Valuation of Finite Maturity Stock Loans
under Regime Switching
and Mean Reverting Stock Models

by

David J. Prager

(Under the direction of Qing Zhang)

Abstract

Stock loans are business contracts between borrowers and lenders in which the borrower
uses shares of stock as collateral for the loan. Since the value of the collateral is subject
to wide and frequent price fluctuations, valuing such a transaction behaves more like an
option pricing problem than a debt valuation problem. This dissertation will list, prove, and
analyze formulas for stock loan valuation with finite horizon when the collateral stock obeys a
classical geometric Brownian motion, a mean reverting, or a two-state regime switching with
both mean reverting and geometric Brownian motion states. Also, existence and uniqueness
of viscosity solutions will be proved for the mean reverting and classical geometric Brownian
motion regime switching with partial information stock models. Numerical examples are
reported to illustrate the results.

Index words: Stock loan, Regime switching, Mean reverting, Wonham filter,
Viscosity solution
Valuation of Finite Maturity Stock Loans
under Regime Switching
and Mean Reverting Stock Models

by

David J. Prager

B.S., Miami University, 1999
M.S., Miami University, 2005

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2010
© 2010
David J. Prager
All Rights Reserved
VALUATION OF FINITE MATURITY STOCK LOANS
UNDER REGIME SWITCHING
AND MEAN REVERTING STOCK MODELS

by

DAVID J. PRAGER

Approved:

Major Professor: Qing Zhang

Committee: Dhandapani Kannan
Robert Varley
Andrew Sornborger
Jingzhi Tie

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2010
ACKNOWLEDGMENTS

I need to thank my advisor, Prof. Qing Zhang, for his tireless patience and assistance, without which this dissertation would not have been possible. I need to thank Prof. Dhandapani Kannan who supported me at the very beginning when I most needed it. I need to thank Prof. Robert Varley for his countless hours of valuable conversation based on a wealth of experience. I need to thank Dr. Andrew Sornborger for having the courage to teach this almost computer illiterate first-year graduate student how to write MATLAB code. I need to thank Dr. Jingzhi Tie for his valuable input and insights. I need to thank my mother, Shirley Prager, for supporting my decision to return to graduate school and every decision I have made since then. Finally, I need to thank my saviour Jesus, the Christ, for giving me strength in my weakness and grace to help in time of need.
# Table of Contents

<table>
<thead>
<tr>
<th>Acknowledgments</th>
<th>iv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>1 INTRODUCTION AND NOTATION</td>
<td>1</td>
</tr>
<tr>
<td>2 LOAN VALUATION: BASE CASES</td>
<td>3</td>
</tr>
<tr>
<td>2.1 CLASSICAL GEOMETRIC BROWNIAN MOTION</td>
<td>3</td>
</tr>
<tr>
<td>2.2 MEAN REVERTING</td>
<td>11</td>
</tr>
<tr>
<td>3 TWO-STATE REGIME SWITCHING MODELS</td>
<td>25</td>
</tr>
<tr>
<td>3.1 TWO GEOMETRIC BROWNIAN MOTION STATES</td>
<td>26</td>
</tr>
<tr>
<td>3.2 ONE MEAN REVERTING STATE</td>
<td>30</td>
</tr>
<tr>
<td>3.3 TWO MEAN REVERTING STATES</td>
<td>57</td>
</tr>
<tr>
<td>4 VISCOSITY SOLUTIONS FOR THE MEAN REVERTING STOCK LOAN EQUATION</td>
<td>80</td>
</tr>
<tr>
<td>4.1 EXISTENCE</td>
<td>81</td>
</tr>
<tr>
<td>4.2 UNIQUENESS</td>
<td>91</td>
</tr>
<tr>
<td>4.3 FINITE DIFFERENCE NUMERICAL SCHEME</td>
<td>97</td>
</tr>
<tr>
<td>5 VISCOSITY SOLUTION FOR PARTIALLY OBSERVED STOCK LOANS</td>
<td>105</td>
</tr>
<tr>
<td>5.1 PROBLEM FORMULATION</td>
<td>105</td>
</tr>
<tr>
<td>5.2 EXISTENCE</td>
<td>107</td>
</tr>
<tr>
<td>5.3 UNIQUENESS</td>
<td>117</td>
</tr>
<tr>
<td>5.4 MARKOV CHAIN APPROXIMATION</td>
<td>124</td>
</tr>
<tr>
<td>5.5 NUMERICAL EXAMPLES</td>
<td>132</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction and Notation

Over the past several years an interesting type of transaction called a stock loan has been receiving increased attention in research and industry. A traditional stock loan involves two parties: the borrower and the lender. The borrower owns one share of an underlying stock. This share is used as collateral in order to obtain a loan from the lender. Unlike conventional loans such as car loans and mortgages in which the collateral has a highly predictable market price, the collateral for a stock loan is subject to wide, frequent, and unpredictable price fluctuations. The potential for significant price depreciation increases the risk borne by the lender, and thus conventional lending rules become inadequate. From a borrower’s perspective, a stock loan provides a source of available cash while reducing the risk of owning the stock. If the stock price increases significantly, the borrower can repay the loan and regain the collateral, but if the stock decreases significantly, the borrower can default and keep the original loan principal.

This relatively new problem has already begun to be studied in academic circles, and several papers have been published. In [19], Xia and Zhou discuss the case in which a perpetual stock loan has American maturity (i.e. the borrower can redeem the collateral at any time), the stock obeys a classical geometric Brownian motion, and the lender charges a fixed loan origination fee. In [25], Zhang and Zhou discuss the case in which a perpetual stock loan has American maturity and the stock obeys a classical geometric Brownian motion with regime switching. In [3], perpetual stock loans are valued when the collateral stock obeys a double exponential model. All of these papers study the case in which the loan is perpetual, i.e. the loan contract never matures. In reality, most loan contracts are written with a fixed, finite
maturity time. Therefore, while these papers provide excellent mathematics and valuable intuition, the need remains for formulas and numerical approximations for the stock loan value in the case of finite maturity. This dissertation intends to begin to meet that need.

Throughout this dissertation, the following notation will be used. Let $V(x, t, \cdot)$ denote the value function of the stock loan at time $t$ and initial stock price (or log stock price, as appropriate) $x$. Let $s$ denote the initial time, $q$ the loan principal, and $\gamma$ the loan interest rate. Let $W_t$ be a Brownian motion, and let $S_t$ denote the price of the collateral stock at time $t$. Let $r$ denote the risk-free interest rate and $\sigma > 0$ the volatility. As shown in [19], the following equation must hold in the absence of arbitrage:

$$V(x, s, \cdot) = S_s - q + c \quad (1.0.1)$$

where $c > 0$ is the service charge required by the bank to enter the loan contract. Thus, our objective will be to determine the initial value $V(x, s, \cdot)$ of the stock loan, as the loan value determines a fair service charge for the loan contract.

This dissertation is organized as follows. Chapter 2 proves and analyzes closed-form solutions for the two base cases for the collateral stock: classical geometric Brownian motion and mean reverting. Chapter 3 builds on Chapter 2 by proving and analyzing closed-form solutions for when the collateral stock obeys a two-state regime switching model in which the second state is absorbing. Chapter 4 returns to a pure mean reverting model but gives a viscosity solution approach. A finite difference numerical scheme is defined, proved and, implemented to approximate the viscosity solution. Chapter 5 proves existence and uniqueness of a viscosity solution when the collateral stock obeys an $m$-state regime switching model in which all states are geometric Brownian motions. Unlike in previous chapters, Chapter 5 will only assume that the switching is partially observable as opposed to fully observable. A Markov chain approximation scheme is defined, proved and, implemented to approximate the viscosity solution. The bibliography concludes the dissertation.
Chapter 2

Loan Valuation: Base Cases

2.1 Classical Geometric Brownian Motion

For this section, we will assume the stock price obeys the classical geometric Brownian motion
\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad t \geq s, \quad S_s = x. \tag{2.1.1}
\]

Since we are assuming European maturity, the value function for the stock loan is
\[
V(x, s) = e^{r(T-s)}E(S_T - qe^{\gamma(T-s)})_+
\]
\[= e^{(\gamma-r)(T-s)}E(S_T e^{-\gamma(T-s)} - q)_+.\]

There are two standard methods for finding the initial value \(V(x, s)\) of the stock loan. For the derivation in the European option case, see [14].

2.1.1 PDE Method

To derive the corresponding PDE, we first write the value function in a form similar to that of a call option. Define \(\tilde{S}_t \equiv S_t e^{-\gamma(t-s)}\). Then \(\tilde{S}_s = x\) and
\[
d\tilde{S}_t = -\gamma e^{-\gamma(t-s)}S_t dt + e^{-\gamma(t-s)}dS_t
\]
\[= -\gamma e^{-\gamma(t-s)}S_t dt + e^{-\gamma(t-s)}S_t (\mu dt + \sigma dW_t)
\]
\[= \tilde{S}_t (\mu - \gamma) dt + \sigma dW_t.
\]
The PDE for this value function \(V(x, s)\) is
\[
0 = \frac{\partial V}{\partial s} + x(\mu - \gamma) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} - (\gamma - r)V \tag{2.1.2}
\]
\[V(z, T) = (z - q)_+.
\]
To solve this equation, first do the substitution

\[ U = e^{-(\gamma - r)s}V \]

so that

\[ V = e^{(\gamma - r)s}U \]

and

\[ \frac{\partial U}{\partial s} = e^{-(\gamma - r)s} \frac{\partial V}{\partial s} - (\gamma - r)e^{-(\gamma - r)s}V. \]

Also, we have

\[ \frac{\partial V}{\partial s} = (\gamma - r)Ue^{(\gamma - r)s} + \frac{\partial U}{\partial s}e^{(\gamma - r)s}. \]

Using these substitutions allows us to rewrite (2.1.2) in terms of the new function \( U \):

\[ 0 = (\gamma - r)e^{(\gamma - r)s}U + e^{(\gamma - r)s}x(\mu - \gamma) \frac{\partial U}{\partial x} + \frac{\sigma^2}{2}x^2e^{(\gamma - r)s}\frac{\partial^2 U}{\partial x^2} - (\gamma - r)e^{(\gamma - r)s}U. \]

Canceling the factors \( (\gamma - r)e^{(\gamma - r)t}U \) and dividing through by \( e^{(\gamma - r)t} > 0 \) gives

\[ 0 = \frac{\partial U}{\partial s} + x(\mu - \gamma) \frac{\partial U}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2 U}{\partial x^2}. \]  

(2.1.3)

To continue, we make the substitutions \( y = \log x \) and \( \rho = T - s \) so that

- \[ \frac{\partial U}{\partial s} = -\frac{\partial U}{\partial \rho}; \]
- \[ \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} \frac{\partial y}{\partial x} = \frac{1}{x} \frac{\partial U}{\partial y}; \]
- \[ \frac{\partial^2 U}{\partial x^2} = \frac{1}{x^2} \frac{\partial U}{\partial y} + \frac{1}{x \partial y^2} \frac{\partial^2 U}{\partial x \partial y} = -\frac{1}{x^2} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2}. \]

Making these substitutions in (2.1.3) and regrouping terms gives

\[ 0 = -\frac{\partial U}{\partial \rho} + \left( \mu - \gamma - \frac{\sigma^2}{2} \right) \frac{\partial U}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial y^2}. \]  

(2.1.4)

We next do a change of coordinates

\[ z = y - \left( \mu - \gamma - \frac{\sigma^2}{2} \right) \tau \]

\[ \tau = \rho \]

so that
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial y} \]
\[ \frac{\partial}{\partial \tau} = -\left( \mu - \gamma - \frac{\sigma^2}{2} \right) \frac{\partial}{\partial y} + \frac{\partial}{\partial \rho}. \]

Thus, in the new coordinates \((z, \tau)\), equation (2.1.4) becomes

\[
0 = -\frac{\partial U}{\partial \tau} - \left( \mu - \gamma - \frac{\sigma^2}{2} \right) \frac{\partial U}{\partial z} + \left( \mu - \gamma - \frac{\sigma^2}{2} \right) \frac{\partial U}{\partial z} + \frac{\sigma^2 \partial^2 U}{2 \partial z^2}
\]

\[= -\frac{\partial U}{\partial \tau} + \frac{\sigma^2 \partial^2 U}{2 \partial z^2}. \tag{2.1.5} \]

Note that, in the new coordinates, the value function becomes

\[ V(z, \tau) = e^{(\gamma-r)t} U(z, \tau) = e^{(\gamma-r)t} U \left( T - \rho, \log x - \left( \mu - \gamma - \frac{\sigma^2}{2} \right) \tau \right). \]

Also note that equation (2.1.5) is simply the heat equation in one dimension. By [6], the fundamental solution to equation (2.1.5) is

\[ \Phi(z, \tau) = \frac{1}{\sqrt{2\pi \tau \sigma^2}} \exp \left( -\frac{z^2}{2\sigma^2 \tau} \right). \]

Thus, the solution \(U(z, \tau)\) with initial condition \(g\) on \((\tau = 0) \times \mathbb{R}\) is

\[ U(\tau, z) = \int_{-\infty}^{\infty} \Phi(\tau, z - w) g(w) dw = \frac{1}{\sqrt{2\pi \tau \sigma^2}} \int_{-\infty}^{\infty} g(w) \exp \left( -\frac{(z - w)^2}{2\sigma^2 \tau} \right) dw. \]

Since \(\rho = T - s\), we have

\[ g(w) \equiv V(w, T) = (e^w - q)_+, \]

with \(e^w\) instead of \(w\) because of the substitution \(y = \log x\). Thus, in terms of the original value function \(V(x, s)\), we have

\[ V(x, s) = \frac{e^{(\gamma-r)(T-s)}}{\sqrt{2\pi \sigma^2 (T - s)}} \int_{-\infty}^{\infty} (e^w - q)_+ e(w) dw \tag{2.1.6} \]

where

\[ e(w) \equiv \exp \left( -\frac{(\log x + (\mu - \gamma - (\sigma^2/2))(T - s) - w)^2}{2\sigma^2 (T - s)} \right). \]
2.1.2 Distribution Method

A more straightforward but less adaptable way to compute the value of the stock loan under the assumptions for this section is to use the distribution of $W_T$, i.e. the fact that $W_{T-s} \sim N(0, T-s)$ where $N(0, T-s)$ denotes the normal distribution with mean 0 and variance $T-s$. In this method, we view the value function as the discounted expected future value. Thus, we will attempt to evaluate the value function

$$V(x, s) = e^{-r(T-s)} E[(S_T - qe^{\gamma(T-s)})_+]$$ \hspace{1cm} (2.1.7)

subject to the terminal condition

$$S_T = x \exp \left( (\mu - (\sigma^2 / 2))(T - s) + \sigma W_{T-s} \right).$$ \hspace{1cm} (2.1.8)

Since $W_{T-s} \sim N(0, T-s)$, we can replace $W_{T-s}$ by $Z \sqrt{T-s}$ where $Z \sim N(0, 1)$ in (2.1.7) in order to obtain, using (2.1.8) for the future value,

$$V(x, s) = e^{-r(T-s)} \left[ \left( x \exp \left( (\mu - \sigma^2 / 2)(T - s) + \sigma \sqrt{T-s} Z \right) - qe^{\gamma(T-s)} \right)_+ \right]$$

$$= \frac{e^{-r(T-s)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( x \exp \left( (\mu - \sigma^2 / 2)(T - s) + \sigma y \sqrt{T-s} \right) - qe^{\gamma(T-s)} \right)_+ e^{-(y^2/2)} dy$$ \hspace{1cm} (2.1.9)

where (2.1.9) is due to the definition of expectation and the probability density function of $Z$. To begin evaluating this integral, note that the integrand will be non-zero if and only if

$$x \exp \left( (\mu - (\sigma^2 / 2))(T - s) + \sigma \sqrt{T-s} y \right) - qe^{\gamma(T-s)} > 0.$$

Solving for $y$ gives

$$y > \log \left( \frac{qe^{\gamma(T-s)} / x - (\mu - \sigma^2 / 2)(T - s)}{\sigma \sqrt{T-s}} \right).$$

Thus, defining

$$m_{GBM} = \frac{\log \left( \frac{qe^{\gamma(T-s)} / x - (\mu - \sigma^2 / 2)(T - s)}{\sigma \sqrt{T-s}} \right)}{\sigma \sqrt{T-s}}$$

allows us to rewrite (2.1.9) as

$$\frac{e^{-r(T-s)}}{\sqrt{2\pi}} \int_{m_{GBM}}^{\infty} \left( x \exp \left( (\mu - \sigma^2 / 2)(T - s) + \sigma \sqrt{T-s} y \right) - qe^{\gamma(T-s)} \right)_+ e^{-(y^2/2)} dy$$
We will now evaluate both of the integrals in (2.1.10).

**First term:**
\[
\int_{m_{GBM}}^{\infty} e^{-r(T-s)} \frac{e^{-r(T-s)} e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \int_{m_{GBM}}^{\infty} x \exp \left( (\mu - \frac{\sigma^2}{2})(T-s) + \sigma \sqrt{T-s} y - \frac{y^2}{2} \right) dy
\]

**Second term:**
\[
\frac{e^{-r(T-s)}}{\sqrt{2\pi}} \int_{m_{GBM}}^{\infty} q e^{\gamma(T-s)} e^{-\frac{y^2}{2}} dy
\]

where \(\Phi(\cdot)\) is the cumulative distribution function of \(N(0,1)\).

To continue, we need to complete the square in the exponent of the integral in (2.1.12):
\[
\sigma y \sqrt{T-s} - \frac{y^2}{2} = \sigma y \sqrt{T-s} - \frac{y^2}{2} + \frac{\sigma^2(T-s)}{2} - \frac{\sigma^2(T-s)}{2}
\]
\[
= -\frac{(y - \sigma \sqrt{T-s})^2}{2} + \frac{\sigma^2(T-s)}{2}.
\]

Substituting into (2.1.12) gives
\[
x \exp \left( -r(T-s) + \left( \mu - \frac{\sigma^2}{2} \right)(T-s) \right) \frac{1}{\sqrt{2\pi}} \int_{m_{GBM}}^{\infty} \exp \left( -\frac{(y - \sigma \sqrt{T-s})^2}{2} + \frac{\sigma^2(T-s)}{2} \right) dy.
\]

Now perform the substitution \(w \equiv y - \sigma \sqrt{T-s}\) to obtain
\[
x \exp \left( -r(T-s) + (\mu - (\sigma^2/2))(T-s) \right)
\]
Letting /u1D464 substitution for the solution from the Distribution Method. Starting with equation (2.1.6), perform the result of the PDE Method, can be reformulated to equal equation (2.1.9), the source above yield the same solution. This will be accomplished by showing that equation (2.1.6),

\[ \frac{1}{\sqrt{2\pi}} \int_{m_{GBM}}^{\infty} \exp \left( -\frac{\left( y - \sigma \sqrt{T-s} \right)^2}{2} + \frac{\sigma^2(T-s)}{2} \right) dy \]

\[ = x \exp \left( -r(T-s) + \left( \mu - \frac{\sigma^2}{2} \right)(T-s) \right) \exp \left( \frac{\sigma^2}{2}(T-s) \right) \]

\[ \frac{1}{\sqrt{2\pi}} \int_{m_{GBM}-\sigma \sqrt{T-s}}^{\infty} \exp \left( \frac{w^2}{2} \right) dw \]

\[ = x \exp \left( \mu - r \right) (T-s) \left[ 1 - \Phi \left( m_{GBM} - \sigma \sqrt{T-s} \right) \right] \]

\[ = xe^{(\mu-r)(T-s)} \Phi \left( \sigma \sqrt{T-s} - m_{GBM} \right). \]  

Combining (2.1.11) and (2.1.13) gives

\[ V(x, s) = e^{-r(T-s)} \left[ xe^{\mu(T-s)} \Phi \left( \sigma \sqrt{T-s} - m_{GBM} \right) - qe^\gamma(T-s) \Phi(-m_{GBM}) \right]. \]  

**2.1.3 Reconciling the Two Methods**

To indicate the potential for a unique solution, we now show the two methods presented above yield the same solution. This will be accomplished by showing that equation (2.1.6), the result of the PDE Method, can be reformulated to equal equation (2.1.9), the source for the solution from the Distribution Method. Starting with equation (2.1.6), perform the substitution

\[ u \equiv M + \frac{w}{\sigma \sqrt{T-s}} \Rightarrow du = -\frac{1}{\sigma \sqrt{T-s}} dw \]

where

\[ M \equiv \frac{\log x + \left[ \mu - \gamma - \left( \frac{\sigma^2}{2} \right)(T-s) \right]}{\sigma \sqrt{T-s}}. \]

Letting \( w = h(u) \equiv -(u-M)\sqrt{\sigma^2(T-s)} \), this substitution gives

\[ V(x, s) = -\frac{e^{(\gamma-r)(T-s)}\sigma \sqrt{T-s}}{\sqrt{2\pi \sigma^2(T-s)}} \int_{-\infty}^{\infty} \left( e^{h(u)} - q \right) \exp \left( -u^2/2 \right) du. \]

By definition of \( y \) and \( M \), we have

\[ h(u) = -(u-M)\sqrt{\sigma^2(T-s)} \]

\[ = -u\sigma \sqrt{T-s} + \log x + \left[ \mu - \gamma - \left( \frac{\sigma^2}{2} \right)(T-s) \right]. \]
This implies that \( \exp(h(u)) = x \exp\left(\left(\mu - \gamma - \frac{\sigma^2}{2}\right)(T - s) - u\sigma\sqrt{T - s}\right) \). Thus, with this calculation and some cancellation, (2.1.15) becomes
\[
V(x, s) = -\frac{e^{(\gamma - r)(T - s)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \exp\left(\mu - \gamma - \left(\frac{\sigma^2}{2}\right)(T - s) - u\sigma\sqrt{T - s}\right) - q\right) e^{-u^2} du.
\]
The substitution \( y = -u \) gives
\[
V(x, s) = \frac{e^{(\gamma - r)(T - s)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \exp\left(\mu - \gamma - \left(\frac{\sigma^2}{2}\right)(T - s) + y\sigma\sqrt{T - s}\right) - q\right) e^{-y^2/2} dy,
\]
and distributing the constant \( e^{\gamma(T - s)} \) through the integrand gives
\[
V(x, s) = \frac{e^{-r(T - s)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T - s) + y\sigma\sqrt{T - s}\right) - q e^{\gamma(T - s)}\right) e^{-y^2/2} dy.
\]
(2.1.16)
This equation is identical to equation (2.1.9) derived using the distribution method.

2.1.4 Numerical Examples

This section gives some numerical examples to illustrate the methods presented above and the relationships they produce among the variables of interest. These examples were produced using MATLABR2006a to evaluate the above formula.

The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values given in Table 2.1. Varying one parameter at a time gives the results indicated in Table 2.2. The formula possesses many of the expected properties for a geometric Brownian motion. As suggested by (1.0.1), all else constant, as the loan principal increases, the value decreases; as the initial stock price increases, the value increases. Also, the loan value is a decreasing function of \( \gamma \): as the loan interest rate increases, the amount to be repaid by the borrower at maturity increases, and as such the value of the loan at the current time decreases. The loan value is an increasing function of maturity time \( T \); this result is expected due to the time value of money. Finally, as the stock’s volatility \( \sigma \) increases, the value of the loan increases: the borrower’s choice at maturity to repay the loan or default becomes more valuable as the stock becomes more volatile.
Table 2.1: Default Values for Geometric Brownian Motion, One Dimensional Examples

<table>
<thead>
<tr>
<th>variable</th>
<th>default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>0.05</td>
</tr>
<tr>
<td>q</td>
<td>40</td>
</tr>
<tr>
<td>S₀</td>
<td>50</td>
</tr>
<tr>
<td>γ</td>
<td>0.1</td>
</tr>
<tr>
<td>s</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>1</td>
</tr>
<tr>
<td>σ</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The second set of examples shows relationships as two of the parameters are changed, all others held constant. The results are displayed in Figures 2.1 and 2.2. Both of these figures confirm the findings in the first set of examples. Figure 2.1 shows that the loan value as a function of $\sigma$ is flatter for small loan principals than large ones. Also, the loan value is increasing as a function of loan principal, but the loan value increases more quickly for small values of $\sigma$ than large ones. Figure 2.2 also confirms the findings in the earlier examples. However, for small values of $\sigma$ and short maturities the loan value changes much more slowly than for large values.
Table 2.2: Numerical Results for Geometric Brownian Motion, One Parameter Varied

<table>
<thead>
<tr>
<th>$q$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>39.49</td>
<td>29.24</td>
<td>20.34</td>
<td>13.59</td>
<td>8.89</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>45</th>
<th>55</th>
<th>65</th>
<th>75</th>
<th>85</th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>10.14</td>
<td>17.36</td>
<td>25.66</td>
<td>34.63</td>
<td>44.00</td>
<td>53.62</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
<th>0.19</th>
<th>0.22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>14.27</td>
<td>13.59</td>
<td>12.91</td>
<td>12.25</td>
<td>11.60</td>
<td>10.96</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>11.84</td>
<td>13.58</td>
<td>16.10</td>
<td>17.09</td>
<td>17.97</td>
<td>19.50</td>
<td>20.81</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>10.34</td>
<td>11.93</td>
<td>13.59</td>
<td>15.26</td>
<td>16.94</td>
<td>18.59</td>
</tr>
</tbody>
</table>

2.2 Mean Reverting

For this section, we assume that the stock price obeys the mean reverting model

\[
S_t = e^{X_t},
\]

\[
dX_t = a(L - X_t)dt + \sigma dW_t, \quad X_s = \log S_0
\]
Figure 2.1: Relationship between $q$, $\sigma$, and Loan Value when collateral stock obeys a geometric Brownian motion.
Figure 2.2: Relationship between $T$, $\sigma$, and Loan Value when collateral stock obeys a geometric Brownian motion.
for \( t \geq s \) where \( a > 0 \) is the rate of reversion and \( L \) is the equilibrium level. Studies that support the mean reversion stock returns can be traced back to the 1930’s (see Cowles and Jones [4]) in empirical literature. The research was furthered by many researchers including Fama and French [7], and Gallagher and Taylor [9] among others. Related results in option pricing with a mean reversion asset include Bos, Ware and Pavlov [2].

