Regime switching market models and applications

by

Moustapha N. Pemy

(Under the direction of Qing Zhang)

Abstract

A regime switching model consists of a set of Black-Scholes models (geometric Brownian motions) coupled by a finite state Markov chain. This model is considered as one of the effective mathematical frameworks to study the valuation of stocks and their derivatives. Under this model the associated PDEs satisfied by the option price are quite involved. In the European option case we have a linear system of PDEs; and in the American option case the corresponding PDE is fully nonlinear. Both equations are difficult to solve, and they may not have classical solutions. In this work, we use the framework of viscosity solution to prove that in both cases the option price can be characterized as a unique viscosity solution of those PDEs. This enables us to construct a numerical scheme to approximate the option price. In addition, this framework is used to treat stock selling rule and search for an optimal selling strategy in order to maximize the reward resulted from a selling transaction.

Index words: Optimal stopping, Regime switching, Viscosity solution

Regime switching market models and applications

by

Moustapha N. Pemy

B.S., The University of Yaounde, 1996

M.S., The University of Yaounde, 1998

D.E.A., The University of Yaounde, 1999

Diploma in Math, International Centre of Theoretical Physics, 2001

A Thesis Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2005

Regime switching market models and applications

by

Moustapha N. Pemy

Approved:

Major Professor: Qing Zhang

Committee: Dhandapani Kannan
Paul Wenston
Andrew Sornborger
Jingzhi Tie

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2005

Dedication

To my parents, brothers, and sisters.

To my wife Fathya, and my son Malik.

Acknowledgments

My deepest gratitude goes to my advisor Professor Qing Zhang for his guidance, advices, and all helpful conversations we had. I feel very lucky to be one of his student. I hope to live up to his expectation. I am also grateful to the committee members, Professor Dhandapani Kannan, Professor Paul Wenston, Professor Andrew Sornborger, and Professor Jingzhi Tie for their helpful suggestions and corrections.

I would also like to thank anyone who helped me during these four years in graduate school.

Finally, I would like to thank Fathya, who stood by me throughout this memorable period of my life.

Table of Contents

Page

Acknowledgments v

List of Figures vii

Chapter

1 **Valuation of the European option under regime switching** 1

1.1 **Viscosity solution** 9

1.2 **Uniqueness** 16

2 **Valuation of American option under regime switching** 24

2.1 **Optimal stopping of a switching diffusion** 25

2.2 **Valuation of American options** 39

3 **Numerical methods** 70

3.1 **Infinite time horizon American option** 70

3.2 **Finite time horizon American options** 80

4 **Optimal stock liquidation under regime switching model with finite time horizon** 95

4.1 **Problem formulation** 97

4.2 **Properties of value functions** 98

4.3 **A Numerical example** 101

4.4 **Future research problems** 106

Bibliography 107

List of Figures

3.1 Perpetual American call option 77
3.2 Perpetual American put option 79
3.3 American call option surfaces 88
3.4 Free boundary graphs 89
3.5 Domain of continuation for the first and second state when we fix t=0 90
3.6 American put option surfaces 92
3.7 Free boundary graphs 93
3.8 Domain of continuation for the first and second state when we fix t=0 94
4.1 IBM: 10/28/2002–08/28/2004 102

Chapter 1

Valuation of the European option under regime switching

Introduction

There has been a tremendous interest in valuation of stock options since their introduction in an organized exchange in seventies. Options as other derivative securities are typically used as a hedging tool by traders in order to reduce their exposure and to protect their portfolio. A *derivative security* is a financial contract whose value is derived from another security such as a stock or a bond. Common derivative securities are *call options*, *put options*, *forward contracts* and *futures contracts* etc.

A *call option* gives the holder the right to buy the underlying asset by a prespecified date for a prespecified price. And a *put option* gives the holder the right to sell the underlying asset by a prespecified date for a prespecified price. The prespecified price in the contract is also known as *strike price* or *exercise price* and the prespecified date is also know as the *expiration date*. *American option* can be exercised any time up to the expiration date, whereas *European option* can only be exercised at expiration date.

There has been a great deal of interest in using mathematical models to study financial derivatives. The major breakthrough occurred in 1973 when F. Black and M. Scholes proposed a model based on geometric Brownian motion with deterministic coefficients such as the rate of return and the volatility. Their model gives a reasonably good description of the market, and also leads to a closed-form formula for evaluating the European option price. Since then, the Black-Scholes model has been widely used in option pricing and portfolio management. However, as noticed by many researchers, it has serious flaws and discrepancies due to its insensibility to random parameter changes such as changes in market trends. In order to circumvent those limitations, various modifications of the model have been proposed in the literature. Merton [30] proposed a model based on diffusions with pure jumps in order to capture stock price discontinuity; Clark [5] studied time-changing Brownian motions; Praetz [34] proposed a hyperbolic model in lieu of the traditional log-normal distribution. Fouque, Papanicolaou, and Sircar [14], Hull [18], and Musiela and Rutkowski [31] studied stochastic volatility models in order to capture random changes of the volatility.

With the constant need to build more realistic models that better reflect the random change of the market environment and that are mathematically tractable, the geometric Brownian motion with regime-switching has been introduced. The regime-switching model was first introduced by Hamilton [17] in 1989 to describe time-series. Roughly speaking, in the regime-switching model stock parameters depend on the trend of the market that switches among a finite number of states. The market regime reflects the state of the economy, the general mood of investor and other major economic factors. Due to the effectiveness of this model, there have been an extensive literature on the regime switching. Di Masi *et al.* [7] develop mean-variance hedging for regime-switching European option pricing. In order to evaluate regime-switching American and European options, Bollen [3] uses lattice method and simulation, whereas Buffington and Elliot [4] use risk neutral pricing and derive a set of partial differential equations for option price. Duan *et al.* [8] establish a class of GARCH option models under regime switching. Yao, Zhang and Zhou [40] establish that the regime switching model captures the volatility smile and the term structure.

In the regime switching models, it is extremely difficult to obtain a closed-form solution for option price. In addition, there is no guarantee that the associated PDEs have classical (smooth) solutions. In order to study those PDEs we have used the concept of viscosity solution which is convenient for treating possible non differential solutions. The regime-switching European option price can be characterized as the unique viscosity solution of a system of linear partial differential equations with variable coefficients.

We consider the regime switching model that consists of a set of geometrical Brownian motions coupled by a finite-state Markov chain. Let $X(t)$ be the stock price satisfying the following equation:

$$\mathrm{d}X(t) = X(t)[\mu(\alpha(t))\mathrm{d}t + \sigma(\alpha(t))\mathrm{d}W(t)], \quad s \le t \le T, \qquad X(s) = x, \tag{1.1}$$

where $\alpha(t)$ is a finite state Markov chain, $\alpha(t) \in \mathcal{M} = \{1, 2, 3, ..., n\}$ with generator Q, $W(t)$ is the standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $W(\cdot)$ and $\alpha(t)$ are independent. Given $\alpha(t) = i$, $\mu(i)$ and $\sigma(i)$ are known parameters. Let $\mathcal{F}_t = \sigma\{\alpha(s), W(s); s \le t\}$ and $r > 0$ be the risk-free rate. Under the given probability space $(\Omega, \mathcal{F}, P)$ the discounted option price $e^{-rt}X(t)$ is not a martingale. This creates a possibility of arbitrage. To circumvent this problem we define an equivalent probability measure $\widetilde{P}$ the risk neutral probability measure. Using the Girsanov Theorem one is able to prove that under the new probability space the discounted price $e^{-rt}X(t)$ becomes a martingale. For more about these results, one is referred to Fouque *et al.* [14] or Yao *et al.* [40]. The space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \widetilde{P})$ defines the risk-neutral world. The price of the European option at time s with the stock price $X(s) = x$, and the state of the Markov chain $\alpha(s) = i$ is defined as follows:

$$p(s, x, i) = \widetilde{E}[e^{-r(T-s)}g(X(T), \alpha(T)) \mid X(s) = x, \alpha(s) = i] \tag{1.2}$$

where g is the payoff function. In fact $g(x, i) = (x - K)^+ = \max(x - K, 0)$ for the call option and $g(x, i) = (K - x)^+ = \max(K - x, 0)$ for the put option, and K is the strike price, and T is the expiration time.

Remark 1.0.1 In the sequel we will use the following notation,

$$\widetilde{E}^{s,x,i}[\xi(X(t), \alpha(t))] = \widetilde{E}[\xi(X(t), \alpha(t)) \mid X(s) = x, \alpha(s) = i].$$

First of all, note that from equation (1.1) we deduce that

$$X(t) = X(s) \exp\left(\int_s^t \left(\mu(\alpha(\xi)) - \frac{1}{2}\sigma(\alpha(\xi))^2\right) \mathrm{d}\xi + \int_s^t \sigma(\alpha(\xi))\mathrm{d}W(\xi)\right). \tag{1.3}$$

This implies that $(X(t), \alpha(t))$ is Markovian, i.e, for any bounded Borel function h we have

$$E[h(X(t), \alpha(t)) \mid \mathcal{F}_s] = E[h(X(t), \alpha(t)) \mid X(s), \alpha(s)] \qquad s < t.$$

Lemma 1.0.2 *For any $\theta \in [s, T]$, we have*

$$p(s, x, \alpha_s) = \widetilde{E}^{s,x_s,\alpha_s}[e^{-r(\theta-s)}p(\theta, X(\theta), \alpha(\theta)]. \tag{1.4}$$

Proof. Let $\theta \in [s, T]$. Note that

$$\begin{aligned} p(\theta, X(\theta), \alpha(\theta)) &= \widetilde{E}^{\theta,X(\theta),\alpha(\theta)}[e^{-r(T-\theta)}g(X(T), \alpha(T)] \\ &= \widetilde{E}[e^{-r(T-\theta)}g(X(T), \alpha(T)) \mid X(\theta), \alpha(\theta)] \\ &= \widetilde{E}[e^{-r(T-\theta)}g(X(T), \alpha(T)) \mid \mathcal{F}_\theta]. \end{aligned} \tag{1.5}$$

In view of this, we have

$$\begin{aligned} &\widetilde{E}^{s,x_s,\alpha_s}[e^{-r(\theta-s)}p(\theta, X(\theta), \alpha(\theta))] \\ = &\ \widetilde{E}^{s,x_s,\alpha_s}[e^{-r(\theta-s)}\widetilde{E}[e^{-r(T-\theta)}g(X(T), \alpha(T)) \mid \mathcal{F}_\theta]] \\ = &\ \widetilde{E}[e^{-r(\theta-s)}\widetilde{E}[e^{-r(T-\theta)}g(X(T), \alpha(T)) \mid \mathcal{F}_\theta] \mid \mathcal{F}_s] \\ = &\ \widetilde{E}[e^{-r(\theta-s)-r(T-\theta)}g(X(T), \alpha(T)) \mid \mathcal{F}_s] \\ = &\ \widetilde{E}[e^{-r(T-s)}g(X(T), \alpha(T)) \mid \mathcal{F}_s] \\ = &\ \widetilde{E}[e^{-r(T-s)}g(X(T), \alpha(T)) \mid X(s) = x_s, \alpha(s) = \alpha_s] \\ = &\ p(s, x_s, \alpha_s). \end{aligned} \tag{1.6}$$

This proves the lemma. □

Formally $p(s, x, i)$ satisfies the following system of partial differential equations.

$$\begin{cases} r\Big(p(s,x,i) - x\dfrac{\partial p(s,x,i)}{\partial x}\Big) - \dfrac{\partial p(s,x,i)}{\partial s} - \dfrac{1}{2}x^2\sigma(i)^2\dfrac{\partial^2 p(s,x,i)}{\partial x^2} - Qp(s,x,\cdot)(i) = 0, \\ p(T,x,i) = g(x,i) \qquad \text{for} \qquad i = 1, ..., m. \end{cases} \tag{1.7}$$

Note that g is piecewise linear. Equation (1.7) may not have a smooth solution. To treat the possible non-differentiability of the solution , we resort to a weak form of solution, viscosity solution. For instance, we will prove that p is a viscosity solution of (1.7).

Lemma 1.0.3 *For each $i \in \mathcal{M}$, p(s,x,i) is continuous in (s, x) and has at most a polynomial growth.*

Proof. We only prove for the case of the European put option, the case of the European call option is similar. In this case,

$$p(s, x, i) = \widetilde{E}[e^{-r(T-s)}(K - X(T))^+ \mid X(s) = x, \alpha(s) = i].$$

Define $g(s, x) = e^{-r(T-s)}(x - K)^+$. We have

$$\begin{aligned} |g(s, x) - g(s', x')| &= |e^{-r(T-s)}(x - K)^+ - e^{-r(T-s')}(x - K)^+| \\ &\leq |e^{-r(T-s)}(x - K)^+ - e^{-r(T-s)}(x' - K)^+| \\ &\quad + |e^{-r(T-s)}(x' - K)^+ - e^{-r(T-s')}(x' - K)^+|. \end{aligned} \tag{1.8}$$

Using the mean value theorem for the function $h(s) = e^{-r(T-s)}$ on the interval $[0, T]$, we obtain the existence of a constant $C' > 0$ such that,

$$|g(s, x) - g(s', x')| \leq |x - x'| + KC'|s - s'|. \tag{1.9}$$

Let $C = KC'$, then we have

$$|g(s, x) - g(s', x')| \leq |x - x'| + C|s - s'|.$$

To show the continuity of $p(s, x, i)$ in x, let X_1 and X_2 be two solutions of (1.1) with initial values $X_1(s) = x_1$ and $X_2(s) = x_2$ respectively. We have

$$\begin{aligned} (X_1(t) - X_2(t))^2 = \Big(&(x_1 - x_2) + \int_s^t (X_1(\xi) - X_2(\xi))\, \mu(\alpha(\xi)) d\xi \\ &+ \int_s^t (X_1(\xi) - X_2(\xi))\, \sigma(\alpha(\xi)) dW_\xi \Big)^2. \end{aligned}$$

For this first part of the proof we assume that all expectations are taken under the condition that $X_1(s) = x_1$, $X_2(s) = x_2$, and $\alpha(s) = i$. Thus for any random variable ζ we denote,

$$\widetilde{E}[\zeta] = \widetilde{E}[\zeta \mid X_1(s) = x_1, X_2(s) = x_2, \alpha(s) = i].$$

Consequently, we have

$$\widetilde{E}\left(X_1(t)-X_2(t)\right)^2 \leq 3\widetilde{E} \mid x_1-x_2 \mid^2 +3\widetilde{E}\left(\int_s^t (X_1(\xi)-X_2(\xi))\mu(\alpha(\xi))d\xi\right)^2$$
$$+3\widetilde{E}\left(\int_s^t \left(X_1(\xi)-X_2(\xi)\right)\sigma(\alpha(\xi))dW_\xi\right)^2.$$

Using the Ito isometry, we obtain

$$\widetilde{E}\left(X_1(t)-X_2(t)\right)^2 \leq 3\widetilde{E} \mid x_1-x_2 \mid^2 +3\widetilde{E}(t-s)\int_s^t \Big((X_1(\xi)-X_2(\xi))\mu(\alpha(\xi)\Big)^2 d\xi$$
$$+3\widetilde{E}\int_s^t \Big(\left(X_1(\xi)-X_2(\xi)\right)\sigma(\alpha(\xi))\Big)^2 d\xi.$$

Since μ and σ are bounded, then there exists C such that,

$$\widetilde{E} \mid X_1(t)-X_2(t) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 +C(1+t)\int_s^t \widetilde{E} \mid X_1(\xi)-X_2(\xi) \mid^2 d\xi$$

Then for $t=T$, we have

$$\widetilde{E} \mid X_1(T)-X_2(T) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 +C(1+T)\int_s^T \widetilde{E} \mid X_1(\xi)-X_2(\xi) \mid^2 d\xi.$$

We set $D=C(1+T)$. By Gronwall's inequality, we have

$$\widetilde{E} \mid X_1(T)-X_2(T) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 e^{DT}.$$

Note that,

$$\widetilde{E} \mid X_1(t)-X_2(t) \mid \leq \left(\widetilde{E} \mid X_1(t)-X_2(t) \mid^2\right)^{\frac{1}{2}} \text{ for all } t\in[s,T],$$

and it follows that

$$\widetilde{E}e^{-r(T-s)} \mid X_1(T)-X_2(T) \mid \leq 3 \mid x_1-x_2 \mid e^{DT}. \tag{1.10}$$

Moreover, we have

$$\begin{aligned}
p(s,x_1,i)-p(s,x_2,i) &= \widetilde{E}\Big[e^{-r(T-s)}\Big((K-X_1(T))^+-(K-X_2(T))^+\Big)\Big] \\
&\leq \widetilde{E}[|g(s,X_1(T))-g(s,X_2(T))|] \\
&\leq \widetilde{E}\Big[|X_1(T)-X_2(T)|\Big] \\
&\leq 3 \mid x_1-x_2 \mid e^{DT}.
\end{aligned} \tag{1.11}$$

Let $\epsilon > 0$, for $\mid x_1 - x_2 \mid \leq \frac{\epsilon}{3} e^{-DT}$ we have,

$$\mid p(s, x_1, i) - p(s, x_2, i) \mid \leq \epsilon.$$

This proves that $p(s, x, i)$ is continuous with respect to x.

We now show the continuity of $p(s, x, i)$ with respect to s. Let $X(t)$ be the solution of (1.1) that starts at $t = s$ with $X(s) = x$ with $\alpha(s) = i$. Let $T \geq s' \geq s$, we define

$$\begin{cases} X'(t) = X(t - (s' - s)), \\ \alpha'(t) = \alpha(t - (s' - s)). \end{cases} \tag{1.12}$$

Let us consider the change of variables $u = t - (s' - s)$, thus we obtain $dt = du$ and $dW_t = dW_u$. Moreover,

$$X(t) = x + \int_s^t X(\xi)\mu(\alpha(\xi))d\xi + \int_s^t X(\xi)\sigma(\alpha(\xi))dW_\xi$$

and

$$X'(t) = x + \int_{s'}^t X'(\xi)\mu(\alpha'(\xi))d\xi + \int_{s'}^t X'(\xi)\sigma(\alpha'(\xi))dW_\xi.$$

With this in mind, we obtain

$$\begin{aligned} (X(t) - X'(t)) &= \int_s^t X(\xi)\mu(\alpha(\xi))d\xi + \int_s^t X(\xi)\sigma(\alpha(\xi))dW_\xi \\ &\quad - \int_{s'}^t X'(\xi)\mu(\alpha'(\xi))d\xi - \int_{s'}^t X'(\xi)\sigma(\alpha'(\xi))dW_\xi \\ &= \int_s^t X(\xi)\mu(\alpha(\xi))d\xi + \int_s^t X(\xi)\sigma(\alpha(\xi))dW_\xi \\ &\quad - \int_s^{t-(s'-s)} X(\xi)\mu(\alpha(\xi))d\xi - \int_s^{t-(s'-s)} X(\xi)\sigma(\alpha(\xi))dW_\xi \\ &= \int_{t-(s'-s)}^t X(\xi)\mu(\alpha(\xi))d\xi + \int_{t-(s'-s)}^t X(\xi)\sigma(\alpha(\xi))dW_\xi. \end{aligned} \tag{1.13}$$

For this second part of the proof we assume that all expectations are taken under the condition that $X(s) = x$, $X'(s') = x$, and $\alpha(s) = i = \alpha'(s)$. Thus for any random variable ζ we denote

$$\widetilde{E}[\zeta] = \widetilde{E}[\zeta \mid X(s) = x = X'(s'), \alpha(s) = i = \alpha'(s')].$$

Consequently, we have

$$\widetilde{E}(X(t)-X'(t))^2 \leq 2\widetilde{E}\Big(\int_{t-(s'-s)}^{t} X(\xi)\mu(\alpha(\xi))d\xi\Big)^2 + 2\widetilde{E}\Big(\int_{t-(s'-s)}^{t} X(\xi)\sigma(\alpha(\xi))dW_\xi\Big)^2.$$

Using Ito's isometry and the fact that μ and σ are bounded, there exists a constant $C > 0$ such that,

$$\widetilde{E}(X(t)-X'(t))^2 \leq 2(s'-s)C\widetilde{E}\int_{t-(s'-s)}^{t} |X(\xi)|^2 d\xi + 2C\widetilde{E}\int_{t-(s'-s)}^{t} |X(\xi)|^2 d\xi$$

In addition, by the existence and uniqueness theorem of solution of stochastic differential equation, and using the Fubini-Tonelli theorem we have,

$$\int_0^T \widetilde{E}|X(\xi)|^2 d\xi = \widetilde{E}\int_0^T |X(\xi)|^2 d\xi < +\infty.$$

Thus, there exists $M > 0$ such that $\widetilde{E}|X(\xi)|^2 < M$ almost everywhere in the interval $[0, T]$. Therefore $\widetilde{E}\int_{t-(s'-s)}^{t} |X(\xi)|^2 d\xi < (s'-s)M$, which implies that

$$\widetilde{E}(X(t)-X'(t))^2 \leq R(s'-s), \qquad \text{for some real number } R > 0. \tag{1.14}$$

Moreover, we have

$$\begin{aligned} |p(s,x,i)-p(s',x,i)| &\leq \widetilde{E}\Big[|e^{-r(T-s)}(K-X(T))^+ - e^{-r(T-s')}(K-X'(T))^+|\Big] \\ &\leq \widetilde{E}[|g(s,X(T))-g(s',X'(T))|] \\ &\leq \widetilde{E}[|X(T)-X'(T)|] + C|s-s'| \\ &\leq \sqrt{R(|s'-s)|} + C|s-s'| \\ &\leq \sqrt{|s'-s|}(\sqrt{R}+C\sqrt{s'-s}). \end{aligned} \tag{1.15}$$

The inequality (1.15) implies the continuity of $p(s,x,i)$ with respect to s.

Now let us prove that $p(s,x,i)$ has at most a polynomial growth. We note from the claim (1.9) that

$$|p(s_1,x_1,i)-p(s_2,x_2,i)| \leq \widetilde{E}|g(s_1,X_1(T))-g(s_2,X_2(T))|,$$

and there exist $R_1, R_2 > 0$, such that

$$\begin{aligned} |p(s_1, x_1, i) - p(s_2, x_2, i)| &\leq \widetilde{E}|g(s_1, X_1(T)) - g(s_2, X_2(T))| \\ &\leq \widetilde{E}|X_1(T) - X_2(T)| + R_2|s_1 - s_2| \\ &\leq R_1|x_1 - x_2| + R_2|s_1 - s_2| \qquad (1.16) \end{aligned}$$

And setting $x_1 = x$, $s_1 = s = s_2$ and $x_2 = 0$ we have,

$$X_2(t) = x_2 e^{\int_0^t (\mu(\alpha(s)) - \frac{1}{2}\sigma(\alpha(s))^2)\mathrm{d}s + \int_0^t \sigma(\alpha(s))\mathrm{d}W_s} = 0 \text{ for all } t.$$

Thus we have a constant $C > 0$ such that $p(s, 0, i) \leq K \leq C$, consequently we obtain,

$$|p(s, x, i)| \leq C|x| + |p(s, 0, i)| \leq C(1 + |x|).$$

This completes the proof of the lemma □

1.1 Viscosity solution

In order to study the possibility of existence and uniqueness of a solution of (1.7), we use a notion of weak solution, namely, the concept of viscosity solution introduced two decades ago by Crandall and Lions [6]. In fact, equation (1.7) is a linear system of second order partial differential equations with variable coefficients and a non-smooth boundary condition boundary. There is no guarantee for existence of classical solution. The framework of viscosity solution is more convenient for treating possible non-differential solutions of (1.7).

Definition 1.1.1 *Given $\mathcal{H} : \mathcal{M} \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We say that $f(s, x, i)$ is a viscosity solution of*

$$\mathcal{H}\Big(i, s, x, f(s, x, i), \frac{\partial f(s, x, i)}{\partial s}, \frac{\partial f(s, x, i)}{\partial x}, \frac{\partial^2 f(s, x, i)}{\partial x^2}\Big) = 0, \qquad (1.17)$$

$$\text{for } i \in \mathcal{M}, \ s \in [0, T], \ x \in \mathbb{R}.$$

If

1. *For each* $i \in \mathcal{M}$, $f(s,x,i)$ *is continuous in* (s,x) *and there exist constants* C *and* n *such that*

   $f(s,x,i) \leq C(1+ \mid x \mid^n)$.

2. *For each* $i \in \mathcal{M}$,

$$\mathcal{H}\Big(i, s_0, x_0, f(s_0,x_0,i), \frac{\partial \phi(s_0,x_0)}{\partial s}, \frac{\partial \phi}{\partial x}, \frac{\partial^2 \phi(s_0,x_0)}{\partial x^2}\Big) \leq 0 \tag{1.18}$$

   whenever $\phi(s,x) \in C^2$ *such that* $f(s,x,i) - \phi(s,x)$ *has local maximum at* $(s,x) = (s_0,x_0)$.

3. *And for each* $i \in \mathcal{M}$,

$$\mathcal{H}\Big(i, s_0, x_0, f(s_0,x_0,i), \frac{\partial \psi(s_0,x_0)}{\partial s}, \frac{\partial \psi(s_0,x_0)}{\partial x}, \frac{\partial^2 \psi(s_0,x_0)}{\partial x^2}\Big) \geq 0 \tag{1.19}$$

   whenever $\psi(s,x) \in C^2$ *such that* $f(s,x,i) - \psi(s,x)$ *has local minimum at* $(s,x) = (s_0,x_0)$.

Let f *be a function that satisfies (1.17). It is a viscosity subsolution (resp. supersolution) if it satisfies (1.18) (resp.(1.19)).*

Definition 1.1.2 *Let* $f(s,x,i) : [0,T] \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ *be a function. We define* $f^*(s,x,i) : [0,T] \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ *by,*

$$f^*(i,x,\alpha) = \limsup_{r \downarrow 0} \{f(t,y,i) : (t,y) \in B((s,x);r)\}$$

$f^*(s,x,i)$ *is called the upper semicontinuous envelop of* $f(s,x,i)$.
Similarly we define $f_*(s,x,i) : [0,T] \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ *the lower semicontinuous envelop of* $f(s,x,\alpha)$ *as follows*

$$f_*(s,x,i) = \liminf_{r \downarrow 0} \{f(t,y,i) : (t,y) \in B((s,x);r)\}$$

Remark 1.1.3 It is easy to show that, $f^*(s,x,i)$ is the smallest upper semicontinuous function such that $f(s,x,i) \leq f^*(s,x,i)$.
And $f_*(s,x,i)$ is the largest lower semicontinuous function such that $f_*(s,x,i) \leq f(s,x,i)$.

Definition 1.1.4 *Let $f(s,x,i):[0,T]\times\mathbb{R}\times\mathcal{M}\to\mathbb{R}$. Define the parabolic superjet by*

$$\mathcal{P}^{2,+}f(s,x,i)=\{(p,q,M)\in\mathbb{R}\times\mathbb{R}: f(t,y,i)\le f(s,x,i)+p(t-s)q(y-x) \\ +\tfrac{1}{2}(y-x)^2M+o(\mid y-x\mid^2) \\ as \qquad (t,y)\to(s,x)\}$$

and its closure is

$$\bar{\mathcal{P}}^{2,+}f(s,x,i)=\{(p,q,M)=\lim_{n\to\infty}(p_n,q_n,M_n) \\ with \qquad (p_n,q_n,M_n)\in\mathcal{P}^{2,+}f(s_n,x_n,i) \\ and \qquad \lim_{n\to\infty}(s_n,x_n,f(s_n,x_n,i))=(x,f(s,x,i))\}.$$

Similarly, we define the parabolic subjet $\mathcal{P}^{2,-}f(s,x,i)=-\mathcal{P}^{2,+}(-f)(s,x,i)$ and its closure $\bar{\mathcal{P}}^{2,-}f(s,x,i)=-\bar{\mathcal{P}}^{2,+}(-f)(s,x,i)$

We have the following result.

Lemma 1.1.5 *$\mathcal{P}^{2,+}f(s,x,i)$ (resp. $\mathcal{P}^{2,-}f(s,x,i)$) consist of the set of $(\frac{\partial\phi(s,x)}{\partial s},\frac{\partial\phi(s,x)}{\partial x},\frac{\partial^2\phi(s,x)}{\partial x^2})$ where $\phi\in\mathcal{C}^2([0,T]\times\mathbb{R})$ and $f-\phi$ has a global maximum (resp. minimum) at (s,x).*

A proof can be found in Fleming and Soner [13].

With this in mind, we have this equivalent formulation of the notion of viscosity solution.

Definition 1.1.6 *A function $u(s,x,i)$ continuous in (s,x) satisfying the polynomial growth condition is a viscosity solution of*

$$\mathcal{H}(i,s,x,u,\frac{\partial u(s,x,i)}{\partial s},\frac{\partial u(s,x,i)}{\partial x},\frac{\partial^2 u(s,x,i)}{\partial x^2})=0,$$

if

1. *for each $i\in\mathcal{M}$, for all $(s,x)\in[0,T]\times\mathbb{R}$, and $(a,p,M)\in\mathcal{P}^{2,+}u(s,x,i)$*

$$\mathcal{H}(i,s,x,u,a,p,M)\le 0, \text{ in this case } u \text{ is a viscosity subsolution,}$$

and

2. for each $i \in \mathcal{M}$, for all $(s,x) \in [0,T] \times \mathbb{R}$, and $(b,q,N) \in \mathcal{P}^{2,-}u(s,x,i)$

$$\mathcal{H}(i,s,x,u,b,q,N) \geq 0, \quad \textit{in this case } u \textit{ is a viscosity supersolution.}$$

Theorem 1.1.7 *The price of the European option is a viscosity solution to the system of equations in (1.7).*

Proof. Note that for $t = T$,

$$p(T,x,i) = \widetilde{E}[g(X(T),\alpha(T)) \mid X(T) = x, \alpha(T) = i] = g(x,i).$$

In the sequel, we use the notation $\widetilde{E}^{s,x,i}[\zeta(X)]$ to denote $\widetilde{E}[\zeta(X) \mid X(s) = x, \alpha(s) = i]$.
It suffices to show that $p(s,x,i)$ is a viscosity subsolution and supersolution.
Let $\alpha_s \in \mathcal{M}$. We want to show that

$$\begin{aligned} r\Big(p(s,x_s,\alpha_s) - x_s\frac{\partial \psi(s,x_s,\alpha_s)}{\partial x}\Big) - \frac{1}{2}x_s^2\sigma(\alpha_s)^2\frac{\partial^2\psi(s,x_s,\alpha_s)}{\partial x^2} \\ -\frac{\partial \psi(s,x_s,\alpha_s)}{\partial t} - Qp(s,x_s,\cdot)(\alpha_s) \geq 0 \end{aligned} \tag{1.20}$$

whenever $\psi \in \mathcal{C}^{1,2}([s,T] \times \mathbb{R}^+)$ and $p(t,x,\alpha_s) - \psi(t,x)$ has a local minimum at $(s,x_s) \in [s,T] \times \mathbb{R}^+$.

