Generation of Non-Isomorphic Cubic Cayley Graphs

by

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(Under the direction of Robert W. Robinson)

Abstract

This thesis investigates the generation of non-isomorphic simple cubic Cayley graphs. The research is motivated indirectly by the long standing conjecture that all Cayley graphs with at least three vertices are Hamiltonian. All simple cubic Cayley graphs of degree ≤ 7 were generated. By a simple Cayley graph is meant one for which the underlying Cayley digraph is symmetric and irreflexive. Put another way, each generator is an involution which is not the identity. Results are presented which show which pairs of non-conjugate triples of generators, up to degree 7, yield isomorphic Cayley graphs. These Cayley graphs range in size up to 5040, and include a number for which hamiltonicity or non-hamiltonicity has not been determined.

In addition to the census results some sufficient (but by no means necessary) conditions are shown for isomorphism between Cayley graphs, and an efficient method of counting non-conjugate triples of involutions is developed.

Index words: Cayley graphs, Hamiltonian graphs, graph isomorphism

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DEDICATION

To dearest Mama and Bapa.

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Chapter 1

Introduction and Literature Review

This thesis is an investigation of the generation of an interesting class of cubic Cayley graphs and of isomorphisms among them. The research is motivated indirectly by the long standing conjecture that all Cayley graphs with at least three vertices are Hamiltonian. This is a special case of the conjecture that with five exceptions all connected vertex-transitive graphs are Hamiltonian. The known non-Hamiltonian connected vertex-transitive graphs are the path graph of length 1, the Petersen graph, the Coxeter graph, and the two cubic graphs derived from the Petersen and Coxeter graphs by replacing each vertex with a triangle. Cayley graphs are always connected and vertex-transitive, the path of length 1 has just 2 vertices, and the other four exceptional graphs are not Cayley graphs. The class of cubic Cayley graphs is of particular interest to researchers since it seems to offer the best chance of containing a non-Hamiltonian member. In this thesis we focus on cubic Cayley graphs for which all three generators are involutions.

In Chapter 2 we give basic information about graphs [9], groups [10], Cayley graphs [11], and concepts needed for exploring isomorphisms among Cayley graphs. In Chapter 3 we start with Burnside's Lemma and then proceed to count the number of non-conjugate unordered singletons, pairs, and triples of permutations in the symmetric group $\mathfrak{S}_n$. Then we adapt the analysis to the cases where all generators are involutions. In Chapter 4 we generate the non-conjugate unordered triples of involutions and then use the generator sets to create adjacency lists for their Cayley graphs. In Chapter 5 we check for isomorphism among cubic Cayley graphs by using the software package `nauty`.

The primary inspiration for this project is Scott Effler's thesis [1]. He was particularly interested in hamiltonicity properties of directed Cayley graphs with two generators and undirected cubic Cayley graphs. He counted and generated Cayley pairs and triples, both in general and when restricted to involutions of degree ≤ 7. In our thesis, we have developed a more efficient way of counting triples by applying Burnside's Lemma directly, extending the way it was applied to enumerate pairs. Effler generated adjacency lists from triples. Finally, he tested for isomorphism using `nauty` to reduce the number of graphs to test for hamiltonicity. This is in view of the fact that if graphs G and H are isomorphic and G is Hamiltonian, then H is Hamiltonian. We follow the same pattern, but obtain different results for degree ≤ 7. We find a number of additional Cayley graphs, and our breakdowns by degree (the minimum degree for which a graph is attainable as a Cayley graph) are different. Scott Effler also delved into the complexities of hamiltonicity testing for Cayley graphs. We have not explored hamiltonicity in our thesis.

The hamiltonicity conjecture has always been of great interest to researchers. The most recent survey of the subject [8] references 88 papers. Most of the work is devoted to proving hamiltonicity in a variety of special cases. However there has been some research on generating all connected vertex-transitive cubic graphs up to N vertices, where N is 20, 22, 24, then 26 (extending the Foster census [14]). By contrast Scott Effler's approach is specific to cubic Cayley graphs, and has produced graphs of order 5040 for which the existence of a Hamilton cycle has not yet been proved or disproved.

CHAPTER 2

BASIC DEFINITIONS AND FACTS

A *graph* [9] G is defined as a set $V(G)$ of *vertices* along with a set $E(G)$ of unordered pairs of vertices called *edges*. Further, a *directed graph* or *digraph* G is a set $V(G)$ of vertices and a set $A(G)$ of ordered pairs of vertices called *arcs*. A vertex $v \in V(G)$ is *incident* to edge $e \in E(G)$ if $e = \{u, v\}$ for some $u \in V(G)$. The *degree* of vertex v in graph G, denoted $d(v)$, is the number of edges incident to v. A graph is *regular* if it is without loops or multiple edges and every vertex has the same degree. A regular graph with vertices of degree k is called a *k-regular* graph. A *cubic* graph is a graph where all vertices have degree 3. Hence, a cubic graph is a 3-$regular$ graph. In this thesis all graphs and digraphs will be *finite*, meaning that $V(G)$ (and hence $E(G)$ or $A(G)$) is finite.

Definition 1. *A* group *[10] Γ is a set of elements together with a binary operation called the* product *that satisfies the following properties.*

1. Closure Law*: For every ordered pair a, b of elements of Γ, the product $ab = c$ exists and is a unique element of Γ.*
2. Associative Law*: For every a, b, c in Γ we have $a(bc) = (ab)c$.*
3. Existence of Identity*: There is an* identity *element I in Γ such that $Ia = aI = a$ for every a in Γ.*
4. Existence of Inverse*: For every a in Γ there exists an element a^{-1}, the* inverse *of a, in Γ such that $aa^{-1} = a^{-1}a = I$.*

It is easy to see that in any group Γ the inverse of a is unique, and that $(a^{-1})^{-1} = a$ for any a in Γ. A *permuation* π of the set $[n] = \{1, 2, \ldots, n\}$ is an ordering of the elements of $[n]$, or a function from $[n]$ to $[n]$ that is both one-to-one and onto (bijective). The *one-line representation* of a permutation π is $\pi_1\pi_2\ldots\pi_n$ where $\pi_i = \pi(i)$ for $1 \leq i \leq n$. The functional notation $\pi(i)$ is abbreviated to πi whenever no confusion will arise. The *cycle representation* is as a product of disjoint cycles. Here a *cycle* of length m is $(a_1\,a_2\,\ldots\,a_m)$, where $a_{i+1} = \pi a_i$ for $1 \leq$ i < m and $a_1 = \pi a_m$. In this case we say that $(a_1\,a_2\,\ldots\,a_m)$ is an *m-cycle*. In this thesis a *cycle* of a permutation always refers to a cycle in its disjoint cycle representation. In a *permutation group* the elements are permutations of a set of objects and the product is function composition. The composition is performed left to right, so that if σ and π are permutations and $\sigma\pi$ is their product then $(\sigma\pi)(i) = \pi(\sigma(i))$ for any object i. The *symmetric group* $\mathfrak{S}_n$ is the group of all permutations on a set of n elements. In a permutation group the identity is always the permutation which leaves every object fixed. An *involution* is a permutation that is its own inverse, such as $(1)(23)(4)$. An example of a permutation which is not an involution is $(123)(4)$.

Lemma 1. *If a nonempty set Δ of permutations on $[n]$ is closed under composition, then Δ is a permutation group.*

Proof: We need to prove that such a set of permutations on $[n]$ satisfies the properties of a group.

1. The composition of bijections is always a bijection.

2. Function composition is always associative.

3. Let π be a permutation in Δ. Then $\pi^i = I$ where i is the least common multiple of π's cycle lengths.

4. To see that π has an inverse, note that $\pi\pi^{i-1} = \pi^{i-1}\pi = I$ where i is as in 3.

Lemma 2. *A permutation π on $[n]$ is an involution if and only if every cycle in its disjoint cycle representation has length 1 or 2.*

Proof: For any i in $[n]$, $\pi\pi = I$ implies that $\pi(\pi(i)) = i$; so either $\pi(i) = i$ (i is in a 1-cycle) or else $(i, \pi(i))$ forms a 2-cycle of π. Conversely, if i is in a 1-cycle or a 2-cycle of π then it is left fixed by $\pi\pi$.

Let X be a subset of a finite group Γ which does not contain the identity element. Then X is a *generating set* for Γ if for any set $S \subseteq \Gamma$, (a) and (b) imply that $S = \Gamma$. Here each $x \in X$ is a *generator*.

(a) $I \in S$

(b) $y \in S$ and $x \in X \implies yx \in S$.

The associated *directed Cayley graph* [11] or *Cayley digraph*, written $\vec{Cay}(X{:}\Gamma)$, is a graph whose vertex set is Γ and where the arcs leaving vertex $g \in \Gamma$ consist of (g, gx) for each $x \in X$. We say that $\vec{Cay}(X{:}\Gamma)$ is the Cayley digraph *generated by X*. The associated *undirected Cayley graph* or simply *Cayley graph*, written $Cay(X{:}\Gamma)$, has vertex set Γ and the edges incident to $g \in \Gamma$ consist of $\{g, gx\}$ and $\{g, gx^{-1}\}$ for each $x \in X$. We say this is the Cayley graph *generated by X*. A Cayley graph with $|X| = k$ and is called a *k-generated* Cayley graph. Here we are only interested in cubic Cayley graphs. Note that a cubic Cayley graph must belong to one of two types. *Class 1* cubic Cayley graphs are 2-generated Cayley graphs for which just one generator is an involution. *Class 2* cubic Cayley graphs are 3-generated Cayley graphs for which all three generators are involutions. In this thesis the focus is on class 2 Cayley graphs, which we call *simple cubic Cayley graphs*.

It should be noted that Cayley graphs and digraphs are normally endowed with the edge coloring which associates the "color" x with the edge $\{g, gx\}$ or the arc (g, gx), for each $g \in \Gamma$ and $x \in X$. In this thesis the edge coloring of a Cayley graph or digraph is ignored unless it is specifically mentioned.