2.2.1 Mean Reverting Stock Loan Formula

We will use the fact that 
\[
\frac{X_t}{X_s} - \frac{L}{L} \sim \mathcal{N}(0, \sigma^2) \tag{2.2.1}
\]
where \( \mathcal{N}(0, \sigma^2) \) denotes the normal distribution with mean 0 and variance \( T - s \). We view the value function as the discounted expected future value, i.e., the value function is

\[
V(x, s) = e^{-\gamma(T-s)} E \left( e^{X_T - qe^{\gamma(T-s)}} \right)_+.
\]

By [23], with initial condition \( X_s = x \), we can write

\[
X_t = e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_t^{-1})
\]

where \( \phi_t \equiv \int_0^t \frac{1}{\alpha(\phi_s, \cdot)} ds \),

\[
\alpha(t, \cdot) = \alpha(t) \equiv \sigma e^{at},
\]

and \( \phi_t^{-1} \) denotes the inverse function of \( \phi_t \). As in [23], we can write

\[
\phi_t^{-1} = \frac{\sigma^2(e^{2at} - 1)}{2a}.
\]

We make the substitutions \( \bar{X}_t \equiv X_t - \gamma(t - s) \). Factoring out \( e^{\gamma(T-s)} \) allows us to write the value function in (2.2.1) in terms of this new process:

\[
V(x, s) = e^{(\gamma-r)(T-s)} E \left( \exp \left( X_T - \gamma(T - s) \right) - q \right)_+.
\]

Then the explicit formula for \( \bar{X}_t \) is

\[
\bar{X}_t = e^{-a(t-s)}(\log x - L) + L - \gamma(t - s) + e^{-a(t-s)}W(\phi_t^{-1})
\]

Thus, the value function becomes

\[
V(x, s) = e^{(\gamma-r)(T-s)} E \left( e^{\bar{X}_t - q} \right)_+.
\]
\[ e^{(\gamma - \tau)(T-s)} E \left[ (\exp \left[ e^{-a(T-s)}(\log x - L) + L - \gamma(T - s) + e^{-a(T-s)}W(\phi_{T-s}^{-1}) \right] - q) \right] \text{.} \]

Since \( W(\phi_{T-s}^{-1}) \sim N(0, \phi_{T-s}^{-1}) \), we can replace \( W(\phi_{T-s}^{-1}) \) by \( Z \sqrt{\phi_{T-s}^{-1}} \) where \( Z \sim N(0, 1) \) in order to obtain

\[
\frac{e^{(\gamma - \tau)(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{-\infty}^{\infty} (\exp \left[ e^{-a(T-s)}(\log x - L) + L - \gamma(T - s) + y e^{-a(T-s)} \right] - q) e^{\frac{-y^2}{2\phi_{T-s}^{-1}}} dy
\]

where (2.2.3) is due to the definition of expectation and the probability density function of \( Z \). To simplify, distribute the constant \( e^{\gamma(T-s)} \) through the integral to obtain

\[
\frac{e^{-r(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{-\infty}^{\infty} \exp \left[ e^{-a(T-s)}(\log x - L) + L + y e^{-a(T-s)} \right] - q e^{\gamma(T-s)} \right] e^{\frac{-y^2}{2\phi_{T-s}^{-1}}} dy.
\] (2.2.4)

To begin evaluating this integral, note that the integrand will be non-zero if and only if

\[
\exp \left( e^{-a(T-s)}(\log x - L) + L + y e^{-a(T-s)} \right) - q e^{\gamma(T-s)} > 0.
\]

Solving for \( y \) gives

\[
y > e^{a(T-s)}(\log q - L + \gamma(T - s)) + L - \log x.
\]

Thus, defining

\[
m_{MR} \equiv e^{a(T-s)}(\log q - L + \gamma(T - s)) + L - \log x
\]

allows us to rewrite (2.2.4) as

\[
\frac{e^{-r(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} \exp \left( e^{-a(T-s)}(\log x - L) + L + y e^{-a(T-s)} - \frac{y^2}{2\phi_{T-s}^{-1}} \right) dy
\]

\[
- q e^{(\gamma - \tau)(T-s)} \frac{1}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} \exp \left( -\frac{y^2}{2\phi_{T-s}^{-1}} \right) dy.
\] (2.2.5)

Now do the substitutions:

- \( A \equiv \frac{1}{2\phi_{T-s}^{-1}} > 0 \)
\( B \equiv e^{-a(T-s)} \)
\( C \equiv e^{-a(T-s)}(\log x - L) + L. \)

These substitutions allow us to write (2.2.5) in the more compact form
\[
\frac{e^{-r(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} e^{-Ay^2 + B y + C} \, dy - \frac{qe^{(\gamma-r)(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} e^{-Ay^2} \, dy. \quad (2.2.6)
\]

We will now evaluate both of the integrals in (2.2.6).

- Second term: let \( y = y\sqrt{2A} \) so that \( y^2 = \frac{w^2}{2A} \) and \( dy = \frac{dw}{\sqrt{2A}} \). With these substitutions, we can write
\[
\frac{qe^{(\gamma-r)(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} e^{-Ay^2} \, dy = \frac{qe^{(\gamma-r)(T-s)}}{\sqrt{\phi_{T-s}}} \left( \frac{1}{\sqrt{2\pi}} \right) \frac{1}{\sqrt{2A}} \int_{m_{MR}\sqrt{2A}}^{\infty} e^{-(w^2/2)} \, dw
\]
\[
= \frac{qe^{(\gamma-r)(T-s)}}{\sqrt{2\pi}} \int_{m_{MR}\sqrt{2A}}^{\infty} e^{-(w^2/2)} \, dw = qe^{(\gamma-r)(T-s)} \left[ 1 - \Phi \left( m_{MR}\sqrt{2A} \right) \right] \quad (2.2.7)
\]
where, as before, \( \Phi(\cdot) \) is the cumulative distribution function of \( N(0,1) \).

- First term: we need to complete the square in the exponent of the first integral:
\[
-Ay^2 + B y + C = -A \left( y^2 - \frac{B}{A} y \right) + C
\]
\[
= -A \left( y^2 - \frac{B}{A} y + \frac{B^2}{4A} \right) + \frac{B^2}{4A} + C
\]
\[
= -A \left( y - \frac{B}{2A} \right)^2 + \frac{B^2}{4A} + C.
\]

Substituting into the first term of (2.2.6) gives
\[
\frac{e^{-r(T-s)}}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} \exp \left( -Ay^2 + B y + C \right) \, dy
\]
\[
\frac{e^{-r(T-s)}}{2\pi \phi_{T-s}^{-1}} \int_{m_{MR}}^{\infty} \exp \left( -A \left( y - \frac{B}{2A} \right)^2 + \frac{B^2}{4A} + C \right) dy \\
= \frac{\exp \left( -r(T-s) + \frac{B^2}{4A} + C \right)}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} \exp \left( -A \left( y - \frac{B}{2A} \right)^2 \right) dy.
\]

Substituting \( w \equiv y - \frac{B}{2A} \) yields
\[
\frac{\exp \left( -r(T-s) + \frac{B^2}{4A} + C \right)}{\sqrt{2\pi \phi_{T-s}^{-1}}} \int_{m_{MR}}^{\infty} e^{-A(\frac{w^2}{2\pi})} dw
\]
\[
= \frac{\exp \left( -r(T-s) + \frac{B^2}{4A} + C \right)}{\sqrt{2\pi \phi_{T-s}^{-1}}} \frac{1}{\sqrt{2\pi}} \int_{m_{MR} - \frac{B}{2\pi}}^{\infty} e^{-Aw^2} dw
\]
\[
= \frac{\exp \left( -r(T-s) + \frac{B^2}{4A} + C \right)}{\sqrt{2\pi \phi_{T-s}^{-1}}} \frac{1}{\sqrt{2A}} \left[ 1 - \Phi \left( \frac{\sqrt{2A \left( m_{MR} - \frac{B}{2A} \right)}}{\sqrt{2A}} \right) \right]
\]
\[
= \frac{\exp \left( -r(T-s) + \frac{B^2}{4A} + C \right)}{\sqrt{2\pi \phi_{T-s}^{-1}}} \frac{1}{\sqrt{2A}} \left[ \Phi \left( \sqrt{2A \left( \frac{B}{2A} - m_{MR} \right)} \right) \right].
\]

(2.2.8)

Combining (2.2.7) and (2.2.8) gives
\[
V(x, s) = \frac{e^{-r(T-s)} + \frac{B^2}{4A} + C}{\sqrt{\phi_{T-s}^{-1}}} \frac{1}{\sqrt{2A}} \left[ 1 - \Phi \left( \frac{\sqrt{2A \left( m_{MR} - \frac{B}{2A} \right)}}{\sqrt{2A}} \right) \right]
\]
\[
- \frac{q e^{(\gamma-r)(T-s)}}{\sqrt{\phi_{T-s}^{-1}}} \frac{1}{\sqrt{2A}} \left[ 1 - \Phi \left( m_{MR} \sqrt{2A} \right) \right]
\]

(2.2.9)

where
\[
C \equiv e^{-a(T-s)} (\log x - L) + L
\]
\[
B \equiv e^{-a(T-s)}
\]
\[
A \equiv \frac{1}{2 \phi_{T-s}^{-1}}
\]
\[ m_{MR} \equiv e^{a(T-s)} \left( \log q - L + \gamma(T - s) \right) + L - \log x, \]
and \[ \phi_{T-s} \equiv \int_0^{T-s} \frac{1}{\alpha(\phi_u, \Gamma^u)} du \Rightarrow \phi_{T-s}^{-1} = \frac{\sigma^2(e^{2a(T-s)} - 1)}{2a}. \]

### 2.2.2 Numerical Examples

This section gives some numerical examples to illustrate properties possessed by the formula (2.2.9) and the relationships they produce among the variables of interest. These examples were produced using MATLABR2006a to evaluate formula (2.2.9).

The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values given in Table 2.3. Varying one parameter at a time gives the results indicated in Table 2.4. Many of these results are similar to the conclusions reached under the geometric Brownian motion model. In the mean reverting case, the loan value is decreasing as functions of \( q \), \( \gamma \); it is increasing as functions of \( S_0 \) and \( \sigma \). With respect to the new parameters in the mean reverting model, the loan value is increasing as a function of the equilibrium log-price \( L \) and decreasing as a function of the rate of reversion \( a \). Interestingly, in contrast to the geometric Brownian motion case, the loan value decreases quickly as the loan maturity time \( T \) increases. A stock loan can be thought of as an option with a time dependent strike price. In particular, as the loan maturity time \( T \) increases, the “strike price” of the loan increases as well. Since the value of a call option decreases as the strike price increases, the stock loan case clashes two properties of an option against each other: the option price as a decreasing function of the strike price, and the option price as an increasing function of expiry/maturity time. These results show that in the geometric Brownian motion case, the upward trend of the stock price sufficiently offsets the effect of a time-dependent strike price and sustains the expected result from option pricing theory, whereas the mean reverting case provides no such trend and hence the loan value succumbs to the time-dependence of the strike price.

The second set of examples shows relationships as two of the parameters are changed, all others held constant. These examples are displayed in Figures 2.3 through 2.6. In Figure
2.3, the loan value is a decreasing function of loan principal. The loan value is a decreasing function of \( a \), but the rate of decrease is not uniform as it exhibits different concavities for different loan principals. Small values of \( a \) cause the loan value to increase to \(+\infty\) due to the \( \phi_{T-s}^{-1} \) term in the loan value. In Figure 2.4, as a function of \( T \) the loan value decreases more quickly for large values of \( a \) than for small ones. Also, the loan value decreases as \( a \) increases, but it decreases more slowly for longer maturity times than for short ones. Figure 2.5, which plots loan value as a function of loan principal and \( \sigma \), shows that the value function decreasing substantially as \( q \) increases. Interesting but small variations can be seen when viewing the loan value as a function of \( \sigma \). Figure 2.6, which plots loan value as a function of \( T \) and \( \sigma \), shows a smooth surface, but some unusually large rates of change can be detected for small values of \( T \) and \( \sigma \). The same phenomena described above regarding \( \phi_{T-s}^{-1} \) explains the bottom left corner of the graph.
Table 2.4: Numerical Results for Mean Reverting, One Parameter Varied

<table>
<thead>
<tr>
<th>$q$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>50.00</td>
<td>43.47</td>
<td>28.56</td>
<td>18.51</td>
<td>13.79</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>45</th>
<th>55</th>
<th>65</th>
<th>75</th>
<th>85</th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>16.83</td>
<td>20.19</td>
<td>23.58</td>
<td>26.97</td>
<td>30.30</td>
<td>33.56</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
<th>0.19</th>
<th>0.22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>19.35</td>
<td>18.50</td>
<td>17.72</td>
<td>16.99</td>
<td>16.31</td>
<td>15.67</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>20.42</td>
<td>18.51</td>
<td>15.08</td>
<td>13.72</td>
<td>12.54</td>
<td>10.55</td>
<td>8.84</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>14.90</td>
<td>16.59</td>
<td>18.51</td>
<td>20.67</td>
<td>23.09</td>
<td>25.80</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 30 $\approx$ 3.40</th>
<th>log 40 $\approx$ 3.69</th>
<th>log 50 $\approx$ 3.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>10.26</td>
<td>15.46</td>
<td>21.86</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 60 $\approx$ 4.09</th>
<th>log 70 $\approx$ 4.25</th>
<th>log 80 $\approx$ 4.38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>29.30</td>
<td>37.31</td>
<td>45.51</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>21.99</td>
<td>20.01</td>
<td>18.51</td>
<td>17.34</td>
<td>16.43</td>
<td>15.13</td>
<td>13.66</td>
<td>12.89</td>
<td>12.42</td>
</tr>
</tbody>
</table>
Figure 2.3: Relationship between $q$, $a$, and Loan Value when collateral stock obeys a mean reverting model.
Figure 2.4: Relationship between $T$, $a$, and Loan Value when collateral stock obeys a mean reverting model.
Figure 2.5: Relationship between $q$, $\sigma$, and Loan Value when collateral stock obeys a mean reverting model.
Figure 2.6: Relationship between $T$, $\sigma$, and Loan Value when collateral stock obeys a mean reverting model.
Chapter 3

Two-State Regime Switching Models

Regime switching models are popular in mathematical finance because they allow model parameters to change over time, thus making the model more realistic than the models considered in the previous chapter. The switching is accomplished by introducing a second source of randomness, typically a Markov chain, in addition to the standard Brownian motion $W_t$. The regime switching model was first introduced by Hamilton [10] in 1989 when describing a regime switching time series.

This chapter will derive and numerically analyze formulas for the value of a stock loan with European maturity when the collateral stock obeys a two-state regime switching model in which the second state is absorbing. We will allow our stock model to have both geometric Brownian motion and mean reverting states in the following four combinations:

- both states are geometric Brownian motions (with different parameters);
- the first state is a geometric Brownian motion but the second absorbing state is mean reverting;
- the first state is mean reverting but the second absorbing state is a geometric Brownian motion;
- both states are mean reverting (with different parameters).

The first case is the subject of this chapter’s first section. The second and third cases are very similar and will be treated in the second section. The final case, which is mathematically similar but notationally different from the previous ones, will be dealt with in the last section. Note that some of these results have been accepted for publication in [17].
3.1 Two Geometric Brownian Motion States

3.1.1 Formula Derivation

For this section, we will assume that the stock obeys a two-state regime switching model in which each state corresponds to a geometric Brownian motion with means $\mu(1)$ and $\mu(2)$ and volatilities $\sigma(1)$ and $\sigma(2)$, respectively. In practice, these parameters would need to be estimated using historical data for the collateral stock. One approach to this estimation is that of Hardy [11], where he developed maximum likelihood estimation using real data from the S&P 500 and TSE 300 indices.

To define the model more precisely, let $\alpha(\cdot) = \{\alpha_t : t \geq s\}$ be a two-state Markov chain with $\alpha_t \in \{1, 2\}$. Let $Q \equiv (q_{ij})$, $1 \leq i, j \leq 2$ defined by

$$
Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}
$$

be the generator of $\alpha$ with $\lambda > 0$. Let the stock price $S_t$ be governed by the equation

$$
dS_t = S_t [\mu(\alpha_t)dt + \sigma(\alpha_t)dW_t],
$$

where $s \leq t \leq T$, $W_t$ as usual is a standard Brownian motion, and $T$ is the maturity time. Also assume $\alpha_t$ is independent of $S_t$. European option valuation for this model was studied in [20].

For initial time $s$, let $h(z, s, t) \equiv (z - qe^{\gamma(t-s)})_+$, and let $\tilde{P}$ denote the (equivalent) risk-neutral probability measure. Then the value function can be written

$$
V(x, s, i) = \tilde{E} \left[ e^{-r(T-s)}h(S_T, s, T)|S(s) = x, \alpha(s) = i \right].
$$

Here $s$, $0 \leq s \leq T$, denotes the initial time, $x$ the initial stock price, $i$ the initial state, and $\tilde{E}$ denotes expectation under $\tilde{P}$.

Let $V^0(x, s, i) \equiv \tilde{E} \left[ e^{-r(T-s)}h(S_T, s, T)|S(s) = x, \alpha(u) = i \text{ for } s \leq u \leq T \right]

= e^{-r(T-s)} \int_{-\infty}^{\infty} h(z, s, T) N(z, m(T-s, i), \Sigma^2(T-s, i))dz$
where $N$ is the Normal density function with mean

$$m(t, i) = \left[ \mu - \frac{1}{2} \sigma^2(i) \right] t$$

and variance $\Sigma^2(t, i) = \sigma^2(i) t$. Let $\tau \equiv \inf (t \geq s : \alpha_t \neq \alpha_s)$, i.e. the time of the first (and only, in our case) jump. Since $q_{11} \equiv -\lambda \neq 0$, for any $u > s$,

$$P(\tau > u | \alpha_s = 1) = e^{-\lambda(u-s)}.$$

Let $S$ denote the Banach space of all bounded measurable functions on $\mathbb{R} \times [s, T] \times \{1, 2\}$. Define a mapping $\Upsilon$ on $S$ by

$$(\Upsilon f)(x, s, 1) = \int_s^T e^{-r(T-s)} \left[ \int_{-\infty}^\infty f(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))\,dz \right] \lambda e^{-\lambda(T-s)}\,dt;$$

$$(\Upsilon f)(x, s, 2) = 0.$$

**Theorem 3.1.1.** $V$ is a solution to the equation

$$V(x, s, i) = \Upsilon V + e^{-q_{ii}(T-s)} V^0(x, s, i) \quad (3.1.1)$$

where $q_{ii} = \lambda$ if $i = 1$ and $q_{ii} = 0$ if $i = 2$.

**Proof.** Let $\tilde{E}_{x,s,i}$ denote the conditional expectation under $\tilde{P}$ with conditions $S_s = x$ and $\alpha_s = i$. Then

$$\tilde{E}_{x,s,1} \left[ e^{-r(T-s)} h(S_T, s, T) I_{\{\tau > T\}} \right] = e^{-\lambda(T-s)} V^0(x, s, 1)$$

because $P(\tau > T | \alpha_s = 1) = e^{-\lambda(T-T)}$ and $\tau$ is independent of $S_t$. Thus,

$$V(x, s, i) = \tilde{E}_{x,s,1} \left[ e^{-r(T-s)} h(S_T, s, T) I_{\{\tau \leq T\}} \right] + e^{-\lambda(T-s)} V^0(x, s, 1). \quad (3.1.2)$$

Working on the first term in (3.1.2), condition additionally on $\tau = t$ and apply the Law of Iterated Expectation to write

$$\tilde{E}_{x,s,1} \left[ e^{-r(T-s)} h(S_T, s, T) I_{\{\tau > T\}} \right] = \int_s^T e^{-r(T-s)} \tilde{E}_{x,s,1} \left[ h(S_T, s, T) | \tau = t \right] \lambda e^{-\lambda(t-s)}\,dt.$$
Given \( \tau = t \), the post-jump distribution of \( \alpha_t \) is \( \frac{\lambda}{|\lambda|} = 1 \). Since \( W_t \) is Gaussian and independent of \( \alpha(\cdot) \), we have, for \( s \leq t \leq T \),

\[
\mathbb{E}_{x,s,1} \left[ h(S_T, s, T) | \tau = t \right] = \mathbb{E}_{x,s,1} \left[ \mathbb{E}_{x,s,1} \left[ h(S_T, s, T) | S_t, \alpha_t \right] | \tau = t \right] = \mathbb{E}_{x,s,1} \left[ e^{-r(T-t)} V(S_t, t, \alpha_t) | \tau = t \right] = \int_{-\infty}^{\infty} e^{-r(T-t)} V(z, t, 2) N(z, m(t - s, i), \Sigma^2(t - s, i)) dz,
\]

where the second equality above is by definition of \( V \) and the third equality above is due to the Gaussian property. Thus, the first term in (3.1.2) is just \( \Upsilon V(t, u, 1) \), and so \( V \) is a solution to (3.1.1).

**Corollary 3.1.2.** Under the assumptions of this section, the value of the stock loan is given by

\[
V(x, s, 1) = \int_{s}^{T} e^{-r(t-s)} \left( \int_{-\infty}^{\infty} V(z, t, 2) N(z, m(t - s, 1), \Sigma^2(t - s, 1)) dz \right) \lambda e^{-\lambda(t-s)} dt + e^{-\lambda(T-s)} V^0(x, s, 1),
\]

\[
V(x, s, 2) = V^0(x, s, 2).
\]

### 3.1.2 Numerical Examples

This section gives some numerical examples to illustrate properties possessed by formula (3.1.3) and the relationships they produce among the variables of interest. In this section we will assume \( \mu = r \). These examples were produced using MATLABR2006a.

The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values given in Table 3.1. Varying one parameter at a time gives the results indicated in Table 3.2. Many of these results are similar to the conclusions reached in the previous chapter. As in the geometric Brownian motion case, the loan value is a decreasing function of \( q \) and \( \gamma \); it is an increasing function of \( S_0 \) and both \( \sigma(1) \) and \( \sigma(2) \). As the switching rate \( \lambda \) or the maturity time \( T \) increases, the loan value appears to increase.
These last two relationships depend on choices for $\sigma(1)$ and $\sigma(2)$ and will be explored further in the next set of examples.

The second set of examples varies two of the parameters at a time, all others held constant. These examples are presented in Figures 3.1 through 3.8. Figures 3.1 through 3.4 assume $\sigma_2 - \sigma_1 > 0$, and Figures 3.5 through 3.8 assume $\sigma_2 - \sigma_1 < 0$. Figure 3.1, “Relationship between $q$, $\lambda$, and Loan Value,” shows a smooth surface remarkable in that the loan value varies only slightly as $\lambda$ is changed. In Figure 3.2, the loan value decreases more slowly as a function of $q$ when $\sigma_2 - \sigma_1$ is large than when it is small. Also, the loan value increases more quickly as a function of $\sigma_2 - \sigma_1$ when $q$ is large as opposed to when it is small. In Figure 3.3, “Relationship between $T$, $\lambda$, and Loan Value,” the loan value is increasing and concave down but decreasing and concave down as a function of $T$ and $\lambda$, respectively. As in the mean reverting case, some loss of smoothness can be detected for small values of $\lambda$ which may be due to large step size in the points chosen for the graph; this is another area for further study. In Figure 3.4, the relationships from the previous set of examples are confirmed. However, the loan value increases quickly for large values of $q$ and $\sigma_2 - \sigma_1$ but appears very flat for small values of these variables.

Moving to the examples that assume $\sigma_2 - \sigma_1 < 0$, Figure 3.5, “Relationship between $q$, $\lambda$, and Loan Value,” and Figure 3.6, “Relationship between $q$, $\sigma_2 - \sigma_1$,” look very similar to Figure 3.1. This similarity indicates that the relationships depicted in these figures do not depend on whether $\sigma_2 - \sigma_1$ is positive or negative. In Figure 3.7, another apparent loss of smoothness occurs for small values of $\lambda$ and $T$. As will be seen later, this figure may show a local extremum in this neighborhood, but further study is required. Finally, Figure 3.8 indicates that the loan value surface is flatter for small values of $\sigma_2 - \sigma_1$ than for large ones. Also, the loan value surface is flatter for longer maturity times than for shorter ones.
Table 3.1: Default Values for GBM to GBM, One Dimensional Examples

<table>
<thead>
<tr>
<th>variable</th>
<th>default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$q$</td>
<td>40</td>
</tr>
<tr>
<td>$S_0$</td>
<td>50</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
</tr>
<tr>
<td>$s$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma(1)$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma(2)$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1</td>
</tr>
</tbody>
</table>

3.2 One Mean Reverting State

3.2.1 Formula Derivation

For this section, we will assume that the stock obeys the following two-state regime switching model in which the second state is absorbing:

$$
\frac{dS_t}{S_t} = dt\left[A(S_t, \alpha_t) + \sigma(\alpha_t)dW_t\right]
$$

$$
A(S_t, i_1) = \mu; \quad \sigma(i_1) = \sigma_1
$$

$$
A(S_t, i_2) = a(L - \log(S_t)); \quad \sigma(i_2) = \sigma_2.
$$

Here we assume that $i_1, i_2 \in \{1, 2\}$ and $i_1 \neq i_2$. Note that state $i_1$ is a geometric Brownian motion while state $i_2$ is mean reverting. As before, let $\alpha(\cdot) = \{\alpha_t : t \geq s\}$ be a two-state Markov chain with $\alpha_t \in \{1, 2\}$. Let $Q \equiv (q_{ij}), 1 \leq i, j \leq 2$ defined by

$$
Q = \begin{pmatrix}
-\lambda & \lambda \\
0 & 0
\end{pmatrix}
$$
Table 3.2: Numerical Results for GBM to GBM, One Parameter Varied

<table>
<thead>
<tr>
<th>$q$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>39.18</td>
<td>29.57</td>
<td>20.94</td>
<td>13.70</td>
<td>8.77</td>
</tr>
<tr>
<td>$S_0$</td>
<td>45</td>
<td>55</td>
<td>65</td>
<td>75</td>
<td>85</td>
</tr>
<tr>
<td>Loan value</td>
<td>10.07</td>
<td>17.72</td>
<td>26.42</td>
<td>35.53</td>
<td>44.87</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.07</td>
<td>0.1</td>
<td>0.13</td>
<td>0.16</td>
<td>0.19</td>
</tr>
<tr>
<td>Loan value</td>
<td>14.27</td>
<td>13.70</td>
<td>13.14</td>
<td>12.59</td>
<td>12.06</td>
</tr>
<tr>
<td>$T$</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
<tr>
<td>Loan value</td>
<td>10.94</td>
<td>13.70</td>
<td>19.36</td>
<td>21.78</td>
<td>23.91</td>
</tr>
<tr>
<td>$\sigma(1)$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>Loan value</td>
<td>14.21</td>
<td>14.79</td>
<td>15.39</td>
<td>16.00</td>
<td>16.60</td>
</tr>
<tr>
<td>$\sigma(2)$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>Loan value</td>
<td>9.49</td>
<td>10.11</td>
<td>10.79</td>
<td>11.50</td>
<td>12.23</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Loan value</td>
<td>11.73</td>
<td>13.70</td>
<td>16.10</td>
<td>17.40</td>
<td>18.60</td>
</tr>
</tbody>
</table>
Figure 3.1: Relationship between $q$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to GBM model.

Figure 3.2: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to GBM model.
Figure 3.3: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to GBM model.
Figure 3.4: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to GBM model.
Figure 3.5: Relationship between \( q \), \( \lambda \), and Loan Value assuming \( \sigma_2 - \sigma_1 < 0 \) when collateral stock obeys a GBM to GBM model.
Figure 3.6: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to GBM model.
Figure 3.7: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to GBM model.
Figure 3.8: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to GBM model.
be the generator of \( \alpha(\cdot) \) with \( \lambda > 0 \). For initial time \( s \), let \( h(z, s, t) \equiv (z - qe^{\gamma(t-s)})_+ \). Then the value function can be written

\[
V(x, s, \alpha) = E \left[ e^{-r(T-s)}h(S_T, s, T) | S_s = x, \alpha_s = \alpha \right].
\]

Here \( s, 0 \leq s \leq T \), denotes the initial time, \( x \) the initial stock price, and \( \alpha \) the initial state.

