video. Jacobs et al.'s (2007) description of professional noticing provided four skills for noticing that were interrelated. The fourth skill, responding, was specifically related to the third research question. "Teachers not only must detect children's ideas that are embedded in comments, questions, notations, and actions but must also make sense of what they observe in meaningful ways to use that information in deciding how to respond" (p. 2). Because responding was related to the third research question the following section further elaborates on responding.

Teachers' Responding in Instructional Situations

Once a teacher recognizes students' mathematical thinking in classroom interactions she or he then makes a decision to respond with an action or not. When a teacher intentially decides to not respond is considered a teacher action. The other three skills identified for professional noticing are directly related to when and how a teacher responds to classroom interactions. Teacher actions in classrooms affect whether the cognitive demand of a mathematics task remains high or declines (Stein, Smith, Henningsen, & Silver, 2000). Teachers can choose a high-level task, but the teachers can lower the level by their actions during classroom interactions. Teachers can take away students' opportunities to think and learn by telling students how to solve the task. In contrast, teachers might maintain the cognitive demand of a task by questioning a student in such a way that it scaffolds student thinking and presses for justification and meaning. In this section, I describe the ways noted in the literature that teachers have responded during classroom interactions.

In a case study about a teacher using students' mathematical thinking, Doerr (2006) found that the teacher responded to students' alternative solution strategies in several ways. The first identified action was the teacher set the expectation for students by continuously telling them the task did not have an immediate and obvious solution and to "work hard." The teacher also continuously focused the students on the difficult aspects of the task by reminding them to find the equation. Third, the teacher listened interpretively as defined by Davis (1997). This interpretive stance allowed the teacher to identify how students were making sense of the task and to focus their attention in a productive way. The fourth action was that the teacher questioned the students, pushing for descriptions and explanations. Lastly, the teacher asked particular students to share solutions by putting the solutions on the board, and then she engaged the class in a discussion about the two solutions.

Fernandez (1997) studied nine teachers' reactions to unanticipated student responses. She found that the teachers acted in five ways—generating counterexamples, following the mathematical thought through to either a contradiction or a solution, suggesting a simpler but similar problem, understanding and incorporating students' thinking, and understanding and incorporating an alternative method (in this final action the teacher does not initially understand the students' thinking). The participants were selected based on their close alignment of beliefs with the *Standards* document of the National Council of Teachers of Mathematics (NCTM, 1989). During the study, some of the participants expressed "Traditional/*Standards*" conflicts. These struggles pertained to whether to emphasize "traditional" or "*Standards*-like" instruction were either within themselves, between them and the students, or among the students. Fernandez related her findings to techniques generally used in problem-solving and suggested that prospective teachers could benefit from considering classroom situations as problem solving opportunities.

In an effort to connect the instructional design theory of Realistic Mathematics Education to teaching practices, Rasmussen and Marrongelle (2006) elaborated on two pedagogical content tools: transformational records and generative alternatives. They defined a *pedagogical content tool* as "a device, such as a graph, diagram, equation, or verbal statement, that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward" (p. 389). Teachers use these pedagogical content tools in their instructional practices, and I consider them to be related to teacher actions. Both transformational records and generative alternatives are tools that teachers intentionally use to support and build on students' mathematical thinking. Transformational records appear when the teacher records student thinking through diagrams and other graphical representations. Generative alternatives appear when the teacher use alternative symbolic expressions to promote particular social norms for explanations. In both of these pedagogical content tools, the intent is for the teacher to initiate the use of the tool to connect and build on students' mathematical thinking while encouraging students to use the tools to solve new problems. In the present study, I wanted to see whether teachers acted in ways that aligned with these pedagogical content tools and whether other pedagogical content tools emerged.

Wood et al. (2006) studied the relationship between instructional patterns and students' mathematical thinking. They found 17 instructional patterns. Of these, there were 13 that required a teacher action, while the others required students to act. The 13 types of interaction patterns are the following: (1) the teacher collects answers (2) initiation, reply, and evaluation (Hoetkey & Ahlbrandt, 1969), the teacher ask questions to see if students know what they were expected to know; (3) funnel (Bauersfeld, 1980), the teacher corrects incorrect student answers by telling the students the correct answer; (4) the teacher explains information to students; (5) the teacher gives a hint that takes away part of the challenge of the mathematics task; (6) the teacher elaborates on a student explanation; (7) focus (Wood, 1994), the teacher brings attention to critical aspects of the task; (8) the teacher or students ask questions to understand a student solution; (9) the teacher uses materials to solve the task or focuses students on using the materials; (10) the teacher selects a student with a correct solution to share with the class; (11) the teacher facilitates discussion to move students to a shared understanding; (12) the teacher checks for questions or comments; and (13) the teacher facilitates conceptual understanding through questioning. Wood et al. also found that "those interaction patterns that required greater involvement from the participants were related to higher levels of expressed mathematical thinking by children" (p. 248).

Hiebert and Wearne (1993) studied the relationship between teaching and learning mathematics in two classes: One used an alternative to the textbook approach, and the other used the traditional textbook approach. They analyzed the nature of the classroom discourse by looking at who spoke and the kinds of questions the teacher asked. They identified four broad categories of teacher questions: "those requesting recitation of previously taught facts or procedures, those asking students to describe invented solution strategies, those requesting students to generate problems, and those asking students to explain why things work like they do" (p. 401).

Heid et al. (1999) found that teachers' actions were rooted in their perceptions of students' educational and emotional needs. LeAnn, one teacher, listened evaluatively (Davis, 1997). Her questioning technique became increasingly directed until a student produced the desired answer. Bill, another teacher, was conscious of the students' feelings and did not want to embarrass students, so he did not probe them further when they struggled. Sarah, the third teacher, valued the process, so she resituated students to help students when they struggled. These three teachers' actions all were actions that a teacher can take when interacting with students.

These studies present a range of responses teachers have made during classroom interactions. Some of the teacher responses lower the cognitive demand of the mathematics task, and others maintain the cognitive demand (Henningsen & Stein, 1997). Furthermore, some of the responses are more "mathematical" than others. For example, Fernandez describes teachers' responses as problem solving (e.g., generate counterexamples). In constrast, telling students to "work hard" is not mathematical but may be necessary for students to realize that some mathematics tasks take significant time and effort to solve. In my analysis of the literature, four categories emerged: teacher telling, getting students to share, questioning, and pedagogical content tools. These four categories are connected and may occur simultaneously. For example, a teacher may ask a directive question while simultaneously telling students the expected answer.

Teacher telling. The kinds of telling range from the teacher explaining a student solution to the teacher describing the steps in a solution. Some telling is specific (e.g., tell students to use certain materials to solve a task), whereas other telling is vague (e.g., tell students to "work hard"). An important way to interpret instances of telling is in terms of what the telling allows or does not allow students to do. For example, Henningsen and Stein (1997) found instances when teacher telling lowered the cognitive demand of a mathematics task. The teacher's hint pointed the students in the direction of a solution, and the hint took away the students' opportunities to engage in "doing mathematics." In the same study, they also found instances that the cognitive demand of a task remained high when the teacher modeled high-level performance, which I consider a form of telling.

There are moments when "teacher telling" is appropriate and necessary. Chazan and Ball (1999) brought attention to the tensions that teachers develop in response to interpretations of the NCTM's (1989, 1991, 2000) vision for classroom discourse. "Teacher telling" has developed an image that represents traditional teaching, which has typically been viewed negatively in the mathematics education community. Chazan and Ball challenged that view and noted instances in their own teaching that they found telling to be appropriate and necessary. For example, they noted that telling can be appropriate to manage disagreements, but the kind of telling they described was different than what one might think. It was not telling students the "correct" answer. Instead, it was for the purpose of contributing to and shaping the discussion. Lobato, Clarke, and Ellis (2005) reformulated telling with three suggestions for examining teacher telling: "(a) in terms of the function rather than the form of teachers' communicative acts, (b) in terms of the conceptual rather than the procedural content of the new information, and (c) in terms of its relationship to other actions rather than as an isolated action" (p. 107). This reformulation moves away from describing the kinds of telling to putting telling in the context of the discussion and focuses attention on developing mathematical understanding.

Getting students to share. There were not as many instances in the literature of teachers getting students to share. Two that stand out are getting students to present the correct answer and getting students to elaborate, explain, and compare their solutions. The first type is for the purpose of making sure everyone knows the correct answer. The latter is to explore a student's solution. Henningsen & Stein (1997) found that when teachers pressed students to explain their answers, high cognitive demand was maintained.

Teacher questioning. Teacher questioning is reported in almost all of the studies. The kinds of questioning include assessing student understanding, learning how students are thinking, pushing students to elaborate and explain, and directing students to an answer. Mason (2000) noted that "questions arise as pedagogic instruments both for engaging students in and assessing students' grasp of, ideas and techniques" (p. 97). He identified three forms of asking: focusing attention, testing, and enquiry. Focusing attention is when the teacher notices something that he or she deems important for students to notice and therefore asks questions to focus students' attention. When teachers focus students' attention, the funneling effect may happen (Mason, 2000). The funneling effect is when the teacher becomes more and more directive in order to get students to see what the teacher sees. Testing is when the teacher asks questions to find out what students know. This type of questioning includes asking students to restate, elaborate, or make connections among their ideas. The last kind of questioning, enquiry, occurs when the teacher is genuinely interested in students' thinking. These questions seem to fit Davis's (1997) hermeneutic listening. The questions engage the students, along with the teacher, in inquiry.

Pedagogical content tools. Rasmussen and Marrongelle (2006) defined the terms *pedagogical content tool* as a device that teachers use to connect to students' mathematical thinking while advancing mathematical ideas. I considered these devices as ways that teachers can respond to students' mathematical thinking in classrooms. Furthermore, teachers use pedagogical content tools, but the intent is for students to develop the capabilities of using these tools themselves to solve future problems. Calling these devices pedagogical content tools focuses one's attention on pedagogy and content. The teachers' responses that represent problem-solving strategies in Fernandez's (1997) study fit in this category. For example, students may be struggling with a complex task. When a teacher offers a simpler but similar task she or he is connecting to the student's thinking while moving the mathematics forward. Using a simpler task to solve a more complex task is a problem-solving technique that teachers wish to instill in their students. When a teacher offers that technique as a way to progress on a task, the teacher hopes that she or he is modeling problem-solving skills that students will develop.

The Mathematics Teaching Cycle: Teachers' Knowledge

It is widely accepted in the field of mathematics education that teachers' knowledge of mathematics influences their instructional practices (Thompson, 1984). Teachers' knowledge is a large domain with many facets. Simon (1995b) acknowledged the importance of teachers' mathematical knowledge on teaching practices by identifying one component of the Mathematics Teaching Cycle as teachers' knowledge. He highlighted five kinds of such knowledge: teachers' knowledge of mathematics, their knowledge of mathematical activities and representations, their models of students' knowledge, their conceptions of mathematics learning and teaching, and their knowledge of student learning of particular content. Simon (1995a) noted, "As a teacher, my perception of students' mathematical understandings is structured by my understandings of the mathematics in question. Conversely, what I observe in the students' mathematical thinking affects my understanding of the mathematical ideas involved and their interconnections" (p. 135).

Shulman (1986) brought attention to the kinds of knowledge that teachers need to have in order to do the work of teaching. Specifically, he mentioned content knowledge, curricular knowledge, and pedagogical content knowledge. A year later, he expanded his construct to include seven areas of knowledge: content knowledge, general pedagogical knowledge, pedagogical content knowledge, knowledge of learners and learning, curriculum knowledge, knowledge of educational contexts, and knowledge of educational philosophies, goals, and objectives (Shulman, 1987). Pedagogical content knowledge received particular attention from teacher educators as it opened the door for many discussions about how knowledge of subject matter was related to the work of teaching, and it prompted research studies to investigate that knowledge. Pedagogical content knowledge included the kinds of conceptions students have of particular subject matter. Shulman stated that pedagogical content knowledge included "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). Grossman (1990) elaborated on pedagogical content knowledge and indicated four categories: one of which focused on knowledge of students' understanding, conceptions, and potential misunderstandings.

Ball is recognized in the mathematics education community as one who has contributed greatly to the work on mathematical knowledge needed for teaching. Ball (1997) stated:

> Teaching is essentially about building bridges between students and important ideas of the discipline, between students' experience and the worlds offered by mathematical thought. Therefore, knowledge of students is as essential a resource for effective teaching as is knowledge of mathematics itself. Moreover, the two are not separate in practice: knowledge of students fundamentally depends on mathematical knowledge. And mathematical knowledge must be mobilized in the context of pedagogical issues. (p. 732)

Ball identified three challenges that teachers face when they learn more about what students know, which included developing skills to notice (listening, watching, and studying written work), being generous and skeptical, and realizing that students' understanding was situated in the context in which they learn.

Ball and Cohen (1999) provided a detailed description of the mathematics needed for teaching. First, teachers need to understand mathematics in a way different than how they learned it as students. For example, mathematics teachers "need to know meanings and connections, not just procedures and information" (p. 7). Second, teachers need to have knowledge of students. This knowledge includes understanding "what children are like, what they are likely to find interesting and to have trouble with, in particular domains" (p. 8). Third, teachers need "to learn that knowing students is not simply a matter of knowing individual children" because "teachers often teach children who come from backgrounds different from their own" (p. 9). With this knowledge of students, it is important for teachers to learn ways to listen to and interpret student's ideas. Fourth, teachers need to know pedagogy. "In order to connect students with content in effective ways, teachers need a repertoire of ways to engage learners effectively and the capacity to adapt and shift modes in response to students" (p. 9). Ball and Cohen believe that all of this knowledge is necessary but not sufficient for teaching in the ways promoted by educators and researchers, because it does not take into account the complex and unpredictable interactions that occur in classrooms. Teachers need to be able to inquire into teaching in the moment of teaching and use this knowledge to improve their practice.

In recent work, Hill, Ball, and Schilling (in press) have detailed four kinds of mathematical knowledge needed for teaching: common content knowledge, specialized mathematical knowledge, knowledge of content and teaching, and knowledge of students and content. The first two are considered subject matter knowledge, and the last two are pedagogical content knowledge. All four have an impact on what teachers notice about and how they act on students' mathematical thinking, but knowledge of students and content is particularly important. Knowledge of students and content is "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill et al., in press). Teacher's knowledge of students and content influences teachers' instructional practices. For instance, suppose a teacher knows that a common student error when finding the area of regular polygons is finding the area of half the triangle created by the center and two consecutive vertices instead of finding the area of the entire triangle. If a teacher knows that common error and notices a student making that error, the teacher can address the error. In teaching mathematics, the teacher must be able to notice and act on students' mathematical thinking. The teacher needs to be able to interpret students' mathematical thinking either in the moment or through more interactions with the students. Hill, et al. found four major categories within knowledge of students and content:

- Common student errors – identifying and providing explanations for errors, having a sense for what errors arise with what content;
- Assessing students' understanding of content – interpreting student productions as sufficient to show understanding, deciding which student productions indicate better understanding, etc.;
- Student developmental sequences – the problem types, topics, or mathematical activities that are easier/more difficult at particular ages, what students typically learn "first," what third graders might be able to do, etc.;
- Common student computational strategies – questions pertaining to landmark numbers, fact families, etc.

These four categories were especially useful for the present study because they describe what teachers might notice about students' mathematical thinking (before, during, and after classroom interactions).

Theoretical Discussion

My goal for this study was to conceptualize the ways that teachers understand and use students' mathematical thinking in their instructional practices. To examine ways that teachers understand and use students' mathematical thinking, I focused my attention on what teachers noticed about students' mathematical thinking and the ways that teachers acted on students' mathematical thinking. The interrelated components of Simon's (1995a, 1995b) Mathematics Teaching Cycle illustrated the relationship between teachers' knowledge, learning trajectories, and classroom interactions. Learning trajectories provided a way for me to consider what teachers notice before, during, and after classroom interactions. I consider teacher noticing and teacher actions aspects of classroom interactions. Professional noticing includes what teachers recognize and deem important during classroom interactions. Once a teacher notices students' mathematical thinking, she or he can either act on it or not. To sum up the interactive framework that I used to situate this study, I illustrate the connections among the different constructs to in a modified diagram of Simon's Mathematics Teaching Cycle (see Figure 3). The component of "teacher's knowledge" has a dotted line around it because it was not directly investigated in the study. It was included because it strongly influences the development of learning trajectories and classroom interactions. In exploring the three research questions for this study, I could not ignore teacher's mathematical knowledge for teaching.

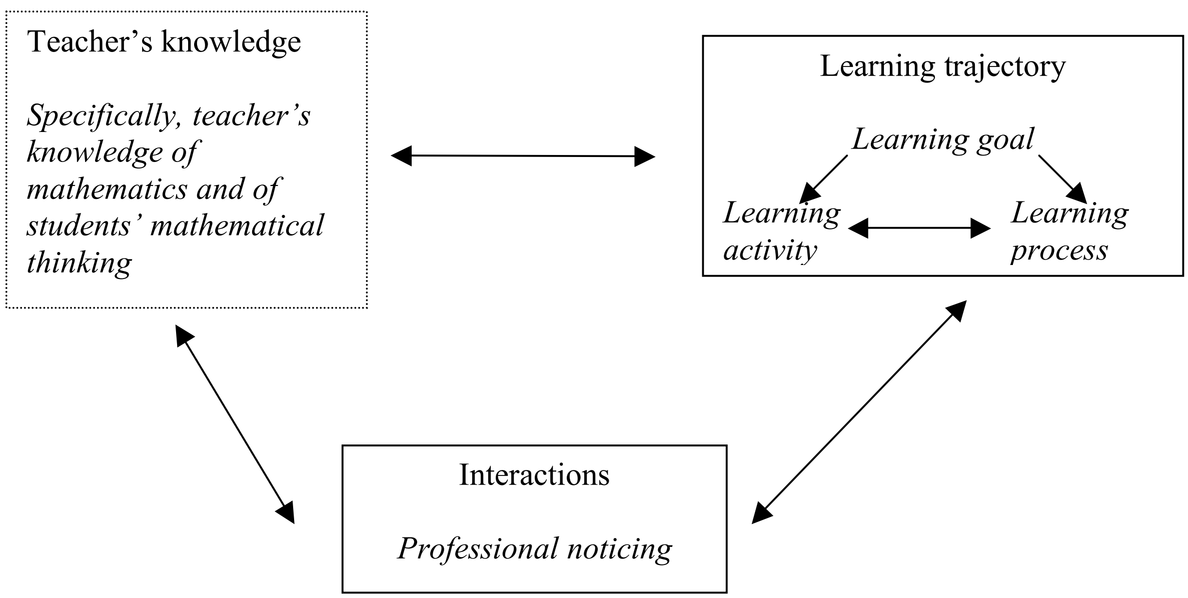

Figure 3. The theoretical framework.

CHAPTER 3

METHOD

For this study, I used learning trajectories and professional noticing to examine how teachers understood and made use of students' mathematics in their teaching practices. To investigate the research questions, I used a qualitative research design with an interpretive approach to data collection and analysis. Interpretive analysis was used to generate learning trajectories for each teacher and to describe the teachers' understanding and use of students' mathematical thinking.

Participant Selection

At the onset of this study, I decided to focus on high school geometry teachers who had at least 5 years of teaching experience and who would be willing to attend working group sessions. It was important to select teachers who were not in their beginning stages of their careers, because I believed that teachers might find it difficult to discuss students' mathematics and limited experience working with students would increase that difficulty. I chose geometry teachers for two reasons. First, I had experience teaching high school geometry, so I was familiar with the mathematics. Second, students in the United States have not been performing well on national and international tests in geometry. Geometry is viewed as an area of need and one that teachers and teacher educators should focus more attention on.

I planned to offer working group sessions every 2 to 3 weeks at one high school, so it would be helpful if the participants worked at the same school. In addition, the teachers needed to be comfortable talking at length with me about their teaching practices and with their colleagues during working group sessions about mathematics and students' mathematical thinking. Furthermore, it was important to select teachers who did not lecture on a regular basis. The teachers needed to interact with their students doing mathematics, and lecturing was not conducive to interaction. To locate participants who met the above description, I did purposeful sampling (Patton, 2002). To assure that I selected teachers with whom I had a strong professional rapport and who would be open and willing to discuss their practice at length, I considered only teachers with whom I had worked for several years. Over the last 3 years, I have supervised student teachers at three high schools. During that time, I developed a professional relationship with a small group of teachers at each of the schools. From that small group, two teachers had the above qualities and were willing to participate.

Data Collection

I collected data for both teachers in one of their semester-long geometry classes during fall 2006. These data included field notes, interviews, videotapes of lessons, teaching artifacts, and a research journal. I transcribed all interviews. I interviewed both teachers at the start of the study to find out background information (see Appendix B for the interview protocol). Over the course of the semester, I observed 13 lessons in one classroom and 16 lessons in the other classroom. The majority of lessons occurred in one day, but there were some that occurred over 2 days. I held six working-group sessions. I took field notes during each classroom observation and working-group session. For six observed lessons in each teacher's classroom, I conducted 30 to 60 minute pre- and post-observation interviews (see Table 1 for dates of all observations and working group sessions). When possible, I observed the teacher's class the day before and the day after the six lessons. The teacher chose four of the six lessons that were connected with the pre- and post-observation interviews. The other two lessons were chosen based on the working-group sessions. In two of the working-group sessions the teachers modified or created tasks to use in their classrooms. The two lessons pertaining to the tasks were part of the six lessons that I conducted interviews before and after the observation. Both the pre- and post-observation interviews were semi-structured and focused on the teachers' descriptions of the components of the learning trajectories and on what the teachers noticed about students' mathematical thinking during classroom interactions (see Appendix C). During the pre-observation interviews, the teachers detailed the learning goal of the lesson and discussed what they planned to have happen in class. They predicted how learning might occur. The teachers also talked about what had happened in class previously and what their students knew about the mathematical ideas in the lesson. The post-observation interviews were based on the classroom observations. In the interviews the teachers reacted to the lesson and discussed their perspectives about students' mathematical thinking during the lesson. The teachers also talked about the direction of the course and what content they planned to address next (sometimes this discussion referred to a lesson, a unit of study, or the course). All of the interviews occurred at the teachers' convenience. The pre-observation interviews occurred on the day of the observation, and the post-observation interviews occurred within 3 days of the observation.

I videotaped the six classroom observations for each teacher and all of the working-group sessions. In addition to the interviews, I digitally recorded and transcribed any conversation that the teachers had with me about their lessons, students, or teaching practices. For instance, one participant's class was split into two time segments, with students going to lunch between the segments. During that lunch break the participant often talked about what she had observed in class, so I recorded those conversations. Also, I collected and analyzed teaching artifacts from the observations and the working-group sessions.