Two unlabeled graphs or digraphs G and H, each having n vertices, are *isomorphic* if the vertices of both can be labeled with the integers $\{1, 2, \ldots, n\}$ so that the edge sets (arc sets) consist of the same unordered (ordered) pairs. This means that the two graphs or digraphs can be given labelings that show them to be identical as labeled graphs or digraphs. Isomorphisms of unlabeled graphs will be important in this thesis. An *automorphism* of a graph G is an isomorphism of G with itself. That is, it is a permutation π of the vertices of G such that $\{x, y\} \in E(G)$ if and only if $\{\pi(x), \pi(y)\} \in E(G)$ for any vertices x and y. For any graph G the collection of all automorphisms of G defines a permutation group, known as the *automorphism group* of G. A group Γ of permutations is said to *act on* $[n]$ if it permutes the elements in $[n]$. Then the *orbit* of $i \in [n]$ under Γ is the set $\Gamma(i) = \{gi : g \in \Gamma\}$. For example, under the permutation group

$$\Gamma_1 = \{(1)(2)(3)(4), (1\,2)(3)(4), (1)(2)(3\,4), (1\,2)(3\,4)\},$$

the orbits of 1 and 2 are $\{1, 2\}$ and the orbits of 3 and 4 are $\{3, 4\}$. A *vertex-transitive* graph is defined to be a graph for which the vertices form a single orbit under its automorphism group.

The research in this thesis is motivated indirectly by the long standing conjecture that all Cayley graphs with at least three vertices are Hamiltonian. This is a special case of the conjecture that with five exceptions all connected vertex-transitive graphs are Hamiltonian. *Hamiltonian* graphs are graphs containing Hamiltonian cycles. A *hamiltonian cycle* is a closed path through a graph that visits each vertex exactly once. The class of cubic Cayley graphs is of particular interest to researchers since it seems to offer the best chance of containing a non-Hamiltonian member. Here, we focus on simple cubic Cayley graphs. These are the Cayley graphs generated by three involutions.

Two elements a and b of group Γ are called *conjugate* in Γ if there is some element σ in Γ with $\sigma a \sigma^{-1} = b$. Writing $\hat{\sigma}(a)$ for $\sigma a \sigma^{-1}$, we note that for fixed σ the map $\hat{\sigma}$ is a group automorphism of Γ. That is, $\hat{\sigma} : \Gamma \to \Gamma$ is bijective, $\hat{\sigma}(ab) = \hat{\sigma}(a)\hat{\sigma}(b)$ for all $a, b \in \Gamma$, and $\hat{\sigma}(I) = I$. These facts are easily checked, and imply that $\hat{\sigma}(a^{-1}) = (\hat{\sigma}(a))^{-1}$ for all $a \in \Gamma$. Furthermore $\hat{\sigma}^{-1} = \widehat{\sigma^{-1}}$ and $\hat{\sigma}\hat{\pi} = \widehat{\sigma\pi}$ for all $\sigma, \pi \in \Gamma$. It is now straightforward to check that conjugacy is an equivalence relation, and so partitions Γ into equivalence classes. The equivalence class that contains the element a in Γ is $Cl(a) = \{\hat{\sigma}(a) : \sigma \in \Gamma\}$ and is called the *conjugacy class* of a. Every element of the group belongs to precisely one conjugacy class, and the classes $Cl(a)$ and $Cl(b)$ are equal if and only if a and b are conjugate.

The map $\hat{\sigma}$ for $\sigma \in \mathit{\Gamma}$ is extended to subsets of $\mathit{\Gamma}$ in the natural way. That is, $\hat{\sigma}(X) = \{\hat{\sigma}(x) : x \in X\}$ for any $X \subseteq \mathit{\Gamma}$. For $X, Y \subseteq \mathit{\Gamma}$ we say that X is *conjugate* to Y if and only if there is some $\sigma \in \mathit{\Gamma}$ such that $Y = \hat{\sigma}(X)$. A *Cayley pair* of degree n is an unordered pair $\{\rho_1, \rho_2\}$ such that $\rho_1, \rho_2 \in \mathfrak{S}_n$, $\rho_1, \rho_2 \neq I$, and $\rho_1 \neq \rho_2$. Similarly, a *Cayley triple* of degree n is an unordered triple $\{\rho_1, \rho_2, \rho_3\}$ such that $\rho_1, \rho_2, \rho_3 \in \mathfrak{S}_n$, $\rho_1, \rho_2, \rho_3 \neq I$, and $\rho_1 \neq \rho_2 \neq \rho_3$. If two Cayley triples are conjugate in $\mathfrak{S}_n$ then their Cayley graphs are isomorphic. This is a special case of the following lemma.

Lemma 3. *$Cay(X{:}\mathit{\Gamma})$ is isomorphic to $Cay(X'{:}\mathit{\Gamma}')$ if X and X' are conjugate sets in some finite group H.*

Proof: Let X, X' be conjugate sets in some finite group H. Then there exists $\sigma \in H$ such that $\hat{\sigma}(X) = X'$. By definition $\mathit{\Gamma} = \langle X \rangle$ and $\mathit{\Gamma}' = \langle X' \rangle$.

Claim: $\hat{\sigma}$ is an isomorphism of $Cay(X{:}\mathit{\Gamma})$ to $Cay(X'{:}\mathit{\Gamma}')$. Strictly speaking it is the restriction of $\hat{\sigma}$ to $\mathit{\Gamma}$ which gives the isomorphism.

1. $\hat{\sigma} : \mathit{\Gamma} \to \mathit{\Gamma}'$.

Let S be the subset of $\mathit{\Gamma}$ which $\hat{\sigma}$ maps into $\mathit{\Gamma}'$.

(a) $I \in S$ since $\hat{\sigma}(I) = I \in \langle X' \rangle$.

(b) If $y \in S$ and $x \in X$ then $yx \in S$.

For $\hat{\sigma}(yx) = \hat{\sigma}(y)\hat{\sigma}(x)$, which is in $\mathit{\Gamma}' = \langle X' \rangle$ since $\hat{\sigma}(x) \in X'$ and $\hat{\sigma}(y) \in \mathit{\Gamma}'$.

Now from (a) and (b) we have $S = \langle X \rangle$, so then $\mathit{\Gamma} = \langle X \rangle$ implies 1.

1′. $\widehat{\sigma^{-1}} : \mathit{\Gamma}' \to \mathit{\Gamma}$.

Since $\widehat{\sigma^{-1}} = \hat{\sigma}^{-1}$ on H we have $\widehat{\sigma^{-1}}(X') = X$, so 1′ is just an instance of 1.

2. From 1 and 1′ along with the fact that $\hat{\sigma}$ and $\widehat{\sigma^{-1}}$ are inverses as maps on H, it follows that the restrictions to $\mathit{\Gamma}$ and $\mathit{\Gamma}'$ (respectively) are inverse to each other, and therefore are both bijections.

3. To see that edges $Cay(X{:}\mathit{\Gamma})$ are mapped to edges in $Cay(X'{:}\mathit{\Gamma}')$ by $\hat{\sigma}$, note that (y, yx) for $x \in X$ and $y \in \mathit{\Gamma}$ is mapped by $\hat{\sigma}$ to $(\hat{\sigma}(y), \hat{\sigma}(yx)) = (\hat{\sigma}(y), \hat{\sigma}(y)\hat{\sigma}(x))$.

$3'$. As in 3. but with $\widehat{\sigma^{-1}}$ in place of σ.

We see that every edge in $Cay(X'{:}\mathit{\Gamma}')$ corresponds under $\hat{\sigma}$ with a unique edge in $Cay(X{:}\mathit{\Gamma})$. Thus $\hat{\sigma}$ (restricted to $\mathit{\Gamma}$) is a graph isomorphism of $Cay(X{:}\mathit{\Gamma})$ to $Cay(X'{:}\mathit{\Gamma}')$, and *vice versa* for $\widehat{\sigma^{-1}}$.

CHAPTER 3

COUNTING

In view of Lemma 3, we are interested in counting and generating non-conjugate Cayley triples. For counting we will apply Burnside's Lemma [12, Thm. 7.3]. Actually, this lemma is more properly attributed to Cauchy and Froebeneus [3]. Let π be a permutation of a finite set A. An element a in set A is said to be *fixed* under π if $\pi(a) = a$. Let $Fix(\pi)$ be the number of elements of A that are fixed under π. If Γ is a group of permutations on A then it induces an equivalence relation on A as follows: for $a, b \in A$, $a \sim b$ if and only if $b = \pi(a)$ for some $\pi \in \Gamma$. The equivalence classes of A under $\sim$ are sometimes called orbits of Γ (as in Chapter 2 when Γ was the automorphism group of a graph acting on its vertex set).

Lemma 4 (Burnside's Lemma). *Let Γ be a group of permutations of a finite set A and let R be the equivalence relation on A induced by Γ. Then the number of equivalence classes in R is given by $\frac{1}{|\Gamma|}\sum_{\pi\in\Gamma} Fix(\pi)$.*

Proof: Suppose that Γ is a group of permutations on set A and a is in A. For each $a \in A$, let $St(a)$, the *stabilizer* of a, be the set of all permutations in Γ under which a is invariant. Let $C(a)$ be the *orbit* of a, the equivalence class containing a under the induced equivalence relation R, that is, the set of all b such that $\pi(a) = b$ for some π in Γ. Below we will show the product identity $|St(a)|\cdot|C(a)| = |\Gamma|$.

Now we prove Burnside's Lemma assuming the product identity. Let

$$\chi(\pi, a) = \begin{cases} 1 & \text{if } \pi(a) = a, \\ 0 & \text{if not.} \end{cases}$$

Then $\sum_{\pi\in\Gamma} Fix(\pi) = \sum_{\pi\in\Gamma}\sum_{a\in A}\chi(\pi,a) = \sum_{a\in A}\sum_{\pi\in\Gamma}\chi(\pi,a) = \sum_{a\in A}|St(a)|$.

We need to show that if $b \in Cl(a)$ then $|St(b)|$ = $|St(a)|$. As $b \in Cl(a)$, then $b = \pi a$ for some $\pi \in \Gamma$. So $St(b) \subseteq \pi^{-1} \cdot St(a) \cdot \pi$ = $\hat{\pi}^{-1}(St(a))$, whence $|St(b)| \leq |St(a)|$, since $\hat{\pi}^{-1}$ is bijective on Γ. The reverse inequality holds in the same way, as $a \in Cl(b)$. Now let O be the set of orbits of A under Γ, and note that

$$\sum_{a\in A}|St(a)| = \sum_{X\in O}\sum_{b\in X}|St(b)| = \sum_{X\in O}|\Gamma|,$$

the latter by the product identity and the fact that the members of an orbit X must all have stabilizer subgroups of Γ of the same cardinality. Burnside's Lemma follows by dividing through by $|\Gamma|$.