Recall that, by writing \( S_T = e^{X_t} \), we can write the mean reverting model explicitly as

\[
X_t = e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_t^{-1})
\]

where \( \phi_t \equiv \int_0^t (1/\alpha(\phi_s, \cdot))ds \), \( \alpha_{t, \cdot} = \alpha_t \equiv \sigma e^{at} \), and \( \phi_t^{-1} \) denotes the inverse function of \( \phi_t \); moreover

\[
\phi_t^{-1} = \frac{\sigma^2(e^{2at} - 1)}{2a}.
\]

For the geometric Brownian motion state \( i = i_1 \), let

\[
V^0(x, s, i_1) \equiv E \left[ e^{-r(T-s)}h(S_T, s, T) | S_s = x, \alpha_u = i_1 \text{ for } s \leq u \leq T \right]
\]

\[
= e^{-r(T-s)} \int_{-\infty}^{\infty} h(z, s, T)N(z, m(T-s, i_1), \Sigma^2(T-s, i_1))dz
\]

where \( N \) is the Normal density function with mean

\[
m(t, i_1) = \left[ \mu - \frac{1}{2} \sigma^2(1) \right] t
\]

and variance \( \Sigma^2(t, 1) = \sigma^2(1)t \). Following the lead of equation (2.2.4), for the mean reverting state \( i = i_2 \), let

\[
V^0(x, s, i_2) \equiv E \left[ e^{-r(T-s)}h(S_T, s, T) | S_s = x, \alpha_u = i_2 \text{ for } s \leq u \leq T \right]
\]

\[
= e^{-r(T-s)} \int_{-\infty}^{\infty} h(z, s, T)N(z, m(T-s, i_2), \Sigma^2(T-s, i_2))dz
\]

where \( N \) is the Normal density function with mean

\[
m(t, i_2) = e^{-at}(\log x - L) + L
\]

and variance \( \Sigma^2(t, i_2) = e^{-2at}\phi_t^{-1} \). Note that the \( \phi_t^{-1} \) adjustment to \( \Sigma^2(t, i_2) \) corrects for the fact that the density in (2.2.6) is \( \exp \left( \frac{-y^2}{2\phi_{T-s}^{-1}} \right) \), not \( \exp \left( \frac{-y^2}{2} \right) \) as in the geometric
Brownian motion case. Let $\tau \equiv \inf (t \geq s : \alpha_t \neq \alpha_s)$, i.e. the time of the first (and only, in our case) jump. Since $q_{11} \equiv -\lambda \neq 0$,

$$P (\tau > u | \alpha_s = 1) = e^{-\lambda(u-s)}.$$

Let $S$ denote the Banach space of all bounded measurable functions on $\mathbb{R} \times [s, T] \times \{1, 2\}$. Define a mapping $\Upsilon$ on $S$ by

$$(\Upsilon f)(x, s, 1) = \int_s^T e^{-r(T-s)} \left[ \int_{-\infty}^{\infty} f(z, m(t-s, 1), \Sigma^2(t-s, 1)) dz \right] \lambda e^{-\lambda(T-s)} dt;$$

$$(\Upsilon f)(x, s, 2) = 0.$$

**Theorem 3.2.1.** $V$ is a solution to the equation

$$V(x, s, i) = \Upsilon V + e^{-q_{ii}(T-s)} V^0(x, s, i) \tag{3.2.1}$$

where $q_{ii} = \lambda$ if $i = 1$ and $q_{ii} = 0$ if $i = 2$.

**Proof.** Let $E_{x, s, i}$ denote the conditional expectation with conditions $S_s = x$ and $\alpha_s = i$. Then

$$E_{x, s, 1} \left[ e^{-r(T-s)} h(S_T, T) I_{\{\tau > T\}} \right] = e^{-\lambda(T-s)} V^0(x, s, 1)$$

because $P(\tau > T | \alpha_s = 1) = e^{-\lambda(T-s)}$ and $\tau$ is independent of $S_t$. Thus,

$$V(x, s, i) = E_{x, s, 1} \left[ e^{-r(T-s)} h(S_T, s, T) I_{\{\tau \leq T\}} \right] + e^{-\lambda(T-s)} V^0(x, s, 1). \tag{3.2.2}$$

Working on the first term in (3.2.2), condition additionally on $\tau = t$ and apply the Law of Iterated Expectation to write

$$E_{x, s, 1} \left[ e^{-r(T-s)} h(S_T, s, T) I_{\{\tau > T\}} \right] = \int_s^T e^{-r(T-s)} E_{x, s, 1} [h(S_T, T) | \tau = t] \lambda e^{-\lambda(t-s)} dt.$$

Given $\tau = t$, the post-jump distribution of $\alpha_t$ is $\frac{\lambda}{|\alpha|} = 1$.

We claim $W(\phi_t^{-1})$ is a Brownian motion. By definition of $W(\phi_0^{-1})$, it is clear that $W(\phi_1^{-1}) = 0$. By [21], it suffices to show that $W(\phi_t^{-1})$ has continuous sample paths and
independent increments. Since $\phi_t^{-1} = \frac{\sigma^2(e^{2at} - 1)}{2a}$, $W(\phi_t^{-1})$ is a composition of continuous functions of $t$. Therefore, $W(\phi_t^{-1})$ has continuous sample paths. Suppose $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$. Again because $\phi_t^{-1} = \frac{\sigma^2(e^{2at} - 1)}{2a}$, $\phi_t^{-1}$ is a strictly increasing function of $t$. Defining $T_i = \frac{\sigma^2(e^{2at_i} - 1)}{2a}$ for $1 \leq i \leq 4$, we have $0 \leq T_1 \leq T_2 \leq T_3 \leq T_4$. Since $W_t$ has independent increments, we have $W(T_4) - W(T_3)$ is independent of $W(T_2) - W(T_1)$, i.e. $W(\phi_{t_4}^{-1}) - W(\phi_{t_3}^{-1})$ is independent of $W(\phi_{t_2}^{-1}) - W(\phi_{t_1}^{-1})$. Therefore, $W(\phi_t^{-1})$ has independent increments. This proves the claim.

Thus, for $s \leq t \leq T$, we have

$$E_{x,s,1}[h(S_T, s, T)|\tau = t] = E_{x,s,1}[E_{x,s,1}[h(S_T, s, T)|S(t), \alpha(t)]|\tau = t]$$

$$= E_{x,s,1}[e^{-r(T-t)}V(S_t, t, \alpha(t))|\tau = t]$$

$$= \int_{-\infty}^{\infty} e^{-r(T-t)}V(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))dz,$$

where the second equality above is by definition of $V$ and the third equality above is due to the Gaussian property. Thus, the first term in (3.2.2) is just $\gamma V(t, u, 1)$, and so $V$ is a solution to (3.2.1). 

\[\square\]

**Corollary 3.2.2.** Under the assumptions of this section, the value of the stock loan is given by

$$V(x, s, 1) = \int_s^T e^{-r(t-s)} \left( \int_{-\infty}^{\infty} V(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))dz \right) \lambda e^{-\lambda(t-s)}dt$$

$$+ e^{-\lambda(T-s)}V^0(x, s, 1)$$

$$V(x, s, 2) = V^0(x, s, 2) \tag{3.2.3}$$

### 3.2.2 Numerical Examples: Mean Reverting State Absorbing

This section gives some numerical examples to illustrate properties possessed by (3.2.3) for the case $i_1 = 1$ and $i_2 = 2$, i.e. the first state is a geometric Brownian motion and the second state is mean reverting. In this section we will assume $\mu = r$ in the geometric Brownian motion state. These examples were produced using MATLABR2006a.
The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values for the indicated parameters given in 3.3. Varying one parameter at a time gives the results indicated in Table 3.4. As in the previous examples, the loan value is a monotone decreasing function of \( q, \gamma, \) and the reversion rate \( a; \) it is a monotone increasing function of \( S_0, \) the equilibrium level \( L, \) as well as both \( \sigma(1) \) and \( \sigma(2). \) As the switching rate \( \lambda \) or the maturity time \( T \) increases, the loan value appears to reach a maximum in finite time. This phenomenon has been documented in previous sections in the maturity time variable but not the switching rate variable. These last two relationships depend on choices for \( \sigma(1), \) \( \sigma(2), \) and \( L \) and will be explored further in the next set of examples.

The second set of examples shows relationships as two of the parameters are changed, all others held constant. These examples are presented in Figures 3.9 through 3.17. Figures 3.9 through 3.12 assume \( \sigma_2 - \sigma_1 > 0, \) and Figures 3.13 through 3.17 assume \( \sigma_2 - \sigma_1 < 0. \) Figure 3.9, “Relationship between \( q, \lambda, \) and Loan Value,” shows a smooth surface with the loan value as an increasing function of lambda. For small values of \( \lambda, \) a finer mesh is needed to produce acceptable results, as indicated in the one-dimensional example for \( \lambda. \) The \( \lambda \) variable acts as an averaging mechanism: when \( \lambda \) is small, the regime switch occurs quickly on average, causing the second (mean reverting, in this case) state to receive a greater weight. This relationship reflects the fact that, for these parameters, the mean reverting model gives a higher loan value that the geometric Brownian motion model. In Figure 3.10, the loan value decreases more slowly as a function of \( q \) when \( \sigma_2 - \sigma_1 \) is small than when it is large.

Figure 3.11, “Relationship between \( T, \lambda, \) and Loan Value,” shows some interesting relationships. For small values of \( T, \) the loan value is increasing and concave down, but the loan value flattens considerably for larger values of \( T. \) This flattening reflects the fact that, under the mean reverting model, the loan value decreases quickly as a function of \( T. \) For small values of \( \lambda, \) the loan value as a function of \( T \) acts much like the geometric Brownian motion case, while for larger values of \( \lambda \) the loan value takes more of a mean reverting shape. This behavior is expected based on our understanding of \( \lambda \) as explained above. In Figure 3.12,
the relationships from the previous set of examples are repeated, albeit with the loan value increasing quickly for large values of $q$ but appearing very flat for small values of $q$.

Moving to the examples that assume $\sigma_2 - \sigma_1 < 0$, Figure 3.13, “Relationship between $q$, $\lambda$, and Loan Value,” looks very similar to Figure 3.9 except when $q$ is large and $\lambda$ is very small. This difference reflects the fact that the loan value under a geometric Brownian motion does not decrease as quickly as a function of $q$ when $\sigma$ is large. The higher volatility creates a higher probability that the stock price will exceed the (large) principal and interest due at maturity. Figure 3.14, “Relationship between $q$, $\sigma_2 - \sigma_1$,” again looks very similar to Figure 3.10, the corresponding figure for the case $\sigma_2 - \sigma_1 > 0$.

Figure 3.15 indicates that the loan value surface is flatter for large maturity times than for small ones. Also, the loan value surface shows the maximum that seems to characterize the loan value as a function of $T$. Finally, Figures 3.16 and 3.17 give the relationship between loan value, loan maturity, and $\lambda$. These figures were created using the same points and parameters with one exception: Figure 3.16 assumes an equilibrium level of $L = \log 45$ while Figure 3.17 assumes an equilibrium level of $L = \log 60$. These figures show that the surface of this graph depends greatly on the parameter $L$ and that this dependence increases as $\lambda$ increases, i.e. as the model switches to the mean reverting state sooner, on average.
Table 3.3: Default Values for GBM to MR, One Dimensional Examples

<table>
<thead>
<tr>
<th>Variable</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>0.05</td>
</tr>
<tr>
<td>( q )</td>
<td>40</td>
</tr>
<tr>
<td>( S_0 )</td>
<td>50</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.1</td>
</tr>
<tr>
<td>( s )</td>
<td>0</td>
</tr>
<tr>
<td>( T )</td>
<td>1</td>
</tr>
<tr>
<td>( \sigma(1) )</td>
<td>0.2</td>
</tr>
<tr>
<td>( \sigma(2) )</td>
<td>0.9</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>1</td>
</tr>
<tr>
<td>( L )</td>
<td>( \log 45 \approx 3.81 )</td>
</tr>
<tr>
<td>( a )</td>
<td>2</td>
</tr>
</tbody>
</table>

3.2.3 Numerical Examples: Geometric Brownian Motion State Absorbing

This section gives some numerical examples to illustrate properties possessed by (3.2.3) for the case \( i_1 = 2 \) and \( i_2 = 1 \), i.e. the first state is mean reverting and the second state is a geometric Brownian motion. In this section we will assume \( \mu = r \) in the geometric Brownian motion state. These examples were produced using MATLABR2006a to evaluate the above formula.

The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values given in Table 3.5. Varying one parameter at a time gives the results indicated in Table 3.6. As expected, the loan value is a monotone decreasing function of \( q \), \( \gamma \), and the reversion rate \( a \); it is a monotone increasing function of \( S_0 \), the equilibrium level \( L \), as well as both \( \sigma(1) \) and \( \sigma(2) \). Unlike the previous case, as the switching rate \( \lambda \) increases, the loan value once again appears to be monotone increasing. Also unlike previous
Table 3.4: Numerical Results for GBM to MR, One Parameter Varied

<table>
<thead>
<tr>
<th>$q$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>45.49</td>
<td>39.32</td>
<td>27.11</td>
<td>17.46</td>
<td>11.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>45</th>
<th>55</th>
<th>65</th>
<th>75</th>
<th>85</th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>14.77</td>
<td>20.40</td>
<td>26.57</td>
<td>32.78</td>
<td>38.90</td>
<td>44.93</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
<th>0.19</th>
<th>0.22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>18.13</td>
<td>17.46</td>
<td>16.81</td>
<td>16.18</td>
<td>15.57</td>
<td>15.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>14.86</td>
<td>17.46</td>
<td>17.99</td>
<td>17.27</td>
<td>16.32</td>
<td>14.28</td>
<td>12.35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma(1)$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>17.97</td>
<td>18.55</td>
<td>19.16</td>
<td>19.76</td>
<td>20.37</td>
<td>20.97</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma(2)$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>13.57</td>
<td>13.85</td>
<td>14.34</td>
<td>14.97</td>
<td>15.70</td>
<td>16.53</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>11.94</td>
<td>14.27</td>
<td>16.07</td>
<td>17.46</td>
<td>20.47</td>
<td>21.79</td>
<td>21.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 30 ≈ 3.40</th>
<th>log 40 ≈ 3.69</th>
<th>log 50 ≈ 3.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>11.55</td>
<td>15.32</td>
<td>19.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 60 ≈ 4.09</th>
<th>log 70 ≈ 4.25</th>
<th>log 80 ≈ 4.38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>24.68</td>
<td>29.90</td>
<td>32.23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>23.55</td>
<td>21.92</td>
<td>20.63</td>
<td>19.60</td>
<td>18.75</td>
<td>17.46</td>
<td>15.82</td>
<td>14.85</td>
<td>14.21</td>
</tr>
</tbody>
</table>
Figure 3.9: Relationship between $q$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to MR model.
Figure 3.10: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to MR model.
Figure 3.11: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to MR model.
Figure 3.12: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys a GBM to MR model.
Figure 3.13: Relationship between $q$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to MR model.
Figure 3.14: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to MR model.
Figure 3.15: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys a GBM to MR model.
Figure 3.16: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ with $L = \log 45$ when collateral stock obeys a GBM to MR model.
Figure 3.17: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ and $L = \log 60$ when collateral stock obeys a GBM to MR model.
cases, the loan value as a function of the maturity time $T$ appears to reach a minimum in finite time as opposed to a maximum. These last two relationships will be the focus of the two-dimensional examples.

The second set of examples shows relationships as two of the parameters are changed, all others held constant. These examples are presented in Figures 3.18 through 3.25. Figures 3.18 through 3.21 assume $\sigma_2 - \sigma_1 > 0$, and Figures 3.22 through 3.25 assume $\sigma_2 - \sigma_1 < 0$.

Figure 3.18, “Relationship between $q$, $\lambda$, and Loan Value,” shows a smooth surface with the loan value as a decreasing function of $q$. As in the one-dimensional example for $\lambda$, for small values of $\lambda$, a finer mesh is needed to produce acceptable results. In contrast to Figure 3.9, the corresponding figure for Case #2, the surface now appears flatter for large values of $q$. The same forces that produced this result in Case #2 now produce the opposite result when the states are switched here in Case #3. In Figure 3.19, the loan value decreases more slowly as a function of $q$ when $\sigma_2 - \sigma_1$ is small than when it is large.

Figure 3.20 shows the relationship between $T$, $\lambda$, and Loan Value. For small values of $T$, the loan value is decreasing and concave up, but the loan value becomes increasing and concave down for larger values of $T$. As expected, this figure exhibits the exact opposite behavior as Figure 3.11, the corresponding figure for Case #2; see that figure for further comments. Figure 3.21 shows some interesting relationships between Loan Value, $T$, and $\sigma_2 - \sigma_1$. While the corresponding figure for the previous case showed, looking at the loan value as a function of $T$, a maximum near $T = 1$, this figure shows a minimum. Recall that the default value for $\lambda$ is $\lambda = 1$. In Case #3, the mean reverting model, which gives loan value as a quickly decreasing function of $T$, is expected to dominate at first, but the second geometric Brownian motion state begins to dominate afterward. This dominance causes the loan value to increase as a function of $T$ as documented for the geometric Brownian motion model. This relationship becomes more pronounced as $\sigma_2$ increases.

Moving to the examples that assume $\sigma_2 - \sigma_1 < 0$, Figure 3.22, “Relationship between $q$, $\lambda$, and Loan Value,” shows a smooth curve in which the loan value is a decreasing function
Table 3.5: Default Values for MR to GBM, One Dimensional Examples

<table>
<thead>
<tr>
<th>Variable</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$q$</td>
<td>40</td>
</tr>
<tr>
<td>$S_0$</td>
<td>50</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
</tr>
<tr>
<td>$s$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma(1)$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma(2)$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1</td>
</tr>
<tr>
<td>$L$</td>
<td>$\log 45 \approx 3.81$</td>
</tr>
<tr>
<td>$a$</td>
<td>2</td>
</tr>
</tbody>
</table>

of $q$ and $\lambda$. In Figure 3.23, the loan value increases as $\sigma_2 - \sigma_1$ takes on larger negative values and decreases as $q$ increases.

Figure 3.24 shows a relatively flat loan value surface, with the surface flatter for large maturity times than for small ones. Finally, Figure 3.25 shows the relationship between Loan Value, $T$, and $\sigma_2 - \sigma_1$. The loan value appears to decrease rapidly as $T$ increases. Recall that, in this figure, $\sigma_2$ is small in comparison to $\sigma_1$. Unlike in Figure 3.21 where $\sigma_2$ is large in comparison to $\sigma_1$, here the geometric Brownian motion loan values generated by the second state are not large enough to cause the “weighted average” loan value to increase after the geometric Brownian motion state is expected to be reached. Thus, the loan value is decreasing as a function of $T$.  


3.3 Two Mean Reverting States

3.3.1 Formula Derivation

For this section, we will assume that the stock obeys the following two-state regime switching model in which the second state is absorbing:

\[
\begin{align*}
    dS_t &= S(t) [A(S_t, \alpha_t)dt + \sigma(\alpha_t)dW_t] \\
    A(S_t, 1) &= a_1(L_1 - \log(S_t)); \quad \sigma(1) = \sigma_1 \\
    A(S_t, 2) &= a_2(L_2 - \log(S_t)); \quad \sigma(2) = \sigma_2.
\end{align*}
\]
Table 3.6: Numerical Results for MR to GBM, One Parameter Varied

<table>
<thead>
<tr>
<th>$q$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>45.42</td>
<td>36.13</td>
<td>27.09</td>
<td>16.54</td>
<td>10.89</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>45</th>
<th>55</th>
<th>65</th>
<th>75</th>
<th>85</th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>13.93</td>
<td>19.29</td>
<td>25.12</td>
<td>31.24</td>
<td>37.53</td>
<td>43.93</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
<th>0.19</th>
<th>0.22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>17.22</td>
<td>16.54</td>
<td>16.01</td>
<td>15.57</td>
<td>15.13</td>
<td>14.62</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>26.08</td>
<td>16.54</td>
<td>21.76</td>
<td>24.21</td>
<td>26.40</td>
<td>30.13</td>
<td>33.06</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma(1)$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>16.79</td>
<td>17.08</td>
<td>17.38</td>
<td>17.68</td>
<td>17.98</td>
<td>18.28</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma(2)$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>12.61</td>
<td>13.12</td>
<td>13.73</td>
<td>14.40</td>
<td>15.10</td>
<td>15.81</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>12.53</td>
<td>13.27</td>
<td>13.94</td>
<td>14.56</td>
<td>15.64</td>
<td>16.54</td>
<td>18.96</td>
<td>21.00</td>
<td>22.15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 30 $\approx$ 3.40</th>
<th>log 40 $\approx$ 3.69</th>
<th>log 50 $\approx$ 3.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>12.34</td>
<td>15.33</td>
<td>19.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>log 60 $\approx$ 4.09</th>
<th>log 70 $\approx$ 4.25</th>
<th>log 80 $\approx$ 4.38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>25.41</td>
<td>30.07</td>
<td>30.76</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>30.64</td>
<td>22.82</td>
<td>19.22</td>
<td>17.74</td>
<td>17.07</td>
<td>16.54</td>
<td>16.24</td>
<td>16.12</td>
<td>16.06</td>
</tr>
</tbody>
</table>
Figure 3.19: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to GBM model.
Figure 3.20: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to GBM model.
Figure 3.21: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to GBM model.
Figure 3.22: Relationship between $q$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to GBM model.
Figure 3.23: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to GBM model.
Figure 3.24: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to GBM model.
Figure 3.25: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to GBM model.
As before, let $\alpha(\cdot) = \alpha_t : t \geq s$ be a two-state Markov chain with $\alpha_t \in \{1, 2\}$. Let $Q \equiv (q_{ij})$, $1 \leq i, j \leq 2$ defined by $Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$ be the generator of $\alpha$ with $\lambda > 0$. Also, assume $\alpha_t$ is independent of $S_t$. For initial time $s$, let $h(z, s, t) \equiv (z - qe^{\gamma(t-s)})_+$. Then the value function can be written

$$V(x, s, i) = E\left[ e^{-r(T-s)}h(S_T, s, T)|S_s = x, \alpha_s = i \right].$$

Here $s, 0 \leq s \leq T$, denotes the initial time, $x$ the initial stock price, and $i$ the initial state.

Recall that we can write the mean reverting model explicitly as

$$X_t = e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_t^{-1})$$

where $\phi_t \equiv \int_0^t (1/\alpha(\phi_s, \cdot))ds$, $\alpha(t, \cdot) = \alpha(t) \equiv \sigma e^{at}$, and $\phi_t^{-1}$ denotes the inverse function of $\phi_t$; moreover

$$\phi_t^{-1} = \frac{\sigma^2(e^{2at} - 1)}{2a}.$$

For $i = 1, 2$, let $V^0(x, s, i) \equiv E\left[ e^{-r(T-s)}h(S_T, s, T)|S_s = x, \alpha_s = i \text{ for } s \leq u \leq T \right]$

$$= e^{-r(T-s)} \int_{-\infty}^{\infty} h(z, s, T)N(z, m(T-s, i), \Sigma^2(T-s, i))dz$$

where $N$ is the Normal density function with mean

$$m(t, i) = e^{-a(t-s)}(\log x - L_i) + L_i$$

and variance $\Sigma^2(t, i) = e^{-2a(t-s)}\phi_t^{-1}$. Note that the inverse function $\phi^{-1}(\cdot)$ is a function of the variance parameter and hence in this model depends on $i$. Let $\tau \equiv \inf(t \geq s : \alpha_t \neq \alpha_s)$, i.e. the time of the first (and only, in our case) jump. Since $q_{11} \equiv -\lambda \neq 0$,

$$P(\tau > u|\alpha_s = 1) = e^{-\lambda(u-s)}.$$

Let $S$ denote the Banach space of all bounded measurable functions on $\mathbb{R} \times [s, T] \times \{1, 2\}$. Define a mapping $\Upsilon$ on $S$ by

$$(\Upsilon f)(x, s, 1) = \int_s^T e^{-r(T-s)} \left[ \int_{-\infty}^{\infty} f(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))dz \right] \lambda e^{-\lambda(T-s)}dt;$$

$$(\Upsilon f)(x, s, 2) = 0.$$
Theorem 3.3.1. $V$ is a solution to the equation

$$V(x, s, i) = \Upsilon V + e^{-q_{ii}(T-s)}V^0(x, s, i)$$  \hspace{1cm} (3.3.1)

where $q_{ii} = \lambda$ if $i = 1$ and $q_{ii} = 0$ if $i = 2$.

Proof. Let $E_{x, s, i}$ denote the conditional expectation with conditions $S_s = x$ and $\alpha_s = i$. Then

$$E_{x, s, 1} \left[ e^{-r(T-s)}h(S_T, T)I_{\{\tau > T\}} \right] = e^{-\lambda(T-s)}V^0(x, s, 1)$$

because $P(\tau > T|\alpha_s = 1) = e^{-\lambda(T-s)}$ and $\tau$ is independent of $S_T$. Thus,

$$V(x, s, i) = E_{x, s, 1} \left[ e^{-r(T-s)}h(S_T, s, T)I_{\{\tau > T\}} \right] + e^{-\lambda(T-s)}V^0(x, s, 1).$$  \hspace{1cm} (3.3.2)

Working on the first term in (3.3.2), condition additionally on $\tau = t$ and apply the Law of Iterated Expectation to write

$$E_{x, s, 1} \left[ e^{-r(T-s)}h(S_T, s, T)I_{\{\tau > T\}} \right] = \int_s^T e^{-r(T-s)}E_{x, s, 1} \left[ h(S_T, T)|\tau = t \right] \lambda e^{-\lambda(t-s)}dt.$$  

Given $\tau = t$, the post-jump distribution of $\alpha_t$ is $\frac{\lambda}{\varphi_{\tilde{\xi}, i}} = 1$.

As in the proof of Theorem 3.2.1, $W(\phi_{\tilde{\xi}, i}^{-1})$ is Gaussian for each $i$. Thus, for $s \leq t \leq T$, we have

$$E_{x, s, 1} \left[ h(S_T, s, T)|\tau = t \right] = E_{x, s, 1} \left[ E_{x, s, 1} \left[ h(S_T, s, T)|S(t), \alpha(t) \right]|\tau = t \right]$$

$$= E_{x, s, 1} \left[ e^{-r(T-t)}V(S_t, t, \alpha(t))|\tau = t \right]$$

$$= \int_{-\infty}^{\infty} e^{-r(T-t)}V(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))dz,$$

where the second equality above is by definition of $V$ and the third equality above is due to the Gaussian property. Thus, the first term in (3.3.2) is just $\Upsilon V(t, u, 1)$, and so $V$ is a solution to (3.3.1).

Corollary 3.3.2. Under the assumptions of this section, the value of the stock loan is given by

$$V(x, s, 1) = \int_s^T e^{-r(t-s)} \left( \int_{-\infty}^{\infty} V(z, t, 2)N(z, m(t-s, 1), \Sigma^2(t-s, 1))dz \right) \lambda e^{-\lambda(t-s)}dt$$

$$+ e^{-\lambda(T-s)}V^0(x, s, 1)$$  \hspace{1cm} (3.3.3)

$$V(x, s, 2) = V^0(x, s, 2)$$
3.3.2 Numerical Examples

This subsection gives some numerical examples to illustrate properties possessed by the formula for the case when both states are mean reverting with different volatility, reversion rate, and equilibrium level. These examples were produced using MATLABR2006a to evaluate (3.3.3).

The first set of examples vary one parameter while holding the others fixed. This set utilizes the default values for the indicated in Table 3.7. Varying one parameter at a time gives the results indicated in Tables 3.8 and 3.9. Many of the relationships illustrated above could be predicted based on earlier findings. The loan value is a monotone decreasing function of $q$, $\gamma$, and both reversion rates $a_1$ and $a_2$. The loan value is a monotone increasing function of $S_0$, both equilibrium levels $L_1$ and $L_2$, as well as both $\sigma(1)$ and $\sigma(2)$. The loan value is also monotone increasing in $\lambda$; this reflects the higher equilibrium level chosen for the second state compared to the first one. Perhaps most interesting is the loan value’s relationship with the maturity time $T$. At first the loan value decreases, then after a momentary and small bump up near $T = 2$, begins decreasing once again. Recall that, for the mean reverting model, the loan value decreases quickly as a function of $T$, explaining everything but the momentary bump. The bump occurs right after $T = 1$, the expected time to switch states (because $\lambda = 1$), and reflects the higher loan value produced by the higher equilibrium level in state # 2. After the expected switch, with no further impetus to force it higher, the loan value resumes its expected downward course.

The second set of examples shows relationships as two of the parameters are changed, all others held constant. These examples are presented in Figures 3.26 through 3.32. Figures 3.26 through 3.29 assume $\sigma_2 - \sigma_1 > 0$, and Figures 3.30 through 3.32 assume $\sigma_2 - \sigma_1 < 0$. Figure 3.26, “Relationship between $q$, $\lambda$, and Loan Value,” shows a smooth surface with the loan value as a decreasing function of $q$ and an increasing function of $\lambda$. These relationships are expected given the difference between the equilibrium levels in the two states. In Figure
3.27, the loan value appears relatively flat as a function of $\sigma_2 - \sigma_1$ with few interesting features.

