WRITING AND THE SECONDARY MATHEMATICS TEACHER

by

SHARON KAY O'KELLEY

(Under the Direction of Patricia S. Wilson)

ABSTRACT

Although much research has focused on the use of writing in K–12 mathematics classes, little has been done on teachers' experiences with writing in mathematics and how those experiences shape their attitudes about incorporating writing into their lessons. A time to explore these responses to writing in mathematics is when teachers are in the preservice phase of their careers or when they are returning to school to advance their careers. In this study, I sought to explore the experiences of five preservice and one inservice teacher in a graduate mathematics education course as they explored mathematics with technology and prepared 11 written reports for Internet publication. I asked the participants to take notes on their work and complete a written reflection after they finished each report. Through the use of questionnaires, three interviews of each participant, field notes based on class meetings, and analysis of all writings, I gained insight into how the participants responded to the writing in terms of their attitudes and beliefs as well as their capabilities in communicating mathematics in writing. I analyzed all responses, except the reports, according to emerging themes in how the participants approached their work. I analyzed the written reports according to a constructed framework based on Leinhardt's (1987) signs of an expert explanation. How the teachers responded was based on the type of writing they were asked to do, their target audience, and the beliefs they held about mathematics and writing in mathematics. Those participants who used writing as a tool to support metacognitive behavior while exploring mathematics tended to respond most favorably to the writing. However, all struggled in their explanations to some degree. Issues of providing clear goals, adequate explanations of graphs, the proper use of mathematical language, and the integration of mathematics and words sometimes interfered with their abilities to effectively communicate the mathematics. In view of these findings, I recommend that mathematics educators use informal and formal writing activities that support metacognitive behavior in the exploration of mathematics. I also recommend a modification to Leinhardt's (1987) framework to help teachers learn how to write to communicate mathematics and how to assess student writing.

INDEX WORDS: Writing in Mathematics, Metacognitive Behavior, Expert Explanation, Mathematical Language, Mathematics Education

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DEDICATION

To the memory of Linda L. O'Kelley. You taught me everything I know about perseverance and courage. You would have loved this journey, and I know you would have been in Athens every Saturday for the football, wearing Carolina colors, and driving me crazy. I miss you, Sis, and I hope you're proud….

To the memory of my parents, Mr. and Mrs. John H. O'Kelley, who taught by example, loved with all their hearts, and who never failed to tell me how proud they were that I chose to be a teacher. Mom and Dad, this one's for you….

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INTRODUCTION

My interest in writing in mathematics began when I was working on my master's degree in mathematics education at Montana State University in 2004. During that time, I was in my eighth year of teaching high school mathematics and had taught secondary English as well. Because of my training, my teaching experience, and the reflection my graduate studies caused, I began to ponder and research possible similarities between the writing process and doing mathematics. As a teacher trained in both disciplines, I had little trouble accepting the characterization of writing as a problem solving process similar in structure to mathematical problem solving, and I began to consider how writing could be used in the mathematics classroom to help students learn.

When I began my doctoral studies, however, the connection between writing and doing mathematics became more than theoretical for me. As I started to take more mathematics courses, I noticed how much I relied on both informal and formal writing to tell me what I knew about the mathematics I was studying. As I worked problems, I realized I informally jotted down notes about what I did and did not understand about the mathematics and then relied on those notes to help me prepare formal written solutions for homework assignments. I also concluded that if I experienced difficulty writing out those solutions, then in most situations, I did not understand the mathematics well enough to communicate what I had learned and the grader usually agreed. Essentially, I realized that I used writing to help me reflect on the mathematics I was exploring and as a gauge to determine how well I understood the material.

Despite my own revelations, I knew from conversation that many of the preservice and inservice teachers with whom I had worked did not share my experiences. In various forms, I often heard the opinion “What does writing have to do with math?” and I realized I wanted to know why my own experiences were different. Specifically, I wanted to know how mathematics teachers respond as students to different types of writing in mathematics. With that desire in mind, I designed this study to explore those responses and to gain insight into teacher beliefs about and attitudes toward mathematics and writing in mathematics. In so doing, I hoped to uncover information that might help mathematics educators design activities that involve teachers in the writing process in a manner those teachers find useful to exploring and communicating mathematics.

Rationale

A “campaign to move writing out of the exclusive domain of the English department” (Bazerman et al., 2005, p. 22) was afoot in American education in the early years of the twentieth century and gained significant strength in the 1970s and 1980s under the influence of British educational reform. In the 1970s, James Britton, a professor at the London School of Education, played an influential role in British reform efforts by advancing the idea that language is a fundamental part of learning. In 1975, he wrote a chapter called “Language Across the Curriculum” for the nationally commissioned Bullock Report in which he advocated that writing needs to move beyond the domain of English classes and into other curricular subjects (Report of the Committee of Enquiry, 1975). He traced this idea back to initiatives coming from the London Association for the Teaching of English in the late 1960s and also referenced “the *Writing Across the Curriculum* project . . . being conducted for the Schools Council at the University of London Institute of Education” (Report of the Committee of Enquiry, 1975, Section 12.10). Around the same time as Britton's work, the phrase "Writing Across the Curriculum" appeared in American educational circles as well and the first documented faculty seminar in its use was conducted at Central College in Pella, Iowa in 1969–1970 (Bazerman et al., 2005).

In one of the earliest reports on writing in mathematics, William Geeslin (1977) described his observations of asking "students of all ages" (p. 112) to write about mathematics. Noting that students generally performed poorly on the assignments, he specifically indicated that middle and high school students tended to exhibit important misconceptions about mathematics that were not revealed on standard achievement tests. He expressed the belief "that writing about mathematics is useful both as a diagnostic tool for the teacher and as a learning device for the student" (p. 113). He also commented that writing helps students "to form more precise ideas about mathematical concepts" (p. 114) and could serve as a way for preservice teachers to learn how to explain mathematics.

A few years after the publication of Geeslin's article, the College Board published *Academic Preparation for College: What Students Need to Know and Be Able to Do* and *Academic Preparation in Mathematics: Teaching for Transition from High School to College.* In the first book, the College Board (1983) described the Basic Academic Competencies, including writing, and defined these competencies as "broad intellectual skills essential to effective work in all fields of study" (p. 7). In the second book, Kilpatrick, the principal writer, addressed the competencies in relation to the study of mathematics and described the usefulness of writing in gaining an understanding of mathematics:

> Through the regular use of writing activities, teachers can demonstrate to students how writing interacts with understanding. We sometimes think we have to understand something before we can write about it, but writing is more often an aid to understanding.

> We may not understand something very well *until* we have written (or spoken) about it. (College Board, 1985, p. 60)

According to Dossey (as cited in McLeod, Stake, Schappelle, Mellissinos, & Gierl, 1996), these two books played an influential role in the creation of the communication standard in the standards documents published by the National Council of Teachers of Mathematics (NCTM). He noted that members of the standards writing teams were aware of these two publications and paid particular attention to the inclusion of writing as a Basic Academic Competency by the College Board as well as to Kilpatrick's description of its usefulness in mathematics. He commented that members of the writing teams received a copy of *Academic Preparation for College* to assist in the drafting of the standards (McLeod et al., 1996).

With the College Board and Kilpatrick's work in mind, NCTM issued the call in its 1989 *Curriculum and Evaluation Standards* that "all students need extensive experience . . . writing about . . . mathematical ideas" (p. 140). At the center of this call is the idea that writing can be used as a method to learn and communicate mathematics. Indeed, since 1989, much research has been conducted to study the use of writing as a valuable tool of student learning in the mathematics classroom (Porter & Masingila, 2001). Despite these urgings, however, many mathematics teachers remain reluctant to use writing as a tool of learning in their lessons particularly at the secondary level. In 2001, Weiss, Banilower, McMahon and Smith reported the results of a national survey of secondary mathematics teachers in the United States in which 55% of the teachers indicated they never use reflective writing in their classrooms, 6% indicated they used it at least once a week, and 1% reported they used it on a daily basis. Therefore, there can be little argument that a gap exists between the lessons of research and the realities of practice in this area of mathematics education.

At the heart of this disconnect may rest a fundamental belief that the process of writing is far removed from the process of mathematical problem solving and yet a close examination of both processes reveals deep similarities between the two. Janet Emig (1977) characterized “writing as heuristic” (p. 122) and maintained that one of its benefits is that it can give “self-provided feedback” (p. 128) as both a process and a product. Her point was that writing can provide a record of the evolution of thought as a process or it can exist as a final product for “review and re-evaluation” (p. 128). The description provided by Emig echoes Polya’s description of problem solving in mathematics. Polya (1945/2004) stated that “modern heuristic endeavors to understand the process of solving problems, especially the mental operations typically useful in this process” (pp. 129–130). In his book *How to Solve It*, he provided guidance on how to move through the problem solving process and suggested questions that students can ask as they engage in the process. Writing, therefore, could offer the “self-provided feedback” (Emig, 1977, p. 128) to assist in the development of questioning skills needed for effective problem solving in mathematics.

In addition to similarities between the processes of writing and problem solving, there are also similarities in how they are characterized by competing views in their respective fields. For example, subscribers to a traditional view of mathematics tend to regard mathematics as a list of rules, formulas, and procedures that must be committed to memory in order to be mastered. In the article “When Good Teaching Leads to Bad Results: The Disasters of ‘Well-Taught’ Mathematics Courses,” Alan Schoenfeld (1988) described the student beliefs that this type of teaching can engender. First, under a traditional perspective, students and others often think that “the processes of formal mathematics . . . have little or nothing to do with discovery or invention” (p. 151). In other words, the primary purpose behind proofs is to confirm knowledge that others with authority, such as teachers, "already knew to be true" (p. 156). Second, the general public tends to believe that mathematical problems should be quickly solved. It follows then that if students cannot accomplish this feat, then they do not adequately understand the material and should give up. Implied in this belief is the idea that the solution is valued more than the process.

Next, under a traditional perspective of mathematics, students tend to "view themselves as passive consumers of others' mathematics" (Schoenfeld, 1988, p. 160). Inherent in this belief is the notion that the authority over mathematics rests with someone other than the student. Therefore, "only geniuses are capable of discovering, creating, or really understanding mathematics" (p. 151). Hence, students have little hope "that they can make sense of it for themselves" (p. 151). Finally, in relation to this last belief, students operating under a traditional view often believe that the only path to success is to imitate what the teacher has demonstrated. Therefore, learning in mathematics is not based upon original thought but upon mimicking the instruction of the teacher.

In the field of composition, a similar view has held sway over the pedagogy used by writing instructors. Richard Young (2009) noted that under the current-traditional rhetoric (CTR) view, composition instruction tends to focus "on the composed product rather than the composing process . . . on usage (syntax, spelling, punctuation) and [on] style (economy, clarity, emphasis)" (p. 398). Young also emphasized that invention is excluded from CTR—that "skills which cannot be formulated as methods cannot be taught" (p. 399). Maxine Cousins Hairston (2009) added to this description. She stated that supporters of CTR "believe that competent writers know what they are going to say before they write; thus their most important task when they are preparing to write is finding a form into which to organize their content" (p. 441). In addition, she added that they “also believe that the composing process is linear, that it proceeds systematically from prewriting to writing to rewriting” (p. 441). In essence, in the CTR classroom, the goal is to write a clear, concise, and well-organized paper that is technically correct.

Both a traditional view of teaching and learning mathematics as well as the CTR method of teaching and learning writing seem to value what is procedural above all else. From these traditional perspectives, mathematics instructors and writing instructors use a pedagogy that analyzes and critiques the product rather than observes and comments on the process. Algorithms and formulas are valued above discovery and invention, and the value of one’s work is judged by the content of one’s final solution. By focusing on the procedural, these teachers convey the message that mathematical problem solving and writing are always linear tasks that can be completed efficiently and in a timely manner. If messiness ensues in either discipline, then the student is missing the point. Essentially, both views seem to perpetuate the notion that those who can write or do mathematics quickly and effortlessly are gifted and that these gifts cannot be taught. Most importantly, both views imply that the ultimate power rests with the teacher to unlock the door to the knowledge of how to correctly write and do mathematics.

If writing is to become a common experience in the mathematics classroom, mathematics educators must find ways to move teachers toward a view of mathematics and writing as related processes rather than as vastly different products. To find these methods, however, researchers must first understand how teachers respond to various forms of writing. Although much research has been done to chronicle the use of writing by mathematics students, little has been done that focuses on teachers’ personal experiences with writing in mathematics and how those experiences shape their attitudes about incorporating writing into their lessons. Logical times to explore these experiences are when teachers are in the preservice phase of their careers, initially being trained in the methodologies of mathematics education, and when they are returning to school in order to advance their careers. These are the times when teachers are exposed to different ways of thinking about the nature of mathematics and what it means to teach it well. These are the times in which teachers are asked to consider ways of approaching mathematics that may be different from what they experienced as students and what they use or may use in their own classrooms. Such times of self-exploration lend themselves well to an examination of attitudes and beliefs about the nature of mathematics and the role writing may play in learning mathematics as well as the extent to which those attitudes and beliefs may be influenced through reflection.

Overview of the Study

In this study, I sought to explore the experiences of secondary preservice and inservice teachers in a graduate mathematics education course as they explored mathematics with technology and prepared written reports of their results for Internet publication. The purpose of the study was to answer the following questions:

- How do secondary teachers of mathematics respond to writing to communicate mathematics?
- How do secondary teachers of mathematics respond to reflective writing?
- What do secondary teachers of mathematics believe about the use of writing in the mathematics classroom?

I followed six participants over the course of a semester as they went through the process of preparing eleven write-ups for Internet publication in a mathematics education course focused on technology. With each write-up, I asked participants to take notes on their work in a specified manner as well as complete a written reflection after they finished each write-up. My goal was to ascertain what the experience of writing to communicate and to reflect upon mathematics revealed about the participants' attitudes and beliefs about mathematics and writing in mathematics. Through the use of an initial and final questionnaire, three interviews of each participant, field notes based on weekly class meetings, and analysis of all writings, I gained insight into how these participants responded to the writing process both in terms of their attitudes and beliefs as well as their capabilities in communicating mathematics in writing. I analyzed all responses, except the write-ups, according to emerging themes in how the participants approached their work and the beliefs and attitudes they held. I analyzed the write-ups according to a constructed framework based on Leinhardt's (1987) signs of an expert explanation.

The information provided by this study can provide insight into ways in which teacher educators can help to prepare and assess teachers as effective written communicators of mathematics as well as offer clarification into how the processes of both expository and reflective writing might be used to promote awareness among students of possible connections between writing and doing mathematics. In so doing, teachers not only can become better communicators of mathematics, but they may also develop an appreciation for writing in mathematics and a deeper understanding of the mathematics itself. In turn, teacher educators can increase the chances that these teachers will become proponents of the writing process with skills they can help their own students to develop.

Overall, writing and doing mathematics can be thought of as similar processes. Both writing and doing mathematics can be described as problem solving processes in which students engage to create products they are often required to communicate. Therefore, both processes involve private as well as public components in which students personally make sense of the problem at hand and then prepare their work for communication with someone else. In the chapters that follow as I examine relevant literature, report the findings of the study, and discuss the results and their implications, I consider writing and doing mathematics as problem solving processes with private and public features.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I review literature that describes writing and doing mathematics as problem solving processes with private and public components. In sections devoted to the private aspects of writing and doing mathematics, I examine the literature relevant to the cognitive aspects of both processes. In sections focused on the public components, I review the literature that addresses the possible influence of public communication on writing and doing mathematics. However, examining the literature on the public components of both processes tends to shift the focus to the product created. Therefore, I specifically examine the literature relevant to the quality of the written product in terms of mathematical language, the unique demands of integrating mathematical notation and figures with words, and the characteristics of an effective written explanation. Finally, I acknowledge that what teachers believe and feel may influence how they respond to writing in mathematics; therefore, I examine the literature relevant to teacher beliefs and attitudes about writing in mathematics and about the nature of mathematics.