To address the product identity, let $C(a) = \{b_1, b_2, \ldots, b_m\}$. Then there is a permutation π_1 which sends a to b_1, a permutation π_2 that sends a to b_2, and so on. Let $P = \{\pi_1, \pi_2, \ldots, \pi_m\}$. Then $|P|$ = $|C(a)|$. The key idea in proving the product identity is that every permutation $\pi \in \Gamma$ can be written in exactly one way as the product of a permutation in P and a permutation in $St(a)$. It then follows by the product rule for counting that $|\Gamma|$ = $|P|\cdot|St(a)|$ = $|C(a)|\cdot|St(a)|$.

We now prove the key idea. Given π in Γ, note that $\pi(a) = b_k$, for some k. Thus $\pi(a) = \pi_k(a)$, so $\pi \circ {\pi_k}^{-1}$ leaves a invariant, showing that $\pi \circ {\pi_k}^{-1}$ is in $St(a)$. But

$$(\pi \circ {\pi_k}^{-1}) \circ \pi_k = \pi \circ ({\pi_k}^{-1} \circ \pi_k) = \pi \circ I = \pi,$$

so π is the product of a permutation in $St(a)$ and a permutation in P. Next, suppose that π can be written in two ways as a product of a permutation in $St(a)$ and a permutation in P. That is, suppose that $\pi = \gamma \circ \pi_k = \delta \circ \pi_l$, where γ, δ are in $St(a)$. Now, $(\gamma \circ \pi_k)(a) = b_k$ and $(\delta \circ \pi_l)(a) = b_l$. Since $\gamma \circ \pi_k = \delta \circ \pi_l$, b_k must equal to b_l, so k = l. Thus $\gamma \circ \pi_k = \delta \circ \pi_k$, and by multiplying by ${\pi_k}^{-1}$ we conclude that $\gamma = \delta$. So in fact there is only one way to write π as a product of a permutation in $St(a)$ and a permutation in P.

We now set about applying Burnside's lemma to counting the numbers of Cayley singletons, pairs, and triples (all excluding I) in $\mathfrak{S}_n$ which are inequivalent under conjugation. Let $P(\boldsymbol{\lambda})$ be the set of permutations of $[n]$ with λ_i i-cycles for $1 \le i \le n$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Here, $\boldsymbol{\lambda}$ is a numerical partition $\boldsymbol{\lambda} \in p(n)$, where $p(n)$ is the set of numerical partitions of n. A *numerical partition* of the positive integer n is a collection of positive integers that sum to n. This can be represented as $n = \sum_{i=1}^{n} i\lambda_i$, where λ_i is the number of *parts* (summands) equal to i. For instance, the integer 4 has the partitions $\{1,1,1,1\}$ $(\lambda_1 = 4, \lambda_2 = \lambda_3 = \lambda_4 = 0)$, $\{1,1,2\}$ $(\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = \lambda_4 = 0)$, $\{2,2\}$ $(\lambda_2 = 2, \lambda_1 = \lambda_3 = \lambda_4 = 0)$, $\{1,3\}$ $(\lambda_1 = \lambda_3 = 1, \lambda_2 = \lambda_4 = 0)$ and $\{4\}$ $(\lambda_4 = 1, \lambda_1 = \lambda_2 = \lambda_3 = 0)$. So if we consider the partition $\{1,3\}$ then $1 \cdot \lambda_1 + 3 \cdot \lambda_3 = 4$, in agreement with the above formula. For $\sigma \in P(\boldsymbol{\lambda})$ we say that σ is a permutation of *cycle type* $\boldsymbol{\lambda}$. Let $P_{1,n}(\boldsymbol{\lambda})$ be the number of fixed points in $\mathfrak{S}_n$ of a permutation of cycle type $\boldsymbol{\lambda}$ acting by conjugacy. Then $P_{1,n}(\boldsymbol{\lambda}) = |\{\pi \in \mathfrak{S}_n : \hat{\sigma}(\pi) = \pi\}|$ where $\sigma \in P(\boldsymbol{\lambda})$. Also let $Q_{1,n}(\boldsymbol{\lambda})$ be the number of different 1-element subsets of $\mathfrak{S}_n$ left fixed by some $\sigma \in P(\boldsymbol{\lambda})$, so that $Q_{1,n}(\boldsymbol{\lambda}) = P_{1,n}(\boldsymbol{\lambda})$. Now note that $\hat{\sigma}(\pi) = \pi$ if and only if

$$\sigma\pi\sigma^{-1} = \pi \iff \sigma\pi = \pi\sigma \iff \sigma = \pi\sigma\pi^{-1} \iff \hat{\pi}(\sigma) = \sigma.$$

Consider the action of $\hat{\pi}$ on any σ in $\mathfrak{S}_n$. Let us represent σ as a product of disjoint cycles. We claim that $\hat{\pi}(\sigma)$ consists of the disjoint cycles of σ relabeled according to π. For instance, suppose that $(a_1\, a_2 \cdots a_m)$ is a m-cycle of σ, so that $\sigma a_i = a_{i+1}$ for $1 \le i < m$ and $\sigma a_m = a_1$. Then $(\pi a_1\, \pi a_2 \cdots \pi a_m)$ is a m-cycle of $\hat{\pi}(\sigma)$, since

$$\hat{\pi}(\sigma)\pi a_i = \pi\sigma\pi^{-1}\pi a_i = \pi\sigma a_i = \pi a_{i+1}$$

for $1 \le i \le m$, with the understanding that $a_{m+1} = a_1$. Applying this analysis to all of the cycles of σ, we have the following fact, which will be of use again later.

Lemma 5. *For π, $\sigma \in \mathfrak{S}_n$ the cycles of $\hat{\pi}(\sigma)$ are obtained from the cycles of σ simply by relabeling the elements according to π.*

We can now evaluate $Fix(\hat{\sigma})$ for σ of cycle type $\boldsymbol{\lambda}$. This is equal to the number of ways to relabel the elements of the cycles of σ in such a way as to produce the same permutation σ. For each m-cycle $(a_1\, a_2\, \cdots\, a_m)$ there are exactly m ways to produce the same m-cycle since a_1 can be mapped to any of $a_1, a_2, \ldots, a_m$. But once a_1 is mapped to a_i then a_2 must be mapped to a_{i+1}, a_3 must be mapped to a_{i+2}, $etc.$ (with $a_{1+m} = a_1$), in order to give the same permutation on $\{a_1, a_2, \ldots, a_m\}$. This is true for all λ_m of the m-cycles of σ, giving m^{λ_m} different ways to relabel the individual m-cycles of σ. In addition, if there are two m-cycles then they can be interchanged by a relabeling and still give the same permutation. In general, there are $\lambda_m!$ permutations of the m-cycles of σ which can be performed without changing the permutation. Obviously an m-cycle of σ can only be mapped to an m-cycle of σ by any relabeling π which yields σ again, so $\lambda_m!\, m^{\lambda_m}$ is the number of ways to relabel the elements of σ's m-cycles in such a way as to give the same permutation as σ on them. The choices for cycle lengths $m = 1, 2, \ldots, n$ are independent, so

$$Q_{1,n}(\boldsymbol{\lambda}) = P_{1,n}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \lambda_i! i^{\lambda_i}. \tag{3.1}$$

The number $J_{1,n}$, of non-isomorphic singletons which are inequivalent by conjugacy can be found by applying Burnside's Lemma to the group $\mathfrak{S}'_n$, the action of $\mathfrak{S}_n$ on itself by conjugacy. This gives

$$J_{1,n} = -1 + \frac{1}{|\mathfrak{S}_n|} \sum_{\sigma \in \mathfrak{S}_n} Fix(\hat{\sigma}) = -1 + \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} N(\boldsymbol{\lambda}) \cdot \prod_{m=1}^{n} \lambda_m! m^{\lambda_m}, \tag{3.2}$$

where $N(\boldsymbol{\lambda})$ is the number of different permutations in $\mathfrak{S}_n$ which have cycle type $\boldsymbol{\lambda}$. The term -1 excludes the set consisting of the identity from the count. We claim that
$N(\boldsymbol{\lambda}) = n!/\prod_{m=1}^{n} \lambda_m! m^{\lambda_m}$. To see this, note that for any $\sigma \in P(\boldsymbol{\lambda})$ the orbit of σ under $\mathfrak{S}'_n$ is $P(\boldsymbol{\lambda})$. For the action of $\mathfrak{S}'_n$ on σ is to relabel it's cycles in all possible ways, as discussed in the proof of equation (3.1). Thus $N(\boldsymbol{\lambda}) = |Cl(\sigma)|$ in $\mathfrak{S}'_n$. From the proof of Burnside's Lemma we have the product identity $|\mathfrak{S}'_n| = |Cl(\sigma)| \cdot |St(\sigma)|$. The claim now follows from equation (3.1) and the fact that $|\mathfrak{S}'_n| = n!$. Substituting the claim in equation (3.2) gives

$$J_{1,n} = -1 + \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} n! = -1 + \sum_{\boldsymbol{\lambda} \in p(n)} 1 = -1 + |p(n)|, \tag{3.3}$$

which is just the number of numerical partitions on n, minus 1.