Figure 3.28 shows the relationship between $T$, $\lambda$, and Loan Value. As in Figure 3.26, the loan value increases quickly at first as a function of $\lambda$ before leveling out. Also, the loan value decreases as a function of $T$ regardless of $\lambda$, suggesting the loan value decreases as a function of maturity time $T$ regardless of equilibrium level $L$, $\sigma$, or $a$. Figure 3.29 shows some interesting relationships between Loan Value, $T$, and $\sigma_2 - \sigma_1$ and gives some more insights into the relationship between loan value and $T$ suggested by the one-variable numerical examples. For large values of $\sigma_2$, the loan value first decreases as a function of $T$, then briefly and slightly increases before resuming its downward trend. Recalling again that the default value for $\lambda$ is $\lambda = 1$, this trend reflects the higher equilibrium level and higher value for $\sigma_2$ associated with the second state. These factors briefly lead to higher loan values before the usual inverse relationship between loan value and $T$ dominates again. For all $T$ the loan value is an increasing function of $\sigma_2 - \sigma_1$, but the increase is greater for large values of $T$ than small ones.

Moving to the examples that assume $\sigma_2 - \sigma_1 < 0$, Figure 3.30, “Relationship between $q$, $\lambda$, and Loan Value,” looks very similar to Figure 3.26, indicating that the sign of $\sigma_2 - \sigma_1$ has little effect on this relationship. In Figure 3.31, the loan value increases as $\sigma_2 - \sigma_1$ takes on larger negative values and decreases as $q$ increases; these results agree with earlier results. Finally, Figure 3.32 shows the relationship between Loan Value, $T$, and $\sigma_2 - \sigma_1$. This figure shows almost the exact opposite relationship as Figure 3.29, the corresponding figure for $\sigma_2 - \sigma_1 > 0$. The same forces that determined the shape of that graph work in the opposite direction to determine the shape of this one.
Table 3.7: Default Values for MR to MR, One Dimensional Examples

<table>
<thead>
<tr>
<th>Variable</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$q$</td>
<td>40</td>
</tr>
<tr>
<td>$S_0$</td>
<td>50</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
</tr>
<tr>
<td>$s$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma(1)$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma(2)$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1</td>
</tr>
<tr>
<td>$L_1$</td>
<td>log 45 $\approx$ 3.81</td>
</tr>
<tr>
<td>$a_1$</td>
<td>2</td>
</tr>
<tr>
<td>$L_2$</td>
<td>log 60 $\approx$ 4.09</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 3.8: Numerical Results for MR to MR, One Parameter Varied

<table>
<thead>
<tr>
<th>q</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>51.74</td>
<td>48.29</td>
<td>39.35</td>
<td>27.50</td>
<td>20.88</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>S₀</th>
<th>45</th>
<th>55</th>
<th>65</th>
<th>75</th>
<th>85</th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>24.94</td>
<td>30.05</td>
<td>35.09</td>
<td>40.04</td>
<td>44.88</td>
<td>49.59</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>γ</th>
<th>0.07</th>
<th>0.1</th>
<th>0.13</th>
<th>0.16</th>
<th>0.19</th>
<th>0.22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>28.31</td>
<td>27.50</td>
<td>26.84</td>
<td>26.27</td>
<td>25.70</td>
<td>25.07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>T</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>37.31</td>
<td>27.50</td>
<td>27.97</td>
<td>27.34</td>
<td>26.33</td>
<td>23.76</td>
<td>20.96</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>σ(1)</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>27.75</td>
<td>28.04</td>
<td>28.34</td>
<td>28.64</td>
<td>28.94</td>
<td>29.24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>σ(2)</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>33.37</td>
<td>30.94</td>
<td>29.39</td>
<td>28.44</td>
<td>27.89</td>
<td>27.61</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>λ</th>
<th>0.125</th>
<th>0.25</th>
<th>0.375</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>14.85</td>
<td>17.56</td>
<td>19.90</td>
<td>21.91</td>
<td>25.13</td>
<td>27.50</td>
<td>31.95</td>
<td>32.78</td>
</tr>
</tbody>
</table>
Table 3.9: Numerical Results for MR to MR, One Parameter Varied (cont.)

<table>
<thead>
<tr>
<th>L₁</th>
<th>log 30 ≈ 3.40</th>
<th>log 40 ≈ 3.69</th>
<th>log 50 ≈ 3.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>23.29</td>
<td>26.29</td>
<td>30.00</td>
</tr>
<tr>
<td>L₁</td>
<td>log 60 ≈ 4.09</td>
<td>log 70 ≈ 4.25</td>
<td>log 80 ≈ 4.38</td>
</tr>
<tr>
<td>Loan value</td>
<td>36.37</td>
<td>41.04</td>
<td>41.74</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a₁</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>41.60</td>
<td>33.78</td>
<td>30.18</td>
<td>28.70</td>
<td>28.03</td>
<td>27.50</td>
<td>27.20</td>
<td>27.08</td>
<td>27.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>L₂</th>
<th>log 30 ≈ 3.40</th>
<th>log 40 ≈ 3.69</th>
<th>log 50 ≈ 3.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>18.65</td>
<td>21.65</td>
<td>24.61</td>
</tr>
<tr>
<td>L₂</td>
<td>log 60 ≈ 4.09</td>
<td>log 70 ≈ 4.25</td>
<td>log 80 ≈ 4.38</td>
</tr>
<tr>
<td>Loan value</td>
<td>27.50</td>
<td>30.30</td>
<td>33.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a₂</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan value</td>
<td>29.44</td>
<td>28.28</td>
<td>27.50</td>
<td>26.96</td>
<td>26.58</td>
<td>26.11</td>
<td>25.74</td>
<td>25.70</td>
</tr>
</tbody>
</table>
Figure 3.26: Relationship between $q$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to MR model.
Figure 3.27: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to MR model.
Figure 3.28: Relationship between $T$, $\lambda$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to MR model.
Figure 3.29: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 > 0$ when collateral stock obeys an MR to MR model.
Figure 3.30: Relationship between \( q, \lambda, \) and Loan Value assuming \( \sigma_2 - \sigma_1 < 0 \) when collateral stock obeys an MR to MR model.
Figure 3.31: Relationship between $q$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to MR model.
Figure 3.32: Relationship between $T$, $\sigma_2 - \sigma_1$, and Loan Value assuming $\sigma_2 - \sigma_1 < 0$ when collateral stock obeys an MR to MR model.
Chapter 4

Viscosity Solutions for the Mean Reverting Stock Loan Equation

In this chapter, we will prove that the value function is a (the) viscosity solution to the PDE corresponding to the case where the collateral stock obeys a mean reverting model

\[ S_t = e^{X_t}, \]
\[ dX_t = a(L - X_t)dt + \sigma dW_t, \quad X_s = \log S_s. \]

For a discussion of viscosity solutions in a Markov chain context, see [21]. For a proof that the value function of a European option under a geometric Brownian motion stock model is a viscosity solution of its corresponding PDE, see [16].

Given maturity time \( T \), let \( g(x, s) \equiv e^{-r(T-s)} (e^x - q e^{\gamma T})_+ \). We will consider the adjusted value function

\[ \tilde{V}(x, s) \equiv E[g(X_T)|X_s = x]. \]

Henceforth in this chapter, for any function \( \zeta \), we will use the notation

\[ \tilde{E}^{s,x}[\zeta(X)] \equiv E[\zeta(X)|X_s = x]. \]

As a consequence of Ito’s Formula, for \( s = 0 \) the adjusted value function must satisfy the differential equation

\[ r\tilde{V}(x, s) = \frac{\partial \tilde{V}}{\partial s} + \frac{\partial \tilde{V}}{\partial x} a(L - x) + \frac{\sigma^2}{2} \frac{\partial^2 \tilde{V}}{\partial x^2} \tag{4.0.1} \]

subject to terminal condition

\[ \tilde{V}(x, T) = (e^x - q e^{\gamma T})_. \]

We begin our discussion of viscosity solution by giving a definition of viscosity solution. In what follows, \( H : \mathbb{R} \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) denotes a function. For a function \( f(x, s) \), consider the following condition:
(*) $f(x,s)$ is continuous in $(x,s)$ and there exist constants $C$ and $n$ such that $f(x,s) \leq C(1 + |x|^n)$.

**Definition 1.** We say $f(x,s)$ is a viscosity subsolution of
\[
H \left( x, s, f(x,s), \frac{\partial f(x,s)}{\partial s}, \frac{\partial f(x,s)}{\partial x}, \frac{\partial^2 f(x,s)}{\partial^2 x} \right) = 0
\]
if $f(x,s)$ satisfies (*) and $H \left( x_0, s_0, f(x_0,s_0), \frac{\partial \phi(x_0,s_0)}{\partial s}, \frac{\partial \phi(x_0,s_0)}{\partial x}, \frac{\partial^2 \phi(x_0,s_0)}{\partial^2 x} \right) \leq 0$ for any $\phi \in C^2(\mathbb{R} \times [0,T])$ such that $f(x,s) - \phi(x,s)$ has a local maximum at $(x,s) = (x_0,s_0)$.

**Definition 2.** We say $f(x,s)$ is a viscosity supersolution of
\[
H \left( x, s, f(x,s), \frac{\partial f(x,s)}{\partial s}, \frac{\partial f(x,s)}{\partial x}, \frac{\partial^2 f(x,s)}{\partial^2 x} \right) = 0
\]
if $f(x,s)$ satisfies (*) and $H \left( x_0, s_0, f(x_0,s_0), \frac{\partial \psi(x_0,s_0)}{\partial s}, \frac{\partial \psi(x_0,s_0)}{\partial x}, \frac{\partial^2 \psi(x_0,s_0)}{\partial^2 x} \right) \geq 0$ for any $\psi \in C^2(\mathbb{R} \times [0,T])$ such that $f(x,s) - \psi(x,s)$ has a local minimum at $(x,s) = (x_0,s_0)$.

**Definition 3.** We say $f(x,s)$ is a viscosity solution of
\[
H \left( x, s, f(x,s), \frac{\partial f(x,s)}{\partial s}, \frac{\partial f(x,s)}{\partial x}, \frac{\partial^2 f(x,s)}{\partial^2 x} \right) = 0
\]
if $f(x,s)$ is both a viscosity subsolution and a viscosity supersolution.

### 4.1 Existence

We will use a series of lemmata to prove that $\tilde{V}(x,s)$ is a viscosity solution of (4.0.1) when $s = 0$. In addition to the functions $g$ and $\tilde{V}$ defined above, we will also have need for the function
\[
g_1(x,s) \equiv e^{-r(T-s)} (x - qe^{\gamma(T-s)})_+.
\]

The corresponding adjusted value function for this payoff is
\[
\tilde{V}_1(x,s) = E [g_1(X_T)|X_s = x].
\]

Note that for any $x \in \mathbb{R}$ and $s \in [0,T]$, $g_1(e^x,s) = g(x,s)$ and $\tilde{V}(x,s) = \tilde{V}_1(e^x,s)$.
Lemma 4.1.1. For any \( x_1, x_2 \in \mathbb{R} \) and \( s_1, s_2 \in [s, T] \),

\[
|g_1(x_1, s_1) - g_1(x_2, s_2)| \leq |x_1 - x_2| + C|s_2 - s_1| \tag{4.1.1}
\]

for some constant \( C > 0 \).

Proof. First note that

\[
|g_1(x_1, s_1) - g_1(x_2, s_2)| \\
\leq |e^{-r(T-s_1)}(x_1 - qe^{\gamma(T-s_1)})_+ - e^{-r(T-s_1)}(x_2 - qe^{\gamma(T-s_1)})_+| \\
+ e^{-r(T-s_1)}(x_2 - qe^{\gamma(T-s_1)})_+ - e^{-r(T-s_2)}(x_2 - qe^{\gamma(T-s_2)})_+|. \tag{4.1.2}
\]

For the first term on the right side of (4.1.2), since \( -r(T-s) \leq 0 \) for any \( s \leq T \), we have

\[
|e^{-r(T-s_1)}(x_1 - qe^{\gamma(T-s_1)})_+ - e^{-r(T-s_1)}(x_2 - qe^{\gamma(T-s_1)})_+| \\
\leq |(x_1 - qe^{\gamma(T-s_1)})_+ - (x_2 - qe^{\gamma(T-s_1)})_+|. 
\]

Next, note that,

\[
| (x_1 - qe^{\gamma(T-s_1)})_+ - (x_2 - qe^{\gamma(T-s_1)})_+ | \leq |x_1 - x_2|. \tag{4.1.3}
\]

To see this, consider cases: the inequality is obvious if both \( x_1 - qe^{\gamma(T-s_1)} \leq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \leq 0 \) or if both \( x_1 - qe^{\gamma(T-s_1)} \geq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \geq 0 \). If \( x_1 - qe^{\gamma(T-s_1)} \geq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \leq 0 \), then \( x_2 \leq qe^{\gamma(T-s_1)} \leq x_1 \) and the claim follows; the case \( x_1 - qe^{\gamma(T-s_1)} \leq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \geq 0 \) is similar.

For the second term on the right side of (4.1.2), we claim that

\[
|e^{-r(T-s_1)}(x_2 - qe^{\gamma(T-s_1)})_+ - e^{-r(T-s_2)}(x_2 - qe^{\gamma(T-s_2)})_+| \leq C|s_2 - s_1| \tag{4.1.4}
\]

for some \( C > 0 \). Again, the proof is by cases. The inequality is obvious if both \( x_2 - qe^{\gamma(T-s_1)} \leq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \leq 0 \). If both \( x_1 - qe^{\gamma(T-s_1)} \geq 0 \) and \( x_2 - qe^{\gamma(T-s_1)} \geq 0 \), then applying the Mean Value Theorem to the differentiable function

\[
h_1(s) = e^{-r(T-s)}(x_2 - qe^{\gamma(T-s)})
\]
Proof. Let $\mathcal{F}_t \equiv \sigma\{W_u : s \leq u \leq t\}$. Then

$$
\tilde{V}(x, s) = \tilde{E}^{s,x} \left[ e^{-r(\theta-s)}\tilde{V}(X(\theta), \theta) \right]
$$

(4.1.5)

**Remark:** (4.1.5) need not hold in general for the actual stock loan value function $V(x, s)$ as defined in previous chapters. If we assume $s > 0$ and $\theta \in [s, T]$, we can only obtain

$$(x, s) \equiv \tilde{E}^{s,x} \left[ e^{-r(T-s)} \left( e^{X_T} - q e^{\gamma(T-s)} \right)_+ \right]

= E \left[ e^{(\gamma-r)(T-\theta+s)} \left( e^{X_T} - q e^{\gamma(T-s)} \right)_+ | \mathcal{F}_s \right]

= E \left[ e^{(\gamma-r)(\theta-s)} E \left[ e^{(\gamma-r)(T-\theta)} \left( e^{X_T} - q e^{\gamma(T-s)} \right)_+ | \mathcal{F}_\theta \right] | \mathcal{F}_s \right]

= E \left[ e^{-r(\theta-s)} E \left[ e^{(\gamma-s) - r(T-\theta)} \left( e^{X_T} - q e^{\gamma(T-s)} \right)_+ | \mathcal{F}_\theta \right] | \mathcal{F}_s \right].

The inside expectation term need not equal $V(X_{\theta}, \theta)$.

**Lemma 4.1.3.** $\tilde{V}(x, s)$ is a continuous function of $x.$
Proof. Let \( x_1, x_2 \in \mathbb{R} \). For any \( s \in [0, T] \), with \( g(x, s) \) as defined above, we have

\[
|g(x_1, s) - g(x_2, s)| = e^{-r(T-s)} \left| (e^{x_1} - qe^{\gamma(T-s)})_+ - (e^{x_2} - qe^{\gamma(T-s)})_+ \right|.
\]

Let \( X_i(t) \) denote the mean reverting process \( X_t \) with initial condition \( X_i(s) = x_i \) so that

\[
dX_1(t) = a(L - X_1(t))dt + \sigma dW_i
\]
\[
dX_2(t) = a(L - X_2(t))dt + \sigma dW_i.
\]

For \( i = 1, 2 \), integrating both sides gives

\[
\int_s^t dX_i(u)du = \int_s^t a(L - X_i(u))du + \int_s^t \sigma dW_u.
\]

Thus,

\[
X_i(t) - X_i(s) = \int_s^t a(L - X_i(u))du + \int_s^t \sigma dW_u,
\]

and consequently

\[
X_i(t) = x_i + \int_s^t a(L - X_i(u))du + \int_s^t \sigma dW_u.
\]

Given a random variable \( Y \), let

\[
\tilde{E}[Y] \equiv E[Y|X_1(s) = x_1, X_2(s) = x_2].
\]

With this notation, we have

\[
\tilde{E}|X_2(T) - X_1(T)|^2 \leq 3|x_2 - x_1|^2 + 3\tilde{E} \left[ \left( \int_s^T a(L - X_2(u)) - a(L - X_1(u))du \right)^2 \right]
\]
\[
= 3|x_2 - x_1|^2 3\tilde{E} \left[ \left( \int_s^T -X_2(u) + X_1(u)du \right)^2 \right]
\]
\[
\leq 3|x_2 - x_1|^2 3 \int_s^T \tilde{E} (X_2(u) - X_1(u))^2 du \tag{4.1.6}
\]
\[
\leq 3|x_2 - x_1|^2 e^{3T} \tag{4.1.7}
\]

Here (4.1.6) is due to Tonelli’s Theorem along with the fact that \( \left( \int h(x) \right)^2 \leq \int (h(x))^2 \), and (4.1.7) is due to Gronwall’s Inequality. Since

\[
\tilde{E}|X_2(T) - X_1(T)| \leq \sqrt{\tilde{E}|X_2(T) - X_1(T)|^2},
\]
we have
\[ \tilde{E}e^{-r(T-s)}|X_2(T) - X_1(T)| \leq 3|x_2 - x_1|e^{-r(T-s)+3T}. \] (4.1.8)

By the previous lemma,
\[ |g_1(X_1(T), s) - g_1(X_2(T), s)| \leq e^{-r(T-s)}|X_1(T) - X_2(T)|. \] (4.1.9)

Thus,
\[ |\tilde{V}_1(x_1, s) - \tilde{V}_1(x_2, s)| = |E[g_1(X_1(T)) - g_1(X_2(T))|X_i(s) = x_i]| \]
\[ \leq E[|g_1(X_1(T)) - g_1(X_2(T))| |X_i(s) = x_i]| \]
\[ \leq E[|X_1(T) - X_2(T)||X_i(s) = x_i]| \]
\[ \leq 3|x_2 - x_1|e^{-r(T-s)+3T}. \] (4.1.10)

Here (4.1.10) is due to (4.1.9) and (4.1.11) is by (4.1.8). Therefore, \( \tilde{V}_1 \) is continuous in \( x \).

Since \( \tilde{V}(x, s) = \tilde{V}_1(e^s, s) \), \( \tilde{V} \) is a composition of continuous functions in \( x \). Therefore, \( \tilde{V} \) is continuous in \( x \).

\[ \boxdot \]

**Lemma 4.1.4.** \( \tilde{V}(x, s) \) is a continuous function of \( s \).

**Proof.** Suppose \( 0 \leq s_1 \leq s_2 \leq T \). Let \( X_t \) denote the solution of the equation that starts at time \( t = s_1 \). Define the process
\[ \tilde{X}_t \equiv X_{t-(s_2-s_1)}. \]

Let \( u \equiv t - (s_2 - s_1) \), and note that \( dt = du \) and \( dW_t = dW_u \). Now
\[ X_t = x + \int_{s_1}^{t} a(L - X(\xi))d\xi + \int_{s_1}^{t} \sigma dW_\xi \]
and
\[ \tilde{X}_t = x + \int_{s_2}^{t} a(L - \tilde{X}(\xi))d\xi + \int_{s_2}^{t} \sigma dW_\xi. \]
Thus,

\[ X_t - \tilde{X}_t = \int_{s_1}^t a(L - X(\xi))d\xi + \int_{s_1}^t \sigma dW_\xi - \int_{s_2}^t a(L - \tilde{X}(\xi))d\xi - \int_{s_2}^t \sigma dW_\xi \]

\[ = \int_{s_1}^t a(L - X(\xi))d\xi + \int_{s_1}^t \sigma dW_\xi \]

\[ - \int_{s_1}^{t-(s_2-s_1)} a(L - \tilde{X}(\xi))d\xi - \int_{s_1}^{t-(s_2-s_1)} \sigma dW_\xi \]

\[ = \int_{t-(s_2-s_1)}^t a(L - X(\xi))d\xi + \int_{t-(s_2-s_1)}^t \sigma dW_\xi. \quad (4.1.12) \]

Here (4.1.12) is by the change of variables \( \xi \rightarrow \xi - s_2 + s_1 \).

Given a random variable \( Y \), let

\[ \tilde{E}[Y] \equiv E[Y | X(s_1) = x = \tilde{X}(s_2)]. \]

Then

\[ \tilde{E} \left( X(t) - \tilde{X}(t) \right)^2 \]

\[ \leq 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t a(L - X(\xi))d\xi \right)^2 + 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t \sigma dW_\xi \right)^2 \]

\[ = 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t a(L - X(\xi))d\xi \right)^2 + 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t \sigma^2 d\xi \right) \quad (4.1.13) \]

\[ = 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t a(L - X(\xi))d\xi \right)^2 + 2\sigma^2(s_2 - s_1) \]

\[ \leq 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t a^2(L - X(\xi))^2d\xi \right) + 2\sigma^2(s_2 - s_1) \quad (4.1.14) \]

\[ \leq 2a^2L^2(s_2 - s_1) - 4\tilde{E} \left( \int_{t-(s_2-s_1)}^t a^2LX(\xi)d\xi \right) \]

\[ + 2\tilde{E} \left( \int_{t-(s_2-s_1)}^t a^2(X(\xi))^2d\xi \right) + 2\sigma^2(s_2 - s_1) \]

\[ \leq 2a^2L^2(s_2 - s_1) - 4a^2L\tilde{E} \langle X(\xi) \rangle (s_2 - s_1) \]

\[ + 2a^2 \int_{t-(s_2-s_1)}^t \tilde{E}|X(\xi)|^2d\xi + 2\sigma^2(s_2 - s_1). \quad (4.1.15) \]

Note that (4.1.13) is by the Ito Isometry, (4.1.14) is because \((\int f)^2 \leq \int f^2\), and (4.1.15) is by Tonelli’s Theorem. Using the explicit formula

\[ X_t = e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_{t-s})^{(1)} \]
where 
\[ \phi_t^{-1} = \frac{\sigma^2(c^{2a} - 1)}{2a}, \]

since \( a, L, \) and \( \sigma \) are finite constants, we can find \( M > 0 \) such that
\[ \tilde{E}|X(\xi)|^2 < M \]

for almost every \( \xi \in [s, T] \). Then we can bound (4.1.15) above by
\[
2a^2 L^2(s_2 - s_1) - 4a^2 L \tilde{E} [X(\xi)] (s_2 - s_1) + 2a^2 M(s_2 - s_1) + 2\sigma^2(s_2 - s_1)
\]
\[
= (s_2 - s_1) \left[ 2a^2 L^2 - 4a^2 L \tilde{E} [X(\xi)] + 2a^2 M + 2\sigma^2 \right].
\]

Let \( R \equiv 2a^2 L^2 - 4a^2 L \tilde{E} [X(\xi)] + 2a^2 M + 2\sigma^2, \) and note that, by taking \( M \) sufficiently large, we can make \( R > 0 \). Thus, we have
\[
\tilde{E} \left( X(t) - \tilde{X}(t) \right)^2 \leq R(s_2 - s_1),
\]

and consequently, by taking square root and applying Jensen’s Inequality, we have
\[
\tilde{E}|X(t) - \tilde{X}(t)| \leq \sqrt{R(s_2 - s_1)}. \tag{4.1.16}
\]

As a result, we have
\[
|\tilde{V}_1(x, s_1) - \tilde{V}_1(x, s_2)| \leq \tilde{E}|g_1(X(T), s_1) - g_1(\tilde{X}(T), s_2)|
\]
\[
\leq \tilde{E}|X(T) - \tilde{X}(T)| + C|s_2 - s_1| \tag{4.1.17}
\]
\[
\leq \sqrt{R(s_2 - s_1)} + C|s_2 - s_1| \tag{4.1.18}
\]
\[
\leq \sqrt{s_2 - s_1} \left( \sqrt{R} + C|s_2 - s_1| \right).
\]

Here (4.1.17) is by Lemma 4.1.1, and (4.1.18) is by (4.1.16). Therefore, \( \tilde{V}_1 \) is continuous in \( s \). Since \( \tilde{V}(x, s) = \tilde{V}_1(e^x, s) \), this implies that \( \tilde{V} \) is continuous in \( s \).

**Remark:** The bound given by the previous lemma is independent of the choice of \( x \). Therefore, the two previous lemmata combine to give that \( \tilde{V} \) is jointly continuous in \((x, s)\) by the triangle inequality, i.e.
\[
|\tilde{V}(x, s) - \tilde{V}(x_1, s_1)| \leq |\tilde{V}(x, s) - \tilde{V}(x_1, s)| + |\tilde{V}(x_1, s) - \tilde{V}(x_1, s_1)|.
\]
Lemma 4.1.5. \( \tilde{V}(x, s) \) has at most polynomial growth.

Proof. Let \( x > 0 \). Using the explicit formula for \( X_t \), we have

\[
|\tilde{V}(x, s)| \leq \tilde{E}\left|e^{X(t)} - q e^{\gamma(T-s)}\right|_+
\leq \tilde{E}\left[e^{X(t)}\right]
= \tilde{E}\left[\exp e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_{t-s}^{-1})\right].
\]

If \( \log x - L < 0 \), then

\[
\tilde{E}\left[\exp e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_{t-s}^{-1})\right] \leq \tilde{E}\left[\exp (L + e^{-a(t-s)}W(\phi_{t-s}^{-1}))\right]
\leq e^{L}\tilde{E}\left[\exp (e^{-a(t-s)}W(\phi_{t-s}^{-1}))\right]
\leq e^{L}C_1
\tag{4.1.19}
\]

with (4.1.19) because \( W(\phi_{t-s}^{-1}) \) is a Brownian motion, as proved in Chapter 3, implying that

\[
\tilde{E}\left[\exp (e^{-a(t-s)}W(\phi_{t-s}^{-1}))\right] < C_1 < \infty
\]

for some \( C_1 > 0 \). If \( \log x - L \geq 0 \), then because \( e^{-a(t-s)} < 1 \), we have

\[
\tilde{E}\left[\exp (e^{-a(t-s)}(\log x - L) + L + e^{-a(t-s)}W(\phi_{t-s}^{-1}))\right]
\leq |x|\tilde{E}\left[\exp (e^{-a(t-s)}W(\phi_{t-s}^{-1}))\right]
\leq |x|C_1
\leq C_1(1 + |x|)
\]

with \( C_1 \) as chosen above. Thus, defining \( C \equiv \max\{C_1, e^{L}C_1\} \) gives

\[
|\tilde{V}(x, s)| \leq C(1 + |x|).
\]

Therefore, \( \tilde{V}(x, s) \) has at most polynomial growth. \( \square \)
Lemma 4.1.6. \( \tilde{V}(x, s) \) is a viscosity supersolution to (4.0.1).