Let $\psi \in C^2([s,T] \times \mathbb{R}^+)$ and $(s,x) \in [s,T] \times \mathbb{R}^+$ such that $p(t,x,\alpha_s) - \psi(t,x)$ has a local minimum at (s,x_s) in a neighborhood $N(s,x_s)$. We define a function φ as follows:

$$\varphi(t,x,i) = \begin{cases} \psi(t,x) + p(s,x_s,\alpha_s) - \psi(s,x_s), & \text{if} \quad i = \alpha_s, \\ p(t,x,i), & \text{if} \quad i \neq \alpha_s. \end{cases} \tag{1.21}$$

Let γ be the first jump time of $\alpha(\cdot)$ from the state α_s, and let $\theta \in [s,\gamma]$ be such that $(t, X(t))$ starts at (s,x_s) and stays in $N(s,x_s)$ for $s \leq t \leq \theta$. Since $\theta \leq \gamma$ we have $\alpha(t) = \alpha_s$, for $s \leq t \leq \theta$. By Dynkin's formula, we have

$$\begin{aligned} &\widetilde{E}^{s,x_s,\alpha_s} e^{-r(\theta-s)}\varphi(\theta,X(\theta),\alpha_s) - \varphi(s,x_s,\alpha_s) \\ = \; &\widetilde{E}^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- r\varphi(t,X(t),\alpha_s) \\ &+\frac{\partial \varphi(t,X(t),\alpha_s))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\varphi(t,X(t),\alpha_s)}{\partial x^2} \\ &+rX(t)\frac{\partial \varphi(t,X(t),\alpha_s)}{\partial x} + Q\varphi(t,X(t),\cdot)(\alpha_s)\Big)dt. \end{aligned} \tag{1.22}$$

Recall that, $(t, X(t)) \in N(s, x_s)$ for $s \le t \le \theta$ and (s, x_s) is the minimum of $p(t, x, \alpha_s) - \psi(t, x)$. Then, for $s \le t \le \theta$, we have

$$p(t, X(t), \alpha_s) \ge \psi(t, X(t)) + p(s, x_s, \alpha_s) - \psi(s, x_s) = \varphi(t, X(t), \alpha_s). \tag{1.23}$$

Using equation (1.21) and (1.23), we have

$$\begin{aligned} & \widetilde{E}^{s,x_s,\alpha_s} e^{-r(\theta-s)} p(\theta, X(\theta), \alpha_s) - p(s, x_s, \alpha_s) \\ \ge\ & \widetilde{E}^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big(- rp(t, X(t), \alpha_s) \\ & + \frac{\partial \psi(t, X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(t, X(t))}{\partial x^2} \\ & + rX(t) \frac{\partial \psi(t, X(t))}{\partial x} + Q\varphi(t, X(t), \cdot)(\alpha_s) \Big) dt \end{aligned} \tag{1.24}$$

the inequality (1.23) can also be written in the following form

$$\psi(t, X(t)) \le p(t, X(t), \alpha_s) - (p(s, x_s, \alpha_s) - \psi(s, x_s)). \tag{1.25}$$

We recall that,

$$Qp(t, x, \cdot)(\alpha_s) = \sum_{\beta \ne \alpha_s} q_{\alpha_s \beta} (p(t, x, \beta) - p(t, x, \alpha_s)).$$

Using equation (1.21), we have

$$\begin{aligned} Q\varphi(t, x, \cdot)(\alpha_s) &= \sum_{\beta \ne \alpha_s} q_{\alpha_s \beta} (\varphi(t, x, \beta) - \varphi(t, x, \alpha_s)) \\ &= \sum_{\beta \ne \alpha_s} q_{\alpha_s \beta} (p(t, x, \beta) - \varphi(t, x, \alpha_s)) \\ &= \sum_{\beta \ne \alpha_s} q_{\alpha_s \beta} \Big(p(t, x, \beta) - [p(s, x_s, \alpha_s) \\ &\quad + \psi(t, x) - \psi(s, x_s)] \Big). \end{aligned} \tag{1.26}$$

From equation (1.23), we obtain

$$\begin{aligned} Q\varphi(t, X(t), \cdot)(\alpha_s) &= \sum_{\beta \ne \alpha_s} q_{\alpha_s \beta} \Big(p(t, X(t), \beta) - [p(s, x_s, \alpha_s) \\ &\quad + \psi(t, X(t)) - \psi(s, x_s)] \Big) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{\beta\neq\alpha_s} q_{\alpha_s\beta}\Big(p(t,X(t),\beta) - [p(s,x_s,\alpha_s) + p(t,X(t),\alpha_s)\\
&\quad -(p(s,x_s,\alpha_s) - \psi(s,x_s)) - \psi(s,x_s)]\Big)\\
&\geq \sum_{\beta\neq\alpha_s} q_{\alpha_s\beta}\left(p(t,X(t),\beta) - p(t,X(t),\alpha_s)\right)\\
&\geq Qp(t,X(t),\cdot)(\alpha_s). \qquad (1.27)
\end{aligned}$$

In view of (1.27) and Lemma 1.0.2, we deduce

$$\begin{aligned}
0 &= \widetilde{E}^{s,x_s,\alpha_s} e^{-r(\theta-s)} p(\theta,X(\theta),\alpha_s) - p(s,x_s,\alpha_s)\\
&\geq \widetilde{E}^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- rp(t,X(t),\alpha_s)\\
&\quad + \frac{\partial\psi(t,X(t))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\psi(t,X(t))}{\partial x^2}\\
&\quad +\mu(\alpha_s)X(t)\frac{\partial\psi(t,X(t))}{\partial x} + Qp(t,X(t),\cdot)(\alpha_s)\Big)dt. \qquad (1.28)
\end{aligned}$$

therefore,

$$\begin{aligned}
&\widetilde{E}^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- rp(t,X(t),\alpha_s) +\\
&\frac{\partial\psi(t,X(t))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\psi(t,X(t))}{\partial x^2} + rX(t)\frac{\partial\psi(t,X(t))}{\partial x}\\
&\quad +Qp(t,X(t),\cdot)(\alpha_s)\Big)dt \leq 0.
\end{aligned}$$

Multiplying both sides by $\frac{1}{\theta} > 0$ and sending $\theta \to s$ gives

$$\begin{aligned}
&rp(s,x_s,\alpha_s) - \frac{\partial\psi(s,x_s)}{\partial t} - \frac{1}{2}x_s^2\sigma^2(\alpha_s)\frac{\partial^2\psi(s,x_s)}{\partial x^2}\\
&\quad -rx_s\frac{\partial\psi(s,x_s)}{\partial x} - Qv(s,x_s,\cdot)(\alpha_s) \geq 0,
\end{aligned}$$

which is the desired supersolution solution inequality (1.20). Next, let us prove the subsolution inequality, namely, that

$$\begin{aligned}
&r\Big(p(s,x_s,\alpha_s) - x_s\frac{\partial\phi(s,x_s)}{\partial x}\Big) - \frac{\partial\phi(s,x_s)}{\partial s}\\
&-\frac{1}{2}x_s^2\sigma^2(\alpha_s)\frac{\partial^2\phi(s,x_s)}{\partial x^2} - Qv(s,x_s,\cdot)(\alpha_s) \leq 0 \qquad (1.29)
\end{aligned}$$

whenever $\phi \in \mathcal{C}^{1,2}([s,T]\times\mathbb{R}^+)$ and $v(t,x,\alpha_s) - \phi(t,x)$ has a local maximum at $(s,x_s) \in [s,T]\times\mathbb{R}^+$.

Let $\phi \in \mathcal{C}^{1,2}([s,T]\times\mathbb{R}^+)$ and $p(t,x,\alpha_s)-\phi(t,x)$ has a local maximum at $(s,x_s)\in[s,T]\times\mathbb{R}^+$ we can assume without loss of generality that $p(s,x_s,\alpha_s)-\phi(s,x_s)=0$. We define

$$\Phi(t,x,i)=\begin{cases}\phi(t,x), & \text{if} \quad i=\alpha_s,\\ p(t,x,i), & \text{if} \quad i\neq\alpha_s.\end{cases} \tag{1.30}$$

Let γ be the first jump time of $\alpha(\cdot)$ from the state α_s and let $\theta_0\in[s,\gamma]$ be such that $(t,X(t))$ starts at (s,x_s) and stays in $N(s,x_s)$ for $s\le t\le\theta_0$. Note that $\alpha(t)=\alpha_s$, for $s\le t\le\theta_0$. Moreover, recall that $p(s,x_s,\alpha_s)-\phi(s,x_s)=0$ and attains its maximum at (s,x_s) in $N(s,x_s)$. It follows that

$$p(\theta,X(\theta),\alpha(\theta))\le\phi(\theta,X(\theta),) \qquad \text{for any } \theta\in[s,\theta_0].$$

Moreover, in view of the definition of Φ in (1.30), we have

$$p(\theta,X(\theta),\alpha(\theta))\le\Phi(\theta,X(\theta)),\alpha(\theta)) \qquad \text{for any } \theta\in[s,\theta_0]. \tag{1.31}$$

Using Dynkin's formula, we have

$$\begin{aligned}
&\widetilde{E}^{s,x_s,\alpha_s}e^{-r(\theta-s)}p(\theta,X(\theta),\alpha_s)\\
\le\;&\widetilde{E}^{s,x_s,\alpha_s}e^{-r(\theta-s)}\Phi(\theta,X(\theta),\alpha_s)\\
=\;&\Phi(s,x_s,\alpha_s)+\widetilde{E}^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Bigg[\\
&\frac{\partial\phi(t,X(t))}{\partial t}-r\Phi(t,X(t),\alpha(t))\\
&+rX(t)\frac{\partial\phi(t,X(t))}{\partial x}+Q\Phi(t,X(t),\cdot)(\alpha_s)\\
&+\frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\phi(t,X(t))}{\partial x^2}\Bigg]dt.
\end{aligned} \tag{1.32}$$

Note that

$$\begin{aligned}
Q\Phi(t,X(t),\cdot)(\alpha_s) &= \sum_{\beta\neq\alpha_s}q_{\alpha_s\beta}(p(t,X(t),\beta)-\phi(t,X(t)))\\
&\le \sum_{\beta\neq\alpha_s}q_{\alpha_s\beta}(p(t,X(t),\beta)-p(t,X(t),\alpha_s))\\
&\le Qp(t,X(t),\cdot)(\alpha_s).
\end{aligned} \tag{1.33}$$

Using (1.30) and (1.33), we obtain

$$\begin{aligned}
& \widetilde{E}^{s,x_s,\alpha_s} e^{-r(\theta-s)} p(\theta, X(\theta), \alpha_s) \\
\leq \;& \widetilde{E}^{s,x_s,\alpha_s} e^{-r\theta} \Phi(\theta, X(\theta), \alpha_s) \\
= \;& \phi(s, x_s) + \widetilde{E}^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big[\frac{\partial \phi(t, X(t))}{\partial t} \\
& + rX(t) \frac{\partial \phi(t, X(t))}{\partial x} - rp(t, X(t), \alpha_s) \\
& + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(t, X(t))}{\partial x^2} + Qp(t, X(t), \cdot)(\alpha_s).
\end{aligned} \tag{1.34}$$

In view of Lemma 1.0.2, we deduce

$$\begin{aligned}
0 \;=\; & \widetilde{E}^{s,x_s,\alpha_s} e^{-r(\theta-s)} p(\theta, X(\theta), \alpha_s) - \phi(s, x_s) \\
\leq \; & \widetilde{E}^{s,x_s,\alpha_s} \int_s^\theta e^{-rt} \Big[- rp(t, X(t), \alpha_s) \\
& + \frac{\partial \phi(t, X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(X(t), \alpha_s)}{\partial x^2} \\
& + rX(t) \frac{\partial \phi(X(t), \alpha_s)}{\partial x} + Qp(t, X(t), \cdot)(\alpha_s) \Big] dt.
\end{aligned} \tag{1.35}$$

Multiplying the right-hand side by $\frac{1}{\theta} > 0$ and sending $\theta \downarrow s$ gives

$$\begin{aligned}
rp(s, X(s), \alpha_s) - \frac{\partial \phi(s, X(s))}{\partial t} - \frac{1}{2} X(s)^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(X(s), \alpha_s)}{\partial x^2} \\
- rX(s) \frac{\partial \phi(X(s), \alpha_s)}{\partial x} - Qp(s, X(s), \cdot)(\alpha_s) \leq 0.
\end{aligned}$$

This inequality implies the subsolution inequality (1.29), thus $p(t, x, \alpha)$ is a viscosity solution of (1.7). This ends the proof of the theorem. □

1.2 Uniqueness

Uniqueness of viscosity solution property is crucial in various analysis of the underlying system dynamics.In this section we prove a comparison principle for solutions of (1.7) and this will lead to the uniqueness of the viscosity solution. Firstly we state the key result for our uniqueness proof.

Theorem 1.2.1 (Crandall, Lions and Ishii [6]) *For $i = 1, 2$, let Ω_i be locally compact subsets of $\mathbb{R}$, and $\Omega = \Omega_1 \times \Omega_2$, let u_i be upper semicontinuous in $[0, T] \times \Omega_i$, and $\bar{J}^{2,+}_{\Omega_i} u_i(t, x)$ the parabolic superjet of $u_i(t, x)$, and ϕ be twice continuously differentiable in a neighborhood of $[0, T] \times \Omega$.*

Set

$$w(t, x_1, x_2) = u_1(t, x_1) + u_2(t, x_2)$$

for $(t, x_1, x_2) \in [0, T] \times \Omega$, and suppose $(\hat{t}, \hat{x_1}, \hat{x_2}) \in [0, T] \times \Omega$ is a local maximum of $w - \phi$ relative to $[0, T] \times \Omega$. Moreover let us assume that, there is an $r > 0$ such that for every $M > 0$ there exists a C such that for $i = 1, 2$

$$b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \bar{J}^{2,+}_{\Omega_i} u_i(t, x_i),$$
$$| x_i - \hat{x}_i | + | t - \hat{t} | \leq r \text{ and } | u_i(t, x_i) | + | q_i | + \|X_i\| \leq M. \tag{1.36}$$

Then for each $\epsilon > 0$ there exists $X_i \in \mathcal{S}(1) = \mathbb{R}$ such that

1. $$(b_i, Dx_i\phi(\hat{t}, \hat{x}), X_i) \in \bar{J}^{2,+}_{\Omega_i} u_i(\hat{t}, \hat{x}_i) \text{ for } i = 1, 2$$

2. $$-\left(\frac{1}{\epsilon} + \|D^2\phi(\hat{x})\|\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2\phi(\hat{x}) + \epsilon(D^2\phi(\hat{x}))^2 \tag{1.37}$$

3. $$b_1 + b_2 = \frac{\partial \phi(\hat{t}, \hat{x}, \hat{y})}{\partial t} \tag{1.38}$$

Theorem 1.2.2 (Comparison Principle) *If $p_1(t, x, i)$ and $p_2(t, x, i)$ are both continuous with respect to the argument (t, x) and are respectively viscosity subsolution and supersolution of (1.7) with at most a linear growth, in other terms, there exist C_1, C_2.*

$$p_k(t, x, i) \leq C_k(1 + x), \text{ for } (t, x, i) \in [s, T] \times \mathbb{R}^+ \times \mathcal{M}, \; k = 1, 2.$$

Then

$$p_1(t, x, i) \leq p_2(t, x, i) \quad \text{for all } (t, x, i) \in [s, T] \times \mathbb{R}^+ \times \mathcal{M}. \tag{1.39}$$

Proof. For any $0 < \delta < 1$ and $0 < \gamma < 1$. Define

$$\Phi(t,x,y,i) = p_1(t,x,i) - p_2(t,y,i) - \frac{1}{\delta} \mid x - y \mid^2 - \gamma e^{(T-t)}(x^2 + y^2),$$

and

$$\phi(t,x,y) = \frac{1}{\delta} \mid x - y \mid^2 + \gamma e^{(T-t)}(x^2 + y^2).$$

In view of the linear growth condition for p_1 and p_2, we have for each

$i \in \mathcal{M}$

$$\lim_{|x|+|y|\to\infty} \Phi(t,x,y,i) = -\infty. \tag{1.40}$$

Note that $\Phi(t,x,y,i)$ is continuous with respect to the arguments (t,x,y) for each $i \in \mathcal{M}$. Therefore, $\Phi(t,x,y,i)$ has a global maximum. Recall that $\mathcal{M}$ is a finite set. There exists a point $(t_\delta, x_\delta, y_\delta, \alpha_0)$ such that $\Phi(t_\delta, x_\delta, y_\delta, \alpha_0)$ is the global maximum of Φ. Observe that

$$\Phi(t_\delta, x_\delta, x_\delta, \alpha_0) + \Phi(t_\delta, y_\delta, y_\delta, \alpha_0) \le 2\Phi(t_\delta, x_\delta, y_\delta, \alpha_0).$$

It implies

$$\begin{aligned} p_1(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, x_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(x_\delta^2) + p_1(t_\delta, y_\delta, \alpha_0) \\ -p_2(t_\delta, y_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(y_\delta^2) \le 2p_1(t_\delta, x_\delta, \alpha_0) - 2p_2(t_\delta, y_\delta, \alpha_0) \\ -\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 - 2\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) \end{aligned}$$

and

$$\begin{aligned} -p_2(t_\delta, y_\delta, \alpha_0) - 2e^{(T-t_\delta)}\gamma(x_\delta^2) + p_1(t_\delta, x_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(y_\delta^2) \\ \le p_1(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, y_\delta, \alpha_0) - \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \\ -2\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \le (p_1(t_\delta, x_\delta, \alpha_0) - p_1(t_\delta, y_\delta, \alpha_0)) \\ +(p_2(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, y_\delta, \alpha_0)). \end{aligned} \tag{1.41}$$

By the linear growth condition, there exist K_1, K_2 such that

$p_1(t,x,i) \le K_1(1+ \mid x \mid)$ and $p_2(t,x,i) \le K_2(1+ \mid x \mid)$. Therefore there exists K such that

$$\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \le K(1+ \mid x_\delta \mid + \mid y_\delta \mid).$$

So

$$| x_\delta - y_\delta |^2 \leq \delta K(1+ | x_\delta | + | y_\delta |). \tag{1.42}$$

In addition, $\Phi(s,0,0,\alpha_0) \leq \Phi(t_\delta, x_\delta, y_\delta, \alpha_0)$ and $| \Phi(s,0,0,\alpha_0) | \leq K(1+ | x_\delta | + | y_\delta |)$. Therefore,

$$\begin{aligned} \gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) &\leq p_1(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, y_\delta, \alpha_0) \\ &\quad -\frac{1}{\delta} | x_\delta - y_\delta |^2 - \Phi(s,0,0,\alpha_0) \\ &\leq 3K(1+ | x_\delta | + | y_\delta |). \end{aligned} \tag{1.43}$$

It follows that

$$\frac{\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2)}{(1+ | x_\delta | + | y_\delta |)} \leq 3K.$$

Consequently, there exists C_γ such that

$$| x_\delta | + | y_\delta | \leq C_\gamma \text{ and } t_\delta \in [s,T]. \tag{1.44}$$

This inequality implies that the sets $\{x_\delta, \delta > 0\}$, and $\{y_\delta, \delta > 0\}$ are bounded by C_γ independent of δ. We can extract convergent subsequences also denote $(x_\delta)_\delta$, $(y_\delta)_\delta$, $(t_\delta)_\delta$. Moreover, from the inequality (1.42) we conclude that there exists x_0 such that

$$\lim_{\delta \to 0} x_\delta = x_0 = \lim_{\delta \to 0} y_\delta \qquad \text{and} \qquad \lim_{\delta \to 0} t_\delta = t_0. \tag{1.45}$$

Using (1.41) and the previous limit, we obtain

$$\lim_{\delta \to 0} \frac{2}{\delta} | x_\delta - y_\delta |^2 = 0. \tag{1.46}$$

Recall that Φ achieves its maximum at $(t_\delta, x_\delta, y_\delta, \alpha_0)$, so by Theorem 1.2.1 for each $\epsilon > 0$ there exists $b_{1\delta}$, $b_{2\delta}$, X_δ, and Y_δ such that

$$(b_{1\delta}, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t)} x_\delta, X_\delta) \in \bar{\mathcal{P}}^{2,+} p_1(t_\delta, x_\delta, \alpha_0) \tag{1.47}$$

and

$$(-b_{2\delta}, -\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t)} y_\delta, -Y_\delta) \in \bar{\mathcal{P}}^{2,+}(-p_2(t_\delta, y_\delta, \alpha_0)).$$

On the other hand, note that

$$\bar{\mathcal{P}}^{2,+}(-p_2(t_\delta, y_\delta, \alpha_0)) = -\bar{\mathcal{P}}^{2,-}p_2(t_\delta, y_\delta, \alpha_0).$$

We obtain

$$(b_{2\delta}, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{(T-t)}y_\delta, Y_\delta) \in \bar{\mathcal{P}}^{2,-}p_2(t_\delta, y_\delta, \alpha_0). \tag{1.48}$$

The equation (1.47) implies by the definition of the viscosity solution

$$rp_1(t_\delta, x_\delta, \alpha_0) - b_{1\delta} - \frac{1}{2}(x_\delta)^2\sigma^2(\alpha_s)X_\delta - x_\delta\mu(\alpha_0)\Big(\frac{2}{\delta}(x_\delta - y_\delta) \\ +2\gamma e^{(T-t_\delta)}x_\delta\Big) - Qp_1(t_\delta, x_\delta, \cdot)(\alpha_0) \le 0.$$

Similarly, (1.48) implies by the definition of the viscosity solution that,

$$rp_2(t_\delta, y_\delta, \alpha_0) - b_{2\delta} - \frac{1}{2}(y_\delta)^2\sigma^2(\alpha_0)Y_\delta - y_\delta\mu(\alpha_0)\Big(\frac{2}{\delta}(x_\delta - y_\delta) \\ -2\gamma e^{(T-t_\delta)}y_\delta\Big) - Qp_2(t_\delta, y_\delta, \cdot)(\alpha_0) \ge 0.$$

Combining the last two inequalities, we obtain

$$r(p_1(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, y_\delta, \alpha_0)) \le \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2X_\delta - (y_\delta)^2Y_\delta\right) + \\ \mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta - y_\delta)^2 + 2\gamma e^{(T-t_\delta)}\Big[(x_\delta)^2 + (y_\delta)^2\Big]\right) \\ Qp_1(t_\delta, x_\delta, \cdot)(\alpha_0) - Qp_2(t_\delta, y_\delta, \cdot)(\alpha_0) + b_{1\delta} - b_{2\delta}.$$

In view of Theorem 1.2.1, we have

$$b_{1\delta} - b_{2\delta} = \frac{\partial\phi(t_\delta, x_\delta, y_\delta)}{\partial t} = \gamma e^{(T-t_\delta)}((x_\delta)^2 + (y_\delta)^2).$$

Therefore, we obtain

$$r(p_1(t_\delta, x_\delta, \alpha_0) - p_2(t_\delta, y_\delta, \alpha_0)) \le \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2X_\delta - (y_\delta)^2Y_\delta\right) \\ +\mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta - y_\delta)^2 + 2\gamma e^{(T-t_\delta)}\Big[(x_\delta)^2 + (y_\delta)^2\Big]\right) \\ Qp_1(t_\delta, x_\delta, \cdot)(\alpha_0) - Qp_2(t_\delta, y_\delta, \cdot)(\alpha_0) + \gamma e^{(T-t_\delta)}((x_\delta)^2 + (y_\delta)^2). \tag{1.49}$$

Using the Maximum principle, we have

$$-\left(\frac{1}{\epsilon} + \|D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta)\|\right) I \le \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \le D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta) + \\ \epsilon(D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta))^2.$$

Moreover,

$$D^2_{(x,y)}\phi(t_\delta,x_\delta,y_\delta)=\frac{2}{\delta}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}+2\gamma e^{(T-t_\delta)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and

$$\begin{aligned}(D^2_{(x,y)}\phi(t_\delta,x_\delta,y_\delta))^2 &= \frac{8}{\delta^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}+\frac{8\gamma e^{(T-t_\delta)}}{\delta}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\\ &\quad +4\gamma^2 e^{2(T-t_\delta)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= \frac{8+8\gamma\delta e^{(T-t_\delta)}}{\delta^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\\ &\quad +4\gamma^2 e^{2(T-t_\delta)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \end{aligned}\tag{1.50}$$

Note that,

$$\begin{aligned}(x_\delta)^2X_\delta-(y_\delta)^2Y_\delta &= (x_\delta,y_\delta)\begin{pmatrix}X_\delta & 0\\ 0 & -Y_\delta\end{pmatrix}\begin{pmatrix}x_\delta\\ y_\delta\end{pmatrix}\\ &\leq (x_\delta,y_\delta)\Bigg[\frac{2}{\delta}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\\ &\quad +\Big(2\gamma e^{(T-t_\delta)}+4\epsilon\gamma^2 e^{2(T-t_\delta)}\Big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &\quad +\epsilon\frac{8+8\gamma\delta e^{(T-t_\delta)}}{\delta^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\Bigg]\begin{pmatrix}x_\delta\\ y_\delta\end{pmatrix}. \end{aligned}\tag{1.51}$$

Letting $\gamma\to 0$, we obtain

$$(x_\delta)^2X_\delta-(y_\delta)^2Y_\delta\leq (x_\delta,y_\delta)\left[(\frac{2}{\delta}+\epsilon\frac{8}{\delta^2})\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right]\begin{pmatrix}x_\delta\\ y_\delta\end{pmatrix}.$$

Taking $\epsilon=\frac{\delta}{4}$, leads to

$$(x_\delta)^2X_\delta-(y_\delta)^2Y_\delta\leq (x_\delta,y_\delta)\left[\frac{4}{\delta}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right]\begin{pmatrix}x_\delta\\ y_\delta\end{pmatrix}=\frac{4}{\delta}(x_\delta-y_\delta)^2.$$

Using (1.46), we have

$$\begin{aligned}\limsup_{\delta\downarrow 0}(x_\delta)^2X_\delta-(y_\delta)^2Y_\delta &\le \limsup_{\delta\downarrow 0}(x_\delta,y_\delta)\left[\frac{4}{\delta}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right]\begin{pmatrix}x_\delta\\ y_\delta\end{pmatrix}\\ &= \limsup_{\delta\downarrow 0}\frac{4}{\delta}(x_\delta-y_\delta)^2=0. \qquad (1.52)\end{aligned}$$

Letting $\gamma\to 0$ in (1.49), we have

$$\begin{aligned}r(p_1(t_\delta,x_\delta,\alpha_0)-p_2(t_\delta,y_\delta,\alpha_0)) &\le \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2X_\delta-(y_\delta)^2Y_\delta\right)\\ +\mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta-y_\delta)^2\right) &+ Qp_1(t_\delta,x_\delta,\cdot)(\alpha_0)-Qp_2(t_\delta,y_\delta,\cdot)(\alpha_0)\end{aligned}$$

and taking the lim sup as δ goes to zero and using (1.52), we obtain

$$r(p_1(t_0,x_0,\alpha_0)-p_2(t_0,x_0,\alpha_0))\le Qp_1(t_0,x_0,\cdot)(\alpha_0)-Qp_2(t_0,x_0,\cdot)(\alpha_0). \qquad (1.53)$$

Since $(t_\delta,x_\delta,y_\delta,\alpha_0)$ is maximum of Φ then, for all $x\in\mathbb{R}$ and for all $i\in\mathcal{M}$ we have

$$\Phi(t,x,x,i)\le\Phi(t_\delta,x_\delta,y_\delta,\alpha_0)$$

we have

$$\begin{aligned}p_1(t,x,i)-p_2(t,x,i)-2\gamma e^{(T-t)}x^2 &\le p_1(t_\delta,x_\delta,\alpha_0)\\ &-p_2(t_\delta,y_\delta,\alpha_0)-2\gamma e^{(T-t_\delta)}(x_\delta^2+y_\delta^2).\end{aligned}$$

Letting $\delta\to 0$, we obtain

$$\begin{aligned}p_1(t,x,i)-p_2(t,x,i)-2\gamma e^{(T-t)}x^2 &\le p_1(t_0,x_0,\alpha_0)\\ &-p_2(t_0,x_0,\alpha_0)-2\gamma e^{(T-t)}x_0^2. \qquad (1.54)\end{aligned}$$

Taking $x=x_0$, and $t=t_0$, we have

$$\begin{aligned}p_1(t_0,x_0,i)-p_2(t_0,x_0,i)-2\gamma e^{(T-t_0)}x_0^2 &\le p_1(t_0,x_0,\alpha_0)\\ &-p_2(t_0,x_0,\alpha_0)-2\gamma e^{(T-t_0)}x_0^2,\end{aligned}$$

so

$$p_1(t_0,x_0,i)-p_2(t_0,x_0,i)\le p_1(t_0,x_0,\alpha_0)-p_2(t_0,x_0,\alpha_0).$$

We recall

$$\begin{aligned} & Qp_1(t_0,x_0,\cdot)(\alpha_0) - Qp_2(t_0,x_0,\cdot)(\alpha_0) \\ = & \sum_{i\neq\alpha_0} q_{\alpha_0\alpha}[p_1(t_0,x_0,i) - p_1(t_0,x_0,\alpha_0) \\ & -p_2(t_0,x_0,i) + p_2(t_0,x_0,\alpha_0)] \leq 0, \end{aligned} \tag{1.55}$$

using (1.53) we have

$$p_1(t_0,x_0,\alpha_0) - p_2(t_0,x_0,\alpha_0) \leq 0.$$

Therefore using (1.54) we conclude that

$$\begin{aligned} p_1(t,x,i) - p_2(t,x,i) - 2\gamma e^{(T-t)}x^2 &\leq \\ p_1(t_0,x_0,\alpha_0) - p_2(t_0,x_0,\alpha_0) - 2\gamma e^{(T-t_0)}x_0^2 &\leq 0. \end{aligned} \tag{1.56}$$

Finally, letting $\gamma \to 0$, we have

$$p_1(t,x,i) \leq p_2(t,x,i).$$

This completes the proof. □

The uniqueness of the viscosity solution of (1.7) follows directly from this theorem because any viscosity solution is both viscosity subsolution and supersolution. What remains is to develop numerical schemes to approximate that solution since it is very difficult to obtain closed form solution even in this case.

Chapter 2

Valuation of American option under regime switching

Introduction

In this chapter, we consider pricing of American options. Unlike European options where the holder can only exercise his or her option at maturity, the holder of an American option can exercise his or her option anytime up to maturity. This level flexibility of American option increases the complexity in the study of its valuation. The holder has to time the best date to exercise the option in order to maximize his or her profit. The valuation of American option is related to optimal stopping. There has been a huge interest in the literature on American option because, there is no analytic formula for American options even in the simple non-switching model. In this connection, McKean [28] published in 1965 was the first to study the relationship between the early exercise feature of American options and optimal stopping problem. Van Moerbeke [38] further studied some properties of related free boundary problems. More recently, Bensoussan [2], Karatzas [20, 21], Kim [25], Myneni [24] and many others have studied various aspects of American options pricing problem.

Particularly, we study American option pricing with a regime switching model. We focus on optimal stopping time of a switching diffusion. Existence result of optimal stopping was studied by Dynkin [9] where he proved the existence of optimal stopping time for Markov processes which are right-continuous and quasi-continuous from the left, and in a martingale context by Snell [36]. Nevertheless, in our case the joint process $(X(t), \alpha(t))$ is not quasi-continuous in $\alpha(t)$.

Moreover, we also study the associated dynamic programming equations that are essential for characterizing American options pricing function as a unique viscosity solution of the associated HJB equations.

This chapter is organized as follows. In Section 1 we use the optimal stopping theory to prove the existence of an optimal stopping time, in Section 2 we prove that the value function is the unique viscosity solution of the HJB equation associated with this optimal stopping problem both in the infinite and finite time horizon cases.