We now apply Burnside's Lemma to counting Cayley pairs and triples in $\mathfrak{S}_n$ which are inequivalent under conjugation, that is, under the action of $\mathfrak{S}'_n$ on $\mathfrak{S}_n$. The formulae were obtained previously by Effler[1]. They are presented here for completeness, and to pave the way for counting pairs and triples of involutions. Our formulae for the latter are more direct and efficient than those derived by Effler. Let $Q_{2,n}(\boldsymbol{\lambda})$ be the number of orbits of distinct unordered pairs which are left fixed by any $\sigma \in P(\boldsymbol{\lambda})$. Thus

$$Q_{2,n}(\boldsymbol{\lambda}) = |\{\pi_1, \pi_2\} : \pi_1, \pi_2 \in \mathfrak{S}_n \;\;\&\;\; \pi_1 \neq \pi_2 \;\;\&\;\; \{\hat{\sigma}(\pi_1), \hat{\sigma}(\pi_2)\} = \{\pi_1, \pi_2\}|$$

for $\sigma \in P(\boldsymbol{\lambda})$. The two cases in which $\{\hat{\sigma}(\pi_1), \hat{\sigma}(\pi_2)\} = \{\pi_1, \pi_2\}$ are
$(a)\, \hat{\sigma}(\pi_1) = \pi_1 \;\;\&\;\; \hat{\sigma}(\pi_2) = \pi_2$, and $(b)\, \hat{\sigma}(\pi_1) = \pi_2 \;\;\&\;\; \hat{\sigma}(\pi_2) = \pi_1$. In either case we have
(c) $\hat{\sigma}^2(\pi_1) = \pi_1 \;\;\&\;\; \hat{\sigma}^2(\pi_2) = \pi_2$. So to fix a pair either both of the permutations remain fixed, or else they map to each other. The number of ways to choose an unordered pair of permutations which satisfy case (a) is $\binom{P_{1,n}(\boldsymbol{\lambda})}{2}$. To count the pairs for case (b), we will use condition (c) and subtract the number of those which belong to case (a). To this end we define $P_{2,n}(\boldsymbol{\lambda})$ to be the number of elements in $\mathfrak{S}_n$ which belong to 2-cycles of $\hat{\sigma}$ for $\sigma \in P(\boldsymbol{\lambda})$. That is,

$$P_{2,n}(\boldsymbol{\lambda}) = |\{\pi \in \mathfrak{S}_n : \hat{\sigma}^2(\pi) = \pi \;\;\&\;\; \hat{\sigma}(\pi) \neq \pi\}|$$

where $\sigma \in P(\lambda)$. To count the fixed points of $\hat{\sigma}^2$ we need to know the cycle type of σ^2 in $\mathfrak{S}_n$, since $\hat{\sigma}^2 = \widehat{\sigma^2}$. When i is odd and there are λ_i i-cycles in σ, then there are also λ_i i-cycles in σ^2. When i is even and there are λ_i i-cycles in σ, then there are $2\lambda_i$ $i/2$-cycles in σ^2. Thus if $\sigma \in P(\boldsymbol{\lambda})$ then $\sigma^2 \in P(\boldsymbol{\lambda}^2)$, where

$$(\lambda^2)_i = \begin{cases} 2\lambda_{2i} & \text{when } i \text{ is even, and} \\ \lambda_i + 2\lambda_{2i} & \text{when } i \text{ is odd.} \end{cases}$$

Now $P_{1,n}(\boldsymbol{\lambda}^2)$ counts case (b) twice, along with case (a) when $\pi_1 = \pi_2$. Therefore
$P_{2,n}(\boldsymbol{\lambda}) = P_{1,n}(\boldsymbol{\lambda}^2) - P_{1,n}(\boldsymbol{\lambda})$ and $Q_{2,n}(\boldsymbol{\lambda}) = \binom{P_{1,n}(\boldsymbol{\lambda})}{2} + \frac{P_{2,n}(\boldsymbol{\lambda})}{2}$. Finally, we find the number $J_{2,n}$ of unordered pairs of distinct permutations in $\mathfrak{S}_n$ which are inequivalent under conjugacy by using

Burnside's Lemma:

$$J_{2,n} = -J_{1,n} + \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} N(\boldsymbol{\lambda}) \cdot Q_{2,n}(\boldsymbol{\lambda}). \tag{3.4}$$

Here the term $-J_{1,n}$ excludes conjugacy classes of distinct pairs containing the identity from the count.

To count the number $Q_{3,n}(\boldsymbol{\lambda})$ of unordered sets of three different permutations of $\mathfrak{S}_n$ which are inequivalent under conjugacy, we start by evaluating

$$P_{3,n}(\boldsymbol{\lambda}) = |\{\pi \in \mathfrak{S}_n : \hat{\sigma}^3(\pi) = \pi \;\;\&\;\; \hat{\sigma}(\pi) \neq \pi\}|$$

where $\sigma \in P(\boldsymbol{\lambda})$. This is the number of $\pi \in \mathfrak{S}_n$ which belong to 3-cycles of $\hat{\sigma}$ for $\sigma \in P(\boldsymbol{\lambda})$. Notice that $\hat{\sigma}^2(\pi) \neq \pi$ is implied by $\hat{\sigma}^3(\pi) = \pi$ and $\hat{\sigma}(\pi) \neq \pi$. For if $\hat{\sigma}^3(\pi) = \pi$ and $\hat{\sigma}^2(\pi) = \pi$ then $\hat{\sigma}(\pi) = \hat{\sigma}(\hat{\sigma}^2(\pi)) = (\hat{\sigma}^2 \circ \hat{\sigma})(\pi) = \hat{\sigma}^3(\pi) = \pi$. Let $Q_{3,n}(\boldsymbol{\lambda})$ be the number of unordered triples of distinct permutations which are left fixed by $\hat{\sigma}$ for any $\sigma \in P(\boldsymbol{\lambda})$. Thus $Q_{3,n}(\boldsymbol{\lambda})$ is equal to

$$|\{\pi_1, \pi_2, \pi_3\} : \pi_1, \pi_2, \pi_3 \in \mathfrak{S}_n \;\;\&\;\; \pi_1 \neq \pi_2 \neq \pi_3 \;\;\&\;\; \{\hat{\sigma}(\pi_1), \hat{\sigma}(\pi_2), \hat{\sigma}(\pi_3)\} = \{\pi_1, \pi_2, \pi_3\}|$$

for $\sigma \in P(\boldsymbol{\lambda})$. It is convenient to classify the triples contributing to $Q_{3,n}(\boldsymbol{\lambda})$ according to the cycle type of $\hat{\sigma}$ acting on the three members of the triple. There are three possibilities: (a) three 1-cycles; (b) one 1-cycle and one 2-cycle; (c) one 3-cycle. In all these cases we have (d) $\hat{\sigma}^3(\pi_1) = \pi_1$, $\hat{\sigma}^3(\pi_2) = \pi_2$ and $\hat{\sigma}^3(\pi_3) = \pi_3$. So to fix a triple, each of the three permutations remain fixed, or one is fixed and the other two map to each other, or all three are swapped with each other in a 3-cycle. The number of ways to choose an unordered triple of permutations satisfying case (a) is $\binom{P_{1,n}(\boldsymbol{\lambda})}{3}$. To count the triples for case (b), we multiply $P_{1,n}(\boldsymbol{\lambda})$ (for the permutation which is fixed) by $P_{2,n}(\boldsymbol{\lambda})/2$ (for the other two permutations which are mapped to each other). For case (c), we will use case (d) and subtract the number which belong to case (a). For case (d), we need to count the fixed points of $\hat{\sigma}^3$, for which we need to know the cycle type of σ^3 in $\mathfrak{S}_n$, since $\hat{\sigma}^3 = \widehat{\sigma^3}$. When i is not divisible by 3 and there are λ_i i-cycles in σ, then there are also λ_i i-cycles in σ^3. When i is a multiple by 3 and there are λ_i i-cycles in σ, then there are $3\lambda_i$ $i/3$-cycles in σ^3. Thus if $\sigma \in P(\boldsymbol{\lambda})$ then $\sigma^3 \in P(\boldsymbol{\lambda}^3)$ where

$$(\lambda^3)_i = \begin{cases} 3\lambda_{3i} & \text{when } i \text{ is a multiple of 3,} \\ \\ \lambda_i + 3\lambda_{3i} & \text{when } i \text{ is not a multiple of 3.} \end{cases}$$

Now $P_{1,n}(\boldsymbol{\lambda}^3)$ counts case (c) thrice along with case (a) with $\pi_1 = \pi_2 = \pi_3$. Therefore $Q_{3,n}(\boldsymbol{\lambda}) = \binom{P_{1,n}(\boldsymbol{\lambda})}{3} + \frac{P_{2,n}(\boldsymbol{\lambda})P_{1,n}(\boldsymbol{\lambda})}{2} + \frac{P_{3,n}(\boldsymbol{\lambda})}{3}$ where $P_{2,n}(\boldsymbol{\lambda}) = P_{1,n}(\boldsymbol{\lambda}^2) - P_{1,n}(\boldsymbol{\lambda})$ (as before) and $P_{3,n}(\boldsymbol{\lambda}) = P_{1,n}(\boldsymbol{\lambda}^3) - P_{1,n}(\boldsymbol{\lambda})$.

Finally, the number $J_{3,n}$ of non-conjugate unordered triples of distinct permutations in $\mathfrak{S}_n$ is a direct application of Burnside's Lemma:

$$J_{3,n} = -J_{2,n} + \frac{1}{n!}\sum_{\boldsymbol{\lambda}\in p(n)} N(\boldsymbol{\lambda}) \cdot Q_{3,n}(\boldsymbol{\lambda}). \tag{3.5}$$

The term $-J_{2,n}$ serves to exclude from the count distinct triples containing the identity permutation.

But in this thesis we are interested in finding non-congruent triples in which all generators are involutions. As noted in Lemma 2, an involution is a permutation such that each of its cycles has length 1 or 2. To apply Burnside's Lemma, note that by Lemma 5 conjugation preserves involutions, as well as the identity. Let $P'_{1,n}(\boldsymbol{\lambda})$ denote the number of involutions on $[n]$ other than the identity which are left fixed by $\hat{\sigma}$ for some $\sigma \in P(\boldsymbol{\lambda})$.

Let π be any involution which is left fixed by $\hat{\sigma}$ for some $\sigma \in \mathfrak{S}_n$, take any $i \in [n]$, let γ be the cycle of σ containing i, and let m be the length of γ. Since π is an involution, i's cycle in π must have length 1 or 2. If (i) is a 1-cycle of π then by Lemma 5 (σi) must also be a 1-cycle of π, as must $(\sigma^2 i)$, $(\sigma^3 i)$, etc. Thus every member of γ has to be a 1-cycle of π. If, on the other hand, $(i\ j)$ is a 2-cycle of π then $(\sigma i\ \sigma j)$, $(\sigma^2 i\ \sigma^2 j)$, $(\sigma^3 i\ \sigma^3 j)$, *etc.*, must all be 2-cycles of π. If j's cycle in σ is δ, $\delta \neq \gamma$, then δ must also be an m-cycle of σ and $(\sigma^k i\ \sigma^k j)$ must be a 2-cycle of π for $0 \leq k < m$. But it may be the case that j belongs to γ. In this situation it can be seen that m is even and $\sigma^{m/2} i = j$. For otherwise there is some r, $1 \leq r < m/2$, such that $\sigma^r i = j$

or $\sigma^r j = i$. Then $(\sigma^r i \ \sigma^r j)$ is a 2-cycle of π which overlaps $(i \ j)$, hence must be the same 2-cycle since distinct cycles of π are disjoint. Because $\sigma^r i = j \neq i$, it must be that $\sigma^r j = i$. Thus $\sigma^{2r} i = \sigma^r \sigma^r i = \sigma^r j = i$; since $2 \leq 2r < m$, this contradicts the hypothesis that i lies on an m-cycle of σ. So when $(i \ j)$ is a 2-cycle of π and both i and j lie on γ then $\sigma^{m/2} i = j$, $\sigma^{m/2} j = i$, and all $m/2$ of the "diagonal" pairs $(i \ \sigma^{m/2} i), (\sigma i \ \sigma^{m/2+1} i), \ldots, (\sigma^{m/2-1} i \ \sigma^{m-1} i)$ form 2-cycles of π.