Proof. Continuity and polynomial growth have already been established. Suppose \( \psi \in C^2(\mathbb{R} \times [0, T]) \) such that \( \tilde{V}(x, s) - \psi(x, s) \) has a local minimum at \( (x, s) = (x_0, s_0) \) in a neighborhood \( N(x_0, s_0) \). Let

\[
\Psi(x, s) \equiv \psi(x, s) + \tilde{V}(x_0, s_0) - \psi(x_0, s_0).
\]

By continuity, we can choose \( \theta \in [s_0, T] \) such that \( (X_t, t) \) starts at \( (x_0, s_0) \) and \( \{(X_t, t)\} \subseteq N(x_0, s_0) \) for all \( t \in [s_0, \theta] \). Then, by Dynkin’s Formula, we have

\[
\tilde{E}^{x_0, s_0} \left[ e^{-r(\theta - s_0)} (\Psi(X_\theta, \theta)) \right]
\]

\[
= \tilde{E}^{x_0, s_0} \int_{s_0}^{\theta} e^{-r(\theta - s_0)} \left[ -r \Psi(X_t, t) + \frac{\partial \Psi(X_t, t)}{\partial t} X_t \frac{\partial \Psi(X_t, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \Psi(X_t, t)}{\partial x^2} \right] dt
\]

(4.1.21)

Since \( (X_t, t) \in N(x_0, s_0) \) for all \( t \in [s_0, \theta] \) and \( \tilde{V}(x, s) - \psi(x, s) \) has a local minimum at \( (x, s) = (x_0, s_0) \), we have

\[
\tilde{V}(X_t, t) - \psi(X_t, t) \geq \tilde{V}(x_0, s_0) - \psi(x_0, s_0).
\]

Thus,

\[
\tilde{V}(X_t, t) \geq \tilde{V}(x_0, s_0) - \psi(x_0, s_0) + \psi(X_t, t) \equiv \Psi(X_t, t).
\]

(4.1.22)

Note from the definition of \( \Psi \) that

\[
\frac{\partial \psi(X_t, t)}{\partial t} = \frac{\partial \psi(X_t, t)}{\partial t},
\]

\[
\frac{\partial \psi(X_t, t)}{\partial x} = \frac{\partial \psi(X_t, t)}{\partial x},
\]

\[
\frac{\partial^2 \psi(X_t, t)}{\partial x^2} = \frac{\partial^2 \psi(X_t, t)}{\partial x^2}.
\]

Hence, by substituting into (4.1.21), we obtain

\[
\tilde{E}^{x_0, s_0} \left[ e^{-r(\theta - s_0)} (\Psi(X_\theta, \theta)) \right]
\]

\[
\geq \tilde{E}^{x_0, s_0} \int_{s_0}^{\theta} e^{-r(\theta - s_0)} \left[ -r \tilde{V}(X_t, t) + \frac{\partial \psi(X_t, t)}{\partial t} \right.
\]

\[
X_t \frac{\partial \psi(X_t, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \psi(X_t, t)}{\partial x^2} \left. \right] dt.
\]

(4.1.23)
Applying Lemma 4.1.2 along with inequality (4.1.23) gives

\[ 0 = \tilde{V}(x_0, t) - \tilde{V}(x_0, s_0) \]

\[ \geq \tilde{E}^{x_0, s_0} \int_{s_0}^{t} e^{-r(t-s_0)} \left[ -r \tilde{V}(X_t, t) + \frac{\partial \psi(X_t, t)}{\partial t} X_t \frac{\partial \psi(X_t, t)}{\partial x} + \frac{\sigma^2 \partial^2 \psi(X_t, t)}{2 \partial x^2} \right] dt. \]

Multiplying both sides by \( \frac{-1}{\theta} \) < 0 and sending \( \theta \downarrow s_0 \) gives

\[ 0 \leq r \tilde{V}(x_0, s_0) - \frac{\partial \psi(x_0, s_0)}{\partial t} - x_0 \frac{\partial \psi(x_0, s_0)}{\partial x} - \frac{\sigma^2 \partial^2 \psi(x_0, s_0)}{2 \partial x^2}. \]

Therefore, \( \tilde{V}(x, s) \) is a viscosity supersolution to (4.0.1).

**Lemma 4.1.7.** \( \tilde{V}(x, s) \) is a viscosity subsolution to (4.0.1).

*Proof.* Continuity and polynomial growth have already been established. Suppose \( \phi \in C^2(\mathbb{R} \times [s, T]) \) such that \( \tilde{V}(x, s) - \phi(x, s) \) has a local maximum at \( (x, s) = (x_0, s_0) \) in a neighborhood \( N(x_0, s_0) \). By continuity, we can choose \( \theta \in [s_0, T] \) such that \( (X_t, t) \) starts at \( (x_0, s_0) \) and \( \{ (X_t, t) \} \subseteq N(x_0, s_0) \) for all \( t \in [s_0, \theta] \). By adjusting \( \phi(x, s) \) by a constant, we can assume without loss of generality that

\[ \tilde{V}(x_0, s_0) - \phi(x_0, s_0) = 0 \]

so that \( \tilde{V}(X_t, t) \leq \phi(X_t, t) \) for any \( t \in [s_0, \theta] \). Then, by Dynkin’s Formula, we have

\[ \tilde{E}^{x_0, s_0} \left[ e^{-r(\theta-s_0)} \left( \tilde{V}(X_{\theta}, \theta) \right) \right] \]

\[ \leq \phi(x_0, s_0) + \tilde{E}^{x_0, s_0} \int_{s_0}^{\theta} e^{-r(t-s_0)} \left[ -r \phi(X_t, t) + \frac{\partial \phi(X_t, t)}{\partial t} X_t \frac{\partial \phi(X_t, t)}{\partial x} + \frac{\sigma^2 \partial^2 \phi(X_t, t)}{2 \partial x^2} \right] dt. \]
Then applying Lemma 4.1.2 along with (4.1.24) gives

\[
0 = \tilde{E}^{x_0,s_0} \left[ e^{-r(\theta-s_0)}\tilde{V}(X_\theta, \theta) \right] - \tilde{V}(x_0, s_0) \\
\leq \tilde{E}^{x_0,s_0} \int_{s_0}^{\theta} e^{-r(t-s_0)} \left[ -r\tilde{V}(X_t, t) + \frac{\partial \phi(X_t, t)}{\partial t} \\
X_t \frac{\partial \phi(X_t, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \phi(X_t, t)}{\partial x^2} \right] dt.
\]

Multiplying both sides by \(-1/\theta < 0\) and sending \(\theta \downarrow s_0\) gives

\[
0 \geq rV(x_0, s_0) - \frac{\partial \phi(x_0, s_0)}{\partial t} \\
- x_0 \frac{\partial \phi(x_0, s_0)}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 \phi(x_0, s_0)}{\partial x^2}.
\]

Therefore, \(\tilde{V}(x, s)\) is a viscosity subsolution to (4.0.1).

The two previous lemmata combine to give the following:

**Theorem 4.1.8.** \(\tilde{V}(x, s)\) is a viscosity solution to (4.0.1).

**Remark:** If we take \(s = 0\), then \(\tilde{V}(x, 0) = V(x, 0)\), where \(V\) as defined before is the value function for a stock loan with European maturity. Thus, for \(s = 0\), \(V(x, 0)\) is a viscosity solution to (4.0.1).

### 4.2 Uniqueness

In this section, we will prove that the viscosity solution to (4.0.1) is unique. We begin with a definition and a major background result.

**Definition 4.** Given a function \(f : \mathbb{R} \times [s, T] \to \mathbb{R}\), we define the parabolic superjet of \(f(x, s)\) to be

\[
J^{2+} f(x, s) \equiv \{(p, q, M) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : f(y, t) \leq f(x, s) + p(t - s) + q(y - x) \\
+ \frac{1}{2}(y - x)^2 M + o(|y - x|^2) \text{ as } (y, t) \to (x, s)\}.
\]
The closure of the parabolic superjet is defined by

\[
\overline{J}^2_+ f(x, s) \equiv \{(p, q, M) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \lim_{n \to \infty} (p_n, q_n, M_n) = (p, q, M) \text{ with } (p_n, q_n, M_n) \in J^2_+ f(x_n, s_n) \text{ and } \lim_{n \to \infty} (x_n, s_n, f(x_n, s_n)) = (x, s, f(x, s))\}.
\]

The parabolic subjet of \( f(x, s) \) is defined by

\[
J^2_- f(x, s) \equiv -J^2_+ (-f)(x, s),
\]

and its closure is defined by

\[
\overline{J}^2_- f(x, s) \equiv -\overline{J}^2_+ (-f)(x, s).
\]

The following theorems can be found in [5].

**Theorem 4.2.1.** Let \( \Omega_1, \Omega_2 \) be locally compact subsets of \( \mathbb{R} \), and define \( \Omega \equiv \Omega_1 \times \Omega_2 \). For \( i = 1, 2 \), let \( u_i \) be upper semicontinuous in \( \Omega_i \times [s, T] \), \( \overline{J}^2_{\Omega_i} u_i(x, t) \) the parabolic superjet of \( u_i(x, t) \), and \( \phi \) be a twice continuously differentiable function in a neighborhood of \( \Omega_i \times [s, T] \). For \( (x_1, x_2, t) \in \Omega \times [s, T] \), define

\[
w(x_1, x_2, t) \equiv u_1(x_1, t) + u_2(x_2, t).
\]

Assume \( (\hat{x}_1, \hat{x}_2, \hat{t}) \) is a local minimum of \( w - \phi \) relative to \( \Omega \times [s, T] \). Further assume that there exists \( r > 0 \) such that, for every \( M > 0 \), there exists \( C \) such that, for \( i = 1, 2 \), we have

\[
b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \overline{J}^2_{\Omega_i} u_i(x_i, t),
\]

\[
|x_i - \hat{x}_i| + |t - \hat{t}| \leq r, \text{ and }
\]

\[
|u_i(x_i, t)| + |q_i| + \|X_i\| \leq M.
\]

Then, for each \( \epsilon > 0 \), there exists \( X_i \in \mathbb{R} \) such that the following are true:

1. \( (b_i, D x_i \phi(\hat{x}_i, \hat{t}), X_i) \in \overline{J}^2_{\Omega_i} u_i(\hat{x}_i, \hat{t}) \) for \( i = 1, 2 \);

2. \(- \left( \frac{1}{\epsilon} + \|D^2 \phi(\hat{x})\| \right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2 \phi(\hat{x}) + \epsilon (D^2 \phi(\hat{x}))^2\);
3. $b_1 + b_2 = \frac{\partial \phi(\hat{x}, \hat{y}, \hat{t})}{\partial t}$.

**Lemma 4.2.2.** The set $J^{2,+} f(x, s)$ (resp. $J^{2,-} f(x, s)$) consists of the set of

\[
\left( \frac{\partial \phi(x, s)}{\partial s}, \frac{\partial \phi(x, s)}{\partial x}, \frac{\partial^2 \phi(x, s)}{\partial x^2} \right)
\]

where $\phi$ is twice continuously differentiable on $\mathbb{R} \times [s, T]$ and $f - \phi$ has a global maximum (resp. minimum) at $(x, s)$.

**Proof.** A proof can be found in [8]. □

We are now ready to state and prove the main theorem of this subsection.

**Theorem 4.2.3.** If $V_1(x, t)$ and $V_2(x, t)$ are viscosity solutions to (4.0.1) with at most linear growth, i.e. for $i = 1, 2$, there exist constants $C_i$ such that

\[V_i(x, t) \leq C_i(1 + |x|) \text{ for } (x, t) \in \mathbb{R} \times [s, T],\]

then

\[V_1(x, t) \leq V_2(x, t) \text{ for all } (x, t) \in \mathbb{R} \times [s, T].\]

**Proof.** Following the idea of [12], given $0 < \delta < 1$ and $0 < \nu < 1$, define

\[
\Phi(x, y, t) \equiv V_1(x, t) - V_2(y, t) - \frac{1}{\delta} |x - y|^2 - \nu e^{(T-t)}(x^2 + y^2)
\]

(4.2.2)

and

\[
\phi(x, y, t) \equiv \frac{1}{\delta} |x - y|^2 + \nu e^{(T-t)}(x^2 + y^2).
\]

Since $V_1(x, t)$ and $V_2(x, t)$ have at most linear growth, the (negative) quadratic terms in (4.2.2) will dominate $V_1(x, t) - V_2(y, t)$ as $|x|, |y| \to \infty$; hence

\[
\lim_{|x| + |y| \to \infty} \Phi(x, y, t) = -\infty.
\]

(4.2.3)

Since $\Phi(x, y, t)$ is continuous, (4.2.3) implies that there exists a point $(x_\delta, y_\delta, t_\delta)$ such that $\Phi(x_\delta, y_\delta, t_\delta)$ is a global maximum of $\Phi$. Since

\[
\Phi(x_\delta, x_\delta, t_\delta) + \Phi(y_\delta, y_\delta, t_\delta) \leq 2\Phi(x_\delta, y_\delta, t_\delta),
\]
we have by definition of $\Phi$

$$V_1(x_\delta, t_\delta) - V_2(x_\delta, t_\delta) - 2\nu e^{(T-t_\delta)} x_\delta^2 + V_1(y_\delta, t_\delta) - V_2(y_\delta, t_\delta) - 2\nu e^{(T-t_\delta)} y_\delta^2$$

$$\leq 2V_1(x_\delta, t_\delta) - 2V_2(y_\delta, t_\delta) - \frac{2}{\delta} |x_\delta - y_\delta|^2 - 2\gamma e^{(T-t_\delta)} (x_\delta^2 + y_\delta^2).$$

This implies

$$-V_2(x_\delta, t_\delta) - 2\nu e^{(T-t_\delta)} x_\delta^2 + V_1(y_\delta, t_\delta) - 2\nu e^{(T-t_\delta)} y_\delta^2$$

$$\leq V_1(x_\delta, t_\delta) - V_2(y_\delta, t_\delta) - \frac{2}{\delta} |x_\delta - y_\delta|^2 - 2\nu e^{(T-t_\delta)} (x_\delta^2 + y_\delta^2),$$

and hence

$$\frac{2}{\delta} |x_\delta - y_\delta|^2 \leq (V_1(x_\delta, t_\delta) - V_1(y_\delta, t_\delta)) + (V_2(x_\delta, t_\delta) - V_2(y_\delta, t_\delta)) \tag{4.2.4}$$

Since $V_1$ and $V_2$ have at most linear growth, (4.2.4) implies that there exists a constant $K_1$ such that

$$\frac{2}{\delta} |x_\delta - y_\delta|^2 \leq K_1 (1 + |x_\delta| + |y_\delta|).$$

Thus, taking $K = 2K_1$, we have

$$|x_\delta - y_\delta|^2 \leq \delta K (1 + |x_\delta| + |y_\delta|). \tag{4.2.5}$$

Since $\Phi(0, 0, s) \leq \Phi(x_\delta, y_\delta, t_\delta)$ and $|\Phi(0, 0, s)| \leq K (1 + |x_\delta| + |y_\delta|)$, we have by definition of $\Phi$

$$\gamma e^{(T-t_\delta)} (x_\delta^2 + y_\delta^2) \leq V_1(x_\delta, t_\delta) - V_2(x_\delta, t_\delta) - \frac{1}{\delta} |x_\delta - y_\delta|^2 - \Phi(0, 0, s)$$

$$\leq 3K (1 + |x_\delta| + |y_\delta|). \tag{4.2.6}$$

Dividing gives

$$\frac{\gamma e^{(T-t_\delta)} (x_\delta^2 + y_\delta^2)}{(1 + |x_\delta| + |y_\delta|)} \leq 3K.$$

Since $x^2 + y^2$ grows faster than $x + y$ as $x, y \to \infty$, this implies that there exists a constant $C_\nu$ such that

$$|x_\delta| + |y_\delta| \leq C_\nu \tag{4.2.7}$$
and \( t_\delta \in [s, T] \). Thus, \( \{x_\delta : \delta > 0\} \) and \( \{y_\delta : \delta > 0\} \) are uniformly bounded with respect to \( \delta \). This boundedness implies that we can find convergent subsequences, which we will denote \((x_\delta)_\delta, (y_\delta)_\delta, (t_\delta)_\delta\) (recall that \( t_\delta \in [s, T] \)). Moreover, by (4.2.5), there exist points \( x_0 \) and \( t_0 \) such that

\[
\lim_{\delta \to 0} x_\delta = x_0 = \lim_{\delta \to 0} y_\delta \quad \text{and} \quad \lim_{\delta \to 0} t_\delta = t_0.
\]

These limits, (4.2.4), and (4.2.5) imply that

\[
\lim_{\delta \to 0} |x_\delta - y_\delta|^2 = 0. \tag{4.2.8}
\]

Since \( \Phi \) has a global maximum at \((x_\delta, y_\delta, t_\delta)\), part 1 of Theorem 4.2.1 implies that, given \( \epsilon > 0 \), there exist \( b_{1\delta}, b_{2\delta}, X_\delta, \) and \( Y_\delta \) such that

\[
(b_{1\delta}, \frac{2}{\delta}(x_\delta - y_\delta) + 2\nu e^{(T-t)x_\delta}, X_\delta) \in J^{2,+}_1(x_\delta, t_\delta) \tag{4.2.9}
\]

and

\[
(-b_{2\delta}, -\frac{2}{\delta}(x_\delta - y_\delta) + 2\nu e^{(T-t)y_\delta}, -Y_\delta) \in J^{2,+} - V_2(y_\delta, t_\delta). \tag{4.2.10}
\]

By definition of parabolic subjet, (4.2.10) implies that

\[
(b_{2\delta}, \frac{2}{\delta}(x_\delta - y_\delta) - 2\nu e^{(T-t)y_\delta}, Y_\delta) \in J^{2,-} - V_2(y_\delta, t_\delta). \tag{4.2.11}
\]

By definition of viscosity solution and Lemma 4.2.2, (4.2.9) implies that

\[
rV_1(x_\delta, t_\delta) - b_{1\delta} - \frac{\sigma^2}{2}X_\delta - a(L - x) \left( \frac{2}{\delta}(x_\delta - y_\delta) - 2\nu e^{(T-t_\delta)x_\delta} \right) \leq 0.
\]

Similarly, (4.2.10) implies that

\[
rV_2(y_\delta, t_\delta) - b_{2\delta} - \frac{\sigma^2}{2}Y_\delta - a(L - x) \left( \frac{2}{\delta}(x_\delta - y_\delta) - 2\nu e^{(T-t_\delta)y_\delta} \right) \geq 0.
\]

Combining these inequalities gives

\[
r(V_1(x_\delta, t_\delta) - V_2(y_\delta, t_\delta)) \leq \frac{\sigma^2}{2}(X_\delta - Y_\delta) + 2a(L - x)\nu e^{(T-t_\delta)}(x_\delta - y_\delta) + b_{1\delta} - b_{2\delta}
\]

By part 3 of Theorem 4.2.1, we have

\[
b_{1\delta} - b_{2\delta} = \frac{\partial \phi(x_\delta, y_\delta, t_\delta)}{\partial t} = -\nu e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).
\]
The right equality is by direct computation from the definition of \( \phi \). Substituting gives

\[
\begin{align*}
\quad r(V_1(x_\delta, t_\delta) &- V_2(y_\delta, t_\delta)) \\
\leq & \frac{\sigma^2}{2}(X_\delta - Y_\delta) + \nu \left( 2a(L - x) e^{(T-t_\delta)}(x_\delta - y_\delta) - e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) \right).
\end{align*}
\] (4.2.12)

Letting \( \nu \to 0 \) in (4.2.12) gives, by boundedness of \( \{x_\delta : \delta > 0\} \) and \( \{y_\delta : \delta > 0\} \),

\[
\begin{align*}
r(V_1(x_\delta, t_\delta) &- V_2(y_\delta, t_\delta)) \\
&\leq \frac{\sigma^2}{2}(X_\delta - Y_\delta). \quad (4.2.13)
\end{align*}
\]

By part 2 of Theorem 4.2.1, we have

\[
\begin{align*}
- \left( \frac{1}{\epsilon} + \|D_{(x,y)}^2 \phi(x_\delta, y_\delta, t_\delta)\| \right) I &\leq \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \\
&\leq D_{(x,y)}^2 \phi(x_\delta, y_\delta, t_\delta) + \epsilon(D^2 \phi(x_\delta, y_\delta, t_\delta))^2.
\end{align*}
\]

Moreover, by direct calculation, we have

\[
D_{(x,y)}^2 \phi(x_\delta, y_\delta, t_\delta) = \frac{2}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]

Thus,

\[
\begin{align*}
X_\delta - Y_\delta &= (1, 1) \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} (1, 1)^T \\
&\leq (1, 1) \left( \frac{2}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \epsilon \frac{4}{\delta^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) (1, 1)^T \\
&= (1,) \left( \frac{2}{\delta} + \epsilon \frac{8}{\delta^2} \right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} (1,1)^T
\end{align*}
\]

Taking \( \epsilon = \delta \) gives

\[
\begin{align*}
(1, 1) \left( \frac{10}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) (1,1)^T &= \left( \frac{10}{\delta} \right) (1,1) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 0.
\end{align*}
\]

Thus,

\[
X_\delta - Y_\delta \leq 0.
\]
Applying this inequality to the right side of (4.2.13) gives

\[ r(V_1(x_0, t_0) - V_2(y_0, t_0)) \leq 0 \] (4.2.14)

Since \( \Phi \) has a global maximum at \((x_\delta, y_\delta, t_\delta)\), we know that for all \( t \in [s, T] \) and \( x \in \mathbb{R} \)

\[ \Phi(x, x, t) \leq \Phi(x_\delta, y_\delta, t_\delta). \]

By definition of \( \Phi \), this gives for all \( t \in [s, T] \) and \( x \in \mathbb{R} \)

\[ V_1(x, t) - V_2(x, t) - 2\gamma e^{(T-t)}x^2 \leq V_1(x_\delta, t_\delta) - V_2(y_\delta, t_\delta) - \frac{1}{\delta}|x_\delta - y_\delta|^2 - \gamma e^{(T-t)}(x_\delta^2 + y_\delta^2). \]

Letting \( \delta \to 0 \) and \( \gamma \to 0 \) gives, via (4.2.8) and (4.2.14),

\[ V_1(x, t) - V_2(x, t) \leq V_1(x_0, t_0) - V_2(y_0, t_0) \leq 0. \]

Thus,

\[ V_1(x, t) \leq V_2(x, t) \]
as desired. □

**Corollary 4.2.4.** Equation (4.0.1) has a unique viscosity solution with at most linear growth.

**Proof.** The solutions \( V_1(x, t) \) and \( V_2(x, t) \) in the previous theorem were arbitrary viscosity solutions to (4.0.1) with at most linear growth. □

### 4.3 Finite Difference Numerical Scheme

In this section, we will develop a numerical scheme for approximating the viscosity solution to (4.0.1) and prove that our scheme converges. For a more general framework, see [21], chapter 10.

#### 4.3.1 Scheme Definition

Recall that for equation (4.0.1), we defined \( H : \mathbb{R} \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[ H \left( x, s, V(x, s), \frac{\partial V(x, s)}{\partial s}, \frac{\partial V(x, s)}{\partial x}, \frac{\partial^2 V(x, s)}{\partial^2 x} \right) \]
\[ \equiv rV(x, s) - \frac{\partial V(x, s)}{\partial s} - a(L - x) \frac{\partial V(x, s)}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 V(x, s)}{\partial^2 x} = 0 \quad (4.3.1) \]

and

\[ g(x, s) = e^{r(T-s)} (e^x - qe^{\gamma T})^+. \]

Let \( h > 0 \) denote the increment in the \( x \)-variable, and let \( k > 0 \) denote the increment in the \( s \)-variable. Define the following finite difference operators:

\[
\Delta_s u(x, s) \equiv \frac{u(x, s + k) - u(x, s)}{k};
\]

\[
\Delta_x u(x, s) \equiv \frac{u(x + h, s) - u(x, s)}{h};
\]

\[
\Delta_x^2 u(x, s) \equiv \frac{u(x + h, s) + u(x - h, s) - 2u(x, s)}{h^2}.
\]

With these operators, the function \( H \) can be discretized as follows:

\[
rV(x, s) - \frac{V(x, s + k) - V(x, s)}{k} - a(L - x) \frac{V(x + h, s) - V(x, s)}{h}
- \frac{\sigma^2}{2} \frac{V(x + h, s) + V(x - h, s) - 2V(x, s)}{h^2} = 0. \quad (4.3.2)
\]

Rearranging terms allows us to rewrite (4.3.2) as

\[ V(x, s) \left( r + \frac{1}{k} + \frac{a(L - x)}{h} + \frac{\sigma^2}{2h^2} \right) - \frac{V(x, s + k)}{k} \]

\[ -V(x + h, s) \left( \frac{a(L - x)}{h} + \frac{\sigma^2}{2h^2} \right) - \frac{\sigma^2}{2h^2} V(x - h, s) = 0. \]

Let \( B(\mathbb{R}^+ \times [0, T]) \) denote the space of bounded functions defined on \( \mathbb{R}^+ \times [0, T] \) and continuous in \((x, s)\). Define \( S : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times (B(\mathbb{R}^+ \times [0, T])) \rightarrow \mathbb{R} \) by

\[
S(x, h, k, y, u) \equiv yh \left( r + \frac{1}{k} + \frac{a(L - x)}{h} + \frac{\sigma^2}{2h^2} \right) - \frac{hu(x, s + k)}{k}
- u(x + h, s) \left( a(L - x) + \frac{\sigma^2}{2h} \right) - \frac{\sigma^2}{2h} u(x - h, s). \quad (4.3.3)
\]

Note that, when \( a(L - x) + \frac{\sigma^2}{2h} > 0 \), which will be true when \( h < \frac{\sigma^2}{2a(L - x)} \), all coefficients of \( u \) in (4.3.3) are negative. Thus, for \( h \) sufficiently small, \( S \) is monotone decreasing, i.e. if \( u \leq v \), then

\[
S(x, h, k, y, u) \geq S(x, h, k, y, v). \quad (4.3.4)
\]
4.3.2 Convergence

We first need to state some definitions related to convergence.

**Definition 5.** We say an approximation scheme $S$ to the equation

$$H \left( x, s, V(x, s), \frac{\partial V(x, s)}{\partial s}, \frac{\partial V(x, s)}{\partial x}, \frac{\partial^2 V(x, s)}{\partial^2 x} \right) = 0$$

is consistent if

$$\lim_{z \to x, \epsilon \to 0, k \to 0, h \to 0} \frac{S(z, h, k, \omega(z, s) + \epsilon, \omega + \epsilon)}{h} = H \left( x, s, \omega(x, s), \frac{\partial \omega(x, s)}{\partial s}, \frac{\partial \omega(x, s)}{\partial x}, \frac{\partial^2 \omega(x, s)}{\partial^2 x} \right)$$

for every $x \in \mathbb{R}^+$, $s \in [0, T]$, and test function $\omega(x, s) \in C^{2,1} \left( \mathbb{R}^+ \times [0, T] \right)$.

**Definition 6.** We say an approximation scheme $S$ to the equation

$$H \left( x, s, V(x, s), \frac{\partial V(x, s)}{\partial s}, \frac{\partial V(x, s)}{\partial x}, \frac{\partial^2 V(x, s)}{\partial^2 x} \right) = 0$$

is stable if, for every $h, k \in \mathbb{R}^+$, there exists a bounded solution $u_{h,k} \in B(\mathbb{R}^+ \times [0, T])$ to the equation

$$S(x, h, k, u(x, t), u) = 0 \quad (4.3.5)$$

with the bound independent of $k$ and $h$.