2.1 Optimal stopping of a switching diffusion

We begin with the existence of optimal stopping for the joint process $(X(t), \alpha(t))$ where $\alpha(t)$ is a finite state Markov chain taking values in the set $\mathcal{M} = \{1, 2, ..., m\}$ and with generator Q, and $X(t)$ follows the dynamics

$$\mathrm{d}X(t) = X(t)\Big(\mu(\alpha(t))\mathrm{d}t + \sigma(\alpha(t))\mathrm{d}W(t)\Big), \tag{2.1}$$

where $W(t)$ the standard Weiner process. Both $W(t)$ and $\alpha(t)$ are independent and defined on the risk neutral probability space $(\Omega, \mathcal{F}, \widetilde{P})$. Given $\alpha(t) = i$, $\mu(i)$ and $\sigma(i)$ are known parameters.

Let g be a reward function. For example $g(t, x, i) = e^{-rt}(K - x)^+$ for put option and $g(t, x, i) = e^{-rt}(x - K)^+$ for call option, where K being the strike price of the option and r is the risk free rate. Let $\mathcal{F}_t = \sigma\{\alpha(s), W(s); s \leq t\}$. The problem is to find an $\mathcal{F}_t$-stopping time τ^* that maximizes $E\left[g(\tau, X(\tau), \alpha(\tau)) \mid X(0) = x, \alpha(0) = i\right]$ over all $\mathcal{F}_t$-stopping time, in the infinite time horizon case and $E\left[g(\tau - s, X(\tau), \alpha(\tau)) \mid X(s) = x, \alpha(s) = i\right]$ over all $\mathcal{F}_t$-stopping time τ such that $s \leq \tau \leq T$, in the finite time horizon case.

We define the value function v by

$$\begin{aligned} v(x, i) &= \sup_{\tau} E\left[g(\tau, X(\tau), \alpha(\tau)) \mid X(0) = x, \alpha(0) = i\right] \\ &= E\left[g(\tau^*, X(\tau^*), \alpha(\tau^*) \mid X(0) = x, \alpha(0) = i\right] \end{aligned} \tag{2.2}$$

for the infinite time horizon case; and

$$\begin{aligned} v(s,x,i) &= \sup_{\tau\in\Lambda_{s,T}} E\left[g(\tau-s,X(\tau),\alpha(\tau)) \mid X(s)=x,\alpha(s)=i\right] \\ &= E\left[g(\tau^*-s,X(\tau^*),\alpha(\tau^*)) \mid X(s)=x,\alpha(s)=i\right] \end{aligned} \tag{2.3}$$

where $\Lambda_{s,T} = \{\tau,\ \mathcal{F}_t - \text{stopping time}\ ; s \leq \tau \leq T\}$ for the finite time horizon case. The stopping time τ^* is called the optimal stopping time.

Definition 2.1.1 *The infinitesimal generator $\mathcal{A}$ of the process $(X(t),\alpha(t))$ in the time-homogeneous case is defined by*

$$\mathcal{A}f(x,i) = \lim_{h\to 0}\frac{E^{x,i}[f(X(h),\alpha(h)] - f(x,i)}{h},\ \textit{for all } x\in\mathbb{R}^+, i\in\mathcal{M}, \tag{2.4}$$

for all functions f such that the limit exists.

Proposition 2.1.2 *The infinitesimal generator of the process $(X(s),\alpha(s))$ is given by*

$$(\mathcal{A}v)(x,i) = \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 v(x,i)}{\partial x^2} + x\mu(i)\frac{\partial v(x,i)}{\partial x} + Qv(x,\cdot)(i),$$

for every function v in the domain of $\mathcal{A}$, where Q is the generator of $(\alpha(t))$.

Proof. The proof follows from the same argument as in Lemma 1 of Yao *et al* [40]. □

We will prove the existence of the optimal stopping time in the time homogeneous case, and the result for the time inhomogeneous case can easily be derived after a change of variable is done.

We already know that $(X(t), \alpha(t))$ is a Markov process, we firstly prove that $(X(t), \alpha(t))$ is a strong Markov process.

Lemma 2.1.3 *Let τ be a discrete $\mathcal{F}_t$-stopping time. Then $(X(t), \alpha(t))$ is strong Markov at τ.*

Proof. First, note that the continuity of $X(t)$ and the right continuity of $\alpha(t)$ imply that $(X(t), \alpha(t))$ is $\mathcal{F}_t$-progressive. The strong Markov property at τ follows from the Markov property of $(X(t), \alpha(t))$ and Proposition 1.3 in Either and Kurtz [12, p159]. $\square$

Proposition 2.1.4 *The process $(X(t), \alpha(t))$ is a strong Markov process,i.e.,*

$$E^{x,\alpha}[f(X(\tau + h), \alpha(\tau + h) \mid \mathcal{F}_\tau] = E^{X_\tau,\alpha_\tau}[f(X(h), \alpha(h))] \quad \textit{for all } \; h \geq 0, \tag{2.5}$$

for all bounded Borel function f.

Proof. First, note that every stopping time can be approximated by a sequence of non increasing discrete stopping times. Let τ be a stopping time and $\{\tau_k\}$ a sequence of discrete stopping times such that $\tau_{k+1} \leq \tau_k$ and $\tau_k \to \tau$. Since the set of continuous functions is dense in the set of Borel functions, thus it suffices to prove that the strong Markovian property holds for any continuous function f. Let f be a continuous function, then we have

$$E^{x,i}[f(X(\tau_k + h), \alpha(\tau_k + h) \mid \mathcal{F}_{\tau_k}] = E^{X_{\tau_k},\alpha_{\tau_k}}[f(X(h), \alpha(h))] \quad \text{for all } \; h \geq 0, \; k \geq 0$$

since $(\alpha(t))$ is right continuous and $(X(t))$ is continuous then passing to the limit using the monotone convergence theorem we obtain 2.5

$\square$

Let $Q^{x,i}$ denote the probability measure of $\{X(t), \alpha(t), \, t \geq 0\}$, for $x \in \mathbb{R}$ and $i \in \mathcal{M}$. The following definition is just a generalization of the definition of *supermeanvalued function* in Oksendal [32, p196].

Definition 2.1.5 *Let $X(t)$ be a diffusion process and $\alpha(t)$ a finite state Markov Chain. A measurable function $f : \mathbb{R}\times\mathcal{M} \to [0,\infty]$ is called supermeanvalued with respect to $(X(t), \alpha(t))$ if*

$$f(x,i) \geq E[f(X(\tau), \alpha(\tau))|X(0) = x, \alpha(0) = i] \tag{2.6}$$

for all stopping time τ and all $x \in \mathbb{R}, i \in \mathcal{M}$.

If, in addition, f is also lower semi-continuous with respect to its first variable, then f is called superharmonic with respect to $X(t)$.

Throughout the thesis, we adopt the following notation

$$E^{x,i}[\zeta] = E[\zeta \mid X(0) = x, \alpha(0) = i]$$

where ζ is any random variable and $E^{x,i}[\zeta]$ is just the expectation of ζ under the condition that $X(0) = x$ and $\alpha(0) = i$.

Remark 2.1.6 Note that if f is superharmonic w.r.t. $X(t)$, then for any sequence $\{\tau_n\}$ of stopping times such that for every n, $\alpha(\tau_n) = i$ and $\tau_n \to 0$, using Fatou's Lemma, we have

$$\begin{aligned} f(x,i) &\leq E^{x,i}[\liminf_{k\to\infty} f(X(\tau_k), \alpha(\tau_k))] \\ &\leq \liminf_{k\to\infty} E^{x,i}[f(X(\tau_k), \alpha(\tau_k))] \\ &\leq \limsup_{k\to\infty} E^{x,i}[f(X(\tau_k), \alpha(\tau_k))] \\ &\leq f(x,i). \end{aligned} \tag{2.7}$$

Consequently,

$$f(x,i) = \lim_{k\to\infty} E^{x,i}[f(X(\tau_k), \alpha(\tau_k))]. \tag{2.8}$$

Lemma 2.1.7

a) If f is superharmonic w.r.t $X(t)$ (supermeanvalued w.r.t. $(X(t),\alpha(t))$) and $\eta > 0$, then ηf is superharmonic w.r.t $X(t)$ (supermeanvalued w.r.t. $(X(t),\alpha(t))$)

b) If f_1, f_2 are superharmonic w.r.t $X(t)$ (supermeanvalued w.r.t. $(X(t),\alpha(t))$), then $f_1 + f_2$ is superharmonic (supermeanvalued w.r.t. $(X(t),\alpha(t))$).

c) If $\{f_j\}_{j\in J}$ is a family of supermeanvalued functions w.r.t. $(X(t),\alpha(t))$, then $f := \inf_{j\in J}\{f_j(x)\}$ is supermeanvalued w.r.t. $(X(t),\alpha(t))$ if it is measurable.

d) If f_1,f_2,... are superharmonic w.r.t $(X(t),\alpha(t))$ (supermeanvalued w.r.t. $(X(t),\alpha(t))$) functions and $f_k \uparrow f$ pointwise, then f is superharmonic w.r.t $(X(t))$ (supermeanvalued w.r.t. $(X(t),\alpha(t))$).

e) Let $r > 0$, if f is supermeanvalued w.r.t. $(X(t),\alpha(t))$, and $\sigma \le \tau$ are $\mathcal{F}_t$- stopping times, then

$$E^{x,i}[e^{-r\sigma}f(X(\sigma),\alpha(\sigma))] \ge E^{x,i}[e^{-r\tau}f(X(\tau),\alpha(\tau))].$$

Proof. a) and b) are straightforward and just imply that the set of supermeanvaled (superharmonic) functions is a vector space.

c) Suppose f_j is supermeanvalued for all $j \in J$. Then

$$f_j(x,\alpha) \ge E^{x,\alpha} f_j(X(\tau),\alpha(\tau)) \ge E^{x,\alpha} \inf_{j\in J} f_j(X(\tau),\alpha(\tau))$$

so $f(x,\alpha) = \inf_{j\in J} f_j(x,\alpha) \ge E^{x,\alpha} f(X(\tau),\alpha(\tau))$.

d) Suppose f_j are supermeanvalued w.r.t. $(X(t),\alpha(t))$ and $f_j \uparrow f$, therefore

$$f(x,\alpha) \ge f_j(x,\alpha) \ge E^{x,\alpha}[f_j(X(\tau),\alpha(\tau))] \text{ for all } j, \text{ then}$$
$$f(x,\alpha) \ge \lim_{j\to\infty} E^{x,\alpha}[f_j(X(\tau),\alpha(\tau))] = E^{x,\alpha}[f(X(\tau),\alpha(\tau))].$$

Consequently, f is supermeanvalued w.r.t. $(X(t),\alpha(t))$.

Suppose that each f_j is superharmonic w.r.t $(X(t))$ then f_j is lower semicontinuous.

Let $(y_k, \alpha_k)_k$ a sequence such that $y_k \to x_0$, and $\alpha_k \to \alpha_0$ then for any $\alpha_0 \in \mathcal{M}$ by the lower semicontinuity of f_j, we have

$$\begin{aligned} f_j(x_0, \alpha_0) &\leq \liminf_{k\to\infty} f_j(y_k, \alpha_k) \\ &\leq \liminf_{k\to\infty} f(y_k, \alpha_k). \end{aligned} \tag{2.9}$$

therefore, f is superharmonic.

e) Suppose f is supermeanvalued w.r.t. $(X(t), \alpha(t))$, by the Markov property we have, for $t > s$

$$\begin{aligned} E^{x,\alpha}[e^{-rt} f(X(t), \alpha(t)) \mid \mathcal{F}_s] = e^{-rt} E^{X(s),\alpha(s)}[f(X(t-s), \alpha(t-s))] &\leq e^{-rt} f(X(s), \alpha(s)) \\ &\leq e^{-st} f(X(s), \alpha(s)). \end{aligned}$$

So the process $\zeta_t = e^{-rt} f(X(t), \alpha(t))$ is a super-martingale w.r.t. $\mathcal{F}_t$. By Doob's optional sampling theorem, we have

$$E^{x,\alpha}[e^{-r\tau} f(X(\tau), \alpha(\tau)) \mid \mathcal{F}_\sigma] \leq e^{-r\sigma} f(X(\sigma), \alpha(\sigma)).$$

In view of this, we obtain

$$\begin{aligned} E^{x,\alpha}[e^{-r\tau} f(X(\tau), \alpha(\tau))] &= E^{x,\alpha} E^{x,\alpha}[e^{-r\tau} f(X(\tau), \alpha(\tau)) \mid \mathcal{F}_\sigma] \\ &\leq E^{x,\alpha}[e^{-r\sigma} f(X(\sigma), \alpha(\sigma))] \end{aligned} \tag{2.10}$$

for all stopping times σ, τ such that $\sigma \leq \tau$ a.s.

□

Definition 2.1.8 *Let h be a real measurable function on $\mathbb{R} \times \mathcal{M}$. If f is a superharmonic (supermeanvalued) function and $f \geq h$ then, f is called a superharmonic (supermeanvalued) majorant of h. The function*

$$\bar{h}(x, \alpha) = \inf_f f(x, \alpha),$$

the inf *taken over all supermeanvalued majorants f of h, is called the least supermeanvalued majorant of h.*

Likewise we define the least superharmonic majorant of h and we denote $\hat{h}$.

Remark 2.1.9 Let g be a reward function and f be a supermeanvalued majorant of g, then for any stopping time τ, we have

$$\begin{aligned} f(x,i) &\geq E^{x,i}[f(X(\tau),\alpha(\tau))] \\ &\geq E^{x,i}[g(X(\tau),\alpha(\tau))]. \end{aligned} \tag{2.11}$$

Therefore, we have

$$f(x,i) \geq \sup_{\tau} E^{x,i}[g(X(\tau),\alpha(\tau))] = \sup_{\tau} E^{x,i}[e^{-r\tau}g(X(\tau),\alpha(\tau))] = v(x,i).$$

This implies that the least supermeanvalued majorant $\bar{g}$ of g satisfies the inequality

$$\bar{g}(x,i) \geq v(x,i).$$

Similarly, we can verify that the least superharmonic majorant of g satisfies

$$\hat{g}(x,i) \geq v(x,i). \tag{2.12}$$

The next result is a generalization of an existence theorem for optimal stopping proved in Oksendal [32], we extent the result to the case of a joint process of a diffusion process and a finite state Markov chain.

Theorem 2.1.10 (Existence of the optimal stopping time) *Let v denote the optimal reward and $\hat{g}$ the least superharmonic majorant of a reward $g(x,i)$ defined on $\mathbb{R} \times \mathcal{M}$ such that $g(x,i)$ is continuous in x, $g(x,i) = f(x) \geq 0$ for all $x \in \mathbb{R}$, $i \in \mathcal{M}$, and for a given function f.*

a) Then

$$v(x,i) = \hat{g}(x,i) \ \textit{for all} \ (x,i) \in \mathbb{R} \times \mathcal{M}. \tag{2.13}$$

b) For $\epsilon > 0$ let

$$\begin{aligned} D &= \{(x,i) \in \mathbb{R} \times \mathcal{M}; g(x,i) < \hat{g}(x,i)\} \ \textit{be the continuation region, and} \\ D_\epsilon &= \{(x,i) \in \mathbb{R} \times \mathcal{M};\ g(x,i) < \hat{g}(x,i) - \epsilon\}. \end{aligned} \tag{2.14}$$

We define

$$\tau_D = \inf\{t > 0 \,:\, (X(t), \alpha(t)) \notin D\},$$
$$\tau_\epsilon = \inf\{t > 0 \,:\, (X(t), \alpha(t)) \notin D_\epsilon\}. \tag{2.15}$$

If we suppose that g is bounded then

$$| \, v(x,i) - E^{x,i}[g(X(\tau_\epsilon), \alpha(\tau_\epsilon))] \, | \leq 2\epsilon \tag{2.16}$$

for all $x \in \mathbb{R}$ and $i \in \mathcal{M}$.

c) For arbitrary g, we define $g_N = \min(g, N)$, $D_N = \{(x,i) : g_N(x,i) < \widehat{g_N}(x,i)\}$, and τ_{D_N} the first exit time from D_N, for all N.

Let us assume that $0 < \tau_D < \infty$ almost surely, and if the sequence $\{g_N(X(\tau_{D_N}), \alpha(\tau_{D_N}))\}$ is uniformly integrable then,

$$\tau_{D_N} \uparrow \tau_D \text{ and } v(x,i) = E^{x,i}[g(X(\tau_D), \alpha(\tau_D))], \tag{2.17}$$

and τ_D is an optimal stopping time.

Proof. To show (a) and (b), we first consider that g is bounded and we define

$$\widetilde{g}_\epsilon(x,i) = E^{x,i}[\hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon))] \qquad \text{for} \quad \epsilon > 0.$$

We first show that $\widetilde{g}_\epsilon$ is supermeanvalued. Let β be a stopping time, by the strong Markov property we have

$$\begin{aligned} E^{x,i}[\widetilde{g}_\epsilon(X(\beta), \alpha(\beta))] &= E^{x,i}E^{X(\beta),\alpha(\beta)}[\hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon))] \\ &= E^{x,i}E^{x,i}[\theta_\beta \hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon)) \mid \mathcal{F}_\beta] \end{aligned} \tag{2.18}$$

where θ_β is the *shift operator*, note that

$$\theta_\beta \hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon)) = \hat{g}(X(\tau_\epsilon^\beta), \alpha(\tau_\epsilon^\beta))$$

with

$$\tau_\epsilon^\beta = \inf\{t > \beta : (X(t), \alpha(t)) \notin D_\epsilon\} \text{ and } \tau_\epsilon^\beta \geq \tau_\epsilon.$$

For more about the properties of θ_β one can refer to Oksendal [32] page 114. Therefore, equation (2.18) becomes

$$\begin{aligned} E^{x,i}[\widetilde{g}_\epsilon(X(\beta),\alpha(\beta))] &= E^{x,i}E^{x,i}[\theta_\beta \hat{g}(X(\tau_\epsilon),\alpha(\tau_\epsilon)) \mid \mathcal{F}_\beta] \\ &= E^{x,i}[\theta_\beta \hat{g}(X(\tau_\epsilon),\alpha(\tau_\epsilon))] \\ &= E^{x,i}[\hat{g}(X(\tau_\epsilon^\beta),\alpha(\tau_\epsilon^\beta))] \quad \text{by Lemma 2.1.7, e) we have} \\ &\leq E^{x,i}[\hat{g}(X(\tau_\epsilon),\alpha(\tau_\epsilon))] \\ &= \widetilde{g}_\epsilon(x,i) \end{aligned} \tag{2.19}$$

which implies that

$$\widetilde{g}_\epsilon(x,i) \geq E^{x,i}[\widetilde{g}_\epsilon(X(\beta),\alpha(\beta))].$$

Thus $\widetilde{g}_\epsilon$ is supermeanvalued. We claim that

$$g(x,i) \leq \widetilde{g}_\epsilon(x,i) + \epsilon \text{ for all } x, i. \tag{2.20}$$

In order to prove the claim, let us suppose that,

$$\lambda = \sup_{(x,i)\in\mathbb{R}\times\mathcal{M}} g(x,i) - \widetilde{g}_\epsilon(x,i) > \epsilon. \tag{2.21}$$

For $\eta > 0$ such that

$$\eta < \epsilon < \lambda, \tag{2.22}$$

we can find (x_0, α_0) so that

$$g(x_0,\alpha_0) - \widetilde{g}_\epsilon(x_0,\alpha_0) \geq \lambda - \eta.$$

Moreover, we know that $\widetilde{g}_\epsilon + \lambda$ is a supermeanvalued majorant of g. So we have

$$\hat{g}(x_0,\alpha_0) \leq \widetilde{g}_\epsilon(x_0,\alpha_0) + \lambda. \tag{2.23}$$

Combining (2.21) and (2.23), we obtain

$$g(x_0,\alpha_0) + \eta \geq \hat{g}(x_0,\alpha_0).$$

We consider two possible cases:

Case 1: $\tau_\epsilon > 0$ almost surely. Then by the last inequality and the definition of D_ϵ we deduce

$$\begin{aligned} g(x_0, \alpha_0) + \eta \geq \hat{g}(x_0, \alpha_0) &\geq E^{x,\alpha}[\hat{g}(X(t \wedge \tau_\epsilon))] \quad \text{for all } t > 0 \\ &\geq E^{x_0,\alpha_0}[[g(X(t), \alpha(t)) + \epsilon] I_{t<\tau_\epsilon}]. \end{aligned} \tag{2.24}$$

Using the lower semicontinuity of g with respect to its first argument, the fact that g is constant in its second argument and Fatou's lemma, we have

$$\begin{aligned} g(x_0, \alpha_0) + \eta &\geq \liminf_{t \downarrow 0} E^{x_0,\alpha_0}[[g(X(t), \alpha(t)) + \epsilon] I_{t<\tau_\epsilon}] \\ &\geq E^{x_0,\alpha_0}[\liminf_{t \downarrow 0}[g(X(t), \alpha(t)) + \epsilon] I_{t<\tau_\epsilon}] \\ &\geq g(x_0, \alpha_0) + \epsilon. \end{aligned} \tag{2.25}$$

(2.25) implies that $\eta \geq \epsilon$ which contradicts (2.22).

Case 2: $\tau_\epsilon = 0$ almost surely. Then we have

$$\begin{aligned} \widetilde{g}_\epsilon(x_0, \alpha_0) &= E^{x_0,\alpha_0}[\hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon))] \\ &= \hat{g}(x_0, \alpha_0) \\ &\geq g(x_0, \alpha_0). \end{aligned} \tag{2.26}$$

(2.26) implies that $0 \geq g(x_0, \alpha_0) - \widetilde{g}_\epsilon(x_0, \alpha_0) \geq \lambda - \eta$ which contradicts (2.22). We then observe that the assumption $\epsilon < \lambda$ leads to contradiction. Thus we must have $\lambda \leq \epsilon$. This proves that $\widetilde{g}_\epsilon(x, i) + \epsilon$ is a supermeanvalued majorant of g. So

$$\begin{aligned} \hat{g}(x, i) &\leq \widetilde{g}_\epsilon(x, i) + \epsilon \\ &\leq E^{x,i}[\hat{g}(X(\tau_\epsilon), \alpha(\tau_\epsilon))] + \epsilon \\ &\leq E^{x,i}[g(X(\tau_\epsilon), \alpha(\tau_\epsilon)) + \epsilon] + \epsilon \\ &\leq v(x, i) + 2\epsilon. \end{aligned} \tag{2.27}$$

Since ϵ is arbitrary then $\hat{g} \leq v$, combining this last inequality with (2.12), we have

$$v = \hat{g}.$$

Moreover, from (2.27) we deduce that

$$\begin{aligned} \hat{g}(x,i) - E^{x,i}[g(X(\tau_\epsilon),\alpha(\tau_\epsilon))] &\leq 2\epsilon \\ v(x,i) - E^{x,i}[g(X(\tau_\epsilon),\alpha(\tau_\epsilon))] &\leq 2\epsilon \\ |v(x,i) - E^{x,i}[g(X(\tau_\epsilon),\alpha(\tau_\epsilon))]| &\leq 2\epsilon. \end{aligned} \tag{2.28}$$

Thus, we have (2.16) and (b) is proved.

Now we assume that g is not bounded. Define

$$g_N = \min(N, g), \qquad N = 1, 2, ...$$

$\widehat{g_N}$ is the least superharmonic majorant of g_N and v_N the optimal reward function associated with g_N. Then

$$v \geq v_N = \widehat{g_N}$$

and the sequence $\{\widehat{g_N}\}$ is an increasing sequence. Let $h = \lim_{N\to\infty} \widehat{g_N}$, thus h is superharmonic by Lemma 2.1.7 d) and h is a majorant of g. Therefore $h \geq \hat{g}$. Moreover, note that $v \geq \hat{g}_N$ for all N. Then $v \geq h \geq \hat{g}$, this implies that $v = \hat{g}$, since $v \leq \hat{g}$. So

$$v = \hat{g} = \lim_{N\to\infty} \widehat{g_N}.$$

And this proves a).

c) We know that $0 < \tau_D < \infty$. Let first assume that g is bounded. We have

$$\tau_\epsilon \uparrow \tau_D \qquad \text{as} \qquad \epsilon \downarrow 0$$

and

$$g(X(\tau_\epsilon),\alpha(\tau_\epsilon)) \to g(X(\tau_D),\alpha(\tau_D)) \qquad \text{as} \ \ \epsilon \downarrow 0$$

because g is continuous in its first argument and constant in its second argument. Therefore, by the bounded convergence theorem, we obtain

$$E^{x,i}[g(X(\tau_\epsilon),\alpha(\tau_\epsilon)) \to E^{x,i}[g(X(\tau_D),\alpha(\tau_D))] \qquad \text{as} \ \ \epsilon \downarrow 0. \tag{2.29}$$

The equations (2.28) and (2.29) imply that

$$v(x,i) = E^{x,i}[g(X(\tau_D),\alpha(\tau_D))]. \tag{2.30}$$

In general, we consider g_N which are bounded and we apply the formula (2.30) and we obtain

$$v_N(x,i) = E^{x,i}[g_N(X(\tau_{D_N},\alpha(\tau_{D_N}))].$$

Note that

$$g_N(X(\tau_{D_N}),\alpha(\tau_{D_N})) \to g(X(\tau_D),\alpha(\tau_D)) \qquad \text{as} \quad N \to \infty$$

and the family $\{g_N(X(\tau_D),\alpha(\tau_D))\}$ is uniformly integrable. Therefore, we have

$$\begin{aligned} v(x,i) = \hat{g}(x,i) &= \lim_{N\to\infty} \widehat{g_N}(x,i) \\ &= \lim_{N\to\infty} E^{x,i}[g_N(X(\tau_{D_N}),\alpha(\tau_{D_N}))] \\ &= E^{x,i} \lim_{N\to\infty} [g_N(X(\tau_{D_N}),\alpha(\tau_{D_N}))] \\ &= E^{x,i}[g(X(\tau_D),\alpha(\tau_D))]. \end{aligned} \tag{2.31}$$

□

Corollary 2.1.11 *The optimal reward v is supermeanvalued, so for all $(x,i) \in \mathbb{R} \times \mathcal{M}$ and $\mathcal{F}_t$-stopping time θ we have:*

$$v(x,i) \geq E^{x,i}[v(X(\theta),\alpha(\theta))] \geq E^{x,i}[e^{-r\theta}v(X(\theta),\alpha(\theta))] \ \ \textit{with } r \geq 0. \tag{2.32}$$

Proof. This comes directly from the fact that $v = \hat{g}$ which is the superharmonic majorant of g. □

Remark 2.1.12 1. We can easily extend these results in the time inhomogeneous case. Given $g = g(t,x,i)$, we consider the new process $Y_t = (s+t, X(t))$ where s is the time we start studying the joint process $(X(t),\alpha(t))$ with $X(t)$ solution of the stochastic differential equation (2.1), and $\alpha(t)$ the Markov chain. The optimal stopping problem is to find $v(s,x,i)$ and a stopping time τ such that

$$\begin{aligned} v(s,x,i) &= \sup_{\theta\in\Lambda} E^{s,x,i}[g(\theta, X(\theta), \alpha(\theta))] = E^{s,x,i}[g(\tau, X(\tau), \alpha(\tau))] \\ &= \sup_{\theta\in\Lambda} E^{s,x,i}[g(Y(\theta), \alpha(\theta))] = E^{s,x,i}[g(Y(\tau), \alpha(\tau))]. \end{aligned} \tag{2.33}$$

The result in Theorem 2.1.10 applies.

2. Let $\mathcal{A}$ be the generator of the joint process $(X(t), \alpha(t))$. Assume g in the domain of $\mathcal{A}$. Define

$$U = \{(x,i) \in \mathbb{R} \times \mathcal{M}, \mathcal{A}g(x,i) > 0\}.$$

Then,

$$U \subset D. \tag{2.34}$$

In order to prove (2.34), let $(x,i) \in U$ and let τ_0 be the first exit time from a bounded open set W containing (x,i), $W \subset U$. Then by Dynkin's formula, for $u > 0$

$$\begin{aligned} E^{x,i}[g(X(\tau_0 \wedge u), \alpha(\tau_0 \wedge u))] &= g(x,i) + E^{x,\alpha}\Big[\int_0^{\tau_0\wedge u} \mathcal{A}g(X(s), \alpha(s))ds\Big] \\ &> g(x,i). \end{aligned} \tag{2.35}$$

If we assume that $g(x,i) = \hat{g}(x,i)$ then (2.35) will contradicts the fact that $\hat{g}$ is supermeanvalued. Therefore, $g(x,i) < \hat{g}(x,i)$ so $(x,i) \in D$. Thus $U \subset D$.

Corollary 2.1.13 *Let $\epsilon \geq 0$, $r \geq 0$, and θ be an $\mathcal{F}_t-$stopping time, such that $\theta \leq \tau_\epsilon$ the $\epsilon-$optimal stopping time.*

Then,

$$v(x,i) = E^{x,i}[e^{-r\theta} v(X(\theta), \alpha(\theta))] \text{ for all } x, i. \tag{2.36}$$

Moreover for $s < \theta < T$ where s and T are positive real numbers, we have

$$v(s,x,i) = E^{s,x,i}[e^{-r(\theta-s)} v(\theta, X(\theta), \alpha(\theta))] \text{ for all } x, i. \tag{2.37}$$

Proof. Consider the continuous reward function $g(s,x,i) = e^{-rs}g_0(x,i)$ where $g_0(x,i) : \mathbb{R} \times \mathcal{M} \to [0,+\infty)$ is continuous in x and bounded. Let $w(s,x,i)$ be the optimal reward function. Note that $v(x,i) = w(0,x,i)$. Using (2.16), we have

$$\begin{aligned} |w(s,x,i) - E^{s,x,i}[e^{-r\tau_\epsilon}g_0(X(\tau_\epsilon),\alpha(\tau_\epsilon))]| &< 2\epsilon \text{ for all } s,x,i \\ |w(0,x,i) - E^{0,x,i}[e^{-r\tau_\epsilon}g_0(X(\tau_\epsilon),\alpha(\tau_\epsilon))]| &< 2\epsilon, \text{ so we have,} \\ |v(x,i) - E^{x,i}[e^{-r\tau_\epsilon}g_0(X(\tau_\epsilon),\alpha(\tau_\epsilon))]| &< 2\epsilon. \end{aligned} \tag{2.38}$$

Note that $g_0(x,i) \le v(x,i)$, then

$$E^{x,i}[e^{-r\tau_\epsilon}g_0(X(\tau_\epsilon),\alpha(\tau_\epsilon))] \le E^{x,i}[e^{-r\tau_\epsilon}v(X(\tau_\epsilon),\alpha(\tau_\epsilon))].$$

Consequently, using Corollary 2.1.11, we have

$$0 \le v(x,i) - E^{x,i}[e^{-r\tau_\epsilon}v(X(\tau_\epsilon),\alpha(\tau_\epsilon)) \le v(x,i) - E^{x,i}[e^{-r\tau_\epsilon}g_0(X(\tau_\epsilon),\alpha(\tau_\epsilon))] \le 2\epsilon. \tag{2.39}$$

Recall that $\theta \le \tau_\epsilon$ and $v(x,i)$ is supermeanvalued using Lemma 2.1.7 e), we have

$$E^{x,i}[e^{-r\tau_\epsilon}v(X(\tau_\epsilon),\alpha(\tau_\epsilon))] \le E^{x,i}[e^{-r\theta}v(X(\theta),\alpha(\theta))].$$

Combining (2.39) and the last inequality we obtain

$$0 \le v(x,i) - E^{x,i}[e^{-r\theta}v(X(\theta),\alpha(\theta))] \le v(x,i) - E^{x,i}[e^{-r\tau_\epsilon}v(X(\tau_\epsilon),\alpha(\tau_\epsilon)) \le 2\epsilon.$$

Therefore, we have

$$0 \le v(x,i) - E^{x,i}[e^{-r\theta}v(X(\theta),\alpha(\theta))] \le 2\epsilon. \tag{2.40}$$

Sending ϵ to zero in (2.40), we have

$$v(x,i) = E^{x,i}[e^{-r\theta}v(X(\theta),\alpha(\theta))].$$

Now let us prove (2.37). We first apply inequality (2.40) for a time inhomogeneous optimal reward $v(s,x,i)$ and we obtain

$$0 \le v(s,x,i) - E^{s,x,i}[e^{-r\theta}v(\theta,X(\theta),\alpha(\theta))] \le 2\epsilon. \tag{2.41}$$

Moreover, we know that

$$E^{s,x,i}[e^{-r\theta}v(\theta,X(\theta),\alpha(\theta))] \le E^{s,x,i}[e^{-r(\theta-s)}v(\theta,X(\theta),\alpha(\theta))].$$

Consequently,

$$0 \le v(s,x,i) - E^{s,x,i}[e^{-r(\theta-s)}v(\theta,X(\theta),\alpha(\theta))] \le v(s,x,i) - E^{s,x,i}[e^{-r\theta}v(\theta,X(\theta),\alpha(\theta))] \le 2\epsilon.$$

Sending ϵ to zero in the last inequality we have

$$v(s,x,i) = E^{s,x,i}[e^{-r(\theta-s)}v(\theta,X(\theta),\alpha(\theta))].$$

Now assume g_0 is unbounded. Define

$$g_{0N} = \min(N, g_0), \qquad g_N(s,t,i) = e^{-rs}g_{0N}(x,i), \;\; N = 1, 2, ...$$

and v_N optimal reward of g_N. For each N, we have

$$v_N(s,x,i) = E^{s,x,i}[e^{-r(\theta-s)}v_N(\theta,X(\theta),\alpha(\theta))].$$

Using the monotone convergence theorem, we obtain

$$v(s,x,i) = E^{s,x,i}[e^{-r(\theta-s)}v(\theta,X(\theta),\alpha(\theta))]. \tag{2.42}$$

This concludes the proof. □

2.2 **Valuation of American options**

2.2.1 **American Options with infinite time horizon**

The infinite time horizon is known as perpetual American option. Most related results in Guo and Zhang [16] who have obtained a closed form solution with a two state Markov chain. It is difficult to find closed-form solution with general Markov chain, we prove that the American perpetual option in the regime switching model can be characterized as a unique viscosity solution of the associated HJB equations. This existence and uniqueness are crucial to develop numerical schemes to approximate the value of the option. In this subsection we only consider perpetual put option. Similar analysis works for perpetual call option.