Let $I_{m,k}$ be the number of involutions on some set S of mk integers which are left fixed by some permutation σ on S which is a product of m cycles of length k. Of course $I_{m,k} = 0$ if $m < 0$ and $I_{m,k} = 1$ if $m = 0$. To obtain a recurrence for $m > 0$, let γ be the k-cycle of σ which contains the element max(S); say $\gamma = (a_1 \, a_2 \, \cdots \, a_k)$. Now consider how the elements of γ can figure in an involution π such that $\hat{\sigma}(\pi) = \pi$. If the cycles of π containing elements of γ do not contain elements of any other cycle of σ, then, as we have seen, either all elements of γ form 1-cycles of π or else k is even and the diagonally opposing pairs on γ form 2-cycles of π. This gives one possibility of k is odd and two if k is even. In this case there are exactly $I_{m-1,k}$ ways to complete π to an involution fixed by $\hat{\sigma}$. The other situation is that the cycles of π containing elements of γ also contain elements of another k-cycle, say $\delta = (b_1 \, b_2 \, \cdots \, b_k)$. There are m-1 choices for δ, and given that choice exactly k ways to choose the cycles of π on the elements of γ and δ. This is because $(a_1 \ b_i)$ can be a 2-cycle for $i = 1, 2, \ldots k$ but the rest of the 2-cycles of π are then forced to be $(a_j \ b_{i+j})$ for $j = 2, 3, \ldots, k$ (the addition taken modulo k) by the requirement that π be fixed by $\hat{\sigma}$. Completing π on the members on the members of S which are not in γ or δ in such a way that $\hat{\sigma}(\pi) = \pi$ can be done in $I_{m-2,k}$ ways. Hence for $m > 0$ we have the recursive condition

$$I_{m,k} = \begin{cases} I_{m-1,k} + (m-1)kI_{m-2,k} & \text{for } k \text{ odd,} \\ 2I_{m-1,k} + (m-1)kI_{m-2,k} & \text{for } k \text{ even.} \end{cases} \tag{3.6}$$

The possibilities for constructing an involution π such that $\hat{\sigma}(\pi) = \pi$ for different lengths of cycles of σ are independent, so we have

$$P'_{1,n}(\boldsymbol{\lambda}) = -1 + \prod_{k=1}^{n} I_{\boldsymbol{\lambda}_k,k}.$$

The term -1 adjusts for the identity permutation, since that is included in the count given by the product.

Applying Burnside's Lemma (Lemma 4) to counting the number $J'_{1,n}$ of conjugacy classes of involutions in $\mathfrak{S}_n$ other than the identity, we find

$$J'_{1,n} = \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} N(\boldsymbol{\lambda}) \cdot P'_{1,n}(\boldsymbol{\lambda}).$$

Since the conjugacy class of an involution just depends on the number m of 2-cycles (as the other $n - 2m$ members of $[n]$ must be 1-cycles) and $m > 0$ to exclude the identity permutation, the number of conjugacy classes is just $J'_{1,n} = \lfloor n/2 \rfloor$.

To find the numbers $J'_{2,n}$ and $J'_{3,n}$ of conjugacy classes of unordered pairs and triples of non-identity involutions, we follow the derivation of equations (3.4) and (3.5) replacing $P_{1,n}(\boldsymbol{\lambda})$ with $P'_{1,n}(\boldsymbol{\lambda})$ everywhere. For the pairs this gives

$$J'_{2,n} = \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} N(\boldsymbol{\lambda}) \cdot Q'_{2,n}(\boldsymbol{\lambda}). \tag{3.7}$$

where $Q'_{2,n}(\boldsymbol{\lambda}) = \binom{P'_{1,n}(\boldsymbol{\lambda})}{2} + \frac{P'_{2,n}(\boldsymbol{\lambda})}{2}$ and $P'_{2,n}(\boldsymbol{\lambda}) = P'_{1,n}(\boldsymbol{\lambda}^2) - P'_{1,n}(\boldsymbol{\lambda})$. For the triples we obtain the equation

$$J'_{3,n} = \frac{1}{n!} \sum_{\boldsymbol{\lambda} \in p(n)} N(\boldsymbol{\lambda}) \cdot Q'_{3,n}(\boldsymbol{\lambda}). \tag{3.8}$$

where $Q'_{3,n}(\boldsymbol{\lambda}) = \binom{P'_{1,n}(\boldsymbol{\lambda})}{3} + \frac{P'_{2,n}(\boldsymbol{\lambda})P'_{1,n}(\boldsymbol{\lambda})}{2} + \frac{P'_{3,n}(\boldsymbol{\lambda})}{3}$ and $P'_{3,n}(\boldsymbol{\lambda}) = P'_{1,n}(\boldsymbol{\lambda}^3) - P'_{1,n}(\boldsymbol{\lambda})$.

In Table 3.1 are shown the first few numbers of conjugacy classes of sets of non-identity involutions as calculated on the basis of equations (3.5), (3.6), and (3.7).

n	$J'_{1,n}$	$J'_{2,n}$	$J'_{3,n}$	$J'_{3,n} - J'_{3,n-1}$
1	0	0	0	0
2	0	0	1	1
3	1	1	1	0
4	2	5	10	9
5	2	8	37	27
6	3	20	197	160
7	3	29	676	479
8	4	60	3094	2418
9	4	83	12022	8928
10	5	151	55912	43890
11	5	206	250278	194366
12	6	352	1234235	983957
13	6	474	6111854	4877619
14	7	767	31939281	25827427
15	7	1028	169610999	137671718

Table 3.1: Enumerations of conjugacy classes of singletons, unordered pairs and triples of non-identity involutions.

CHAPTER 4

GENERATING

First we need to generate involutions on the set $[n] = \{1,\ 2,\ \ldots,n\}$. This can be done recursively, following the same logic used for determining $I_{n,1}$ in the previous chapter. This is because every involution is left fixed by the identity under conjugation, so $I_{n,1}$ is the total number of involutions over $[n]$. Specialized to $k = 1$, the counting formula (3.6) becomes

$$I_{n,1} = I_{n-1,1} + (n-1)\, I_{n-2,1} \tag{4.1}$$

for $n \geq 1$ with $I_{0,1} = 1$. The first few values of $I_{n,1}$ are shown in Table 4.1.

The first term on the right counts involutions in which (n) is a 1-cycle. The second is the number of ways to choose i to participate in the 2-cycle $(i\ n)$, for $1 \leq i \leq n-1$, times the number of involutions on the set $[n-1] - \{i\}$ of cardinality $n-2$.

`Maple`'s *group* library was used for generation; in this library the disjoint cycle format omits 1-cycles. Let It_n be the set of strings representing the involutions on $[n]$ over the alphabet $\Sigma_n = [n] \cup \{), (\}$ after the manner of `Maple`'s disjoint cycle format, and let ε denote the empty string. Then corresponding to equation (4.1) we have

$$It_n = It_{n-1} \cup \bigcup_{i=1}^{n-1} (i\ n) subs(i = n-1, It_{n-2}) \tag{4.2}$$

for $n \geq 1$ with $It_0 = \varepsilon$. Here $subs(i = n-1, It_{n-2})$ is the notation in `Maple` for the result of substituting $n-1$ for i in all the strings of It_{n-2}.

Involutions are compared and represented by their $rank$ in the recursive generation process. For a member σ of It_n the rank is denoted by $rank(n, \sigma)$, or just $rank(\sigma)$ if n is determined by

n	$I_{n,1}$
0	1
1	1
2	2
3	4
4	10
5	26
6	76
7	232
8	764
9	2620
10	9496
11	35696
12	140152

Table 4.1: Numbers of involutions over $[n]$.

the context. Inductively we have that $rank(0, \varepsilon) = 0$, and for $n \geq 1$ $rank(n, \sigma) = rank(n-1, \sigma)$ if n is fixed by σ (so n does not appear in the `Maple` representation of σ), and

$$rank(n, (i, n)\sigma) = I_{n-1,1} + (i-1)I_{n-2,1} + rank(n-2, \sigma')$$

for σ' an involution on $[n-2]$, $1 \leq i \leq n-1$, and $\sigma = subs(i = n-1, \sigma')$.

For the purposes of identifying a canonical representative of each congruence class of triples of involutions, we define a linear order on the triples of degree n as follows.

Definition 2. *For unordered triples of distinct involutions* $\{\rho_1,\ \rho_2,\ \rho_3\}$ *and* $\{\rho_1',\ \rho_2',\ \rho_3'\}$ *labeled so that* $rank(\rho_1) < rank(\rho_2) < rank(\rho_3)$ *and* $(rank(\rho_1') < rank(\rho_2') < rank(\rho_3'))$ *we define the* lexicographic-rank *order by* $\{\rho_1,\ \rho_2,\ \rho_3\} < \{\rho_1',\ \rho_2',\ \rho_3'\}$ *if and only if* $rank(\rho_1') < rank(\rho_1)$ *or*
$rank(\rho_1') = rank(\rho_1)\ \ \&\ \ rank(\rho_2') < rank(\rho_2)$ *or*
$rank(\rho_1') = rank(\rho_1)\ \ \&\ \ rank(\rho_2') = rank(\rho_2)\ \ \&\ \ rank(\rho_3') < rank(\rho_3)$.

Now a conjugacy class of triples of involutions in $\mathfrak{S}_n$ can be represented by the *canonical triple* $\{\rho_1,\ \rho_2,\ \rho_3\}$, which is minimal with respect to lexicographic-rank order. Our generation method is simply to examine all unordered triples of non-identity involutions over $\mathfrak{S}_n$, retaining only those which are canonical. Given the candidate $\{\rho_1,\ \rho_2,\ \rho_3\}$ we run through $\sigma \in \mathfrak{S}_n$ until it is rejected or else, if never rejected, it is accepted as canonical. To consider σ, let $\{\rho_1', \rho_2', \rho_3'\} = \{\sigma\rho_1\sigma^{-1}, \sigma\rho_2\sigma^{-1}, \sigma\rho_3\sigma^{-1}\}$ and reject $\{\rho_1,\ \rho_2,\ \rho_3\}$ if $\{\rho_1',\ \rho_2',\ \rho_3'\} < \{\rho_1,\ \rho_2,\ \rho_3\}$. In this case $\{\rho_1', \rho_2', \rho_3'\}$ is a witness that $\{\rho_1, \rho_2, \rho_3\}$ is not canonical, hence is rejected when σ is considered.