**Lemma 4.3.1.** The scheme $S$ for $H$, with $S$ and $H$ as defined in (4.3.3) and (4.3.1) respectively, is consistent.

**Proof.** Let $\omega(x, s) \in C^{2,1} \left( \mathbb{R}^+ \times [0, T] \right)$. Then

$$S(z, h, k, \omega(z, t) + \epsilon, \omega + \epsilon) = (\omega(z, s) + \epsilon) h \left( r + \frac{1}{k} + \frac{a(L-z)}{h} + \frac{\sigma^2}{h^2} \right) - \frac{h(\omega(z, s + k) + \epsilon)}{k}$$

$$- (\omega(z + h, s) + \epsilon) \left( a(L-z) + \frac{\sigma^2}{2h} \right) - \frac{\sigma^2}{2h} (\omega(z - h, s) + \epsilon).$$

Thus,

$$\lim_{z \to x, \epsilon \to 0, k \to 0, h \to 0} \frac{S(z, h, k, \omega(z, s) + \epsilon, \omega + \epsilon)}{h} = \lim_{z \to x, \epsilon \to 0, k \to 0, h \to 0} \frac{(\omega(z, s) + \epsilon) \left( r + \frac{1}{k} + \frac{a(L-z)}{h} + \frac{\sigma^2}{h^2} \right) - (\omega(z, s + k) + \epsilon)}{k}$$
\[-\frac{(\omega(z+h,s)+\epsilon)(a(L-z)+\frac{\sigma^2}{2h})}{h} - \frac{\sigma^2}{2h^2}(\omega(z-h,s)+\epsilon)\]
\[= \lim_{z \to x, \epsilon \to 0, k \downarrow 0} r(\omega(z,t)+\epsilon) - \frac{\omega(z,s+k)+\omega(z,t)}{k} - a(L-z)\frac{\omega(z+h,s)-\omega(z,t)}{h} - \frac{\sigma^2(\omega(z+h,s)+\epsilon)+\sigma^2\omega(z-h,s)-2\sigma^2\omega(z,s)}{2h^2}.\]

Clearly this equation is continuous in $\epsilon$; it is continuous in $z$ because $\omega$ is continuous. Thus, by applying the convergence of the difference formulas, we get that the previous limit is
\[r\omega(x,t) - \frac{\partial \omega(x,s)}{\partial s} - a(L-x)\frac{\partial \omega(x,s)}{\partial x} - \sigma^2 \frac{\partial^2 \omega(x,s)}{\partial x^2} = H\left(x, s, \omega(x,s), \frac{\partial \omega(x,s)}{\partial s}, \frac{\partial \omega(x,s)}{\partial x}, \frac{\partial^2 \omega(x,s)}{\partial x^2}\right)\]
as desired. 

Let $c(h,k) \equiv \frac{rh^2 + \frac{h^2}{k} + ha(L-x) + \sigma^2}{h}$. Using the technique in [21], we can rewrite $S$ in the equivalent form
\[
\omega(z,t) = \frac{1}{c(h,k)} \left[ -h\omega(z,s+k) + \omega(z+h,s) \left( a(L-z) + \frac{\sigma^2}{2h} \right) - \omega(z-h,s) \frac{\sigma^2}{2h} \right].
\]

Define an operator $\Upsilon_{h,k}$ on $B(\mathbb{R}^+ \times [0,T])$ by
\[
\Upsilon_{h,k}\omega(z,t) \equiv \frac{1}{c(h,k)} \left[ -h\omega(z,s+k) + \omega(z+h,s) \left( a(L-z) + \frac{\sigma^2}{2h} \right) - \omega(z-h,s) \frac{\sigma^2}{2h} \right].
\]

**Lemma 4.3.2.** For $h,k > 0$ sufficiently small, $\Upsilon_{h,k}$ is a strict contraction mapping.

**Proof.** Let $f, g \in B(\mathbb{R}^+ \times [0,T])$. Then $\Upsilon_{h,k}f - \Upsilon_{h,k}g$
\[
= \frac{1}{c(h,k)} \left[ \frac{-h}{k} \left( f(z,s+k) - g(z,s+k) \right) - \left( f(z+h,s) - g(z+h,s) \right) \left( a(L-z) + \frac{\sigma^2}{2h} \right) \right.
\]
\[
\quad \left. - \left( f(z-h,s) - g(z-h,s) \right) \frac{\sigma^2}{2h} \right].
\]
As described when showing that $S$ is monotone decreasing, for $h$ sufficiently small, all coefficients inside the brackets are negative. Thus,
\[
|\Upsilon_{h,k}f - \Upsilon_{h,k}g| \leq \frac{1}{c(h,k)} \left[ \frac{-h}{k} - a(L-z) - \frac{2\sigma^2}{2h} \right] \cdot \|f - g\|.\]
We claim \( \frac{1}{c(h, k)} \left[ \frac{-h}{k} - a(L - z) - \frac{2\sigma^2}{2h} \right] < 1 \). By definition of \( c(h, k) \), we have

\[
\left| \frac{1}{c(h, k)} \left[ \frac{-h}{k} - a(L - z) - \frac{2\sigma^2}{2h} \right] \right| = \frac{h}{rh + \frac{h}{k} + \frac{a(L-z)\sigma^2}{h}} < 1,
\]

with the inequality holding when \( r, h > 0 \). Thus, taking

\[
\beta \equiv \frac{h}{rh + \frac{h}{k} + \frac{a(L-z)\sigma^2}{h}},
\]

we have

\[
\| \Upsilon_{h,k}f - \Upsilon_{h,k}g \| \leq \beta \| f - g \|
\]

with \( \beta < 1 \). Therefore, \( \Upsilon_{h,k} \) is a strict contraction mapping.

Given a positive integer \( N \), let \( B^N(\mathbb{R}^+ \times [0, T]) \equiv \{ u \in B(\mathbb{R}^+ \times [0, T]) : \| u \| \leq N \} \). Let \( \Upsilon_{h,k}^N \) denote the operator on \( B^N(\mathbb{R}^+ \times [0, T]) \) that corresponds to \( \Upsilon_{h,k} \). As in the previous lemma, for each \( N \), \( \Upsilon_{h,k}^N \) is a strict contraction mapping for \( h \) sufficiently small. Thus, if \( \omega \in B^N(\mathbb{R}^+ \times [0, T]) \), then \( \| \omega \| \leq N \) implies that \( \| \Upsilon_{h,k}^N \omega \| \leq N \). Let \( S_N : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times (B^N(\mathbb{R}^+ \times [0, T])) \to \mathbb{R} \) denote the approximation scheme \( S \) with \( B(\mathbb{R}^+ \times [0, T]) \) replaced by \( B^N(\mathbb{R}^+ \times [0, T]) \). By a proof identical to that of Lemma 4.3.1, \( S_N \) is consistent. Moreover, we have:

**Corollary 4.3.3.** For each \( N \), \( S_N \) is stable.

**Proof.** Since \( \Upsilon_{h,k}^N \) is a strict contraction mapping for \( h \) sufficiently small, \( \Upsilon_{h,k}^N \) has a unique fixed point, which we will denote \( u_{h,k}^N \). Since \( u_{h,k}^N \in B^N(\mathbb{R}^+ \times [0, T]) \), we have \( \| u_{h,k}^N \| \leq N \) for each \( h, k \) sufficiently small. Since \( N \) does not depend on \( h \) or \( k \), this proves \( S_N \) is stable.

We now arrive at the main result of this section.

**Theorem 4.3.4.** As \( h \downarrow 0 \), \( k \downarrow 0 \), and \( N \to \infty \), \( u_{h,k}^N \) converges locally uniformly to the unique viscosity solution of \( (4.0.1) \).
Proof. With \( g(x) = e^{-r(T-s)} (e^x - qe^{\gamma(T-s)})_+ \) and \( V(x, s) \equiv E[g(X_T)|X_s = x] \) as defined earlier, let
\[
g^N(x) \equiv \min\{e^{-r(T-s)} (e^x - qe^{\gamma(T-s)})_+, N\}
\]
and \( V^N(x, s) \equiv E[g^N(X_T)|X_s = x] \). Since \( g^N \to g \) as \( N \to \infty \), \( V^N \to V \) as \( N \to \infty \). Thus, it suffices to prove that \( u^N_{h,k} \to V^N \) as \( h \downarrow 0, k \downarrow 0 \).

Define
\[
u_-(x, s) = \liminf_{z \to x, h_0, k_0} u^N_{h,k}(z, s),
\]
and
\[
u_+(x, s) = \limsup_{z \to x, h_0, k_0} u^N_{h,k}(z, s).
\]
We claim \( \nu_- \) (resp. \( \nu_+ \)) is a viscosity supersolution (resp. subsolution) to
\[
H\left(x, s, u(x, s), \frac{\partial u(x, s)}{\partial s}, \frac{\partial u(x, s)}{\partial x}, \frac{\partial^2 u(x, s)}{\partial x^2}\right) = 0
\]
with \( H \) as defined above. We will give details for the supersolution case; the subsolution case is similar.

Suppose \( \psi \in C^2(\mathbb{R}^+ \times [0, T]) \) and \( \nu_-(x, s) - \psi(x, s) \) has a local minimum at \( (x, s) = (x_0, s_0) \). By adjusting \( \psi(x, s) \) by a constant, we can assume without loss of generality that \( \nu_-(x_0, s_0) - \psi(x_0, s_0) = 0 \). Moreover, by stability, we can assume that \( \psi \geq 2 \sup_{k,h} \|u^N_{h,k}\| \) outside of the ball \( B((x_0, s_0), r) \) where \( r \) is chosen such that
\[
u_-(x, s) - \psi(x, s) \leq 0 = \nu_-(x_0, s_0) - \psi(x_0, s_0) \text{ in } B((x_0, s_0), r).
\]
Then we can choose sequences \( k_n > 0, h_n > 0 \), and \( (y_n, s_n) \in [0, C] \times [0, T] \) such that, as \( n \to \infty \),
\[
k_n \downarrow 0, h_n \downarrow 0, y_n \to x_0, s_n \to s_0, u^N_{h_n,k_n}(y_n, s_n) \to \nu_-(x_0, s_0) \quad \quad (4.3.7)
\]
and \( (y_n, s_n) \) is a global minimum of \( u^N_{h_n,k_n} - \psi \). Let \( \epsilon_n \equiv u^N_{h_n,k_n}(y_n, s_n) - \psi(y_n, s_n) \). Then by (4.3.7), \( \epsilon_n \to 0 \). Moreover, by the global minimum property, we have for all \( (x, s) \in \mathbb{R}^+ \times [0, T] \),
\[
u^N_{h_n,k_n}(x, s) - \psi(x, s) \geq \nu^N_{h_n,k_n}(y_n, s_n) - \psi(y_n, s_n) = \epsilon_n.
\]
This implies that $u_{h_n,k_n}^N(x,s) \geq \psi(x,s) + \epsilon_n$ for all $(x,s) \in \mathbb{R}^+ \times [0,T]$. Since $u_{h_n,k_n}^N(x,s)$ is a solution to $S_N(y_n, h_n, k_n, y, u) = 0$ and $S_N$ is monotone decreasing, we have

$$S_N(y_n, h_n, k_n, \psi(y_n, s_n) + \epsilon_n, \psi + \epsilon_n) \geq S_N(y_n, h_n, k_n, u_{h_n,k_n}^N(x,s), u_{h_n,k_n}^N) = 0.$$ 

Thus,

$$\lim_{n \to \infty} \frac{S_N(y_n, h_n, k_n, \psi(y_n, s_n) + \epsilon_n, \psi + \epsilon_n)}{h_n} \geq 0.$$

By consistency,

$$H \left( x_0, s_0, u_-, \frac{\partial u_-}{\partial s}, \frac{\partial u_-}{\partial x}, \frac{\partial^2 u_-}{\partial^2 x} \right) = \lim_{z \to x, \epsilon \to 0, k \to 0, h \to 0} \frac{S_N(z, h, k, \psi(z, s) + \epsilon, \psi + \epsilon)}{h} \geq 0.$$

Thus, $u_-^N$ is a viscosity supersolution. Moreover, by uniqueness of viscosity solution, we have $u_-^N = u_+^N = V^N$. By definition of $u_-^N$ and $u_+^N$, we have that $u_{h,k}^N$ converges locally uniformly to $V^N$ as desired.

4.3.3 Numerical Examples

In this section, we will present numerical examples that illustrate and implement the approximation scheme developed in the previous section. These examples use the parameters given in Table 4.1. For boundary conditions on $x$, we take $u(x,s) \equiv 0$ for the lower boundary and $u(x,s) = (e^x - qe^{\gamma T})_+$ for the $s = T$ boundary. For the upper boundary, we use the Neumann boundary condition

$$\frac{u(x+h,s) - u(x,s)}{h} = \frac{\partial V}{\partial x},$$

with the partial derivative on the right hand side calculated numerically from the explicit formula proved in Chapter 1.

The results can be seen in Figure 4.1; the actual graph shows values generated by the explicit formula, while the numerical graph shows values generated by the numerical scheme. As expected, the numerical scheme gives good approximations for small values for $x$, but errors grow as we get further away from the lower and right boundaries.
Table 4.1: Parameters for Finite Difference Numerical Scheme

<table>
<thead>
<tr>
<th>variable</th>
<th>default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$q$</td>
<td>5</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
</tr>
<tr>
<td>$T$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
</tr>
<tr>
<td>$L$</td>
<td>$\log 45 \approx 3.81$</td>
</tr>
<tr>
<td>$a$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 4.1: Comparison of analytical versus numerical results for mean reverting stock loan.
Viscosity Solution for Partially Observed Stock Loans

For this chapter, the collateral stock is assumed to obey a regime switching model with any finite state space $M = \{1, ..., m\}$. All states will be geometric Brownian motions with means $\mu_1, \mu_2, ..., \mu_m$ respectively and volatility $\sigma$. The states in the embedded Markov chain will now be assumed to be only partially observable. We will use a Wonham filter approach as outlined in Zhang, Yin, and Moore in [22] to handle the partial observability property. This chapter will formally formulate the problem, prove existence of a unique viscosity solution, and give a numerical scheme to approximate this solution.

5.1 Problem Formulation

For $i = 1, 2, ..., m$, let

$$p_i(t) \equiv P(\mu_t = \mu_i | S_u : s \leq u \leq t).$$

Here $\mu_i$ is a constant for each $i$, and as indicated in the introduction, $\mu_t$ jumps according to an $m$-state Markov chain with generator $Q$. We assume the Markov chain is independent of $S_t$. A two-state version of this model is considered in [18]. With this notation, we will assume that the collateral stock price obeys the stochastic differential equation

$$dS_t = S_t \left( \sum_{i=1}^{m} \mu_i p_i(t) dt + \sigma dW_t \right)$$

with initial condition $S_s = x$. In (5.1.1), $x$ and $\sigma$ are positive constants, and $W_t$ is a standard Brownian motion. Let $X_t \equiv \log S_t$ so that (5.1.1) becomes

$$dX_t = \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$
Define the \( \mathbb{R}^m \) vector \( P_t \equiv (p_1(t), ..., p_m(t)) \). As in [22], the Wonham filter is defined by

\[
dP_t = P_t Q dt - \frac{1}{\sigma^2} \left( \sum_{i=1}^{m} \left( \mu_i - \frac{\sigma^2}{2} \right) p_i(t) \right) P_t A_t dt + \frac{1}{\sigma^2} P_t A_t dX_t. \tag{5.1.3}
\]

In (5.1.3), \( A_t \) is the \( m \times m \) matrix given by

\[
A_t \equiv \text{diag}(\mu_1, ..., \mu_m) - \sum_{i=1}^{m} \mu_i p_i(t) I
\]

where \( I \) is the \( m \times m \) identity matrix. Next, define the innovations process by

\[
dv_t \equiv \frac{dX(t) - \sum_{i=1}^{m} \left( \mu_i - \frac{\sigma^2}{2} \right) p_i(t) dt}{\sigma} \tag{5.1.4}
\]

with initial condition \( v_0 = 0 \). As shown by Lirong Yu in [24], \( v_t \) is a Brownian motion, and \( P_t \) is a solution of

\[
dP_t = P_t Q dt + (1/\sigma) P_t A_t dv_t \tag{5.1.5}
\]

with initial condition \( P = (p_1(0), ..., p_m(0)) \).

By [15], the HJB equation corresponding to the stock loan value function for this process is

\[
0 = \frac{\partial V}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} - P Q \frac{\partial V}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
+ 2 P A_t \frac{\partial^2 V}{\partial p \partial x} + \text{tr} \{ A_t P^T P A_t \frac{\partial^2 V}{\partial p^2} \} \tag{5.1.6}
\]

subject to terminal condition

\[
V(x, p, T) = (e^x - q e^{\gamma T})_+.
\]

The following sections will be devoted to studying this equation.

For the definition of viscosity subsolution, viscosity supersolution, viscosity solution, see the introduction of the previous section. Also, recall the following notation: given maturity time \( T \), let \( g(x, s) \equiv e^{-r(T-s)} (e^x - q e^{\gamma T})_+ \) and \( g_1(x, s) \equiv e^{-r(T-s)} (x - q e^{\gamma T})_+ \). Note that \( g_1(e^x, s) = g(x, s) \). We will consider the adjusted value functions

\[
\tilde{V}(x, p, s) \equiv E[g(X_T)|X_s = x]
\]
\[ 
V_1(x, p, s) = E[g_1(X_T)|X_s = x]. 
\]

Also, for any \( x \in \mathbb{R}, p \in [0, 1], \) and \( s \in [0, T], \) let \( V(x, p, s) = V_1(e^x, p, s). \)

### 5.2 Existence

In a structure similar to that employed in the previous chapter, we will use a series of lemmata to prove that \( \tilde{V}(x, p, s) \) is a viscosity solution of (5.1.6) when \( s = 0. \) The first two lemmata are properties of the functions \( g \) and \( g_1 \) and hence do not depend on the underlying process. These lemmata were proved in a previous chapter, but for convenience we will restate them here without proof.

**Lemma 5.2.1.** For any \( x_1, x_2 \in \mathbb{R} \) and \( s_1, s_2 \in [0, T], \)

\[
|g_1(x_1, s_1) - g_1(x_2, s_2)| \leq |x_1 - x_2| + C|s_2 - s_1|. \tag{5.2.1}
\]

**Lemma 5.2.2.** For any stopping time \( \theta \in [0, T], \) we have

\[
\tilde{V}(x, p, s) = \tilde{E}^{s, p, x} \left[ e^{-r(\theta-s)}\tilde{V}(X_\theta, P_\theta, \theta) \right]. \tag{5.2.2}
\]

**Lemma 5.2.3.** \( \tilde{V}(x, p, s) \) is a continuous function of \( x. \)

**Proof.** Let \( x_1, x_2 \in \mathbb{R}. \) Then for any \( s \in [0, T] \) and \( p \in [0, 1], \) with \( g(x, s) \) as defined above, we have

\[
|g(x_1, s) - g(x_2, s)| = e^{-r(T-s)} \left| \left( e^{x_1} - q e^\gamma(T-s) \right)_+ - \left( e^{x_2} - q e^\gamma(T-s) \right)_+ \right|. 
\]

For \( i = 1, 2, \) let \( X_i(t) \) denote the process defined by (5.1.2) with initial condition \( X_i(s) = x_i. \) Then

\[
dX_1(t) = \left( \sum_{i=1}^m \mu_i p_i(t) - \frac{\sigma^2}{2} \right) dt + \sigma dv, \quad X_1(s) = x_1, \ P_s = p, \\
dX_2(t) = \left( \sum_{i=1}^m \mu_i p_i(t) - \frac{\sigma^2}{2} \right) dt + \sigma dv, \quad X_2(s) = x_2, \ P_s = p. 
\]
Lemma 5.2.4. Then for $u$, and $\mu_i$, $p_i$.

By the previous lemma, we have

Thus,

Therefore, $\tilde{\mu}$ is continuous in $u$. Thus, $\tilde{\mu}$ is a continuous function of $\alpha_t$ and $X_t$ gives

Thus,

and

By the previous lemma,

Thus,

Therefore, $\tilde{V}_1$ is continuous in $x$. Since $\tilde{V}(x, p, s) = \tilde{V}_1(e^x, p, s)$, $\tilde{V}$ is a composition of continuous functions in $x$. Therefore, $\tilde{V}$ is continuous in $x$.

Lemma 5.2.4. $\tilde{V}(x, p, s)$ is a continuous function of $p$.

Proof. Let $p_1, p_2 \in [0, 1]^m$. Then for any $s \in [0, T]$ and $x \in \mathbb{R}$, with $g(x, s)$ as defined above, we have

For $i = 1, 2$, let $P^i(t)$ denote the process defined by (5.1.5) with initial condition $P^i_s = p_i$.

Then substituting into the process $X_t$ gives the corresponding processes

Then for $i = 1, 2$, integrating both sides and applying independence of $\alpha_t$ and $X_t$ gives

Therefore, $\tilde{X}_1(t) = \tilde{X}_2(t)$,

and

By the previous lemma,

Thus,

Thus,

Therefore, $\tilde{V}_1$ is continuous in $x$. Since $\tilde{V}(x, p, s) = \tilde{V}_1(e^x, p, s)$, $\tilde{V}$ is a composition of continuous functions in $x$. Therefore, $\tilde{V}$ is continuous in $x$.

Lemma 5.2.4. $\tilde{V}(x, p, s)$ is a continuous function of $p$.

Proof. Let $p_1, p_2 \in [0, 1]^m$. Then for any $s \in [0, T]$ and $x \in \mathbb{R}$, with $g(x, s)$ as defined above, we have

For $i = 1, 2$, let $P^i(t)$ denote the process defined by (5.1.5) with initial condition $P^i_s = p_i$.

Then substituting into the process $X_t$ gives the corresponding processes

$$dX_1(t) = \left( \sum_{i=1}^m \mu_i p_i(t) - \frac{\sigma^2}{2} \right) dt + \sigma dv_t, X_1(s) = x, P^1_s = p_1$$

$$dX_2(t) = \left( \sum_{i=1}^m \mu_i p_i(t) - \frac{\sigma^2}{2} \right) dt + \sigma dv_t, X_2(s) = x, P^2_s = p_2.$$
For $i = 1, 2$, integrating both sides and applying independence of $\alpha_t$ and $X_t$ gives

$$X_i(t) = x + \int_s^t \left( \sum_{i=1}^m \mu_i p_i(u) - \frac{\sigma^2}{2} \right) du + \int_s^t \sigma dv_u.$$ 

Thus,

$$|X_1(t) - X_2(t)| = \left| \int_s^t \sum_{i=1}^m \mu_i (p_i^2(u) - p_i^1(u)) du \right| \leq m \max \{|\mu_i|\} \int_s^t \sum_{i=1}^m |p_i^2(u) - p_i^1(u)| du.$$ 

Taking expectation and applying Fubini’s Theorem gives

$$\tilde{E}|X_1(t) - X_2(t)| \leq m \max \{|\mu_i|\} \int_s^t \sum_{i=1}^m \tilde{E}|p_i^2(u) - p_i^1(u)| du. \quad (5.2.5)$$

To bound the right hand side, note that, for $i = 1, 2$, we have by integrating (5.1.5),

$$P_i = p_i - \int_s^t P_i^Q du + \int_s^t (1/\sigma) P_i^A dv_u.$$ 

Thus,

$$\|P_i^1 - P_i^2\| \leq \|p_1 - p_2\| + \|Q\| \int_s^t \|P_i^1 - P_i^2\| du + (1/\sigma) \int_s^t \|P_i^A\| dv_u.$$ 

Here $\|\cdot\|$ denotes the infinity norm. Squaring both sides and taking expectation gives, when applying the fact that $v_t$ is a Brownian motion and $\|Q\| \leq 1$,

$$\tilde{E}\|P_i^1 - P_i^2\|^2 \leq 3\|p_1 - p_2\|^2 + 3\tilde{E}\int_s^t \|P_i^1 - P_i^2\|^2 du \leq 3\|p_1 - p_2\|^2 \exp (3(t - s)) \quad (5.2.6)$$

$$\leq 3\|p_1 - p_2\|^2 \exp (3(T - s))$$

$$= \tilde{D}\|p_1 - p_2\|^2.$$ 

Here (5.2.6) is due to Gronwall’s Inequality. Consequently, we have

$$\tilde{E}\|P_i^1 - P_i^2\| \leq \sqrt{\tilde{D}}\|p_1 - p_2\|.$$ 

Applying this bound to the right side of (5.2.5) gives,

$$\tilde{E}|X_1(t) - X_2(t)| \leq D\|p_1 - p_2\| \quad (5.2.7)$$
for some constant $D$. Thus, by (5.2.1), we have

$$|\tilde{V}_1(x, p_1, s) - \tilde{V}_1(x, p_2, s)| = \left| \tilde{E} \left[ g_1(X_1(T), s) - g_1(X_2(T), s) \right] |X_i(s) = x_i| \right|$$

$$\leq \tilde{E} \left[ |X_1(T) - X_2(T)| \right] |X_i(s) = x_i|$$

$$\leq D\|p_1 - p_2\|.$$

Therefore, $\tilde{V}_1$ is continuous in $p$. Since $\tilde{V}(x, p, s) = \tilde{V}_1(e^x, p, s)$, $\tilde{V}$ is also continuous in $p$. □

**Lemma 5.2.5.** $\tilde{V}(x, p, s)$ is a continuous function of $s$.

**Proof.** Suppose $0 \leq s_1 \leq s_2 \leq T$. Let $X_t$ denote the solution of (5.1.2) that starts at time $t = s_1$. Define the process

$$\tilde{X}_t \equiv X(t - (s_2 - s_1)).$$

Let $u \equiv t - (s_2 - s_1)$, and note that $dt = du$ and $dv_t = dv_u$. Integrating gives

$$X_t = x + \int_{s_1}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi + \int_{s_1}^{t} \sigma dv_\xi$$

$$\tilde{X}_t = x + \int_{s_2}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi + \int_{s_2}^{t} \sigma dv_\xi.$$

Then

$$X_t - \tilde{X}_t = \int_{s_1}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi + \int_{s_1}^{t} \sigma dv_\xi$$

$$- \int_{s_2}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi - \int_{s_2}^{t} \sigma dv_\xi$$

$$= \int_{s_1}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi + \int_{s_1}^{t} \sigma dv_\xi$$

$$- \int_{s_1}^{t-(s_2-s_1)} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi - \int_{s_1}^{t-(s_2-s_1)} \sigma dv_\xi$$

$$= \int_{t-(s_2-s_1)}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right) d\xi + \int_{t-(s_2-s_1)}^{t} \sigma dv_\xi.$$

Here (5.2.8) is by the change of variables $\xi \to \xi - s_2 + s_1$. 
Recall that, for a random variable $Y$,

$$
\tilde{E}[Y] \equiv E[Y | X_{s_1} = x = \hat{X}_{s_2}].
$$

Then

$$
\tilde{E}\left(X_t - \hat{X}_t\right)^2 \leq 2\tilde{E}\left(\int_{t-(s_2-s_1)}^t \left( \sum_{i=1}^m \mu_ip_i(\xi) - \frac{\sigma^2}{2} \right) d\xi \right)^2
$$

$$
+ 2\tilde{E}\left(\int_{t-(s_2-s_1)}^t \sigma d\nu_\xi \right)^2
$$

$$
= 2\tilde{E}\left(\int_{t-(s_2-s_1)}^t \left( \sum_{i=1}^m \mu_ip_i(\xi) - \frac{\sigma^2}{2} \right) d\xi \right)^2
$$

$$
+ 2\tilde{E}\left(\int_{t-(s_2-s_1)}^t \sigma^2 d\xi \right)
$$

(5.2.9)

$$
\leq 2\tilde{E}\left(\int_{t-(s_2-s_1)}^t \left( \sum_{i=1}^m \mu_ip_i(\xi) - \frac{\sigma^2}{2} \right)^2 d\xi \right)
$$

$$
+ 2\sigma^2(s_2 - s_1).
$$

(5.2.11)

Note that (5.2.9) is by the Ito Isometry, and (5.2.11) is because $(f^2)^2 \leq \int f^2$.