Table 1

Dates of Observations and Working-Group Sessions

Date	Working-group session	Barbara's lessons	Judy's lessons
Thursday, Aug. 24		X	
Thursday, Sept. 7		X	X
Friday, Sept. 8	X – Do mathematics		
Thursday, Sept. 14		X	
Thursday, Sept. 21		X	X
Friday, Sept. 22	X – Watch video		
Monday, Sept. 25		X	X
Tuesday, Sept. 26		X – Area of sectors and arc lengths	X – Area of regular polygons
Thursday, Sept. 28		X	X
Monday, Oct. 2		X	
Monday, Oct. 9	X – Modify task		
Thursday, Oct. 12		X	X
Thursday, Oct. 23		X – Window Task	X – Window Task
Friday, Oct. 24		X – Window Task continued	X – Window Task continued
Monday, Oct. 26	X – Discuss lesson		
Friday, Nov. 3	X – Modify task		
Wednesday, Nov. 8			X
Thursday, Nov. 9			X – Similar figures
Thursday, Nov. 16			X
Friday, Nov. 17			X – Three-dimensional shapes
Monday, Nov. 27		X – Angle and arc relationships in circles	
Tuesday, Nov. 28		X	X – Box Task
Wednesday, Nov. 29		X – Equations of circles and angles outside the circle	X – Box Task continued
Monday, Dec. 4		X	
Tuesday, Dec. 5		X – Transformations	
Thursday, Dec. 14			X – Pi Day
Tuesday, Dec. 19	X – Discuss lesson	X – Box task	

Note. The X identifies the dates that data were collected. The six lessons that I observed and for which I interviewed the teachers are labeled with a lesson title, and the other observed lessons are marked with an X. There are 17 and 15 days of observations for Barbara and Judy, respectively, but only 16 and 13 lessons observed because some lessons occurred over 2 days.

A final data source was a research journal that I kept throughout the study. In the journal I kept track of my thoughts and conjectures as a way to begin the data analysis. I took notes of specific methodological and pedagogical decisions that I made throughout study. The journal documented my hypotheses development and testing throughout the study.

Participant and School Descriptions

Barbara and Judy[2] were the participants in the study. Both teachers were National Board for Professional Teaching Standards certified in high school mathematics and had advanced college degrees. Barbara and Judy taught geometry multiple times over their teaching careers. The two teachers regularly participated in professional activities beyond their high school teaching; for example, they both taught mathematics courses for practicing teachers and hosted student teachers. These two teachers were highly involved in activities at the high school and in the school district. Both teachers missed anywhere from 10 to 16 school days during the study because of illness or professional and family responsibilities. These absences affected the study by limiting the times that I could observe, interview, and hold working-group sessions with the teachers.

The high school was a small public 9–12 high school located in a suburban community in a southern state. The school year was divided into two 18-week terms, and the school operated on a 4 by 4 block (i.e., students took four courses each semester and each class period was 1 ½ hours long). The mathematics classes that the school offered ranged from Prealgebra to Advanced Placement Calculus. There were three tracks for geometry: Informal Geometry, Advanced Euclidean Geometry, and Honors Euclidean Geometry. The major differences between the lower track and the upper two tracks were the pacing and the presence of proofs.

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[2] All names of participants and students are pseudonyms.

The lower track operated at a slower pace than the other two tracks, and students in the lower track were not expected to construct or present proofs of geometric conjectures or theorems. Most of the students in the Honors Euclidean Geometry course were freshmen, whereas most of the students in Advanced Euclidean Geometry were sophomores. The Honors course covered content at a deeper and more advanced level than the Advanced course. Students were placed in the tracks based on their grades in Algebra I and teacher recommendations. Barbara taught an Honors Euclidean Geometry course, and Judy taught a Advanced Euclidean Geometry course.

Barbara

In geometry classes, Barbara believed it was important for students to conceptualize quantities and concepts with a picture. She wanted students to "see" mathematical relationships. Barbara described a typical day in her geometry class as having a set structure: go over homework by getting students to share their solutions and answer each other's questions, use some activating strategy to motivate the lesson (the problem that initiates the concept), begin new material, and give students time to try some problems with their peers or on their own. She described her philosophy of teaching as wanting students to "conceptualize so they don't have to memorize" (interview, 8/24). Barbara wanted students to develop their own knowledge, and her role in the classroom was to help students in that process. She said that a textbook keeps teachers focused narrowly and inhibits them from noticing the big picture. She thought that geometry proofs were difficult for students and that textbooks start too quickly with formal proofs. Barbara deviated from the textbook at the beginning of the school year by starting with relationships in right triangles. Later in the semester this deviation became a big source of frustration because she felt that she had strayed too far from the book.

Barbara tended to use a conceptual approach[3] to teaching, but she sporadically tried an investigative approach (Baroody, Cibulskis, Lai, & Li, 2004; Baroody & Coslick, 1998). During the majority of class time, Barbara's instruction was semi-authoritarian and teacher centered. According to Barbara, in teacher-centered lessons, the teacher communicates directly with the students, and in student-centered lessons, the students collaborate with each other. Her goal for students was to develop procedural and conceptual understanding. Barbara would pose a leading question and would hope she had at least one student who would answer it in a way that she believed was best. She welcomed multiple responses to the questions, but ultimately she was searching for particular answers. Similarly, she welcomed multiple student solutions, and there were times that multiple solutions were equally valued. For example, in one of the lessons the students found the area under a curve. All the groups shared their solution strategies. Barbara did not emphasize one solution over another. She did, however, want students to notice different solution paths. At other times, she encouraged students to share multiple solutions, though she hoped that someone would share a particular solution that she expected as a sort of standard solution or one that she saw as the curricular emphasis of the lesson. For example, in a lesson about the volume of the box she wanted a student or group of students to come up with the formula for the volume. Barbara asked leading questions to push students to think about an algebraic solution instead of numerical solutions. For example, she asked the students if they could represent the surface area using x. Barbara welcomed other solutions, but she emphasized on the student solution that included an equation.

[3] In the conceptual approach the teacher serves as a guide and wants students to understand prescribed material (Baroody et al., 2004; Baroody & Coslick, 1998). In this approach the teacher listens interpretively to assess understanding. In the investigative approach the teacher wants to foster mathematical power through inquiry. In this approach the teacher listens hermeneutically to understand students' thinking.

Barbara thought students needed to investigate and to have hands-on experiences to learn geometry. She wanted to be more student-centered, but she felt pressure to cover the content and had limited time to plan her lessons. She noted that planning investigative activities took extra planning time and extra class time, both of which she felt she did not have enough of. She tried an investigative approach in a lesson about maximizing the perimeter and area of a rectangle (the Window Task). Her approach to instruction was more in line with the investigative approach on the first day. She allowed the students to explore as she walked from group to group observing students, offering comments, or posing questions to help students. Her questioning was open and not directive. She decided that the students did not get as far as she had hoped on the first day, nor did they go in the direction that she thought they would. So she shifted her approach on the second day to be more directive. She was leading with her questions and comments. Barbara wanted the students to understand the algebraic equation and the graphical representation for the task.

I observed Barbara's classroom for 16 lessons (17 days). The summaries of the six observations with pre- and post-observation interviews are included in Table 2. Barbara missed 2 weeks of school in November, which affected the days I observed her class. She taught an Honors Euclidean Geometry class with a total of 24 students (11 boys and 13 girls). The desks were aligned in pairs facing a whiteboard, and the students were allowed and encouraged to work with their partner (a person sitting next to them) on most activities.

Table 2

Dates and Summaries of Observations with Interviews of Barbara's Lessons

Date	Lesson
September 26	*Area of sectors and arc lengths*: Class started with Barbara and some students solving homework problems on the board. Then Barbara led the class through developing the formula for finding the area of a sector and thearc length of that sector for any given angle in a circle. Barbara passed out the area investigation (see Appendix D) that asked the students to find the area of a shaded region and the nonshaded region. Barbara led a discussion about the task and brought up the idea of geometric mean.
October 23–24	*Window Task*: Barbara started class by allowing students to complete their midterm tests. Then Barbara introduced the Window Task (see Appendix E). The remainder of the first day was for students to solve the task in groups. On the second day Barbara led a group discussion about the table of measurements and the algebraic solution.
November 27	*Angle and arc relationships in circles*: Barbara began class by asking students to draw a circle. Afterwards Barbara asked students to tell her everything that they knew about circles. From there she developed the vocabulary of *central angle*. Barbara asked students to inscribe a regular hexagon in a circle. She asked leading questions to discuss various theorems pertaining to angles and chords in circles.
November 29	*Equation of circles and angles outside the circle*: Barbara started class by going over homework. Then she asked students to find the equation of a circle when given the equation of a line tangent to the circle, the coordinates of the point of tangency, and the coordinates of the center. The class discussed the task and was given time to work text book problems. Then, the class was interrupted by a fire drill.
December 5	*Transformations*: Students started working on a vector problem at the beginning of class. Barbara introduced transformational geometry by asking students what they have previously learning about transformations. She passed out a worksheet (see Appendix F) and the class went back and forth between Barbara leading a discussion and the class working on the worksheet.
December 19	*Box Task*: Barbara wanted students to participate in a mathematical investigation and to recognize relationships between geometry and algebra. She introduced the task by talking about design professions and the need for companies to maximize volume for specific surface area. Then she gave students the task and asked each student to create one box, find the length measurements, calculate the area and volume for their box. Once students completed the task, Barbara led the students through the solution by asking specific questions.

Judy

Judy thought that geometry was much more concrete than algebra and noted that geometric thinking included thinking pictorially and finding relationship through diagrams. A typical day in Judy's class began by going over homework. During that time she would try to figure out what the students understood and what they were struggling with. After going over homework, she would try to do an exploratory activity with students. Judy stated, "I can tell them [students] how to do something much quicker than they could figure it out, but they don't remember it, and they don't always understand" (interview, Judy, 8/24/06). She thought that a hands-on approach was more interesting and that students were able to remember and understand more that way. She wanted students to understand mathematics conceptually. Judy said she had faith that her students could learn geometry and that it was her role to provide opportunities for them to learn and to support them through the process. "Learning mathematics should be fun" (interview, Judy, 8/24/06). Judy wanted to create fun activities for students to explore. When planning, she thought about where she wanted her students to go, and then she tried to come up with activities that would get them there. Judy said it was important to make planning decisions based on students' thinking. She thought her geometry courses were different each semester because the students were different, and thus her instruction was different. Students and their solutions intrigued Judy. During interviews she said she was often excited and curious about how students might solve a task. This excitement was also visible in post-observation interviews as she discussed students' solutions.

Judy's approach to teaching mathematics was an investigative approach (Baroody et al., 2004; Baroody & Coslick, 1998) for the majority of her lessons. Her lessons were semi-democratic and student centered. In other words, the students influenced the direction of the lesson. She wanted her students to develop procedural and conceptual understanding. She often talked about finding mathematics tasks to use in class that would motivate a procedure. Typically, Judy set up a task and allowed the students to explore it. She valued multiple solutions. If she noticed students approaching a task in a way that she did not think was viable, she posed questions or gave another task that encouraged the students to reconsider their solution method. For example, in one lesson the students were learning about how to find the area of regular polygons. Judy did not give a formula or instructions. Instead, she asked students to use what they knew about area to find the area of various regular polygons. The students solved the tasks and shared their solutions. During the whole-class discussion, the students determined which solutions were viable. Judy did not favor one solution over another. The students did not learn the typical formula for the area of regular polygons ($A = ½ P a$), because she did not think they were ready to make that generalization. Judy did think that some students were showing evidence of leading to the generalization, but the students fell short.

I observed 13 (15 days) lessons in Judy's classroom. The summaries of the six observations with pre- and post-observation interviews are included in Table 3. Judy taught an Advanced Euclidean Geometry class with a total of 21 students (12 boys and 9 girls). The desks were aligned in pairs facing a whiteboard, and the students were allowed and encouraged to work with their partner (a person sitting next to them) on most activities.

Table 3

Dates and Summaries of Observations with Interviews of Judy's Lessons

Date	Lesson
September 26	*Area of regular polygons*: Judy began class by asking students what the minimum amount of information was needed to find the area of a regular polygon. Students made hypotheses. Then Judy asked students to find the area of three regular polygons, each of which had different given information (apothem length, side length, and radius length). Afterwards she asked students to put their solutions on the board, and the class discussed the solutions. Students took a quiz for the remainder of class.
October 23–24	*Window Task*: Judy introduced the Window Task (see Appendix E). The remainder of the first day was for students to solve the task in groups. On the second day students used Geometer's Sketchpad (GSP) (Jackiw, 1991) to explore the task further. Afterwards, Judy changed the task to finding the maximum area when the perimeter was 24.
November 9	*Similar figures*: Judy started class by giving a warm-up and checking homework. Students put solutions to warm-up on board. Judy passed out a worksheet for students to decide if pairs of figures were similar (see Appendix G). Afterwards, students put solutions on board, and Judy led a discussion of those solutions. From that discussion she asked students to write similarity statements for the figures that were similar, and she asked some questions about when figures were similar.
November 17	*Three-dimensional shapes*: Judy gave all students 10 cubes to work with. She showed students a three-dimensional figure quickly and asked students to make that figure with their cubes. She asked students to describe their figures and asked how many cubes were necessary to create the figure. Then, she asked students to draw the figures from different perspectives (top, bottom, front, back, left, right). Students were then asked to build a figure that represented the given views. The last few minutes of class students worked on a worksheet that asked them to tell how many cubes were in the figure.
November 28–29	*Box Task*: Judy began class by going over homework. Then she introduced the Box Task. She explained to students that they needed to use a sheet of paper to create a box. Then students needed to find the length, width, and height of the box that can be created with the largest volume. Students worked in groups to solve the task. On the second day the class created a table of measurements and looked at the graphs for the area and for the volume. For homework Judy asked students if they could think of a way to generalize what they were doing by creating an equation.
December 14	*Pi Day*: Judy had three tasks that she wanted to work on. The first one was not mathematical. The second one was determining the size of a tree from a newspaper article, and the third one related to putting a cord around the earth. The students worked on the tasks for the majority of the class and Judy went around to the different groups to observe and help students. Toward the end of class Judy led a class discussion about each task.

Working-Group Sessions

The complexities of the classroom make it difficult for teachers to focus specifically on students' mathematics. With that in mind, I held six working-group sessions that lasted 60 to 90 minutes every 2 to 3 weeks. All 7 mathematics teachers at the school were invited to the sessions, but only Barbara and Judy attended. The 2 to 3 weeks between the working-group sessions allowed time for me to plan future sessions as well as to engage in ongoing analysis. I planned and led all of the sessions, and the curriculum for those sessions developed over the course of the semester as teacher needs and my needs as a researcher surfaced. The purpose of the working group sessions was not to change the two teachers' instructional practices, but to support them in their efforts to discuss their students' mathematical thinking. The tasks in the working-group sessions were based on the literature on professional development (Ball & Cohen, 1999; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003; Smith, 2001). These sessions encouraged the two teachers to collaborate and focused on building their knowledge about content and students' mathematical thinking. My planning for and my actions in the working-group sessions were informed by the constructs learning trajectories and professional noticing, but these ideas were not explicitly discussed. In the sessions the teachers solved mathematics tasks, watched a video of students solving a task, create task to use in their own classrooms, and discussed their students' solutions to the tasks (see Table 4 for short summaries). The overall idea in the working-group sessions was to move the teachers from discussing their own mathematics to discussing their students' mathematics.

Table 4

Dates and Summaries of Working-Group Sessions

Date	Session
September 8	The purpose was for the teachers to discuss mathematics by engaging in a mathematics task. The teachers worked on a task that asked them to compare the ratio of a smaller square to a larger square (see Appendix H). The small square was inside the larger square and was created using the vertices of the larger square. The teachers solved the task using Pythagorean theorem, GSP, and an algebra equation. Judy was quick to look for a geometric solution and Barbara wanted an algebraic solution.
September 22	The purpose of this session was to engage teachers in mathematics again, but then to observe students in a video solving the same task (WGBH Education Foundation, 1996). The task asked teachers to draw a scalene triangle and cut the triangle one time to make a parallelogram. The teachers were asked to share their solutions and to share strategies that students might use to solve this task. The teachers watched a video and discussed their observations.
October 9	The purpose of this session was for the teachers to create or modify a task to use in their classrooms. The teachers selected a task from *The Mathematics Teacher* (NCTM, 1998) to modify. The modified task was called the Window Task (see Appendix E).
October 27	The purpose of this session was to analyze student work from the Window Task. The teachers shared what happened in their lessons. In addition, the teachers shared three student solutions and discussed what their students were thinking mathematically.
November 3	The purpose of this session was for the teachers to create or modify a task to use in their classrooms. They decided to use a task that Barbara typically uses in her geometry classes, which we called the Box Task. In the box task students are asked to find the maximum volume of a box created by cutting squares out of the four corners of a sheet of paper.
December 19	The purpose of this session was to analyze student work from the Box Task. The teachers shared what happened in their lessons. In addition, the teachers shared student solutions and discussed what their students were thinking mathematically.

In Sessions 3 and 5 the teachers created or modified a task to use in their own classrooms. The two lessons that included those tasks were part of the six lessons that included pre- and post-interviews. For both teachers, the lessons on those two tasks were productive and are cited often in the results chapter, so I give more detail in the next two sections.

The Window Task

The Window Task that the teachers worked on is given in Figure 4. In the working-group session the teachers modified the task to use with their students and created a handout for students (see Appendix E). This task explored the maximum perimeter and area of a rectangle inscribed in a triangle. The two solution paths that were discussed by the teachers included a measurement solution and an algebraic solution (see Appendix I).

Given an equilateral triangle ABD with side length 10.5 and a rectangle EFGC. Suppose point C is allowed to assume different positions on the line segment BD (as C changes positions the points E, F, and G also change accordingly.) Is there a position for point C along line segment BD where the perimeter of the window EFGC is the largest? … area largest?

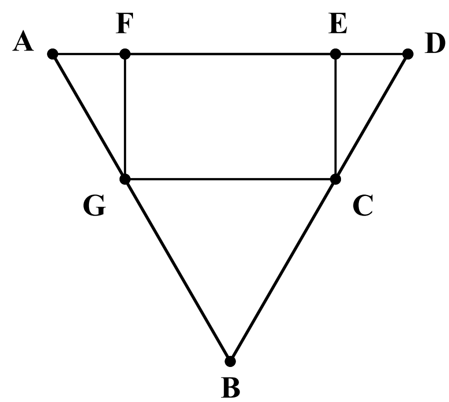

Figure 4. The window task.

This lesson spanned 2 days in both teachers' classrooms. The learning goals varied for the two teachers. Barbara's goal for her lesson was for students to engage in a culminating activity that incorporated a lot of concepts from the semester. She expected her stronger students to quickly generalize and use an algebraic representation to support their answers. In addition, she expected some of her students to struggle when beginning the task. There was another group of students who she expected would use a measurement strategy. Even though she thought they would use measurement, she also thought they would realize that they needed an algebraic solution in order to prove their answer. These different solution approaches signified that Barbara was considering three learning trajectories for her students. She thought her students would think differently about the task and would vary in what they learned from engaging in the task.

Judy wanted her students to explore the idea of maximum area and perimeter with some organized investigation. She expected her students to use a measurement strategy, and she hoped they would justify their solutions. Judy acknowledged that some of her students would approach the task at a deeper level, but the multiple solutions she described all dealt with the same solution strategy and the same learning trajectory (just at a different pace).

During the first day of the lessons the two teachers' classrooms looked similar, with corresponding teacher moves and corresponding student solutions. The teachers introduced the task in a similar manner by putting a transparency of the Window Task on the overhead projector. They described the task and posed questions to help students begin. Both teachers gave very little guidance about possible solution paths. Students were asked to work individually for 10 minutes to think about possible solutions. After that time the students worked in groups to solve the task. The students solved the task with some variation of a measurement strategy. On the second day there were noticeable differences between the two classes.

Barbara was disappointed that no students had approached the task using algebra and that many students solved the task with very little data to support their solutions. On the second day She became more prescriptive in her interactions with students. She created a table of student measurements on the board. She commented to the students that a measurement solution required a lot of numerical data. Afterwards she told the students that the difficulty with measurement was that it does not prove something were true. Then she asked students if there was a way to represent the changing length of segment CE. The students quickly answered by labeling it x. The class came up with an equation and graphed that equation on a calculator to see the maximum value for the function.

Judy's class on the second day was different. She gave the students 10 minutes to work again in their groups. Then, students explored the task using a teacher-prepared sketch on Geometer's Sketchpad (GSP) (Jackiw, 1991). During that time Judy walked from group to group listening to conversations. Following this activity, she asked particular students to share their solutions. The class looked at a table of values and graphed the values. Judy noticed that many of her students were not connecting the table values to the graphed values, so she decided to change the task to finding the maximum area for a rectangle with a constant perimeter of 24 (the perimeter in the Window Task varied).

The Box Task

The Box Task is shown in Figure 5. The task asks for the maximum surface area and volume of a box created by a sheet of paper.

On the last day of the semester Barbara gave her students the Box Task. The class period was extended because of a modified schedule, so the class lasted 2 hours. She wanted the students to see a problem from the initial statement, to an informal investigation, and then to a formal investigation. She thought the students needed to experience the connections between geometry and algebra, and she wanted them to use a graphing calculator. Barbara began class by discussing professions that require knowledge of design and the possibility that one of the students in class might one day work one of those professions. Then she posed the task. In

Given a sheet of paper with length 8 inches by width 11 inches. What is the maximum surface area and volume of a box that is created by cutting squares from the corners and folding the sides of the box?

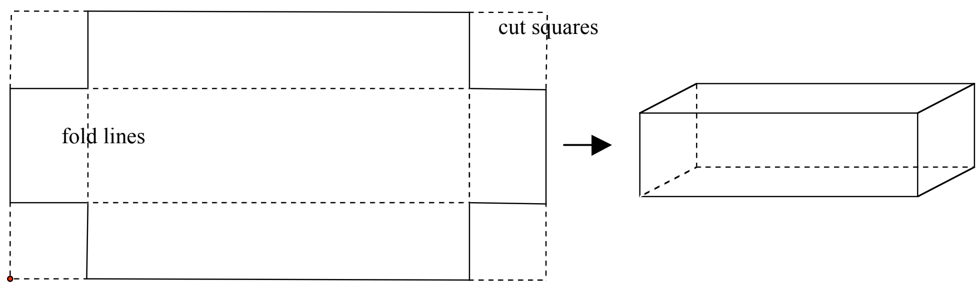

Figure 5. The box task.

posing it, she asked each student to create one box. Once students had created the box, they were supposed to find the measurements for the length, width, and height in order to find the volume. After students had their measurements, they recorded those measurements on the board. Then Barbara led the class through a set of instructions to enter the data in a graphing calculator. She explained how to use a graphing calculator to graph the data and to find regression equations for the data. From there, Barbara asked, "Can you represent the surface area using x" (field notes, Barbara, 12/19/06). The class created the equation for surface area and for volume. The class entered their equations in the calculator and looked at the graph. Barbara led the class in a discussion about domain and range. For the majority of this class period Barbara led the discussion and decided the direction of the lesson.