Describing Writing and Doing Mathematics as Problem Solving

Over the years, scholar and practitioner alike have noted the similarities between writing and mathematics. Renowned psychologist Jerome Bruner (1966) stated in his book *Toward a Theory of Instruction* that both writing and mathematics can be thought of as “devices ordering thoughts about things and thoughts about thoughts” (p. 112). In her book *Writing to Learn Mathematics*, Joan Countryman (1992) specifically addressed the similarity between writing and learning mathematics. She stated that students learn mathematics "by exploring, justifying, representing, discussing, using, describing, investigating, [and] predicting" (p. 2), all processes for which "writing is an ideal activity" (p. 2). The National Council of Teachers of Mathematics (NCTM, 1989) also noted that the "view of writing as a process [that] emphasizes brainstorming, clarifying, and revising . . . can readily be applied to solving a mathematical problem" (p. 142). Implicit in these views is the notion that in order for educators to connect writing to mathematics they must focus on the process rather the product.

Cuoco, Goldenberg, and Mark (1996) asserted "that much more important than specific mathematical results are the habits of mind used by the people who create those results" (pp. 375–376). They specified that they were concerned with those "mental habits that allow students to develop a repertoire of general heuristics and approaches that can be applied in many different situations" (p. 378). Some of the mental habits Cuoco et al. listed were detecting patterns, experimenting with a problem, tinkering with an idea by taking it apart and reassembling it in a different way, being able to describe what is going on in a problem, conjecturing about possible methods and solutions, and simply guessing. In referring to the development of "a repertoire of general heuristics" (p. 378), Cuoco et al. called to mind the pivotal work of George Polya (1945/2004) who described four phases of problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. Within these four phases, Polya described the use of mental operations such as experimenting and conjecturing that are similar to what Cuoco et al. described in their work. In other writings, Polya (1954) also indicated that he, too, believed that guessing plays a role in learning mathematics. He noted that "the result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing" (p. vi).

As Cuoco et al. (1996) and Polya (1945/2004, 1954) implied, Schoenfeld (1994) succinctly stated that the learning of mathematics should be about “learning to think mathematically” (p. 60). Much like Cuoco et al., Schoenfeld (1992) maintained that mathematics students should be “flexible thinkers with a broad repertoire of techniques and perspectives for dealing with novel problems and situations” (p. 335). He placed an emphasis on the affective aspect of learning mathematics as well. He stated in *Mathematical Thinking and Problem Solving* that “learning to think mathematically” (1994, p. 60) means, in part, “developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them” (1994, p. 60). In other words, mathematics teachers should want students to develop an appreciation not only for the mathematics but also for their ability to understand it. In essence, by focusing on the habits of mind encapsulated in problem solving and honed through the development of learning to think mathematically, mathematics educators move away from the traditional notion of learning mathematics as a passive experience in which only the teachers have the power to give knowledge. In addition, educators who focus on these habits challenge the belief that doing mathematics should be quick and simple. Rather, their students discover the powers of their own mind in taking time to experiment, tinker, devise, and execute their own plans of action. In essence, Cuoco et al., Polya, and Schoenfeld give the power of learning back to the student.

Borrowing mathematical phrasing to describe a view similar to Cuoco et al. and Polya’s view of mathematics as a process of discovery, Linda Flower and John Hayes (2009) described “writing as a problem-solving, cognitive process” (p. 468) in which students need to get from point A to point B. Unlike the emphasis on product in current-traditional rhetoric, however, the emphasis in this problem solving approach is on the journey between points A and B. Like the view of problem solving in mathematics offered by Cuoco et al., Polya, and Schoenfeld, Flower and Hayes emphasized that the writing process is often not one of instantaneous enlightenment but rather a process that requires time and attention. They stressed that “a writer in the act of discovery is hard at work searching memory, forming concepts, and forging a new structure of ideas” (p. 467). Therefore, this view of writing engenders the notion that, like problem solving in mathematics, writing can be a messy endeavor that requires patience and perseverance.

Similar to Schoenfeld who emphasized the affective aspect of learning mathematics, Flower and Hayes (2009) emphasized that “writers discover what they want to do by insistently, energetically exploring the entire problem before them and building for themselves a unique image of the problem they want to solve” (p. 477). Echoing the first of Polya’s four phases, understanding the problem, they asserted that “the most crucial part of [this] process [is] the act of finding or defining the problem to be ‘solved’” (p. 468). They were careful to note, however, that instructors must “teach students to explore and define their own problems, even within the constraints of an assignment” (p. 477) because in so doing students “create inspiration instead of wait for it” (p. 477). As in the case of mathematical problem solving, such exploration in writing blurs the boundary between teacher and student and empowers students to construct their own knowledge and to have faith in their own experiences.

Supporting Metacognitive Behaviors

In his work, Schoenfeld (1992) acknowledged that the term *metacognition* has a wide range of meanings but essentially entails “individuals’ declarative knowledge about their cognitive processes . . . [and] self-regulatory procedures, including monitoring and ‘on-line’ decision-making” (p. 347). In essence, it encompasses the ability to describe how one is processing information and the skill to make adjustments accordingly. As noted by Schoenfeld, these abilities can play an integral role in students' development as successful mathematical thinkers and, hence, as problem-solvers. To aid in the development of student thinking, he advocated that teachers use "explicit instruction that focuses on metacognitive aspects of mathematical thinking" (p. 356). Therefore, teachers must have at their disposal various methods of instruction that help to bring the students' thinking to the forefront of their own minds. Research indicates that writing may be one of these methods (Pugalee, 2001).

From a Vygotskian point of view, the strength of writing as a tool of learning rests in "the fact that it is planned and conscious" (Sierpinska, 1998, p. 45) and that it is "a valuable way of reflecting on and solidifying what one knows" (NCTM, 2000, p. 351). Writing serves as a way for students to create a record of what they know and how their pieces of knowledge can fit together. Pugalee (2001) stated that this type of metacognitive behavior "or the monitoring of one's mental activities" (p. 237) is "essential to employing the appropriate information and strategies during problem solving" (p. 237). This process of writing need not be made public, however, in order to be considered a useful tool of learning. Essentially, students arrive at a personal articulation of what the problem is, of their own mathematical knowledge, and finally of their solution to the problem. Indeed, Sfard (2001) argued that mathematics educators need to consider the very act of thinking as a form of communication:

> Let me then argue that thinking has not been excluded from my communicational account of learning. This point becomes immediately clear when we realize that the traditional split between thinking and communicating is untenable, and that *thinking is a special case of the activity of communicating.* Indeed, a person who thinks can be seen as communicating with herself. This is true whether the thinking is in words, in images, or in any other symbols. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our response. (pp. 4–5)

By using writing as a means of communicating with oneself, as a way to unpack the mathematics one brings to a problem, a student of mathematics can become a better problem solver. Research tends to support this claim.

In a recent study, David Pugalee (2001) examined the writings of students in a ninth-grade algebra class “to investigate whether students’ written descriptions of their problem solving methods [showed] evidence of metacognitive behaviors, and if so to describe the types of behaviors that [were] evident” (p. 237). Over 6 days, twenty students were allowed time in class to solve one problem per day during which they recorded their thought processes about the problem through writing. The writing sessions typically lasted 10 minutes and the students were encouraged to write whatever was going on in their minds. When students paused for long periods of time, they were encouraged to keep writing. In addition, to prevent the students from editing their work, erasers were removed from the pencils.

At the end of the 6 days, Pugalee (2001) collected the work and subjected it to an analysis in which he employed qualitative methodologies to look “for convergence or determining which pieces of data were similar” (p. 238). He then classified the results into the four categories of orientation, organization, execution, and verification as specified by a metacognitive framework. Pugalee concluded that “the data showed students' use of metacognitive behaviors in the orientation, organization, execution, and verification phases of problem solving” (p. 243) and that “this study raises important questions regarding the potential for writing to function as a vehicle in supporting metacognitive behaviors identified as crucial to mathematical problem solving” (p. 242). Overall, Pugalee concluded that his “study supports reform efforts promoting writing in mathematics” (pp. 242–243).

The Role of Public Communication

Although writing can help students articulate on a personal level what they know mathematically, writing for public communication may heighten the need for clarity. Huang, Normandia, and Greer's (2005) study of precalculus students over 3 months demonstrated that the verbal discourse used in the classroom had done little to move the students to a higher level of conceptual understanding; however, Huang et al. did note one area of success. They noticed that placing students in the role of teacher seemed to move them to a deeper understanding of the material. Their findings imply that when a student bears the burden of publicly communicating what he or she has learned that student makes a conscious choice to move deeper into the subject matter. This power of "eliciting and making public student thinking" (Franke, Kazemi, & Battey, 2007, p. 243) has been noted in other research as well and also applies to the process of writing. Cohen and Riel (1989) reported a study in which the writing of students improved when they "wrote to communicate with peers as compared to when they wrote to demonstrate their skill for their teacher's evaluation" (p. 155). The implication of Cohen and Riel's research is that when a student writes for a targeted audience for communicative rather than evaluative purposes, he or she tends to give greater care to the articulation of the information.

Writing, therefore, can serve both personal and public purposes in the mathematics classroom. Its greatest power, however, may come to fruition when the personal and the public converge into one process—when writing for *publication* or to make public one's thinking becomes the fifth phase to the problem solving process. This final phase can occur when students endeavor to organize their processes and solutions into a final document that others can read for understanding. When taken as a whole, when viewed as a process through which students move as they go from personal understanding to public presentation, writing may hold the power to propel students into a deeper, richer understanding of mathematics.

The Language of Mathematics

To write mathematics properly, teachers and students must give attention to the language of mathematics, which is often defined as the mathematics register. Foley (2008) described the mathematics register as “the formal academic approach to mathematical speaking and writing” (p. 1). Schleppegrell (2007) divided the mathematics register into two categories: multiple semiotic representations and grammatical patterns. The category of multiple semiotic representations includes symbolic notation, oral and written language, as well as graphs and other visual displays. The category of grammatical patterns consists of technical vocabulary, dense noun phrases, and “implicit logical relationships” (p. 141). In this study, I focused on those aspects of the mathematics register unique to written language: the integration of symbolic notation and figures into sentences as well as the use of technical vocabulary. Higham (1993) implied that this integration requires “the mathematical writer . . . to be aware of a number of matters specific to mathematical writing . . . such as choice of proper notation [and] how to punctuate mathematical expressions” (p. 12). He noted that “mathematical expressions are part of the sentence and so should be punctuated” (p. 24).

The use of mathematical language in an explanation can evoke differing opinions from mathematics teachers in terms of how technically precise they think they should be when presenting mathematical information. Despite these differences, the argument can be made that the proper use of mathematical language is as important as the proper use of mathematical procedures and concepts. Indeed, Ball and Sleep (2007) stated that “mathematical language is both mathematical content to be learned and [a] medium for learning mathematical content” (p.

13). Nevertheless, teachers can feel a pull in two opposing directions— the urge to use informal language that students know and the need to cultivate the proper use of mathematical language. Jill Adler (1997) described this tension as one of the "dilemmas of mediation" (p. 235) in which mathematics teachers have the responsibility of "shaping informal, expressive and sometimes incomplete and confusing language, while aiming towards the abstract and formal language of mathematics" (p. 236).

While helping students learn mathematical language, teachers must also be able to produce effective explanations. Based upon a study of an expert teacher, Leinhardt (1987) identified the features of an effective explanation:

1. Identification of the goal.
2. Signal monitors indicating progress toward the goal.
3. Examples of the case or instance.
4. Demonstrations that include parallel representations, some level of linkage of these representations, and identification of conditions of use and nonuse.
5. Legitimization of the new concept or procedure in terms of one or more of the following—known principles, cross-checks of representations, and compelling logic.
6. Linkage of new concepts to old through identification of familiar, expanded, and new elements. (pp. 226–227)

Although Leinhardt offered her list in regards to teachers and their oral explanations, Shield and Galbraith (1998) found it useful in analyzing the expository writings of eighth-grade mathematics students in Australia. Inspired by Shield and Galbraith's use of Leinhardt's list, I adapted the features of the list and constructed a framework through which to evaluate the quality of the expository writings prepared by the participants during this project or study. The framework is presented in the data analysis section of chapter 3.

Beliefs, Attitudes and Mathematical Understanding

In this section, the word *belief* is being used in a broad sense similar to what Thompson (1992) called "teachers' conceptions—mental structures, encompassing both beliefs and any aspect of the teachers' knowledge that bears on their experience, such as meanings, concepts, propositions, rules, mental images, and the like" (p. 141). These conceptions play a role in the attitudes that teachers exhibit in their learning and teaching of mathematics. Philipp (2007) defined attitudes as "manners of acting, feeling, or thinking that show one's disposition or opinion" (p. 259). Arguably, beliefs and attitudes are closely related, and although the relationship may not necessarily be one of cause-and-effect, what people believe does influence how they act and feel.

Mewborn and Cross (2007) implied that "teachers' beliefs about the nature of mathematics" (p. 260) sets off a chain reaction. Teachers' beliefs about the nature of mathematics "influence their beliefs about what it means to learn and do mathematics" (p. 260) and these beliefs in turn "influence . . . instructional practices" (p. 260). These practices then "dictate the opportunities that students have to learn mathematics" (p. 260). Therefore, if preservice teachers believe that mathematics is a compilation of rules and procedures to be mastered these teachers will perpetuate that belief in their students. Mewborn and Cross, however, asserted that "helping teachers . . . become aware of their beliefs is a significant step toward improving students' opportunities to learn mathematics" (p. 262). They noted that journal-writing may be an activity that helps preservice teachers become aware of their beliefs about the nature of mathematics.

As Mewborn and Cross (2007) indicated, how teachers view mathematics often dictates the structure and content of their lessons. Hiebert (2003) characterized a traditional style of teaching as one of "demonstration-practice" (p.11) in which the teacher demonstrates the topic to be learned with explanations and examples and then gives the students time to practice what they have been taught. Under this traditional view, mathematics is often regarded as a list of rules, formulas, and procedures that must be committed to memory in order to be mastered. Indeed, in a survey of mathematics teachers throughout the country, Weiss, Smith, and Malzahn (2001) found that "teachers nationally tend to emphasize objectives related to basic mathematics skills such as learning computational skills, algorithms/procedures, and preparing for standardized tests" (p. 48). Such data suggest that teachers tended to favor the acquisition of procedural knowledge over the building of conceptual knowledge.

Hiebert (1986) characterized procedural knowledge as consisting of "rules, algorithms, or procedures to solve mathematical tasks" (p. 6) and "step-by-step instructions that prescribe how to complete tasks" (p. 6). To Hiebert, if procedural knowledge focused on the steps, then conceptual knowledge focused on the links between those steps:

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some other network. In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (pp. 3–4)

In essence, conceptual knowledge fills in the gaps that a sole focus on procedural knowledge leaves behind. Hiebert carefully noted, however, that "mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge" (p. 9). Therefore, the two types of knowledge should be treated equally, and students are not considered "fully competent in mathematics if either kind of knowledge is deficient or if they have been acquired but remain separate entities" (p. 9). Hence, an integration of procedural and conceptual knowledge is crucial to attaining a rich understanding of mathematics.

Hiebert's discussion of procedural and conceptual knowledge is similar to Skemp's description of instrumental and relational understanding. According to Skemp (1987), an instrumental understanding of mathematics is characterized by a knowledge of “rules without reasons” (p. 153) and, in part, a dependence “on outside guidance” (p. 163) for help. On the other hand, a relational understanding of mathematics is marked by an understanding of “knowing not only what method worked but why” (p. 158) and engenders confidence and independence during the problem solving process.

Arguably, those teachers who tend to promote the acquisition of procedural knowledge in their lessons may themselves have an instrumental understanding of the mathematics. Such a belief in promoting procedural knowledge coupled with an instrumental understanding of mathematics may affect what these teachers believe about the use of writing in the mathematics classroom. If they believe in promoting rules and algorithms because procedures are what they understand, then they may be less inclined to allow students to explore mathematics through writing.

In teacher education, beliefs play an important role in how preservice and inservice teachers approach teacher training and what they take away from it. Beliefs about education are formed early in life and are reinforced through years of observation; therefore, they can be difficult to change (Pajares, 1992). Cooney (1998) stated that the consideration of such beliefs, however, “enables us to create activities that encourage teachers to wonder, to doubt, to consider what might be, to reflect, and most important, to be adaptive” (p. 332). Pajares suggested that more research is needed to accurately define the beliefs of preservice teachers and what it would take to alter those beliefs. Such research could have a substantial impact on the structure and content of teacher education programs. Therefore, if writing is to become an accepted method of teaching and learning mathematics, research studies need to address the beliefs that preservice teachers hold about the use of writing as a teaching and learning tool in mathematics.