Because

$$
\sum_{i=1}^m \mu_ip_i(\xi) = \sum_{\mu_i < 0} \mu_ip_i(\xi) + \sum_{\mu_i \geq 0} \mu_ip_i(\xi)
$$

and $0 \leq P_\xi \leq 1$, we have

$$
C_L \equiv \sum_{\mu_i < 0} \mu_i \leq \sum_{i=1}^m \mu_ip_i(\xi) \leq \sum_{\mu_i \geq 0} \mu_i \equiv C_U.
$$

Since $\sigma$ is a constant, this implies

$$
\left( \sum_{i=1}^m \mu_ip_i(\xi) - \frac{\sigma^2}{2} \right)^2 \leq C
$$
for some constant \( C \). Applying this bound to (5.2.11) gives

\[
2 \tilde{E} \left( \int_{t -(s_2 - s_1)}^{t} \left( \sum_{i=1}^{m} \mu_i p_i(\xi) - \frac{\sigma^2}{2} \right)^2 d\xi \right) + 2\sigma^2 (s_2 - s_1)
\]

\[
\leq 2 \tilde{E} \int_{t -(s_2 - s_1)}^{t} C^2 d\xi + 2\sigma^2 (s_2 - s_1)
= 2C^2 (t - (t - (s_2 - s_1))) + 2\sigma^2 (s_2 - s_1)
= 2C^2 (s_2 - s_1) + 2\sigma^2 (s_2 - s_1)
= (s_2 - s_1) \cdot 2(C^2 + \sigma^2).
\]

Taking square root and applying Jensen’s Inequality gives

\[
\tilde{E}|X_t - \bar{X}_t| \leq \sqrt{(s_2 - s_1) \cdot 2(C^2 + \sigma^2)}.
\]  

(5.2.12)

As a result, we have

\[
|\tilde{V}_1(x, p, s_1) - \bar{V}_1(x, p, s_2)| \leq \tilde{E}|g_1(X_T, s_1) - g_1(\bar{X}_T, s_2)|
= \tilde{E}|X_T - \bar{X}_T| + \tilde{C}|s_2 - s_1|
\]

(5.2.13)

\[
\leq \sqrt{(s_2 - s_1) \cdot 2(C^2 + \sigma^2)} + \tilde{C}|s_2 - s_1|
\]

(5.2.14)

Here (5.2.13) is by Lemma 5.2.1, and (5.2.14) is by (5.2.12). Therefore, \( \tilde{V}_1 \) is continuous in \( s \). Since \( \tilde{V}(x, p, s) = \tilde{V}_1(e^x, p, s) \), this implies that \( \tilde{V} \) is continuous in \( s \). \( \Box \)

**Remark:** The bound given by the previous lemma is independent of the choice of \( x \). Therefore, the two previous lemmata combine to give that \( \tilde{V} \) is jointly continuous in \( (x, p, s) \) by the triangle inequality.

**Lemma 5.2.6.** \( \tilde{V}(x, p, s) \) has at most polynomial growth.

**Proof.** Using the definition of the value function yields

\[
|\tilde{V}(x, p, s)| \leq E\left( e^{X_T} - q e^{\gamma(T-s)} \right)_+ \leq E\left[ e^{X_T} \right] = E\left[ |S_T| \right].
\]
By Itô’s Lemma, (5.1.1) implies that $S_t$ can be written as

$$S_t = x \exp \left( \int_0^t \sum_{i=1}^m \mu_i p_i(\xi) d\xi + \sigma W_t - \frac{\sigma^2}{2} t \right).$$

Applying this formula gives

$$E[|S_T|] = E \left[ |x| \exp \left( \int_0^T \sum_{i=1}^m \mu_i p_i(\xi) d\xi + \sigma W_T - \frac{\sigma^2}{2} T \right) \right] = |x| \exp \left( -\frac{T \sigma^2}{2} \right) E \left[ \exp \left( \int_0^T \sum_{i=1}^m \mu_i p_i(\xi) d\xi + \sigma W_T \right) \right]. \quad (5.2.15)$$

Since $\mu_i$ is constant for each $i$, taking $M \equiv \max_i |\mu_i|$ gives $\mu_i p_i(\xi) \leq M$ for all $\xi$. Applying this bound in (5.2.15) gives

$$|x| \exp \left( -\frac{T \sigma^2}{2} \right) E \left[ \exp \left( \int_0^T \sum_{i=1}^m \mu_i p_i(\xi) d\xi + \sigma W_T \right) \right] \leq |x| \exp \left( -\frac{\sigma^2}{2} (T + mMT) \right) E \left[ \exp (\sigma W_T) \right] \leq |x| C_1 \leq C_1 (1 + |x|)$$

with $C_1 \equiv \exp \left( -\frac{\sigma^2}{2} T + mMT \right) E \left[ \exp (\sigma W_T) \right]$. Therefore, $\tilde{V}(x, p, s)$ has at most linear, a fortiori, polynomial growth. □

Lemma 5.2.7. $\tilde{V}(x, p, s)$ is a viscosity supersolution to (5.1.6).

Proof. Continuity and polynomial growth have already been established. Suppose $\psi \in C^2(\mathbb{R} \times [0, 1] \times [0, T])$ such that $\tilde{V}(x, p, s) - \psi(x, p, s)$ has a local minimum at $(x, p, s) = (x_0, p_0, s_0)$ in a neighborhood $N(x_0, p_0, s_0)$. Let

$$\Psi(x, p, s) \equiv \psi(x, p, s) + \tilde{V}(x_0, p_0, s_0) - \psi(x_0, p_0, s_0).$$
By continuity, we can choose $\theta \in [s_0, T]$ such that $(X_t, P_t, t)$ starts at $(s_0, p_0, x_0)$ and $(X_t, P_t, t) \subseteq N(x_0, p_0, s_0)$ for all $t \in [s_0, \theta]$. Then, by Dynkin’s Formula, we have

$$
\tilde{E}^{x_0, p_0, s_0} \left[ e^{-r(\theta - s_0)} \Psi(X_\theta, P_\theta, \theta) \right] \\
= \tilde{E}^{x_0, p_0, s_0} \int_{s_0}^{\theta} e^{-r(\theta - s_0)} \left[ \frac{\partial \Psi(X_t, P_t, t)}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \Psi(X_t, P_t, t)}{\partial x} \right. \\
- P Q \frac{\partial^2 \Psi(X_t, P_t, t)}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 \Psi(X_t, P_t, t)}{\partial x^2} \\
+ 2 P A_t \frac{\partial^2 \Psi(X_t, P_t, t)}{\partial p \partial x} + tr \{ A_t P^T P A_t \frac{\partial^2 \Psi(X_t, P_t, t)}{\partial p^2} \} \right] dt.
$$

Since $(t, P_t, X_t) \in N(x_0, p_0, s_0)$ for all $t \in [s_0, \theta]$ and $\tilde{V}(x, p, s) - \psi(x, p, s)$ has a local minimum at $(x, p, s) = (x_0, p_0, s_0)$, we have

$$
\tilde{V}(X_t, P_t, t) - \psi(X_t, P_t, t) \geq \tilde{V}(x_0, p_0, s_0) - \psi(x_0, p_0, s_0).
$$

Thus,

$$
\tilde{V}(X_t, P_t, t) \geq \tilde{V}(x_0, p_0, s_0) - \psi(x_0, p_0, s_0) + \psi(X_t, P_t, t) \equiv \Psi(X_t, P_t, t).
$$

Note from the definition of $\Psi$ that

$$
\frac{\partial \Psi(X_t, P_t, t)}{\partial t} = \frac{\partial \psi(X_t, P_t, t)}{\partial t}, \\
\frac{\partial \Psi(X_t, P_t, t)}{\partial x} = \frac{\partial \psi(X_t, P_t, t)}{\partial x}, \\
\frac{\partial^2 \Psi(X_t, P_t, t)}{\partial x^2} = \frac{\partial^2 \psi(X(t), P(t), t)}{\partial x^2}.
$$

Hence, by substituting into (5.2.16), we obtain

$$
\tilde{E}^{x_0, p_0, s_0} \left[ e^{-r(\theta - s_0)} \Psi(X_\theta, P_\theta, \theta) \right] \\
\geq \tilde{E}^{x_0, p_0, s_0} \int_{s_0}^{\theta} e^{-r(\theta - s_0)} \left[ \frac{\partial \psi(X_t, P_t, t)}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \psi(X_t, P_t, t)}{\partial x} \right. \\
- P Q \frac{\partial \psi(X_t, P_t, t)}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 \psi(X_t, P_t, t)}{\partial x^2} \\
+ 2 P A_t \frac{\partial \psi(X_t, P_t, t)}{\partial p \partial x} + tr \{ A_t P^T P A_t \frac{\partial^2 \psi(X_t, P_t, t)}{\partial p^2} \} \right] dt.
$$

(5.2.17)
Then applying Lemma 5.2.2 along with inequality (5.2.17) gives

\[
0 = \bar{E}^{0,p,x} \left[ e^{-r\theta} \tilde{V}(X_\theta, P\theta, \theta) \right] - \tilde{V}(x, p, 0)
\]

\[
\geq \bar{E}^{x_0,p_0,0} \int_{s_0}^{\theta} e^{-r(\theta-s_0)} \left[ \frac{\partial \psi(X_t, P_t, t)}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \psi(X_t, P_t, t)}{\partial x} ight.
\]

\[
- PQ \frac{\partial \psi(X_t, P_t, t)}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 \psi(X_t, P_t, t)}{\partial x^2}
\]

\[
+ 2 PA_t \frac{\partial^2 \psi(X_t, P_t, t)}{\partial p \partial x} + \text{tr} \left\{ A_t P^T P A_t \frac{\partial^2 \psi(X_t, P_t, t)}{\partial p^2} \right\} dt.
\]

Multiplying both sides by \( \frac{1}{\theta} < 0 \) and sending \( \theta \downarrow s_0 \) gives

\[
0 \leq - \frac{\partial \psi(X_t, P_t, t)}{\partial t} - \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \psi(X_t, P_t, t)}{\partial x}
\]

\[
+ PQ \frac{\partial \psi(X_t, P_t, t)}{\partial p} - \frac{\sigma^2}{2} \frac{\partial^2 \psi(X_t, P_t, t)}{\partial x^2}
\]

\[
- 2 PA_t \frac{\partial^2 \psi(X_t, P_t, t)}{\partial p \partial x} - \text{tr} \left\{ A_t P^T P A_t \frac{\partial^2 \psi(X_t, P_t, t)}{\partial p^2} \right\}.
\]

Therefore, \( \tilde{V}(x, p, s) \) is a viscosity supersolution to (5.1.6). \[\square\]

**Lemma 5.2.8.** \( \tilde{V}(x, p, s) \) is a viscosity subsolution to (5.1.6).

**Proof.** Continuity and polynomial growth have already been established. Suppose \( \phi \in C^2(\mathbb{R} \times [0, 1] \times [0, T]) \) such that \( \tilde{V}(x, p, s) - \phi(x, p, s) \) has a local maximum at \( (x, p, s) = (x_0, p_0, s_0) \) in a neighborhood \( N(x_0, p_0, s_0) \). By continuity, we can choose \( \theta \in [s_0, T] \) such that \( (X_t, P_t, t) \) starts at \( (x_0, p_0, s_0) \) and \( \{(X_t, P_t, t)\} \subseteq N(x_0, p_0, s_0) \) for all \( t \in [s_0, \theta] \). By adjusting \( \phi(x, p, s) \) by a constant, we can assume without loss of generality that

\[
\tilde{V}(x_0, p_0, s_0) - \phi(x_0, p_0, s_0) = 0 \tag{5.2.18}
\]
so that \( \tilde{V}(X_t, P_t, t) \leq \phi(X_t, P_t, t) \) for any \( t \in [s_0, \theta] \). Then, by Dynkin’s Formula, we have

\[
\tilde{E}^{x_0, p_0, s_0} \left[ e^{-r(t-s_0)} \tilde{V}(X_{s_0}, P_{s_0}) \right] 
\leq \phi(x_0, p_0, s_0) + \tilde{E}^{x_0, p_0, s_0} \int_{s_0}^{\theta} e^{-r(t-s_0)} \left[ \frac{\partial \phi(X_t, P_t, t)}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \phi(X_t, P_t, t)}{\partial x} \right. \\
- PQ \frac{\partial \phi(X_t, P_t, t)}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 \phi(X_t, P_t, t)}{\partial x^2} \\
\left. + 2PA_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p \partial x} + tr\{A_t P^T P A_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p^2} \} \right] dt.
\] (5.2.19)

Then applying Lemma 5.2.2 along with (5.2.18) gives

\[
0 = \tilde{E}^{x_0, p_0, s_0} \left[ e^{-r(s_0-s_0)} \tilde{V}(X_{s_0}, P_{s_0}, s_0) \right] - \tilde{V}(x_0, p_0, s_0) 
\leq \tilde{E}^{x_0, p_0, s_0} \int_{s_0}^{\theta} e^{-r(t-s_0)} \left[ \frac{\partial \phi(X_t, P_t, t)}{\partial t} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \phi(X_t, P_t, t)}{\partial x} \right. \\
- PQ \frac{\partial \phi(X_t, P_t, t)}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 \phi(X_t, P_t, t)}{\partial x^2} \\
\left. + 2PA_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p \partial x} + tr\{A_t P^T P A_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p^2} \} \right] dt.
\]

Multiplying both sides by \( \frac{1}{\theta} < 0 \) and sending \( \theta \downarrow s_0 \) gives

\[
0 \geq - \frac{\partial \phi(X_t, P_t, t)}{\partial t} - \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial \phi(X_t, P_t, t)}{\partial x} \\
+ PQ \frac{\partial \phi(X_t, P_t, t)}{\partial p} - \frac{\sigma^2}{2} \frac{\partial^2 \phi(X_t, P_t, t)}{\partial x^2} \\
- 2PA_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p \partial x} - tr\{A_t P^T P A_t \frac{\partial^2 \phi(X_t, P_t, t)}{\partial p^2} \}.
\]

Therefore, \( \tilde{V}(x, p, s) \) is a viscosity subsolution to (5.1.6). \[\square\]

The two previous lemmata combine to give the following:

**Theorem 5.2.9.** \( \tilde{V}(x, p, s) \) is a viscosity solution to (5.1.6).
5.3 Uniqueness

For this section, we will work with the related PDE

\[ 0 = rV(x, p, s) - \frac{\partial V}{\partial t} - \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} + PQ \frac{\partial V}{\partial p} - \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} \]

\[-2PA_i \frac{\partial^2 V}{\partial p \partial x} - \text{tr}\{A_i P^T P A_i \frac{\partial^2 V}{\partial p^2}\} \]  

(5.3.1)

subject to terminal condition

\[ V(x, p, T) = (e^x - qe^{\gamma T})_+. \]

In the notation from [21], Appendix 4, we choose \(\rho = r\).

The concepts of parabolic superjet and subjet defined next are similar to those defined in the previous section, but we will restate the definitions here due to some additional complexities and notation.

**Definition 7.** Let \(S(k)\) denote the set of real \(k \times k\) symmetric matrices, and let \(xp \equiv (x, p) \in \mathbb{R}^{m+1}\). Given a function \(f : \mathbb{R} \times [0, 1]^m \times [s, T] \to \mathbb{R}\), we define the parabolic superjet of \(f(x, p, s)\) to be

\[ J^{2,+} f(\tilde{x}, \tilde{p}, s) \equiv \{(p_0, xp_0, M) \in \mathbb{R} \times \mathbb{R}^{m+1} \times S(m + 1) : f(x, p, t) \leq f(\tilde{x}, \tilde{p}, s) + p_0(t - s) \]

\[ + q_0 \cdot (\tilde{x}p - xp) + \frac{1}{2} M(\tilde{x}p - xp) \cdot (\tilde{x}p - xp) + o(\|\tilde{x}p - xp\|^2) \text{ as } (\tilde{x}, \tilde{p}, t) \to (x, p, s)\}. \]

The closure of the parabolic superjet is defined by

\[ \bar{J}^{2,+} f(x, p, s) \equiv \{(p_0, xp_0, M) \in \mathbb{R} \times \mathbb{R}^{m+1} \times S(m + 1) : \lim_{n \to \infty} (p_n^0, x p_n, M_n) = (p_0, xp_0, M) \]

with \((p_n^0, x p_n, M_n) \in J^{2,+} f(x_n, p_n, s_n)\)

and \(\lim_{n \to \infty} (x_n, p_n, s_n, f(x_n, p_n, s_n)) = (x, p, s, f(x, p, s))\).\}

The parabolic subjet of \(f(x, p, s)\) is defined by

\[ J^{2,-} f(x, p, s) \equiv -J^{2,+}(-f)(x, p, s), \]
and its closure is defined by

\[ J^{2,-} f(x, p, s) \equiv -J^{2,+}(-f)(x, p, s). \]

The main background theorems from [5] now state as follows:

**Theorem 5.3.1.** Let \( \Omega_1, \Omega_2 \) be locally compact subsets of \( \mathbb{R}^{m+1} \), and define \( \Omega \equiv \Omega_1 \times \Omega_2 \). For \( i = 1, 2 \), let \( u_i \) be upper semicontinuous in \( \Omega_i \times [0, T] \), \( J^{2,+}_{\Omega_i} u_i(x, p, t) \) the parabolic superjet of \( u_i(x, p, t) \), and \( \phi \) be a twice continuously differentiable function in a neighborhood of \( \Omega \times [0, T] \). For \( (x_1, p_1, x_2, p_2, t) \in \Omega \times [0, T] \), define

\[ w(x_1, p_1, x_2, p_2, t) \equiv u_1(x_1, p_1, t) + u_2(x_2, p_2, t). \]

Assume \( \hat{x} \equiv (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \hat{t}) \) is a local minimum of \( w - \phi \) relative to \( \Omega \times [0, T] \). Further assume that there exists \( R > 0 \) such that, for every \( K > 0 \), there exists \( C \) such that, for \( i = 1, 2 \), we have

\[ b_i \leq C \text{ whenever } (b_i, q_i^0, M_i^0) \in J^{2,+}_{\Omega_i} u_i(x_i, p_i, t), \]

\[ |x_i - \hat{x}_i| + |t - \hat{t}| \leq R, \text{ and} \]

\[ |u_i(x_i, t)| + |q_i| + \|M\| \leq K. \]

Then, for each \( \epsilon > 0 \), there exist symmetric \( m + 1 \times m + 1 \) real matrices \( X_1, X_2 \) such that the following are true:

1. \( (b_i, D_{x_i} \phi(\hat{x}_i, \hat{p}_i, \hat{t}), M) \in J^{2,+}_{\Omega_i} u_i(\hat{x}_i, \hat{p}_i, \hat{t}) \) for \( i = 1, 2 \);

2. \(-\left(\frac{1}{\epsilon} + \|D^2 \phi(\hat{x})\|\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2 \phi(\hat{x}) + \epsilon(D^2 \phi(\hat{x}))^2;\)

3. \( b_1 + b_2 = \frac{\partial \phi(\hat{x})}{\partial \hat{t}}. \)

**Lemma 5.3.2.** The set \( J^{2,+} f(x, p, s) \) (resp. \( J^{2,-} f(x, p, s) \)) consists of the set of

\[ \left(\frac{\partial \phi(x, p, s)}{\partial s}, D\phi(x, p, s), D^2 \phi(x, p, s)\right) \]

where \( \phi \) is twice continuously differentiable on \( \mathbb{R} \times \mathbb{R}^m \times [0, T] \) and \( f - \phi \) has a global maximum (resp. minimum) at \( (x, p, s) \).
We now state and prove the main result of this section.

**Theorem 5.3.3.** If $V_1(x, p, t)$ and $V_2(x, p, t)$ are viscosity solutions to (5.3.1) with at most linear growth, i.e. for $i = 1, 2$, there exist constants $C_i$ such that

$$V_i(x, p, t) \leq C_i(1 + |x|) \text{ for } (x, p, t) \in \mathbb{R} \times [0, 1]^m \times [0, T],$$

then

$$V_1(x, p, t) \leq V_2(x, p, t) \text{ for all } (x, p, t) \in \mathbb{R} \times [0, 1]^m \times [0, T].$$

(5.3.2)

**Proof.** Again following the idea of Ishii in [12], given $0 < \delta < 1$ and $0 < \eta < 1$, define

$$\Phi(x, y, p, q, t) \equiv V_1(x, p, t) - V_2(y, q, t) - \frac{1}{\delta} |x - y|^2 - \eta e^{(T-t)}(x^2 + y^2)$$

(5.3.3)

and

$$\phi(x, y, t) \equiv \frac{1}{\delta} |x - y|^2 + \eta e^{(T-t)}(x^2 + y^2).$$

Since $V_1(x, p, t)$ and $V_2(y, q, t)$ have at most linear growth, the (negative) quadratic terms in (5.3.3) will dominate $V_1(x, p, t) - V_2(y, q, t)$ as $|x|, |y| \to \infty$; hence

$$\lim_{|x| + |y| \to \infty} \Phi(x, y, p, q, t) = -\infty.$$  

(5.3.4)

Since $\Phi(x, y, p, q, t)$ is continuous and $p, q \in [0, 1]^m$, a compact set, (5.3.4) implies that there exists a point $(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta)$ such that $\Phi(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta)$ is a global maximum of $\Phi$. Since

$$\Phi(x_\delta, x_\delta, p_\delta, p_\delta, t_\delta) + \Phi(y_\delta, y_\delta, q_\delta, q_\delta, t_\delta) \leq 2\Phi(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta),$$

we have by definition of $\Phi$

$$V_1(x_\delta, p_\delta, t_\delta) - V_2(x_\delta, p_\delta, t_\delta) - 2\eta e^{(T-t_\delta)} x_\delta^2 + V_1(y_\delta, q_\delta, t_\delta) - V_2(y_\delta, q_\delta, t_\delta) - 2\eta e^{(T-t_\delta)} y_\delta^2$$

$$\leq 2V_1(x_\delta, p_\delta, t_\delta) - 2V_2(y_\delta, q_\delta, t_\delta) - \frac{2}{\delta} |x_\delta - y_\delta|^2 - 2\eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).$$

This implies

$$-V_2(x_\delta, p_\delta, t_\delta) - 2\eta e^{(T-t_\delta)} x_\delta^2 + V_1(y_\delta, q_\delta, t_\delta) - 2\eta e^{(T-t_\delta)} y_\delta^2$$

$$\leq 2V_1(x_\delta, p_\delta, t_\delta) - 2V_2(y_\delta, q_\delta, t_\delta) - \frac{2}{\delta} |x_\delta - y_\delta|^2 - 2\eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).$$
\[ \leq V_1(x_\delta, p_\delta, t_\delta) - V_2(y_\delta, q_\delta, t_\delta) - \frac{2}{\delta} |x_\delta - y_\delta|^2 - 2\eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2), \]

and hence
\[ \frac{2}{\delta} |x_\delta - y_\delta|^2 \leq (V_1(x_\delta, p_\delta, t_\delta) - V_1(y_\delta, q_\delta, t_\delta)) + (V_2(x_\delta, p_\delta, t_\delta) - V_2(y_\delta, q_\delta, t_\delta)). \quad (5.3.5) \]

Since \( V_1 \) and \( V_2 \) have at most linear growth, (5.3.5) implies that there exists a constant \( K_1 \) such that
\[ \frac{2}{\delta} |x_\delta - y_\delta|^2 \leq K_1 (1 + |x_\delta| + |y_\delta|). \]

Thus, taking \( K \equiv 2K_1 \), we have
\[ |x_\delta - y_\delta|^2 \leq \delta K (1 + |x_\delta| + |y_\delta|). \quad (5.3.6) \]

Since \( \Phi(0, 0, p, q, s) \leq \Phi(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta) \) and \( |\Phi(0, 0, p, q, s)| \leq K (1 + |x_\delta| + |y_\delta|) \), we have by definition of \( \Phi \)
\[ \eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) \leq V_1(x_\delta, p_\delta, t_\delta) - V_2(x_\delta, q_\delta, t_\delta) - \frac{1}{\delta} |x_\delta - y_\delta|^2 - \Phi(0, 0, p, q, s) \]
\[ \leq 3K (1 + |x_\delta| + |y_\delta|). \quad (5.3.7) \]

Dividing gives
\[ \frac{\eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2)}{(1 + |x_\delta| + |y_\delta|)} \leq 3K. \]

Since \( x^2 + y^2 \) grows faster that \( x + y \) as \( x, y \to \infty \), this implies that there exists a constant \( C_\eta \) such that
\[ |x_\delta| + |y_\delta| \leq C_\eta \quad \text{and} \quad t_\delta \in [s, T]. \quad (5.3.8) \]

Thus, \( \{x_\delta : \delta > 0\} \) and \( \{y_\delta : \delta > 0\} \) are uniformly bounded with respect to \( \delta \). By compactness, \( \{p_\delta : \delta > 0\} \) and \( \{q_\delta : \delta > 0\} \) are uniformly bounded with respect to \( \delta \). This boundedness implies that we can find convergent subsequences, which we will denote \((x_\delta)_\delta, (y_\delta)_\delta, (p_\delta)_\delta, (q_\delta)_\delta, (t_\delta)_\delta\). Moreover, by (5.3.6), there exist points \( x_0 \) and \( t_0 \) such that
\[ \lim_{\delta \to 0} x_\delta = x_0 = \lim_{\delta \to 0} y_\delta \quad \text{and} \quad \lim_{\delta \to 0} t_\delta = t_0. \]
These limits, (5.3.7), and (5.3.8) imply that
\[
\lim_{\delta \to 0} |x_\delta - y_\delta|^2 = 0. \tag{5.3.9}
\]

Since \( \Phi \) has a global maximum at \((x_\delta, y_\delta, p_\delta, q_\delta, t_\delta)\), part 1 of Theorem 5.3.1 implies that, given \( \varepsilon > 0 \), there exist \( b_{1\delta}, b_{2\delta}, M^{X(\delta)}, \) and \( M^{Y(\delta)} \) such that
\[
(b_{1\delta}, \left( \frac{2}{\delta} (x_\delta - y_\delta) + 2\eta e^{(T-t_\delta)x_\delta}, 0, ..., 0 \right), M^{X(\delta)}) \in \mathcal{J}^2_+ V_1(x_\delta, p_\delta, t_\delta) \tag{5.3.10}
\]
and
\[
(-b_{2\delta}, \left( -\frac{2}{\delta} (x_\delta - y_\delta) + 2\eta e^{(T-t_\delta) y_\delta}, 0, ..., 0 \right), -M^{Y(\delta)}) \in \mathcal{J}^2_+ - V_2(y_\delta, p_\delta, t_\delta). \tag{5.3.11}
\]

By definition of parabolic subject, (5.3.11) implies that
\[
(b_{2\delta}, \left( -\frac{2}{\delta} (x_\delta - y_\delta) + 2\eta e^{(T-t_\delta) y_\delta}, 0, ..., 0 \right), M^{Y(\delta)}) \in \mathcal{J}^2_2 V_2(y_\delta, p_\delta, t_\delta). \tag{5.3.12}
\]

By definition of viscosity solution and by Lemma 5.3.2, (5.3.10) implies that
\[
rV_1(x_\delta, p_\delta, t_\delta) + b_{1\delta} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \left( \frac{2}{\delta} (x_\delta - y_\delta) + 2\eta e^{(T-t_\delta)x_\delta} \right)
\]
\[-\frac{\sigma^2}{2} M^{X(\delta)} - 2 PA_t \cdot (M^{X(\delta)})_{2 \leq i \leq m+1} - tr\{A_t P^T A_t (M^{X(\delta)})_{2 \leq i, j \leq m+1} \} \leq 0.
\]

Similarly, (5.3.12) implies that
\[
rV_2(y_\delta, q_\delta, t_\delta) + b_{2\delta} + \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \left( \frac{2}{\delta} (x_\delta - y_\delta) + 2\eta e^{(T-t_\delta) y_\delta} \right)
\]
\[-\frac{\sigma^2}{2} M^{Y(\delta)} - 2 P_t A_t \cdot (M^{Y(\delta)})_{2 \leq i \leq m+1} - tr\{A_t P^T A_t (M^{Y(\delta)})_{2 \leq i, j \leq m+1} \} \geq 0.
\]

Combining these inequalities gives
\[
r(V_1(y_\delta, p_\delta, t_\delta) - V_2(x_\delta, q_\delta, t_\delta)) \leq \frac{\sigma^2}{2} \left( M^{X(\delta)} - M^{Y(\delta)} \right) + 2 PA_t \cdot \left( M^{X(\delta)} - M^{Y(\delta)} \right)_{2 \leq i \leq m+1}
\]
\[+ tr\{A_t P^T A_t (M^{X(\delta)} - M^{Y(\delta)})_{2 \leq i, j \leq m+1} \}
\]
\[+ 2\eta \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) e^{(T-t_\delta)} (x_\delta - y_\delta) + b_{1\delta} - b_{2\delta}.
\]
By boundedness of \( b_{1\delta} - b_{2\delta} = \frac{\partial \phi(x_\delta, y_\delta, t_\delta)}{\partial t} = -\eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) \).