Judy's lesson on the Box Task lasted 2 days. During the pre-interview Judy had not completely planned her lesson and was worried she would not do a good job introducing the task. She wanted students to organize the data and make connections between dimensions of the box and the size of the surface area and volume. Judy introduced the task by asking students how they might create a box with a sheet of paper. The discussion brought up ideas about nets from the previous day. Eventually, one student suggested cutting the corners in order to fold the paper to get a box. Judy demonstrated what would happen if rectangles were cut instead of squares. Then she asked students to predict what they think happens to the surface area and volume when the box changed sizes. Judy asked students to explore with their partners and she gave them paper and scissors. She noticed that some students quickly realized that they did not have to cut squares in order to find the measurements. One pair of students became the center of attention in class because they debated whose box had the greatest volume – one student had a tall box with a small base, and the other student had a short box with a large base. At the end of the first day, Judy asked the students to use the process they were using in class to find the volume of five boxes for the next day. The second day began with students putting some of their measurements on the board and sharing conjectures they made about the surface area and volume. The pair of students who were debating their volumes became the center of attention again. The two had decided which one had the larger volume, and the class discussed the reasons for the larger volume. The class entered the data in the calculator and looked at the graph. One student mentioned that the volume was a cubic function because it was length times width times height. Another student shared his idea that you did not need to cut the squares as long as you knew the size of the paper and the size of the square. Judy asked students questions about a way to generalize what they were saying. For example, Judy asked, "So you said you don't need to cut out the squares. Well, how do you know what the length and width are?" (field notes, Judy, 11/30/06). A few groups began coming up with equations to represent the surface area and volume. For homework, Judy asked the students to see if they could come up with equations to represent the surface area and volume. The next day Judy collected these equations but did not discuss them.

Data Analysis

Data were analyzed based on the learning trajectory framework and an interpretative stance in order to examine "how individuals experience and interact with their social world, [and] the meaning it has for them" (Merriam & Associates, 2002, p. 4). In this study, my objective was to explain the teacher's perspective from the researchers' perspective, which involved the following:

> The researcher attempts to understand and articulate teachers' approaches to the problems of practice: how and what the teachers perceive and how they make sense of, think about, and respond to situations as they perceive them. However, the result may be different from what the teachers would say about their own practice. (Simon & Tzur, 1999, p. 254)

I considered various analysis techniques, such as the popular grounded theory approach (Glaser & Strauss, 1967). After these considerations, I decided to approach data analysis at two levels: ongoing analysis, which took place during and between interactions with teachers; and retrospective analysis, which focused on sequences of events. During ongoing analysis, "the merged role of researcher and teacher provides an opportunity for the researcher to develop knowledge through multiple iterations of a reflection-interaction cycle" (Simon, 2000, p. 339). The research journal was instrumental in documenting this ongoing analysis. The contents of the journal included my reactions to lessons, ideas about possible answers to the research questions, and themes that were emerging. The following two excerpts are from the research journal and document some of the initial analysis:

> I also had the initial reaction that teachers who are open and listen to their students are likely to go in directions that students come up with…. Also teachers who listen can take tasks in directions that are not predicted in advance (for example, both teachers did not solve this task before using the task, and the solution was not as obvious as both teachers originally thought, but because they listen to students they were able to capitalize on the interesting mathematical ideas within the task). (Research journal, 10/25/06)
>
> In responses to how students approach the task, her [Barbara's] response is very general and seems to be based on how the mathematics naturally flows (or how she sees the mathematics naturally flowing), which seems to be a very obvious answer to the question. So the responses seem to be focused more on how mathematics develops versus how the students develop – but there may be a lot of overlap between these two things. (Research journal, 11/28/06)

After data collection I began a retrospective analysis in which I constantly rethought and reformulated my hypotheses. I identified themes and patterns, and I constantly compared them across data. The process was cyclical, and the themes evolved though coding, reorganizing data, comparing categories, looking for confirming and disaffirming evidence, and reassessing categories. During this process I progressed through four stages sequentially, but I continuously revisited previous stages to reformulate and reconsider my previous work. To begin with, I coded field notes and interviews. Data that I noted as meaningful were compared with each other to develop categories. I grouped data in categories to consider emerging themes. Next, using interview data, field notes, and teaching artifacts, I created the teachers' learning trajectories. In order to distinguish between what the teacher planned to happen during the class and what the teacher said about what actually happened during the class, I created two learning trajectories for each of the six lessons—the projected learning trajectory (PLT) and the enacted learning trajectory (ELT). When necessary, I used video data to fill in missing components of field notes. Third, I coded the created learning trajectories. Again, I compared data and grouped data according to categories. In the process of coding the learning trajectories, I also compared the PLTs and ELTs for individual lessons. Finally, I looked for confirming and disconfirming evidence for each of the themes that emerged.

Throughout the study I consulted my major professor about data collection, data analysis, and the working group curriculum. This action benefited my study in two ways. First, and perhaps most important, it contributed to my learning as a researcher-teacher by providing support, guidance, and feedback. Second, it contributed to the rigor of my study by providing another perspective on the interactions and ongoing analysis. This other perspective helped introduce alternative explanations, challenged my interpretations, and encouraged me to articulate my evolving hypotheses.

Strategies for Validity

In qualitative inquiry, "there are multiple, changing realities and … individuals have their own unique construction of reality" (Merriam & Associates, 2002, p. 25). The reality that I presented was my interpretations of participants' understandings. I incorporated four strategies to make my findings trustworthy. First, I used triangulation. Specifically, I examined multiple sources of data, including interviews, observations, and teaching artifacts. For example, I compared what each teacher said in an interview with what she did in the classroom or working-group session to find similarities and differences. Through triangulation I created rich descriptions of the teachers' understanding and use of students' mathematical thinking. Triangulation strengthened my findings. Second, I incorporated peer reviews. I stated earlier that

I consulted my major professor throughout the data collection and analysis process. I also consulted other colleagues, such as committee members, about findings. Third, I kept a research journal that documented my assumptions, experiences, and interpretations. My research journal provided an audit trail that documented how the study was conducted and the data were analyzed. Lastly, I was immersed in data collection for a semester (17 weeks); therefore I interacted with my participants for an extended period of time. These four strategies, taken in conjunction, added to the trustworthiness of my findings.

Researcher Role and Subjectivities

During the study, my role as a researcher depended on the situation. Throughout the interviews I engaged in conversations with the participants about the research questions and other issues that arose. In the conversations I made a conscious effort to not place value on or evaluate their responses, which strengthened the level of trust between the participants and me. In the working group sessions I posed tasks, asked clarifying questions, and observed the teachers solve mathematics tasks, discuss students' mathematics, and share their teaching practices. During classroom observations I positioned myself as an onlooker by sitting quietly in the back of the room taking field notes and operating the video camera. My goal was to minimize the influence that my presence had on the activities in the classroom. I observed almost weekly and often visited multiple times in a week, so the students became comfortable with me in the classroom. They often talked to me and asked the typical questions that a student their age would ask. I did engage in conversations with them at appropriate times.

As I stated earlier, I had a professional relationship with both teachers prior to this study and that relationship had its advantages and disadvantages for the study. One advantage was that the teachers were open and willing to share details about their teaching practices and about their students' thinking, some of which I would classify as personal. At times, the disadvantage was that the teachers wanted to know my thoughts and reactions to their lessons, and I thought the teachers were seeking advice or approval. Although I felt comfortable having discussions with the teachers about my thoughts about the lessons, I respected the teachers' limited time.

In a qualitative research study, it is important to take note of "subjectivities" of the researcher. I take the perspective that people cannot remove themselves from their subjectivities, but they can recognize and develop an understanding of them. "The goal is to become more reflective and conscious of how who you are may shape and enrich what you do, not to eliminate it" (Bogdan & Biklen, 2003, p. 34). In order to recognize and understand my subjectivities I used two strategies. First, before this study I wrote a teacher educator autobiography in which I detailed my professional development and how my theoretical perspectives fit within that development. In that autobiography I detailed my interests, beliefs, experiences, and emotions, all of which I consider to be pieces of who I am as a researcher, a teacher educator, and, ultimately, as a person. Many of the comments in the autobiography were personal as they detail my own development, including successes and frustrations. After writing the document I analyzed my experiences for common themes. The following two excerpts are from the analysis portion of my autobiography and relate specifically to this study:

> The struggles that I had about how teachers made sense of student thinking led me to the University of Georgia. It was during my doctoral work that I began to debate teacher preparation programs. I struggle about the purpose of a preparation program. On one hand I believe we need to train prospective teachers to be reflective thinkers (Dewey, 1933/1989) and to notice various aspects of teaching and student learning, while on the

> other teaching is complex and includes being able to keep up with grades and interact with parents. (Autobiography, p. 22)
>
> In my autobiography, I consistently discuss how I learned to notice student thinking and make sense of student thinking. I think we [teacher educators] need to contemplate more about the ways to encourage a focus on student thinking in preparation programs. I also think we need to understand better how teachers interpret student thinking and how that impacts instructional practices. (Autobiography, p. 26)

The second strategy that I employed was to record my reactions to the study, my beliefs, and my biases from the initial stages of writing the prospectus to final stages of writing this report. I kept these recordings in my research journal that I discussed earlier.

Limitations

In the design of this study I made decisions based on my goal and the research questions for the study. In making those decisions I created opportunities to learn from my participants as well as limitations to what I might learn. For example, there were only two participants in this study. The small number of participants afforded me the opportunity to closely examine their thoughts and actions. Yet these two teachers' experiences were not representative of those of high school geometry teachers. A large, diverse group of participants would allow researchers to make general statements about high school geometry teachers.

Another limitation to this study was that I focused on lessons throughout a geometry course instead of focusing on multiple lessons within a unit of study. I intentionally decided to focus on lessons throughout the two geometry courses, which afforded me opportunities to examine the notion of learning trajectories and professional noticing over 4 months. Another advantage to this decision was I could recognize commonalities and differences among the learning trajectories that were not related to similarities in content. However, the decision limited my ability to notice trends within one unit of study. The teachers used multiple tasks for many of the lessons, but those tasks were related to learning goals for one lesson. Focusing on one unit of study would have highlighted how the teachers sequenced tasks pertaining to one topic and the ways that teachers think about students' mathematical thinking across those tasks.

A fourth limitation of this study was I did not create my own learning trajectories of students' mathematical thinking to compare with the teachers' trajectories. Again, I intentionally made that decision. First, the purpose of this study was to describe and understand the teachers' thinking and not to compare their thinking against some predetermined measure. I wanted to listen to the teachers without preconceived notions of what I thought they should or should not say. I acknowledge that it is not possible to be completely removed from subjectivities as a researcher (Peshkin, 1988), but I made efforts to document my subjectivities to attempt to be as objective as possible. Second, I had limited access to students and limited knowledge of the students. It would be unfair for me to create learning trajectories for those students when I had such limited interactions with them. Nonetheless, making comparisons between the teachers' learning trajectories and a researchers' learning trajectories has advantages. The comparisons would allow researchers to compare similarities and differences between the trajectories, which may highlight particular teacher understandings that were missed in this study.

This study was about how teachers understand students' mathematical thinking, and therefore I focused on teachers. However, teaching is about students. A limitation to this study was I did not make any direct connection between what the teachers noticed in classrooms and what the students learned. I made the assumption that when teachers notice the ways that students' think mathematically, they will make informed and positive decisions about learning experiences for students.

CHAPTER 4

RESULTS

As stated earlier, the fundamental goal for this study was to begin to conceptualize the ways that teachers understand and use students' mathematical thinking in their instructional practices. Specifically, this study examined teachers' "informal" learning trajectories, what teachers noticed about students' mathematical thinking, and how teachers responded to students' mathematical thinking. These three ideas taken together provided a representation of teachers' understanding of the ways that students think mathematically. In this chapter, I respond to the three research questions. I begin by describing the learning trajectories that I created for the teachers. Learning trajectories were used to identify what teachers notice about students' mathematical thinking before, during, and after classroom interactions. Next, I detail what teachers noticed about students' mathematical thinking. Following that discussion, I present the ways that teachers responded to students' mathematical thinking. Last, I synthesize the results.

Learning Trajectories: What Do They Look Like for Teachers?

As discussed earlier, researchers have multiple views on learning trajectories. Steffe (2004) describes learning trajectories as models of students' thinking created by teacher-researchers and students. He believes the teacher-researcher needs to do extensive work with small groups of students to develop learning trajectories. Simon's (1995a) hypothetical learning trajectories are teachers' informed predictions about students' learning determined by an intended goal and the mathematics tasks that will be used. I used a combination of these two ideas to create two learning trajectories for each teacher's six lessons.

The learning trajectories that I created for each teacher's six lessons were complex because of the multiple factors that influenced their development. The projected learning trajectory (PLT) and the enacted learning trajectory (ELT) contained the teacher's learning goals, students' starting points, mathematical activity, learning process, and students' ending points. The students' starting points were the students' mathematical understanding at the beginning of the lesson and the students' ending points were their understanding at the end of the lesson. The difference between the two trajectories was the PLT described what the teacher planned to happen during class and the ELT was the teacher's thoughts about what actually happened. I developed the teachers' PLTs from the pre-observation interviews and I developed the ELTs from the post-observation interviews and my classroom observations.

The PLTs were the teachers' thoughts about the initial learning goals, the planned learning activities, and the projected learning process. The PLTs included the teachers' hypotheses about students' starting points and the projected ending points. The ELT had the same components as the PLT, but it was based on the teachers' thoughts after the lesson. The ELT included the teachers' thoughts about the learning goals, the students' mathematical understanding at the beginning of the lesson, what happened during the learning activities, the changes that occurred in students' mathematical understanding, and the students' mathematical understanding at the end of the lesson.

I illustrate the learning trajectory in Figure 6. The figure is used metaphorically to show particular points about the learning trajectories as I present findings. I want to emphasize that I am not claiming that learning is continuous, nor do I claim that learning fits the curve shown in the figure. In other words, this figure is a visual aid and is not an illustration of the actual learning process. There are questions in the literature about whether learning trajectories are ladder-like sequences or multiple-path or branching-tree trajectories (Lesh & Yoon, 2004). This study was not about how students learn mathematics. This was a study about teachers' learning trajectories of students' mathematical thinking, and therefore questions about learning trajectories were addressed as they pertained to the teachers' thoughts about students' mathematical thinking. In this report, the learning trajectories represent learning for a class or a subgroup of a class. I acknowledge that learning trajectories can represent the learning of individual students, but I focused on the ways that teachers described learning trajectories. The two teachers in this study mostly discussed groups of students. In specific instances, a teacher might have talked about one student, but in many of those instances the teachers used the thinking of one student to represent the thinking of a group of students.

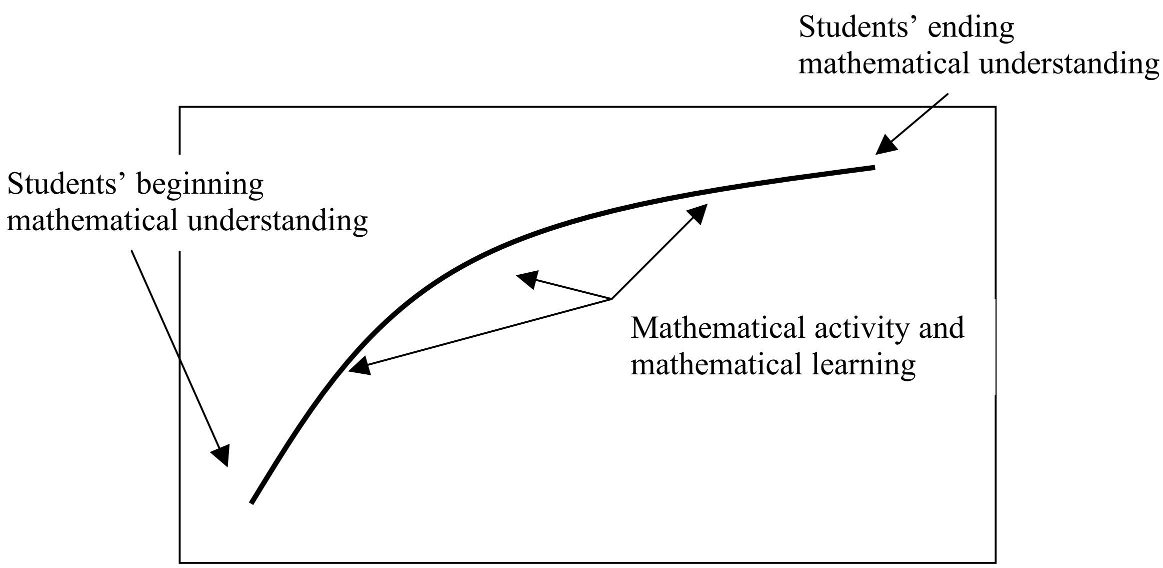

Figure 6. A learning trajectory.

There was a significant difference in the way that I used learning trajectories and the way that learning trajectories were discussed in the literature. Researchers typically use learning trajectories to represent students' mathematical thinking over multiple and consecutive days for a concept. The way that I defined learning goal influenced the way that I created the teachers' learning trajectories. The teachers had goals for individual lessons, multiple lessons, units of study, and courses of study. I was interested in the learning goals that the teachers created for the six individual lessons, but I did not ignore the larger goals that these were situated within. In order to situate the lessons within a unit of study and the course, I observed the teachers' classes the day before and the day after some of the lessons, and I asked interview questions that addressed what happened in class before and after the lesson. For example, I observed Judy's class 3 days before, 1 day before, and 1 day after her lesson on finding the area of regular polygons. The multiple observations and interview questions about previous and future classes were helpful for creating detailed learning trajectories that were situated within the larger unit of study.

The learning trajectories were created for individual lessons, but *learning trajectory* was not synonymous with a lesson. The lessons spanned 1 to 3 days. Some lessons contained multiple tasks, and others had one central task with multiple parts (e.g., the Window Task). The learning trajectories did not span units or courses of study. Many times the teachers would refer to larger goals when they described aspects of the learning trajectories (e.g., when Judy discussed area of regular polygons, she also mentioned its relationship to finding the volume of three-dimensional shapes). In some lesson observations, the lesson covered the entire learning trajectory. In other observations, the lesson I observed was among multiple lessons that were created for the same learning goal.

In the following two sections, I discuss the PLTs and ELTs. Following that discussion, I examined how these two trajectories compared to each other within the same lesson.

Projected Learning Trajectories

The PLT described the direction of learning that the teacher projected for a particular learning goal. The PLT took into consideration the teachers' planning and initial thoughts about the learning process for students. The teachers' lesson plans were an important component of the PLT because lesson plans include the learning goals and the learning activities. However, the lesson plan was not the PLT. The PLTs went beyond the teachers' lesson plans to include the ways that teachers thought students' learning might occur and how the mathematical activity might influence that learning. The PLT included hypotheses about how learning might occur. The PLTs included specific mathematical understanding that students need to understand the concept. The teachers described multiple paths in the PLTs for the same learning goal (see Figure 7). There were several reasons for the different paths. First, the teachers sometimes had different mathematical activities planned, and a decision about which activity to use was made during the actual lesson. For example, Judy had several tasks planned for her lesson on three-dimensional shapes. She decided which task to use once she observed how her students were engaging with and understanding the mathematics in the beginning tasks. Second, the teachers described multiple learning processes for the ways that different subgroups of students progress through the lesson. The teacher sometimes described students starting at different points and ending at different points (see Figure 8). The way students engaged in the mathematical activity or the way the teacher posed a question or comment influenced the teachers' perception of the students' learning process. For the Window Task, Barbara identified multiple starting points for her students and noted that not all of her students would reach the same learning goal. She thought that some of her students would use an algebraic solution, whereas others would use a measurement solution. She thought these two solutions represented significantly different learning. Even though Barbara had two learning trajectories in mind for the Window Task, she predominately spoke of one.

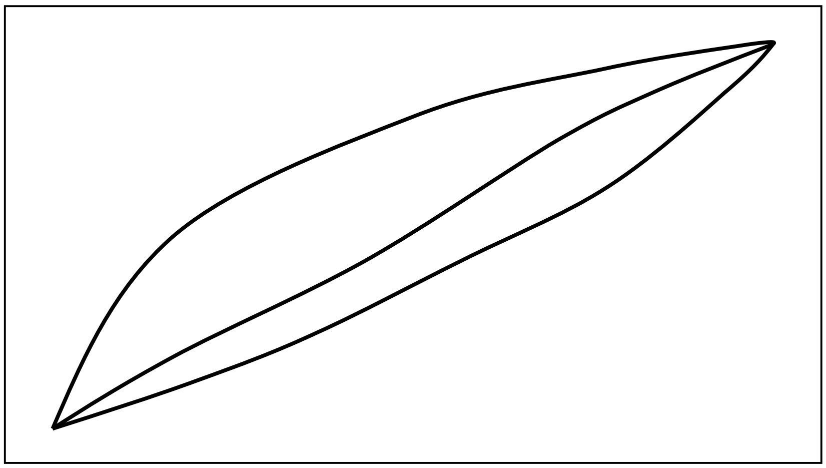

Figure 7. Multiple learning trajectories, but same learning goal.

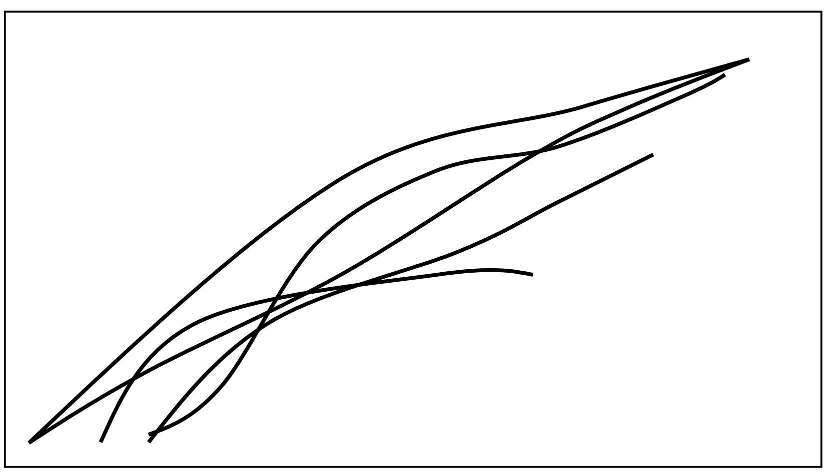

Figure 8. Multiple learning trajectories, but different starting points and different learning goals.