Given a strike price K and a risk free rate r, the price of perpetual American option is given by,

$$v(x,i) = \sup_{\tau} E^{x,i}[e^{-r\tau}(K - X(\tau))^+]. \tag{2.43}$$

The supremum is taken over all $\mathcal{F}_t$-stopping time. Recall that $\mathcal{A}$ the generator of $(X(t), \alpha(t))$ is defined as follows:

$$(\mathcal{A}v)(x,i) = \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 v(x,i)}{\partial x^2} + x\mu(i)\frac{\partial v(x,i)}{\partial x} + Qv(x,\cdot)(i),$$

where $Qv(x,.)(i) = \sum_{j\neq i} q_{ij}(v(x,j) - v(x,i))$.

Formally, v satisfies

$$\begin{aligned}\mathcal{H}\left(i,x,v,\frac{\partial v}{\partial x},\frac{\partial^2 v}{\partial x^2}\right) &= \min\left[rv(x,i) - \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 v(x,i)}{\partial x^2}\right. \\ &\left. -x\mu(i)\frac{\partial v(x,i)}{\partial x} - Qv(x,\cdot)(i), v(x,i) - (K-x)^+\right] = 0.\end{aligned} \tag{2.44}$$

Lemma 2.2.1. *$v(x,i)$ is continuous in x and $| v(x,i) | \leq K$ for some K.*

Proof. From the definition of v, it follows that

$$E[e^{-r\tau}(K - X(\tau))^+; X(0) = x, \alpha(0) = i] \leq K \qquad \text{for all stopping time } \tau.$$

So

$$v(x,i) = \sup_{\tau} E^{x,i}\left[e^{-r\tau}\left(K - X(\tau)\right)^+\right] \leq K.$$

To show the continuity of $v(x,i)$ in x, let X_1 and X_2 be solutions of (2.1) with initials $X_1(0) = x_1$ and $X_2(0) = x_2$ respectively. We have

$$\begin{aligned}(X_1(t) - X_2(t))^2 = \Big((x_1 - x_2) + \int_0^t (X_1(s) - X_2(s))\,\mu(\alpha(s))ds + \\ \int_0^t (X_1(s) - X_2(s))\,\sigma(\alpha(s))dW_s\Big)^2\end{aligned}$$

and therefore,

$$E\left(X_1(t)-X_2(t)\right)^2 \leq 3E \mid x_1-x_2 \mid^2 + 3E\left(\int_0^t (X_1(s)-X_2(s))\mu(\alpha(s))ds\right)^2$$
$$+3E\left(\int_0^t \left(X_1(s)-X_2(s)\right)\sigma(\alpha(s))dW_s\right)^2.$$

Using the Ito isometry, we obtain

$$E\left(X_1(t)-X_2(t)\right)^2 \leq 3E \mid x_1-x_2 \mid^2 + 3Et\int_0^t \Big((X_1(s)-X_2(s))\mu(\alpha(s)\Big)^2 ds$$
$$+3E\int_0^t \Big(\left(X_1(s)-X_2(s)\right)\sigma(\alpha(s))\Big)^2 ds.$$

Since μ and σ are bounded, then there exists C such that

$$E \mid X_1(t)-X_2(t) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 + C(1+t)\int_0^t E \mid X_1(s)-X_2(s) \mid^2 ds$$

Let $\epsilon > 0$. We can find T large enough so that, $Ke^{-rT} < \frac{\epsilon}{2}$. Then for $t \leq T$, we have

$$E \mid X_1(t)-X_2(t) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 + C(1+T)\int_0^t E \mid X_1(s)-X_2(s) \mid^2 ds.$$

We set $D = C(1+T)$. By Gronwall's inequality, we have

$$E \mid X_1(t)-X_2(t) \mid^2 \leq 3 \mid x_1-x_2 \mid^2 e^{Dt}.$$

Note that

$$E \mid X_1(t)-X_2(t) \mid \leq \left(E \mid X_1(t)-X_2(t) \mid^2\right)^{\frac{1}{2}}.$$

In view of this, it follows that

$$Ee^{-rt} \mid X_1(t)-X_2(t) \mid \leq 3 \mid x_1-x_2 \mid e^{Dt}. \tag{2.45}$$

For all stopping time τ, we have

$$E\Big[e^{-r\tau}\Big((K-X_1(\tau))^+ - (K-X_2(\tau))^+\Big)\Big]$$
$$\leq E\Big[e^{-r\tau}\Big((K-X_1(\tau))^+ - (K-X_2(\tau))^+\Big)I_{\tau \leq T}$$
$$+e^{-r\tau}\Big((K-X_1(\tau))^+ - (K-X_2(\tau))^+\Big)I_{\tau > T}\Big]$$

and

$$E\Big[e^{-r\tau}\Big((K - X_1(\tau))^+ - (K - X_2(\tau))^+\Big) I_{\tau\le T}\Big]$$
$$\le E\Big[e^{-rs} \mid X_1(\tau) - X_2(\tau) \mid I_{\tau\le T}\Big].$$

Moreover, we have

$$\begin{aligned} E\left[e^{-r\tau}\left((K - X_1(\tau))^+ - (K - X_2(\tau))^+\right) I_{\tau>T}\right] &\le Ke^{-rT} \\ &< \frac{\epsilon}{2}. \end{aligned} \tag{2.46}$$

Using equation (2.87), we obtain

$$E\left[e^{-r\tau} \mid (X_1(\tau) - X_2(\tau)) \mid I_{\tau\le T}\right] \le 3 \mid x_1 - x_2 \mid e^{DT}.$$

For

$$\mid x_1 - x_2 \mid \le \frac{\epsilon}{6} e^{-DT},$$

we have

$$\begin{aligned} E\left[e^{-r\tau} \mid (X_1(\tau) - X_2(\tau)) \mid I_{\tau\le T}\right] &\le 3 \mid x_1 - x_2 \mid e^{DT} \\ &\le \frac{3\epsilon}{6} e^{-DT} e^{DT} \\ &= \frac{\epsilon}{2}. \end{aligned} \tag{2.47}$$

Adding the two inequalities (2.46) and (2.47), for all τ,we obtain

$$E\left[e^{-r\tau}((K - X_1(\tau))^+ - (K - X_2(\tau))^+)I_{\tau>T}\right]$$
$$+E\left[e^{-r\tau} \mid (X_1(\tau) - X_2(\tau)) \mid I_{\tau\le T}\right] \le \epsilon.$$

Therefore,

$$\sup_{\tau} \mid E\left[e^{-r\tau}\left((K - X_1(\tau))^+ - (K - X_2(\tau))^+\right)\right] \mid \le \epsilon$$
$$\mid v(x_1, i) - v(x_2, i) \mid \le \epsilon.$$

This proves that $v(x, i)$ is continuous with respect to x. □

Theorem 2.2.2 *The value function*

$$v(x, i) = \sup_{\tau} E^{x,i}\left[e^{-r\tau}(K - X(\tau))^+\right]$$

is a viscosity solution of equation (2.44).

Proof. Let $\alpha_0 \in \mathcal{M}$, we need to prove (1.18) and (1.19). First of all, let us prove the inequality (1.19).

We have, for $\tau = 0$, $E^{x_0,\alpha_0}[e^{-r\tau}(K - X(\tau))^+] = (K - x_0)^+$. This implies that

$$v(x_0, \alpha_0) - (K - x_0)^+ \geq 0. \tag{2.48}$$

Let $\psi(\cdot) \in C^2(\mathbb{R})$ and $x_0 \in \mathbb{R}^+$ such that $v(x, \alpha_0) - \psi(x)$ has local minimum at $x = x_0$ in a neighborhood $N(x_0)$. Define the function φ as follows:

$$\varphi(x, i) = \begin{cases} \psi(x) + v(x_0, \alpha_0) - \psi(x_0), & \text{if} \quad i = \alpha_0, \\ v(x, i), & \text{if} \quad i \neq \alpha_0. \end{cases} \tag{2.49}$$

Let γ be the first jump time of $\alpha(\cdot)$ and let $\theta \in (0, \gamma]$ be such that $X(t)$ starts at x_0 and stays in $N(x_0)$ for $0 \leq t \leq \theta$. Note that $\theta \leq \gamma$. We have $\alpha(t) = \alpha_0$, for $0 \leq t \leq \theta$. By Dynkin's formula, we have

$$\begin{aligned} E^{x_0,\alpha_0} e^{-r\theta} \varphi(X(\theta), \alpha_0) - \varphi(x_0, \alpha_0) = E^{x_0,\alpha_0} \int_0^\theta e^{-rt} \Big(- r\varphi(X(t), \alpha_0) \\ + \frac{1}{2} X(t)^2 \sigma^2(\alpha_0) \frac{\partial^2 \varphi(X(t), \alpha_0)}{\partial x^2} + X(t)\mu(\alpha_0) \frac{\partial \varphi(X(t), \alpha_0)}{\partial x} \\ + Q\varphi(X(t), \cdot)(\alpha_0) \Big) dt. \end{aligned} \tag{2.50}$$

Recall that, for $0 \leq t \leq \theta$, $X(t) \in N(x_0)$, and x_0 is the minimum of $v(x, \alpha_0) - \psi(x)$. Then for $0 \leq t \leq \theta$, we have

$$v(X(t), \alpha_0) \geq \psi(X(t)) + v(x_0, \alpha_0) - \psi(x_0) = \varphi(X(t), \alpha_0). \tag{2.51}$$

Using equations (2.49) and (2.51), we have

$$\begin{aligned} E^{x_0,\alpha_0} e^{-r\theta} \Big(\psi(X(\theta)) + v(x_0, \alpha_0) - \psi(x_0) \Big) - v(x_0, \alpha_0) \\ \geq E^{x_0,\alpha_0} \int_0^\theta e^{-rt} \Big(- rv(X(t), \alpha_0) + \frac{1}{2} X(t)^2 \sigma^2(\alpha_0) \frac{\partial^2 \psi(X(t))}{\partial x^2} \\ + X(t)\mu(\alpha_0) \frac{\partial \psi(X(t))}{\partial x}, \alpha_0) + Q\varphi(X(t), \cdot)(\alpha_0) \Big) dt. \end{aligned} \tag{2.52}$$

The inequality (2.51) can also be written in the following form

$$\psi(X(t)) \leq v(X(t), \alpha_0) - (v(x_0, \alpha_0) - \psi(x_0)). \tag{2.53}$$

Recall that

$$Qv(x,\cdot)(\alpha_0) = \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}(v(x,\beta) - v(x,\alpha_0)).$$

Using equation (2.49), we have

$$\begin{aligned} Q\varphi(x,\cdot)(\alpha_0) &= \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}(\varphi(x,\beta) - \varphi(x,\alpha_0)) \\ &= \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}(v(x,\beta) - \varphi(x,\alpha_0)) \\ &= \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}\Big(v(x,\beta) - [v(x_0,\alpha_0) + \psi(x) - \psi(x_0)]\Big). \end{aligned} \tag{2.54}$$

Combining with equation (2.51), we obtain

$$\begin{aligned} Q\varphi(X(t),\cdot)(\alpha_0) &= \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}\Bigg(v(X(t),\beta) - [v(x_0,\alpha_0) + \psi(X(t)) - \psi(x_0)]\Bigg) \\ &\geq \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}\Bigg(v(X(t),\beta) - [v(x_0,\alpha_0) + v(X(t),\alpha_0) \\ &- (v(x_0,\alpha_0) - \psi(x_0)) - \psi(x_0)]\Bigg) \qquad (2.55) \\ &\geq \sum_{\beta\neq\alpha_0} q_{\alpha_0\beta}\left(v(X(t),\beta) - v(X(t),\alpha_0)\right) \\ &\geq Qv(X(t),\cdot)(\alpha_0). \qquad (2.56) \end{aligned}$$

Then, we obtain

$$\begin{aligned} &E^{x_0,\alpha_0} e^{-r\theta}\Big(\psi(X(\theta)) + v(x_0,\alpha_0) - \psi(x_0)\Big) - v(x_0,\alpha_0) \\ &\geq E^{x_0,\alpha_0}\int_0^\theta e^{-rt}\Bigg(- rv(X(t),\alpha_0) + \frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\psi(X(t))}{\partial x^2} \\ &\qquad + X(t)\mu(\alpha_0)\frac{\partial\psi(X(t))}{\partial x} + Qv(X(t),\cdot)(\alpha_0)\Bigg)dt. \end{aligned}$$

Using equation(2.51), we have

$$\begin{aligned} &E^{x_0,\alpha_0} e^{-r\theta}\Big(\psi(X(\theta)) + v(X(\theta),\alpha_0) - \psi(X(\theta))\Big) - v(x_0,\alpha_0)) \\ &\geq E^{x_0,\alpha_0}\int_0^\theta e^{-rt}\Big(- rv(X(t),\alpha_0) + \frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\psi(X(t))}{\partial x^2} \\ &\qquad + X(t)\mu(\alpha_0)\frac{\partial\psi(X(t))}{\partial x} + Qv(X(t),\cdot)(\alpha_0)\Big)dt. \end{aligned}$$

Which leads to,

$$E^{x_0,\alpha_0} e^{-r\theta}(v(X(\theta),\alpha_0)) - v(x_0,\alpha_0) \geq$$
$$E^{x_0,\alpha_0} \int_0^\theta e^{-rt}\Big(\frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\psi(X(t))}{\partial x^2} - rv(X(t),\alpha_0)$$
$$+X(t)\mu(\alpha_0)\frac{\partial\psi(X(t))}{\partial x} + Qv(X(t),\cdot)(\alpha_0)\Big)dt.$$

Noticing that v is supermeanvalued by Corollary 2.1.11, we have

$$E^{x_0,\alpha_0} e^{-r\theta} v(X(\theta),\alpha_0) \leq v(x_0,\alpha_0)$$

therefore,

$$0 \geq E^{x_0,\alpha_0} \int_0^\theta e^{-rt}\left(\frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\psi(X(t))}{\partial x^2} - rv(X(t),\alpha_0)\right.$$
$$\left.+X(t)\mu(\alpha_0)\frac{\partial\psi(X(t))}{\partial x} + Qv(X(t),\cdot)(\alpha_0)\right)dt.$$

Dividing both sides by θ, we obtain

$$0 \geq \frac{1}{\theta}E^{x_0,\alpha_0} \int_0^\theta e^{-rt}\left(\frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\psi(X(t))}{\partial x^2}\right.$$
$$\left.+X(t)\mu(\alpha_0)\frac{\partial\psi(X(t))}{\partial x} - rv(X(t),\alpha_0) + Qv(X(t),\cdot)(\alpha_0)\right)dt.$$

By letting $\theta \to 0$, we obtain

$$0 \geq \frac{1}{2}x_0^2\sigma^2(\alpha_0)\frac{\partial^2\psi(x_0)}{\partial x^2} + x_0\mu(\alpha_0)\frac{\partial\psi(x_0)}{\partial x} - rv(x_0,\alpha_0) + Qv(x_0,\cdot)(\alpha_0).$$

So,

$$rv(x_0,\alpha_0) - \frac{1}{2}x_0^2\sigma^2(\alpha_0)\frac{\partial^2\psi(x_0)}{\partial x^2} - x_0\mu(\alpha_0)\frac{\partial\psi(x_0)}{\partial x} - Qv(x_0,\cdot)(\alpha_0) \geq 0. \tag{2.57}$$

Combining (2.48) and (2.57), we obtain

$$\mathcal{H}(\alpha_0, x_0, v(x_0,\alpha_0), \frac{\partial\psi(x_0)}{\partial x}, \frac{\partial^2\psi(x_0)}{\partial x^2}) \geq 0$$

which gives (1.19). Therefore, $v(x,\alpha)$ is a viscosity supersolution.

Next, let us prove the inequality (1.18). Let $\phi(\cdot) \in C^2(\mathbb{R})$ and $x_0 \in \mathbb{R}^+$ such that $v(x,\alpha_0) - \phi(x)$ has local maximum at $x = x_0$ in a neighborhood $N(x_0)$. Let θ_0 be a stopping time less than γ the first jump time of the process $\alpha(\cdot)$ and such that $X(t)$ starts at x_0 and stays in

$N(x_0)$ for $0 \le t \le \theta_0$. We can assume without loss of generality that $v(x_0, \alpha_0) - \phi(x_0) = 0$. Let $\epsilon > 0$ and τ_ϵ be the $\epsilon-$optimal stopping time, using Corollary 2.1.13 for $0 \le \theta \le \min(\tau_\epsilon, \theta_0)$ we have

$$v(x_0, \alpha_0) \le E^{x_0,\alpha_0}[e^{-r\theta} v(X(\theta), \alpha_0))]. \tag{2.58}$$

Note that $v(x_0, \alpha_0) - \phi(x_0) = 0$ and attains its maximum at x_0 in $N(x_0)$ therefore

$$v(X(\theta), \alpha_0) \le \phi(X(\theta)).$$

Define

$$\varphi(x, i) = \begin{cases} \phi(x), & \text{if} \quad i = \alpha_0, \\ v(x, i), & \text{if} \quad i \ne \alpha_0. \end{cases} \tag{2.59}$$

Thus, we also have

$$v(X(\theta), \alpha_0) \le \varphi(X(\theta)), \alpha_0). \tag{2.60}$$

Using Dynkin's formula, we obtain

$$\begin{aligned} E^{x_0,\alpha_0} e^{-r\theta} v(X(\theta), \alpha_0) &\le E^{x_0,\alpha_0} e^{-r\theta} \varphi(X(\theta), \alpha_0) \\ &= \varphi(x_0, \alpha_0) + E^{x_0,\alpha_0} \int_0^\theta e^{-rt} \Big[Q\varphi(X(t), \cdot)(\alpha_0) \\ &\quad + X(t)\mu(\alpha_0) \frac{\partial \phi(X(t))}{\partial x} - r\varphi(X(t), \alpha(t)) \\ &\quad + \frac{1}{2} X(t)^2 \sigma^2(\alpha_0) \frac{\partial^2 \phi(X(t))}{\partial x^2} \Big] dt. \end{aligned} \tag{2.61}$$

Moreover,

$$\begin{aligned} Q\varphi(X(t), \cdot)(\alpha_0) &= \sum_{\beta \ne \alpha_0} q_{\alpha_0 \beta}(v(X(t), \beta) - \phi(X(t))) \\ &\le \sum_{\beta \ne \alpha_0} q_{\alpha_0 \beta}(v(X(t), \beta) - v(X(t), \alpha_0) \\ &\le \sum_{\beta \ne \alpha_0} q_{\alpha_0 \beta}(v(X(t), \beta) - v(X(t), \alpha_0)) \\ &\le Qv(X(t), \cdot)(\alpha_0). \end{aligned} \tag{2.62}$$

Using (2.60) and (2.62), we obtain

$$E^{x_0,\alpha_0}e^{-r\theta}v(X(\theta),\alpha_0) \le \phi(x_0) + E^{x_0,\alpha_0}\int_0^\theta e^{-rt}\Big[\frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\phi(X(t),\alpha_0)}{\partial x^2}$$
$$+X(t)\mu(\alpha_0)\frac{\partial\phi(X(t),\alpha_0)}{\partial x} - rv(X(t),\alpha(t)) + Qv(X(t),\cdot)(\alpha_0)\Big]dt. \tag{2.63}$$

From (2.58) we deduce

$$0 = v(x_0,\alpha_0) - \phi(x_0) \le E^{x_0,\alpha_0}\int_0^\theta e^{-rt}\Big[\frac{1}{2}X(t)^2\sigma^2(\alpha_0)\frac{\partial^2\phi(X(t),\alpha_0)}{\partial x^2}$$
$$+X(t)\mu(\alpha_0)\frac{\partial\phi(X(t),\alpha_0)}{\partial x} - rv(X(t),\alpha(t)) + Qv(X(t),\cdot)(\alpha_0)\Big]dt.$$

Dividing the last inequality by $\theta > 0$ and sending $\theta \downarrow 0$, we have

$$\frac{1}{2}x_0^2\sigma^2(\alpha_0)\frac{\partial^2\phi(x_0)}{\partial x^2} + x_0\mu(\alpha_0)\frac{\partial\phi(x_0)}{\partial x} - rv(x_0,\alpha_0)$$
$$+Qv(x_0,\cdot)(\alpha_0) \ge 0.$$

Thus

$$rv(x_0,\alpha_0) - \frac{1}{2}x_0^2\sigma^2(\alpha_0)\frac{\partial^2\phi(x_0)}{\partial x^2} - x_0\mu(\alpha_0)\frac{\partial\phi(x_0)}{\partial x} - Qv(x_0,\cdot)(\alpha_0) \le 0.$$

The last inequality implies (1.18). Finally, $v(x,i)$ is a viscosity subsolution of (2.44). This concludes the proof. □

In order to prove the uniqueness result of the viscosity solution we need the following maximum principle for semicontinuous function, which is stated in a suitable form for our application.

Theorem 2.2.3 (Crandall, Lions and Ishii [6]) *For $i = 1, 2$, let Ω_i be locally compact subsets of $\mathbb{R}$ and u_i be upper semicontinuous in Ω_i, and $\bar{J}^{2,+}_{\Omega_i}u_i(x)$ the parabolic superjet of $u_i(x)$, and ϕ be twice continuously differentiable in a neighborhood of Ω.*
Set

$$w(x) = u_1(x_1) + u_2(x_2)$$

for $x = (x_1, x_2) \in \Omega$, and suppose $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \Omega$ is a local maximum of $w - \phi$ relative to Ω. Then for each $\epsilon > 0$ there exists $X_i \in \mathcal{S}(1) = \mathbb{R}$ such that

$$(Dx_i\phi(\hat{x}), X_i) \in \bar{J}^{2,+}_{\Omega_i}u_i(\hat{x}_i)$$

for $i = 1, 2$ and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\epsilon} + \|D^2\phi(\hat{x})\|\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2\phi(\hat{x}) + \epsilon(D^2\phi(\hat{x}))^2. \tag{2.64}$$

Theorem 2.2.4 (Comparison Principle) *If $v_1(x, i)$ and $v_2(x, i)$ are respectively viscosity subsolution and supersolution of (2.44) and are continuous with respect to x and have at most a linear growth. Then*

$$v_1(x, i) \leq v_2(x, i) \quad \textit{for all } (x, i) \in \mathbb{R}^+ \times \mathcal{M}. \tag{2.65}$$

Proof. For any $0 < \delta < 1$ and $0 < \gamma < 1$, we define

$$\Phi(x, y, i) = v_1(x, i) - v_2(y, i) - \frac{1}{\delta} \mid x - y \mid^2 - \gamma(x^2 + y^2),$$

and

$$\phi(x, y) = \frac{1}{\delta} \mid x - y \mid^2 + \gamma(x^2 + y^2).$$

Since $v_1(x, i)$ and $v_2(x, i)$ satisfy the linear growth, we have for each $i \in \mathcal{M}$

$$\lim_{|x|+|y| \to \infty} \Phi(x, y, i) = -\infty.$$

Moreover the continuity of Φ in (x,y) implies that its has a global maximum at a point $(x_\delta, y_\delta, \alpha_0)$ because $\mathcal{M}$ is finite. Therefore,

$$\Phi(x_\delta, x_\delta, \alpha_0) + \Phi(y_\delta, y_\delta, \alpha_0) \leq 2\Phi(x_\delta, y_\delta, \alpha_0).$$

So,

$$\begin{aligned} v_1(x_\delta, \alpha_0) - v_2(x_\delta, \alpha_0) - 2\gamma(x_\delta^2) + v_1(y_\delta, \alpha_0) - v_2(y_\delta, \alpha) - 2\gamma(y_\delta^2) \leq 2v_1(x_\delta, \alpha_0) \\ -2v_2(y_\delta, \alpha) - \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 - 2\gamma(x_\delta^2 + y_\delta^2). \end{aligned}$$

It follows that

$$\begin{aligned} -v_2(y_\delta, \alpha_0) - 2\gamma(x_\delta^2)) + v_1(x_\delta, \alpha_0) - 2\gamma(x_\delta^2)) \leq v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0) \\ -\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 - 2\gamma(x_\delta^2 + y_\delta^2). \end{aligned}$$

Finally, we have

$$\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \leq (v_1(x_\delta, \alpha_0) - v_1(y_\delta, \alpha_0)) + (v_2(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0)). \tag{2.66}$$

By the linear growth condition, we know that there exist constants K_1, K_2 such that $v_1(x, i) \leq K_1(1+ \mid x \mid)$ and $v_2(x, i) \leq K_2(1+ \mid x \mid)$. Therefore there exists C such that

$$\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \leq C(1+ \mid x_\delta \mid + \mid y_\delta \mid),$$

which implies

$$\mid x_\delta - y_\delta \mid^2 \leq \delta C(1+ \mid x_\delta \mid + \mid y_\delta \mid). \tag{2.67}$$

We also have, $\Phi(0, 0, \alpha_0) \leq \Phi(x_\delta, y_\delta, \alpha_0)$ and $\mid \Phi(0, 0, \alpha_0) \mid \leq C(1+ \mid x_\delta \mid + \mid y_\delta)$. Therefore,

$$\begin{aligned} \gamma(x_\delta^2 + y_\delta^2) &\leq v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0) - \frac{1}{\delta} \mid x_\delta - y_\delta \mid^2 - \Phi(0, 0, \alpha_0) \\ &\leq 3C(1+ \mid x_\delta \mid + \mid y_\delta \mid) \end{aligned} \tag{2.68}$$

so

$$\frac{\gamma(x_\delta^2 + y_\delta^2)}{(1+ \mid x_\delta \mid + \mid y_\delta \mid)} \leq 3C.$$

Therefore, there exists C_γ such that

$$\mid x_\delta \mid + \mid y_\delta \mid \leq C_\gamma. \tag{2.69}$$

The inequality (2.69) implies the sets $\{x_\delta, \delta > 0\}$, and $\{y_\delta, \delta > 0\}$ are bounded by C_γ independent of δ so we can extract convergent subsequences. Moreover, the inequality (2.67) implies the existence x_0 such that

$$\lim_{\delta \to 0} x_\delta = x_0 = \lim_{\delta \to 0} y_\delta. \tag{2.70}$$

Using (2.66) with the last result, we deduce that

$$\lim_{\delta \to 0} \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 = 0. \tag{2.71}$$

The point $(x_\delta, y_\delta, \alpha_0)$ is the maximum of Φ, so by the Crandall Ishii and Lions's Maximum principle for each $\epsilon > 0$ there exist X_δ and Y_δ such that

$$(\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma x_\delta, X_\delta) \in \bar{\mathcal{P}}^{2,+} v_1(x_\delta, \alpha_0) \tag{2.72}$$

and

$$(-\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma y_\delta, -Y_\delta) \in \bar{\mathcal{P}}^{2,+}(-v_2(y_\delta, \alpha_0)) = -\bar{\mathcal{P}}^{2,-} v_2(y_\delta, \alpha_0).$$

So

$$(\frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma y_\delta, Y_\delta) \in \bar{\mathcal{P}}^{2,-} v_2(y_\delta, \alpha_0). \tag{2.73}$$

(2.72) implies by the definition of the viscosity solution that

$$\min \Big[r v_1(x_\delta, \alpha_0) - \frac{1}{2} x_\delta^2 \sigma^2(\alpha_0) X_\delta - x_\delta \mu(\alpha_0) \left(\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma x_\delta \right) \\ - Q v_1(x_\delta, \cdot)(\alpha_0), v_1(x_\delta, \alpha_0) - (K - x_\delta)^+ \Big] \leq 0.$$

In view of these, we have two cases: either

$$v_1(x_\delta, \alpha_0) - (K - x_\delta)^+ \leq 0$$

or

$$r v_1(x_\delta, \alpha_0) - \tfrac{1}{2} x_\delta^2 \sigma^2(\alpha_0) X_\delta - x_\delta \mu(\alpha_0) \left(\tfrac{2}{\delta}(x_\delta - y_\delta) + 2\gamma x_\delta \right) \\ - Q v_1(x_\delta, \cdot)(\alpha_0) \leq 0.$$

Next, we assume that $v_1(x_\delta, \alpha_0) - (K - x_\delta)^+ \leq 0$. Similarly, (2.73) implies by the definition of the viscosity solution that,

$$\min \Big[r v_2(y_\delta, \alpha_0) - \frac{1}{2} y_\delta^2 \sigma^2(\alpha_0) Y_\delta - y_\delta \mu(\alpha_0) \left(\frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma y_\delta \right) \\ - Q v_2(y_\delta, \cdot)(\alpha_0), v_2(y_\delta, \alpha_0) - (K - y_\delta)^+ \Big] \geq 0.$$

Therefore, we have

$$(K - y_\delta)^+ - v_2(y_\delta, \alpha_0) \leq 0.$$

We obtain

$$v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0) - (y_\delta - x_\delta)^+ \leq 0.$$