Some researchers argue that teacher beliefs about the use of writing will not change unless preservice teachers are trained to use writing in their lessons. Flores and Brittain (2003) suggested that "teachers probably will not use this tool . . . unless they have had the experience themselves of writing in relation to mathematics" (p. 112). These authors maintained that "writing serves as a tool to organize the thoughts of prospective teachers about issues related to the teaching of mathematics" (p. 114) and allows "teachers to look back at their thoughts and reflect on their growth" (p. 114). Therefore, writing can serve both preservice and inservice teachers on several fronts. It can help them to learn mathematics, to reflect on that learning, and to reflect on how their own learning experiences can shape their teaching.

As a teacher of preservice teachers in elementary mathematics education, Dianne McCarthy (2008) routinely found that only half of her students had experienced writing in their mathematics lessons while in elementary school. Concerned that teachers often teach the way they were taught, she conducted a "teacher development experiment" (p. 335) in which she taught her students how to use a graphic organizer that essentially provided a written way for them to organize what they knew about the mathematics involved in a problem. McCarthy defined a teacher development experiment as a study that involves the teacher educator (researcher), the preservice teachers, and the preservice teachers' students. After practicing with the organizer, the students then took the tool out into the field, taught elementary students how to use it, and reported their experiences back in McCarthy's class. McCarthy found "that all but one of the preservice teachers who did writing in the mathematics class found it helped their understanding of the mathematics they were learning" (p. 336). She also found that "not a single preservice teacher indicated that [he or she] would not use writing to help students learn math" (p. 339). She concluded that her experiment "illustrated that preservice teachers can develop the skills and positive attitudes toward writing in mathematics even if they had limited experience writing in mathematics when they were in elementary school" (p. 339).

Studies such as McCarthy's indicate that exposure to writing in mathematics education classes can influence the beliefs and attitudes of preservice elementary mathematics teachers about using writing as a tool of learning. What is lacking, however, is research that indicates that the same result can be achieved at the secondary level where the mathematics and hence the writing can be more complex. One of the goals of this study was to address that gap in the research.

CHAPTER 3

METHODOLOGY

This study was born out of my desire to understand how secondary mathematics teachers respond to various forms of writings in mathematics and what that information might reveal about how to engage them in writing that they would find useful to the exploration and communication of mathematics. In this study, I define *participant response* to include attitudes toward and beliefs about mathematics and writing in mathematics as well as the quality of the writing produced. Because I wanted to explore depth of experience, I chose a qualitative design based on interviews and written responses of a small number of participants. In so doing, I sought to fully explore how and what these participants felt and the quality of their writings.

Setting and Participant Selection

Maxwell (2005) stated in his book *Qualitative Research Design: An Interactive Approach* that "the typical way of selecting settings and individuals" (p. 88) is "purposeful selection" (p. 88). He noted that "this is a strategy in which particular settings, persons, or activities are selected deliberately in order to provide information that can't be gotten as well from other choices" (p. 88). In studying attitudes and beliefs of teachers about the use of writing in a mathematics class, researchers can find it difficult to ascertain what a participant thinks about these topics. When the researcher is the instructor, issues of not wanting to express an unpopular opinion may create reticence on the part of the participants to speak their mind during the course of the study. Therefore, it is beneficial to seek out a mathematics-intensive class in mathematics education in which writing is already heavily used. In so doing, it diminishes the risk that the opinions and biases of the researcher are routinely conveyed to the participants.

In this study, I observed a graduate course for preservice and inservice secondary mathematics teachers that featured an exploration of mathematics with technology and the preparation of 11 reports written for Internet publication. Two of the primary objectives for this course were for students "to solve mathematics problems using application software"[1] and "to communicate mathematics ideas that arise from mathematics applications." The reports students prepared were based on 11 activities that covered topics in algebra, geometry, data analysis, precalculus, and calculus and presented a wide range of tasks that students could explore using software such as Geometer's Sketchpad (Version 4.07) and Graphing Calculator (Version 3.5). Within each activity, students had the choice of tasks they could explore and about which they could write a report. The professor referred to these reports as "write-ups" and they could be posted on the Internet at any time during the semester. In addition, the write-ups could be taken down, refined, and reposted at any time. The professor's only requirement was that all 11 assignments be completed by the end of the semester. Over the course of the semester, the class met once a week for 3 hours. During that time, the professor would spend the first part of each class introducing an activity or responding to student questions or concerns and then the students would spend the remainder of the time working independently on assignments at computer stations while the professor and a teaching assistant answered individual questions. Although this class met weekly, the students could work on the assignments at their own pace.

Near the beginning of the semester, the professor encouraged the students to "play with [the mathematics] to see where it goes" (Field Notes, September 28) and to write reports on what they found interesting. Frequently, he offered examples of explorations and ways in which

[1] To protect the identities of the participants, I do not cite the course web-page.

students could argue or support a claim. He also left the decision of writing format to the students but provided a description of “What is a Write-Up?” on the course web-page. In that description, he instructed the students to write so that “it convincingly communicates what you have found to be important from the investigation.” He also suggested that “the hypothetical audience might be your students, your classmates, or classroom mathematics teachers.” He also mentioned during a lesson at the beginning of the semester that students might use an article from *The Mathematics Teacher* as a model for writing. Other than these introductory comments, however, the professor offered no specific instruction on style or grammar and did not emphasize the writing portion of the assignments in class during the rest of the semester. He did, however, monitor postings and informed students when they needed to correct some portion of their mathematical work or to clarify some aspect of their writing.

At the first class meeting, I asked all master’s students in a class of 31 graduate students to complete an initial questionnaire to be returned to me by the next class meeting. Twenty-three students signed the consent form in class and 18 students returned the questionnaire by email or by hand at the next class meeting. Of these 18 students, I asked 10 if they would volunteer to participate in the study. I chose the participants so that they would have a range of experience and opinion about mathematics and writing in mathematics as indicated by their responses on the initial questionnaire (see Table 1). For example, I chose Amy[2] because she indicated that she had struggled with writing in college and would only consider using reflections in her own classroom someday. In contrast to Amy, I chose Claire because she thought writing in mathematics was a “wonderful idea” (Initial Questionnaire, August 19) and stated that she “fully intends” (Initial Questionnaire, August 19) to use it in her own classroom. By choosing such a range in

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[2] All names of participants are pseudonyms.

experiences, beliefs and attitudes, I hoped to be able to compare and contrast their experiences with the course and project.

Table 1

Participant Responses to Initial Questionnaire

Participant	Undergraduate Degree	Is doing mathematics like following a recipe?	Experiences with Writing	How do you feel about writing in mathematics?	How do you feel about using writing in your classroom?
Grace	Mathematics Education	"Math is about logic and thinking and solving problems. It's a whole lot deeper than a recipe."	"I did very well in my high school and college writing classes." "I have done many long proofs and papers in math classes in college. In high school I did not."	"I think that writing is a good practice in math classes."	"I think that journals in a math class are a really good idea. "
Lisa	Architecture	"Pure mathematics involves creativity and understanding, and this is what makes math fun."	"I am confident in my writing skills." "I can't recall any math writing in high school. In [college] I had to put my design ideas (and many times these were mathematical) into written words."	"I think it's an important concept. I am excited to start on it."	"Many . . . students don't like when words and mathematical concepts mix. I believe students would be less hesitant to approach [word problems] if they themselves were used to integrating writing and mathematics."
Gwen	Business Management	"Yes, and no. I believe it can be, but I also believe people have different viewpoints and processes they go through to solve the same problem and come to the same conclusion."	"I was extremely comfortable writing in both my high school and college classes." "I have never really used writing in any of my mathematics classes."	"I feel that the writing component of this course will be an area of strength for me, and I am glad I will be able to have something that comes more naturally for me in the course."	"I'm fairly early in my degree program, and I have a lot to learn when it comes to teaching. I do believe writing is important in any subject…."
Kim	Major: Ecology Minor: Mathematics	"I think it takes more than just a recipe…. It takes thinking and skills to find the answer to most questions. If one doesn't know the formulas to apply then a recipe is all you have."	"I had many struggles in my writing classes." "I have only done proofs and I don't really remember them. I took most of my mathematics classes over 10 years ago."	"I am not too concerned. With age I am more confident in my abilities to express myself."	"While I know writing is important in every class, I am not too comfortable with too much required writing. I do make them write out step by step instructions in Algebra I, but other than that nothing more. I do not feel I am qualified enough to grade their writing to help them."
Amy	Mathematics	"I disagree. There is more than one way to solve a problem. The standard way is not always the easiest way to teach or learn mathematics."	"It was a struggle to me at first. I failed the second part of composition two." "We did proofs, but they were written in shorthand and symbols. "	"I don't mind writing. I don't have any concerns yet."	"The only writing I would consider is reflections on what was learned."
Claire	Statistics	"I view doing math as a puzzle that requires the student to use techniques that they already know to solve problems that are unfamiliar to them."	"I have not had to take any English class in college and, as a result, I have become less comfortable with my writing." "In . . . college I have been required to write many proofs…. I do not recall doing any writing in my high school math classes."	"I feel that writing in mathematics classes is a wonderful idea."	"I fully intend to use writing in my mathematics classroom."

After the second class meeting, I met with all the participants, explained the requirements of the project, and outlined my expectations. During this meeting, I emphasized that I wanted them to work at a steady pace throughout the course. After the meeting, one of the participants withdrew from the project and I replaced that participant with an alternate. The final participant list comprised three inservice teachers and seven preservice teachers. Eight of the participants were female and two male.

Data Collection

I collected data through the use of an initial and final questionnaire, write-ups, reflective writing, interviews, and field notes. All participants were asked to complete a voluntary questionnaire at the beginning of the semester and a different questionnaire at the end of the course. The primary purpose of the questionnaires was to track any changes in attitudes about mathematics and writing in mathematics that might have occured during the course of the semester as the students prepared their formal reports for Internet publication. The questionnaires asked the participants to consider prompts such as “How do you feel about writing in a mathematics class?” (See Appendix A for the Initial Questionnaire and Appendix B for the Final Questionnaire).

In an attempt to examine both the private and public aspects of writing and doing mathematics, I asked the participants to complete three different types of writing. First, the course required that the participants publicly communicate their work in writing. Throughout the semester-long course, all students explored mathematics and reported their results in 11 expository write-ups and a final exam which were posted on the Internet. I subjected each of the write-ups to a textual analysis to ascertain how well the participants explained the mathematics.

To examine how the participants privately responded to doing mathematics and writing, I asked the participants to complete a write-up worksheet and post write-up reflection guide for each report. The write-up worksheet was a graphic organizer designed to guide the students through the mathematical explorations (see Appendix C for the Write-Up Worksheet). The worksheet comprised the four headings *goal, exploration, findings, and conclusion*. During the first meeting, I emphasized to the participants that the purpose behind the worksheet was to have the students make an organized record of what they were doing and thinking during the exploration of the mathematics; therefore, I encouraged them to write as much as possible. If the participants found the worksheet too limiting in terms of space, I encouraged them to use the headings from the worksheet on their own notes. After the participants completed a write-up, they completed a post write-up reflection guide (see Appendix D for the guide). The guide contained a series of prompts asking students to reflect on the processes of mathematical exploration and of completing the write-ups. On some ocassions, I asked the participants to elaborate on their responses to the guide and encouraged them to explain their responses as fully as possible.

With each activity, I expected the participants to engage in a pre-writing or informal writing phase by taking notes aided by the use of the write-up worksheet, in an expository writing phase by completing the write-ups, and in a reflective phase by completing the post write-up reflection guides. I considered the compilation of these three writings a packet of writing. Therefore, by the end of the semester, I expected the participants to have completed eleven packets. Because the participants' write-ups are posted on the Internet, I do not provide references to their work in this document.

In addition to examining what the participants wrote, I also interviewed them for approximately 30 minutes three times during the semester to chart their progress with preparing the write-ups as well as with completing the reflective writings. The interviews occurred at the beginning, middle, and end of the semester. Interview questions centered on their experiences with and attitudes about the mathematical explorations, taking notes, preparing the write-ups and completing the post write-up reflection guides. These recorded interviews were semi-structured in design. I interviewed the participants in accordance with an interview guide (see Appendix E). I asked additional interview questions in response to what participants said during the interviews or wrote in their questionnaires, reflective writings, and write-ups. I conducted each interview separately and in as private a location as circumstances would allow. In chapter 4, I annotate references to interviews by the order of the interview and the date it occurred. I refer to the first interview as FI, the second as SI, and the third as TI. For example, a participant's response in the final interview on December 10 is cited as (TI, December 10).

In addition to collecting data through formal instruments, I also kept field notes based on informal interviews I conducted with participants in class each week. As students worked independently at their computer stations, I talked to each one about his or her progress in class. During this time, I also collected notes and reflections for any write-up the participants had completed during the week and inquired if they had made any changes to the work they had done. During these interviews, I also sometimes asked for clarification on responses to completed post write-up reflection guides. To preserve my neutrality as an observer, however, I endeavored throughout the semester to refrain from commenting on the quality of either their mathematical work or their written work.

Near the end of the semester as I examined the manner in which the participants were completing the course and project, I realized that four of the participants were completing both at a questionable pace. At the time of the last class meeting of the semester, four of the participants had less than half of the write-up packets completed. That meant they would complete the majority of the course and project in 2 weeks which ran counter to my initial request that they work at a steady pace throughout the semester so they could have a sustained amount of time for reflection. In good faith, I could not compare the work of those who finished the bulk of the class and the project in 2 weeks with that of those who took the time to generally pace themselves through the course and the project. For these reasons, I eliminated four participants from my study. Therefore, the remainder of this document addresses the work of five preservice teachers and one inservice teacher given the pseudonyms of Grace, Lisa, Gwen, Kim, Amy, and Claire. Kim was the inservice teacher.

Data Analysis

Once I completed data collection and transcribed the interviews, I began an analysis of the data. To specifically track how the particpants completed the class and project, I first compiled a time line for each participant noting her progression through the course. For example, I noted what write-ups the participants had posted by each class meeting, when they turned in the worksheets and guides, emails I sent asking for elaboration, and any pertinent comments they had made during class. Once these timelines were complete, I began an analysis of each participant's contribution to the project according to when she completed it. In addition, I reviewed each participant's work in chronological order of the three stages of the project (see Table 2).

Table 2

Stages of the Project

Stage 1: Background
• Initial questionnaire • First interview
Stage 2: Mid-semester
• Completed writing packets • Second interview
Stage 3: End of the semester
• Completed writing packets • Final questionnaire • Final interview

During this phase of the analysis, I read all documents and analyzed participant responses according to topics, such as background and beliefs, that allowed me to situate the participants according to their experiences. For each participant, I compiled a report in outline form (see Table 3). In addition, at this stage of the analysis I examined each write-up to ascertain the soundness of both the mathematics and of the overall writing in terms of style and grammar. After I completed a report for each participant, I reviewed the reports to note emerging themes across the documents. I was specifically looking for issues related to attitudes and beliefs about mathematics and writing in mathematics. Once I identified these themes from the reports, I reexamined the questionnaires, interview transcriptions, notes, and reflections, taking note of any new evidence that came to light in support of these major ideas. These major ideas and the evidence supporting them became the basis for my conclusions and recommendations as reported in chapter 4.

Table 3

Template for Participant Report

I. Background A. Education 1. Secondary 2. College 3. Currently B. Prior Experiences 1. Mathematics 2. Writing
II. Logistics of the Course
III. Experiences with the Course A. Mathematics B. Writing 1. Write-up worksheet 2. Write-ups a. Process b. Audience c. What if written for the professor d. General comments 3. Post write-up reflection guide
IV. Beliefs about Mathematics A. Beginning of the semester B. End of the semester
V. Beliefs about Writing in Mathematics A. Beginning of the Semester B. End of the Semester 1. General comments 2. Use of writing in the classroom
VI. Analysis of Write-Up's

Next, I completed a textual analysis of each write-up using Leinhardt's list which I introduced in chapter 2. In her article "Development of an Expert Explanation," Leinhardt (1987) listed what she considered to be the critical features of an effective explanation:

1. Identification of the goal.
2. Signal monitors indicating progress toward the goal.
3. Examples of the case or instance.
4. Demonstrations that include parallel representations, some level of linkage of these representations, and identification of conditions of use and nonuse.