Enacted Learning Trajectories

The ELT was similar to the PLT, but it represented the actual learning trajectory based on the teacher's thoughts about the students' learning process during the enacted lesson. The ELTs were created from the teacher's thoughts about how the lesson progressed, how students engaged in the mathematics, and what students learned from the engagement. The ELT included moments in class that the teachers noticed and identified as important in developing the students' mathematical understandings. The ELT also accounted for moments in the lesson that the teacher did not anticipate.

Much like the PLT, the teacher described multiple paths in the ELTs for lessons (see Figures 7 and 8). In Figure 7, the multiple paths represent multiple learning routes that students took to reach the same goal. The ways that students understood and solved the mathematics task influenced the teachers' identification of multiple learning trajectories. There were instances in which the way the teacher interacted with the students caused the student to go in a different direction than other students. For the Window Task, Judy identified more than one ELT for her students. She noticed that some of her students used a measurement strategy to figure out what was happening to the perimeter of the rectangle using a ruler to measure the various lengths and widths, whereas another group used their knowledge of segment relationships in triangles to decide what was happening to the perimeter. The two groups of students solved the task differently, and ultimately their learning process was different. Figure 8 represents different starting points and different learning routes that led to different endpoints. The teachers commented that there were some students who approached mathematics tasks at a higher level than others. Furthermore, the teachers expected these students to advance quicker with a deeper understanding.

How Do the PLT and ELT Compare?

To identify and compare themes among the learning trajectories as well as to help identify instances that teachers noticed students' mathematical thinking, I compared the teachers'

PLT and ELT that I created for each lesson. For example, I compared Barbara's PLT for her lesson on the Box Task to her ELT for that same lesson. In some of the lessons the PLT and ELT were similar, whereas in other lessons the PLT and ELT were different. When I made decisions about the relationship between the two trajectories, I examined whether the learning goal, learning activities, and learning process were similar between the PLT and the ELT or whether they changed during the enacted lesson. I operated under the assumption that none of the ELTs were exact replicas of the PLTs. When I claim that the PLT and the ELT were similar, I am claiming that the data suggests that what the teacher projected to happen during the lesson did happen. I did not consider the PLT and ELT similar when the learning goal, the learning activities, or the learning process was different than originally projected.

In my analysis I considered what the teachers said in interviews about both the PLT and the ELT. In the post-observation interviews, the teachers identified whether there was a difference in what they anticipated would happen and what actually happened. Once teachers identified a difference, I looked at whether that difference influenced a change in the learning goal or a change in the mathematical activity. A change in plans did not automatically suggest a direction change in the learning trajectory or a change in the mathematical activity. For example, on the first day of the Window Task, Judy felt that there were some students who were not making progress or making the intended connections. On the second day, she created a GSP sketch for students to explore. Even though the sketch was not originally planned on the first day, the ELT and the PLT were similar. The overall idea of the task did not change; only the method for solving the task was modified. Instead of students physically measuring the side lengths of the rectangles, the computer program GSP measured the sides. I considered the two tasks similar because the learning goal, the learning process, and the actually task was the same.

The solution method was slightly different. In this teaching episode the ELT and the PLT were similar because the change in method did not change the learning process, nor did it change the overall goal. The overall ELT was not similar to the PLT for the Window Task, but the change was not due to the use of GSP. There is a slight difference in learning pertaining to whether students measure or the computer program measures, but I do not consider this difference was not significant enough to call it different ELTs. Ultimately, the learning goal was the same, and the learning process shifted slightly. These two examples also bring attention to the idea that within lessons there can be moments when the two learning trajectories were similar and when the two learning trajectories were not similar. When I compared the PLT and the ELT for a lesson I decided to compare the entire PLT and ELT instead of moment-by-moment pieces of the PLT and ELT.

In all the lessons the teacher identified student responses or actions that she did not anticipate. I did not automatically identify those as changes in the learning trajectory unless it noticeably changed the direction of the learning trajectory. In other words, I compared the teacher's comments about unexpected student responses or actions to what he or she projected to happen in the class. If the unexpected responses or actions aligned with the original learning goal, learning activity, and learning process I did not consider it an instance when the PLT and ELT were not similar. However, I did take note of instances in which these student responses or actions led the students to engage in the mathematical task notably different than anticipated and therefore changed the student's learning and mathematical thinking. I also took note of instances in which the students' responses influenced the teacher to act in a way that changed the learning trajectory.

In comparing the two trajectories for one lesson, I noted some instances in which the PLT and ELT were similar, whereas in other instances these trajectories were different. Specifically, there were two cases that happened within the 12 models created: (a) the PLT was similar to the ELT, and (b) the PLT was not similar ELT.

Projected Learning Trajectory Followed

The PLT that I created for the teacher included the teacher's lesson plan and her idea of how the students' learning would progress through the lesson. The ELT followed the PLT in seven of the lessons. In these instances the teacher played an important role in determining whether the students led each other through the PLT or whether the teacher led them. This analysis was related to Nemirovsky and Monk's (2000) idea of *path-following* and *trail-making*. For *path-following* you begin with a set of directions to get from one point to another point. Within these directions were markers, which bring attention to important aspects of the directions. In *path-following* you follow the sequence of directions given. On the other hand, in *trail-making* there is an endpoint in mind, but no set path to reaching that goal. Nemirovsky and Monk applied these terms to approaches to solving mathematics tasks. These terms were helpful to describe the comparisons between the PLTs and ELTs. There were lessons in which the teachers' actions followed their identified path for students' learning. With few exceptions, in these lessons the ELTs were similar to the PLTs. Furthermore, while the teachers in this study always had a projected path planned, there were lessons that the teachers were creating a path based on what they noticed and how they acted on students' mathematical thinking during class. In seven lessons the ELT was similar to the PLT, and in the other five lessons they were not similar.

Students Led the Way

In three lessons the ELT was similar to the PLT, and the teacher's actions and instructional moves were led by student's thinking. Thus, the direction of the ELT was strongly influenced by what the students said and did in the classroom.

For example, in one of Judy's lessons she wanted students to find the area of a regular polygon. She also wanted students to recognize angle relationships in the isosceles triangles in the regular polygons and to justify intuitively why all the isosceles triangles were congruent. According to Judy, at the end of the lesson the majority of the students could find the area of any regular polygon. She said, "Initially I was worried that there were a lot of them that weren't really getting it and weren't there, but then they did. When they were doing this on their own, they did it really, really well" (interview, 10/3/06). Judy also acknowledged that students progressed through learning process as she projected. Before the lesson, Judy thought students would divide the regular polygon into equal triangles, and then drop the heights in the triangles to find the area. When describing the students' approach to solving the task after the lesson, she noted, "They divided up into the triangles… and they get the isosceles triangle on the side, and they drop their height" (interview, 10/3/06). This is an example in which the ELT was similar to the PLT because the students progressed through the learning process as Judy had predicted. Her instructional plans did not change, and, according to Judy, the students accomplished the larger goal.

In the enacted lesson the students helped guide the direction of the lesson and ultimately influenced their learning trajectory. To begin with, as Judy set up the task, she revisited student solutions from the day before. She asked several students to share their solutions. After the task was set up Judy encouraged the students to work together to solve the task. As the students worked on the task, they helped one another instead of asking Judy for guidance. She walked around the class to observe the students and to occasionally pose questions. In the whole-group discussion, she asked students to put their solutions on the board. She selected particular student solutions to share, but she encouraged other students to share their solutions as well. In this lesson, it was apparent that the teacher's role was essential in the way the class progressed through the tasks. In this lesson, however, the teacher's role was not to do the mathematics for the students. I identified this lesson as students leading the ELT because it was they who were doing the mathematics and presenting solutions. The teacher did take an active role in the class by acting on students' mathematical thinking, but the ways she acted support the ways the students were currently thinking and did not change the learning path they were on. In other words, the teacher responded to students in ways that allowed the teacher and the students to lead the way.

Teacher Led the Way

This case differs from the first one because the PLT and ELT were similar, and the teacher led the students through the ELT. The teacher was more directive in her comments and leading in the wording of her questions. Even though the teacher led the way, the students did influence some of the teachers' actions and instructional decisions.

For example, in one of Barbara's lessons her goal was for students to recognize angle relationships in circles and to be able to find any angle measure in a circle. In addition, she thought that in this introductory lesson on circles the students needed to revisit previously learned definitions and to extend those definitions in order to learn more definitions and solve various tasks pertaining to the circle. She described the learning activity as student centered and exploratory. She wanted to use the regular hexagon to make connections between what students already knew and what she wanted them to learn about the circle. “We’ll start with vocabulary, some known vocabulary and some new vocabulary. And from that vocabulary we will move towards some mathematical relationships regarding angles” (interview, 11/27/06). After the lesson Barbara noted that the lesson had gone in the direction she intended to go. “Actually, it went, to me, how I envisioned it…. I kind-of planned on working off of their knowledge…. I don’t think there was any direction I took today that I didn’t intend to take” (interview, 11/27/06). Barbara thought that the majority of her students could follow and understand the lesson. She identified four students who might not have been with her, but overall she noted that “they have the essence of what I want them to walk away with. I think they have that. I kind of think that most of them have it right now” (interview, 11/27/06). This lesson was an example of a similar PLT and ELT.

Barbara noted several times that she would take an active role in guiding students through the learning process. “I haven’t thought about leaving them on their own to investigate it. So, it will be a guided investigation with very key questions and drawings that I would want them to measure and infer” (interview, 11/27/06). The following excerpt from class was representative of her interactions with students during this lesson.

> [Students were instructed to draw a circle A with a diameter DC and radius AB. Students measured angle BAC and Barbara chose to use a student’s response of 60 degrees as her own angle BAC measure. See Figure 9.]
>
> B: I want us to look at this. Do you know how many degrees is in this portion here if that portion has 60 degrees? [Barbara points to the obtuse angle BAC.]
>
> S: 300.
>
> B: 300, right? Do you know how many degrees are right here in comparison to all the way around? [Barbara points to the angle BAC.]
>
> S: 60.

B: So, this piece is 60 degrees out of? [Barbara points to the sector BAC.]

S: 360.

B: 360 degrees. If this angle is 60 degrees right here, what is the measure of this arc?

S: 60.

B: 60 degrees. Is that right? We call this angle here cut from the center with two radii, we call that a central angle.

B: And the central angle has the same measure of the arc that it cuts off on a circle. Do you believe that? I mean that's easy to believe, right, to understand based on your prior knowledge.

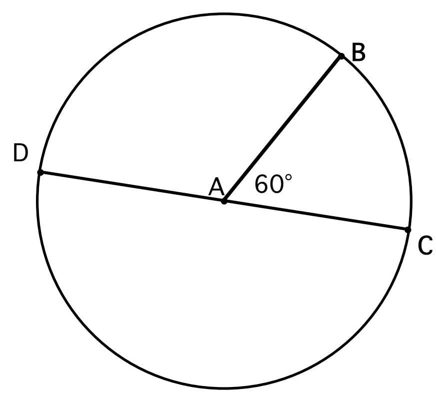

Figure 9. The circle drawn on the board.

Notice that Barbara was directive and posed questions to the whole class. She hoped that someone in the group would be able to answer her questions. "They approached it like I guided them to approach it. And this one time that there wasn't any big surprise because I got someone in the class to see what I wanted them to see and to say it or articulate it. And we sort of verified it with another example" (interview, 11/27/06). Once a student answered the question, Barbara moved on to another question. Furthermore, she asked questions in such a way to encourage particular answers. Notice in the following excerpt how Barbara asked students what concentric circles are and then provided students with an example.

> B: We can talk about circles being tangent. What do you think it means for two circles to be tangent?
>
> S: Just touch each other.
>
> B: They just touch each other at one point. They don't overlap, right? They don't intersect more than once. So we can talk about them being externally tangent or internally tangent. You know? Not a big, big deal for us, but we can talk about circles being tangent. We could talk about concentric circles. Do you know what concentric circles are? You might know that from some other arena of knowledge. Concentric circles. Imagine a bull's eye, a dartboard. What do you think concentric circles are?
> [Multiple student responses.]
>
> B: Parallel circles. I like that. I think that's a good description. They're circles that would never intersect. What would they have to share? What's the one thing they would have to share?
> [Multiple student responses.]
>
> S: The center.
>
> B: They share the same center.

Once a student responded with an answer that Barbara believed was acceptable, she made comments to define the terminology or to help students make connections. After a student responded with "parallel circles," Barbara noted that concentric circles never intersect and asked the students what concentric circles share. These two excerpts are examples of the ways that Barbara led students through the PLT.

As mentioned above, at times the students influenced the questions the teacher asked, as in the following example:

> S: I have a question. Was it a fluke that the diagonals equaled 180? [See Figure 10 for the diagram that student was referring to.]
>
> B: Say that again.

S: Was it a fluke that both the diagonals equal 180?

B: Listen to [She speaks to whole class], Jacob has a question for you. Is it a fluke that these opposite angles, is that what you meant?

S: Yes.

B: Is it a fluke that these opposite angles equal 180?
[Multiple student responses, some saying yes and others saying no.]

S: It's in a quadrilateral.

S: I just saw it and I thought it ….

B: Well, let's look. Think about this one first. It intercepts this arc [points to angle C and arc BAD]. Right? The other one, it's opposite intercepts this arc [points to arc BCD]. What do the two together intercept?

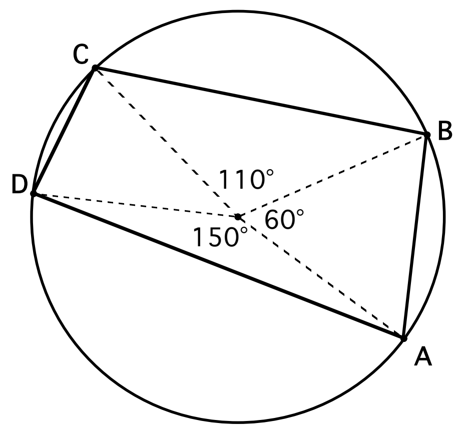

Figure 10. The figure about which a student asked a question.

In this excerpt John posed a question that Barbara explored with her students. She later noted that John's question "was a nice moment" in which she was able to work off his question (interview, 11/27/06). She "had not thought about that relationship necessarily coming out today" (interview, 11/27/06). The question was not one she intended to ask, but she was willing to look at it with the class. She was willing to address the question, but it did not change the learning trajectory. After responding to the question, she shifted her questions back to her goals for the lesson.

The active role that Barbara mentioned earlier was revealed in the way that she questioned students during class. The whole-group discussion consisted of mathematical discussions between Barbara and her students. The discussions did not occur between students and students. The students answered the teacher's questions but did not actively solve any mathematics task without being led through it. This lesson was an example of the teacher leading students through the PLT.

The Projected Learning Trajectory and the Enacted Learning Trajectory Were Not Similar

Of the 12 teaching episodes, there were 5 in which the ELT was not similar to the PLT. In those teaching episodes there were times when the PLT and ELT were similar at the beginning of the lesson and not similar at the end of the lesson. There were also lessons when the ELT was not similar to the PLT at the beginning of the lesson, and similar at the end of the lesson. Those episodes fell into two categories: (1) teacher intervention, and (2) management and planning constraints.

Teacher Intervention

In two teaching episodes, the teacher decided to change the direction of what was happening in the class. This intervention caused the ELT to change direction. In one instance, the intervention caused the ELT to take another course whereas in the other the intervention caused the ELT that was not similar to the PLT to map back onto the PLT.

A different direction. In this first case the PLT and the ELT were similar initially. At some point in the lesson the teacher decided to change the direction of the lesson, and consequently the ELT was no longer similar to the PLT. Figure 11 shows the two trajectories close at the beginning of the lesson and then a point at which the ELT took a different direction. The figure is use metaphorically to illustrate multiple paths, and therefore the two directions do not represent anything more than the trajectory going a different direction. That point was the moment that the teacher made an intervention that changed the direction of the ELT.

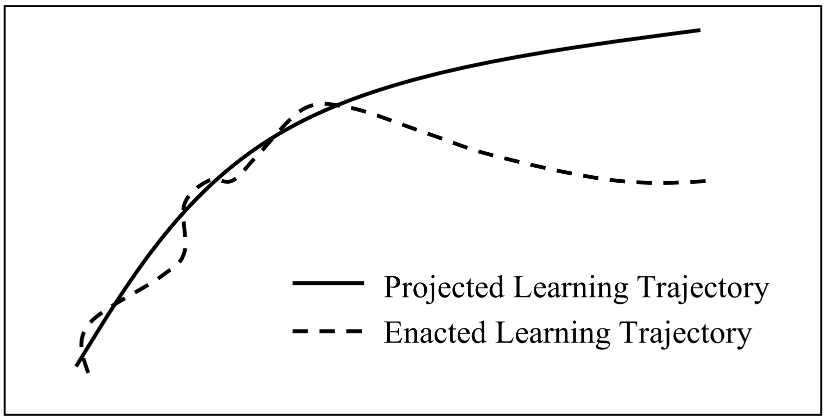

Figure 11. A different direction.

In this case, Judy wanted her students to explore the maximum area and perimeter in a rectangle during the Window Task. The rectangle was constrained by an equilateral triangle. She wanted the students to do an organized investigation and justify their solutions. She thought that the students knew how to find area and perimeter of a rectangle. Furthermore, she thought, "They [students] want the area and perimeter to always go together. So, if the area was the same, then the perimeter was always the same" (interview, 10/23/06). The lesson took place over 2 days. I created one PLT and ELT for this lesson; I did not create separate learning trajectories for the different days. During the first day Judy introduced the Window Task and allowed the students to solve it n groups. As they worked on the task, Judy walked around to observe and sometimes pose questions to students. She noted that the students approached the task numerically, as she had projected. On this first day, the PLT and the ELT were similar. The students began solving the task as Judy had anticipated, which signified that the class was following the PLT. The change in the trajectory occurred during the second day in the GSP investigation, when the students led Judy to showing them the graph of the data points. Judy said, "I realized that the parabola just wasn't going to make sense if we just left it there" (interview, 10/25/06). She changed the task to investigate area of a rectangle when the perimeter stayed constant.

> I made that decision in the middle of the lesson. It was actually when I started listening to what they were saying about the parabola…. Most of them couldn't remember the word *parabola*…. I realized that if we left it there, it would have been like we've got this graph and we have no idea why. And I wanted there to be a little more understanding of how we got it. I had to change the problem when we came in here because I wanted to make the perimeter manageable. (interview, 10/25/06)

Judy noticed a gap in the students' mathematical understanding. Because of the gap, she changed the mathematical task to address it and therefore changed the direction of the ELT. She also noticed that the mathematical idea in the original Window Task was complicated, and she thought a change would allow the students to notice the symmetrical shape of the parabola and how the shape related to the length and width of the rectangle.

This example shows the way that one teacher began a lesson with the intention of students reaching a particular goal and ended the lesson with a modified goal. Judy said, "The goal kind of changed. When we went to the lab and started looking at it…. I didn't expect to go into what we did when we came back to the class with the rectangles, but I just felt like the parabola just magically appeared" (interview, 10/25/06). The ELT and PLT were similar until the teacher noticed the way her students were thinking about parabola. She intervened to change the task, which ended up changing the direction of the trajectory.

Let's get back on track. In this case the teacher noticed that what was actually happening in class was different than what she intended. The ELT began to go in a different direction than the PLT. At some point the teacher intervened to bring the students back so that they were following the PLT (see Figure 12). The teacher intervention was immediate, and the mathematical activity was changed. That change caused the direction of the ELT to change.

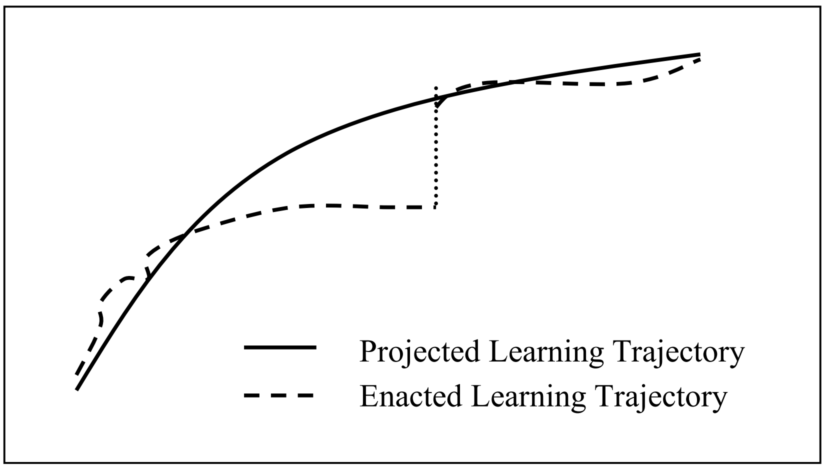

Figure 12. Let's get back on track.

In the lesson on the Window Task, Barbara's goal was for the students to engage in an exploratory activity in which they would generalize and generate a mathematical proof. The lesson spanned 2 days. On the first day, Barbara introduced the task, and the students had the remaining class time to work on it. During that time, Barbara walked around to observe students and to pose questions or comments to help them. After the first day, Barbara noted that all students were using a measurement strategy. She said, "My big surprise is no one used a variable, and I'm thinking, why did they do that? And I'm thinking the first question about measurement was what kind of sent them in that direction of some pragmatic justification" (interview, 10/23/06). She also mentioned that none of her students had a proof for his or her solution.

On the second day Barbara intervened to change the direction the students were going. First, she led the discussion to push the students in the direction of a mathematical proof. She thought the first day had been "mediocre because they [the students] didn't really try to push justifying," and she "loved" the second day. She also thought the second day was "a little teacher centered" (interview, 10/25/06). Barbara noted, "I wanted to take them to that algebra place…. I defiantly wanted to get that in Tuesday…. I made sure it happened on Tuesday, you know, what I wanted to happen in that problem" (interview, 10/25/06). On the second day Barbara asked leading questions and selected students to share solutions to push the justification and the algebraic solution.

Management and Planning Constraints

In three lessons the ELT partially followed the PLT for reasons other than students' mathematical thinking. Teachers work within various constraints that sometimes influence classroom practices in noticeable ways. In this case, aspects other than students' mathematical thinking influenced the ELT. The PLT was followed, but the constraints inhibited students from progressing completely through the PLT. Figure 13 shows the ELT was similar to the PLT up to a point. At that point, the ELT ended because constraints influenced what happened in the class.