Finally, an adjacency list is generated from each canonical triple $\{\rho_1, \rho_2, \rho_3\}$ of non-identity involutions. For a triple of degree n, the adjacency list is initially represented by an $n! \times 3$ array. Each row corresponds to a member σ of $\mathfrak{S}_n$, the three entries in the row for σ corresponding to $\sigma\rho_1$, $\sigma\rho_2$ and $\sigma\rho_3$. Here σ, $\sigma\rho_1$, $\sigma\rho_2$ and $\sigma\rho_3$ are represented by their ranks, which are numbers lying in the range $[0, n! - 1]$. We have used the concept of unranking and ranking of permutations[4] to generate the adjacency lists. Rank is a one-to-one function from $\mathfrak{S}_n$ to $[0, n! - 1]$ used here to find the rank of each permutation ρ_1, ρ_2 and ρ_3. Similarly the unranking function is the inverse, returning the permutation of given rank in the range $[0, n! - 1]$. Then, we need to find whether the adjacency list generates a connected graph. The Cayley graph is the connected component containing I, which we find by breadth-first search [13, Sec. 22.2]. If it has $size$ vertices and $size < n!$ then the adjacency list is relabeled to $[0, size - 1]$ in order to create a standard representation of the Cayley graph.

Chapter 5

Isomorphism Testing

Two unlabeled graphs G and H, each having n vertices are considered *isomorphic* if the vertices of both can be labeled with the integers $1, 2, \ldots, n$ so that the edge sets consist of the same unordered pairs. In other words, two graphs are isomorphic if the vertices of one can be relabeled to match the vertices of the other in a way that preserves adjacency. An *automorphism* of a graph G is an isomorphism of G with itself. In other words, it is a permutation of the vertices of G that preserves adjacency. Cayley graphs and digraphs are normally endowed with the edge coloring which associates the "color" x with the edge $\{g, gx\}$ or the arc (g, gx), for each $g \in \Gamma$ and $x \in X$. Two colored Cayley graphs G and H are *color isomorphic* if there exists a bijection σ between the sets of vertices and a bijection η between the generating sets such that
$E(H) = \{\{\sigma(g), \sigma(g)\eta(x)\} : g \in \Gamma \ \ \& \ \ x \in X\}$.

If two Cayley graphs are color isomorphic then they are isomorphic. In most cases for which non-congruent triples generate isomorphic cubic simple Cayley graphs, we know of no way to anticipate this in advance. In the rare exceptions, which are discussed below, the proof actually shows the two Cayley graphs to be color isomorphic.

In this thesis we use `nauty` [5], [6], a software package written by Brendan McKay that computes the automorphism groups of graphs and digraphs. It can also produce canonical labelings. We use `nauty` in this research due to its ability to check isomorphism between large graphs quickly. `Nauty` introduced sophisticated backtrack methods which have since been applied to computational group theory problems [7]. `Nauty` takes as input the adjacency list generated earlier by a set of three generators and converts it to a canonically labeled graph. Two graphs are isomorphic if and only if their canonical labelings are identical. So once the simple Cayley graphs of given degree have been labeled canonically, it is very quick to test for isomorphism.

As noted by Scott Effler [1, page 13], if all three generators contain a common 1-cycle, say (i), then an isomorphic Cayley graph of lower degree is obtained by simply removing i from the set of objects being permuted. Accordingly, such triples have been omitted from our generation of non-congruent Cayley triples. The resulting numbers of degree-minimal triples are shown in Table 3.1.

The observation on the common fixed point can be generalized to other situations in which non-congruent triples generate isomorphic Cayley graphs. For example if $X = \{(1\,2),\ (2\,3),\ (1\,3)\}$ and $X' = \{(1\,2)(4\,5),\ (2\,3)(4\,5),\ (1\,3)(4\,5)\}$ then X and X' both generate Cayley graphs isomorphic to the complete bipartite graph $K_{3,3}$. This is an instance of the following: X is a set of three distinct involutions, not including the identity, on a set S of objects, such that at least one member of X is odd, and σ is an involution which is not the identity on a set T of objects which is disjoint from S. In our example $S = \{1,\ 2,\ 3\}$, $T = \{4,\ 5\}$, $\sigma = (4\,5)$, and all three members of X are odd. Recall that a permutation is odd or even according as the number of its even length cycles is odd or even.

In general we define a map ϕ from permutations of S to permutations of $S \cup T$ by the rule

$$\phi(\gamma) = \begin{cases} \gamma & \text{if } \gamma \text{ is even,} \\ \sigma\gamma & \text{if } \gamma \text{ is odd.} \end{cases}$$

Then $X' = \phi(X)$ generates a Cayley graph which is isomorphic to the one generated by X. That is, if $\Gamma = \langle X \rangle$ and $\Gamma' = \langle X' \rangle$ then $Cay(\Gamma{:}X)$ is isomorphic to $Cay(\Gamma'{:}X')$. In fact the two Cayley graphs are color-isomorphic. The vertex isomorphism is given by the map ϕ restricted to Γ, with the color isomorphism given by ϕ restricted to X. The verification follows easily from the fact that the product $\alpha\beta$ of the two permutations $\alpha,\ \beta$ of S is odd if and only if one of $\alpha,\ \beta$ is even and the other is odd.

CHAPTER 6

RESULTS

Most of the functions were written using `Maple` 14. But as `nauty` is written in `C`, the functions written to call it are in `C`.

Below is a list of all functions in `Maple`.

Isocount: Counts the number of non-conjugate unordered singletons, pairs, and triples of permutations in the symmetric group $\mathfrak{S}_n$.

Invgen: Counts involutions and generates them on the set $[n]$.

rank: Gives the rank for an involution, in order of generation.

Comptriple: Determines the lexicographic order of two triples of ranks.

Trplgen: Generates the canonical triples with respect to lexicographic-rank order.

rank1: Gives the rank of each permutation $\mathfrak{S}_n$ as a number in the range $[0, n! - 1]$.

unrank1: Gives the permutation in $\mathfrak{S}_n$ for a specific rank in the range $[0, n! - 1]$.

Adjlist: Generates the adjacency list for a given canonical triple of non-identity involutions.

Adjcon: Checks whether the adjacency list corresponds to a connected graph, and if not relabels the list to create one.

Filegen: Creates the files of adjacency lists for a specific degree, grouped by order of graphs generated.

Below is a list of all programs in `C`.

tiso.c: Calls `nauty` to check which files in a directory correspond to isomorphic graphs.

canfgen.c: Creates a list of unique graphs which are not isomorphic to any other graph of a specific order.

rr-correspgen.c: Calls `nauty` to compare our list of unique graphs with Scott Effler's result graphs to check for isomorphism.

Table 6.1 shows the number of non-isomorphic simple cubic Cayley graphs with respect to a specific order and minimum degree. We first counted non-congruent Cayley pairs and triples by extending Burnside's Lemma directly. The numbers of degree up to 12 are shown in Table 3.1. Canonical triples of involutions without common fixed point (one for each congruence class) were then generated for degree ≤ 7. The number of triples generated for each degree was checked against the number calculated by formula, and which is shown in the last column of Table 3.1. For instance, exactly 479 canonical triples of degree 7 without common fixed points were generated. For each canonical triple the adjacency list for its Cayley graph was generated. For each order encountered all isomorphisms among these Cayley graphs, and between them and Effler's Cayley graphs of class 2, were found by use of `nauty`.

$\|V\|$	n=3	4	5	6	7	Total
4	0	1	0	0	0	1
6	1	0	0	0	0	1
8	0	2	0	0	0	2
10	0	0	1	0	0	1
12	0	0	3	0	1	4
14	0	0	0	0	2	2
16	0	0	0	2	0	2
18	0	0	0	1	0	1
20	0	0	0	0	4	4
24	0	4	0	0	6	10
36	0	0	0	4	0	4
48	0	0	0	8	4	12
60	0	0	6	0	0	6
72	0	0	0	4	2	6
120	0	0	11	0	27	38
144	0	0	0	0	12	12
168	0	0	0	0	3	3
240	0	0	0	0	31	31
360	0	0	0	7	0	7
720	0	0	0	11	0	11
2520	0	0	0	0	9	9
5040	0	0	0	0	253	253
Total	1	7	21	37	354	420

Table 6.1: Numbers of Non-Isomorphic simple cubic Cayley graphs of degree ≤ 7.

Now, we show the canonical Cayley triples by degree of the permutations. The triples are generated with respect to lexicographic-rank order. The label for a triple has two subscripts. The first subscript is the degree, and the second subscript gives the order of generation. So for example $F_{4,5}$ is the fifth canonical triple of degree 4 having no common fixed point among its members. Then any of Effler's class 2 cubic Cayley graphs which are isomorphic to that of our canonical triple are listed.