5. Legitimization of the new concept or procedure in terms of one or more of the following—known principles, cross-checks of representations, and compelling logic.
6. Linkage of new concepts to old through identification of familiar, expanded, and new elements. (pp. 226–227)

Although Leinhardt derived her list from the analysis of the verbal explanations of an expert teacher, I modified it to construct a framework through which to examine the written works of the participants. The purpose behind the modification was to ascertain how well the participants wrote to communicate mathematics. From this perspective, I broke apart the elements of the list and reassembled them into two categories that classify the elements according to their emphasis on the general writing aspect of the explanation or on the writing aspect specific to mathematics (see Figure 1).

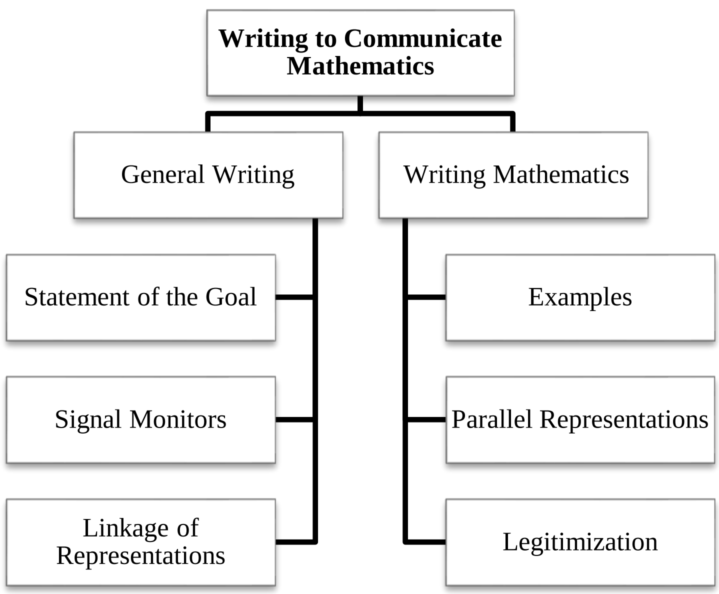


Figure 1. Framework for writing to communicate mathematics using the features of an expert explanation described by Leinhardt. Adapted from "Development of an Expert Explanation: An Analysis of a Sequence of Subtraction Lessons," by G. Leinhardt, 1987, *Cognition and Instruction, 4*, pp. 226-227. Copyright 1987 by Lawrence Erlbaum Associates.

The parts of the framework that highlight the general writing aspect can be thought of in terms of how well the participants presented the information independent of the mathematics. If their general writing was clear, then their work would include an "identification of the goal" near the beginning and "signal monitors indicating progress toward the goal" throughout the work. These monitors could be as simple as the use of transition words such as *therefore*. The third element in the category of general writing is part of the fourth feature of Leinhardt's list. The fourth feature states that a sound mathematical explanation should provide "demonstrations that include parallel representations, some level of linkage of these representations, and identification of conditions of use and nonuse" (Leinhardt, 1987, p. 227). For example, parallel representations of a mathematical situation occur in a written explanation when the writer offers an equation and its graph. Arguably, the graph and the equation are the mathematical components of the communication, but the presence and quality of the explanation that links them together is determined by the text. Therefore, the linkage of representations is more aptly included under the category of general writing. Overall, a statement of the goal, signal monitors, and linkage of representations provide a textual map that tends to make the written explanation clear and cohesive.

The category of writing mathematics contains those elements of Leinhardt's list that are unique to the communication of mathematics. These include the presentation of examples, parallel representations, and legitimization. I have not included in the framework, however, two elements from Leinhardt's list: a linkage between old and new concepts and the conditions of use and nonuse. Unlike oral explanations given in the classroom in which teachers attempt to connect the present lesson to previous lessons, written explanations are often meant to stand alone; therefore, writers seldom link new concepts to old ones. This was particularly true for the write-ups in this study because the course involved focused on 11 individual activities for each participant. In addition, Leinhardt (1987) interpreted the conditions of use and nonuse in terms of the procedure being explained as well as the use of parallel representations. In other words, Leinhardt looked for evidence that the teacher had discussed when and when not to use a particular procedure. Since the writings in this study were less about procedures and more about exploring mathematical topics, the condition of use and nonuse was not generally applicable.

Although examples and parallel representations are important parts of a written mathematical explanation, the legitimization of the concept is arguably at the heart of the explanation. Often, whether or not a written explanation is considered mathematically sound is primarily based on how well the author justifies his or her work. As specified by Leinhardt (1987), this justification can be created by the use of "known principles, cross-checks of representations, [or] compelling logic" (p. 227) in which proof is the highest form. How well participants justified their work figured heavily into how strong I considered their mathematical work to be.

After I constructed the framework, I created a chart based on the categories of the framework and reexamined the write-ups in light of the framework (see Table 4). After I completed each chart, I went back and color coded the cells according to strengths and weaknesses I perceived in the write-ups. From there, I was able to draw conclusions as to whether the participants were strong or weak overall in their abilities to communicate mathematics.

Viewed through the lens of the framework, the written work of teachers can be assessed to pinpoint where the weaknesses and strengths of their written explanations are located. For example, a teacher may effectively identify the goal, provide transition points, and provide text that links multiple representations; however, various aspects of the mathematical writing may be weak, such as using less-than-compelling logic to legitimize a mathematical claim. If this pattern

Table 4

Gwen – Analysis of Write-Ups According to Leinhardt's (1987) Framework

Assignment #	Identification of the Goal	Signal monitors indicating progress toward the goal	Linkage of Parallel Representations	Legitimization of the new concept
1 (Algebra)	States "Let's investigate the graph activity for functions involving…."	"First…." "Now, let's examine…."	Links equation to graph with words; color-coding.	Reports results of exploration; superficial math; does see the patterns (does not discuss domain or range or bring in other math like Grace).
2 (Algebra)	Missing	"First…." "At this point…." "Now…."	Links equation to graph with words; identifies the shape as a parabola; color-coding.	Reports results of exploration; superficial math but some justification through algebra. (Tells students to discover mathematical explanation.)
3 (Algebra)	Uses the title…. (Implied) "Examining the Activity of Quadratic Equations…."	"First…." "Finally…."	Confusing – Not enough explanation; color-coding.	Reports results of exploration; pulls out some math; some justification through algebra – i.e., uses Quadratic Formula.
4 (Geometry)	Plainly Stated – Nice Introduction; "Today, we will focus on comparing…."	"First…." "Now, let's explore…." "Next…."	Graph needs to be broken down somewhat. Picture too condensed; needs more words to go with graph.	Reports results of exploration; superficial but pulls out some math; some justification through logic about centroid.
6 (Geometry)	Missing	"We can first notice…."	Needs to explain the graphs in pieces – would be clearer.	Uses measurements for proof. (Does reference theorems.)
10 (Parametric)	Implied in the introduction: "We can now notice several different observations."	"Next…." "To further illustrate this behavior…."	Color-coding in addition to words; good break down of explanation.	Reports results of exploration; some justification through logic about the coefficient and how it affects the domain and range of sine.
11 (Polar)	Implied in the introduction: "To see what effect k's value has on the behavior…."	Sections "We can now hypothesize…." "Now that we understand…."	Confusing – What is the basic equation? What are the values? But good separation of text and pictures.	Reports results of exploration; draws out the patterns but nothing more.
12 (Data)	Direct – "We will attempt to predict future values for stamp prices in the United States…."	"Now…." "First…."	Not clear how the graphs work. How is the trend line derived?	Offers justification through compelling logic
8 (Geometry)	Plainly states the goal toward the end of an introduction: "Our goal is to determine the angle…."	"The first thing we will do…." "Next…." "What do we know…."	Does a decent job of breaking up text and graphs. (Issues of notation with arcs).	Offers a good proof but mixes in measurement; issue of precision in language.
9 (Geometry)	Direct – "During this activity, we will explore the behavior of pedal triangles."	"The first thing…." "Now…." Asks questions. "So what could be the…?"	Too much information condensed in the graph.	Reports results of exploration; pulls out some math; Refers to circumcircle and Simson Line.

continues over a series of writings, one can argue that the teacher is stronger at writing text in general than at writing mathematics. On the other hand, another teacher may be stronger at writing mathematics than at writing in general. In addition, a teacher may be weak at both or strong at both. If a teacher is strong at both the general writing and the writing of mathematics, he or she may be regarded as an expert in writing to communicate mathematics. Therefore, there are four possible interactions between the two categories which are outlined in more detail in chapter four.

Quality of Evidence

I took considerable care in designing and conducting this study to insure there were several methods of collecting evidence that would provide as much information as possible about the participants and their experiences with the course, mathematics, and writing in mathematics. For example, what the participants believed and how they felt about writing in mathematics were tracked in the questionnaires, interviews, the post write-up reflection guides, as well as in informal conversations during class meetings. My goal was to study their responses from as many angles as possible. I must note, however, that the majority of this project was based upon self-reporting. Therefore, I sought out a course in which writing was not overtly promoted and in which I could be an observer.

In addition, I strove throughout the semester to remain as neutral as possible about writing in mathematics when I was in contact with the participants despite my own personal biases about the topic. I routinely encouraged the participants to speak freely and often reminded them that there were no right answers to my questions. Because of this encouragement and the neutrality both the professor of the course and I offered, I believe the participants were as candid as possible during our encounters. In hindsight, I also realize that an exploratory study was the best design to handle my bias about writing in mathematics. Because I was not looking to affirm a hypothesis but rather to explore the experiences of the participants, I found it easier to remain open to what they reported. In addition, the evaluations of the write-ups were primarily based on my judgment. However, not only did I endeavor to follow the framework based on Leinhardt's (1987) list, but I also worked to follow established mathematical and grammatical principles. Because of these measures and precautions, I therefore believe the quality of the evidence I obtained is sound.

CHAPTER 4

RESULTS

Writing to communicate mathematics in an expert manner entails those features that are endemic to general writing and those that are common to writing mathematics (see Figure 1 on p. 35). In this chapter, the results for each participant are reported with respect to each of the three elements in the category of general writing. The results analyzed according to the features under the category of writing mathematics, however, are reported in a different manner. Because the class observed in this study focused on explorations with technology, the presentation of examples and parallel representations was an inherent part of each write-up. Consequently, all six participants in the study generally demonstrated the use of examples and parallel representations in their writings. Therefore, the focus in this analysis was not on the presence of these features but rather on the quality of their presentation as well as on the presence and quality of the legitimization of mathematical concepts.

In the sections that follow, the six participants are grouped according to the four interactions of the two categories established by the framework: (a) those participants who were stronger in writing mathematics, (b) those who were stronger in the general writing, (c) those who struggled with both, and (d) those who were strong in both general writing and writing mathematics and thus exhibiting traits of producing expert explanations in their work (see Table 5). Although the boundaries between these categories are not necessarily clear-cut, they nevertheless provide a useful lens through which to view the participants, their backgrounds and beliefs, and their contributions to the project.

Table 5

Categorization of the Participants Based on the Framework

Stronger in Writing Mathematics	Stronger in General Writing	Struggled with Both Types of Writing	Strong in Both Types of Writing
Grace Lisa	Gwen	Kim Amy	Claire

Stronger in Writing Mathematics

Grace

Background.

Of the six participants, Grace was the only one with an undergraduate degree in mathematics education. After graduating from a small southern college, Grace began her master's program. She stated in the first interview that she wanted to finish the master's degree before she started her teaching career and chose the current program because it suited her best. Because of her undergraduate work, she had started the master's program already licensed to teach secondary mathematics.

By the time Grace entered the program, she also had compiled an impressive academic record culminating in a perfect grade point average in college. She stated in the interviews that she was a Type-A personality who liked to learn, worked hard, was very organized, and always strove to do her best on each task she was given. She also acknowledged that although she obviously liked mathematics, she had to work hard to learn it and this characteristic, she believed, would make her a better teacher. From the first interview, it was clear Grace was passionate about teaching.

Despite her dedication to her studies, however, Grace reported during the first interview that whether or not she liked a subject depended upon the quality of the teaching she received. When asked what made a teacher memorable to her, she stated that she liked teachers who were "not just there to give the lecture and leave" (FI, August 31). She noted that she responded best to teachers who cared enough to get to know her and to be there to answer her questions after class. She referred to one teacher in particular who stood out in her mind. Her high school English teacher had helped to transform her attitude about English and writing. Grace stated that she "had always seen grammar and spelling as the evil subject" (FI, August 31) but this teacher had "made it doable and understandable" (FI, August 31) to the point that she now enjoyed "writing and . . . having the correct grammar" (FI, August 31).

Unlike the majority of the participants, Grace entered the master's program with a solid background in writing in mathematics. She reported in the first interview that she wrote extensively in her mathematics and mathematics education courses in college. In her mathematics courses, she primarily wrote proofs that she described as "closer to a paragraph proof" (FI, August 31) rather than a list of statements. She stated that her instructors were specific about how to properly punctuate the proofs. In her mathematics education courses, Grace stated that she and other students "would have a specific problem and then write all about the problem and how we would use it, and how we would use it in a lesson" (FI, August 31). In addition, she indicated that she wrote major papers in both disciplines including a 40-page paper for a logic class as well as a paper she presented at a regional conference sponsored by the National Council of the Teachers of Mathematics.

Process.

Grace completed her work for the project in a fairly consistent manner. She completed 4 of the 11 packets in the first half of the semester and the remaining 7 during the second half. Toward the end of the course, however, she began to fall behind in submitting her post write-up reflection guides for the project. When asked at the end of November about the delay in submitting three reflections, she reported that she really did not have enough time to complete the project but that she would honor her commitment. She turned in her final four reflections and completed the final questionnaire during the last 2 weeks of the semester.

As a self-professed Type-A personality, Grace followed a fairly systematic process in exploring the mathematics and writing up the reports. In the second interview, she described how she chose which task to pursue in a given assignment:

> Most of the time I'll go through, just open my notebook and start kind of jotting down ideas about each one, not going into depth on any of them, just kind of playing with each one a little bit and then if there's one after that that kind of triggers my interest then I'll pursue it in more depth. (October 19)

Once she chose a task she started working on it by writing down everything she knew about the topic and "exploring with it on paper" (SI, October 19). After exploring the mathematics with pencil and paper, she moved on to the appropriate technology to examine graphs and sketches. When asked during the second interview if she took notes while exploring with the technology, she stated she would "write stuff down and jot stuff down" (October 19) that she observed. Out of the 11 packets she completed, she presented handwritten notes for 8 of the assignments. In general, these notes reflected the results of her observations and what she wanted to indicate in the write-up. By the time she reached the end of the course, however, the notes became less detailed. She commented during the final interview that in most instances she would move from exploring the mathematics with the technology to directly preparing the write-ups on the computer, but that was not entirely the case. Although her notetaking did lessen as the semester progressed, she was still using some form of notetaking that was reflected in her finished write-up. Therefore, there is some indication that Grace was unaware of the extent to which she used informal writing to process her work.

During the final interview, Grace reported that she rarely used the write-up worksheet before she prepared her final write-up for the class. When asked what she thought the purpose of the worksheet was, she stated that it was "to make sure I was doing all of those things and to organize what I was doing so that you knew what I was doing . . . kind of so you knew the process I was going through, I guess" (TI, December 7). She indicated that using the worksheet did not help her at all because she had her own process of writing. She commented in the final interview that she liked to "play around" (December 7) in her mind:

> The playing around kind of made me see what directions I could go in and then I would write down, you know, what I thought and then I would kind of just put it on the paper in an organized way. (December 7)

After the seventh write-up, Grace abandoned the write-up worksheet and turned in only her handwritten notes for the last four assignments.