Barbara's lesson on geometric probability was an example of this category. She described the lesson goal as students doing an area investigation and extending their probability vocabulary to solve geometric probability tasks. Barbara noted that the students would start with the basic definition of *probability*, *area*, and *circumference* and extend that knowledge to the idea of geometric probability. In this lesson the students did not follow the PLT for several reasons.

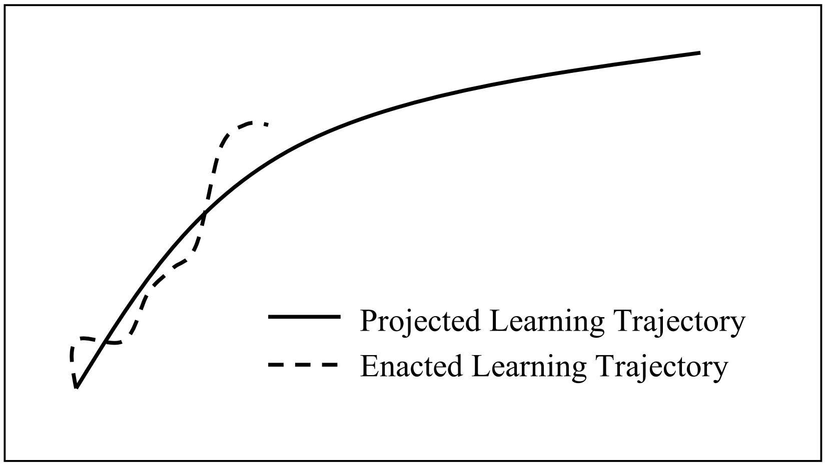

Figure 13. Management and planning constraints – not quite there.

First, a lot of time was spent reviewing homework because not all students completed the homework assignment. Barbara noted, “There was a little bit of long time on homework” (interview, 9/26/06). She liked the homework portion of the lesson and thought that it was necessary. Second, several students were disruptive. “It’s homecoming and there’s just different stuff going on, so they’re just a little hyper-er” (interview, 9/26/06). There was one student in particular that Barbara was frustrated with, and she spent class time speaking to him about his behavior. Third, Barbara was rushed during her planning time and created a mathematics task that could be solved differently than she anticipated. She said, “I had not seen the trapezoid minus the circle. Then it was like, oh, shoot. They don’t have to go this way…. But I did not see, I didn’t have much time too. I was rushing to finish it. I mean time just ran out. I even ran and copied it between class” (interview, 9/26/06). The students found the task trivial and solved it quickly, whereas Barbara had intended that they would students to struggle some and apply knowledge from a previous triangle investigation. The students did not make connections to the previous investigation. In her reaction to the overall lesson, she stated, “I lost what I wanted to gain today" (interview, 9/26/06). In this example, Barbara thought that students did not reach her goal for the lesson and that the students did not progress through the learning process as she had projected.

One might question whether this was truly an example of when the PLT and ELT were not similar, because the teacher could pick up where the class left off on the following day, and the lesson could become a multi-day lesson. That did not happen in the example presented. The teacher chose not to continue the lesson on the following day but instead allowed more time on homework questions. For the most part, the teacher moved on as if the students had met the projected learning goal. Barbara felt pressure to cover the state curriculum and thought moving on would be a better use of their class time. I noted this example as a case in which the learning trajectories were not similar because it highlights the way that management and planning constraints significantly influence the direction of the learning in classrooms.

Summary

Comparing the two learning trajectories for each lesson brought attention to whether the PLT and ELT were similar or not. In seven cases the PLT was similar to the ELT and in five cases the PLT and ELT were not similar. The comparisons also helped to identify lessons in which the teacher acted in ways that led the students to follow an anticipated path, and lessons that the teacher acted in ways that allowed the students to make their own trails. Table 5 summarizes how each PLT was or was not similar to the ELT.

Table 5

Lessons in which the PLT and ELT Were Similar or Not

Barbara

Lessons	Similar	Not similar
Area of Sectors and Arc Lengths		X – Management and planning constraints
Window Task		X – Let's get back on track
Angle and Arc Relationships in Circles	X – Teacher led	
Equation of Circles and Angles Outside the Circle		X – Management and planning constraints
Transformations	X – Teacher led	
Box Task	X – Teacher led	

Judy

Lessons	Similar	Not similar
Area of Regular Polygons	X – Student led	
Window Task		X – A different direction
Similar Figures	X – Student led	
Three-Dimensional Shapes	X – Student led	
Box Task	X – Teacher led	
Pi Day		X – A different direction

What Do Teachers Notice?

The two teachers in this study viewed their students as mathematically literate students who had a lot of geometry knowledge and were intuitive about mathematical relationships. The teachers made comments such as, "I'm always amazed at how students see things and do things. I think they will often be faster than me at seeing a way to show something" (interview, Barbara, 9/26/06). Before and after each lesson the teachers were asked to describe the ways their students were thinking mathematically. Both teachers found this question to be difficult to answer, and as the study progressed the teachers began to question how they assessed their students' mathematical thinking during lessons. Barbara noted, "You know you've asked me that so many times that I'm like, I really don't know that I assess that very well" (interview, Barbara, 12/5/06). Before I begin to describe what teachers noticed I want to provide a brief description about the way they viewed geometry students, and, specifically, their geometry students.

These Geometry Students and Geometry Students in General

The teachers described students' mathematical thinking in terms of their current students and in terms of their general knowledge of geometry students. Before a lesson the teachers spoke more about geometry students in general. Several times during the interviews, I pushed the teachers to talk about their students, and that was difficult for the teachers. Their comments about specific students were generally the same: This student will do well, and that student will struggle. During the post-observation interviews, the teachers made comments about their current students. These comments dealt with individual students or groups of students.

Through Years of Working with Students

The teachers described students' mathematical thinking in light of their experiences with students over the years. Those descriptions related to students learning the subject geometry and learning specific concepts in geometry. The teachers thought that students needed to visualize relationships and have concrete experiences to support their transition to abstract thinking (this view relates to one of Piaget and Inhelder's themes). The teachers thought students needed to draw diagrams and physically touch three-dimensional objects. They noted that such experiences were necessary to help students notice mathematical relationships and to develop abstract ways of thinking. The comments that the teachers made about students' mathematical thinking that related to specific concepts were about student struggles or misconceptions. For example, Judy noted that students commonly had the idea that when the area of a rectangle gets larger so does the perimeter (which has been confirmed in research by Tierney et al., 1990).

Through Getting to Know These Students

The teachers described their students' mathematics based on their prior and current experiences with these particular students. The teachers made statements about individual students' overall mathematical learning ability, specific student solution strategies or mathematical understanding, and the class's mathematical understanding.

The teachers made reference to individual students' ability to learn mathematics. For example, Barbara frequently made comments about specific students understanding or not understanding the mathematics. Many of these comments were related to mathematics in general and not to specific mathematical concepts. Barbara thought there were students who were "seers" and others who struggle to see. This identification of students relates to Battista's (2007) comment, "Underlying most geometric thought is spatial reasoning, which is the ability to "see," inspect, and reflect on spatial objects, images, relationships, and transformations" (p. 843). A difference between Battista's and Barbara's thoughts about seers is that Battista notes that students' mathematical conceptions affects perception. He also states that experiences can influence student conceptions and thus influence perception. Barbara did not make statements that suggested she believed that mathematical experiences would influence the nonseers' conceptions. In contrast, Barbara identified a subgroup of her students as seers and made comments such as, "Mathew, Monica, Laura, um, I think will be my brightest people" (interview, 12/5/06). She expected those students to solve tasks in more sophisticated ways than other students. For example, in the Window Task she noted, "My ultimate goal is that my better students are going to generalize this immediately and will say, "Yes, I have a mathematical proof, and yes, I'm sure this is the answer'" (interview, 10/23/06). There was another group of four students that Barbara consistently thought would struggle. She made comments such as:

> They struggle with seeing relationships, and they struggle with explaining it. And Savannah will come back and say that you confused me when I come to help her. But that's because she's confused anyways, and she doesn't realize it. She doesn't realize how, sometimes. I don't know. She can't see the forest through the trees or the trees through the forest. (interview, 9/26/06)

Barbara thought these students would always struggle to see and often would not realize that they did not understand. Barbara did not make comments about the ways that different experiences might support these nonseers' learning.

Judy also categorized students, but in a slightly different way. She consistently identified the same group of students who she considered to be the "bright" students. She made comments such as, "Kids like Andrew, Nick, and, probably, Amy and Lauren, Tyler, and the other Lauren, I think they look at it at a slightly higher level" (interview, 11/17/06). Judy did not identify a group of students who she thought would always struggle, but she did comment on students who were having trouble with specific mathematical ideas. The students that she noted were struggling were relatively the same each time, even though she did not categorize the students together.

At times, the teachers would make comments about specific students, and at other times they would talk about their classes as a whole. When the teachers commented on specific students they would say things about how a student solved a task. For example, Judy described the way that Joseph found the area of a square. In addition to making comments about specific students, the teachers would also comment on how the entire class was thinking mathematically. In many of these instances, the teachers would use the word “they” to represent the class. For example, Judy thought her class was doing better on the Window Task than she originally thought they would. She mentioned that her students were all using a measurement strategy to solve the task.

Professional Noticing

The interview questions prompted the teachers to discuss all four skills of professional noticing: (1) identifying noteworthy aspects of instructional situations, (2) describing instructional situations, (3) interpreting instructional situations, and (4) responding (Jacobs et al., 2007). In particular, the teachers were asked about students’ mathematical thinking during the instructional situations. As noted by Simon (1995b), classroom interactions are influenced by teachers’ knowledge and their learning trajectories. The ways that teachers discussed what they noticed during the lessons was complex. Many times the teachers intertwined their descriptions of what they noticed with their interpretations of what they noticed. In those instances I allowed the teachers to make their comments without pressing for one over another. In the following sections I describe what the teachers noticed. The examples were selected because they represent the intended point well. Furthermore, both teachers exhibited the characteristics mentioned, but the frequency of each and the degree to which each happened varied.

Teachers Noticed Their Uncertainties About Student Thinking

When the teachers described their students' mathematical thinking, they shared uncertainties about what they noticed pertaining to that thinking. For example, one of Barbara's students thought his answer to the Box Task was wrong. She brought up this student's solution when answering a question about how she would describe the ways her students were thinking about the Box Task. She noted that the student explained his answer a couple of times, but she did not understand what he was saying. When I asked her to describe what he thought, she stated, "It was something about, he thought, I don't know what he thought" (interview, 12/19/06). In this instance, Barbara did not understand what the student was thinking, but she did notice the thinking and attempted to understand it.

In some instances, the teachers made comments such as, "I don't know what they were thinking," and they made decisions to move forward without understanding a student's solution. For instance, Barbara commented that she felt bad about not addressing student thinking because she did not understand the student's solution, and she did not take class time to try to understand. She stated, "It felt incomplete. I really didn't want to, but I knew I had to make some progress today. So, I had to leave it…. I knew I was not with her so I needed to move on" (interview, 9/26/06).

The teachers' descriptions of student solutions were limited by what the teachers noticed. They often identified instances when they did not understand something that a student said or wrote in class, but then they provided a limited description about the student solutions.

Teachers Noticed Student Struggles

The teachers made comments about struggles that students had with particular content or tasks. From the teacher-identified student struggles, I have created four categories: mechanical mistake, students struggled to begin a task, error with no connection to conceptual understanding, and conceptual error. When the teacher identified a mechanical mistakes or a conceptual error, the teacher described and interpreted students' mathematical thinking. When the teacher discussed a student error with no connection to conceptual understanding, the teacher was simply describing the error. I claim that when the teachers described and interpreted the student error, the teachers noticed more during the instructional situation than when they did not describe or could not interpret the error.

The category of mechanical mistake included teacher-identified instances in which students knew how to do a computation, procedure, or solution, but made an error. The teacher did not think the error represented a misconception but that it prevented the student from getting the correct answer. These errors included a student writing the incorrect number on his or her paper, typing the incorrect number in the calculator, or not reading both parts of a question. An example was when Barbara thought that one group of students did not have an answer to a question about the Window Task because they had not read the question, not because they did not know how to solve the task. Another example was when Judy thought that one student had the incorrect area of a regular polygon because he had entered the numbers incorrectly in the calculator. She noted that she knew he had entered incorrect numbers because he had the correct numbers on his paper. In these two examples, Barbara and Judy thought the students had made a mechanical mistake that was not based on misconceptions. Furthermore, in these examples the teachers noticed a reason for the mechanical mistake.

In lessons that incorporated higher-level tasks, the teachers often commented that some students would struggle to develop a method for beginning a task. The teachers only described the struggles; they did not identify why beginning a task was difficult for some students. Both teachers mentioned such struggles for the Window Task and the Box Task. For both tasks, the teachers worried that the students would not know how to start the task, but once the students developed a method the teachers thought the task would be easy. The teachers did not describe their interpretations of why starting a task was difficult for students. In pre-observation interviews, Judy would reference the same group of students who struggled on tasks. Barbara generally talked about her "nonseers" as the students who would struggle, and she assumed the other students would have a method to begin tasks.

Some struggles the teachers identified were related to students not doing something correctly, but there was no connection made to whether the student understood or not. In other words, the teachers did not indicate whether those struggles were related to conceptual or mechanical errors. The teachers described the student error, but did not interpret it. For instance, Barbara's student Michael incorrectly simplified the expression $\frac{1}{2}\left(2\sqrt{2}\right)\left(2\sqrt{2}\right)$ when he used the distributive property to get 2. Barbara was not sure whether he had the incorrect answer because he used distributive property by mistake or whether he did not understand the property. Another example was when Judy mentioned that one of her students did not use the correct trigonometric function to solve for a side length in a right triangle. In this instance it was unclear whether the student made a mistake or whether the student did not know how to identify which sides of the triangle represented the opposite side, adjacent side, and hypotenuse. The teachers did not interpret the student error. This represents an example when the teachers described the error, but did not have a reason for it.

The majority of teacher-identified instances of student struggles were conceptual. They included students not recognizing mathematical relationships, students struggling to go from the concrete to the abstract, and students not justifying their solutions. For example, referring to the surface area of rectangular prisms, Judy stated, “They [the students] wanted all of the sides going around to be the same, and they’re not. And so, when they started to think about surface area, it was a problem. When they drew a net for a rectangular prism, it looked like a rectangle with a bunch of squares going through it” (interview, 11/17/06). Judy thought that the students did not recognize the relationship between the lateral faces of a rectangular prism and the net of the prism. The teachers often noticed instances in which students struggled to go from concrete to the abstract thinking. For example, during the Window Task, Judy noted that several students did not make the connection between the data they collected and the overall idea of what was happening in the task. She remarked that the students lacked a conceptual understanding and needed more experience with concrete examples before transitioning to abstract thinking. The teachers noticed instances in which students did not justify their solutions. Barbara was frustrated with her students when they struggled to justify their answers in the Window Task. She thought they had all the necessary knowledge to make the transition from inductive reasoning to deductive reasoning. In these three examples the teachers noticed student struggles that were related to conceptual understanding. The teachers provided details in their descriptions and made interpretations of the student errors.

Teachers Noticed Surprises About Student Thinking

The teachers noticed students' mathematical thinking in positive and negative surprises. The positive surprises consisted of students' mathematical thinking that teachers were impressed with because they had not thought about that solution or did not expect that solution from their students. For example, Judy was impressed with her two students Morgan and Andrew for the justification they gave of their answer to the Window Task. Judy had not thought about that justification for this task. In addition, she did not expect that kind of justification from Morgan. She noted, "Morgan explained that to me incredibly well, especially for her. I kept looking at it and saying, 'Oh, my god, is this even right?' So, I didn't expect any of that" (interview, 11/23/06). Barbara also identified positive surprises. For example, in her reflection on her lesson on circles, she noted, "Monica's response was a pleasant surprise. Because I thought that one might be hard for someone to come up with. So, Monica's moment was probably the highlight of the day" (interview, 11/27/06).

The negative surprises consisted of students' mathematical thinking that the teachers thought were not a viable way of solving the task. Barbara and Judy both identified negative surprises, but Barbara often placed value on the surprises whereas Judy did not. For example, Judy noticed some of her students were picking numbers to represent the lengths and widths of the rectangle in the Window Task. She was surprised by this method and did not think it would help them solve the task. Judy accepted this approach as the way the students were thinking about the task and did not place positive or negative value on the students' way of thinking. However, Barbara stated that she was disappointed with her students because she expected more out of them. For example, she was highly surprised by and disappointed with the way her students approached the Window Task. She expected some students to solve the task using algebra, and she was disappointed that all of them approached the task with a measurement strategy. Furthermore, Barbara was surprised that students used 2 to 5 cases to justify their solution. She did not think that was a reasonable number of cases, and she was not happy with their solution method. In looking over the students' written work, she made the comment, "I'm not sure I have a great one in this whole stack" (interview, 10/23/06).

Teachers Noticed Student Thinking Through Mathematics Tasks

Many times the teachers noticed aspects of their students' mathematical thinking that was connected to a task. More specifically, the teachers detailed the steps in solving the task when they described their students' mathematics. The teachers talked about what students did or did not do with the task but rarely made comments that were connected to specific mathematical concepts or ideas. For instance, when Barbara talked about how her students were thinking mathematically about the Box Task, she talked about the steps to solve the task. She talked about the specifics of how the table was created and the steps necessary to move towards the algebraic solution. Barbara stated, "Students found the measurements by cutting out squares, and we compiled a table. Then, you hope that someone is going to come up with a quick area formula, and somebody was thinking in that direction. So, we put it in the calculator" (interview, 12/19/06). She did not talk about how the students were thinking about the concepts of area and volume, nor did she refer to understanding that students had that allowed them to transition from the concrete to the abstract. Likewise, Judy often noticed students' mathematical thinking as it related to the steps in the task. For instance, when she described how students found the area of the regular polygon, she detailed the steps to finding the area by finding the area of triangles. She noted, "They divided up into the triangles…. They get the isosceles triangle on the side, and they drop the height…. They find the area of the little triangle and multiply it by twice the number of sides" (interview, 10/3/06).

There were rare cases in which the teachers talked about students' understanding of specific mathematics concepts related to the mathematics task instead of about the steps in the tasks. As an example, Judy talked about the conceptual understanding that some students had when they noticed they could find the area without cutting paper or measuring. She described the way these students were thinking in terms of the task, but it was related to concepts and not steps. She noted, "A few of the kids didn't cut it out or anything. They figured out the pattern and went with it." The students noticed "that this one piece of paper that you're cutting four squares out of, you don't have to measure to understand what's happening" (interview, 12/7/06). She continued to describe what understanding these students had about surface area: "They were making the connection that the surface area was going down and why it was going down" (interview, 12/7/06). In most of the instances related to conceptual understanding the teachers typically referred to student struggles or mathematical misconceptions (these were discussed earlier). For instance, Judy noted that students typically believe that when perimeter of a rectangle gets larger, so does its area.

Teachers Noticed Student Thinking Through Their Own Mathematical Knowledge

The way that teachers described what they noticed about their students' mathematics was based on their own mathematical knowledge. The flexibility of that knowledge determined the teachers' ability to notice and articulate the complexities of their students' mathematical thinking.

Barbara thought her own mathematical thinking was rigid. She stated, "I think they [the students] will often be faster than me at seeing a way to show something. I'm too rigid mathematically thinking. I'm rigid. I just don't have that freedom" (interview, p. 6). rigidity influenced how Barbara described her students' mathematical thinking. When she described her their thinking about the Window Task, she used words that compared the ways the students solved the task and the way she solved the task. She was listening interpretively. She made very few statements indicating what students did do; however, she made statements in terms of what they did not do. Specifically, what the students did not do was solve the Window Task the way that she had. Barbara acknowledged that the students solved the task using measurement, but in her description she did not provide detail. The lack of detail was evidence that Barbara was unable to notice the complexity of her students' mathematical thinking. She did not think the measurement strategy was a sophisticated approach to solving the task because it was not a formal proof. Barbara was also disappointed that the students only used three to five cases to support their claims. She thought they did not use enough cases, whereas the students were satisfied with and convinced by the three to five cases. She felt this response showed a lack of student understanding of what it means to support an answer with a justification.

Barbara's rigid mathematical thinking often influenced what she noticed about student developmental sequences for various tasks. For example, with the Box Task Barbara expected the students to quickly transition from physically cutting and measuring to labeling a side of the box with a variable. She did not discuss the understanding that students needed to have in order to make that transition, nor did she recognize that there was a developmental step between the physical measurement and the algebraic strategy. Barbara did not mention that students needed to recognize that they could find the measurements of the box without physically cutting it.

In addition, there were several instances in class in which Barbara was unable to understand a solution a student was presenting. Many of these instances included visual representations that Barbara did not understand. In one case she noted that she did not understand what a student was doing until she thought about it algebraically. She originally thought the student was doing something incorrectly because it did not make sense to her. Barbara took the student's explanation and created an algebra equation. After she explored the solution using algebra she realized how the student was thinking and what she could do to help the student.

Barbara's rigid mathematics pertaining to certain mathematical ideas also led her to limited descriptions of her students' mathematics and caused her to notice what students did not do. Barbara was flexible in her thinking when it came to mathematical concepts that she understood more deeply, and consequently she described her students' mathematics in more depth in those instances. Specifically, she was more flexible with algebraic solutions and with angle and arc relationships in circles.

Judy was flexible in her mathematical knowledge. She was able to adapt her thinking to the ways her students were thinking. She did not make statements that implied that she was measuring her students' mathematics against her own mathematics, which allowed her to be flexible in thinking about her students' thinking. Judy could look at a figure or an algebraic solution and interpret her students' thinking, which was evident in the ways that she interacted with students in class and described her students' mathematics. She rarely discussed what students could not do, but rather what they did do. The way that Judy described her students' thinking showed that she thought it was complex. Furthermore, she provided details of students' developmental sequences and gave evidence from her classroom observations. For example, when Judy described her students' solution to the Box Task, she commented on how some students had to physically cut each square and measure, whereas other students recognized that they could find the box measurements without actually cutting squares. She was impressed by her students' ability to generalize but noted that the students who did not generalize were not lacking in understanding.