Order 4

$F_{4,5} : (12), (34), (12)(34) : none$

Order 6

$F_{3,1} : (12), (23), (13) : none$

Order 8

$F_{4,6} : (12), (34), (13)(24) : result6.1228$

$\mathrm{F}_{4,7} : (12), (12)(34), (13)(24) : none$

Order 10

$F_{5,26} : (12)(34), (23)(45), (14)(35) : result6.1893$

Order 12

$F_{5,1} : (12), (23), (45) : result5.15, result6.10, result6.128, result6.2144, result7.36230$

$\mathrm{F}_{5,2} : (12), (23), (12)(45) : none$

$F_{5,6} : (12), (34), (12)(45) : result7.10101, result7.11786, result7.80995$

$F_{7,125} : (12)(34), (12)(45), (35)(67) : none$

Order 14

$F_{7,474} : (12)(34)(56), (23)(45)(67), (14)(36)(57) : result7.73090$

$\mathrm{F}_{7,479} : (12)(34)(56), (23)(45)(67), (16)(24)(37) : result7.78930, result7.81839$

Order 16

$F_{6,11} : (12), (34), (35)(46) : none$

F$_{6,24}$: $(12), (12)(34), (15)(26)$: $none$

Order 18

$F_{6,91}$: $(12)(34), (12)(45), (53)(26)$: $none$

Order 20

$F_{7,68}$: $(12), (34)(56), (45)(67)$: $result7.1089, result7.44254, result7.61526, result7.80661$

F$_{7,69}$: $(12), (34)(56), (12)(45)(67)$: $none$

$F_{7,148}$: $(12)(34), (23)(45), (12)(34)(67)$: $none$

$F_{7,155}$: $(12)(34), (23)(45), (14)(35)(67)$: $result7.44263, result7.75727, result7.75734$

Order 24

$F_{4,1}$: $(12), (23), (34)$: $result4.1, result7.10085, result7.10608$

F$_{4,2}$: $(12), (23), (12)(34)$: $result6.1056, result6.1181, result6.1213,$
$result6.1520, result6.1696, result6.2546$

$F_{4,3}$: $(12), (23), (24)$: $result6.1119, result7.12183, result7.18434,$
$result7.61902, result7.68048$

$F_{4,4}$: $(12), (23), (13)(24)$: $result6.1123, result6.1384, result6.1507$

$F_{7,6}$: $(12), (34), (45)(67)$: $result7.1073, result7.3130, result7.63449,$
$result7.72539, result7.78487$

$F_{7,13}$: $(12), (12)(34), (25)(67)$: $result7.49392, result7.55966, result7.68466$

$F_{7,87}$: $(12), (13)(24)(56), (13)(24)(67)$: $result7.11640, result7.13557,$
$result7.4341, result7.68994$

$F_{7,143}$: $(12)(34), (12)(45), (16)(34)(27)$: $none$

$F_{7,185}$: $(12)(34), (12)(56), (13)(24)(67)$: $result7.11492, result7.17254, result7.33574$

$F_{7,289}:(12)(34),(12)(35)(46),(63)(45)(27):result7.49386, result7.80473, result7.80489$

Order 36

$F_{6,13}:(12),(34),(25)(46):result6.100, result6.1102, result6.127$

$\mathrm{F}_{6,22}:(12),(12)(34),(25)(46):result6.1870, result6.1882, result6.2564$

$F_{6,40}:(12),(23)(45),(23)(56):result6.1413, result6.383, result6.551$

$F_{6,41}:(12),(23)(45),(13)(56):result6.1372, result6.2635$

Order 48

$F_{6,1}:(12),(23),(34)(56):result6.1002, result6.137, result6.1840,$
$result6.192, result6.568$

$\mathrm{F}_{6,2}:(12),(23),(12)(34)(56):result6.1043, result6.112$

$F_{6,3}:(12),(23),(24)(56):result6.1014, result6.1146, result6.1443,$
$result6.154, result6.1765, result6.2479$

$F_{6,8}:(12),(34),(23)(56):result6.134, result6.141, result6.1527$

$F_{6,16}:(12),(12)(34),(23)(56):result6.1021, result6.1482, result6.1515,$
$result6.1525, result6.2631, result6.502$

$F_{6,26}:(12),(13)(24),(23)(56):result6.1020, result6.1151, result6.501$

$F_{6,33}:(12),(13)(24),(12)(35)(46):result6.1150, result6.1743$

$F_{6,39}:(12),(13)(24),(15)(34)(26):result6.1733, result6.1841, result6.1848$

$F_{7,9}:(12),(34),(26)(35)(47):result7.10466, result7.11300, result7.134,$
$result7.15438, result7.17156, result7.17253$

$F_{7,17}:(12),(12)(34),(16)(25)(47):result7.12296, result7.13559, result7.19775,$
$result7.53254, result7.61703$

$F_{7,45}:(12),(23)(45),(23)(46)(57):result7.3559, result7.43255$

$F_{7,46}$: $(12), (23)(45), (13)(46)(57)$: $result7.35306, result7.40459, result7.40460$

Order 60

$F_{5,20}$: $(12)(34), (13)(24), (12)(45)$: $result6.1044, result6.1077, result6.1087,$
$result6.1114, result6.1241, result6.1242$

$\mathrm{F}_{5,21}$: $(12)(34), (13)(24), (23)(45)$: $result6.1012, result6.1079, result6.1516$

$F_{5,22}$: $(12)(34), (12)(45), (23)(45)$: $result5.27, result5.30, result5.41$

$F_{5,24}$: $(12)(34), (12)(45), (24)(35)$: $result5.28, result5.32, result6.1005$

$F_{5,25}$: $(12)(34), (12)(45), (13)(25)$: $result5.34, result5.38, result5.69$

$F_{5,27}$: $(12)(34), (23)(45), (14)(25)$: $result5.43$

Order 72

$F_{6,5}$: $(12), (23), (15)(24)(36)$: $result6.164, result6.1761, result6.2547$

$\mathrm{F}_{6,14}$: $(12), (34), (13)(25)(46)$: $result6.160$

$F_{6,23}$: $(12), (12)(34), (13)(25)(46)$: $result6.1523, result6.163, result6.2553, result6.573$

$F_{6,52}$: $(12), (23)(45), (15)(24)(36)$: $result6.1149, result6.1355, result6.165,$
$result6.167, result6.1812, result6.2527$

$F_{7,135}$: $(12)(34), (12)(45), (26)(57)$: $result7.10107, result7.10612, result7.10622$

$F_{7,139}$: $(12)(34), (12)(45), (26)(47)$: $result7.10680$

Order 120

$F_{5,4}$: $(12), (23), (24)(35)$: $result5.11, result5.21, result5.24,$
$result5.4, result6.1358, result6.2548$

$\mathrm{F}_{5,5}$: $(12), (23), (14)(35)$: $result5.17, result6.1607, result6.2549$

$F_{5,7}$: $(12), (34), (23)(45)$: $result5.13, result5.16, result5.19, result5.26,$
$result7.10098, result7.11420$

$F_{5,9} : (12), (12)(34), (23)(45) : result6.1147, result6.1909, result6.2001,$

$result6.2518, result6.2555, result6.839$

$F_{5,10} : (12), (12)(34), (14)(25) : result6.1282, result6.1405, result6.1526,$

$result6.2506, result6.2554, result6.364$

$F_{5,12} : (12), (13)(24), (23)(45) : result5.10, result5.31, result6.1232,$

$result6.1588, result7.11163, result7.65779$

$F_{5,13} : (12), (13)(24), (13)(45) : result5.14, result5.29, result6.1145,$

$result6.1231, result7.1075, result7.70232$

$F_{5,14} : (12), (13)(24), (14)(25) : result5.18, result5.25, result5.33,$

$result6.1390, result6.1414, result6.2514$

$F_{5,15} : (12), (13)(24), (43)(25) : result5.12, result5.36, result6.1328,$

$result6.1357, result6.1498, result6.2528$

$F_{5,16} : (12), (13)(24), (13)(25) : result6.1185, result7.1076, result7.1078,$

$result7.29106, result7.66263$

$F_{5,18} : (12), (23)(45), (24)(35) : result6.1152, result6.1519, result6.2526$

$F_{7,115} : (12)(34), (13)(24), (12)(45)(67) : none$

$F_{7,123} : (12)(34), (12)(45), (13)(24)(67) : result7.11140, result7.11208, result7.34345$

$F_{7,124} : (12)(34), (12)(45), (23)(45)(67) : result7.1077, result7.1080, result7.27452,$

$result7.29299, result7.43835$

$F_{7,128} : (12)(34), (12)(45), (13)(25)(67) : result7.11212, result7.43329, result7.88620$

$F_{7,150} : (12)(34), (23)(45), (13)(24)(67) : result7.1105, result7.16825, result7.34339$

$F_{7,152} : (12)(34), (23)(45), (32)(14)(67) : result7.11520, result7.11737, result7.43930$

$F_{7,153} : (12)(34), (23)(45), (12)(45)(67) : result7.11525, result7.11527, result7.27477,$

$result7.29309, result7.43849, result7.86167$

$F_{7,154} : (12)(34), (23)(45), (13)(45)(67) : result7.11167, result7.26381, result7.27457,$

$result7.66319, result7.72664$

$F_{7,159}$: $(12)(34), (23)(45), (23)(15)(67)$: $result7.11169, result7.11523, result7.29116,$
$result7.29149, result7.43321, result7.86175$

$F_{7,255}$: $(12)(34), (13)(24)(56), (12)(65)(47)$: $result7.10417, result7.11894, result7.27570$

$F_{7,256}$: $(12)(34), (13)(24)(56), (23)(65)(47)$: $result7.10420, result7.13740, result7.27562$

$F_{7,257}$: $(12)(34), (13)(24)(56), (13)(65)(47)$: $result7.10265, result7.12053, result7.35316$

$F_{7,363}$: $(12)(34), (12)(45)(67), (23)(45)(67)$: $result7.10268, result7.11895, result7.27478,$
$result7.27499, result7.27515, result7.29303$

$F_{7,365}$: $(12)(34), (12)(45)(67), (24)(35)(67)$: $result7.10199, result7.11891, result7.27458,$
$result7.29110, result7.35008, result7.73255$

$F_{7,366}$: $(12)(34), (12)(45)(67), (14)(25)(67)$: $none$

$F_{7,367}$: $(12)(34), (12)(45)(67), (43)(25)(67)$: $result7.11896, result7.27453, result7.35011$

$F_{7,368}$: $(12)(34), (12)(45)(67), (13)(25)(67)$: $result7.10458, result7.11947, result7.29155,$
$result7.29211, result7.35018, result7.88621$

$F_{7,383}$: $(12)(34), (23)(45)(67), (13)(45)(67)$: $result7.10197, result7.27505$

$F_{7,384}$: $(12)(34), (23)(45)(67), (24)(35)(67)$: $none$

$F_{7,386}$: $(12)(34), (23)(45)(67), (14)(25)(67)$: $result7.10264, result7.29313, result7.29342$

$F_{7,388}$: $(12)(34), (23)(45)(67), (23)(15)(67)$: $result7.10201, result7.29120, result7.29237$

$F_{7,426}$: $(12)(34)(56), (13)(24)(56), (12)(65)(47)$: $none$

$F_{7,427}$: $(12)(34)(56), (13)(24)(56), (23)(65)(47)$: $none$

$F_{7,439}$: $(12)(34)(56), (12)(34)(67), (12)(45)(67)$: $result7.27500, result7.27516, result7.29336$

$F_{7,443}$: $(12)(34)(56), (12)(34)(67), (12)(46)(57)$: $result7.27506, result7.29205$

$F_{7,445}$: $(12)(34)(56), (12)(34)(67), (12)(35)(47)$: $result7.29215, result7.29243, result7.88147$

$F_{7,466}$: $(12)(34)(56), (12)(34)(67), (26)(35)(47)$: $result7.29346$

Order 144

$F_{7,47}$: $(12), (23)(45), (36)(57)$: $result7.10, result7.1001, result7.11219,$