Grace was clear from the beginning of the course that she enjoyed exploring the mathematics. During the second interview, she expressed dismay at the fact that she was exploring the mathematics more than the technology but that she was enjoying the activities. There was one aspect of the explorations, however, with which she struggled. During the final interview, Grace commented that her explorations of the mathematics had gone well but that she sometimes had a difficult time knowing when to stop "because you can always go deeper" (December 7). When asked how she determined whether or not she had come to a stopping point in an exploration, she implied that as long as she had reached some sort of mathematical conclusion to report, she believed she had sufficiently explored the material.

The Write-Ups.

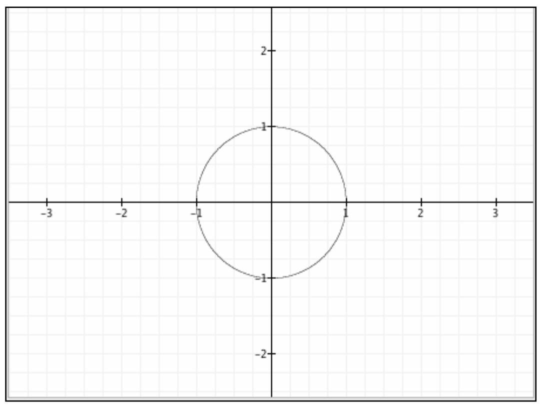
Overall, Grace's writing skills were strong. Except for an occasional spelling error, she produced writings that were grammatically correct and easy to read. She did tend to write sentences, however, as lists in bullet form rather than paragraph form. She stated in the first interview that she struggled with whether or not to write in bullet or paragraph form so she "tried to do a little bit of both" (August 31) in her writing. Although Grace's write-ups were generally strong overall, she tended to be less effective in the general writing category of the framework. In general, Grace did not offer a clear textual map of what she was presenting in her write-ups. For example, in 6 of the 11 write-ups, the statement of the goal was either non-existent, indirectly stated, or placed in the middle of the writing. Since the statement of the goal was often unclear, it was difficult for Grace to signal progression toward that goal. She did attempt on most occasions, however, to help the reader understand where she was headed in her reporting. For example, she paused in one write-up and asked "But what is true for the vertices of each of these graphs?" In another write-up, she wrote the phrase "before I make my conclusions" to ask the reader to pause with her and consider additional information. In addition, in 5 of the write-ups, the explanation behind or the linkage between the representations was missing or confusing at best.

As a case in point of Grace's difficulties with providing a textual map in her work, consider the passage at the beginning of her first write-up as shown in Figure 2. In reference to the elements of the framework, the statement of the goal is missing in this passage. It is not clear from this section of the write-up what the purpose of the writing is, nor is there any textual linkage between the equation and graph. Essentially, there is no statement indicating that the graph is a representation of the equation. In addition, the sentence below the graph describes the graph but does not explicitly offer a link to the equation. Arguably, much is implied in this passage and assumed to be understood by the reader.

Circle to Square
by Grace

Graphs

$$x^2 + y^2 = 1$$



This graph is a circle with radius 1 and center (0,0).

Figure 2. Grace's description of a circle using an equation, graph, and sentence but providing no textual linkages among them.

The lack of a clear textual map in Grace's work may be explained, however, by the audience Grace had in mind when she wrote the essays. She commented in the second interview that "I think about how I can use [the write-ups] when I teach and I think about writing things that would be applicable to me as a teacher and other teachers" (October 19). She also stated on the final questionnaire that "I think the most beneficial part of the write-ups is having my exploration process written in a way that later down the road I can go back and understand what I did and why I did it" (December 3). In essence, she viewed the write-ups as a resource for her and other teachers to use; therefore, she may not have felt it necessary to provide the depth of explanation in words that she would have used with students. In addition, Grace seemed to view her write-ups not as polished essays but rather as reports of what she did and what she found. She stated in her final interview that "I just kind of wrote up . . . what I thought and the way it went and my conclusions" (December 7). This statement implies that the organization of the writing as an explanation was not of primary concern to Grace. She did indicate in the final interview, however, that she always tried to do her best on assigned tasks, so the fact that the write-ups were posted on the Internet instead of given only to the professor was of little concern to her. She stated on the final questionnaire that "the opinion of my Professor matters just as much as strangers seeing my work" (December 3).

Although Grace did not tend to provide a textual map in her writings, she was consistent in drawing out the mathematics in her work. In 10 of the 11 write-ups, she did more than simply report what she observed. In these writings, she drew out the mathematics involved and offered it as a part of her report. For example, in several writings, she noted how changes to equations and their corresponding graphs may or may not influence domain and range. She also discussed the major and minor axes of ellipses as well as the axis of symmetry of parabolas. In 6 of her write-ups, however, she provided the level of legitimization that Leinhardt (1987) described as occurring through the use of "known principles, cross-checks of representations, [or] compelling logic" (p. 227). For example, Grace provided proofs in 4 of the 5 geometry write-ups. In the fifth geometry write-up, she offered logic regarding the centers of triangles to explain the conditions under which the pedal triangle is inside or outside the original triangle. In the final example of legitimization, Grace followed the tips in the activity's instructions to use the known principle of completing the square to verify that $y = 1 - x^2$ is the locus of the vertices of the parabola represented by $y = x^2 + bx + 1$ as the value of *b* is varied.

In addition to legitimizing several of her findings, Grace also indicated in her work that she was aware that some form of justification was necessary to add substance to her explorations. For example, she used the measurement feature of the Geometer's Sketchpad (Version 4.07) software in an assignment to first explore the relationships between a triangle and the triangle formed by the first triangle's medians, but then she stated that she needed a proof to verify her observations, which she then provided. In the only write-up in which she simply reported what she observed without drawing out the mathematics, she ended the writing with an admission that she needed something more to support her conclusion. She stated "I then concluded that these ratios would converge to these numbers . . . . However, I just provided examples where this holds, not a formal proof."

Although Grace was clear that she had enjoyed exploring the mathematics, she conveyed mixed messages concerning how she felt about writing up her reports. In the final interview, she stated that "I think I still would have learned a lot of math just exploring and not actually formally writing [the reports]" (December 7) but conceded on the final questionnaire that she found the write-ups useful for her as a teaching resource. She did remark, however, that doing the write-ups may have had some merit for her as a student:

> I think it did help me organize my thoughts a lot more and so that in itself is a very helpful math tool. I mean math is all about reasoning and logic and so having to logically place it all together and doing it formally does help me to do better work. (TI, December 7)

These statements suggest that Grace operated from two different perspectives about the writing. As a teacher, she saw the write-ups as a valuable resource for her future career, but as a student, she was not sure how important writing was to learning mathematics.

Attitudes and Beliefs.

Grace entered the project with what she described in the final interview as an established set of beliefs about mathematics. From the beginning, she expressed her beliefs in promoting a conceptual understanding of mathematics. When asked if she thought doing mathematics was like following a recipe, she stated she felt that was “a common misconception of math and that especially students . . . think that's all you have to do is to find the formula, plug it in [and] there's your answer” (FI, August 31). She noted, however, that because of her background in mathematics education she had learned about mathematical concepts and “how so many of the pieces fit together” (FI, August 31). She indicated that she thought understanding concepts was more important than memorizing formulas and that writing in mathematics could help students move beyond the formulas. When describing in the first interview her experience with learning how to write proofs in college, she stated that the process had helped her thinking “because once you actually get your thoughts on paper, it kind of helps you flow more and you go deeper into [the mathematics] instead of just writing down the basics of what [you] know” (August 31).

Grace was careful to note, however, in the final interview that she felt having the students talk about the mathematics was just as effective as having them write about it but that writing was particularly useful as a homework assignment when class time was limited. Essentially, she felt that the crucial part of student communication was having students justify their answers whether in writing or in discussions. At the end of the semester, she indicated on the final questionnaire that her beliefs about the use of writing in the mathematics classroom had not changed over the course of the semester:

> I still think that having students write is a good practice for math learners. It connects math to other disciplines and lets them see that the subject of math is not all numbers and memorizing formulas. Some students may have writing as a strength and this may be just

> what they need. Plus, writing is a way to put all of your jumbled thoughts down in a somewhat organized manner. (December 3)

Grace also stressed on the final questionnaire, however, that writing "is a method that should be used in teaching math, not all the time, but one of the many" (December 3) methods that can be used.

During the project, Grace also offered several ideas about how she thought writing could be used in a mathematics class. She stated on the initial questionnaire that she thought using journals in the mathematics class was a good idea in order to have students write about ways they use mathematics on a daily basis or to write "out longer thought problems" (August 21). When asked in the first interview what she meant by the phrase "longer thought problems," Grace stated she was referring to "problems that have more of a thought process with it, a word problem, or something of that nature" (August 31). She noted that with those types of problems "actually sitting down and writing down…thoughts is important and…actually going through that thought process sometime[s] is very helpful" (FI, August 31). In response to being asked about the necessity of writing out one's thought processes, Grace qualified her comments. She stated that writing could be helpful to struggling students because "it can . . . help them go through what they're thinking but it's not always necessary" (FI, August 31). In addition to helping the student, Grace noted in the final interview that writing can also be an assessment tool for the mathematics teacher because it can help the teacher see what the students are thinking.

When asked in the final interview about the use of reflective writing in the mathematics classroom, Grace stated that she thought it was a good idea to have students not only reflect on class activities but also on the "math in the world around [them] and the ways that [they] see math" (December 7). Grace, however, did not seem to have this same experience with the reflections she wrote for the project. Her responses to the post write-up reflection guides were minimal, culminating with her submission of an incomplete guide toward the end of the semester. When asked in the final interview if she thought completing the guides had helped her, she responded “not really” (December 7). Grace did indicate, however, on both the final questionnaire and in the final interview that the question on the guide most useful to her was the one that asked how she might be able to use the activity in her classroom.

Summary.

Grace came to this project with the most experience in writing in mathematics of all the participants, but her feelings about the writing were somewhat ambivalent. To Grace, writing in mathematics was useful but not necessary. Her underlying message seemed to be that writing in mathematics is more for the struggling student or the student who identifies more with the writing than the mathematics. Grace as a student did not seem to fit into either category. Perhaps these attitudes and beliefs explain why Grace tended to be somewhat equivocal about the value of the writing she did for the course and the project. She did not struggle with the mathematics nor did she seem aware of the amount of notes she took as she explored the mathematics. Therefore, in Grace’s mind and in her words she “still would have learned a lot of math just exploring and not actually formally writing [the reports]” (TI, December 7).

On some levels, it seemed as if Grace had moved on from being a student. On the whole, she approached the class from the perspective of a teacher and considered the write-ups a valuable contribution to her repertoire as a teacher. Because of this perspective, Grace produced reports that tended to be theoretical without much background information. She drew out the mathematics involved in her work and attempted to justify her observations in the majority of her write-ups. She did not, however, provide many definitions. Perhaps because of her choice of teachers as her audience, she did not appear to be interested in providing detailed explanations that would include background information such as the use of definitions or the use of a textual roadmap in her writing.

Lisa

Background.

Lisa started the master's program after graduating with a degree in architecture from a prominent southern university. She indicated at the beginning of the first interview that she had decided to become a mathematics teacher because she realized during her final year of college that the job outlook in the architectural field was not good and that she did not want to pursue a graduate degree in that field. She also commented that although she liked creating designs and models she "just didn't want to design buildings and worry about all the . . . code aspects" (FI, September 2). Lisa was quite honest about her decision to become a teacher. She stated she was unsure why she first considered becoming a teacher but noted that her stepmother had served as an influence in her life. She indicated that the fact her stepmother had been recognized as a teacher of the year made her realize that teaching was something she could do as well. Because she had always loved mathematics, she also believed she was well-suited to being a mathematics teacher. When asked why she chose architecture as her primary field of study, she described that she had both a mind for mathematics and a desire to be creative. She stated that "I just thought that architecture would be a good way to be creative but still be mathematical and do it all" (FI, September 2).

Lisa was also a good student. She was tracked as a gifted student during her middle school years and graduated from high school with a 4.0 grade point average. When asked what her favorite classes were in school, she indicated that she enjoyed her mathematics classes but also admitted that her "least favorite was English every year" (FI, September 2). When questioned further about her feelings, she stated that she did not like literature-based classes but did not mind those that allowed her to write about topics that interested her. She noted that she was "more of a writing grammar person than . . . a reading person" (FI, September 2) because she could be exact with grammar rather than with "interpreting what someone else did" (FI, September 2). When asked if she had used writing in any of her high school mathematics classes, Lisa stated that perhaps she had written some proofs in geometry but nothing more. She did, however, state on the initial questionnaire that "in my architectural courses I had to put my design ideas (and many times these were mathematical) into written words" (August 19). Lisa also commented on the initial questionnaire that she was confident in her writing abilities and that her work in one of her architectural classes had been used as an example of how writing should be done.

Process.

Lisa completed the bulk of the project during the second half of the semester. At the time of the second interview, she had completed only 3 packets; however, she explained in the interview that she took a long time with the first write-up and realized that she may have gone a little deeper than was needed so she was trying to learn how "to cut back" (October 12) on her writing. Despite the difficulties in learning how to pace her work, Lisa was consistent in turning in her notes and reflection guides and did not need to be reminded to do so. By the time of her final interview in December, Lisa had completed 10 of the 11 packets.

In addition to being consistent with turning in her notes and reflection guides, Lisa also approached the process of exploring the mathematics and preparing the write-ups in a reasonably organized manner. She reported in the second interview that she liked to first read through an assignment and pick an activity in which she was interested. After selecting an activity, she would then explore the mathematics with the technology and take notes as she investigated. She stated in the second interview that if she had "created a lot of questions" (October 12) for herself during the exploration then she would perform an Internet search to aid in her investigation. She stated on the final questionnaire that she had "to learn about the subjects enough to be able to organize them and write accurate information about them" (December 13). Therefore, she commented that the organization of the write-ups took some time. After the exploration was complete, she would prepare the write-ups.

As Lisa took notes on her own paper for each of the ten activities, she used the headings of the write-up worksheet. Her notes were a combination of sketches, mathematical notation, short phrases, and sentences. On 8 of the 10 sets of notes, she indicated questions that arose for her during the exploration phase. In 6 sets of notes, she asked direct questions such as "Why does this intersection work?" On the remaining 2 she simply placed a question mark after a mathematical expression. She reported during the final interview that most of her notes were about her explorations and that she would identify her findings as she explored and would group them together. After grouping the findings, she stated that she "went back for the conclusion" (TI, December 14). According to Lisa, this process of taking notes greatly assisted her in preparing and writing up the activities:

> I would say that doing the notes for you helped me in my work. I don't know if I would have done notes otherwise, but it was helpful to write down what I was doing and be able to reference it later when I went to write it up. (Final Questionnaire, December 13)

As Lisa implied on the final questionnaire, the process of taking notes did not come naturally to her. For example, I observed her in class near the end of September exploring an activity on the computer without taking notes, so I reminded her to get her thought processes down on paper. When asked in the final interview if she thought taking the notes had helped her mathematically, she responded "Yeah, especially with the proof writing I did more of trying to figure it [out] on the notes and then I just did it on the computer" (December 14). Therefore, Lisa implied that this process of notetaking had helped her organize her write-ups.

The Write-Ups.

During the second interview, Lisa stated that she approached her write-ups as if she were explaining the material to a high school student. Because of this perspective, she stated that preparing the write-ups was time-consuming because she was "trying to think of how to introduce [the topic and] how to come across" (SI, October 12) to students; therefore, she would "start out more basic and then build on [the explanation]" (SI, October 12). Nonetheless, Lisa tended to write in brief statements that were condensed to fit in charts. When she wrote in paragraph form, however, her writing was generally grammatically correct and flowed well except for sporadic errors.

Despite Lisa's dedication to the organization of her write-ups, her general writing presented several problems in terms of the framework. For example, Lisa had difficulty identifying the goal in 5 of her 10 write-ups. In 3 of the write-ups, the statements of the goal were missing. In 2 of the write-ups, the statements were implied. For example, in her fourth write-up, Lisa began with the statement "Consider the equation $x^2 + bx + 1 = 0$" without any other introductory material. I considered this an implied statement of the goal because Lisa did not indicate where her writing was headed; only that it concerned the stated equation. It is worth noting, however, that Lisa improved in the development of her goal statements. Although not ideally stated, the goals she expressed in the last 5 write-ups were more direct. For example, she provided a restatement of the problem as an introduction to her sixth write-up: "Given two circles and a point on one of the circles, construct a circle tangent to the two circles with one point of tangency being the designated point." Despite the fact that this sentence is not necessarily a direct statement of introduction, it nevertheless is a stronger statement of direction than the earlier example. In addition, like Grace, once Lisa began her write-ups she generally provided signals to indicate transition points and direction in her work. She effectively used formatting techniques such as blocking to indicate sections and also used words and questions such as "Why is this?" to indicate direction.