Letting $\delta \to 0$, we have

$$v_1(x_0, \alpha_0) - v_2(x_0, \alpha_0) \le 0. \tag{2.74}$$

Recall that, the function Φ reaches its maximum at $(x_\delta, y_\delta, \alpha_0)$. It follows that, for all x and all $i \in \mathcal{M}$,

$$\begin{aligned} v_1(x,i) - v_2(x,i) - 2\gamma x^2 = \Phi(x,x,i) &\le \Phi(x_\delta, y_\delta, \alpha_0) \\ &\le v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0) - \gamma(x_\delta^2 + y_\delta^2). \end{aligned}$$

Again letting $\delta \to 0$ and using (2.74), we obtain

$$v_1(x,i) - v_2(x,i) - 2\gamma x^2 \le v_1(x_0, \alpha_0) - v_2(x_0, \alpha_0) - 2\gamma(x_0)^2 \le 0.$$

So, we have

$$v_1(x,i) - v_2(x,i) \le 2\gamma(x^2). \tag{2.75}$$

Now, let us assume that

$$\begin{aligned} \frac{1}{2} x_\delta^2 \sigma^2(\alpha_0) X_\delta + x_\delta \mu(\alpha_0) \left(\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma x_\delta \right) \\ + Q v_1(x_\delta, \cdot)(\alpha_0) - r v_1(x_\delta, \alpha_0) \ge 0. \end{aligned}$$

Then from (2.73), we have

$$\begin{aligned} \frac{1}{2} y_\delta^2 \sigma^2(\alpha_0) Y_\delta + y_\delta \mu(\alpha_0) \left(\frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma y_\delta \right) \\ + Q v_2(y_\delta, \cdot)(\alpha_0) - r v_2(y_\delta, \alpha_0) \le 0, \end{aligned}$$

and

$$\begin{aligned} r(v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0)) \le \frac{1}{2}\sigma^2(\alpha_0) \left((x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \right) \\ + \mu(\alpha_0) \left(\frac{2}{\delta}(x_\delta - y_\delta)^2 + 2\gamma((x_\delta)^2 + (y_\delta)^2) \right) \\ + Q v_1(x_\delta, \cdot)(\alpha_0) - Q v_2(y_\delta, \cdot)(\alpha_0). \end{aligned} \tag{2.76}$$

Moreover, from the Maximum Principle, we have

$$\begin{aligned} -\left(\frac{1}{\epsilon} + \|D^2\phi(x_\delta, y_\delta)\| \right) I \le \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \le D^2\phi(x_\delta, y_\delta) \\ + \epsilon (D^2\phi(x_\delta, y_\delta))^2. \end{aligned} \tag{2.77}$$

Note that

$$D^2\phi(x_\delta, y_\delta) = \frac{2}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2\gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} (D^2\phi(x_\delta, y_\delta))^2 &= \frac{8}{\delta^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{8\gamma}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &+ 4\gamma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{8+8\gamma\delta}{\delta^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 4\gamma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.78}$$

We remark that

$$\begin{aligned} (x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta &= (x_\delta, y_\delta)\begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix}\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} \\ &\leq (x_\delta, y_\delta)\Big[\frac{2}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \epsilon\frac{8+8\gamma\delta}{\delta^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &+ (2\gamma + 4\epsilon\gamma^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big]\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix}. \end{aligned} \tag{2.79}$$

Letting $\gamma \to 0$ in the last expression, we obtain

$$(x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \leq (x_\delta, y_\delta)\left[(\frac{2}{\delta} + \epsilon\frac{8}{\delta^2})\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix}.$$

Take $\epsilon = \delta/4$, this leads us to

$$(x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \leq (x_\delta, y_\delta)\left[\frac{4}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} = \frac{4}{\delta}(x_\delta - y_\delta)^2.$$

Sending $\delta \to 0$ and using (2.70), we obtain

$$(x_0)^2 X_0 - (y_0)^2 Y_0 \leq 0. \tag{2.80}$$

Moreover, using (2.70), (2.71) and letting $\delta, \gamma \to 0$ in (2.76) imply that

$$r(v_1(x_0, \alpha_0) - v_2(x_0, \alpha_0)) \leq Qv_1(x_0, \cdot)(\alpha_0) - Qv_2(x_0, \cdot)(\alpha_0). \tag{2.81}$$

Recall that $(x_\delta, y_\delta, \alpha_0)$ is maximum of Φ then, for all $x \in \mathbb{R}$ and for all $i \in \mathcal{M}$ we have

$$\Phi(x, x, i) \leq \Phi(x_\delta, y_\delta, \alpha_0)$$

in order terms, we have

$$v_1(x, i) - v_2(x, i) - 2\gamma x^2 \leq v_1(x_\delta, \alpha_0) - v_2(y_\delta, \alpha_0) - 2\gamma(x_\delta^2 + y_\delta^2).$$

Letting $\delta \to 0$, we obtain

$$v_1(x, i) - v_2(x, i) - 2\gamma x^2 \leq v_1(x_0, \alpha_0) - v_2(y_0, \alpha_0) - 2\gamma x_0^2 \tag{2.82}$$

taking $x = x_0$, we have

$$v_1(x_0, i) - v_2(x_0, i) - 2\gamma x_0^2 \leq v_1(x_0, \alpha_0) - v_2(y_0, \alpha_0) - 2\gamma x_0^2.$$

Consequently,

$$v_1(x_0, i) - v_2(x_0, i) \leq v_1(x_0, \alpha_0) - v_2(x_0, \alpha_0).$$

We recall that

$$\begin{aligned} Qv_1(x_0, \cdot)(\alpha_0) - Qv_2(x_0, \cdot)(\alpha_0) = \sum_{\beta \neq \alpha_0} q_{\alpha_0 \beta}[v_1(x_0, \beta) - v_1(x_0, \alpha_0) \\ - v_2(x_0, \beta) + v_2(x_0, \alpha_0)] \leq 0, \end{aligned}$$

using (2.81), we have

$$v_1(x_0, \alpha_0) - v_2(x_0, \alpha_0) \leq 0.$$

Therefore using (2.82), we conclude that

$$v_1(x, i) - v_2(x, i) - 2\gamma x^2 \quad \leq \quad v_1(x_0, \alpha_0) - v_2(y_0, \alpha_0) - 2\gamma x_0^2 \leq 0. \tag{2.83}$$

Letting $\gamma \to 0$ in (2.75) and the previous inequality, we obtain

$$v_1(x, i) \leq v_2(x, i).$$

This completes the proof of the theorem. $\square$

This result implies the uniqueness of the viscosity solution of (2.44) because any viscosity solution is both viscosity supersolution and subsolution.

2.2.2 Finite Time Horizon American Option

In reality the perpetual America option is not tradable, only the finite time horizon American option is traded on financial markets. As for the perpetual one, the valuation of the finite time horizon American option is concerned with optimal stopping with finite time horizon. As a result, the associated PDE is parabolic. There is substantial literature on American option pricing. We refer to Bensoussan [2], Bollen [3], [4], Duan [8], Kim [25], among many others for related results.

Recall that the value of the American option put is given by

$$v(s,x,i) = \sup_{\tau \in \Lambda_{s,T}} E\left[e^{-r(\tau-s)}(K - X(\tau))^{+} \mid X(s) = x, \alpha(s) = i\right], \tag{2.84}$$

where $\Lambda_{s,T} = \{\tau,\ \mathcal{F}_t - \text{stopping time}\ ; s \leq \tau \leq T\}$ K is the strike price of the option, and $T < \infty$ the expiration date. The generator of the process $(X(s), \alpha(s))$ is defined as follows,

$$(\mathcal{A}f)(s,x,i) = \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 f(s,x,i)}{\partial x^2} + x\mu(i)\frac{\partial f(s,x,i)}{\partial x} + Qf(s,x,\cdot)(i)$$

where

$$Qf(s,x,\cdot)(i) = \sum_{j \neq i} q_{ij}(f(s,x,j) - f(s,x,i)).$$

We define the following Hamiltonian,

$$\begin{aligned}\mathcal{H}(s,x,i,u,\mathrm{D}_s u,\mathrm{D}_x u,\mathrm{D}_x^2 u) = \min\Big[ru(s,x,i) - \frac{\partial u(s,x,i)}{\partial s} \\ -(\mathcal{A}u)(s,x,i), u(s,x,i) - (K-x)^{+}\Big] = 0.\end{aligned} \tag{2.85}$$

Formally, the value function $v(s,x,i)$ satisfies

$$\begin{cases} \mathcal{H}(s,x,i,v,\mathrm{D}_s v,\mathrm{D}_x v,\mathrm{D}_x^2 v) = 0 \ \text{ for } (s,x,i) \in [s,T) \times \mathbb{R}^{+} \times \mathcal{M}, \\ v(T,x,\alpha(T)) = (K-x)^{+}. \end{cases} \tag{2.86}$$

As for the infinite horizon case we will firstly prove that the value function is continuous in terms of x, so we have the following lemma.

Lemma 2.2.5 *The value function $v(s,x,i)$ defined in (2.84) is continuous in (s,x) and satisfies $\mid v(s,x,i) \mid \leq K$.*

Proof. Note that $|v(s,x,i)| \leq K$. Given x_1 and x_2, let X_1 and X_2 be two solutions of (1.1) with $X_1(s) = x_1$ and $X_2(s) = x_2$, respectively. Applying Gronwall's inequality, we have

$$E \mid X_1(t) - X_2(t) \mid^2 \leq C \mid x_1 - x_2 \mid^2 e^{Dt}, \qquad \text{for some } C > 0,\ D > 0.$$

This implies, in view of Cauchy-Schwarz inequality, that

$$E \mid X_1(t) - X_2(t) \mid \leq C \mid x_1 - x_2 \mid e^{Dt}. \tag{2.87}$$

Using this inequality, we have

$$\begin{aligned} v(s,x_1,i) - v(s,x_2,i) &\leq \sup_{\tau \in \Lambda_{s,T}} E\Big[e^{-r(\tau-s)}\Big|(K - X_1(\tau))^+ - (K - X_2(\tau))^+\Big|\Big] \\ &\leq \sup_{\tau \in \Lambda_{s,T}} E\Big[|X_1(\tau) - X_2(\tau)|\Big] \\ &\leq C \mid x_1 - x_2 \mid e^{DT}. \end{aligned} \tag{2.88}$$

This implies the (uniform) continuity of $v(s,x,i)$ with respect to x.

We next show the continuity of $v(s,x,i)$ with respect to s. Let $X(t)$ be the solution of (1.1) that starts at $t = s$ with $X(s) = x$ and $\alpha(s) = i$. Let $0 \leq s \leq s' \leq T$, we define

$$\begin{cases} X'(t) = X(t - (s' - s)), \\ \alpha'(t) = \alpha(t - (s' - s)). \end{cases} \tag{2.89}$$

It is easy to show that

$$E(X(t) - X'(t))^2 \leq C(s' - s).$$

Given $\tau \in \Lambda_{s,T}$, let $\tau' = \tau + (s' - s)$. Then $\tau' \geq s'$ and $P(\tau' > T) \to 0$ as $s' - s \to 0$.

Let $g(t,x) = e^{-rt}(K - x)^+$. Then $v(s,x,i) = e^{rs} \sup_{\tau \in \Lambda_{s,T}} E^{s,x,i} g(\tau, X(\tau))$. It is easy to show that

$$|g(s,x) - g(s',x')| \leq |x - x'| + C|x' - K||s - s'|,$$

for some constant C.

We define

$$J(s,x,i,\tau) = e^{rs} E^{s,x,i} g(\tau, X(\tau)).$$

We have

$$\begin{aligned} J(s,x,i,\tau) = &\ e^{rs}Eg(\tau' - (s'-s), X'(\tau')) \\ = &\ e^{rs'}Eg(\tau', X'(\tau')) + o(1) \\ = &\ e^{rs'}Eg(\tau', X'(\tau'))I_{\{\tau' \le T\}} + e^{rs'}Eg(\tau', X'(\tau'))I_{\{\tau' > T\}} + o(1) \\ = &\ J(s', x, i, \tau' \wedge T) + o(1), \end{aligned}$$

where $o(1) \to 0$ as $s' - s \to 0$. It follows that

$$|v(s',x,i) - v(s,x,i)| \le \sup_{\tau \in \Lambda_{s,T}} |J(s',x,i,\tau') - J(s,x,i,\tau)| \to 0.$$

Therefore, we have

$$\lim_{s'-s \to 0} |v(s',x,i) - v(s,x,i)| = 0. \tag{2.90}$$

This gives the continuity of v with respect to s.

The joint continuity of v follows from (2.88) and (2.90). This completes the proof. □

Theorem 2.2.6 *The value function $v(s,x,i)$ defined in equation(2.84) is a viscosity solution of equation (2.86).*

Proof. We note that $v(s,x,i)$ satisfies the boundary condition since

$$v(T, x, \alpha(T)) = (K - x)^+.$$

Using Corollary 2.1.13 we know that, for $s \le \theta \le \tau_\epsilon$ where τ_ϵ is the $\epsilon-$optimal stopping for some $\epsilon > 0$, we have

$$v(s,x,i) = E^{s,x,i}[e^{-r(\theta - s)}v(\theta, X(\theta), \alpha(\theta))]. \tag{2.91}$$

And from Corollary 2.1.11, we have

$$v(s,x,i) \ge E^{s,x,i}[e^{-r(\theta - s)}v(\theta, X(\theta), \alpha(\theta))] \tag{2.92}$$

for any stopping time $\theta \in \Lambda_{s,T}$.

First we want to prove that $v(t,x,i)$ is a viscosity supersolution of (2.86), namely, for any $i \in \mathcal{M}$ we have

$$\min\Big[rv(s,x_s,i) - \frac{\partial \psi(s,x_s)}{\partial s} - \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2\psi(s,x_s)}{\partial x^2} - x\mu(i)\frac{\partial \psi(s,x_s)}{\partial x} - Qv(s,x_s,\cdot)(i), v(s,x_s,i) - (K-x_s)^+\Big] \geq 0 \tag{2.93}$$

whenever $\psi \in \mathcal{C}^{1,2}([s,T]\times\mathbb{R}^+)$ and $v(t,x,\alpha_s)-\psi(t,x)$ has a local minimum at $(s,x_s) \in [s,T]\times\mathbb{R}^+$.

From the definition of v, we have

$$\begin{aligned} v(s,x_s,\alpha_s) &\geq (K-x_s)^+ \\ v(s,x_s,\alpha_s) - (K-x_s)^+ &\geq 0. \end{aligned} \tag{2.94}$$

Let $\psi \in C^2([s,T]\times\mathbb{R}^+)$ and $(s,x) \in [s,T]\times\mathbb{R}^+$ such that $v(t,x,i)-\psi(t,x)$ has local minimum at (s,x_s) in a neighborhood $N(s,x_s)$. We define a function φ as follows:

$$\varphi(t,x,i) = \begin{cases} \psi(t,x) + v(s,x_s,\alpha_s) - \psi(s,x_s), & \text{if} \quad i = \alpha_s, \\ v(t,x,i), & \text{if} \quad i \neq \alpha_s. \end{cases} \tag{2.95}$$

Let γ be the first jump time of $\alpha(\cdot)$ after the state α_s, and let $\theta \in [s,\gamma]$ be such that $(t,X(t))$ starts at (s,x_s) and stays in $N(s,x_s)$ for $s \leq t \leq \theta$. Since $\theta \leq \gamma$ we have $\alpha(t) = \alpha_s$, for $s \leq t \leq \theta$, by Dynkin's formula, we have

$$\begin{aligned} & E^{s,x_s,\alpha_s} e^{-r(\theta-s)}\varphi(\theta,X(\theta),\alpha_s) - \varphi(s,x_s,\alpha_s) \\ = \; & E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- r\varphi(t,X(t),\alpha_s) \\ & + \frac{\partial \varphi(t,X(t),\alpha_s))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\varphi(t,X(t),\alpha_s)}{\partial x^2} \\ & + X(t)\mu(\alpha_s)\frac{\partial\varphi(t,X(t),\alpha_s)}{\partial x} + Q\varphi(t,X(t),\cdot)(\alpha_s)\Big)dt \end{aligned} \tag{2.96}$$

Since for $s \leq t \leq \theta$, $(t,X(t)) \in N(s,x_s)$, and (s,x_s) is the minimum of $v(t,x,\alpha_s)-\psi(t,x)$, then for $s \leq t \leq \theta$ we have,

$$v(t,X(t),\alpha_s) \geq \psi(t,X(t)) + v(s,x_s,\alpha_s) - \psi(s,x_s) = \varphi(t,X(t),\alpha_s) \tag{2.97}$$

Using equation (2.95) and (2.97), we have

$$\begin{aligned}
& E^{s,x_s,\alpha_s} e^{-r(\theta-s)} v(t,X(t),\alpha_s) - v(s,x_s,\alpha_s) \\
\geq\ & E^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big(- rv(t,X(t),\alpha_s) \\
& + \frac{\partial \psi(t,X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(t,X(t))}{\partial x^2} \\
& + X(t)\mu(\alpha_s) \frac{\partial \psi(t,X(t))}{\partial x} + Q\varphi(t,X(t),\cdot)(\alpha_s) \Big) dt \qquad (2.98)
\end{aligned}$$

the inequality (2.97) can also be written in the following form

$$\psi(t,X(t)) \leq v(t,X(t),\alpha_s) - (v(s,x_s,\alpha_s) - \psi(s,x_s)). \qquad (2.99)$$

Recall that

$$Qv(t,x,\cdot)(\alpha_s) = \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} (v(t,x,\beta) - v(t,x,\alpha_s)).$$

Using equation (2.95), we have

$$\begin{aligned}
Q\varphi(t,x,\cdot)(\alpha_s) &= \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} (\varphi(t,x,\beta) - \varphi(t,x,\alpha_s)) \\
&= \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} (v(t,x,\beta) - \varphi(t,x,\alpha_s)) \\
&= \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} \Big(v(t,x,\beta) - [v(s,x_s,\alpha_s) \\
&\quad + \psi(t,x) - \psi(s,x_s)] \Big) \qquad (2.100)
\end{aligned}$$

and from equation (2.97), we obtain

$$\begin{aligned}
Q\varphi(t,X(t),\cdot)(\alpha_s) &= \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} \Big(v(t,X(t),\beta) - [v(s,x_s,\alpha_s) \\
&\quad + \psi(t,X(t)) - \psi(s,x_s)] \Big) \\
&\geq \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} \Big(v(t,X(t),\beta) - [v(s,x_s,\alpha_s) + v(t,X(t),\alpha_s) \\
&\quad - (v(s,x_s,\alpha_s) - \psi(s,x_s)) - \psi(s,x_s)] \Big) \\
&\geq \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta} \left(v(t,X(t),\beta) - v(Xt,X(t),\alpha_s) \right) \\
&\geq Qv(t,X(t),\cdot)(\alpha_s). \qquad (2.101)
\end{aligned}$$

Taking these into account, we deduce

$$
\begin{aligned}
& E^{s,x_s,\alpha_s} e^{-r(\theta-s)} v(t,X(t),\alpha_s) - v(s,x_s,\alpha_s) \\
\geq\ & E^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big(- rv(t,X(t),\alpha_s) \\
& + \frac{\partial\psi(t,X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(t,X(t))}{\partial x^2} \\
& + X(t)\mu(\alpha_s) \frac{\partial\psi(t,X(t))}{\partial x} + Qv(t,X(t),\cdot)(\alpha_s) \Big) dt. \qquad (2.102)
\end{aligned}
$$

Using equation (2.92), we obtain

$$
\begin{aligned}
& E^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big(- rv(t,X(t),\alpha_s) \\
& + \frac{\partial\psi(t,X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(t,X(t))}{\partial x^2} + X(t)\mu(\alpha_s) \frac{\partial\psi(t,X(t))}{\partial x} \\
& + Qv(t,X(t),\cdot)(\alpha_s) \Big) dt \leq 0.
\end{aligned}
$$

Multiply the last inequality by $\frac{1}{\theta} > 0$ and send $\theta \to s$, gives

$$
\begin{aligned}
& -rv(s,x_s,\alpha_s) + \frac{\partial\psi(s,x_s)}{\partial t} + \frac{1}{2} x_s^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(s,x_s)}{\partial x^2} \\
& + x_s \mu(\alpha_s) \frac{\partial\psi(s,x_s)}{\partial x} + Qv(s,x_s,\cdot)(\alpha_s) \leq 0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& rv(s,x_s,\alpha_s) - \frac{\partial\psi(s,x_s)}{\partial t} - \frac{1}{2} x_s^2 \sigma^2(\alpha_s) \frac{\partial^2 \psi(s,x_s)}{\partial x^2} \\
& - x_s \mu(\alpha_s) \frac{\partial\psi(s,x_s)}{\partial x} - Qv(s,x_s,\cdot)(\alpha_s) \geq 0. \qquad (2.103)
\end{aligned}
$$

The supersolution inequality (2.93) is deduced just by combining (2.94) and (2.103).
Now, let us prove the subsolution inequality, namely, that:

$$
\begin{aligned}
& \min \Big[rv(s,x_s,\alpha_s) - \frac{\partial\phi(s,x_s)}{\partial s} - \frac{1}{2} x^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(s,x_s)}{\partial x^2} \\
& - x\mu(\alpha_s) \frac{\partial\phi(s,x_s)}{\partial x} - Qv(s,x_s,\cdot)(\alpha_s), v(s,x_s,\alpha_s) - (K - x_s)^+ \Big] \leq 0 \qquad (2.104)
\end{aligned}
$$

whenever $\phi \in \mathcal{C}^{1,2}([s,T] \times \mathbb{R}^+)$ and $v(t,x,\alpha_s) - \phi(t,x)$ has a local maximum at $(s,x_s) \in [s,T] \times \mathbb{R}^+$.

Let $\phi \in \mathcal{C}^{1,2}([s,T]\times\mathbb{R}^+)$ and $v(t,x,\alpha_s)-\phi(t,x)$ has a local maximum at $(s,x_s)\in[s,T]\times\mathbb{R}^+$ we can assume without loss of generality that $v(s,x_s,\alpha_s)-\phi(s,x_s)=0$. Define

$$\Phi(t,x,i)=\begin{cases}\phi(t,x), & \text{if} \quad i=\alpha_s,\\ v(t,x,i), & \text{if} \quad i\neq\alpha_s.\end{cases} \tag{2.105}$$

Let γ be the first jump time of $\alpha(\cdot)$ from the state α_s, and let $\theta_0 \in [s,\gamma]$ be such that $(t,X(t))$ starts at (s,x_s) and stays in $N(s,x_s)$ for $s\le t\le\theta_0$. Since $\theta_0\le\gamma$ we have $\alpha(t)=\alpha_s$, for $s\le t\le\theta_0$, and let τ_D be the optimal stopping time, and for $s\le\theta\le\min(\tau_D,\theta_0)$ we have from (2.91)

$$v(s,x_s,\alpha_s)\le E^{s,x_s,\alpha_s}[e^{-r(\theta-s)}v(\theta,X(\theta),\alpha(\theta))]. \tag{2.106}$$

Moreover, since $v(s,x_s,\alpha_s)-\phi(s,x_s)=0$ and attains its maximum at (s,x_s) in $N(s,x_s)$ then

$$v(\theta,X(\theta),\alpha(\theta))\le\phi(\theta,X(\theta)).$$

Thus, we also have

$$v(\theta,X(\theta),\alpha(\theta))\le\Phi(\theta,X(\theta)),\alpha(\theta)). \tag{2.107}$$

This implies, using Dynkin's formula, that

$$\begin{aligned} & E^{s,x_s,\alpha_s}e^{-r(\theta-s)}v(\theta,X(\theta),\alpha_s)\\ \le\ & E^{s,x_s,\alpha_s}e^{-r(\theta-s)}\Phi(\theta,X(\theta),\alpha_s)\\ =\ & \Phi(s,x_s,\alpha_s)+E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\\ & \bigg[\frac{\partial\phi(t,X(t))}{\partial t}-r\Phi(t,X(t),\alpha(t))\\ & +X(t)\mu(\alpha_s)\frac{\partial\phi(t,X(t))}{\partial x}+Q\Phi(t,X(t),\cdot)(\alpha_s)\\ & +\frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\phi(t,X(t))}{\partial x^2}\bigg]dt. \end{aligned} \tag{2.108}$$

Note that

$$Q\Phi(t,X(t),\cdot)(\alpha_s) \;=\; \sum_{\beta\neq\alpha_s} q_{\alpha_s\beta}(v(t,X(t),\beta)-\phi(t,X(t)))$$

$$
\begin{aligned}
&\leq \sum_{\beta \neq \alpha_s} q_{\alpha_s \beta}(v(t, X(t), \beta) - v(t, X(t), \alpha_s)) \\
&\leq Qv(t, X(t), \cdot)(\alpha_s). \qquad (2.109)
\end{aligned}
$$

Using (2.105) and (2.109), we obtain

$$
\begin{aligned}
& E^{s,x_s,\alpha_s} e^{-r(\theta - s)} v(\theta, X(\theta), \alpha_s) \\
\leq\ & E^{s,x_s,\alpha_s} e^{-r\theta} \Phi(\theta, X(\theta), \alpha_s) \\
=\ & \phi(s, x_s) + E^{s,x_s,\alpha_s} \int_s^\theta e^{-r(t-s)} \Big[\frac{\partial \phi(t, X(t))}{\partial t} \\
& + X(t)\mu(\alpha_s) \frac{\partial \phi(t, X(t))}{\partial x} - rv(t, X(t), \alpha_s) \\
& + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(t, X(t))}{\partial x^2} + Qv(t, X(t), \cdot)(\alpha_s). \qquad (2.110)
\end{aligned}
$$

Recall that, $v(s, x_s, \alpha_s) = \phi(s, x_s)$ by assumption. From (2.106), we deduce

$$
\begin{aligned}
0 \ \leq\ & E^{s,x_s,\alpha_s} e^{-r(\theta - s)} v(\theta, X(\theta), \alpha_s) - \phi(s, x_s) \\
\leq\ & E^{s,x_s,\alpha_s} \int_s^\theta e^{-rt} \Big[- rv(t, X(t), \alpha_s) \\
& + \frac{\partial \phi(t, X(t))}{\partial t} + \frac{1}{2} X(t)^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(X(t), \alpha_s)}{\partial x^2} \\
& + X(t)\mu(\alpha_s) \frac{\partial \phi(X(t), \alpha_s)}{\partial x} + Qv(t, X(t), \cdot)(\alpha_s) \Big] dt. \qquad (2.111)
\end{aligned}
$$

Multiplying the last inequality by $\frac{1}{\theta} > 0$ and sending $\theta \downarrow s$, gives

$$
\begin{aligned}
rv(s, x_s, \alpha_s) - \frac{\partial \phi(s, x_s)}{\partial t} - \frac{1}{2} x_s^2 \sigma^2(\alpha_s) \frac{\partial^2 \phi(x_s, \alpha_s)}{\partial x^2} \\
- x_s \mu(\alpha_s) \frac{\partial \phi(x_s, \alpha_s)}{\partial x} - Qv(s, x_s, \cdot)(\alpha_s) \leq 0.
\end{aligned}
$$

This last inequality implies the subsolution inequality (2.104), thus $v(t, x, \alpha)$ is a viscosity solution of (2.86). This ends the proof of the theorem. □

In order to have the uniqueness of the viscosity solution we will follow almost the same process as in the infinite horizon case. Let us first give the useful lemma which is proved in P.L. Lions [27]

Lemma 2.2.7 *Let $v(t, x)$ defined in $[0, T] \times \mathbb{R}^n$ the parabolic superjet is*

$$
\bar{J}^{2,+} v(t, x) = \Big\{ \Big(\frac{\partial \phi(t, x)}{\partial t}, D_x \phi(t, x), D_x^2 \phi(t, x) \Big), \phi(t, x) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n) \\
\text{and } v - \phi \ \text{ has a global maximum at } (t, x) \Big\},
$$

and the parabolic subjet is

$$\bar{J}^{2,-}v(t,x) = \Big\{ \Big(\frac{\partial \phi(t,x)}{\partial t}, D_x\phi(t,x), D_x^2\phi(t,x) \Big), \phi(t,x) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n) \\ \text{and } v - \phi \text{ has a global minimum at } (t,x) \Big\}.$$

In order to prove the uniqueness result for the viscosity solution we need the following maximum principle for semicontinuous function, which is stated in a suitable form for our application.