The flexibility of mathematical knowledge came across strongly in the two teachers' descriptions of what they noticed, but there were other aspects of their mathematical knowledge that were important those descriptions. For example, the teachers thought that it was important for students to make connections among mathematical ideas and to provide fully developed justifications for their solutions. These connections and justifications were related to the connections and justifications that the teachers noted when they discussed the ways they would solve the task and the mathematical ideas of the lesson. The teachers noticed instances in which students were or were not making these mathematical connections or justifications. Before the lesson on the Box Task, Barbara discussed her solution to the task. She noted that she would create an algebraic equation and then analyze the graph. Creating a table was not part of her solution, and she thought that the equation and graph were viable and justifiable solutions. She thought her students would start with a table and graph, and then move to the algebra. She noted, "I hoped that they would understand, or start to recognize, if they hadn't already… that numerical data in a chart is sometimes void in terms of interpretation" (interview, 12/19/06).

Teachers Noticed Student Thinking Through Their Own Actions

The teachers talked about students' mathematics based on their own actions in the classroom. The teachers would reference an action they did in class that influenced the way the students were thinking about or solved a task. One type of influence that they mentioned was missed opportunities, which were situations in which the teacher did not take some action that would have led the students down a particular path. For example, Barbara thought she had not provided students with a good reason to develop a formula for the Box Task. She felt that her action influenced the way her students approached and solved the task.

The teachers also mentioned instances that something that they said or did influenced the ways their students were thinking about the task. For instance, Barbara thought that she influenced some students to solve the trapezoid task the same way that she solved it. She said, "I misled [them] by making them think they had to see it that way. So, that probably predisposed them to not see the trapezoid-and-circle relationship. I was sorry that I had done that" (interview, 9/26/06).

Teachers Noticed Noteworthy Aspects of Classroom Interactions

There was an interview question that prompted teachers to identify important moments during classroom interactions—which was one of the skills of professional noticing. The two teachers answered this question differently. Barbara struggled to answer the question and consistently changed the question to what were “interesting” aspects of the lesson. She made the following statements: “It may come to me later” (interview, 9/26/06), “Nothing leaping into my mind about any positive academic interactions that were outstanding” (interview, 11/27/06), “I don’t remember anything outstanding or significant,” and “Oh gosh, I just think this lesson is probably jam packed with some interesting interactions, but, um, interesting is relative” (interview, 12/19/06). During the interviews, I never used the word *interesting* and often tried to focus Barbara on “important” moments from class. When she made comments other than the ones just mentioned, she said something like, “Any time you can get a kid to go there easily” (interview, 10/25/06). She gave an example from class when a student, John, explained a solution to another student, Katie. After the explanation, Katie said that the problem was easy and that she now understands what she did wrong. Barbara thought this was an important moment because Katie caught onto John’s explanation easily. Judy, on the other hand, made comments related to the students' struggle, the interactions that helped students get past that struggle, and students developing conceptual understanding. For instance, Judy noted, “I think one thing that was important was looking at when people got different models to decide which one’s right, and why they’d do this” (interview, 11/20/06). Another example Judy identified was a debate between two students while solving the Box Task. One student cut large squares out of the paper and made a tall, skinny box, whereas the other student cut small squares out of the paper and made a short, fat box. The two students debated which box had the larger volume. The debate became a topic that was discussed in the whole group. Judy thought this debate was important because students in the class were drawn into it, and the debate encouraged the students to develop a conceptual understanding of volume.

Factors that Influence Teachers' Noticing

As I began this study, I was interested only in the ways that teachers make sense of their students' mathematical thinking and how teachers work off of that thinking in class. I realized that teachers do not have many opportunities to verbalize their interpretations of their students' mathematical thinking, so I hypothesized that verbalization would be difficult for teachers. That hypothesis was confirmed. It was clear that for many reasons the teachers struggled to notice their students' mathematical thinking. This study provides insights into some of those reasons, so I considered the following question when responding to the second research question: What factors influence a teacher's capacity to notice students' mathematical thinking?

The complexity of the classroom was one obvious and predictable factor that was evident in the data. That complexity includes managing behavior issues, the number of students in the class, and class interruptions. Both Barbara and Judy had behavior issues that took them away from their teaching. Furthermore, the teachers' classes were interrupted for various reasons by other faculty members or the school office. Those interruptions caused the teachers to lose time observing and listening to their students. The teachers also noted that it was difficult to attend to every student when there were 20 to 24 students in a class. They acknowledged that on some days the students that they noticed were the ones who spoke up in class, whether the students were making comments, answering a teacher question, or asking questions of the teacher. If the teacher did not interact with a student doing mathematics, then it was difficult for her to notice how that student was thinking mathematically.

Limited time was a major issue for both teachers. The teachers felt they had limited class time and limited planning time. Barbara often made comments like, "I feel like I'm racing the clock" (interview, 11/27/06). Both teachers were concerned about having enough time to cover all the concepts in the state curriculum. This concern influenced their instructional practices, which in turn influenced the ways they noticed their students' mathematical thinking. For example, Barbara felt that she was very teacher centered during the lesson on arc and angle relationships in circles. She noted that she did not have additional class time to use student-centered lessons. She felt that student-centered lessons consumed more class time than teacher-centered lessons. Another example of time constraints was when one student noticed an error he had made in his work, but he was not sure what he had done. Barbara was unable to describe the error and noted, "I really hadn't looked at the data to analyze his thought. I was too rushed for that" (interview, 12/19/06).

In addition to time constraints because of the curriculum, the teachers' lack of planning time influenced their noticing. In several instances, the teachers had not planned their lesson when I interviewed them during their planning period the day of the lesson. Barbara reflected on how she would change one lesson, saying, "I would hope that I didn't have to go get my son from school and spend my whole planning period almost doing other things" (interview, 11/28/06). I often interviewed Judy by the copy machine as she was finishing her plans and making copies of a worksheet for class. The lack of planning time caused the teachers to create tasks that did not match the intended goal for the lesson. For example, Barbara created an area task quickly and did not realize that it had multiple solutions. Her students solved in a different way than she had intended. She noticed the overall approach students took with the task, but she did not notice the idiosyncrasies or the details of their solutions. A lack of planning caused her discomfort and influenced what she noticed.

Another factor that influenced the teachers' capacity to notice students' mathematical thinking was the extent to which they interacted with the students. If the teacher did not interact with students while they were engaged in doing mathematics, the teacher was unable to describe the their mathematical thinking. Two factors influenced whether the teacher interacted with students engaging in mathematics: the mathematics tasks incorporated in the lesson and the type of listening. The choice of mathematics task was important in determining whether the teachers noticed the students' mathematical thinking. The higher the cognitive demand of the task, the more likely the teacher would interact with multiple students who were engaged in mathematics. In other words, the teachers were less likely to notice the students' mathematical thinking when the students were engaged in procedural tasks. In addition, the type of listening the teachers engaged in determined what they noticed about students' mathematical thinking. When the teachers engaged in evaluative listening, they could not describe what they had noticed about students' mathematical thinking beyond the ways they intended students to think. When the teacher was listening interpretively, she noticed some aspects of students' mathematical thinking, but still in terms of whether those aspects measured up to the ways she intended students to think. Lastly, when the teachers listened hermeneutically, they provided elaborate detail about what they noticed.

As mentioned in the previous section, the flexibility of the teachers' knowledge influenced their ability to notice students' mathematical thinking. When the teacher was not as comfortable with the mathematics, she was less likely to notice students' mathematical thinking. Barbara made several comments about not understanding her students. In those instances, she was not flexible enough in her own mathematical thinking to consider the ways the students were thinking.

Teacher Responding

Previously I detailed what teachers noticed about their students' mathematical thinking. Responding was one of the four skills in professional noticing, but I chose to address it separately because it was the only skill that required the teacher to act. In order to address the third research question—In what ways do teachers respond to students' mathematical thinking in classrooms?—I revisited the way that I am use the term *students' mathematical thinking*. In the analysis, I considered instances in which the teachers responded to observable student actions that represented abstraction and generalization of mathematical ideas. In addition to thinking about the way that I described students' mathematical thinking, I also considered the ways the teachers described their students' mathematical thinking. I began by examining how they responded to students' mathematical thinking during classroom interactions. I considered instances when the teacher noticed students' mathematical thinking and acted on that thinking during class. During my classroom observations I had access only to what the teacher did in front of the whole class, and therefore most of my data came from instances that affected the whole class, not smaller groups of students. From the interview data I also considered teacher-identified instances of acting on students' mathematical thinking.

In the interviews the teachers brought up aspects of students' mathematical thinking from their observations during group and individual work, whole-class discussions, student presentations of solutions, and student questions or comments. After the teachers recognized students' mathematical thinking, they had two choices: they could act on that thinking or not. The ways the teachers acted on that thinking fell into five categories: the teacher posed questions; the teacher asked students to present solutions or mathematical ideas; the teacher posed another task; the teacher told what to do; or the teacher used a pedagogical content tool (see Table 6 for examples). The table represents the range of ways that the teachers acted on students' mathematical thinking. The examples are not exhaustive, but they do show the differences between categories. For instance, the range in responses for the category "teacher posed questions" included the teacher leading students to a mathematical idea, the teacher assessing what students understand, and the teacher trying to understand what the students were thinking. These types of questions fall into Mason's (2000) three forms of asking: focusing attention, testing, and enquiry, respectively. Furthermore, the three forms were helpful in considering the other ways teachers acted on students' mathematical thinking. For example, the different types of "teacher telling" and "teacher used a pedagogical content tool" fell into the category "focus attention."

These categories overlapped in many instances. For example, sometimes the teachers' questions were actually used as telling. Consider the following two questions from Barbara: "What do you know about this distance right here [she drew a chord connecting two consecutive points on a circle]?" and "Is it the same as your radius?" (video, 9/26/06) This is an example of the teacher acting on students' mathematical thinking because of the teacher-student exchange prior to the two questions. The student made a conjecture about a polygon inscribed in a circle. It represents an instance when the teacher was questioning and telling. The students did not answer the first question and most of them responded yes to the second question. The second question clarified the first question but also revealed the kind of response Barbara wanted.

Table 6

Ways Teachers Acted on Students' Mathematical Thinking

Action	Example
Teacher poses questions	1. During the Window Task Barbara wanted her students to justify their solutions with an algebraic proof. When students did not solve the task with algebra, Barbara asked questions to led students to the algebraic proof.
	2. Judy asked questions to understand a student's justification for his or her answer about the largest perimeter on the Window Task. Judy made some mathematical connections for herself by asking the student questions.
	3. Judy asked students to describe how they were visualizing the 3-dimensional object in their mind.
Teacher asks students to share solutions or mathematical ideas	4. During the circle lesson Barbara noticed two groups of students who approached finding the measure of an angle in different ways. She asked both groups of students to share their method.
	5. Barbara asked a student to share her equation for the perimeter of the Box Task.
Teacher tells	6. During the Box Task Judy noticed several students trying to create a box using nets instead of cutting squares from the four corners. Judy was concerned students would get stuck on the nets and lose sight of the original task, so she told the students to cut squares from the corners to create the box.
	7. Barbara introduced the words *major arcs* during the lesson on circles. She also defined the words.
Teacher poses another mathematical task	8. During the Window Task Judy thought students did not have an understanding of where the parabola came from, so she changed the task. She asked students to find the area of a rectangle if the perimeter was 24.
Teacher uses a pedagogical content tool	9. On the second day of the Window Task, Barbara recorded student data on the board to illustrate a way to organize data and to move the mathematical discussion forward.
	10. Judy wrote the equation on the board that students were beginning to develop for area of regular polygons. This move organized the students' mathematical thinking.

It is important that just because a teacher performed one of the five ways to respond to students' mathematical thinking did not mean that she was responding to the students' mathematical thinking. The defining factor included the teacher noticing students' mathematical thinking during class and acting on that thinking. I did not consider instances in which I did not think the students' were thinking mathematically. For example, during the interviews the teachers would share their thoughts about the lesson, and sometimes they identified instances in which they used students' mathematical thinking. When I asked Barbara if there was anything that she wanted to add to her interview about the circle lesson, she noted, "I really thought that I worked off of their prior knowledge and was able to take them to a new place easily" (interview, p. 11/27/06). During the lesson Barbara had asked students the question "What other words do you know?" (field notes, 11/27/06). One student responded by saying, "An arc." Barbara then showed an arc on a circle and asked the students if they had ever heard the words *major arc* and *minor arc*. No student responded, so she proceeded to define the terms. This teacher-students interaction was not an example of the teacher responding to students' mathematical thinking. The students were not engaged in mathematical activity and were not thinking mathematically. If a student had elaborated on the meaning of the word *arc*, then that would have been an example of students' mathematical thinking.

There was a cyclic relationship between students' mathematical thinking and the teachers' actions in the classroom. After the teachers noticed students' mathematical thinking, they chose whether to act on that thinking or not. If the teacher acted on the thinking, that action influenced the students' mathematical thinking,[4] which in turn influenced whether the teacher

[4] I did not use the word *influence* to indicate that students' mathematical thinking changed in significant ways. I was not analyzing students' mathematical thinking, and therefore I did not look for instances in which the students' mathematical thinking changed. I did, however, look at

acted again. In some teacher-students interactions, the students and teachers went through this cyclic relationship several times, whereas in other interactions the students and teachers went through it once. Consider the previous examples (Table 6). When Barbara wanted students to solve the Window Task using algebra, she intervened when the students used only measurement. In the following excerpt notice the cyclic relationship between Barbara's actions and the students' mathematical thinking.

B: Okay. Guys, do we have a proof?

S: [Various responses.]

B: We know this numerical data is not a proof. That's the problem with numerical data. But what do we need to, um, to have a proof? [No students speak loudly for Barbara to hear.] Well, let's look at this. What do we call this length when we don't know it?

S: *X* [Several students speak]

B: What's *X*? Is this side *X*? [Barbara points to segment GB.]

S: Yes

B: So, what do we need to find? Do we know a way to find the other piece? I want you to put *X* on this side and figure this other side. What do you know about this other side?

S: It's 10.5 minus *X*, and that makes that other side half of it.

B: So, you used your 30-60-90 relationships. You guys go ahead and find that other side. What kind of triangle is this? What kind of side is this? Okay, you guys try this.

Notice the multiple interactions between the teacher and the students. Barbara was scaffolding questions to influence her students' mathematical thinking. She noticed students' mathematical thinking and acted on it, which influences their subsequent mathematical thinking. This cycle continued until the teacher decided not to respond further. In contrast, in Example 6 from a lesson on the Box Task (Table 6), Judy acted on her students' mathematical thinking once. She

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instances in which the teachers made an action for the purpose of influencing students' mathematical thinking.

told the students to cut squares out of a sheet of paper. Once the students began cutting squares, she did not act any further on her students' mathematical thinking pertaining to creating the box.

It was clear that the teachers acted on their students' mathematical thinking for particular reasons. When the teachers chose to intervene in some manner, it was to influence the students' mathematical thinking. In the following section I elaborate on why the teachers acted on students' mathematical thinking.

Why the Teachers Responded to Students' Mathematical Thinking

Lobato et al. (2005) suggested that the response of teacher telling should be reconsidered in terms of the function of the telling instead of the types of telling. Furthermore, they suggested that telling should be considered in terms of conceptual content and should be situated within the surrounding teacher actions. I applied their suggestions to consider the reasons that the teachers responded to students' mathematical thinking. Teachers' intentions are not easy to infer from simply looking at the action, but the reasons become clear when I examined the action in the context of other actions and with interview data pertaining to the action. The majority of the instances in which the teachers acted on students' mathematical thinking were for the purpose of influencing that thinking. Sometimes the teachers wanted to influence individual students' mathematical thinking, and sometimes they wanted to influence subgroups or the entire class of students. This influence fell into three categories: to expand or elaborate on students' mathematical thinking, to change students' mathematical thinking, or to support students' mathematical thinking.

At times the teachers acted on students' mathematical thinking for the purpose of expanding and elaborating that thinking. These were instances in which the teacher wanted to build on her students' mathematical thinking, and they typically occurred when the teacher noticed an interesting solution or thought students were on the right path. For example, a student in Barbara's class asked whether it was a coincidence that the opposite angles in a quadrilateral inscribed in a circle were supplementary. This question surprised Barbara. She did not anticipate that conjecture. She realized the conjecture was valid and decided to address the question in class. She led the students through a series of questions that addressed the reasons that the conjecture was valid. In this example, Barbara decided to use the student question as a way to prove a conjecture. She asked the class to pay attention and to think about the conjecture. The series of questions built on what she believed the students already knew. After the lesson Barbara talked very positively about this teacher-to-students interaction. Another example from Barbara's class was Example 4 in Table 6. In this example, Barbara asked two students to share their solutions in order for the class to make connections between the solutions.

In some instances the teachers wanted to change the students' mathematical thinking because the teachers thought the students were on a path that would not lead to a viable solution or the "correct" solution. In other words, the teacher intervened to change the students' mathematical thinking to either the desired way of thinking or away from an incorrect way of thinking. It was the teacher's intent to encourage students to pursue another line of thought either to prevent unsuccessful work or to focus on particular mathematical ideas. In Example 6 from Table 6 mentioned earlier, Judy told her students to cut squares from the corners of a sheet of paper. She did not think using nets to solve the task would give a viable solution because she thought it would not allow the students to maximize volume and surface area. She told the students to create the box a different way to keep them from spending time solving the task another way that probably would not work. In this intervention the teacher wanted to prevent the student from pursuing an unsuccessful method. In Example 1 from Table 6, Barbara wanted to focus students' attention on a particular solution. She wanted the students to solve the Window Task with algebra, and therefore she intervened with questions when students approached the task a different way.

There were instances in which the teacher did not want to change what the students were thinking, nor did the teacher want to expand their mathematical thinking. During those instances the teacher wanted to support students in their solutions. In many cases, they wanted to help the students with some procedural step in the task. For example, during individual and small group work, the teachers generally walked around to observe and help students. When the teachers noticed several students having difficulty with the same procedural task, the teachers addressed the issue to the whole class. Consider Barbara's interaction with her students about simplifying $\sqrt{98}$. She was helping students work on a textbook assignment when she noticed that several students had difficulty simplifying $\sqrt{98}$. The book asked students to find the area of a given region, and as part of the task the students needed to simplify $\sqrt{98}$. She decided to have a whole class discussion about the difficulty. She told the students that she did not care how they simplified the radical and showed three methods for finding factors of 98—using perfect squares, using a factor tree, and using what she called the upside-down cake way. In this example, Barbara used students' mathematical thinking to acknowledge and address a difficulty students were having. When she addressed the difficulty, she did not build on the students' current understanding. She simply identified that simplifying $\sqrt{98}$ was something that many students did not remember how to do and explained the steps to them. She wanted to help students remember how to simplify radicals, but she was not interjecting to expand or change the students' current mathematical thinking.

As I mentioned earlier, in the majority of the instances in which the teachers acted on students' mathematical thinking, the purpose was to influence that thinking. In some cases the teachers acted on students' mathematical thinking without the intent of influencing that thinking. I identified two categories: (1) teachers acted to understand students' mathematical thinking, and (2) teachers acted to assess students' mathematical thinking.

The teachers rarely acted to understand students' mathematical thinking. Example 2 from Table 6 represents an instance in which the teacher acted for that purpose. In this example Judy was intrigued by the student's solution and wanted to understand it. She asked questions because she wanted to learn from the student, which she did.

When I used the term assess, I refer to teachers measuring students' mathematical thinking against some criterion that the teacher had in mind. The teachers were constantly assessing students formally and informally. When a teacher acted on students' mathematical thinking for the purpose of assessing that thinking, I considered the teacher to be informally assessing the student. There was a lot of overlap between the teachers acting on students' mathematical thinking for the purpose of assessing and the other described reasons for teacher actions. For instance, in the example just mentioned (Example 2 from Table 6) Judy was asking questions to understand the student, but she was also assessing the student's thinking throughout the interaction. She was trying to figure out whether the solution was mathematically correct.

Concluding Remarks

I began this chapter by mentioning the goal for the study was to begin to conceptualize the ways that teachers understood and used students' mathematical thinking in their instructional practices. To examine that goal I focused on teachers' "informal" learning trajectories, what teachers noticed about students' mathematical thinking, and the ways that teachers responded to students' mathematical thinking. I have looked at each of these pieces separately, but the pieces were interconnected. The Mathematics Teaching Cycle brought attention to the interconnected relationship between teachers' learning trajectories and classroom interactions. In this section I began to synthesize the responses to the three research questions in order to connect ideas between teachers' learning trajectories and professional noticing.

As I expected, the learning trajectories that the teachers created for students' mathematical thinking were informal. I did not ask the teachers to specifically create trajectories, but I prompted them to describe each aspect of the learning trajectory, and I created the trajectories based on their responses. The teachers elaborated on the mathematical activities, but they struggled to describe aspects of the learning process. This struggle happened in both the pre-observation and post-observation interviews. The struggle implied that Barbara and Judy were more comfortable discussing their own mathematics than their students' mathematics. The teachers' least favorite question was, "If you were talking to a beginning teacher, how would describe the ways students think about *insert topic of lesson*?" I asked that question in every interview, and I asked similar questions, but the teachers continued to struggle with their responses. The teachers' learning trajectories focused more on the mathematical activity than on the learning process. Before the lessons, the teachers described some aspects of the task that students might find difficult, but many of those difficulties were not connected to conceptual understanding (e.g., the student might use the wrong trigometric ratio). After the lessons, the teachers described student struggles that were due to a lack of conceptual understanding.

As mentioned earlier the ELT[5] was sometimes led by the teacher and sometimes led by the students. In any given lesson the agent who was leading the ELT might change. It was always clear, however, which agent was leading the ELT for the majority of the lesson. When teachers were the leaders of the ELT, they were influenced less, and in different ways, by students' mathematical thinking than when the students co-led the ELT (see Figure 14). When the students led the ELT, the teachers had more opportunities to engage with students doing mathematics. From those interactions the teachers were able to notice students' mathematical thinking, which affected the ways the teachers could respond to students' mathematical thinking. Consider the lessons pertaining to the Window Task in both teachers' classrooms.

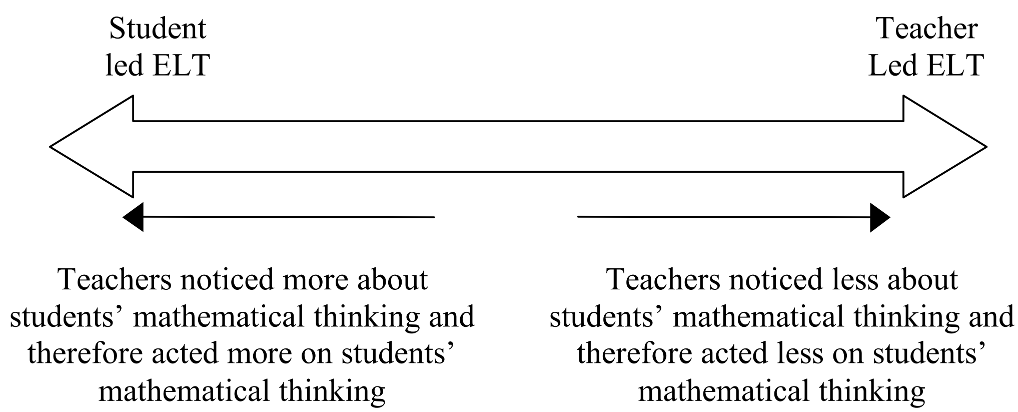

Figure 14. Analysis of learning trajectories.