Although Lisa showed improvement in articulating the goal of her write-ups, she consistently demonstrated difficulties in linking or clarifying the representations she used in 8 of the 10 write-ups. For example, as indicated earlier, Lisa offered a lesson in her sixth write-up on how to "construct a circle tangent to . . . two circles with one point of tangency being [a] designated point." Once the construction is complete, she states that by "animating C and tracing point F, either an ellipse or a hyperbola is formed." To demonstrate her explanation, she provided several constructions, including one that generates an ellipse (see Figure 3). Lisa did not specify, however, that the red figure in the construction is the ellipse. That is left for the reader to infer after comparing it to previous sketches. In addition, she provided a hyperlink to a Geometer's Sketchpad (GSP) (Version 4.07) file of the construction so readers could download the file to animate and trace the given points. She provided no instructions, however, on how to execute those commands in GSP. Therefore, it is implied that the audience must have a working knowledge of GSP to see the ellipse being formed.

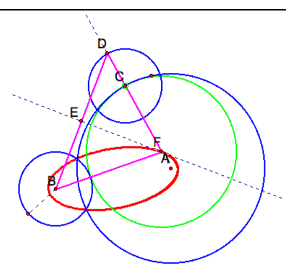


Figure 3. Lisa's construction of an ellipse using tangent circles.

Like Grace, Lisa was stronger in writing the mathematics than in general writing. In each of the ten write-ups that she completed during the project, Lisa drew out the mathematics in the activities. For example, she frequently provided definitions for mathematical terms such as *phase shift*, *amplitude*, and *period*. In addition, she provided justifications in 6 of the 10 write-ups that rose to the level of legitimization called for by the framework. Of those, 3 contained geometric proofs and the other 3 provided legitimization by known principles such as completing the square or using the quadratic formula.

On the whole, Lisa's mathematical work was strong except for one misstep that must be noted. On Lisa's seventh write-up, she presents a proof about the ratio of segments created by the perpendiculars of an acute triangle. At the end of the proof, she offers a diagram and states that "by looking at the measurements in the diagram, we can see the above [proof] holds true." In essence, Lisa is using the measurement feature of GSP (Version 4.07) to verify the work in her proof which implies that measurement is as strong a form of justification as proof. As Grace correctly noted in her work, however, measurement should be used as a tool in geometry to indicate where a proof is needed rather than to substantiate a proof already provided. Arguably, the use of measurement to substantiate a proof is redundant.

Reflecting on her work at the end of the semester, Lisa indicated on the final questionnaire that she had experienced a change of mind about writing in mathematics:

> I didn't know how I would like writing in this class, but I enjoyed it. I felt like it was necessary in explaining myself and my work in these write-ups. Before I guess I didn't really see the usefulness of it. (December 13)

When asked in the final interview to discuss this statement, Lisa commented that she liked writing something that could be useful to her as a teacher and that she had already used two of her write-ups in tutoring situations. She also noted that the writing usually took more work than the explorations because she kept her audience in mind and knew that she “had to say why something was happening when [she] did it” (TI, December 14). She said that the writing had helped her hone her skills as an educator who needs “to be able to verbalize the mathematics and make it clear for students” (TI, December 14). She indicated, however, on the final questionnaire that her position would have been different if the write-ups had only been written for the professor. In that situation, she indicated that she would not have started “with basic ideas and tried to build them up” (Final Questionnaire, December 13). In addition, Lisa acknowledged that a mathematics teacher’s own written notes and assignments need to be clear as well. When asked if she felt preparing the write-ups had helped her mathematically, she stated that she thought the writing had helped her to better commit the mathematical topics to memory.

Attitudes and Beliefs.

Lisa’s reason for studying architecture in college provided a glimpse into her attitudes toward and beliefs about mathematics. As stated earlier, she wanted to study a subject in which she could blend both her aptitude for mathematics and her talent for design. It is no surprise then that she reported on the initial questionnaire that doing mathematics is far from following a recipe:

> My interpretation of the expression “following a recipe” is something one does mindlessly without understanding or caring about. Pure mathematics involves creativity and understanding, and this is what makes math fun. The concept of a recipe limits one to a single process when there may be several ways to come up with a solution! (August 19)

By the end of the semester, Lisa was expressing a belief in the conceptual nature of mathematics. In response to the same question on the final questionnaire, she stated that “I don’t feel like math should be following steps but rather making connections across different ways of doing things” (December 13). Arguably, a subtle shift had occurred in Lisa’s beliefs about what it means to do mathematics. At the beginning of the semester, she implied that it is important to understand that there may be multiple ways to solve a problem. By the end of the semester, she had shifted the importance to making connections among those different ways.

Like Grace, Lisa hinted that she thought the use of writing could be used to foster conceptual understanding in a mathematics classroom. On her initial questionnaire, she stated that she thought writing in mathematics was “an important concept” (August 19) and that she was “excited to start on it” (August 19). When asked in the first interview why she thought writing in mathematics was important, she stated “that a different type of understanding gets brought to [the activity] when you write in math” (September 2). She clarified her belief by stating that she felt that it could help students better understand word problems. On the initial questionnaire, she stated that many of the students she tutored did not “like when words and mathematical concepts mix” (August 19) and so she “believed students would be less hesitant to approach [word problems] if they themselves were used to integrating writing and mathematics” (August 19). She was careful to note, however, that she did not believe “in sending students home with homework and saying ‘write about what we learned today’” (FI, September 2). To Lisa, writing in mathematics had no generative power. If students did not understand the topic in the first place, she believed they could not “bring the understanding” (FI, September 2) from the writing.

At the end of project, Lisa noted on the final questionnaire “I think I’m more likely to have students write in math class than I was at the beginning of the semester” (December 13). She credited the change in her attitude to this course and her other master’s courses. She commented that she now appreciated how writing could help her assess what her students understood. For example, she described a lesson plan she drafted for another class in which she asked students to write about the connection between the distance formula and the Pythagorean Theorem in order to "demonstrate how they knew it" (TI, December 14). When asked in the final interview what kinds of writing she might use in her own classroom someday, she stated proofs and "summary writing" (December 14) in which students would summarize the material or "come up with conclusions" (December 14). According to Lisa, this type of writing would highlight for her and the students how much they understood about the material. She did note, however, that she did not want the writing to be a "time-waster" (TI, December 14) and conceded that she did not know how often she would use writing in her classroom. The implications were that the writing had to serve a purpose for both teacher and student and it was something that she might not regularly use.

It must also be noted that Lisa did not advocate the use of journals or any type of reflective writing in her classroom which corresponds to her response to the post write-up reflection guides. When asked in the final interview if completing the guides had helped her during the semester, she replied "I don't think so" (December 14) because she "really didn't use them from one lesson to the next" (December 14). She did comment on the final questionnaire, however, that it was good to look back over her work through the use of the guides and make sure she was satisfied with what she had done.

Summary.

Like Grace, Lisa tended to approach this course from the perspective of a teacher. She found completing the write-ups useful as a future educator who needed the resource the repertoire of write-ups provided and who could also use the practice in providing explanations that the writing offered. And also like Grace, Lisa plainly admitted that she did not know how often she would use the writing in her own classroom. Again like Grace, she seemed unaware of the role informal writing had played in the compilation and organization of her notes. Lisa did acknowledge that completing the notes using the headings of the writing worksheet was helpful to her, but she did not express an awareness of the self-questioning that the notes captured. She did express that she sometimes created questions for herself during the explorations but she did not indicate that she connected the questions to the notetaking process.

The most distinct difference between Grace and Lisa was the choice of audience. Grace wrote to report to peers. Lisa wrote to explain to students. Because of her audience, Lisa was focused on the preparation and organization of her write-up, and although her textual maps were often lacking, she did attempt to provide background information such as definitions. Therefore, Lisa's write-ups tended to be more theoretical in nature than Grace's.

Stronger in General Writing

Gwen

Background.

Of the six participants, Gwen was one of two who did not directly enter the master's program after graduating from college. Gwen graduated from college in 2007 with a degree in business management. She stated during the first interview that she had worked in management for 2 years before entering the program but decided the job was not what she wanted to do long-term. She indicated that she had decided to make the career change to teaching because of the influence of her mother-in-law who had made Gwen privy to "what it's like to be out there in the field" (FI, September 1). In addition, Gwen commented that she had watched the documentary *Two Million Minutes* in which schools in India, China, and the United States were compared. She noted that the comparison had inspired her to want to be a part of "bettering [the] standard" (FI, September 1) of education in U.S. schools.

When asked why she had chosen mathematics as her specialty, Gwen noted that mathematics had always been her favorite subject and that it was personally important to her because of her business background of “making decisions [based] on efficiency . . . liability . . . cost and profit” (FI, September 1). She indicated that that she “saw not only the academic benefits of math education but . . . the real life benefits” (FI, September 1) as well. She also commented that she preferred “the math fields that are more real life and applicable to everyone” (FI, September 1) because she believed that “people learn [mathematics] better that way” (FI, September 1).

Gwen also stated in the first interview that she had done well academically in both high school and college. However, her mathematics background was somewhat limited. She indicated that the third year of algebra was the highest level of mathematics she had studied in high school and that the second course in calculus was the highest mathematics she had studied in college. She commented that she liked her algebra courses in high school but did not care for geometry because she was not “huge on proofs” (FI, September 1) and tended to solve problems algebraically before resorting to other methods. In addition, she noted that she liked studying statistics because of its “real-life applicability” (FI, September 1).

Although Gwen’s experience with mathematics was somewhat limited, she reported in the first interview that she had done a substantial amount of writing in high school. She particularly recalled her freshman year when she was placed in gifted English. She remarked “it was insane how hard [the class] was” (FI, September 1) and that her “college English class was nothing compared to [that] class” (FI, September 1). She did acknowledge, however, that the class had taught her “the rules of grammar and . . . how to write, to be proper, and to be understood” (FI, September 1). She also noted that in college she had taken a writing course based on argumentation and research that she enjoyed because it taught her “how to construct an argument and defend [a] line of reasoning” (FI, September 1). She indicated she preferred structured writing because she considered herself “very much analytical” (FI, September 1) and unable to “think of things . . . outside the box” (FI, September 1). Therefore, she tended to have difficulties with writing assignments that were not well-defined.

Despite her experience with general writing, Gwen reported that the only writing she remembered using in a mathematics class before entering the master’s program was creating two-column proofs in high school geometry. Indeed, Gwen noted that she was surprised by the amount of writing her linear algebra course in her graduate program was requiring:

> I’m used to just . . . writing out whatever I did and circling the answer but [the professor] says “No, do your scratch work on another sheet of paper. I don’t want to see your scratch work. I want you to write up [your] claim and whatever your answer is that you come to and then put “proof” and then explain how you got there and don’t give me your scratch work” so it’s a little bit different from what I’ve done before, but I think it’ll be more writing because of that, which, you know, is probably a good thing. (FI, September 1)

She then stated that she felt like the process would be a “good thing” (FI, September 1) because she believed that “you understand things better when you communicate them more fully” (FI, September 1).

Although Gwen had limited experience with both mathematics and writing in mathematics, she stated on the initial questionnaire that she felt the writing component of the class would be “an area of strength” (August 18) for her and that she was glad she would “have something that comes more naturally for [her] in the course” (August 18). During the first interview, she commented that she was aware that she did not have the same mathematical background as most of the students in the class but she took comfort in the fact that she knew how to write. She stated that “once I understand the concept, I generally know how to communicate it easily” (FI, September 1).

Process.

Gwen completed 10 of the 11 packets before completing the final questionnaire and sitting for the final interview. Of the six participants, Gwen was one of the most consistent in posting her write-ups and turning in her documentation for the project. Of the 10 write-ups, Gwen completed 3 during the first half of the project. She completed 5 of the remaining 7 write-ups on a weekly basis over the rest of the semester. Gwen was also consistent in the manner she turned in her documentation. Except for the final 2 write-ups which she completed the last week of the project, she would generally submit her handwritten documentation for each write-up in class each week.

In addition to submitting her work in a timely manner, Gwen also followed a somewhat systematic process in approaching her work. She reported in the final interview that she would first use the technology and investigate several tasks in an activity and then she would choose the one that she found to be the “most thought-provoking” (December 16) about which to write. During this stage of the process, she reported that she took notes while she was choosing which task to explore further. During the second interview, she also commented that she looked at the work of other students in past classes to gain inspiration and guidance about how she could begin the lesson and then build on it. Also during the second interview, Gwen described that she would go back and forth between exploring with the software, taking notes, and preparing her write-up. She indicated that if the exploration yielded something interesting, she recognized a pattern, or a question arose for her, she would write it down in her notes and then formalize those notes for the write-up. She then offered a distinction between her notes and the formal write-up:

> It's kind of like . . . my notes are . . . "Well, what will happen if I do this?" and, you know, just sort of like things that I'm thinking of in my head that I wouldn't necessarily just write out on the Website. (SI, October 12)

During the second interview, Gwen also noted that she was "focusing more on asking questions" (October 12) because she felt that sometimes she was not as inquisitive as she needed to be so she was "learning a lot about how to ask questions to discover more things and make the lessons more meaningful" (October 12). She stated that the professor had gone over the first assignment in class after she had completed it and spoke about the different ways the lesson could have been explored. From that experience, she learned that she needed to expand her exploring beyond what the questions in the activity required.

During the final interview, however, Gwen reported that a change had occurred over the course of the project at the notetaking stage in her process. She noted that she had begun the semester writing up her reports as she investigated but then realized that if she was not "knowledgeable enough" (TI, December 16) about the mathematics then she usually had to go back and change what she had written as the investigation proceeded. Essentially, she realized that sometimes she needed to first get a handle on exploring the mathematics with the software before taking notes. She stated in the final interview that "I [had to] wait until I figured out sort of what I [was] doing before I just [wrote] a bunch of notes" (December 16). She implied that this practice involved knowing how to do the work on the software, like constructing various geometric figures, before she began the writing process.

Gwen also reported in the final interview that she found the write-up worksheet useful as a guide when she took notes and when she prepared the write-ups. She stated that during the explorations the headings of the worksheet helped her to "focus on . . . the different aspects of what [she] was doing" (TI, December 16) and to divide and organize her notes. She then commented that this process, in turn, helped her to structure the write-ups. She indicated that without the write-up worksheet she "probably just would have written notes on paper and then ciphered through it" (TI, December 16).

In addition, Gwen commented at the end of the semester that the post write-up reflection guides had also played a role in helping her to prepare the write-ups. She noted on the final questionnaire that "describing the process I took in completing the write-ups helped [me] to internalize my steps so I could focus more on the material being investigated " (December 14). During the final interview, she stated that the reflections had helped her "as a writer or as a presenter of information" (December 16) because they "enhanced" (December 16) the procedure she followed and helped her to think about the best way to present the information.

The Write-Ups.

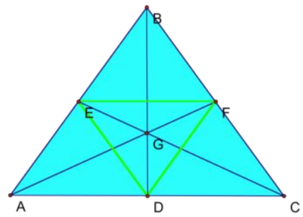
Unlike Grace and Lisa, Gwen composed her write-ups in paragraph form. As Gwen indicated at the beginning of the project, writing was "an area of strength" (Initial Questionnaire, August 18) for her and it showed in her write-ups. Her write-ups contained no major grammatical errors and they flowed well. Although the statement of the goal was missing or indirectly stated in 5 of the write-ups, Gwen effectively used signal monitors such as the words *next* and *now* to indicate transition and movement in her writing. Like Grace and Lisa, Grace also demonstrated problems with the representations she used in 6 of her write-ups. Although she used color coding to link equations to their respective graphs in several of her write-ups, Gwen sometimes tended to let complicated sketches stand alone with little or no explanation.