The right equality is by direct computation from the definition of \( \phi \). Substituting gives

\[
r(V_1(y_\delta, p_\delta, t_\delta) - V_2(x_\delta, q_\delta, t_\delta)) \leq \frac{\sigma^2}{2} \left( M_{11}^{X(\delta)} - M_{11}^{Y(\delta)} \right) + 2PA_t \cdot \left( M_{1i}^{X(\delta)} - M_{1i}^{Y(\delta)} \right)_{2 \leq i \leq m+1}
+ tr\{ A_t P^T P A_t \left( M_{ij}^{X(\delta)} - M_{ij}^{Y(\delta)} \right)_{2 \leq i,j \leq m+1} \}
+ 2\eta \left( \sum_{i=1}^{m} \mu_i \rho_i(t) - \frac{\sigma^2}{2} \right) e^{(T-t_\delta)}(x_\delta - y_\delta) + \eta e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).
\] \hspace{1cm} (5.3.13)

By boundedness of \( (x_\delta^2 + y_\delta^2) \) and \( (x_\delta - y_\delta) \), letting \( \eta \to 0 \) in (5.3.13) gives

\[
r(V_1(y_\delta, p_\delta, t_\delta) - V_2(x_\delta, q_\delta, t_\delta)) \leq \frac{\sigma^2}{2} \left( M_{11}^{X(\delta)} - M_{11}^{Y(\delta)} \right) + 2PA_t \cdot \left( M_{1i}^{X(\delta)} - M_{1i}^{Y(\delta)} \right)_{2 \leq i \leq m+1}
+ tr\{ A_t P^T P A_t \left( M_{ij}^{X(\delta)} - M_{ij}^{Y(\delta)} \right)_{2 \leq i,j \leq m+1} \}.
\]

By boundedness of \( A_t \) and \( P_t \), we can find a constant \( \nu \) such that

\[
\frac{\sigma^2}{2} \left( M_{11}^{X(\delta)} - M_{11}^{Y(\delta)} \right) + 2PA_t \cdot \left( M_{1i}^{X(\delta)} - M_{1i}^{Y(\delta)} \right)_{2 \leq i \leq m+1}
+ tr\{ A_t P^T P A_t \left( M_{ij}^{X(\delta)} - M_{ij}^{Y(\delta)} \right)_{2 \leq i,j \leq m+1} \}
\leq \nu \sum_{i,j} \left( M_{ij}^{X(\delta)} - M_{ij}^{Y(\delta)} \right).
\] \hspace{1cm} (5.3.14)

By part 2 of Theorem 5.3.1, we have

\[
- \left( \frac{1}{\epsilon} + \| D^2_{(x,y)} \phi(x_\delta, y_\delta, t_\delta) \| \right) I \leq \begin{pmatrix} M^{X(\delta)} & 0 \\ 0 & -M^{Y(\delta)} \end{pmatrix}
\leq D^2_{(x,y)} \phi(x_\delta, y_\delta, t_\delta) + \epsilon(D^2 \phi(x_\delta, y_\delta, t_\delta))^2.
\]

In this expression, the variables \( p_\delta \) and \( q_\delta \) are suppressed because they do not appear in the definition of \( \phi \). By direct calculation, letting \( \eta \to 0 \), we have

\[
D^2_{(x,y)} \phi(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta) = \frac{2}{\delta} \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}
\]
where
\[ Z \equiv \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \]

Thus,
\[
\sum_{i,j} \left( M_{i,j}^{X(\delta)} - M_{i,j}^{Y(\delta)} \right)
\]
\[
= (1, 1, ..., 1) \begin{pmatrix} M^{X(\delta)} & 0 \\ 0 & -M^{Y(\delta)} \end{pmatrix} (1, 1, ..., 1)^T
\]
\[
\leq (1, 1, ..., 1) \left( \frac{2}{\delta} \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} + \epsilon \frac{8}{\delta^2} \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right) (1, 1, 1)^T
\]
\[
= (1, 1, ..., 1) \left( \left( \frac{2}{\delta} + \epsilon \frac{8}{\delta^2} \right) \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right) (1, 1, ..., 1)^T
\]

Taking \( \epsilon = \delta \) gives
\[
(1, 1, ..., 1) \left( \frac{10}{\delta} \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right) (1, 1, ..., 1)^T = 0.
\]

Thus,
\[
\sum_{i,j} \left( M_{i,j}^{X(\delta)} - M_{i,j}^{Y(\delta)} \right) \leq 0.
\]

Applying this inequality to the right side of (5.3.14) gives
\[
r(V_1(x_0, p_0, t_0) - V_2(y_0, q_0, t_0)) \leq 0 \quad (5.3.15)
\]

Since \( \Phi \) has a global maximum at \((x_\delta, y_\delta, p_\delta, q_\delta, t_\delta)\), we know that for all \( t \in [s, T] \), \( p \in [0, 1]^m \), and \( x \in \mathbb{R} \)
\[
\Phi(x, x, p, p, t) \leq \Phi(x_\delta, y_\delta, p_\delta, q_\delta, t_\delta).
\]

By definition of \( \Phi \), this gives for all \( t \in [s, T], p \in [0, 1]^m, \) and \( x \in \mathbb{R} \)
\[
V_1(x, p, t) - V_2(x, p, t) - 2\eta e^{(T-t)}x^2 \leq V_1(x_\delta, p_\delta, t_\delta) - V_2(y_\delta, q_\delta, t_\delta) - \frac{1}{\delta} |x_\delta - y_\delta|^2 - \eta e^{(T-t)}(x_\delta^2 + y_\delta^2).
\]

Letting \( \delta \to 0 \) and \( \eta \to 0 \) gives, via (5.3.9) and (5.3.14),
\[
V_1(x, p, t) - V_2(x, p, t) \leq V_1(x_0, p_0, t_0) - V_2(y_0, q_0, t_0) \leq 0.
\]
Thus,

\[ V_1(x, p, t) \leq V_2(x, p, t) \]

as desired. \hfill \Box

**Corollary 5.3.4.** Equation (5.3.1) has a unique viscosity solution with at most linear growth.

**Proof.** The solutions \( V_1(x, p, t) \) and \( V_2(x, p, t) \) in the previous theorem were arbitrary viscosity solutions to (5.3.1) with at most linear growth. \hfill \Box

### 5.4 Markov Chain Approximation

In this section, we will derive a computationally feasible Markov chain approximation to the viscosity solution determined in the previous section and prove the convergence of the approximation. We will use Kushner’s Markov chain method as described in [13]. This method has been used in numerous other applications; for this method applied to a portfolio optimization problem, see [24].

Kushner’s approach is necessary because equation (5.3.1) with covariance matrix

\[ \Sigma = \frac{A_t P}{\sigma} \]

may be degenerate, and hence a finite difference approximation may not converge. For example, take \( m = 2 \). Then with \( \hat{\alpha} \equiv \sum_{i=1}^{2} \mu_i p_i(t) \), we have

\[ \Sigma = \begin{pmatrix} \frac{1}{\sigma} (\mu_1 - \hat{\alpha}) p_1 \\ \frac{1}{\sigma} (\mu_2 - \hat{\alpha}) p_2 \end{pmatrix} . \]

Thus,

\[ \Sigma \Sigma^T = \begin{pmatrix} \sigma^2 & (\mu_1 - \hat{\alpha}) p_1 & (\mu_2 - \hat{\alpha}) p_2 \\ (\mu_1 - \hat{\alpha}) p_1 & \frac{1}{\sigma^2} (\mu_1 - \hat{\alpha})^2 p_1^2 & \frac{1}{\sigma^2} (\mu_1 - \hat{\alpha}) (\mu_2 - \hat{\alpha}) p_1 p_2 \\ (\mu_2 - \hat{\alpha}) p_2 & \frac{1}{\sigma^2} (\mu_1 - \hat{\alpha}) (\mu_2 - \hat{\alpha}) p_1 p_2 & \frac{1}{\sigma^2} (\mu_2 - \hat{\alpha})^2 p_2^2 \end{pmatrix} . \]
Consequently, for $\Sigma \Sigma^T$ to be diagonally dominant, we would need

$$\frac{1}{\sigma^2} (\mu_1 - \hat{\alpha})^2 p_1^2 \ge \frac{1}{\sigma^2} (\mu_1 - \hat{\alpha}) (\mu_2 - \hat{\alpha}) p_1 p_2$$

and

$$\frac{1}{\sigma^2} (\mu_2 - \hat{\alpha})^2 p_2^2 \ge \frac{1}{\sigma^2} (\mu_1 - \hat{\alpha}) (\mu_2 - \hat{\alpha}) p_1 p_2.$$ 

Canceling terms reduces these inequalities to

$$(\mu_1 - \hat{\alpha}) p_1 \ge (\mu_2 - \hat{\alpha}) p_2$$

and

$$(\mu_2 - \hat{\alpha}) p_2 \ge (\mu_1 - \hat{\alpha}) p_1,$$

thus implying $(\mu_2 - \hat{\alpha}) p_2 = (\mu_1 - \hat{\alpha}) p_1$. This will not be true if, for example, $p_1 = p_2 = \frac{1}{2}$, $\mu_1 = 4$, and $\mu_2 = 2$. In these cases, a finite difference scheme is not guaranteed to converge.

To begin constructing the Markov chain approximation, define $\mathbf{C} \equiv (\mathbf{C}, \mathbf{C})$ so that we can rewrite (5.1.2) and (5.1.5) as

$$dY_t = f(t, Y_t) dt + \Sigma(t, Y_t) d\nu_t. \quad (5.4.1)$$

For $h > 0$, we will denote the approximating chain $Y_{t, h}$. Next, the approximating Markov chain must satisfy Kushner’s local consistency conditions, i.e. the approximating chain should have local properties that are consistent with the original chain. In particular, we need

$$E_{x,p,n}^h \Delta Y_n^h = f(t, Y_t) \Delta t^h(Y) + o\left(\Delta t^h(Y)\right) \quad (5.4.2)$$

and

$$COV_{x,p,n}^h \Delta Y_n^h = \Sigma(t, Y_t) \Sigma(t, Y_t)^T \Delta t^h(Y) + o\left(\Delta t^h(Y)\right). \quad (5.4.3)$$

Since the noise covariance matrix is degenerate, the part of the transitions of an approximating Markov chain which approximates the effects of the noise will move the chain in the $\Sigma(s, Y)$ or $-\Sigma(t, Y)$ direction. Thus, the state space $S_h$ consisting of the approximating chains needs to possess the following properties:
1. if \( \gamma \in S_h \), then \( \gamma + h\Sigma(s, \gamma) \in S_h \) and \( \gamma - h\Sigma(s, \gamma) \in S_h \);

2. if \( \gamma \in S_h \), then \( \gamma + e_i h \in S_h \) and \( \gamma - e_i h \in S_h \) for \( i \in \{1, ..., m + 1\} \).

The strategy is to find a transition probability \( P^h(Y, Z) \) and time step function \( \Delta t^h(Y) \) for an approximating chain \( Y^h_t \) that satisfies (5.4.2) and (5.4.3) and then compute the value function \( V(x, p, s) \) for this approximating chain. Note that in this context the phrase “transition probability” refers to transitions from one approximating chain to another, not transitions between states in a chain \( Y^h_t \) for a fixed \( h \). We will execute this strategy by approximating the pieces \( dY_t = f(t, Y_t)dt \) and \( dY_t = \Sigma(t, Y_t)dv_t \) individually and then combining these approximations to obtain an approximation for the original chain \( Y_t \).

- For \( dY_t = \Sigma(t, Y_t)dv_t \), consider the transition probabilities

\[
P_1^h(Y, Y + h\Sigma(s, \gamma)) = P_1^h(Y, Y - h\Sigma(s, \gamma)) = \frac{1}{2}.
\]

Under these transition probabilities, the covariance of the state transition is

\[
\sum_{Z}(Z - Y)(Z - Y)^TP_1^h(Y, Z) = \frac{1}{2}(Y + h\Sigma(s, Y) - Y)(Y + h\Sigma(s, Y) - Y)^T
\]

\[
+ \frac{1}{2}(Y - h\Sigma(s, Y) - Y)(Y - h\Sigma(s, Y) - Y)^T
\]

\[
= h^2\Sigma(s, Y)\Sigma(s, Y)^T.
\]

Thus, defining the interpolation interval by \( \Delta t_1^h(Y) = h^2 \), the local consistency conditions are satisfied for this piece.

- For \( dY_t = f(t, Y_t)dt \) and \( i \in \{1, ..., m + 1\} \), consider the transition probabilities

\[
P_2^h(Y, Y + e_i h) = f_i^+(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)}
\]

and

\[
P_2^h(Y, Y - e_i h) = f_i^-(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)}
\]
where \( f^+ \equiv \max\{f, 0\} \) and \( f^- \equiv \max\{-f, 0\} \). Under these transition probabilities, the expectation of the state transition is

\[
\sum_Z (Z - Y) P_2^h(Y, Z) = \sum_{i=1}^{m+1} (Y + e_i h - Y) f_i^+(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)} \\
+ \sum_{i=1}^{m+1} (Y - e_i h - Y) f_i^-(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)}
\]

\[
= h f(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)}.
\]

Moreover, the covariance of the state transition is

\[
\sum_Z (Z - Y)(Z - Y)^T P_2^h(Y, Z) = h^2 \sum_{i=1}^{m+1} e_i e_i^T f_i^+(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)} \\
+ h^2 \sum_{i=1}^{m+1} e_i e_i^T f_i^-(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)}
\]

\[
= h^2 f(t, Y) \times \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y)} \\
= \sigma \left( \frac{h}{\sum_{i=1}^{m+1} f_i(t, Y)} \right).
\]

Thus, defining the interpolation interval by \( \Delta t_2^h(Y) = \frac{h}{\sum_{i=1}^{m+1} f_i(t, Y)} \), the local consistency conditions are satisfied for this piece.

Let \( \tilde{Q}^h(Y) \equiv 1 + h \left( \sum_{i=1}^{m+1} |f_i(t, Y)| \right) \). Combining the two parts above gives the following transition probabilities:

\[
P^h(Y, Y + h \Sigma(s, Y)) = P^h(Y, Y - h \Sigma(s, Y)) = \frac{1}{2 \tilde{Q}^h(Y)}
\]

\[
P^h(Y, Y + e_i h) = f_i^+(t, Y) \frac{h}{\tilde{Q}^h(Y)}
\]

\[
P^h(Y, Y - e_i h) = f_i^-(t, Y) \frac{h}{\tilde{Q}^h(Y)}.
\] (5.4.4)

These transition probabilities will satisfy Kushner’s local consistency conditions with

\[
\Delta t^h(Y) = \frac{h^2}{\tilde{Q}^h(Y)},
\]
but they do not give a constant interpolation interval $\Delta t^h$. Thus, let

$$Q^h \equiv \sup_p \tilde{Q}^h(Y),$$

$$\Delta t^h \equiv \frac{h^2}{Q^h},$$

and instead consider the following adjusted transition probabilities:

$$P^h(Y, Y + h\Sigma(s, Y)) = P^h(Y, Y - h\Sigma(s, Y)) = \frac{1}{2Q^h}$$

$$P^h(Y, Y + e_i h) = f^+_i(t, Y) \frac{h}{Q^h}$$

$$P^h(Y, Y - e_i h) = f^-_i(t, Y) \frac{h}{Q^h}. \quad (5.4.5)$$

**Lemma 5.4.1.** The interpolation interval $\Delta t^h$ and transition probabilities $P^h$ as defined in (5.4.5) satisfy Kushner’s local consistency conditions.

**Proof.** The proof is by computation. For the expectation, we have

$$\sum_{Z \in S_h} (Z - Y) P^h(Y, Z) = \sum_{i=1}^{m+1} he_i \frac{h f^+_i}{Q^h} - \sum_{i=1}^{m+1} he_i \frac{h f^-_i}{Q^h} + h\Sigma - \frac{h\Sigma}{2Q^h}$$

$$= \frac{h^2}{Q^h} \left( \sum_{i=1}^{m+1} e_i f^+_i - \sum_{i=1}^{m+1} e_i f^-_i \right)$$

$$= \frac{h^2}{Q^h} f(t, Y).$$

For the covariance, we have

$$\sum_{Z \in S_h} (Z - Y)(Z - Y)^T P^h(Z, Y) = \frac{h^2}{Q^h} \sum_{i=1}^{m+1} e_i e_i^T f^+_i(t, Y) - \frac{h^2}{Q^h} \sum_{i=1}^{m+1} e_i e_i^T f^-_i(t, Y)$$

$$+ h^2 \Sigma(s, Y) \Sigma(s, Y)^T \frac{1}{2Q^h} + h^2 \Sigma(s, Y) \Sigma(s, Y)^T \frac{1}{2Q^h}$$

$$= o(\Delta t) + \frac{h^2}{Q^h} \Sigma(s, Y) \Sigma(s, Y)^T.$$
We now turn to approximating the value function. By Lemma 5.2.2 with \( s = s - \Delta t^h \) and \( \theta = s \), we have

\[
\tilde{V}(Y_0, s - \Delta t^h) = E \left[ e^{-r\Delta t^h} \tilde{V}(Y(s), s) \big| Y(s - \Delta t^h) = Y_0 \right].
\]

The Markov chain approximation to the value function \( V^h \) should have the same property:

\[
\tilde{V}^h(Y_0, s - \Delta t^h) = E \left[ e^{-r\Delta t^h} \tilde{V}^h(Y(s), s) \big| Y(s - \Delta t^h) = Y_0 \right].
\]

The numerical scheme for the value function is

\[
\tilde{V}^h(Y, s - \Delta t^h) = \sum_{Z \in S_h} P^h(Y, Z)\tilde{V}^h(Z, s). \tag{5.4.6}
\]

Let \( \Upsilon_h(\phi)(Y) \equiv \sum_{Z \in S_h} P^h(Y, Z)\phi(Z) \) so that

\[
\tilde{V}^h(Y_0, s) = \Upsilon_h \left( \tilde{V}^h(\cdot, s + \Delta t^h) \right) (Y), Y \in S_h
\]

with terminal condition \( \tilde{V}^h(x, p, T) = g(X_T, T) \). Let

\[
H(s, V, DV, D^2V) \equiv \left( \sum_{i=1}^{m} \mu_i p_i(t) - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} - PQ \frac{\partial V}{\partial p} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
+ 2PA_i \frac{\partial^2 V}{\partial p \partial x} + tr \{ A_i P^T P A_i \frac{\partial^2 V}{\partial p^2} \}.
\]

**Lemma 5.4.2.** The function \( \Upsilon_h(\phi) \) has the following properties:

1. **(monotonicity)** If \( \phi_1 \leq \phi_2 \), then \( \Upsilon_h(\phi_1) \leq \Upsilon_h(\phi_2) \).

2. **(stability)** If \( 0 < h < 1 \), then there exists a solution \( v^h \) to the computation scheme and a constant \( K \) such that \( \|v^h\| \leq K \).

3. **(consistency)** If \( w \in C^{2,1}(\mathbb{R}^{m+1}) \), then

\[
\lim_{(q,t,h) \to (p,s,0)} \frac{1}{h} \left[ \Upsilon_h(\cdot, w(t + h))(q) - w(t, q) \right] = \frac{\partial w}{\partial s} + H(s, Y, DY, D^2Y).
\]
Proof. Monotonicity is clear because $P^h(Y, Z) \geq 0$. For stability, note that for any sufficiently smooth $\phi_1$ and $\phi_2$, we have
\[
\| \Upsilon_h(\phi_1)(Y) - \Upsilon_h(\phi_2)(Y) \| = \| \sum_{Z \in S_h} P^h(Y, Z) [\phi_1(Z) - \phi_2(Z)] \|
\leq \| \phi_1 - \phi_2 \| \sum_{Z \in S_h} P^h(Y, Z)
= \| \phi_1 - \phi_2 \|.
\]
Thus, $\Upsilon_h$ is a contraction mapping, and so the fixed point $v^h$ of $\Upsilon_h$ is a solution of (5.4.6). This proves stability.

For consistency, let $w \in C^{2,1}(\mathbb{R}^{m+1})$. For such $w$, we have
\[
\lim_{(q, t, h) \to (p, s, 0)} \frac{1}{h} \left[ \Upsilon_h(\cdot, w(t + h))(q) - w(t, q) \right]
= \frac{1}{h} \left[ \sum_{Z \in S_h} P^h(q, Z) w(t + h, Z) - w(t, q) \right]
= \frac{1}{h} \left[ \sum_{Z \in S_h} P^h(q, Z) (w(t + h, Z) - w(t + h, q)) + w(t + h, q) - w(t, q) \right]
= H(s, Y, DY, D^2Y) + \frac{\partial w}{\partial s}.
\]
This proves consistency. □

The properties proved in the previous lemma allow us to apply the Barles-Sougandis method from [1] to prove convergence. Let
\[
\tilde{V}^*(Y_0, s) \equiv \limsup_{(q, t, h) \to (p, s, 0)} \tilde{V}^h(t, Z)
\]
and
\[
\tilde{V}_*(Y_0, s) \equiv \liminf_{(q, t, h) \to (p, s, 0)} \tilde{V}^h(t, Z).
\]

**Theorem 5.4.3.** $\tilde{V}^*$ is a viscosity subsolution for (5.1.6) and $\tilde{V}_*$ is a viscosity supersolution for (5.1.6).
Proof. To verify the definition of viscosity subsolution, let $\phi \in C^{2,1}(\mathbb{R}^{m+1})$ such that $\tilde{V}^* - \phi$ has a local maximum at $(s, Y)$. By definition of $\tilde{V}^*$, choose a sequence $v^h$ such that $v^h - \phi$ has a local maximum at $(t^h, Y^h)$ and $(t^h, Y^h) \to (s, Y)$ as $h \downarrow 0$. In this case, we have

$$v^h(t^h, Y^h) - \phi(t^h, Y^h) \geq v^h(t^h + h, Y^h) - \phi(t^h + h, Y^h),$$

which implies

$$\phi(t^h + h, Y^h) - \phi(t^h, Y^h) \leq v^h(t^h + h, Y^h) - v^h(t^h, Y^h).$$

Applying monotonicity gives

$$\Upsilon(\phi(t^h + h, \cdot))(Y^h) - \phi(t^h, Y^h) \geq \Upsilon(v^h(t^h + h, \cdot))(Y^h) - v^h(t^h, Y^h). \tag{5.4.7}$$

Since $v^h$ is a fixed point of $\Upsilon$, the right side of (5.4.7) is zero. Thus, dividing through by $h > 0$ and letting $h \downarrow 0$ gives

$$\frac{\partial \phi}{\partial s} + H(s, \phi, D\phi, D^2\phi) \geq 0.$$

Therefore, $\tilde{V}^*$ is a viscosity subsolution for (5.1.6).

Similarly, to verify the definition of viscosity supersolution, let $\psi \in C^{2,1}(\mathbb{R}^{m+1})$ such that $\tilde{V}^* - \psi$ has a local minimum at $(s, Y)$. By definition of $\tilde{V}^*$, choose a sequence $v^h$ such that $v^h - \psi$ has a local minimum at $(t^h, Y^h)$ and $(t^h, Y^h) \to (s, Y)$ as $h \downarrow 0$. In this case, we have

$$v^h(t^h, Y^h) - \psi(t^h, Y^h) \leq v^h(t^h + h, Y^h) - \psi(t^h + h, Y^h),$$

which implies

$$\psi(t^h + h, Y^h) - \psi(t^h, Y^h) \geq v^h(t^h + h, Y^h) - v^h(t^h, Y^h).$$

Applying monotonicity gives

$$\Upsilon(\psi(t^h + h, \cdot))(Y^h) - \psi(t^h, Y^h) \leq \Upsilon(V^h(t^h + h, \cdot))(Y^h) - V^h(t^h, Y^h). \tag{5.4.8}$$

Since $V^h$ is a fixed point of $\Upsilon$, the right side of (5.4.8) is zero. Thus, dividing through by $h > 0$ and letting $h \downarrow 0$ gives

$$\frac{\partial \psi}{\partial s} + H(s, \psi, D\psi, D^2\psi) \leq 0.$$

Therefore, $\tilde{V}^*$ is a viscosity supersolution for (5.1.6).
**Corollary 5.4.4.** As $h \downarrow 0$, $V^h$ converges locally uniformly to the unique viscosity solution of (5.1.6).

**Proof.** Applying the previous theorem and Theorem 5.3.3 gives

$$\bar{V}^* \leq \tilde{V} \leq \tilde{V}_s \leq \tilde{V}^*.$$ 

Thus,

$$\lim_{(q,t,h) \to (p,s,0)} \tilde{V}^h(t, Z) = \tilde{V}(x, p, s).$$

\[\square\]

### 5.5 Numerical Examples

In this section, we will present numerical examples that illustrate and implement the approximation scheme for the value function given by (5.4.6). These examples assume the parameters given in Table 5.1. The results can be seen in Figures 5.1 through 5.3. Figure 5.1 shows the surface generated by allowing $x$ and $s$ to vary while holding $p$ constant at $p = 0.25$. The value of the loan increases as $x$ increases. Because $T = 0.4$ is constant, the length of the loan will increase as we take smaller values for $s$. Thus, the loan value also increases as the length of the loan increases. These results are consistent with the loan value for a stock price obeying a geometric Brownian motion with positive drift as shown in earlier chapters.

Figure 5.2 shows the surface generated by allowing $p$ and $s$ to vary while holding $x$ constant at $x = 2.5$, which implies $S_0 = e^{2.5} \approx 12.2$. As expected, the loan value increases as $s$ decreases, again implying the length of the loan increases. The value function generally decreases as $p$ increases. This trend makes sense in the case when $\mu_2 > \mu_1$, which is the case for the parameters in these examples. An apparent lack of smoothness appears at $p = 0.5$. Notice that the effect is more pronounced for small values of $s$ than for larger ones. This effect is probably due to computational constraints relating to step size. Recall that the time step $\Delta t^h$ is of order $h^2$. Due to the approximation scheme (5.4.6), the value functions are computed backward in time, and the number of intermediate value functions needed to
Table 5.1: Parameters for Kushner Approximation

<table>
<thead>
<tr>
<th>variable</th>
<th>default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>3</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
</tr>
<tr>
<td>$T$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>2</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>4</td>
</tr>
</tbody>
</table>

compute the value function at time $s = 0$ grows exponentially as the number of time steps increases. In light of these constraints, these figures were produced taking $h = 0.2$. This lack of smoothness would likely disappear if $h$ could be taken sufficiently small.

Figure 5.3 shows the surface generated by allowing $p$ and $s$ to vary while holding $s$ constant at $s = 0.2$. As expected, the loan value increases as $x$ increases. The same apparent lack of smoothness at $p = 0.5$ seen in the previous figure can be seen here as well, likely with the same source.
Figure 5.1: Markov Chain Approximation to Value Function: x and s vary.
Figure 5.2: Markov Chain Approximation to Value Function: p and s vary.
Figure 5.3: Markov Chain Approximation to Value Function: $p$ and $x$ vary.