During the first day of the Window Task both classes spent the class period solving the task in groups with little direction from the teachers. Both teachers could describe student solutions relatively well. On the second day, both teachers made instructional decisions based on

[5] The word "lesson" could replace ELT in this instance as the teachers also led the lessons. My point is to emphasize the connection between learning activities and learning process, so I use the term ELT.

their interpretations of their students' mathematical thinking on the second day that changed the direction of the lesson and consequently the ELT. The classes looked very different on the second day. On the second day, Judy decided to further explore the direction the students were currently going. The students co-led the class on both days. As on the first day, Judy described students' mathematical thinking well. She noticed several important interactions in class and described her interpretations of students' mathematical thinking. She responded to that thinking in various ways on both days, including posing a new task and questioning students for her own understanding.

Barbara wanted the students to develop algebraic solutions, and she led the class on the second day. The students' mathematical thinking did not have a significant impact on the direction of the lesson on the second day because Barbara had a set goal in mind. She led the discussion; she used student responses to advance students toward her goal. She struggled in the interview to describe students' mathematical thinking and did not note any important interactions. She did act on students' mathematical thinking, but only for the purpose of reaching the intended goal. Her responses were limited in number, and she only acted on the mathematical thinking of a few students who spoke in class.

The Window Task example illustrates the connection between who led the ELT, what the teacher noticed about students' mathematical thinking, and how the teacher acted on students' mathematical thinking.

CHAPTER 5

CONCLUSIONS

> The problem of the pupils is found in *subject matter*; the problem of teachers is *what the minds of pupils are doing with this subject matter*. (Dewey, 1933/1989, p. 338)

A large body of literature in mathematics education suggests that teachers make instructional changes when they focus on student thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Lubinski & Jaberg, 1997). In addition, research also indicates that instructional practices that support and build on students' thinking will promote students' mathematical understanding (Fennema et al., 1993; Hiebert & Wearne, 1993). To support and build on students' mathematical thinking, one must consider the relationship between the learning activity and learning process. Teachers have learning goals for their lessons and select learning activities based on those goals and their students' current mathematical understanding. In other words, when planning lessons one must consider how a student might engage in a mathematics task and what learning might happen because of that engagement. A major task for teachers is to plan, organize, and manage learning activities for all students that build on and extend students' knowledge. For mathematics teachers to create meaningful learning activities for all students, they must know ways that students think mathematically. One way that teachers gain such knowledge is through interactions with students doing mathematics. The goal for this study was to describe and understand the ways that teachers make sense of students' mathematical thinking during classroom interactions. Specifically, the research questions for this study were the following:

1. What learning trajectories do teachers describe of students' mathematical thinking when they participate in a professional development experience that focuses their attention on their own geometric thinking and their students' geometric thinking?
2. What do teachers notice about students' mathematical thinking during their teaching practices?
3. In what ways do teachers respond to students' mathematical thinking in mathematics classrooms?

The Mathematics Teaching Cycle (Simon, 1995a, 1995b) provided a lens for considering the complex ways that teachers make sense of and use students' mathematical thinking in their teaching. The Mathematics Teaching Cycle is composed of three interrelated components: teacher's knowledge, hypothetical learning trajectories, and classroom interactions. The teacher's knowledge influences her or his development and modification of learning trajectories, and learning trajectories influences what happens in classrooms. Furthermore, what happens in classrooms influences the teacher's knowledge. I consider professional noticing an important component of classroom interactions. Professional noticing and learning trajectories were used in the present study to examine ways that teachers' understand students' mathematical thinking.

Steffe (2004) defined learning trajectories as models of students' mathematics that represent students' starting points, the changes that occur because of the mathematical activity, and the interactions that are involved in those changes. The learning trajectories that Steffe creates are the products of extensive interactions with students. In Simon's (1995b) discussion of learning trajectories, he noted that learning trajectories are hypothetical because they are "the teacher's predictions to the path by which learning might proceed" (p. 135). The hypothetical learning trajectory includes three components: the intended direction, the learning activities, and the hypothetical learning process. Both Steffe and Simon thought that teachers refine and modify existing learning trajectories through their interactions with children.

In Simon's Mathematics Teaching Cycle (1995b), the component classroom interactions includes the teacher facilitating discourse, posing problems, and inquiring into students' mathematical thinking. I claim that professional noticing is an implicit part of classroom interactions and one that I believe is important to consider when examining the ways which teachers make sense of students' mathematical thinking. Teachers must notice students' mathematical thinking in order to act in meaningful ways. Jacob et al. (2007) argued that new ways of noticing can help teachers develop new ways of reasoning and those experiences should be a part of professional development. They defined the term *professional noticing* as "how professionals view and make sense of complex situations" (p. 5). They also described four interrelated skills for expertise in professional noticing: identifying, describing, interpreting, and responding. All four of these skills were examined in this study. Identifying requires the teacher to take note of important aspects of the classroom interactions. Describing yields a detailed but objective account of what happened during classroom interactions. Interpreting takes describing a step further by making sense of what individuals observed (Jacob et al., p. 7).

The last skill, responding, refers to the teacher acting on what he or she sees in classroom interactions. Teachers can respond in various ways, such as asking questions, telling, asking students to share, or using a pedagogical content tool. Lobato et al. (2005) suggested that educators should reformulate teacher telling by considering it in terms of conceptual understanding, the function of telling, and the context of the telling. These suggestions are applicable for considering teacher responses in classroom interactions. In other words, why did the teacher respond, what happened in class before and after the response, and how does that action relate to students' conceptual understanding?

Underlying the four professional noticing skills identified is listening. The ways that teachers listen to students influence what teachers' notice about students' mathematical thinking. If a teacher is listening for a particular answer, she or he may not notice alternative student solutions. D'Ambrosio (2004) defined listening to be more than hearing what students say in classrooms. She considered listening as being aware of various products produced by children (e.g., student work). In Davis's (1997) work with a beginning teacher, he noted the teacher transitioned through three types of listening. The first type of listening is evaluative, which is when the teacher is only listening for a particular response and does not hear other solutions. Interpretive listening occurs when the teacher is listening for a particular response but also hears other responses. The final type of listening is hermeneutic. In hermeneutic listening the teacher is listening not for a particular response but for the purpose of understanding how the students are thinking. The teacher's type of listening influences what she hears and consequently what she notices.

The participants in the present study were two high school geometry teachers. They were selected because they were experienced teachers with advanced college degrees who were willing to discuss their instructional practices and attend working group sessions. Data were collected for both teachers during one semester in one of their semester-long geometry classes. Over the course of the semester I observed 13 to 16 lessons and held six working-group sessions. I took field notes for each observation. Some of the lessons lasted 1 day, and other lessons spanned 2 days. I interviewed the each teacher for 30 to 60 minutes before and after six lessons. The focus of the interviews was to obtain the teachers' descriptions of the components of the learning trajectories and what they noticed about students' mathematical thinking. When possible, the classes were observed the day before and the day after these six lessons. The six lessons for each teacher and all of the working-group sessions were videotaped. In addition, teaching artifacts for the observations and the working-group sessions were collected and analyzed.

Many teachers do not regularly discuss the ways students' think mathematically with their colleagues. With that in mind, I held working-group sessions that occurred approximately every 2 weeks. The purpose of the sessions was to support the teachers in their efforts to discuss their students' mathematical thinking. I planned and led the working-group tasks, which included solving a mathematics task, watching videos of students solving tasks, creating tasks to use in their own classrooms, and discussing their students' solutions to the tasks they developed. In two of the sessions, the teachers created or modified a task to use in their own classrooms. The two lessons that included those tasks were part of the six lessons for which pre- and post-interviews were conducted. Learning trajectories and the skills of professional noticing informed my planning for and my actions in the working-group sessions, but these ideas were not explicitly discussed with the teachers.

I used an interpretive stance towards the data analysis. I began the initial analysis during data collection through a research journal. The contents of the journal included my reactions to the lessons, ideas about possible answers to the research questions, and themes that were emerging. After data collection, I began a retrospective analysis with four stages: coded the data, created learning trajectories, revisited data with a coding scheme, and looked for confirming and disaffirming evidence. After I developed a coding scheme, I revisited the literature to consider existing coding schemes. I created a projected learning trajectory (PLT) and an enacted learning trajectory (ELT) based on interview and observation data for every lesson. Then I revisited the coding scheme to confirm and disaffirm the findings. The process was cyclical, and themes evolved though coding, reorganizing data, comparing categories, looking for confirming and disaffirming evidence, and reassessing categories.

Findings

The three research questions examined the teachers' learning trajectories, what the teachers noticed in classrooms, and ways the teachers responded in classrooms. As hypothesized, the teachers had ideas about the ways that the students think mathematically, but these were often undeveloped ideas. The teachers' learning trajectories in this study documented that conclusion. Furthermore, the teachers' noticing skills affected their development of learning trajectories and vice versa.

I developed the projected learning trajectories (PLT) and the enacted learning trajectories (ELT) for the teachers based on the data, and consequently they were my perceptions of the teachers' learning trajectories. The PLTs described the teachers' learning goals, learning activities, and predicted learning processes for students. The learning processes included the teachers' predictions about students' starting points, learning that might occur, and endpoints. The ELTs illustrated the teachers' perceptions about the learning goal, the learning activities, and the learning processes after the enacted lesson. The teachers' instructional approach influenced the teachers' ELT. In some lessons, the students led the ELT, whereas in other lessons the teacher led the ELT. When the students led the lessons, the teacher gave more detailed accounts of the ELTs. In addition, when students led the ELTs, the teacher gave more detailed accounts of what she noticed during classroom interactions.

The PLT and ELT were similar in some instances but not others. When the trajectories were not similar, the teacher noticed some aspect of students' mathematical thinking that caused her to respond in a particular way. The teacher's response either supported the students going in a different direction or pulled them back to the intended direction.

In lessons in which the ELTs were led by students, the teacher listened either interpretively or hermeneutically. The teacher noticed students' mathematical thinking and responded in ways that reflected her interpretation of that thinking. When the teacher led the ELT, she listened either evaluatively or interpretively. In many of those lessons, the teacher struggled to notice students' mathematical thinking beyond the intended thinking and responded in ways that encouraged students to react in particular ways.

As mentioned earlier, there were four skills to professional noticing: identifying, describing, interpreting, and responding. The teachers described and interpreted what they noticed in terms of their uncertainties and surprises about students' mathematical thinking. Furthermore, the teachers described what they noticed about students' mathematical thinking in terms of the mathematics tasks, their own mathematical knowledge, and their actions with students in the classroom. When I asked the teachers how their students were thinking, they often responded by detailing the steps in the mathematics task. One teacher would sometimes give an analysis of student thinking related to student conceptual understanding, but that occurred only with this participant. In most instances the teachers described what students "didn't do correctly" instead of what they did do.

When the teachers identified important moments in classroom interactions, they emphasized aspects of the interaction that they deemed valuable for student learning. One participant often interpreted “important” moments as “interesting” moments. She rarely identified “interesting” moments in class because she typically wanted specific solutions from students. The teacher listened evaluatively or interpretively, which influenced what she identified as important. When she pushed for the specific solutions, she was often disappointed in the lesson, the students, and herself. In contrast, the other participant identified several important moments in each lesson. The important moments typically represented instances in which a student struggled and then made a breakthrough. At times, the teacher listened to her students for the purpose of understanding and learning. The teacher was listening hermeneutically. There were other times that the teacher was listening evaluatively.

The fourth skill to professional noticing was responding. In this study, the teachers typically responded in five ways: posed a question, asked students to share, told the students something about the mathematics, posed another task, and used a pedagogical content tool. Taking a cue from Lobato, Clarke and Ellis (2005), I examined the reasons for these teacher responses. The most common reason was the teacher wanted to influence students’ mathematical thinking in relation to completing a task or understanding a mathematical idea. The teacher wanted to change the students’ thinking, help them with their thinking, or elaborate or expand their thinking.

Overall, what the teachers noticed in the classroom interactions and the ways that they responded to students affected their development of PLTs and ELTs, and vice versa. When the students led the lessons, they consequently led the ELT. Furthermore, when the students led the ELT, the teachers gave detailed descriptions for the learning trajectories and gave more details for what they noticed during classroom interactions. Underlying these results are the teachers’ instructional approaches to teaching. When the instructional approach was investigative and the teacher was listening interpretively, she was able to give more details about the ways students were thinking mathematically. In contrast, when the instructional approach was not investigative and the teacher was listening evaluatively, she gave less detail about the ways students were thinking mathematically.

The teachers often made comments in interviews about whether students approached the tasks as they predicted or not. When the students approached tasks differently than the teacher anticipated, she would often make a remark about the student's solution during the interviews and she usually mentioned how it surprised her. The teacher expected the students to approach the task in a particular way, and she noticed when students didn't. This suggests that the teachers' learning trajectories influenced what they noticed during classroom interactions.

Implications for Teacher Education and Professional Development

Part of the value of this study is the insights that it gives teacher educators for creating meaningful learning experiences for teachers. Underlying the study is my assumption that teachers need to understand how students think mathematically in order to make informed and productive decisions about learning experiences for students. If we apply this assumption to teacher educators, then it suggests that they need to understand how teachers think about mathematics, students, and pedagogy in order to make informed and productive decisions about learning experiences for teachers. If teacher educators plan to create meaningful learning experiences that build on and support teachers, then they must have knowledge of teachers' thinking. While this study focuses on two teachers, it contributes to the body of literature about how teachers think about their instructional practices and specifically how they think about students' mathematical thinking.

Teachers interact with students every day, and they develop knowledge about how students think mathematically through those interactions. However, that knowledge is disjointed and informal. Learning trajectories and professional noticing are constructs that can be used in professional development experiences to help prospective and practicing teachers build connections between their own mathematical knowledge, their students' mathematical knowledge, and learning activities for students.

The advantage to considering learning trajectories in professional development experiences is that learning trajectories go beyond lesson planning to consider the ways students learn. Learning trajectories are comparable to lessons because they provide opportunities for teachers to identify key ideas in learning goals and examine possible learning activities that address those goals. Lesson plans that teachers create in their day-to-day work do not make an explicit connection between the learning activity and the learning process. Learning trajectories encourage teachers to identify students' learning process when engaging in particular activities. One way to incorporate learning trajectories in professional development experiences would be for teachers to examine existing learning trajectories created by researchers. Once teachers have developed an understanding of the learning trajectory, they can use the learning activities in their classrooms. Then the teachers can reflect on their lessons and make modifications to how students learn the concept based on their classroom observations. Another possible way is for teachers to create their own learning trajectory for a key concept in a course they teach. The learning trajectory can contain several possible tasks and links to what students might learn from the task. I am not suggesting that teachers should create learning trajectories like they create lesson plans. Teachers do not have time to create learning trajectories for all mathematical ideas that they teach. However, the experience of explicitly connecting the learning process and learning activity can help teachers notice and respond to students' mathematical thinking during classrooms interactions. When the teachers create and modify learning trajectories based on lessons, they need to go beyond describing the steps in the mathematics tasks. The teachers need to connect what students do with what students understand. A third way to incorporate learning trajectories is for the teacher to examine several tasks for a particular learning goal. Teachers can describe what they anticipate that students will learn from the task, including possible struggles and important mathematical relationships that students need to recognize, and then observe students solving the tasks. Afterwards, the teachers can make modifications to their understanding of how students think mathematically. All of these activities encourage teachers to make explicit connections between learning activities and the learning process, and can help teachers develop their knowledge about students' mathematical thinking. Some of those activities may be difficult for teachers and guidance from teacher educators may be helpful. In addition, teachers may need support to go beyond describing what students do in class to include analyzing and making generalizations about their students' mathematical thinking.

Much of the current literature on professional noticing focuses on teachers examining artifacts of practice (e.g., videos of classrooms). The advantage of this type of work is that teachers are removed from the actual act of teaching, which allows them to be an observer rather than the teacher making moment-to-moment decisions. The teacher is able to analyze students' mathematical thinking without personal connections to the students and having limited understanding of the classroom context. For example, when a teacher describes and interprets a student's response, her comments are related to what she observed in the teaching artifact, instead of what she already knew of the student. Her noticing is not limited by her biases of the student. The other advantage to this kind of experience is communities of teachers can analyze the artifacts and discuss what they notice. The different perspectives can help teachers develop their noticing skills. The four described skills are useful in considering ways to develop professional noticing. One implication for professional development is to provide opportunities for teachers to go beyond describing and interpreting students' mathematical thinking in terms of the mathematics tasks to consider larger conceptual student understanding. In other words, going beyond answering questions such as "What makes this particular task difficult for students?" to include questions such as "What makes this particular concept difficult for students?" In addition, teachers need opportunities to examine student struggles in terms of conceptual understanding. The teachers in this study identified many instances in which students incorrectly solved a task because of gaps in conceptual understanding and there were less instances that the teacher described students' conceptual understanding. In instances such as this, there is an opportunity for teacher educators to encourage the teacher to re-examine the student's thinking in terms of student understanding.

Another implication for teacher education is to transition teachers from developing professional noticing through use of teaching artifacts to noticing within their own teaching. While the use of teaching artifacts are useful for developing professional noticing, teachers still need to be able to notice in the act of teaching. Teachers have more opportunities to notice students' mathematical thinking when the learning activities are at a higher level and teachers engage with students doing mathematics. In professional development experiences, the teachers need to consider what tasks provide opportunities to notice student thinking and what tasks might limit what they notice. Furthermore, the ways teachers listen influence what they notice. This implies that while teachers are developing their expertise in professional noticing, they also need to be developing expertise in listening. Teacher educators should be aware of this and plan activities for developing noticing while addressing the ways teachers are listening.

As a final note about implications for practice I want to highlight the potential of incorporating both learning trajectories and professional noticing in professional development experiences. For the present study I used learning trajectories to help identify instances of teacher noticing. Similarly, professional development experiences can use learning trajectories as a way to support teachers' development of professional noticing skills.

Implications for Further Research

This research study introduces the idea of using learning trajectories and professional noticing as a way to investigate teachers' understanding of how students think mathematically. In the Mathematics Teaching Cycle, Simon (1995a, 1995b) noted three components—teacher's knowledge, learning trajectories, and classroom interactions. For the purpose of this study, I have focused specifically on professional noticing in classroom interactions and learning trajectories. Studies that address other aspects of the cycle in relation to learning trajectories would be helpful in examining teachers' understanding of how students think mathematically. This study documents that teachers notice students' mathematical thinking in terms of the mathematics task and their own mathematical knowledge, among other ways. One question to pursue is what aspects of teachers' mathematical knowledge influence teachers' understanding of how students think mathematically? In addition to examining other parts of the Mathematics Teaching Cycle, it would also be useful to examine parts that are not explicitly discussed, such as teacher beliefs. Teacher educators need to learn more about how teacher beliefs influence teachers' understanding of students' mathematical thinking. For example, how do teachers' beliefs about mathematics influence their development of learning trajectories? Furthermore, how do those beliefs influence what teachers notice during classroom interactions?

This study focuses on teachers who have continued their education beyond undergraduate degrees and have more than eight years of teaching experience. It is my belief that one way that teachers develop knowledge about students is through interactions with students. These interactions can happen directly as well as indirectly (e.g., analyzing student work). Many teacher preparation programs incorporate field experiences in coursework to provide preservice teachers with classroom experiences. Examining the ways that teachers' knowledge of students develops over time would contribute to mathematics educators' understanding of teacher development. Researchers need to examine multiple ways to develop this type of knowledge to determine effective ways to support teachers developing this knowledge. In addition to thinking about how the knowledge develops, it would be helpful for research to make connections between the development of teachers' knowledge of students and classroom practices.

The lessons I observed for this study occurred throughout a semester long course. I found that in the six lessons I observed there were overlapping mathematical ideas. For example, area of figures seemed to be visible in many of the 12 lessons. Another productive avenue for future research is to consider the same kind of study, but focus on learning trajectories for a unit of study rather than lessons throughout a course. There are several large topics in high school geometry that would provide a rich content focus, such as area, angle and triangle relationships, and volume.

I hold the assumption that teachers' knowledge of students and mathematics strongly influences instructional practices, which ultimately affects student learning. At the onset of this study I wanted to know the specifics about how teachers made sense of students' mathematical thinking. Through the process of developing a framework for studying my focus and analyzing the data, I realized the value of teachers' developing professional noticing skills. Jacob et al. (2007) reported, "through this study, we have become further convinced that professional noticing of children's mathematical thinking is critical, but often hidden, set of skills needed by elementary school teachers who are working to teach in ways consistent with reform recommendations" (p. 46). I agree with that statement and extend their comments to include high school mathematics teachers. While the construct professional noticing has potential for professional development, the mathematics education community needs to understand how to develop it and how expertise in it influences student learning. For example, one would assume that teachers' classroom noticing is different when they observe a class versus teach their own class. How are they different? Based on this study, I hypothesize that teachers' listening skills are directly related to their noticing skills. I suggest the following research questions for future studies:

1. How do teachers' skills in professional noticing develop over time?
2. What is the relationship between teachers' level of skill in professional noticing and their instructional practices?
3. In what ways does developing skills in professional noticing influence teachers' knowledge of how students think mathematically?
4. In what ways does developing skills in professional noticing influence teachers' instructional practices?
5. In what ways does developing skills in professional noticing influence student learning?

Concluding Remarks

I began this report with a quote describing a teacher's realization about the sophisticated ways that her students were thinking mathematically. I would like to end with a comment about my personal realization about the sophisticated ways that teachers think about their students. As mentioned earlier, teacher educators should encourage teachers to describe their students' conceptual understanding, along with recognizing gaps in students' conceptual understanding. As a researcher, I wanted to analyze data and present findings that documented the teachers' understanding of how students think mathematically, as well as recognizing gaps in teachers' understanding. Teaching is complex, and noticing students' mathematical thinking within that complexity is difficult. There were several moments in this study that the teachers became frustrated answering interview questions pertaining to what they noticed about students' mathematical thinking. I am amazed at the teachers' willingness, even during the frustrating moments, to share their understanding of students' mathematical thinking. Furthermore, as I listened to their thoughts, I was often surprised at what the teachers knew about their students' mathematical thinking, and what they knew about the other aspects of the students' lives.

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Appendix A: Van Hiele Levels

Level 0: The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines.