Theorem 2.2.8 (Comparison Principle) *If $v_1(t,x,i)$ and $v_2(t,x,i)$ are continuous in (t,x) and are respectively viscosity subsolution and supersolution of (2.86) with at most a linear growth. Then*

$$v_1(t,x,i) \leq v_2(t,x,i) \quad \text{for all } (t,x,i) \in [0,T] \times \mathbb{R}^+ \times \mathcal{M}. \tag{2.112}$$

Proof. For any $0 < \delta < 1$ and $0 < \gamma < 1$, we define

$$\Phi(t,x,y,i) = v_1(t,x,i) - v_2(t,y,i) - \frac{1}{\delta} \mid x - y \mid^2 - \gamma e^{(T-t)}(x^2 + y^2),$$

and

$$\phi(t,x,y) = \frac{1}{\delta} \mid x - y \mid^2 + \gamma e^{(T-t)}(x^2 + y^2).$$

Note that $v_1(t,x,i)$ and $v_2(t,x,i)$ satisfy the linear growth. Then, we have for each $i \in \mathcal{M}$

$$\lim_{|x|+|y| \to \infty} \Phi(t,x,y,i) = -\infty$$

and since Φ is a continuous in (t,x,y), therefore its has a global maximum at a point $(t_\delta, x_\delta, y_\delta, \alpha_0)$. Observe that

$$\Phi(t_\delta, x_\delta, x_\delta, \alpha_0) + \Phi(t_\delta, y_\delta, y_\delta, \alpha_0) \leq 2\Phi(t_\delta, x_\delta, y_\delta, \alpha_0).$$

It implies

$$\begin{aligned} v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, x_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(x_\delta^2) + v_1(t_\delta, y_\delta, \alpha_0) \\ -v_2(t_\delta, y_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(y_\delta^2) \leq 2v_1(t_\delta, x_\delta, \alpha_0) - 2v_2(t_\delta, y_\delta, \alpha_0) \\ -\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 - 2\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2). \end{aligned}$$

Then

$$-v_2(t_\delta, y_\delta, \alpha_0) - 2e^{(T-t_\delta)}\gamma(x_\delta^2) + v_1(t_\delta, x_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(y_\delta^2)$$
$$\leq v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0) - \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2$$
$$-2\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).$$

Consequently, we have

$$\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \leq (v_1(t_\delta, x_\delta, \alpha_0) - v_1(t_\delta, y_\delta, \alpha_0)) + (v_2(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0)). \quad (2.113)$$

By the linear growth condition, we know that there exist K_1, K_2 such that $v_1(t, x, i) \leq K_1(1 + |x|)$ and $v_2(t, x, i) \leq K_2(1 + |x|)$. Therefore, there exists C such that we have

$$\frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 \leq C(1+ \mid x_\delta \mid + \mid y_\delta \mid).$$

So

$$\mid x_{\delta 1}^0 - x_{\delta 2}^0 \mid^2 \leq \delta C(1+ \mid x_{\delta 1}^0 \mid^{\kappa_1} + \mid x_{\delta 1}^0 \mid^{\kappa_2}). \quad (2.114)$$

We also have $\Phi(s, 0, 0, \alpha_0) \leq \Phi(t_\delta, x_\delta, y_\delta, \alpha_0)$ and $\mid \Phi(s, 0, 0, \alpha_0) \mid \leq K(1+ \mid x_\delta \mid + \mid y_\delta \mid)$. This leads to

$$\begin{aligned} \gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2) &\leq v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0) - \frac{1}{\delta} \mid x_\delta - y_\delta \mid^2 - \Phi(s, 0, 0, \alpha_0) \\ &\leq 3C(1+ \mid x_\delta \mid + \mid y_\delta \mid). \end{aligned} \quad (2.115)$$

It comes that

$$\frac{\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2)}{(1+ \mid x_\delta \mid + \mid y_\delta \mid)} \leq 3C,$$

therefore there exists C_γ such that

$$\mid x_\delta \mid + \mid y_\delta \mid \leq C_\gamma \text{ and } t_\delta \in [s, T]. \quad (2.116)$$

The inequality (2.116) implies the sets $\{x_\delta, \delta > 0\}$, and $\{y_\delta, \delta > 0\}$ are bounded by C_γ independent of δ, so we can extract convergent subsequences that we also denote $(x_\delta)_\delta$,

$(y_\delta)_\delta$, $(t_\delta)_\delta$. Moreover, from the inequality (2.114), it comes that the exists x_0 such that

$$\lim_{\delta\to 0} x_\delta = x_0 = \lim_{\delta\to 0} y_\delta \qquad \text{and} \qquad \lim_{\delta\to 0} t_\delta = t_0. \tag{2.117}$$

Using (2.113) and the previous limit, we obtain

$$\lim_{\delta\to 0} \frac{2}{\delta} \mid x_\delta - y_\delta \mid^2 = 0. \tag{2.118}$$

Φ achieves its maximum at $(t_\delta, x_\delta, y_\delta, \alpha_0)$, so by the theorem 1.2.1 for each $\epsilon > 0$ there exists $b_{1\delta}$, $b_{2\delta}$, X_δ, and Y_δ such that

$$(b_{1\delta}, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t)} x_\delta, X_\delta) \in \bar{\mathcal{P}}^{2,+} v_1(t_\delta, x_\delta, \alpha_0) \tag{2.119}$$

and

$$(-b_{2\delta}, -\tfrac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t)} y_\delta, -Y_\delta) \in \bar{\mathcal{P}}^{2,+}(-v_2(t_\delta, y_\delta, \alpha_0)).$$

But we know that

$$\bar{\mathcal{P}}^{2,+}(-v_2(t_\delta, y_\delta, \alpha_0)) = -\bar{\mathcal{P}}^{2,-} v_2(t_\delta, y_\delta, \alpha_0).$$

Therefore, we obtain

$$(b_{2\delta}, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{(T-t)} y_\delta, Y_\delta) \in \bar{\mathcal{P}}^{2,-} v_2(t_\delta, y_\delta, \alpha_0). \tag{2.120}$$

The equation (2.119) implies by the definition of the viscosity solution that

$$\min \Big[r v_1(t_\delta, x_\delta, \alpha_0) - b_{1\delta} - \frac{1}{2}(x_\delta)^2 \sigma^2(\alpha_s) X_\delta - x_\delta \mu(\alpha_0) \Big(\frac{2}{\delta}(x_\delta - y_\delta) \\ + 2\gamma e^{(T-t_\delta)} x_\delta \Big) - Q v_1(t_\delta, x_\delta, \cdot)(\alpha_0), v_1(t_\delta, x_\delta, \alpha_0) - (K - x_\delta)^+ \Big] \le 0.$$

Consequently, we have two cases; either

$$v_1(t_\delta, x_\delta, \alpha_0) - (K - x_\delta)^+ \le 0$$

or

$$r v_1(t_\delta, x_\delta, \alpha_0) - b_{1\delta} - x_\delta \mu(\alpha_0) \Big(\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t_\delta)} x_\delta \Big) \\ - \frac{1}{2}(x_\delta)^2 \sigma^2(\alpha_s) X_\delta - Q v_1(t_\delta, x_\delta, \cdot)(\alpha_0) \le 0.$$

First of all, we assume that $v_1(t_\delta, x_\delta, \alpha_0) - (K - x_\delta)^+ \le 0$. And similarly, (2.120) implies by the definition of the viscosity solution that,

$$\begin{aligned} \min \Big[& r v_2(t_\delta, y_\delta, \alpha_0) - b_{2\delta} - \frac{1}{2}(y_\delta)^2 \sigma^2(\alpha_0) Y_\delta - y_\delta \mu(\alpha_0) \Big(\frac{2}{\delta}(x_\delta - y_\delta) \\ & -2\gamma e^{(T-t_\delta)} y_\delta \Big) - Q v_2(t_\delta, y_\delta, \cdot)(\alpha_0), v_2(t_\delta, y_\delta, \alpha_0) - (K - y_\delta)^+ \Big] \ge 0. \end{aligned}$$

Therefore, we have

$$v_2(t_\delta, y_\delta, \alpha_0) - (K - y_\delta)^+ \ge 0.$$

It comes that

$$v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0) - (x_\delta - y_\delta)^+ \le 0.$$

Letting $\delta \to 0$, we obtain

$$v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) \le 0 \tag{2.121}$$

Note that the function Φ reaches its maximum at $(t_\delta, x_\delta, y_\delta, \alpha_0)$. It follows that for all $x \in \mathbb{R}$, $t \in [s, T]$, and $i \in \mathcal{M}$, we have

$$\begin{aligned} v_1(t, x, i) - v_2(t, x, i) - 2\gamma e^{(T-t)} x^2 &= \Phi(x, x, i) \le \Phi(t_\delta, x_\delta, y_\delta, \alpha_0) \\ &\le v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0) - \gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2). \end{aligned}$$

Again letting $\delta \to 0$ and using (2.121), we obtain

$$\begin{aligned} & v_1(t, x, i) - v_2(t, x, i) - 2\gamma e^{(T-t)} x^2 \\ \le\ & v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) - 2\gamma e^{(T-t_0)}(x_0)^2 \le 0. \end{aligned}$$

so, we have

$$v_1(t, x, i) - v_2(t, x, i) \le 2\gamma e^{(T-t)} x^2. \tag{2.122}$$

Second of all, let assume that

$$\begin{aligned} r v_1(t_\delta, x_\delta, \alpha_0) - b_{1\delta} - x_\delta \mu(\alpha_0) \Big(\frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{(T-t_\delta)} x_\delta \Big) \\ -\frac{1}{2}(x_\delta)^2 \sigma^2(\alpha_s) X_\delta - Q v_1(t_\delta, x_\delta, \cdot)(\alpha_0) \le 0 \end{aligned}$$

and from (2.120), we have

$$rv_2(t_\delta, y_\delta, \alpha_0) - b_{2\delta} - \frac{1}{2}(y_\delta)^2\sigma^2(\alpha_0)Y_\delta - y_\delta\mu(\alpha_0)\Big(\frac{2}{\delta}(x_\delta - y_\delta) \\ -2\gamma e^{(T-t_\delta)}y_\delta\Big) - Qv_2(t_\delta, y_\delta, \cdot)(\alpha_0) \geq 0.$$

Combining the last two inequalities, we obtain

$$r(v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0)) \leq \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2X_\delta - (y_\delta)^2Y_\delta\right) \\ +\mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta - y_\delta)^2 + 2\gamma e^{(T-t_\delta)}\Big[(x_\delta)^2 + (y_\delta)^2\Big]\right) \\ +Qv_1(t_\delta, x_\delta, \cdot)(\alpha_0) - Qv_2(t_\delta, y_\delta, \cdot)(\alpha_0) + b_{1\delta} - b_{2\delta}.$$

Note that from the equation (1.38), we have

$$b_{1\delta} - b_{2\delta} = \frac{\partial\phi(t_\delta, x_\delta, y_\delta)}{\partial t} = \gamma e^{(T-t_\delta)}((x_\delta)^2 + (y_\delta)^2).$$

Therefore, we have

$$r(v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0)) \leq \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2X_\delta - (y_\delta)^2Y_\delta\right) \\ +\mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta - y_\delta)^2 + 2\gamma e^{(T-t_\delta)}\Big[(x_\delta)^2 + (y_\delta)^2\Big]\right) \\ +Qv_1(t_\delta, x_\delta, \cdot)(\alpha_0) - Qv_2(t_\delta, y_\delta, \cdot)(\alpha_0) + \gamma e^{(T-t_\delta)}((x_\delta)^2 + (y_\delta)^2). \tag{2.123}$$

We know from the Maximum principle that

$$-\left(\frac{1}{\epsilon} + \|D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta)\|\right)I \leq \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leq D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta) \\ +\epsilon(D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta))^2.$$

Moreover,

$$D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta) = \frac{2}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2\gamma e^{(T-t_\delta)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$(D^2_{(x,y)}\phi(t_\delta, x_\delta, y_\delta))^2 = \frac{8}{\delta^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{8\gamma e^{(T-t_\delta)}}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$
\begin{aligned}
& +4\gamma^2 e^{2(T-t_\delta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
= \quad & \frac{8+8\gamma\delta e^{(T-t_\delta)}}{\delta^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
& +4\gamma^2 e^{2(T-t_\delta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (2.124)
\end{aligned}
$$

Note that

$$
\begin{aligned}
(x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \quad = \quad & (x_\delta, y_\delta) \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} \\
\leq \quad & (x_\delta, y_\delta) \Big[\frac{2}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
& + \Big(2\gamma e^{(T-t_\delta)} + 4\epsilon\gamma^2 e^{2(T-t_\delta)} \Big) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
& + \epsilon \frac{8+8\gamma\delta e^{(T-t_\delta)}}{\delta^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Big] \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix}. \qquad (2.125)
\end{aligned}
$$

Letting $\gamma \to 0$, we obtain

$$
(x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \leq (x_\delta, y_\delta) \left[(\frac{2}{\delta} + \epsilon \frac{8}{\delta^2}) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix}.
$$

Take $\epsilon = \frac{\delta}{4}$, this leads to

$$
(x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \leq (x_\delta, y_\delta) \left[\tfrac{4}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} = \tfrac{4}{\delta} (x_\delta - y_\delta)^2.
$$

Using (2.118), we obtain

$$
\begin{aligned}
\limsup_{\delta \downarrow 0} (x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta \quad \leq \quad & \limsup_{\delta \downarrow 0} (x_\delta, y_\delta) \left[\frac{4}{\delta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} \\
= \quad & \limsup_{\delta \downarrow 0} \frac{4}{\delta} (x_\delta - y_\delta)^2 = 0. \qquad (2.126)
\end{aligned}
$$

Letting $\gamma \to 0$ in (2.123), we have

$$r(v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0)) \leq \frac{1}{2}\sigma^2(\alpha_0)\left((x_\delta)^2 X_\delta - (y_\delta)^2 Y_\delta\right)$$
$$+\mu(\alpha_0)\left(\frac{2}{\delta}(x_\delta - y_\delta)^2\right) + Qv_1(t_\delta, x_\delta, \cdot)(\alpha_0) - Qv_2(t_\delta, y_\delta, \cdot)(\alpha_0)$$

and taking the lim sup as δ goes to zero and using (2.126), we obtain

$$r(v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0)) \leq Qv_1(t_0, x_0, \cdot)(\alpha_0) - Qv_2(t_0, x_0, \cdot)(\alpha_0). \tag{2.127}$$

Recall that $(t_\delta, x_\delta, y_\delta, \alpha_0)$ is maximum of Φ. Then, for all $x \in \mathbb{R}$, $t \in [s, T]$, and for all $i \in \mathcal{M}$, we have

$$\Phi(t, x, x, i) \leq \Phi(t_\delta, x_\delta, y_\delta, \alpha_0).$$

It comes that

$$v_1(t, x, i) - v_2(t, x, i) - 2\gamma e^{(T-t)} x^2 \leq$$
$$v_1(t_\delta, x_\delta, \alpha_0) - v_2(t_\delta, y_\delta, \alpha_0) - 2\gamma e^{(T-t_\delta)}(x_\delta^2 + y_\delta^2).$$

Letting $\delta \to 0$, we obtain

$$v_1(t, x, i) - v_2(t, x, i) - 2\gamma e^{(T-t)} x^2 \leq$$
$$v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) - 2\gamma e^{(T-t)} x_0^2. \tag{2.128}$$

Taking $x = x_0$, and $t = t_0$, we have

$$v_1(t_0, x_0, i) - v_2(t_0, x_0, i) - 2\gamma e^{(T-t_0)} x_0^2 \leq$$
$$v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) - 2\gamma e^{(T-t_0)} x_0^2.$$

Consequently,

$$v_1(t_0, x_0, i) - v_2(t_0, x_0, i) \leq v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0).$$

We recall

$$\begin{aligned} & Qv_1(t_0, x_0, \cdot)(\alpha_0) - Qv_2(t_0, x_0, \cdot)(\alpha_0) \\ = & \sum_{\beta \neq \alpha_0} q_{\alpha_0 \alpha}[v_1(t_0, x_0, \beta) - v_1(t_0, x_0, \alpha_0) \\ & -v_2(t_0, x_0, \beta) + v_2(t_0, x_0, \alpha_0)] \leq 0. \end{aligned} \tag{2.129}$$

Using (2.127), we have

$$v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) \leq 0.$$

Therefore using (2.128), we conclude that

$$\begin{aligned} v_1(t, x, i) - v_2(t, x, i) - 2\gamma e^{(T-t)}x^2 &\leq \\ v_1(t_0, x_0, \alpha_0) - v_2(t_0, x_0, \alpha_0) - 2\gamma e^{(T-t_0)}x_0^2 &\leq 0. \end{aligned} \tag{2.130}$$

Letting $\gamma \to 0$ in (2.122) and the previous inequality, we have

$$v_1(t, x, i) \leq v_2(t, x, i).$$

This completes the proof of the theorem. □

The uniqueness of the viscosity solution follows from this theorem because any viscosity solution is both supersolution and subsolution. It is difficult to find closed-form solutions the associated PDEs. In the next chapter, we develop numerical schemes to approximate the value of the American options.

Chapter 3

Numerical methods

In this chapter we consider numerical approximations to the solutions of the HJB equations associated with the valuation of American option under switching regime. We use explicit finite difference methods and establish the convergence of those schemes.

3.1 Infinite time horizon American option

In this section we construct a scheme to approximate the value of a perpetual American put option. Similar analysis works for a perpetual call option.

We want to solve numerically the equation

$$\begin{aligned}\mathcal{H}\left(i, x, v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right) &= \min\Big[rv(x,i) - \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 v(x,i)}{\partial x^2} \\ -x\mu(i)\frac{\partial v(x,i)}{\partial x} &- Qv(x,\cdot)(i), v(x,i) - (K-x)^+\Big] = 0. \qquad (3.1)\end{aligned}$$

We use the method of Barles-Souganidis to construct our scheme. Let $h \in (0,1]$ denote the length of the finite difference interval of variable x and let $B_K(\mathbb{R}^+ \times \mathcal{M})$ denote the space of bounded functions $u(x,i)$ defined on $\mathbb{R}^+ \times \mathcal{M}$ continuous in terms of the first variable x such that $\sup|u(x,i)| \leq K$. We will approximate the value function $v(x,i)$ by a sequence of functions $v^h(x,i)$ and the partial derivatives $\dfrac{\partial v(x,i)}{\partial x}$ and $\dfrac{\partial^2 v(x,i)}{\partial x^2}$ by

$$\Delta_x v^h(x,i) = \frac{v^h(x+h,i) - v^h(x,i)}{h}$$

and

$$\Delta_x^2 v^h(x,i) = \frac{v^h(x+h,i) + v^h(x-h,i) - 2v^h(x,i)}{h^2}.$$

Using Δ_x and Δ_x^2,

$$\begin{aligned}(\mathcal{A}v^h)(x,i) &= \frac{1}{2}x^2\sigma^2(i)\Delta_x^2 v^h(x,i) + x\mu(i)\Delta_x v^h(x,i) + Qv^h(x,\cdot)(i)\\ &= \frac{1}{2}x^2\sigma^2(i)\frac{v^h(x+h,i)+v^h(x-h,i)-2v^h(x,i)}{h^2}\\ &+ x\mu(i)\frac{v^h(x+h,i)-v^h(x,i)}{h} + Qv^h(x,\cdot)(i). \qquad (3.2)\end{aligned}$$

Recall that $Qf(x,\cdot)(i) = \sum_{j\neq i} q_{ij}(f(x,j)-f(x,i))$. Rearranging these terms, we have

$$\begin{aligned}(\mathcal{A}v^h)(x,i) = \frac{1}{h^2}\Bigg[&\left(\frac{1}{2}x^2\sigma^2(i)+x\mu(i)h\right)v^h(x+h,i)\\ &+\frac{1}{2}x^2\sigma^2(i)v^h(x-h,i) - v^h(x,i)(x^2\sigma^2(i)\\ &+x\mu(i)h)\Bigg] + Qv^h(x,\cdot)(i).\end{aligned}$$

Define

$$\begin{aligned}\mathcal{A}_h(f,i)(x) = \frac{1}{h^2}\Bigg[&\left(\frac{1}{2}x^2\sigma^2(i)+x\mu(i)h\right)f(x+h,i)\\ &+\frac{1}{2}x^2\sigma^2(i)f(x-h,i) - f(x,i)(x^2\sigma^2(i)\\ &+x\mu(i)h)\Bigg] + Qf(x,\cdot)(i).\end{aligned}$$

In discrete form we have the equation

$$\begin{aligned}\min\Bigg\{&u^h(x,i)\frac{1}{h^2}\left(rh^2 + x^2\sigma^2(i) + x\mu(i)h + h^2\sum_{j\neq i} q_{ij}\right)\\ &-\frac{1}{2}x^2\sigma^2(i)\left(\frac{u^h(x+h,i)+u^h(x-h,i)}{h^2}\right) - x\mu(i)u^h(x+h,i)\frac{1}{h}\\ &-\sum_{j\neq i} q_{ij}u^h(x,\beta), u^h(x,i)-(K-x)^+\Bigg\} = 0. \qquad (3.3)\end{aligned}$$

In order to approximate the solution of (3.3), we define the following scheme.

$S : \mathbb{R}^+ \times \bar{\Omega} \times \mathcal{M} \times \mathbb{R} \times B(\mathbb{R}^+ \times \mathcal{M}) \to \mathbb{R}$ such that,

$$\begin{aligned}S(h,x,i,t,u) = \min\Bigg\{&\frac{t}{h}\left(rh^2 + x^2\sigma^2(i) + x\mu(i)h + h^2\sum_{j\neq i} q_{ij}\right)\\ &-\frac{1}{2}x^2\sigma^2(i)\left(\frac{u(x+h,i)+u(x-h,i)}{h}\right) - x\mu(i)u(x+h,i)\\ &-h\sum_{j\neq i} q_{ij}u(x,j), h\left(t-(K-x)^+\right)\Bigg\}. \qquad (3.4)\end{aligned}$$

We want to prove that the solution of the following equation whenever it exists is an approximation of the viscosity solution of (3.1)

$$S(h,x,i,u_h(x,i),u_h) = \min\Big\{\frac{u_h(x,i)}{h}\Big(rh^2 + x^2\sigma^2(i) + x\mu(i)h$$
$$+h^2\sum_{j\neq i} q_{ij}\Big) - \frac{1}{2}x^2\sigma^2(i)\left(\frac{u_h(x+h,i)+u_h(x-h,i)}{h}\right) - x\mu(i)u_h(x+h,i)$$
$$-h\sum_{j\neq i} q_{ij}u_h(x,j), h\left(u_h(x,i)-(K-x)^+\right)\Big\} = 0. \tag{3.5}$$

In order to use the Barles-Souganidis theorem to prove the convergence we need to check the following hypotheses:

- *Monotonicity*

$$S(h,x,i,t,u) \le S(h,x,i,t,v) \quad \text{if} \qquad v \le u \quad \text{for all } h \le 0, x \in \Sigma^h,$$
$$i \in \mathcal{M}, t \in \mathbb{R} \text{ and } u, v \in B(\mathbb{R}^+ \times \mathcal{M}).$$

  Note that the coefficients of u in $S(h,x,i,t,u)$ are negative, this implies the monotonicity of our scheme S.

- *Consistency*

$$\lim_{\substack{y\to x\\ \epsilon\to 0\, h\to 0}} \frac{S(h,y,i,\omega(y,i)+\epsilon,\omega+\epsilon)}{h} = \mathcal{F}(\mathrm{D}^2\omega(x,i),\mathrm{D}\omega(x,i),\omega(x,i),x)$$

  for every test function $\omega(\cdot,i) \in \mathcal{C}^2(\mathbb{R})$, for every $i \in \mathcal{M}$.

  We have the consistency because,

$$\lim_{\substack{y\to x\\ \epsilon\to 0\, h\to 0}} \frac{S(h,y,i,\omega(y,i)+\epsilon,\omega+\epsilon)}{h} = \lim_{\substack{y\to x\\ \epsilon\to 0\, h\to 0}} \min\Big\{\frac{\omega(y,i)+\epsilon}{h^2}\Big(rh^2 + y^2\sigma^2(i)$$
$$+y\mu(i)h + h^2\sum_{j\neq i} q_{ij}\Big) - \frac{1}{2}y^2\sigma^2(i)\left(\frac{\omega(y+h,i)+\omega(y-h,i)+2\epsilon}{h^2}\right)$$
$$-\frac{y\mu}{h}(i)(\omega(y+h,i)+\epsilon) - \sum_{j\neq i} q_{ij}(\omega(y,j)+\epsilon), \omega(y,i)+\epsilon-(K-y)^+\Big\}$$

$$\lim_{\substack{y\to x\\ \epsilon\to 0\, h\to 0}} \frac{S(h,y,i,\omega(y,i)+\epsilon,\omega+\epsilon)}{h} = \min\Big\{r\omega(x,i)$$

$$-\frac{1}{2}x^2\sigma^2(i)D^2\omega(x,i) - x\mu(i)D\omega(x,i)$$
$$-Q\omega(x,i), \omega(x,i) - (K-x)^+\Big\}$$
$$= \mathcal{F}(\mathrm{D}^2\omega(x,i), \mathrm{D}\omega(x,i), \omega(x,i), x) \tag{3.6}$$

- *Stability*

  We want to show that for all $h > 0$ there exists a solution v_h to (3.5) and a constant C independent of h such that $\|v_h\| \leq C$.

We next construct a strict contraction mapping to show the existence and uniqueness of solution to (3.5). Let

$$\begin{aligned} a_h(x,i) &= \frac{1}{h}\left(\frac{1}{2}x^2\sigma^2(i) + x\mu(i)h\right) \\ b_h(x,i) &= \frac{1}{2h}x^2\sigma^2(i) \\ c_h(x,i) &= \left(\frac{1}{h}\left[x^2\sigma^2(i) + x\mu(i)h + rh^2 + h^2\sum_{j\neq i} q_{ij}\right]\right). \end{aligned} \tag{3.7}$$

Then equation (3.5) can be written as follows:

$$u(x,i) = \max\Big\{a_h(x,i)u(x+h,i) + b_h(x,i)u(x-h,i)$$
$$+h\sum_{j\neq i} q_{ij}u(x,j), c_h(x,i)(K-x)^+\Big\}c_h(x,i)^{-1}.$$

Let $\mathcal{T}_h$ be the operator on $B_K(\mathbb{R}^+ \times \mathcal{M})$ defined by:

$$\mathcal{T}_h u(x,i) = \max\Big\{a_h(x,i)u(x+h,i) + b_h(x,i)u(x-h,i)$$
$$h\sum_{j\neq i} q_{ij}u(x,j), c_h(x,i)(K-x)^+\Big\}c_h(x,i)^{-1}.$$

Theorem 3.1.1. *$\mathcal{T}_h$ is a contraction mapping.*

Proof. We want to show that there exists $\beta \in (0,1)$ such that

$$\|\mathcal{T}_h f - \mathcal{T}_h g\| \leq \beta\|f-g\| \qquad \text{for all} \qquad f,g \in B(\mathbb{R}^+ \times \mathcal{M})$$

where $\|\cdot\|$ is the sup norm. Note that,

$$\begin{aligned}
\mathcal{T}_h f(x,i) - \mathcal{T}_h g(x,i) &= c_h(x,i)^{-1} \max \Big[a_h(x,i) f(x+h,i) \\
&\quad + b_h(x,i) f(x-h,i) + \\
&\quad h \sum_{j\neq i} q_{ij} f(x,j), c_h(x,i)(K-x)^+ \Big] \\
&\quad - c_h(x,i)^{-1} \max \Big[a_h(x,i) g(x+h,i) \\
&\quad + b_h(x,i) g(x-h,i) \\
&\quad + h \sum_{j\neq i} q_{ij} g(x,j), c_h(x,i)(K-x)^+ \Big] \\
&\leq c_h(x,i)^{-1} \Big[a_h(x,i)(f(x+h,i) - g(x+h,i)) \\
&\quad + b_h(x,i)(f(x-h,i) - g(x-h,i)) \\
&\quad + h \sum_{j\neq i} q_{ij} (f(x,j) - g(x,i)) \Big]. \qquad (3.8)
\end{aligned}$$

Recall that $q_{ij} \geq 0$ for $j \neq i$ and $a_h(x,i) \geq 0$ and $b_h(x,i) \geq 0$ for all $(x,i) \in B_K(\mathbb{R}^+ \times \mathcal{M})$. Then, we have

$$\mathcal{T}_h f(x,i) - \mathcal{T}_h g(x,i) \leq \frac{1}{c_h(x,i)} \left(a_h(x,i) + b_h(x,i) + h \sum_{j\neq i} q_{ij} \right) \|f-g\|.$$

Finally, note that

$$\begin{aligned}
&\frac{\Big(a_h(x,i) + b_h(x,i) + h \sum_{j\neq i} q_{ij} \Big)}{c_h(x,i)} \\
&= \frac{x^2\sigma^2(i) + x\mu(i)h + h^2 \sum_{j\neq i} q_{ij}}{x^2\sigma^2(i) + x\mu(i)h + rh^2 + h^2 \sum_{j\neq i} q_{ij}} < 1. \qquad (3.9)
\end{aligned}$$

Therefore, we have

$$\|\mathcal{T}_h f - \mathcal{T}_h g\| \leq \|f-g\|.$$

This ends the proof. □

Remark 3.1.2 Because of the contraction mapping principle, the fixed point u_h of the contraction $\mathcal{T}_h$ is a solution of the equation (3.5) and $\|u_h\| \leq K$ since $\mathcal{T}_h$ is defined on $B_K(\mathbb{R}^+ \times \mathcal{M})$. Therefore, we have the stability of our scheme. We can now prove the convergence of our scheme.

Theorem 3.1.3 *As $h \to 0$, the solution v_h of (3.5) converges locally uniformly to the unique continuous viscosity solution of (3.1).*

Proof. Define

$$v^*(x,i) = \limsup_{\substack{y \to x \\ h \downarrow 0}} v_h(y,i) \qquad \text{and} \qquad v_*(x,i) = \liminf_{\substack{y \to x \\ h \downarrow 0}} v_h(y,i). \tag{3.10}$$

We first claim that v^* and v_* are respectively sub- and supersolutions of (2.44). To prove this claim, we only consider the subsolution case, because the argument for the supersolution is similar. Let x_0 be a strict local maximum of $v^*(x,i) - \Phi(x)$ for some $\Phi \in \mathcal{C}_b^\infty(\mathbb{R}^+)$. Without loss of generality, we may also assume that $v^*(x_0,i) = \Phi(x_0)$ and that $\Phi \geq 2\sup_h \|v_h\|$ outside the ball $B(x_0,r)$, where $r > 0$ is such that

$$v^*(x,i) - \Phi(x) \leq 0 = v^*(x_0,i) - \Phi(x_0) \qquad \text{in } B(x_0,r).$$

Then, there exist sequences $h_n > 0$ and $y_n \in \mathbb{R}^+$ such that: as $n \to \infty$

$$h_n \to 0, \qquad y_n \to x_0, \qquad v_{h_n}(y_n,\alpha) \to v^*(x_0,i), \text{ and} \tag{3.11}$$
$$y_n \text{ is a global maximum point of } v_{h_n}(\cdot,i) - \Phi(\cdot).$$

Denote $\epsilon_n = v_{h_n}(y_n,i) - \Phi(y_n)$, we have obviously $\epsilon_n \to 0$ and $v_{h_n}(x,i) \leq \Phi(x) + \epsilon_n$ for all $x \in \mathbb{R}^+$.

Recall that

$$S(h_n, y_n, v_{h_n}(x,i), v_{h_n}) = 0.$$

The monotonicity of S and (3.11) imply that

$$S(h_n, y_n, \Phi(y_n) + \epsilon_n, \Phi + \epsilon_n) \leq S(h_n, y_n, v_{h_n}(x,i), v_{h_n}) = 0. \tag{3.12}$$

Therefore,

$$\lim_n \frac{S(h_n, y_n, \Phi(y_n) + \epsilon_n, \Phi + \epsilon_n)}{h_n} \leq 0.$$

It comes that,

$$\begin{aligned}\mathcal{F}(\mathrm{D}^2\Phi(x,i),\mathrm{D}\Phi(x,i),\Phi(x,i),x) &= \\ \lim_{\substack{y\to x_0 \\ \epsilon\to 0\, h\to 0}} \frac{S(h,y,i,\omega(y,i)+\epsilon,\omega+\epsilon)}{h} &\leq 0.\end{aligned} \tag{3.13}$$

This proves that v^* is a viscosity subsolution. By Theorem 2.2.4, we see that our sequence converges locally uniformly to the unique viscosity solution of (3.1). □

3.1.1 Numerical examples

In our examples we use a two state Markov chain, and we use the expression *first state* for the state when $\alpha(t) = 1$ and *second state* for the state when $\alpha(t) = 2$.

Example 1. Perpetual American call option

First we consider a perpetual American call option with exercise price $K = 70$, discount rate $r = 0.06$, the return vector $\mu = (-0.8, -0.2)$, the volatility vector is $\sigma = (0.7, 0.5)$ and the generator

$$Q = \begin{pmatrix} -5 & 5 \\ 9 & -9 \end{pmatrix}$$

In Figure 3.1 we have the corresponding graphs of value and reward functions. This example corresponds to the case when the stock has negative returns. In the first state the return is -0.8 and in the second state the return is -0.2. We see from the first picture of Figure 3.1 that the value function and the reward function intersect when the stock price is approximatively 80 and this is the first time the two curves meet, then the optimal policy is to exercise the option at that time for a payoff of 10. In the second picture, the two curves intersect when the stock price is approximatively 84 and the payoff of exercising the option at that time is 14.