Despite Gwen's tendency to condense too much information into a graph or sketch, she nevertheless attempted to explain her processes as much as possible. She stated in the final interview that she tried to write the reports so that high school students could understand the “aspect of math” (December 16) she was trying to explain. She also noted in the final interview that she thought it was important for students to see the investigative process. She wanted to write about her explorations in such a manner that students could see where she succeeded and failed so they could learn from the process. In addition, she indicated on the final questionnaire that she would not have included as much background information if the write-ups had been written only for the professor.

Although Gwen’s clarity in writing made her work easy to read, it also exposed in greater detail the struggles she had with the mathematics. Of the 10 write-ups Gwen completed before the final interview, 5 demonstrated cases of legitimization at the level required by Leinhardt’s framework. Unlike those of Grace and Lisa, however, none of the write-ups contained formal proofs that were mathematically sound. This difference is not surprising, however, since Gwen admitted in the first interview that she was “not huge on proofs” (September 1). What these 5 write-ups did provide, in contrast, were forms of legitimization based on “known principles . . . [or] compelling logic” (Leinhardt, 1987, p. 227). For example, Gwen used the known principle of the discriminant to justify the absence of a real number solution to a quadratic equation.

In 4 of the 5 remaining write-ups, however, Gwen tended to have difficulty drawing out the mathematics at the level Grace and Lisa demonstrated. For example, in 2 cases, Gwen offered geometric proofs but they did not rise to an adequate level of mathematical rigor because they relied on the measurement feature of Geometer’s Sketchpad (Version 4.07). For example, in the first case, Gwen set up her fifth write-up with a construction of a triangle and its medians and indicated their measurements (see Figure 4). From here, Grace concludes that the triangles are similar according to the Side-Side-Side Theorem of Similarity (SSS) but does not attempt to prove this fact independent of the measurements.

Our construction using Geometer's Sketchpad appears to look like this:



Our measurements for each of the sides are calculated using GSP:

$m\ \overline{AB} = 10.4292$ cm
$m\ \overline{BC} = 10.4785$ cm
$m\ \overline{CA} = 12.3120$ cm
$m\ \overline{FD} = 5.2146$ cm
$m\ \overline{ED} = 5.2392$ cm
$m\ \overline{EF} = 6.1560$ cm

We can first notice that both triangles appear to be roughly the same shape. So we will hypothesize that there could be a similarity relationship. In order to prove this, one method we can use is the SSS Similarity Theorem, which states that if there is a constant ratio between the corresponding side measurements of the 2 triangles, then the triangles are in fact similar. Our calculations would appear as follows:

$$\frac{\overline{AB}}{\overline{FD}} = \frac{10.4292}{5.2146} = 2 \qquad \frac{\overline{BC}}{\overline{ED}} = \frac{10.4785}{5.2392} = 2 \qquad \frac{\overline{CA}}{\overline{EF}} = \frac{12.3120}{6.1560} = 2$$

Figure 4. Gwen's explanation of why two triangles are similar based on measurement.

In the second case, Gwen offered a sound geometrical proof to establish relationships among angles of acute triangles inscribed on a circle but used measurement to establish that the relationships would hold for obtuse triangles as well. In the remaining two cases, Gwen drew out the mathematics but not to the same depth as Grace or Lisa achieved. For example in her first write-up in which the participants were to explore changes to the graph of $x^n + y^n = 1$ as the values of *n* change, Grace explored possible connections to the standard equation of the circle in addition to reporting patterns she observed. Gwen, in contrast, simply reported the patterns. It must be noted, however, that Gwen was aware that her first write-up was not extensive. She stated in the second interview that she thought she had "probably just skimmed the surface of what [she] could have and at the time . . . didn't realize it" (October 12). She then reported that she believed her later write-ups were getting more detailed in their explanations because she was getting "a little bit more inquisitive" (SI, October 12) in her approach. She also stated that she wanted to go back at some point and "strengthen up that first one" (SI, October 12) but she had not done so by the end of the project.

From the beginning, Gwen seemed to be aware of a connection between her writing process and the exploration of the mathematics. She made a reference on her second post write-up reflection guide that she found it "informative . . . to dig a little deeper" during the activity. When I asked in class what she meant by the phrase "to dig a little deeper," she indicated that verbalizing what she was discovering drove her deeper into the material. I then requested that she be more specific in her responses on future reflections. On the next reflection, she offered the requested specificity and described the connection between writing and mathematics:

> I felt like writing about the activity pushed me towards obtaining a deeper understanding of the problem. When my writing is vague or unclear, I tend to realize this and discover more about the graph so I can better explain the behavior. (Post Write-Up Reflection Guide #3)

She also commented in the final interview that "as you're writing up [your results], you sometimes realize that there's things that you didn't think to ask when you were just investigating" (December 16). In other words, Gwen was aware that writing about the mathematics forced her to face how well she understood the mathematics.

Attitudes and Beliefs.

During the first interview, Gwen commented that she was experiencing a shift in her beliefs about mathematics. She stated that she used to like mathematics because she thought it was "very black and white" (FI, September 1) but she was "starting to learn that there's a little more gray than what's there on the surface" (FI, September 1). Because of the mathematics education courses she was taking in the master's program, she noted that she was starting to learn that there was “just a lot more interpretation of [the mathematics]” (FI, September 1) than she had realized. During the final interview at the end of the semester, she described her original beliefs about mathematics:

> I thought that math . . . hadn’t really changed much, you know, since I’d been in high school and that it’s kind of how it’d always been and I didn’t really expect to learn anything really new that I could apply to high school math that I didn’t already learn in high school . . . other than more advanced topics. (December 16)

Because of her course work in mathematics education, however, she commented that she was “impressed . . . and surprised . . . that there [had been] so much change” (TI, December 16) in secondary mathematics. She also implied that she now appreciated the “focus on . . . conceptual understanding rather than just algorithms” (TI, December 16) which was a subtle shift in her perspective from the beginning of the semester.

This shift can also be found in the questionnaires Gwen completed for the project. When asked on the initial questionnaire if she thought doing mathematics was like following a recipe, she replied “yes and no” (August 18) because she believed that “although there may be a ‘standard procedure’ (or recipe) for solving a given problem, some people will solve it differently (or deviate from the recipe)” (August 18). By the end of the semester, she had somewhat refined her position. She stated on the final questionnaire that she thought “doing mathematics is more like cooking in general” (December 14) in which some people choose to follow a recipe and some do not. According to Gwen, those who do not follow a recipe usually “taste test and adjust their ingredients along the way” (December 14) which is similar to those who take “more of an exploratory approach to mathematics, as they try one process to solve a problem, and then reason through why it makes sense or doesn’t” (December 14). A comparison of the two descriptions seems to indicate that Gwen’s awareness of the exploratory nature of mathematics grew as the semester progressed.

In addition to learning a new way to view the study of mathematics, Gwen noted during the first interview that the idea of writing in mathematics was also new to her:

> Before going into this program, I never really considered math as a subject that you would write for. I just thought of it as a bunch of numbers . . . . I've only been in the program now 2 weeks and it's already kind of pushed me to see the subject as a whole in a different light and already I feel like . . . I see the importance of it. (September 1)

She repeated this sentiment during the final interview in which she stated that she had "always believed that writing was powerful" (December 16) and "always really enjoyed writing" (December 16) but had not realized that it could have a place in mathematics before entering the master's program. She noted her experience and beliefs on the final questionnaire:

> This course in combination with all my other courses . . . this semester [has] reinforced the need to include writing in mathematics courses. I believe it should be required of even high school students, because it guides them through developing a more conceptual understanding of the process of solving a problem rather than simply having them to write down calculations. Going in to my first semester in my graduate program, I had viewed mathematics mostly as calculations, and I now believe it is actually much more based on logic than performing operations with numbers. (December 14)

Essentially, Gwen suggested that writing helps to promote a conceptual understanding of mathematics by emphasizing the logic of the mathematics rather than the procedures.

During the final interview and on the final questionnaire, Gwen was emphatic that she would use writing in her own classroom someday. She stated that she believed writing in the mathematics classroom could be powerful for two reasons. First, "it forces the students to think more about what it is they're doing and not just follow through procedures" (TI, December 16) and second "it helps . . . a teacher to understand more and assess what level [the students] are at as far as [their] conceptual understanding" (TI, December 16). She also stated on the final questionnaire, "I absolutely would have my students write in my mathematics class because it demonstrates that they have a true understanding of the curriculum" (December 14). When asked in the final interview what types of writings she might use, she implied that she still had much to learn from the master's program before being able to give a definitive answer but that she would like to work with textbooks that promote problem-solving activities that call for writing and that "give a real world aspect to what it is [students are] doing" (December 16). She also indicated that she would like to use something similar to the write-up worksheet so that students would "be required to explain how it is that they go about solving a problem" (December 16) to insure they are fully exploring each activity.

Summary.

Gwen stood out among the six participants for several reasons. First, using a paragraph form of writing set the stage for Gwen to clearly expose what she did and did not know, and she was acutely aware of this dilemma. She stated early on that if her writing was "vague or unclear" (Post Write-Up Reflection Guide #3) she knew she had to go deeper into the mathematics in order to be able to explain it on a competent level. Gwen was also one of the few to experience a fundamental shift in her beliefs about mathematics. She had entered the master's program with a procedural view of mathematics and regarded the study of it as nothing more than performing calculations. Indeed, she confessed that she had liked this "black and white" (FI, September 1) view of mathematics; therefore, the idea of using writing in the mathematics classroom made little sense to her in the beginning. Already in the first interview after 2 weeks in the program, however, she was starting to realize that there is much more to mathematics than simply following procedures. By the end of the semester, she was integrating the notion of writing with promoting a conceptual understanding of mathematics.

Despite Gwen's lack of a strong mathematical background in comparison with some of her peers, the write-ups she completed were generally correct mathematically. Although her two geometric proofs were somewhat lacking because she used measurement to substantiate her claims, she nevertheless presented sound mathematics in the majority of her work. In addition, she knew she had much to review and much to learn mathematically and the writing seemed to clarify this realization for her. In the process of exploring the mathematics, taking notes, preparing the write-ups, and reflecting on her work, Gwen experienced a connection between the mathematics and the writing. In essence, if she did not know the mathematics well enough, then she knew she could not write clearly and effectively about it and if she could not write about it, then she needed to "dig a little deeper" (Post Write-Up Reflection Guide #2) into it.

Struggles with General Writing and Writing Mathematics

Kim

Background.

Of the six participants, Kim was the only inservice teacher. At the time of the project, she taught mathematics at a local high school and noted on the initial questionnaire that she had experience teaching first- and second-year algebra as well as geometry. Nine years after graduating from college, Kim was returning to school to obtain a master's degree in order to exchange her provisional teaching certificate for a conventional license. Currently in her fifth and final year of teaching on the provisional certificate, Kim chose to obtain the regular license through the master's program. Therefore, during the project, she had to balance the demands of being a mathematics teacher at a local high school with the demands of being a student.

Although Kim reported during the first interview that she enjoyed teaching mathematics, she had not taken a direct route to becoming a teacher. She described in the first interview that she had initially started out in college majoring in mathematics but switched to ecology with only three mathematics courses left to take. She stated that she switched majors because she did not like the professors she would have at the end of her program. She did, however, have enough mathematics courses to declare a minor in the subject. After spending some time traveling after graduation, obtaining an additional degree in criminal justice, and working "little assistant jobs here and there" (FI, September 3), Kim decided to substitute teach to supplement her income from a part-time job. In essence, she believed substitute teaching was the ideal job to fit around her work schedule. Persuaded by an administrator who knew about her mathematics background, Kim moved into a long-term substitute position and applied for her provisional certificate.

Even though Kim did not directly state it, whether or not she liked a course as a student generally seemed to depend on whether or not she liked the teacher. She reported both favorable and difficult experiences with both mathematics and English teachers throughout her academic career. However, she did note on the initial questionnaire that she "had many struggles" (August 22) in her writing courses and did not understand how to express her thoughts until she took an independent study course in history while in college. She reported that she had to write 17 papers for that class and that her writing improved with each paper. She also expressed that she was not too concerned about the writing required for the current course because her confidence had grown over the years about her "abilities to express [herself]" (Initial Questionnaire, August 22). However, she did state on the initial questionnaire that the only writing she had done in her mathematics classes were proofs and that she did not "really remember them" (August 22) because she took most of her mathematics courses over 10 years ago. She did clarify in the first interview that she might have done two-column proofs in her high school geometry class but not paragraph proofs.

Process.

Kim started the project slowly because of technical difficulties with posting her work to the Internet. After approximately a month into the semester, she posted her first assignment and turned in the documentation for the project. Three weeks later, at approximately the mid-point of the semester, she had completed 3 additional packets. Over the course of the rest of the semester, however, Kim tended to turn in the remaining 7 packets in chunks. For example, it was not uncommon for her to post 2 or 3 write-ups at a time without giving me her notes and reflections. On two separate occasions, I sent out an email to remind her of what I was missing in terms of the documentation. On one occasion, she told me that she had lost her notes and the post write-up reflection guide for one of the assignments and would have to recreate them; therefore, she did not know how reliable they would be. Despite these issues, Kim completed the 11 packets by the time of the final interview.

To explore the mathematics and complete the write-ups, Kim followed a process structured by the demands of her hectic life as a student, teacher, wife, and mother. She relayed in the second interview that she began the exploration of each activity by choosing a task that could easily be done in steps so she could leave it and come back to it when needed. In the final interview, she also indicated that it was important to her to choose tasks that were relevant to her teaching and that she could potentially use in her classroom. After choosing a task, Kim would put pencil to paper to sketch out what she needed to do before she started exploring with the technology. She stated in the second interview, “I start out with my own thoughts on where I think I'm supposed to go” (October 15). During the final interview, Kim indicated that she also took notes as she explored the mathematics with the technology but stated that she looked at the notes as she wrote up the material only if they contained a substantial amount of material. It must be pointed out, however, that Kim turned in notes for only 6 of the 11 assignments. The omissions were sometimes due to the fact that she did not take notes or she forgot to turn them in with her write-ups. On the other hand, it must also be stressed that for each of the 11 assignments, Kim did turn in either notes, or the write-up worksheet, or both.

For 9 of the 11 write-ups, Kim completed the write-up worksheet. She reported in the final interview that she would always first state her goal on the worksheet and then allow that statement to guide her explorations. She would then keep the rest of the headings in mind as she explored the mathematics and as she wrote up her report. She stated that she found the headings helpful and that the worksheet gave a structure to her work:

> It kept from me just having a blank sheet in front of me and not knowing where to go. It helped me know that there were four steps I needed to cover on each write-up and . . . I knew that I at least had a consistency throughout each one and it made . . . . It was kind of like a map, you know? (TI, December 16)

Although Kim mentally used the worksheet headings to guide both her explorations and writings, she tended to treat the document itself as a reflection and often wrote statements about how she felt about a particular activity. These statements were most often found in the section designated for the conclusion. For example, she stated that she found the first activity easy but “the difficulty . . . was learning how to post it” (Write-Up Worksheet #1).

The Write-Ups.

Unlike Grace, Lisa, and Gwen, Kim demonstrated problems with both general writing and writing mathematics. In terms of the framework, 5 of her write-ups had missing goals or goals that were indirectly stated or placed near the middle of the work. In addition, 7 write-ups demonstrated problems with representations. In general, too much information was condensed into graphs or sketches with little or no explanation provided. However, Kim did attempt to keep the reader abreast of changes in her writing. She used phrases such as “let’s look” as well as questions to signal where she was headed in the description of her exploration. In addition to demonstrated problems with the elements of the framework, Kim also struggled with the mechanics of the writing. Her writings were often riddled with any combination of spelling errors, run-on sentences, sentence fragments, or poorly punctuated statements. For example, in her eighth write-up, Kim described some of her observations about pedal triangles:

> Now we can see that the triangles appear to be similar but are they, but with a little investigation we see that it takes three pedal triangles to create similar triangles.

Although readers can infer what Kim intends, the awkwardness of integrating the question into the sentence without proper punctuation is enough to give readers pause.