$result7.11548, result7.11656, result7.13165$

F$_{7,48}$: $(12), (23)(45), (12)(36)(57) : none$

$F_{7,51} : (12), (23)(45), (26)(57) : result7.102, result7.10621, result7.10744,$
$result7.11221, result7.11495, result7.27852$

$F_{7,54} : (12), (23)(45), (13)(26)(57) : result7.10767, result7.11666, result7.14908,$
$result7.15121, result7.27332, result7.85237$

$F_{7,55} : (12), (23)(45), (16)(57) : result7.10104, result7.1123, result7.15091$

$F_{7,57} : (12), (23)(45), (23)(16)(57) : result7.10459, result7.1053, result7.11655,$
$result7.15252, result7.27333, result7.85065$

$F_{7,137} : (12)(34), (12)(45), (43)(26)(57) : result7.10720, result7.12809, result7.27814,$
$result7.29140, result7.34703$

$F_{7,141} : (12)(34), (12)(45), (26)(35)(47) : result7.10725, result7.35779$

$F_{7,236} : (12)(34), (12)(34)(56), (26)(47) : none$

$F_{7,283} : (12)(34), (12)(35)(46), (63)(27) : result7.10948, result7.10996,$
$result7.12867, result7.27365$

$F_{7,285} : (12)(34), (12)(35)(46), (64)(27) : result7.10502, result7.12865,$
$result7.27366, result7.30801$

$F_{7,306} : (12)(34), (25)(46), (43)(15)(67) : result7.10611, result7.10749,$
$result7.25589, result7.27819, result7.29163, result7.29995$

Order 168

$F_{7,208} : (12)(34), (23)(56), (36)(47) : result7.10022$

F$_{7,210}$: $(12)(34), (23)(56), (26)(47) : result7.10009, result7.10021, result7.10917$

$F_{7,220} : (12)(34), (23)(56), (16)(37) : result7.10015, result7.10614, result7.10950$

Order 240

$F_{7,4}: (12), (23), (24)(35)(67) : result7.1, result7.1037, result7.11322, result7.27164$

$\mathrm{F}_{7,5}: (12), (23), (14)(35)(67) : result7.10472$

$F_{7,8}: (12), (34), (23)(45)(67) : result7.10083, result7.10492, result7.11306, result7.27169$

$F_{7,12}: (12), (12)(34), (23)(45)(67) : result7.10198, result7.10327, result7.11417,$
$result7.27449, result7.27808, result7.64012$

$F_{7,14}: (12), (12)(34), (14)(25)(67) : result7.10326, result7.10403, result7.13257,$
$result7.27189, result7.66887, result7.88614$

$F_{7,18}: (12), (13)(24), (45)(67) : result7.100, result7.1003, result7.1069,$
$result7.11281, result7.11660$

$F_{7,19}: (12), (13)(24), (12)(45)(67) : none$

$F_{7,20}: (12), (13)(24), (23)(45)(67) : result7.10156, result7.10196, result7.12013,$
$result7.12965, result7.27454, result7.27553$

$F_{7,21}: (12), (13)(24), (13)(45)(67) : result7.10154, result7.10470, result7.10904,$
$result7.11893, result7.27462, result7.27473$

$F_{7,22}: (12), (13)(24), (25)(67) : result7.10254, result7.1067, result7.1116,$
$result7.11288, result7.11493$

$F_{7,23}: (12), (13)(24), (14)(25)(67) : result7.10149, result7.11989, result7.14879,$
$result7.27162, result7.66912$

$F_{7,24}: (12), (13)(24), (43)(25)(67) : result7.10151, result7.10263, result7.10913,$
$result7.29136, result7.35229, result7.64039$

$F_{7,25}: (12), (13)(24), (13)(25)(67) : result7.10153, result7.11325, result7.12003,$
$result7.14881, result7.27177, result7.90169$

$F_{7,31}: (12), (23)(45), (12)(34)(67) : none$

$F_{7,33}: (12), (23)(45), (13)(24)(67) : result7.10195, result7.11311, result7.11661,$
$result7.15462, result7.71355$

$F_{7,34} : (12), (23)(45), (14)(67) : result7.10112, result7.1119, result7.3162$

$F_{7,35} : (12), (23)(45), (32)(14)(67) : result7.10457, result7.10935, result7.11663,$
$result7.15263, result7.71351$

$F_{7,40} : (12), (23)(45), (24)(35)(67) : result7.101, result7.10419, result7.1042, result7.13617$

$F_{7,41} : (12), (23)(45), (14)(35)(67) : result7.10100, result7.10121, result7.10200,$
$result7.10479, result7.29115, result7.34959$

$F_{7,42} : (12), (23)(45), (14)(25)(67) : result7.10247, result7.10361, result7.10944,$
$result7.15466, result7.71331$

$F_{7,61} : (12), (23)(45), (26)(45)(37) : result7.1020, result7.11319, result7.1289,$
$result7.15728, result7.27161, result7.85097$

$F_{7,62} : (12), (23)(45), (16)(45)(37) : result7.1004, result7.15708, result7.27459$

$F_{7,72} : (12), (34)(56), (43)(25)(67) : result7.1000, result7.1006, result7.11303, result7.27167$

$F_{7,82} : (12), (12)(34)(56), (43)(25)(67) : none$

$F_{7,83} : (12), (12)(34)(56), (16)(34)(27) : none$

$F_{7,96} : (12), (13)(24)(56), (23)(65)(47) : result7.11314, result7.15795, result7.27558$

$F_{7,97} : (12), (13)(24)(56), (13)(65)(47) : result7.1023, result7.15753, result7.27511$

$F_{7,104} : (12), (13)(24)(56), (14)(65)(27) : result7.1005, result7.15969, result7.27165,$
$result7.29210, result7.85144$

$F_{7,105} : (12), (13)(24)(56), (43)(65)(27) : result7.1024, result7.15792, result7.27567$

$F_{7,106} : (12), (13)(24)(56), (13)(65)(27) : result7.16499, result7.85006$

$F_{7,108} : (12), (23)(45)(67), (24)(35)(67) : none$

Order 360

$F_{6,80} : (12)(34), (12)(45), (23)(56) : result6.1035, result6.1038, result6.1180$

$\mathrm{F}_{6,83} : (12)(34), (12)(45), (24)(56) : result6.1010, result6.1036, result6.1078,$
$result6.1116, result6.1154, result6.1317$

$F_{6,86}: (12)(34), (12)(45), (23)(46) : result6.1040, result6.1071, result6.1126$

$F_{6,90}: (12)(34), (12)(45), (15)(26) : result6.1000, result6.1075, result6.1157,$
$result6.1275, result6.1859, result6.1957$

$F_{6,94}: (12)(34), (12)(45), (14)(26) : result6.1007, result6.1009, result6.1188$

$F_{6,100}: (12)(34), (23)(45), (24)(56) : result6.1006, result6.1073, result6.1155$

$F_{6,102}: (12)(34), (23)(45), (14)(56) : result6.1080, result6.1183, result6.1383$

Order 720

$F_{6,34}: (12), (13)(24), (25)(46) : result6.1015, result6.110, result6.1143,$
$result6.1245, result6.2489, result6.569$

$\mathrm{F}_{6,35}: (12), (13)(24), (13)(25)(46) : result6.111, result6.2458, result6.630$

$F_{6,36}: (12), (13)(24), (15)(46) : result6.1018, result6.1098, result6.1128,$
$result6.1138, result6.114, result6.2556$

$F_{6,37}: (12), (13)(24), (32)(15)(46) : result6.1233, result6.161, result6.1729,$
$result6.2482, result6.2517, result6.648$

$F_{6,42}: (12), (23)(45), (34)(56) : result6.1011, result6.1076, result6.133, result6.171$

$F_{6,43}: (12), (23)(45), (12)(34)(56) : none$

$F_{6,44}: (12), (23)(45), (24)(56) : result6.1004, result6.101, result6.1070,$
$result6.1099, result6.125, result6.1361$

$F_{6,45}: (12), (23)(45), (13)(24)(56) : result6.1144, result6.162, result6.1846$

$F_{6,46}: (12), (23)(45), (14)(56) : result6.1148, result6.1153, result6.1156,$
$result6.1247, result6.144, result6.151$

$F_{6,48}: (12), (23)(45), (25)(36) : result6.1008, result6.1101, result6.121$

$F_{6,49}: (12), (23)(45), (15)(36) : result6.1100, result6.1118, result6.147$

Order 2520

$F_{7,160}$: (12)(34), (23)(45), (46)(57) : $result7.10003$

$F_{7,164}$: (12)(34), (23)(45), (36)(57) : $none$

$F_{7,168}$: (12)(34), (23)(45), (26)(57) : $none$

$F_{7,172}$: (12)(34), (23)(45), (16)(57) : $result7.10106, result7.10110$

$F_{7,175}$: (12)(34), (23)(45), (36)(47) : $result7.10010$

$F_{7,177}$: (12)(34), (23)(45), (26)(47) : $none$

$F_{7,201}$: (12)(34), (23)(56), (45)(67) : $result7.10001$

$F_{7,212}$: (12)(34), (23)(56), (16)(47) : $result7.10105, result7.10108$

$F_{7,298}$: (12)(34), (25)(46), (35)(67) : $none$

CHAPTER 7

CONCLUSION

In this thesis we have developed a more efficient way then previously known to count non-congruent Cayley pairs and triples by extending Burnside's Lemma directly. Canonical triples of involutions (one for each congruence class) were generated for degree ≤ 7. Then for each such triple the adjacency list for its Cayley graph was generated. All isomorphisms between these Cayley graphs were found by use of `nauty`. The results are summarized by order and minimum degree in Table 6.1.

Comparing our results with Effler's Table 3.3 [1], it is seen that our census includes a number of simple cubic Cayley graphs of degree ≤ 7 which he had missed. Also we find a lower minimum degree for a number of others. We believe that our methods could be extended to encompass degree 8. Probably degrees 9 and 10 could be included too, but for that `Maple` code would need to be rewritten in a more efficient language such as `C`.

We have made a first step toward anticipating cases in which non-congruent triples generate isomorphic simple cubic Cayley graphs, but it seems likely that much more could be done along these lines by someone with a good knowledge of group theory.

An obvious future direction for research is to ascertain the hamiltonicity of the non-isomorphic simple cubic Cayley graphs we have generated. The hope is that a counterexample to the hamiltonicity conjecture for Cayley graphs of order ≥ 3 can be found among those that are simple and cubic.

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