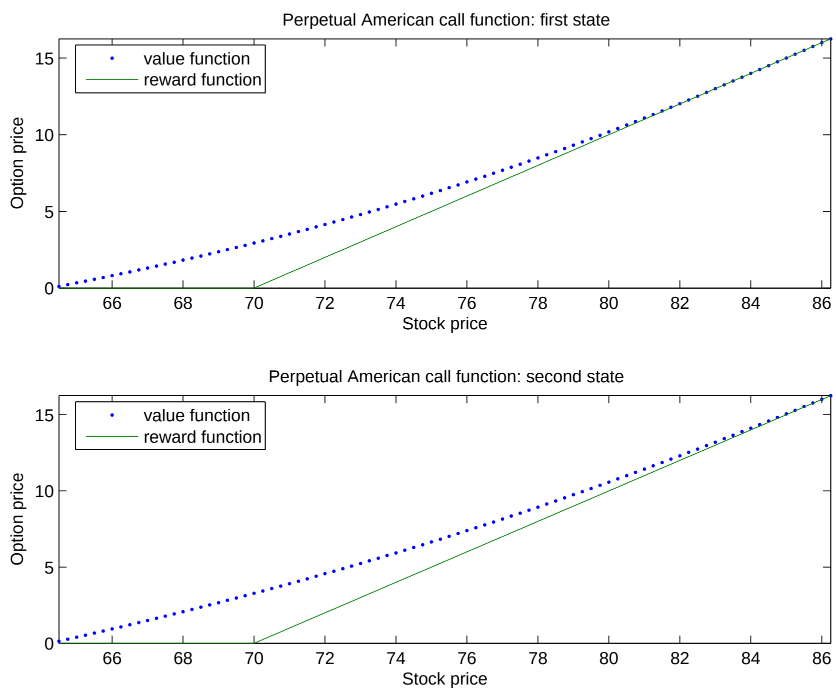


Figure 3.1: Perpetual American call option

Example 2. Perpetual American put

We study a perpetual American put option with exercise price $K = 75.5$ discount rate $r = 0.06$, the return vector $\mu = (0.6, 0.2)$, the volatility vector is $\sigma = (0.5, 0.7)$ and the generator

$$Q = \begin{pmatrix} -5 & 5 \\ 9 & -9 \end{pmatrix}$$

In Figure 3.2 we have the corresponding graphs of value functions and reward function. In this case the stock has positive returns. In the *first state*, the return is 0.6 and the first time the reward function is different from value function is when the stock price is 68, so the optimal policy is to exercise the option at that time for a payoff of 7.5. In the *second state*, the return is 0.2 and the first time the value function is different from the reward function is when the stock price is 61, so the optimal policy is to exercise the option at that time for a payoff of 14.5.

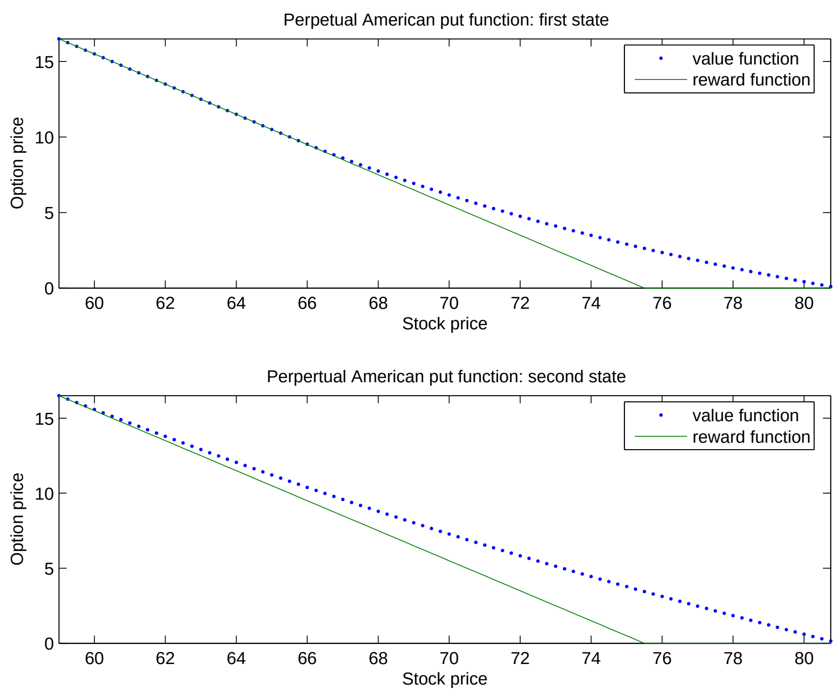


Figure 3.2: Perpetual American put option

3.2 Finite time horizon American options

In this section, we consider American options with finite time horizon.

3.2.1 Convergence

As we have done for infinite time horizon we will prove the convergence of finite difference scheme to the unique viscosity solution of the equation

$$\begin{cases} \mathcal{H}(t,x,i,v,\mathrm{D}_t v,\mathrm{D}_x v,\mathrm{D}_x^2 v)=0 \\ v(T,x,\alpha(T))=(X(T)-K)^+ \end{cases}, \tag{3.14}$$

where $\mathcal{H}$ is defined as follows

$$\begin{aligned} \mathcal{H}(t,x,i,u,\mathrm{D}_t u,\mathrm{D}_x u,\mathrm{D}_x^2 u) = \min\Big[& ru(t,x,i)-\frac{\partial u(t,x,i)}{\partial t} \\ & -\frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 u(t,x,i)}{\partial x^2}-x\mu(i)\frac{\partial u(t,x,i)}{\partial x} \\ & -Qu(t,x,\cdot)(i),u(t,x,i)-(K-x)^+\Big]=0. \end{aligned} \tag{3.15}$$

Given a positive integer N, let g_N denote the truncated function $g_N(x,i)=\min((x-K)^+,N)$ of $g(x,i)=(x-K)^+$, where g is the payoff of American call option. We consider the corresponding optimal stopping problem with the reward function g_N in lieu of g for N large enough. Let v_N denote the corresponding value function, i.e.,

$$v_N(s,x,i)=\sup_{\tau\in\Lambda_{s,T}} E\left[e^{-r(\tau-s)}g_N(\alpha(X(\tau),i)\mid X(s)=x,\alpha(s)=i\right].$$

We can show, as in the untruncated case, that v_N is the unique viscosity solution of the equation

$$\begin{cases} \mathcal{H}_N(i,s,x,v,\mathrm{D}_s v,\mathrm{D}_x v,\mathrm{D}_x^2 v)=0, \quad \text{for } (s,x,i)\in[0,T)\times\mathbb{R}^+\times\mathcal{M}, \\ v(T,x,\alpha(T))=g_N(x) \end{cases} \tag{3.16}$$

where $\mathcal{H}_N$ is the following Hamiltonian

$$\mathcal{H}_N(i,s,x,u,\mathrm{D}_s u,\mathrm{D}_x u,\mathrm{D}_x^2 u)=\min\Big[ru(s,x,i)-\frac{\partial u(s,x,i)}{\partial s} \tag{3.17}$$

$$-(\mathcal{A}u)(s,x,i),u(s,x,i)-g_N(x,i)\Big]=0. \tag{3.18}$$

Moreover, note that $g_N \to g$ as $N \to \infty$. It follows that $v_N \to v$, for all (s, x, i). In view of this, we only need to find a numerical solution for v_N.

Let $B([0,T] \times \mathbb{R}^+ \times \mathcal{M})$ denote the space of bounded functions $u(t, x, i)$ defined on $[0,T] \times \mathbb{R}^+ \times \mathcal{M}$ and continuous in (t, x). Let $h > 0$ denote the spatial step and $k > 0$ the time step. We consider the finite difference operators Δ_t, Δ_x and Δ_x^2 defined by

$$\begin{aligned}\Delta_t u(t,x,i) &= \frac{u(t+k,x,i) - u(t,x,i)}{k},\\ \Delta_x u(t,x,i) &= \frac{u(t,x+h,i) - u(t,x,i)}{h}.\end{aligned}$$

and

$$\Delta_x^2 u(t,x,i) = \frac{u(t,x+h,i) + u(t,x-h,i) - 2u(t,x,i)}{h^2}.$$

The corresponding discrete version of the Hamiltonian $\mathcal{H}_N$ is given by

$$\begin{aligned}\min\Big[ru(t,x,i) - \Delta_t u(t,x,i) - \frac{1}{2}x^2\sigma^2(i)\Delta_x^2 u(t,x,i) - x\mu(i)\Delta_x u(t,x,i)\\ -Qu(t,x,\cdot)(i), u(t,x,i) - g_N(x,i)\Big] = 0.\end{aligned} \tag{3.19}$$

Rearranging these terms, we obtain

$$\begin{aligned}\min\Big[&u(t,x,i)\Big(r + \frac{1}{k} + \frac{x^2\sigma(i)}{h^2} + \frac{x\mu(i)}{h}\Big) - \frac{u(t+k,x,i)}{k}\\ &-u(t,x+h,i)\Big(\frac{x^2\sigma^2(i)}{2h^2} + \frac{x\mu(i)}{h}\Big) - u(t,x-h,i)\frac{x^2\sigma^2(i)}{2h^2}\\ &-Qu(t,x,\cdot)(i), u(t,x,i) - g_N(x,i)\Big] = 0.\end{aligned}$$

Define a mapping $S_N : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{M} \times \mathbb{R} \times B_K([0,T] \times \mathbb{R}^+ \times \mathcal{M}) \to \mathbb{R}$

$$\begin{aligned}S_N(k,h,x,i,y,u) = \min\Big[&yh\Big(r + \frac{x^2\sigma^2(i)}{h^2} + \frac{1}{k} + \frac{x\mu(i)}{h} + \sum_{j\neq i} q_{ij}\Big)\\ &-\frac{hu(t+k,x,i)}{k} - u(t,x+h,i)\Big(\frac{x^2\sigma^2(i)}{2h} + x\mu(i)\Big) - u(t,x-h,i)\frac{x^2\sigma^2(i)}{2h}\\ &-h\sum_{j\neq i} q_{ij}u(t,x,j), hy - hg_N(x,i)\Big].\end{aligned}$$

Then, (3.19) is equivalent to $S_N = 0$.

Moreover, note that all coefficients of u in S_N are negative. This implies that S_N is monotone, i.e., for all $u, v \in B_K([0,T] \times \mathbb{R}^+ \times \mathcal{M})$, $k, h \in \mathbb{R}^*$, $x \in \mathbb{R}^+$, $y \in \mathbb{R}$, and $i \in \mathcal{M}$, we have

$$S_N(k,h,x,i,y,u) \leq S_N(k,h,x,i,y,v) \text{ whenever } u \geq v.$$

Definition 3.2.1 *The scheme S_N is said to be consistent if, for every $i \in \mathcal{M}$, $x \in \mathbb{R}^+$, $t \in [0,T]$ and for every test function $\omega(\cdot,\cdot,i) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ we have*

$$\lim_{z\to x,\, k\to 0,\, \epsilon\to 0,\, h\to 0} \frac{S_N(k,h,z,i,\omega(t,z,i)+\epsilon,\omega+\epsilon)}{h} = \mathcal{H}_N(t,x,i,\omega,\mathrm{D}_x\omega,\mathrm{D}_x^2\omega).$$

Lemma 3.2.2 *The scheme S_N is consistent.*

Proof. For $i \in \mathcal{M}$, let $\omega(\cdot,\cdot,i) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. We write

$$\begin{aligned}\frac{S_N(k,h,z,i,\omega(t,z,i),\omega)}{h} &= \min\Big\{r\omega(t,z,i) - \frac{\omega(t+k,z,i)-\omega(t,z,i)}{k}\\ &-\frac{1}{2}z^2\sigma^2(i)\frac{\omega(t,z+h,i)+\omega(t,z-h,i)-2\omega(t,z,i)}{h^2}\\ &-z\mu(i)\frac{\omega(t,z+h,i)-\omega(t,z,i)}{h} - Q\omega(t,z,\cdot)(i), \omega(t,z,i)-g_N(z,i)\Big\}.\end{aligned}$$

Sending $z\to x$, $k\to 0$, $\epsilon\to 0$, and $h\to 0$, we can show that

$$\frac{S_N(k,h,z,i,\omega(t,z,i)+\epsilon,\omega+\epsilon)}{h} \to \mathcal{H}_N(t,x,\omega,\mathrm{D}_t\omega,\mathrm{D}_x\omega,\mathrm{D}_x^2\omega).$$

The consistence follows. □

Note that the equation $S_N(k,h,z,i,\omega(t,z,i),\omega)=0$ is equivalent to the equation

$$\begin{aligned}\omega_N(t,z,i) = \ &\max\Big[\frac{1}{c_{h,k}(z,i)}\Big(\frac{h\omega(t+k,z,i)}{k} + \omega(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h}+z\mu(i)\Big)\\ &+\omega(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\omega(t,z,j)\Big), g_N(z,i)\Big]\end{aligned}$$

where

$$c_{h,k}(z,i) = \frac{1}{h}\Big(rh^2 + \frac{h^2}{k} + z^2\sigma^2(i) + z\mu(i)h + h^2\sum_{j\neq i} q_{ij}\Big).$$

We define an operator $\mathcal{T}^N_{h,k}$ on $B([0,T]\times\mathbb{R}^+\times\mathcal{M})$ as follows,

$$\begin{aligned}\mathcal{T}^N_{h,k}\omega(t,z,i) = \max\Big[&\frac{1}{c_{h,k}(z,i)}\Big(\frac{h\omega(t+k,z,i)}{k} + \omega(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h}\\ &+z\mu(i)\Big) + \omega(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\omega(t,z,i)\Big), g_N(z,i)\Big]\end{aligned}$$

Lemma 3.2.3 *For each N, k, and h, $\mathcal{T}^N_{k,h}$ is a contraction map.*

Proof. To show that $\mathcal{T}_{k,h}^N$ is a contraction, we need to show that there exists $0 < \beta < 1$ such that

$$\|\mathcal{T}_{k,h}^N f - \mathcal{T}_{k,h}^N g\| \le \beta \|f - g\| \qquad \text{for all} \qquad f, g \in B([0,T] \times \mathbb{R}^+ \times \mathcal{M})$$

where $\|\cdot\|$ is the sup norm. Note that

$$\begin{aligned}
&\mathcal{T}_{k,h}^N \omega(t,z,i) - \mathcal{T}_{k,h}^N u(t,z,i) \\
&\quad = \max\Bigg[c_{h,k}(z,i)^{-1}\Big(\frac{h\omega(t+k,z,i)}{k} + \omega(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h} + z\mu(i)\Big) \\
&\qquad + \omega(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\omega(t,z,j)\Big), g_N(z,i)\Bigg] - \max\Bigg[c_{h,k}(z,i)^{-1}\Big(\frac{hu(t+k,z,i)}{k} \\
&\qquad + u(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h} + z\mu(i)\Big) + u(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}u(t,z,j)\Big), g_N(z,i)\Bigg].
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathcal{T}_{k,h}^N \omega(t,z,i) - \mathcal{T}_{k,h}^N u(t,z,i) \\
&\quad \le \Bigg|\Bigg[c_{h,k}(z,i)^{-1}\Big(\frac{h\omega(t+k,z,i)}{k} + \omega(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h} + z\mu(i)\Big) \\
&\qquad + \omega(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\omega(t,z,j)\Big)\Bigg] - \Bigg[c_{h,k}(z,i)^{-1}\Big(\frac{hu(t+k,z,i)}{k} \\
&\qquad + u(t,z+h,i)\Big(\frac{z^2\sigma^2(i)}{2h} + z\mu(i)\Big) + u(t,z-h,i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}u(t,z,j)\Big)\Bigg]\Bigg|.
\end{aligned} \tag{3.20}$$

Moreover, we know that $q_{ij} \ge 0$ for $j \neq i$ and the other coefficients are also nonnegative, then we have

$$\mathcal{T}_{k,h}^N \omega(t,z,i) - \mathcal{T}_{k,h}^N u(t,z,i) \le c_{h,k}(x,i)^{-1}\left[\frac{h}{k} + \frac{z^2\sigma^2(i)}{2h} + z\mu(i) + \frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\right]\|\omega - u\|.$$

In addition, note that

$$c_{h,k}(z,i) = \left[\frac{h}{k} + \frac{z^2\sigma^2(i)}{2h} + z\mu(i) + \frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\right] + rh,$$

which implies that,

$$\beta = c_h(z,i)^{-1}\left[\frac{h}{k} + \frac{z^2\sigma^2(i)}{2h} + z\mu(i) + \frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\right] < 1.$$

Therefore,

$$\|\mathcal{T}_{k,h}^N \omega - \mathcal{T}_{k,h}^N u\| < \beta\|\omega - u\|.$$

□

Definition 3.2.4 *The scheme S_N is said to be stable if for every $h, k \in \mathbb{R}^*$, there exists a bounded solution $u_{h,k} \in B([0,T] \times \mathbb{R}^+ \times \mathcal{M})$ to the equation*

$$S_N(k, h, x, i, u(t, x, i), u) = 0. \tag{3.21}$$

with the bound independent of k, and h.

Let us denote $B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$ subset $B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$, such that for every $u \in B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$, $\|u\| \leq N$.

Lemma 3.2.5 *If $\omega \in B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$, Then $\|\mathcal{T}_{k,h}^N \omega\| \leq N$.*

Proof. Note that

$$\begin{aligned}\mathcal{T}_{h,k}^N \omega(t, z, i) = \quad & \max\Bigg[\frac{1}{c_{h,k}(z,i)}\Bigg(\frac{h\omega(t+k, z, i)}{k} + \omega(t, z+h, i)\Big(\frac{z^2\sigma^2(i)}{2h} + z\mu(i)\Big) \\ & + \omega(t, z-h, i)\frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\omega(t, z, j)\Bigg), g_N(z, i)\Bigg],\end{aligned}$$

which implies,

$$\mathcal{T}_{k,h}^N \omega(t, z, i) \leq c_{h,k}(z,i)^{-1}\left[\frac{h}{k} + \frac{z^2\sigma^2(i)}{2h} + z\mu(i) + \frac{z^2\sigma^2(i)}{2h} + h\sum_{j\neq i} q_{ij}\right] \max\Big[\|\omega\|, g_N(z,i)\Big].$$

Therefore, we have

$$\|\mathcal{T}_{k,h}^N \omega(t, z, i)\| \leq \max\Big[\|\omega\|, \|g_N\|\Big] \leq N.$$

□

Remark 3.2.6 *The result of this last lemma shows that for any $h, k \in \mathbb{R}^*$, $\mathcal{T}_{k,h}^N$ is an operator on $B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$. And since $\mathcal{T}_{k,h}^N$ is a strict contraction, it has a unique fixed point in $B_N([0,T] \times \mathbb{R}^+ \times \mathcal{M})$ that we denote $u_{k,h}^N$*

Lemma 3.2.7 *The scheme S_N is stable.*

Proof. Note that the solution of the equation (3.21) is just a fixed point of the contraction $\mathcal{T}_{h,k}^N$ and by the contraction mapping principle such a fixed point exists. And using Lemma 3.2.5 we conclude that the equation (3.21) has a unique solution $u_{k,h}^N$ in the set $B_N([0,T]\times\mathbb{R}^+\times\mathcal{M})$ thus we have, $\|u_{k,h}\|\leq N$. □

We are now in the position of proving the convergence of our scheme to the unique viscosity solution. For all k, and h in $\mathbb{R}^+$, $x\in\mathbb{R}^+$, and $i\in\mathcal{M}$ we define,

$$v_{k,h}^N(t,x,i)=\begin{cases} u_{k,h}^N(t,x,i), & \text{if } t\in[0,T),\\ g_N(x,i), & \text{if } t=T.\end{cases}$$

Theorem 3.2.8 *As $h\to 0$, and $k\to 0$ the sequence $v_{k,h}^N$ converges locally uniformly on $[0,T]\times\mathbb{R}^+\times\mathcal{M}$ to the unique viscosity solution of (3.16).*

Proof. Define

$$v_N^*(t,x,i)=\limsup_{y\to x,k\downarrow 0,h\downarrow 0} v_{k,h}^N(t,y,i) \text{ and } v_{*N}(t,x,i)=\liminf_{y\to x,k\downarrow 0,h\downarrow 0} v_{k,h}^N(t,y,i). \tag{3.22}$$

We claim that v_N^* and v_{*N} are respectively sub- and supersolutions of (3.16).
To prove this claim. We only consider the v_N^* case, since the argument for v_{*N} is similar. Namely we want to prove that for any $i\in\mathcal{M}$, we have

$$\mathcal{H}_N(t_0,x_0,i,v_N^*,\mathrm{D}_t\Phi,\mathrm{D}_x\Phi,\mathrm{D}_x^2\Phi)\leq 0$$

for any test function $\Phi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^+)$ such that (t_0,x_0) is a strict local maximum of $v_N^*(t,x,i)-\Phi(t,x)$. Without loss of generality we may also assume that $v_N^*(t_0,x_0,i)=\Phi(t_0,x_0)$ and because of the stability of our scheme we can also assume that, $\Phi\geq 2\sup_{k,h}\|v_{k,h}^N\|$ outside of the ball $B((t_0,x_0),r)$ where $r>0$ is such that

$$v_N^*(t,x,i)-\Phi(t,x)\leq 0=v_N^*(t_0,x_0,i)-\Phi(t_0,x_0) \qquad \text{in } B((t_0,x_0),r).$$

This implies that there exist sequences $k_n>0$, $h_n>0$ and $(t_n,y_n)\in[0,T]\times[0,C]$ such that; as $n\to\infty$ we have

$$k_n\to 0,\;\; h_n\to 0,\;\; y_n\to x_0,\;\; t_n\to t_0,\;\; v_{k_n,h_n}^N(t_n,y_n,i)\to v_N^*(t_0,x_0,i), \tag{3.23}$$

$$\text{and } (t_n, y_n) \text{ is a global maximum point of } v^N_{k_n,h_n}(\cdot,\cdot,i) - \Phi(\cdot,\cdot).$$

We denote $\epsilon_n = v^N_{k_n,h_n}(t_n, y_n, i) - \Phi(t_n, y_n)$, we have obviously $\epsilon_n \to 0$ and $v^N_{k_n,h_n}(t,x,i) \leq \Phi(t,x) + \epsilon_n$ for all $(t,x) \in [0,T] \times \mathbb{R}^+$.

Recall that

$$S_N(k_n, h_n, y_n, \alpha, v^N_{k_n,h_n}(t,x,i), v^N_{k_n,h_n}) = 0.$$

The monotonicity of S and (3.23) imply that

$$\begin{aligned} S_N(k_n, h_n, y_n, i, \Phi(t_n, y_n) + \epsilon_n, \Phi + \epsilon_n) &\leq S_N(k_n, h_n, y_n, \alpha, v^N_{k_n,h_n}(t,x,i), v^N_{k_n,h_n}) \\ &= 0 \end{aligned} \tag{3.24}$$

therefore

$$\lim_n \frac{S_N(k_n, h_n, y_n, i, \Phi(t_n, y_n) + \epsilon_n, \Phi + \epsilon_n)}{h_n} \leq 0$$

Consequently,

$$\begin{aligned} \mathcal{H}_N(t_0, x_0, i, v^*_N, \mathrm{D}_t\Phi, \mathrm{D}_x\Phi, \mathrm{D}^2_x\Phi) &= \\ \lim_{y\to x_0, k_n\to 0, \epsilon\to 0, h\to 0} \frac{S(k,h,y,i,\Phi(y,i)+\epsilon, \Phi+\epsilon)}{h} &\leq 0. \end{aligned} \tag{3.25}$$

This proves that v^*_N is a viscosity subsolution and, using the uniqueness of the viscosity solution, we see that $v_N = v^*_N = v_{*N}$. Therefore we conclude that the sequence $(v^N_{h,k})_N$ converges locally uniformly to v_N and we already know that $(v_N)_N$ converges locally uniformly to v the unique viscosity solution of (2.86), finally we have

$$\lim_{N\to\infty} \lim_{h\to 0, k\to 0} v^N_{h,k} = v.$$

□

3.2.2 Numerical examples

In our examples we use two state Markov chains, and we use the expression *first state* for the state when $\alpha(t) = 1$ and *second state* for the state when $\alpha(t) = 2$.

Example 1. American call option

For this example we study an American call option with expiration $T = 0.25$ (1 quarter of a year), exercise price $K = 70$ discount rate $r = 0.06$, the return vector $\mu = (-.6, 0.4)$, the volatility vector is $\sigma = (0.3, 0.4)$ and the generator

$$Q = \begin{pmatrix} -.5 & .5 \\ .9 & -.9 \end{pmatrix}.$$

The corresponding option prices are given in Figure 3.3. We see that when the market is bad as it is in the *first state* the American call option price is relatively smaller than when the market is good as we see on the second figure. Figure 3.4 describes the free boundary for the two states. And Figure 3.5 gives the continuation domain when we fix $t = 0$ for both the first and second states. We then deduce that, in the *first state* when the market is bad, the first time the value function $v(0, x, 1)$ and reward function $g(x, 1)$ intersect is when the stock price is $x = 80$ for and option price of 10 dollars. And when the market is good those two curves intersect at the point $(83, 13)$. So the optimal policy is to exercise the option as soon as those curves intersect.

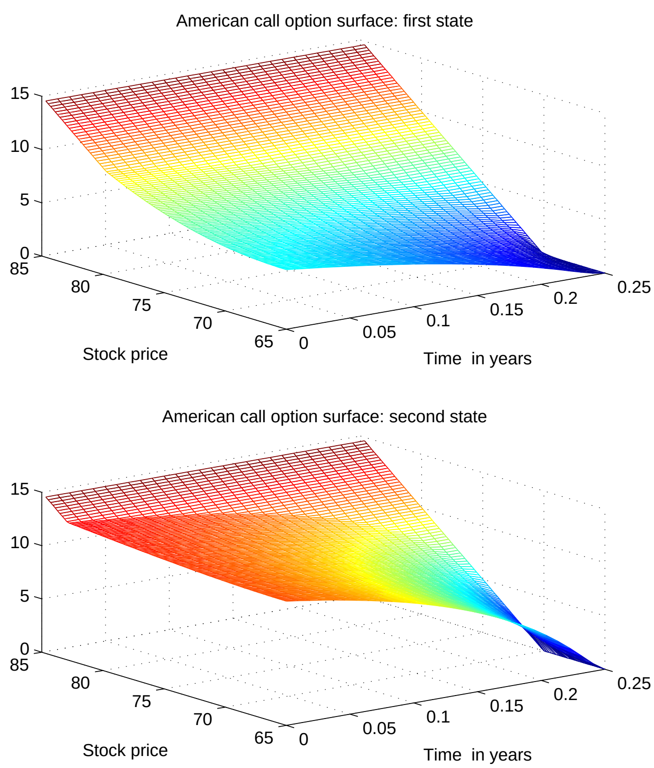


Figure 3.3: American call option surfaces

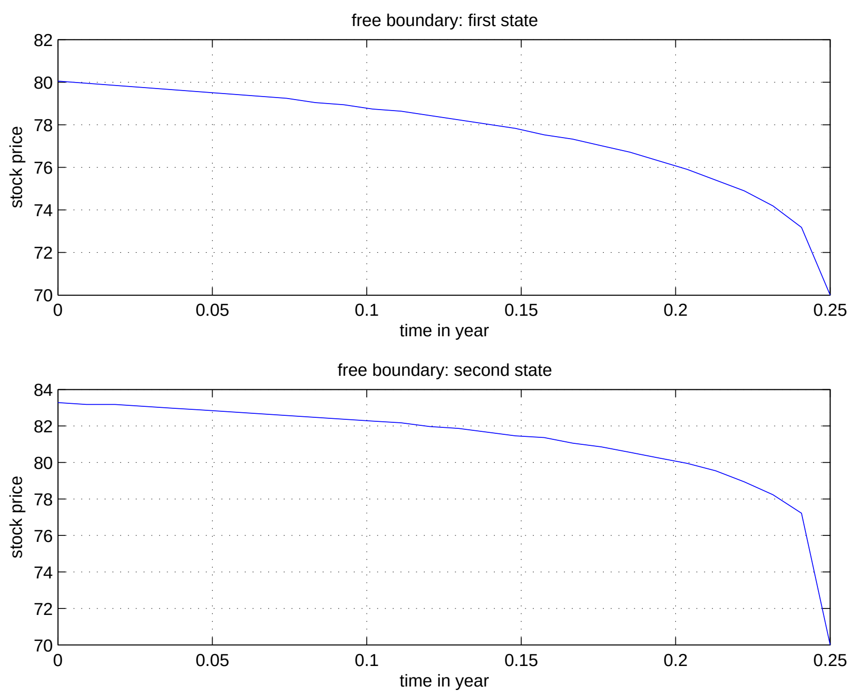


Figure 3.4: Free boundary graphs

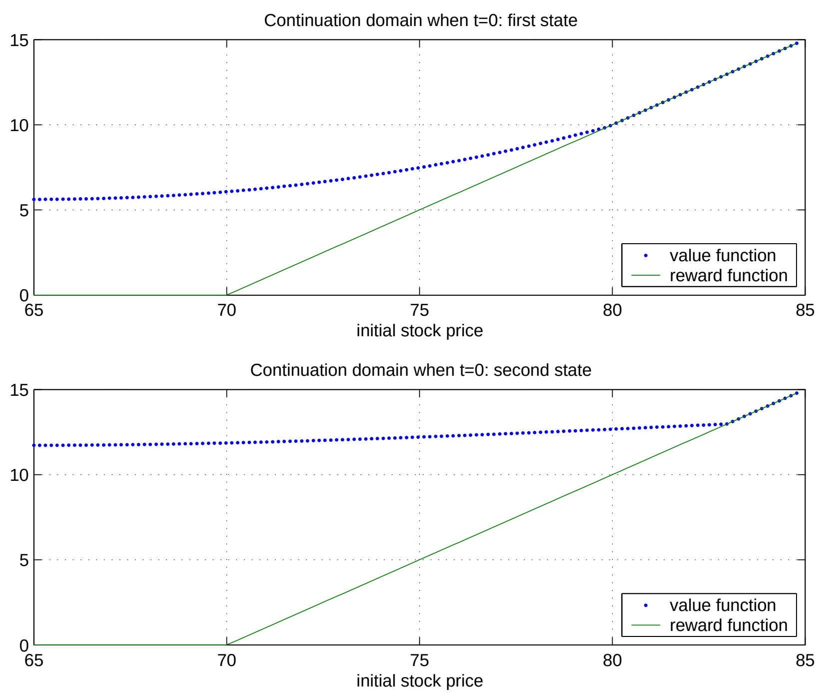


Figure 3.5: Domain of continuation for the first and second state when we fix t=0

Example 2. American put option

For second example we consider an American put option with expiration $T = 0.25$ years, exercise price $K = 80$ discount rate $r = 0.06$, the return vector $\mu = (-0.6, 0.4)$, the volatility vector is $\sigma = (0.3, 0.4)$ and the generator

$$Q = \begin{pmatrix} -1.6 & 1.6 \\ 2 & -2 \end{pmatrix}.$$

Figure 3.6 describes the put option price surface. We see that when the market is bad as it is in the *first state* the American put option price, unlike what we have for the call option, is relatively greater than when the market is good as we see on the *second state*'s graph. Figure 3.7 describes the free boundary for the two states. And Figure 3.8 gives the continuation domain when we fix $t = 0$ for both the first and second states. We then deduce that, in the *first state* when the market is bad, the first time the value function $v(0, x, 1)$ and reward function $g(x, 1)$ intersect is when the stock price is $x = 65.50$ for and option price of 14 dollars. And when the market is good those two curves intersect when the stock price is 68 and the option price is 12 dollars. So the optimal policy is to exercise the option as soon as those curves intersect.