Level 1: The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).

Level 2: The student logically interrelates previously discovered properties/rules by giving or following informal arguments.

Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems.

Level 4: The student establishes theorems in different postualational systems and analyzes/compares these systems. (Fuys, Geddes, & Tischler, 1988, p. 5)

Appendix B: Background Interview Protocol

First Interview

Rhodes
August 24, 2006

1. Can you give a brief history of your teaching experience and education?

2. What is geometric thinking to you?

3. What does a typical day look like in your geometry classroom? How is this different from a typical day in your algebra classroom? How is it the same?

4. What teaching practices do you use and prefer?

5. How do you go about planning for the year? For a chapter? For a particular day? What role do the QCC, the county curriculum guide, the textbook, your students, your colleagues play in your planning?

6. What is your philosophy of teaching? In other words, describe how you teach and why you teach that way?

7. In what ways do you think you facilitate the learning process in your geometry students? (What role do you have in the classroom?)

8. Over the years, what insights have you developed about the ways that students think geometrically? (or learn geometry?)

Appendix C: Interview Protocol

Pre-Observation

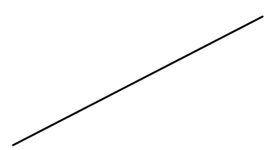
1. What is the goal of this lesson?

2. What do students already know about *insert task*? (What mathematics came before today's lesson? How does today's lesson connect to what students have done before?)

3. How does this *insert task* relate to your lesson goal?

4. How would you solve this *insert task*? Do you think students might solve this differently? How so? (I don't know about this question – I want to know whether the teachers can articulate the differences, if any, in the way that she and the students approach the task)

5. Suppose a new teacher asked you how students make sense of (learn about?) *insert concept/topic/goal.* What would you say?
    a. How do you think students will progress through this activity?
    b. Suppose a student progresses in a linear fashion.



Can you describe where your students are starting, and the changes that you anticipate as students engage in the activity?

6. What about *insert task* do you think some of your students will have trouble? What misconceptions do students bring with them to this topic? To what extent are these misconceptions major obstacles to learning the content of this lesson? Minor obstacles? How did you take these misconceptions into account when you planned the lesson? What do students typically struggle with in this lesson? How have you changed your approach to this topic over the years to try to deal with students' difficulties?

7. How will you know whether your students achieve the goal of the lesson? What will cause you to change your approach to this lesson midstream?

Post-Observation

1. How did the lesson go? Did the students meet the goal of the lesson? How do you know? All of them? Were there any that didn't? Which ones? How do you know?

2. Can you think of any instances that the lesson did not go as planned?

3. How did students approach this task? Did students have more than one approach to the task? Describe one approach you saw. Describe another approach you saw.

4. Were you surprised by any of the student responses? (give example)

5. Were there any interactions between students or students and you that you thought were particularly important for student understanding? In what sense were they important?

6. Did you change any of your plans during this class? Why?
   a. Can you think of any instances that you changed something you planned to do because of something that a student said or did?

7. Before the lesson you said that you would tell a beginning teacher that students make sense of *insert concept/topic/goal* like this *insert teacher response*? Did you observe any instances that your students were thinking that way? (ask for an example) Did you find any instances that your students were not thinking that way? (ask for an example)
   a. Do you want to make any changes to the trajectory you created before the lesson? (what are they and why)

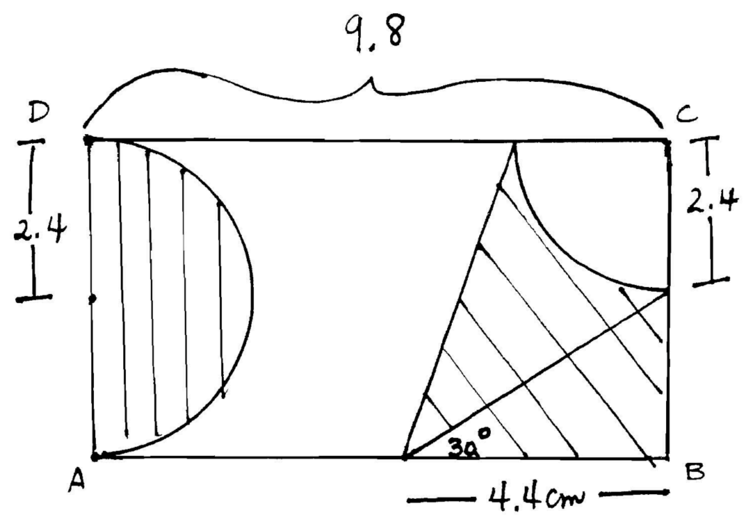
8. If you could do this lesson over again what would you do differently?

9. What mathematics follows today's lesson?

Appendix D: Barbara's Handout on Area

9/26/07

Students were asked to find (1) the area of the shaded region, (2) the ratio of the shaded region to the un-shaded region, and (3) the ratio of the shaded region to the rectangle.



assume all arcs are arcs of a circle.

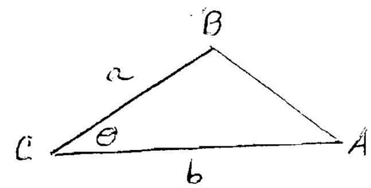
quad ABCD is a rectangle.

Honors Geometry   Special Assignment   Name ____________

Due next Friday

Given triangle ABC with sides lengths a and b and the angle between them is the Greek letter theta, Θ. Using trigonometry prove,

Area of Triangle ABC = ½ ab ~~cos~~ sin Θ



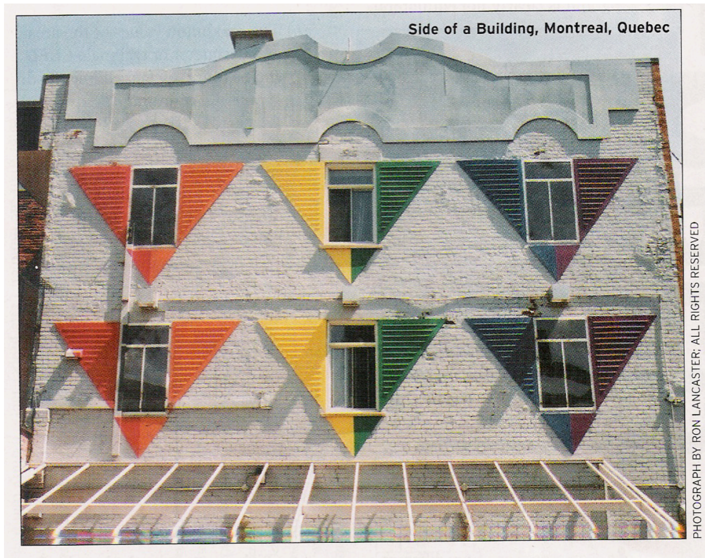
Use good notation and include enough steps so your presentation is absolutely clear. Remember you can add lines or segment if needed.

Given a Rhombus (look up the definition) and the fact that the diagonals, d_1 and d_2 are perpendicular bisectors of each other, prove

Area of Rhombus = $\frac{1}{2} d_1 d_2$

Appendix E: The Window Task

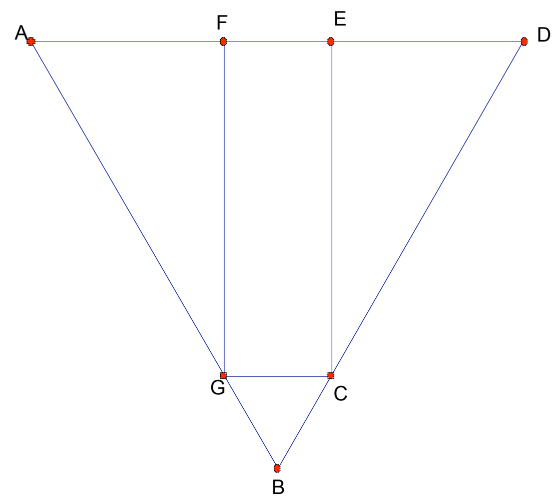
Window Investigation Name ______________________________
Individual Response



1. A gentleman stood on the sidewalk across the street form the building when he took the above picture. What type of triangles appears in this picture? Make any necessary measurements on the photograph to help you decide. Explain how you arrived at your answer.

In the remaining questions, assume that an equilateral triangle has been formed around each of the windows. Let rectangle CEFG represent one of the windows.

2. Suppose that point C in the figure is allowed to assume different positions along the line segment BD, starting at B and moving toward and ending at D. Explain how the perimeter and the area of the rectangular window changes as C moves from B to D. (Be specific and give as much detail and supporting information as you can!)



3. Is there a position for C along line segment BD where the area of the window is the largest? Is there a position where the perimeter is the largest? Are these positions the same?

Window Investigation
Group Response

Names ______________________________
____
____

In the remaining questions, assume that an equilateral triangle has been formed around each of the windows.

1. Let rectangle CEFG represent one of the windows. Suppose that point C in the figure is allowed to assume different positions along the line segment BD, starting at B and moving toward and ending at D. Explain how the perimeter and the area of the rectangular window changes as C moves from B to D. (Be specific and give as much detail and supporting information as you can!)

2. Is there a position for C along line segment BD where the area of the window is the largest? Is there a position where the perimeter is the largest? Are these positions the same? Justify your answers.

Appendix F: Barbara's Handout on Transformations

12/5/07

Vectors and Transformational Geometry

Find the components, magnitude and direction of the vector from Athens to Savannah. Place the scale drawing on this paper with the appropriate details.

Graph A (-2, 3), B (-6, 1) and C (-4, 6). Draw triangle ABC. Reflect the triangle through the x-axis.

Point A's reflection has ordered pair ____________
Point B's reflection has ordered pair ____________
Likewise C's reflection has ordered pair ____________

These three new points will be called A', B' and C' respectively.

Is $\triangle ABC$ congruent to $\triangle A'B'C'$? ______________

Consider another transformation of $\triangle ABC$ called a translation.

This transformation takes the original points, (X,Y) and changes them by adding ~~three~~ six to each X and subtracting 4 from each Y.

In other words each $(X,Y) \rightarrow (X + 6, Y - 4)$.

The new triangle $A''B''C''$, is it congruent to $\triangle ABC$? ≅ $\triangle A'B'C'$

Determine the equation of the line of reflection that maps A (-2,3) to D (2,5).

Estimate the image of B and C as they are reflected through this line and name them respectively E and F.

An isometry is

____

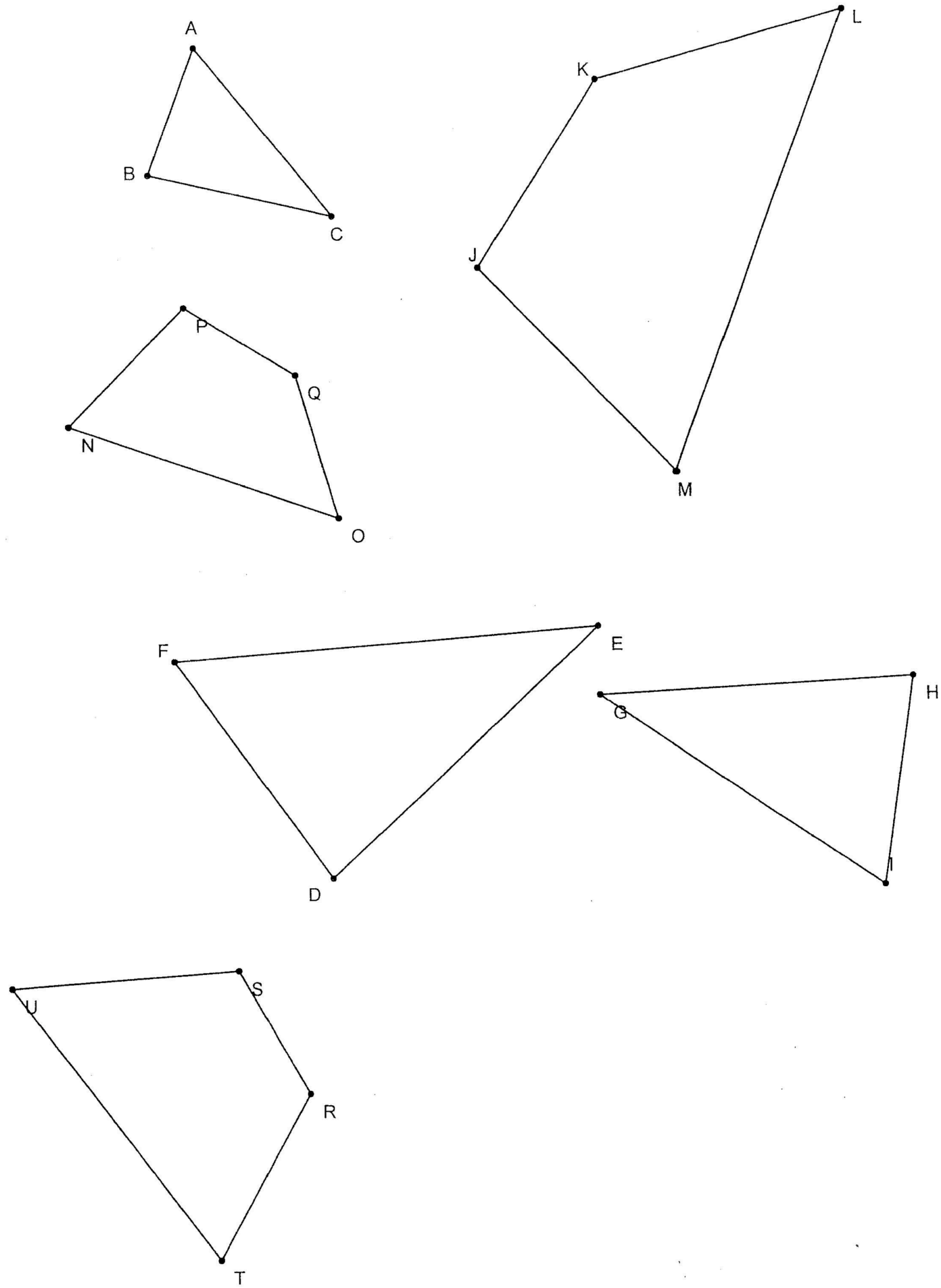
____

On another graph plot $\triangle ABC$ again; together we will rotate this triangle about the origin through various turns.

Appendix G: Judy's Handout on Similar Polygons

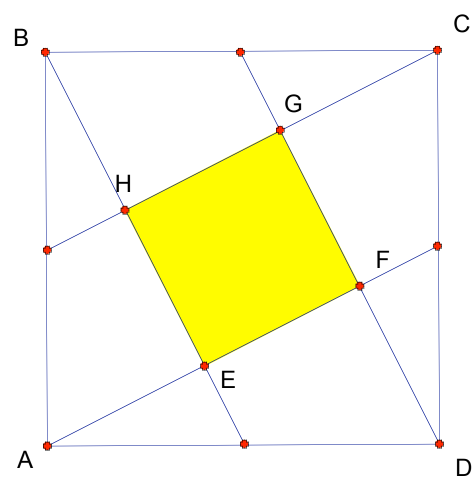
11/9/07

Determine which, if any, of the polygons are similar to each other.

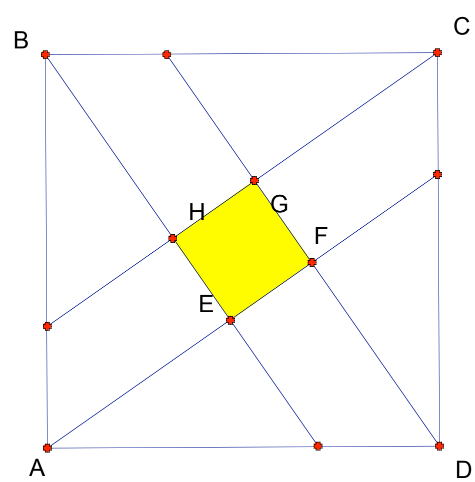


Appendix H: Ratios of Areas

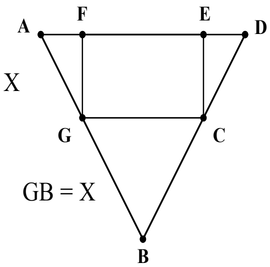
Given a square of side s and segments drawn from each vertex to the midpoint of the second side, counting counter-clockwise, as follows. The segments intersect to form a new square. What is the ratio of the area of the shaded square (HGFE) to the area of the original square?



Let's consider the points on the sides of the original square dividing each side into thirds. One of the resulting squares is the following. What is the ratio of the shaded square (HGFE) to the original square?



Appendix I: Possible Solutions to Window Task



Let GB = X, then AG = 10.5 - X.

ΔAGF is a 30-60-90 Δ because $\angle A = 60^\circ$ and $\angle AFG = 90^\circ$.

So, the short leg is 1/2 of the hypothenuse.

$AF = \dfrac{10.5 - X}{2}$.

Since AD = 10.5 and AF + ED = 10.5 - X, FE = X.

FG is the long leg in a 30-60-90 Δ, therefore it is $\sqrt{3}$ times the short leg.

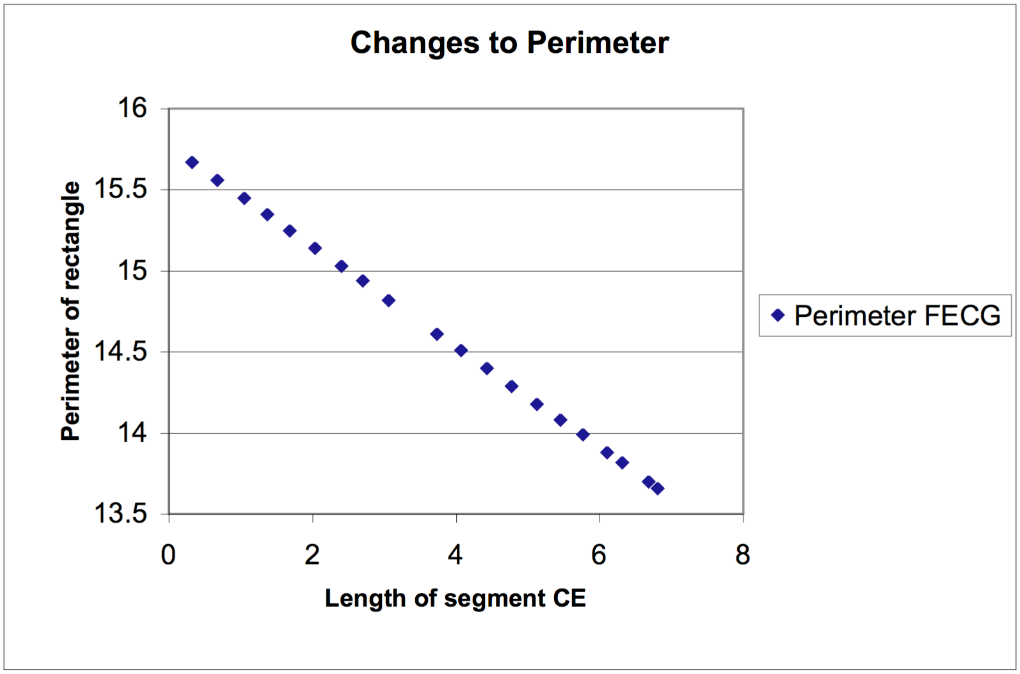
So, $FG = \dfrac{10.5 - X}{2}\sqrt{3}$.

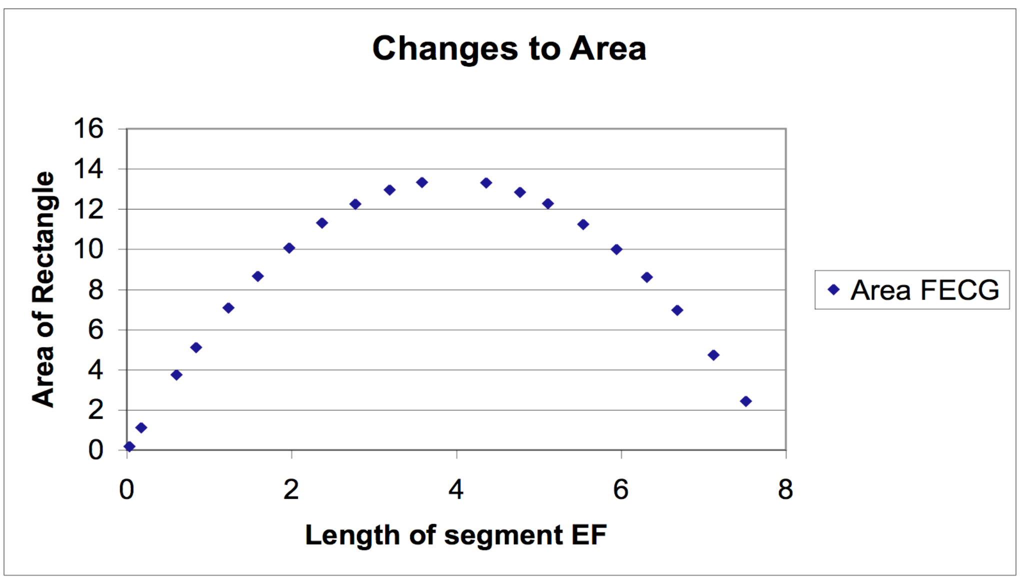
$$P(X) = X + \frac{10.5 - X}{2}\sqrt{3} + X + \frac{10.5 - X}{2}\sqrt{3}$$

So, $P(X) = 2X + (10.5 - X)\sqrt{3}$ and

$$A(X) = X\left(\frac{10.5 - X}{2}\right)\sqrt{3}$$

m CE	m EF	Perimeter FECG	Area FECG
0.32	7.51	15.67	2.44
0.67	7.12	15.56	4.74
1.05	6.68	15.45	6.98
1.37	6.31	15.35	8.62
1.68	5.94	15.25	10
2.03	5.54	15.14	11.26
2.4	5.11	15.03	12.28
2.7	4.77	14.94	12.86
3.06	4.36	14.82	13.31
3.1	4.3	14.81	13.35
3.73	3.58	14.61	13.34
4.07	3.19	14.51	12.97
4.43	2.77	14.4	12.26
4.77	2.37	14.29	11.33
5.12	1.97	14.18	10.08
5.45	1.59	14.08	8.67
5.76	1.23	13.99	7.09
6.1	0.84	13.88	5.12
6.31	0.6	13.82	3.76
6.68	0.17	13.7	1.13
6.8	0.03	13.66	0.19


Changes to Perimeter
Perimeter of rectangle
16
15.5
15
14.5
14
13.5
0
2
4
6
8
Length of segment CE
Perimeter FECG


Changes to Area
Area of Rectangle
16
14
12
10
8
6
4
2
0
0
2
4
6
8
Length of segment EF
Area FECG