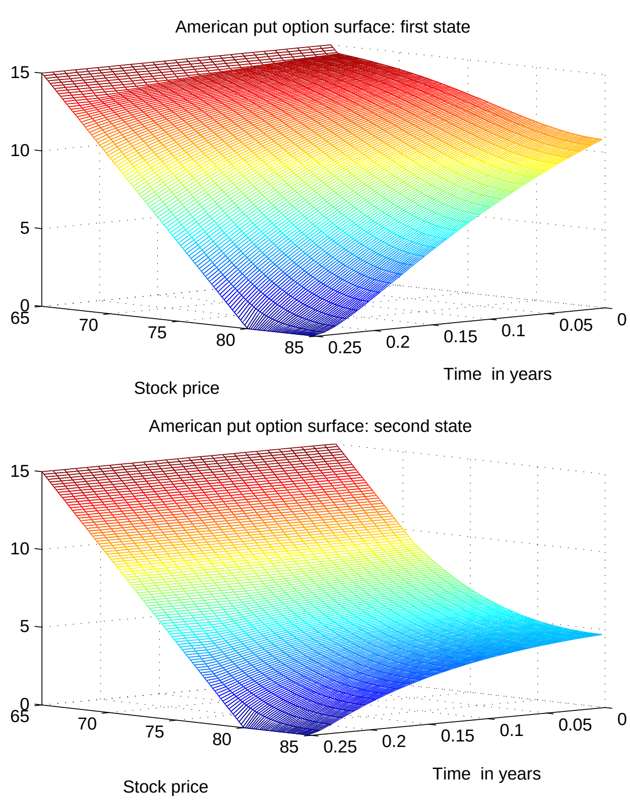


Figure 3.6: American put option surfaces

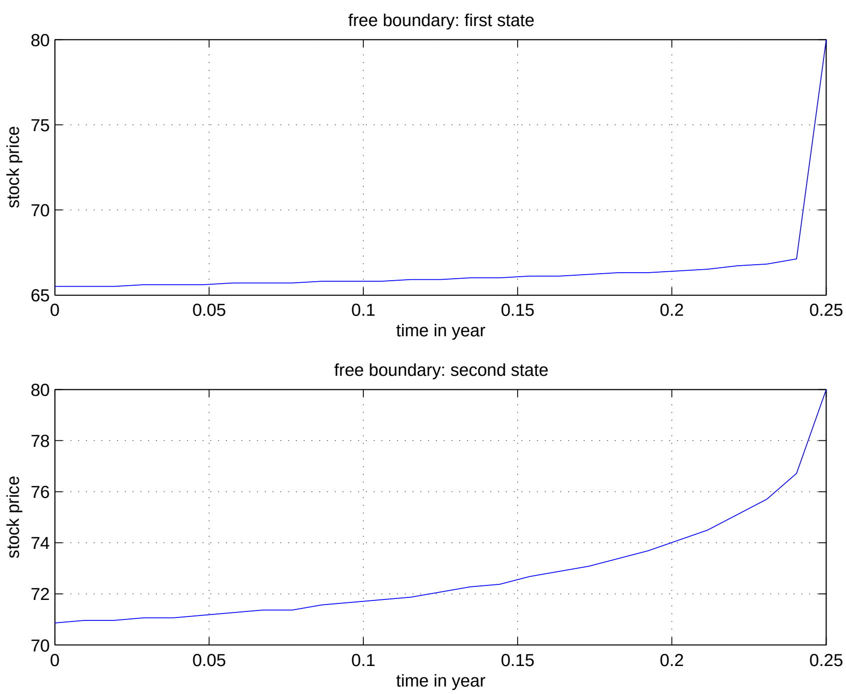


Figure 3.7: Free boundary graphs

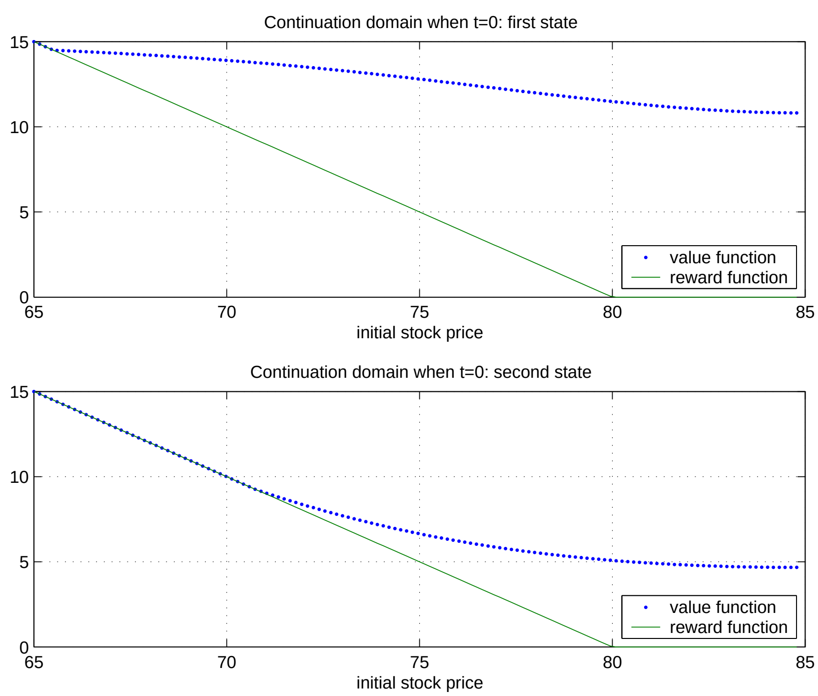


Figure 3.8: Domain of continuation for the first and second state when we fix t=0

Chapter 4

Optimal stock liquidation under regime switching model with finite time horizon

Introduction

Decision making in stock liquidation is crucial in successful trading and portfolio management. One of the main factors that affects decision making in a marketplace is the trend of the stock market. It is necessary to incorporate such trends in modeling to capture detailed stock price movements. In a recent paper of Zhang [42] the regime switching model is proposed and developed. Such switching processes can be used to represent market trends or the trends of an individual stock. In addition, various economic factors such as interest rates, business cycles etc. can also be easily incorporated in the model. In [42], a selling rule determined by two threshold levels, a target price and a stop-loss limit is considered. One makes a selling decision whenever the price reaches either the target price or the stop-loss limit. The objective is to choose these threshold levels to maximize an expected return function. In [42], such optimal threshold levels are obtained by solving a set of two-point boundary value problems.

In this chapter, we consider an optimal selling rule among the class of almost all stopping times under a regime switching model, this is a more general type of selling. We study the case when the stock has to be sold within a pre-specified time limit. Given a fixed transaction cost, the objective is to choose a stopping time so as to maximize an expected return. The optimal stopping problem was studied by McKean [28] back to the 1960's when there is no switching, see also Samuelson [35] in connection with derivative pricing and Øksendal [32] for optimal stopping in general. In models with regime switching, Guo and Zhang [15] considered the model with a two-state ($m = 2$) Markov chain. Using a smooth-fit technique, they were able to convert the optimal stopping problem to a set of algebraic equations under certain smoothness conditions. Closed-form solutions were obtained in these cases. However, it can be shown with extensive numerical tests that the associated algebraic equations may have no solutions in some cases. This suggests that the smoothness (C^2) assumption may not hold in these cases. Moreover, the results in [15] and [32] are established on an infinite time horizon setup. However, in practice, an investor often has to sell his stock holdings by a certain date due to various non-price related consideration such as year-end tax deduction or the need for raising cash for major purchases. In these cases, it is necessary to consider the corresponding optimal selling with a finite horizon. It is the purpose of this chapter to treat the underlying finite horizon optimization problem with possible non-smoothness of the solutions to the associated HJB equations. We resort to the concept of viscosity solutions and show that the corresponding value is indeed the only viscosity solution to the HJB equation. We also establish the convergence of a (explicit) finite-difference scheme for solving the HJB equations. The main results of the chapter include treatment of an optimal stopping in a general regime switching model and the corresponding numerical investigations of these solutions. It is well known that the optimal stopping rule can be determined by the corresponding value function; see, for example, Krylov [26] and ØKsendal [32] for diffusions, Pham [33] for jump diffusions, and Guo and Zhang [15] for regime switching diffusions. A main focus of this paper is to completely characterize the value function in terms of viscosity solutions.

The chapter is organized as follows. In the next section, we formulate the problem under consideration and then present the associate HJB equations and their viscosity solutions. In Section 3, we obtain the continuity property of the value function and show that it is the only viscosity solution to the HJB equations. In Section 4, we construct the corresponding finite difference method for solving the HJB equation and establish its convergence. In Section 5, we give a numerical example. Real market data is used to illustrate our results.

Our main objective is to find the optimal reward function of a selling transaction when movement of the market a driven by a hidden the Markov chain with n arbitrary states.

4.1 Problem formulation

Given an integer $m \geq 2$, let $\alpha(t) \in \mathcal{M} = \{1, 2, ..., m\}$ denote a Markov chain with an $m \times m$ matrix generator $Q = (q_{ij})_{m,m}$, i.e., $q_{ij} \geq 0$ for $i \neq j$ and $\Sigma_{j=1}^{m} q_{ij} = 0$ for $i \in \mathcal{M}$. Let $X(t)$ denote the price of a non-dividend stock. It satisfies the following stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = X(t) \left(\mu(\alpha(t))\mathrm{d}t + \sigma(\alpha(t))\mathrm{d}W(t) \right), \\ X(s) = x, \quad s \leq t \leq T, \end{cases} \tag{4.1}$$

where x is the initial price, T is a finite time, $\mu(i)$ is the rate of return, $\sigma(i)$ is the volatility, and $W(t)$ is the standard Wiener process. Both $W(\cdot)$ and $\alpha(\cdot)$ are defined on a probability space $(\Omega, \mathcal{F}, P)$ and $W(\cdot)$ is independent of $\alpha(\cdot)$.

In this chapter, we consider the optimal selling rule with a finite horizon T. Given the transaction cost $a > 0$, the objective of the problem is to sell the stock by time T so as to maximize $E[e^{-r(\tau - s)}(X(\tau) - a)]$, where $r > 0$ is a discount rate.

Let $\mathcal{F}_t = \sigma\{\alpha(s), W(s); s \leq t\}$ and let $\Lambda_{s,T}$ denote the set of $\mathcal{F}_t$-stopping times such that $s \leq \tau \leq T$ a.s. The value function can be written as follows

$$v(s, x, i) = \sup_{\tau \in \Lambda_{s,T}} E\left[e^{-r(\tau - s)}(X(\tau) - a) \mid X(s) = x, \alpha(s) = i \right]. \tag{4.2}$$

Given the value function $v(s, x, i)$, it is typical that an optimal stopping time τ^* can be determined by the following continuation region

$$D = \{(t, x, i) \in [0, T) \times \mathbb{R} \times \mathcal{M};\, v(t, x, i) > x - a\},$$

as follows:

$$\tau^* = \inf\{t > 0;\, (t, X(t), \alpha(t)) \notin D\}.$$

We know from Corollary 2.1.13 that, if $\tau^* < +\infty$ then

$$v(s,x,i) = E^{s,x,i}[e^{-r(\tau^*-s)}(X(\tau^*)-a)]. \tag{4.3}$$

Thus τ^* is the optimal stopping time.

Let $\mathcal{A}$ denote the generator of $(X(t),\alpha(t))$. Then, we have

$$(\mathcal{A}f)(s,x,i) = \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 f(s,x,i)}{\partial x^2} + x\mu(i)\frac{\partial f(s,x,i)}{\partial x} + Qf(s,x,\cdot)(i)$$

where

$$Qf(s,x,\cdot)(i) = \sum_{j\neq i} q_{ij}(f(s,x,j)-f(s,x,i)).$$

The corresponding Hamiltonian has the following form

$$\begin{aligned}\mathcal{H}(i,s,x,u,\mathrm{D}_s u,\mathrm{D}_x u,\mathrm{D}_x^2 u) = \min\Big[ru(s,x,i) - \frac{\partial u(s,x,i)}{\partial s} - \\ (\mathcal{A}u)(s,x,i), u(s,x,i)-(x-a)\Big] = 0.\end{aligned} \tag{4.4}$$

Note that $X(t) > 0$ for all t. Let $\mathbb{R}^+ = (0,\infty)$. Formally, the value function $v(s,x,i)$ satisfies the HJB equation

$$\begin{cases} \mathcal{H}(i,s,x,v,\mathrm{D}_s v,\mathrm{D}_x v,\mathrm{D}_x^2 v) = 0, & \text{for } (s,x,i) \in [0,T)\times\mathbb{R}^+\times\mathcal{M}, \\ v(T,x,\alpha(T)) = (x-a). \end{cases} \tag{4.5}$$

We will prove that in fact the value function is a viscosity solution of (4.5).

4.2 Properties of value functions

In this section, we study the continuity of the value function; show that it satisfies the associated HJB equation as a viscosity solution; and establish the uniqueness. We first show the continuity property.

Lemma 4.2.1 *For each $i \in \mathcal{M}$, the value function $v(s,x,i)$ is continuous in (s,x). Moreover, it has at most linear growth rate, i.e., there exists a constant C such that $\mid v(s,x,i) \mid \leq C(1+|x|)$.*

Proof. for the continuity in (s,x) the argument is similar as in Lemma 2.2.5 and for the linear growth the argument is similar as in Lemma 1.0.3.

□

Theorem 4.2.2 *The value function $v(s,x,i)$ is the unique viscosity solution of equation (4.5).*

Proof. First we prove that $v(s,x,i)$ is a viscosity supersolution of (4.5). Given $(s,x_s) \in [0,T]\times\mathbb{R}^+$, let $\psi \in C^2([0,T]\times\mathbb{R}^+)$ such that $v(t,x,\alpha)-\psi(t,x)$ has local minimum at (s,x_s) in a neighborhood $N(s,x_s)$. We define a function

$$\varphi(t,x,i) = \begin{cases} \psi(t,x) + v(s,x_s,\alpha_s) - \psi(s,x_s), & \text{if} \quad i = \alpha_s, \\ v(t,x,i), & \text{if} \quad i \neq \alpha_s. \end{cases} \tag{4.6}$$

Let $\gamma \geq s$ be the first jump time of $\alpha(\cdot)$ from the initial state α_s, and let $\theta \in [s,\gamma]$ be such that $(t,X(t))$ starts at (s,x_s) and stays in $N(s,x_s)$ for $s \leq t \leq \theta$. Moreover, $\alpha(t)=\alpha_s$, for $s \leq t \leq \theta$. Using Dynkin's formula, we have,

$$\begin{aligned} & E^{s,x_s,\alpha_s} e^{-r(\theta-s)}\varphi(\theta,X(\theta),\alpha_s) - \varphi(s,x_s,\alpha_s) \\ = \; & E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- r\varphi(t,X(t),\alpha_s) \\ & + \frac{\partial\varphi(t,X(t),\alpha_s))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\varphi(t,X(t),\alpha_s)}{\partial x^2} \\ & + X(t)\mu(\alpha_s)\frac{\partial\varphi(t,X(t),\alpha_s)}{\partial x} + Q\varphi(t,X(t),\cdot)(\alpha_s)\Big)dt. \end{aligned} \tag{4.7}$$

Recall that (s,x_s) is the minimum of $v(t,x,\alpha_s)-\psi(t,x)$ in $N(s,x_s)$. For $s \leq t \leq \theta$, we have

$$v(t,X(t),\alpha_s) \geq \psi(t,X(t)) + v(s,x_s,\alpha_s) - \psi(s,x_s) = \varphi(t,X(t),\alpha_s). \tag{4.8}$$

Using equation (4.6) and (4.8), we have

$$\begin{aligned} & E^{s,x_s,\alpha_s} e^{-r(\theta-s)} v(\theta,X(\theta),\alpha_s) - v(s,x_s,\alpha_s) \\ \geq \; & E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(- rv(t,X(t),\alpha_s) \\ & + \frac{\partial\psi(t,X(t))}{\partial t} + \frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\psi(t,X(t))}{\partial x^2} \\ & + X(t)\mu(\alpha_s)\frac{\partial\psi(t,X(t))}{\partial x} + Q\varphi(t,X(t),\cdot)(\alpha_s)\Big)dt. \end{aligned} \tag{4.9}$$

Moreover, we have

$$\begin{aligned} Q\varphi(t,X(t),\cdot)(\alpha_s) &= \sum_{\beta\neq\alpha_s} q_{\alpha_s\beta}\Big(\varphi(t,X(t),\beta)-\varphi(t,X(t),\alpha_s)\Big) \\ &\geq \sum_{\beta\neq\alpha_s} q_{\alpha_s\beta}\Big(v(t,X(t),\beta)-v(t,X(t),\alpha_s)\Big) \\ &\geq Qv(t,X(t),\cdot)(\alpha_s). \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\begin{aligned} &E^{s,x_s,\alpha_s} e^{-r(\theta-s)} v(\theta,X(\theta),\alpha_s) - v(s,x_s,\alpha_s) \\ &\quad\geq E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(-rv(t,X(t),\alpha_s) \\ &\quad+\frac{\partial\psi(t,X(t))}{\partial t}+\frac{1}{2}X(t)^2\sigma^2(\alpha_s)\frac{\partial^2\psi(t,X(t))}{\partial x^2} \\ &\quad+X(t)\mu(\alpha_s)\frac{\partial\psi(t,X(t))}{\partial x}+Qv(t,X(t),\cdot)(\alpha_s)\Big)dt. \end{aligned} \tag{4.11}$$

It follows from Corollary 2.1.11 that

$$\begin{aligned} E^{s,x_s,\alpha_s}\int_s^\theta e^{-r(t-s)}\Big(-rv(t,X(t),\alpha_s)+\frac{\partial\psi(t,X(t))}{\partial t}+\frac{1}{2}X^2(t)\sigma^2(\alpha_s)\frac{\partial^2\psi(t,X(t))}{\partial x^2} \\ +X(t)\mu(\alpha_s)\frac{\partial\psi(t,X(t))}{\partial x}+Qv(t,X(t),\cdot)(\alpha_s)\Big)dt \leq 0. \end{aligned}$$

Dividing both sides by $\theta > 0$ and sending $\theta \to s$ lead to

$$\begin{aligned} -rv(s,x_s,\alpha_s)+\frac{\partial\psi(s,x_s)}{\partial t}+\frac{1}{2}x_s^2\sigma^2(\alpha_s)\frac{\partial^2\psi(s,x_s)}{\partial x^2} \\ +x_s\mu(\alpha_s)\frac{\partial\psi(s,x_s)}{\partial x}+Qv(s,x_s,\cdot)(\alpha_s)\leq 0. \end{aligned} \tag{4.12}$$

By definition, $v(s,x,i) \geq x-a$. The supersolution inequality follows from this inequality and (Corollary 2.1.13).

The proof for the viscosity subsolution inequality is similar to the supersolution part except that we need to treat points (t,x) such that $v(t,x,i) > x-a$, for $i \in \mathcal{M}$.
In this case, take $\varepsilon = (v(t,x,i)-(x-a))/2 > 0$ and let

$$\tau^\varepsilon = \inf\{s \leq t \leq T:\ v(t,X(t),\alpha(t)) \geq (X(t)-a)+\varepsilon\} \wedge \gamma,$$

where γ is the first jump time of $\alpha(t)$ with $\alpha(s)=\alpha_s$. It can be shown as in Pham [33] that $E\tau^\varepsilon > 0$. Following Corollary 2.1.11, Dynkin's formula with $\tau = \theta\wedge\tau^\varepsilon$, and let $\theta \to 0$ in the resulting inequality, we have

$$rv(s,X(s),\alpha_s) - \frac{\partial \phi(s,X(s))}{\partial t} - \frac{1}{2}X(s)^2\sigma^2(\alpha_s)\frac{\partial^2\phi(X(s),\alpha_s)}{\partial x^2}$$
$$-X(s)\mu(\alpha_s)\frac{\partial\phi(X(s),\alpha_s)}{\partial x} - Qv(s,X(s),\cdot)(\alpha_s) \le 0.$$

This gives the subsolution inequality. Therefore, $v(t,x,i)$ is a viscosity solution of (4.5).

Finally, the uniqueness is just similar to the uniqueness of the finite time horizon American option obtain in chapter 2. □

Remark 4.2.3 In the infinite time horizon case the optimal reward or value function is defined as follows,

$$v(x,i) = \sup_{\tau} E\left[e^{-r\tau}(X(\tau)-a) \mid X(0)=x, \alpha(0)=i\right] \tag{4.13}$$

where the supremum taking over all possible stopping time τ. We can similarly prove that $v(x,i)$ is the unique viscosity solution of the nonlinear PDE

$$\min\left\{rv(x,i) - \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2 v(x,i)}{\partial x^2} - x\mu(i)\frac{\partial v(x,i)}{\partial x} - Qv(x,\cdot)(i), v(x,i)-(x-a)\right\} = 0 \tag{4.14}$$

Remark 4.2.4 *Due to the similarities of this problem and the finite time horizon American option we can just use the scheme we developed in order to approximate the value of finite time horizon American option to approximate the value of the viscosity solution of equation (4.5).*

4.3 A Numerical example

In this section, we present a numerical example using the IBM stock daily closing from October 28, 2002 to August 28, 2004. We consider the switching process $\alpha(t)$ where $\alpha(t) \in \mathcal{M} = \{1,2\}$ represents the trends of IBM stock. In particular, $\alpha(t) = 1$ stands for the up-trend and $\alpha(t) = 2$ the down-trend. The generator of $\alpha(t)$ is given by

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}. \tag{4.15}$$

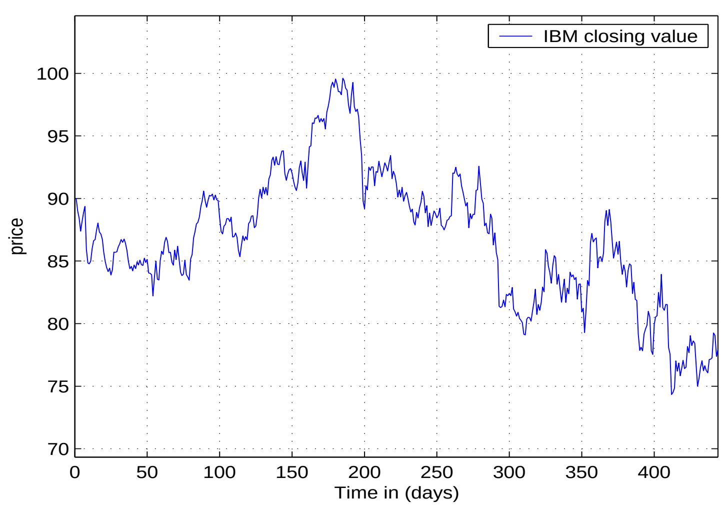


Figure 4.1: IBM: 10/28/2002–08/28/2004

The IBM daily closing prices during this period are plotted in Figure 1.

Based on these closing prices, we calibrate the model using the method introduced in [42] and obtain

$$\lambda = 2.0367, \qquad \mu = 1.9821.$$

The corresponding stationary distribution $\nu = (\nu_1, \nu_2) = (\mu/(\lambda + \mu), \lambda/(\lambda + \mu)) = (0.4932, 0.5068)$.

And for the volatility and return we have, we have the following vectors

$$\sigma = (\sigma(1), \sigma(2)) = (0.3478, 0.3385) \quad r = (r(1), r(2)) = (0.2501, -0.3570),$$

where $\sigma(1)$ represents the volatility when IBM is up and $\sigma(2)$ represent the volatility when IBM is down. The same holds for the return vector.

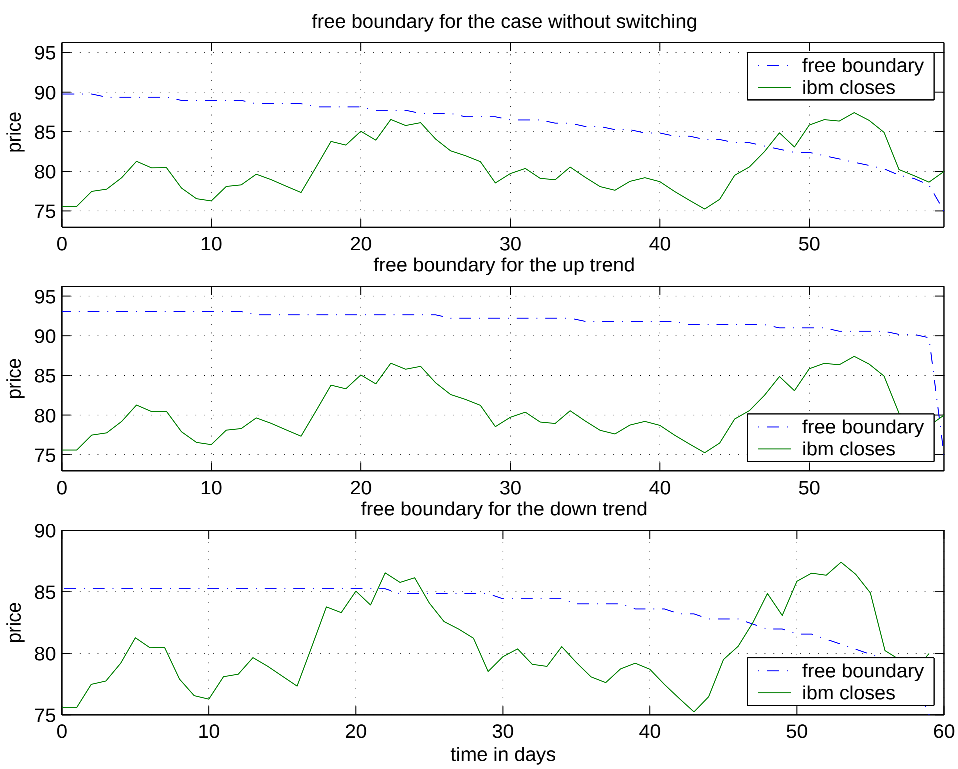
Then the averaged volatility $\overline{\sigma} = \sqrt{\nu_1\sigma^2(1) + \nu_2\sigma^2(2)} = 0.3431$. and the average return $\overline{r} = \nu_1 r(1) + \nu_2 r(2) = -0.0576$.

We take the transaction fee $a = 0.05$ per share and the discount rate is $r_0 = 0.06$.

In Figure 2, the first picture gives the continuation region with the dished line when the averaged volatility and return rate is used in a model without switching. In this case, the selling action should take place when $t = 48$ at 83 per share which corresponds to 9.2% return in 48 days and normalized to 48% annual return.

The second and third pictures in Figure 2 give the continuation regions when the market is in uptrend and downtrend, respectively. Suppose we can detect a trend change in two to three days, which is typical with the help of Wonham filter. Then one should be able to detect a trend change from up to down near $t = 25$. In this case, one should sell at $t = 25$ at 84.7 per share which amounts a 11.4% gain in 25 days and equals 114.9% annual return.

This example shows that by differentiating different market modes (up or down trends), a better selling decision can be made to achieve a higher return.


free boundary for the case without switching
price
free boundary
ibm closes
free boundary for the up trend
free boundary for the down trend
time in days

4.4 Future research problems

1. Find a formula to evaluate European options under regime switching model.

2. Study the analytical properties of the free boundary problem associated to the optimal stopping of a switching diffusion.

3. Develop numerical methods to approximate the free boundary of a switching diffusion.

Bibliography

[1] G. Barles and P.E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptot. Anal.* **4**, (1991), 271-283

[2] A. Bensoussan, On the theory of option pricing, *Acta Appl. Math.* **2**, 139-158.

[3] N. P. B. Bollen, Valueing options in regime-switching models, *Journal of Derivatives,* **6**, (1998), 38-49.

[4] J. Buffington and R. J. Elliot, American option with regime switching, *International Journal of Theoretical and Applied Finance*, **5**, (2002), 497-514.

[5] P. Clark, A subordinated stochastic process model with finite variance for speculative prices, *Econometrica*, **41**, (1973), 135-155.

[6] M.G. Crandall, H. Ishii, and P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, **27**, (1992), 1-67.

[7] G. B. Di Masi, Y. M. Kabanov and W. J. Runggaldier, Mean variance hedging of options on stocks with Markov volatility, *Theory of Probability and Applications*, **39**, (1994), 173-181. .

[8] J.C. Duan, I. Popova and P. Ritchken, Option pricing under regime switching,*Quantitative Finance*, **2**,(2002), 116-132.

[9] E. B. Dynkin, The optimum choice of the instant for stopping a Markov process, *Soviet Mathematics*, **4**,(1963), 627-629.

[10] E. B. Dynkin, *Markov Processes* Vol. 1 and 2, Springer-Verlag.

[11] R. J. Elliott and P. E. Kopp, *Mathematics of Financial Markets*, Springer-Verlag, New York, 1998.

[12] N.S. Ethier and T. Kurtz, *Markov processes: Characterization and Convergence*, J. Wiley, New York, 1985.

[13] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1933.

[14] J. P. Fouque, G. Papanicolaou, and R. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University press, 2000.

[15] X. Guo and Q. Zhang, Optimal selling rules in a regime switching model, working paper.

[16] X. Guo and Q. Zhang, Closed-from solution for perpetual American put options with regime switching, *SIAM J. Appl. Math.* **64**, 2034-2049.

[17] Hamilton, A new approach to the economic analysis of nonstationary time series, *Econometrica*,**57**, (1989), 357-384.

[18] J.C. Hull, *Options, Futures, and Other Derivatives*, 3rd Ed., Prentice-Hall, Upper Saddle River, NJ, 1997.

[19] H. Ishii,, Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations, *Indiana Univ. Math. J.* **33**, (1984), 721-748.

[20] I. Karatzas, On the pricing of American options, *Appl. Math. Optim.*, **17**, (1988), 411-424.

[21] I. Karatzas, Optimization problems in the theory of continuous trading, *SIAM J. Control Optim.*, **27**,(1989) 1221-1259.

[22] I. Karatzas, *Lectures on the Mathematics of Finance*, American Mathematical Society, Providence, RI, 1996.

[23] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer, New York, 1998.

[24] R. Myneni,*The pricing of American option*, Anan. Appl. Probab. **2**,(1992) 1-23.

[25] I.J. Kim, The analytical valuation of American options, *Rev. Financial Stud.*, **3**, (1990), 547-572.

[26] N.V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, Berlin, 1980.

[27] P.L. Lions, Generalized Solutions of Hamilton-Jacobi Equations, *Research Notes in Mathematics*, *Pitman, Boston, MA*,**69**, 1982.

[28] H.P. McKean, A free boundary problem for the heat equation arising from a problem in mathematical economics, *Indust. Management Rev.*, **60**, (1965), 32-39.

[29] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time case, *Rev. Econom. Statist.*, **51**, (1969), 247-257.

[30] R.C. Merton, Option pricing when underlying stock returns are discontinuous, *J. Financial Economics*, **3**, (1976), 125-144.

[31] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag, New York, 1997.

[32] B. ØKsendal, *Stochastic Differential Equations*, Springer, New York, 1998

[33] H. Pham, Optimal stopping of controlled jump diffusion processes: A viscosity solution approach, *J. Math. Syst. Estimation Control*, **8**, (1998), 1-27.

[34] P. Praetz, The distribution of share price changes, *J. Business*, **45**, (1972), 49-55.

[35] P.A. Samuelson, Rational theory of warrant pricing, *Industr. Management Rev.*, **6**, (1995), 13-32.

[36] J.L. Snell *Application of martingale system theorems*, Trans. Amer. Math. Soc. **73**, 293-312.

[37] A.N. Shiryayev, *Optimal Stopping Rules*, Springer-Verlag, New York, 1978.

[38] P. Van Moerbeke *Optimal stopping and free boundary problem*, Arch. Rational Mech. Anal.**60**, 101-148.

[39] H.M. Soner, Optimal control with state space constraints II, *SIAM J. Control Optim.*, **24**, (1986), 1110-1122.

[40] D.D. Yao, Q. Zhang, X.Y. Zhou, A Regime-Switching Model for European Options, working paper.

[41] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer-Verlag, New York, 1998.

[42] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Control Optim.*, **40**, (2001), 4-87.