In terms of the framework, Kim also struggled with writing the mathematics. For example, in 10 of the 11 write-ups Kim offered little or no explanation or justification for what she reported; therefore, her work did not rise to the level of legitimization required by Leinhardt’s framework. For example, in her fourth write-up, Kim asked what it would take algebraically “to make the parent function flip upside down.” She then stated that “one must know how to complete the square of the equation” which she implies is “a complicated process.” Instead of providing that process in her write-up, however, she simply stated the outcome.

In addition to struggling with the mathematical elements of the framework, Kim also produced a substantial number of mathematical errors in 5 of the 11 write-ups unlike Grace, Lisa, or Gwen. For example, in 3 of the 5 geometry activities, Kim used the measurement function of Geometer’s Sketchpad (Version 4.07) to verify or support her claims. A case in point is her third write-up concerning equilateral triangles. She defined an equilateral triangle and then provided an example with measurements for each of its sides (see Figure 5).

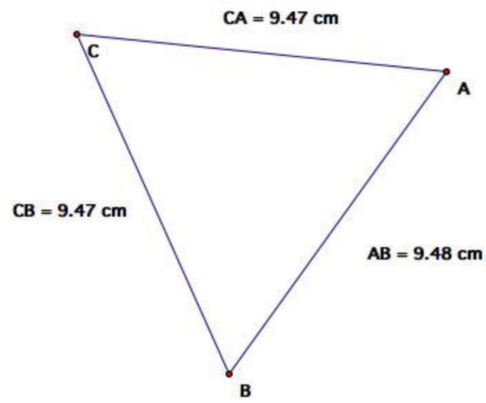


Figure 5. Kim's example of an equilateral triangle based on the measurement of its sides.

Kim does not call attention in her writing to the fact that the sides of the triangle in the figure are not equal according to the given measurements. Therefore, the triangle is technically not equilateral but rather isosceles. After incorrectly stating, "A midpoint of a segmant [*sic*] in a triangle is known as the median," she then postulates that the medians of an equilateral triangle will form equilateral triangles. She uses measurements to verify her claim (see Figure 6). Again, some of the measurements do not support her claim and she offers no proof of the claim other than the measurements.

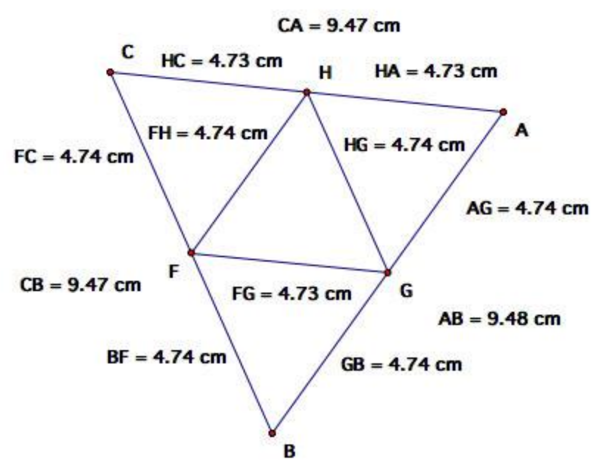


Figure 6. Kim's use of measurement to justify that the medians of an equilateral triangle form another equilateral triangle.

In another example that may indicate a possible gap in Kim's mathematical understanding, she presented information in her first write-up about the different operations of two functions such as the addition and multiplication of functions. She chose linear functions as her examples and noted that the multiplication of these two functions yields a quadratic function. At the end of the operations, however, she performed a composition of the two linear functions and stated that the resulting linear function was "not as interesting as I would like it to be." Next, she tried two different linear functions and ran them through the same operations. Again, at the end of the write-up, she performed the composition of the two functions, obtained a new linear function, and stated, "I was hoping for something more than just a linear function." She then implored the readers to explore with their own functions so they "can create something interesting." Whether or not Kim intended it, she leaves the reader with the impression that she was somewhat surprised by the simple idea that the composition of two linear functions yields another linear function.

Although Kim stated on the final questionnaire that she enjoyed "writing and creating projects for everyone to see" (December 15), she also stated that "writing about the mathematics was a little more difficult" (December 15) than she expected. She stated that she wrote or discussed mathematics every day for her students but posting her write-ups on the Internet "for everyone in the world to see" (Final Questionnaire, December 15) was somewhat stressful for her. She was not entirely clear, however, who that *everyone* was. During the final interview, she noted that she was writing for her students at first but then switched to her colleagues. She reported, however, that she soon realized that her colleagues would know the information so she once again changed her audience to a more general population of "anybody who just wanted to know a quick thing about [the mathematics]" (TI, December 16). This evolution of audience may account for changes in her writings. She also indicated during the second interview that she felt like she "over-explained" (October 15) the concepts in her first write-up and was repetitive in her ideas; therefore, she tried in the later write-ups to reduce the number of sentences she used. In addition, she indicated on the final questionnaire that she would not have laid out the mathematics "step by step" (December 15) if the write-ups had been written only for the professor.

Attitudes and Beliefs.

Kim offered what appeared to be conflicting beliefs about what it means to do mathematics. She stated on the initial questionnaire that she thought doing mathematics "takes more than just a recipe" (August 22) because "sometimes there is not always just a formula to follow in order to solve a problem" (August 22). For Kim, doing mathematics requires "thinking and skills to find the answer to most questions" (Initial Questionnaire, August 22). However, she qualified her thoughts during the first interview and on the final questionnaire. She stated on the final questionnaire that she thought "some basic level of mathematics is like following a recipe, but as one dives deeper into math you have to know how to cook in order to make a meal" (December 15). She described in the first interview where she thought the basic mathematics ended and the advanced began:

> We all know once you get past the eighth grade, ninth grade, just algebra part and geometry part, it's conceptual. It's not about formulas any more. It's thinking outside the box. There's no formula. You just got to do it. You got to try it out. You got to figure it out. So once you give [the students] the knowledge to have a little bit of background, it's . . . . And you know that the best cooks in the world don't follow a recipe. They see three ingredients and they know what to make with it. (September 3)

Therefore, Kim implied that basic mathematics, to include freshman algebra and geometry, requires only a procedural understanding for one to fully comprehend the material.

In addition to acknowledging that learning mathematics “takes more than just a recipe” (Initial Questionnaire, August 22), Kim also acknowledged on the initial questionnaire that she knew “writing is important in every class” (August 22) but stated that she was “not too comfortable with too much required writing” (August 22) in her own classes. When asked about this belief in the first interview, she stated that if she were “required to do it, then it puts a burden on [her]” (September 3). She noted that she could not always fit writing into her lessons because the students were not used to it. Kim expressed the belief that students struggled “enough with mathematics” (FI, September 3) without adding in the writing component. She did report, however, that she had tried writing in her classes before and met with little success. She also commented on the initial questionnaire that she did not “feel . . . qualified enough to grade their writing to help them” (August 22). She explained in the first interview that she felt qualified to grade the mathematics but not the grammar. She stressed, “I don't think I'm truly, truly qualified to say [an answer is] not a very good answer unless it's just mathematics” (FI, September 3).

Not only did Kim question her ability to evaluate her students’ writings, she also questioned their ability to do the writing. She stated on the final questionnaire that she would like for her students to be able to write about mathematics, but she did not believe they were on “the cognitive level to discuss intelligently . . . the mathematics they [were] learning” (December 15). She thought perhaps they could write about basic mathematical topics but conceded that “it would be a stretch” (Final Questionnaire, December 15). She also implied in the final interview that it would be hard to integrate writing into mathematics because students have to know the mathematics well to write about it; therefore, “it’s not something you could [do] right after the lesson” (December 16). She also lamented that her students had problems with writing in general and that if they were required to do detailed writing they would “get lost in their words and they'd focus too much on the words" (TI, December 16). To Kim, her students did not have the vocabulary for the writing and would "lose the mathematics" (TI, December 16) in search of the words.

Although Kim expressed hesitation in using writing in her classroom, she did state in the final interview that she had plans to implement writing in her second-year algebra classes by having "the stronger students write about [the mathematics] and show [their work] to the weaker students" (December 16). She also implied that she might have them write about concepts they had previously studied as a type of review. In addition, Kim had used her write-up on function composition from the course as a teaching aid in her second-year algebra class.

Summary.

Kim seemed to be a source of contradictions throughout the semester. For example, she stated on the final questionnaire and in the final interview that she enjoyed completing the post write-up reflection guides and found them helpful in going back over her work, but she also felt completing them "was like explaining to my boss why I chose each step" (Final Questionnaire, December 15). There was also a certain degree of disconnect for Kim between what she demonstrated in her writing and mathematics and her assessment of where she was with both. When asked in the final interview if she thought completing the write-ups had helped her mathematically, she responded negatively because she liked to think she was already at her "mastery level" (December 16) in mathematics. In her words, she noted "I'm there" (TI, December 16). Her mathematical work, however, did not necessarily communicate that mastery. In addition, she commented in the final interview that "she was able to write intelligently" (December 16) on the write-ups because she knew mathematics and English, but again, her work tended not to substantiate this claim.

To Kim, writing and mathematics seemed to be disconnected concepts in the first place. She implied that she was hesitant to use writing in her own classroom because it had the potential to distract student attention away from the mathematics. To Kim, students could write about mathematics only if they knew it well in the first place. Yet, she relied on "pencil and paper" (SI, October 15) to get her started in her own write-ups and reported in the second interview that she had compiled notebooks of her own lesson notes in her teaching career. For Kim, writing in mathematics seemed to conjure up notions of form and formality with a focus on the end result rather than on the process.

Amy

Background.

Amy entered the master's program after obtaining an undergraduate degree in mathematics from a small southern college. When asked in the first interview why she had chosen to obtain a master's degree in mathematics education, she indicated that she had wanted to teach mathematics since she was in middle school but chose to focus on the study of mathematics during her undergraduate career. She also stated that her first choice had been to study for a doctorate in mathematics education but she discovered that she needed teaching experience to gain entrance into the program. It was at that point that she opted for the master's program. She stated during the interview that she would be satisfied teaching either advanced secondary or college mathematics.

Although Amy was an honor graduate in both high school and college, she described instances in the first interview in which she had struggled with both writing and mathematics in high school and college. She described in detail her experience with writing papers for a high school English class:

> Never really liked it. I used to write my papers on flashcards and I used to cut the flashcard in half, and to make it seem like I was writing a lot cause I hated it, I would write on these flashcards and I'd just keep going and label the numbers. So I got, like, this many flashcards but the paper was only like two pages but it was psychological to me. It was like "Okay, all right, I still got bunch of cards and I can fill this up easily" versus a paper that I got to fill up and that worked better. (FI, September 3)

Despite this struggle, she stated that she left high school believing she was a good writer but after failing a composition course in college, she realized her writing needed work. She noted that she was glad she later retook the class because it made her seek out tutoring and helped her "understand the five paragraph process" (FI, September 3) of writing essays.

In addition to describing her struggles with writing, Amy also noted that she had some difficulties with high school mathematics because of poorly run classrooms. She noted particular frustration with her geometry and advanced algebra/trigonometry classes. These experiences, she believed, had left her deficient in some basic areas of mathematics. She reported in the first interview that she was "still paying for it" (September 3) and had "to backtrack and get that stuff now" (September 3). She also indicated that she seldom wrote in any of her mathematics classes at the secondary or collegiate level except for proof writing and a major senior paper in college. In addition, she commented in the final interview that she had limited exposure to technology in both her high school and college mathematics classes.

Process.

Of the six participants, Amy was the first to complete the 11 write-up packets for the project. She reported during the first interview that she did not like to procrastinate and followed through on that claim by working in a fairly consistent manner throughout the course. She completed 5 of the write-up packets during the first half of the semester and the remaining 6 during the second half. She completed the assignments with 3 weeks left in the semester. In addition, it must be noted that although Amy initially completed the 11 packets, she replaced 1 of her write-ups with another at some point during the semester. She did so without completing the supporting documentation for the project and without reporting the change when I asked in the final interview if there had been any changes to the write-ups during the semester. When asked after the semester had ended why she had replaced the write-up, she stated that she had figured out how to make a particular geometric construction and wanted to post that report instead. She also stated she had simply forgotten to turn in the revised documentation. Because of this change, I used only 10 of her write-ups in the analysis.

Although Amy worked at a steady pace throughout the course, she had experienced a pivotal shift in how she approached the activities by the time she reached the second interview. Because of her frustrations with geometry, she reported that she was no longer planning out her write-ups. Instead, she chose tasks that would not give her “the most hassle” (SI, October 13). She then researched the topics via the Internet or by looking at the work of former students, and completed the write-ups as she explored. Unlike Gwen, who made it a point to explore beyond what the questions on the activity asked her, Amy implied that she simply answered the questions that the activity posed as she explored with the technology.

Before this shift, Amy said she used the write-up worksheet as a planning tool for the first two assignments because she knew enough about the mathematics to plan out what she wanted to write. Indeed, her worksheets for both assignments were handwritten rough drafts of the finished products. When she reached the third assignment, however, she realized that she did not know enough about the mathematics to plan out her writing so she abandoned taking notes and completing the worksheet as a planning tool and started to use the worksheet as a report of what she had done; therefore, she filled out the worksheet after she completed the write-ups.

During the final interview, she commented that she did not find the write-up worksheet helpful and that it was simply a rewrite of what she had posted on the Internet.

The Write-Ups.

Like Kim's, Amy's write-ups indicated problems with both general writing and writing the mathematics. On the surface, however, Amy's write-ups seemed to comply with most of the framework's requirements for general writing. For example, most of Amy's write-ups started with introductory statements and ended with phrases such as "in conclusion." In the body of the write-ups, Amy also provided indicators of transition within her writing such as the use of words like *next* and *now*. Therefore, there seemed to be a structure to her writing. There was, however, a lack of depth to her work reflecting the pivotal shift in her approach to the course that occurred after her second write-up. For example, in her first two write-ups, the goals were clear, concise, and had a degree of specificity. For example, she stated in her first write-up that "the write-up is for students who are looking to learn the graphical behaviors of the equation $x^a + y^a = 1$ when a is a positive integer." After the shift, however, Amy's stated goals tended to lose their specificity. For example, she stated in the fifth write-up that "this write-up is for students learning about tangent circles." There was no indication, however, of what it was about tangent circles the reader would be learning. This loss of specificity can be traced back to the fact that Amy admitted in the second interview that after the second write-up she chose to no longer plan her write-ups but rather to write as she explored. Hence, her goal statements could not specify where she was headed because she did not know.

Although Amy sometimes color coded parts of graphs and broke down the steps in constructions, she struggled in 9 of the 10 write-ups with providing adequate explanations behind her representations and with providing parallel representations and the linkages between them.

For example, Amy attempted in one write-up to explain the construction of a triangle from the medians of another triangle (see Figure 7). In this example, Amy provided an incorrect description of what she had done. Essentially, she had translated the medians of triangle GBD rather than its sides to create triangle ECD. The proper construction is implied in the color coding of her figure; nevertheless, without the benefit of additional explanation, the reader is left to infer what the translation is and how it has occurred.

A median triangle is formed by taking the medians of all sides and then translating the sides to form a triangle see picture below.

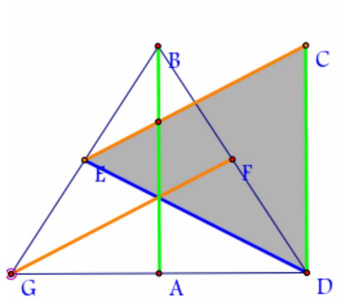


Figure 7. Amy's construction of a triangle from the medians of another triangle.

In another example, Amy provided numerous animations of the graphs of polar equations via Geometer's Sketchpad (Version 4.07) and described how they change as various values in a generic equation, such as $r = a + bcos(n\theta)$, change. Unlike Grace and Lisa, however, Amy did not provide specific examples of the equations. She simply described in general what is happening and provided the graphs. Therefore, it can be argued Amy has omitted parallel representations for the graphs.

In addition to struggling with the writing, Amy did mathematical work that often lacked the depth found in Emily and Lisa's work. In general, Amy offered write-ups that were simple reports of what she observed without much explanation or justification of her results. For example, in one of her geometry write-ups, she offered conjectures such as "the area of the hexagon is twice the area of the original triangle" but offered no explanation or justification to support the conjecture. Because of this lack of mathematical depth, Amy's write-ups did not generally rise to the level of legitimization required by Leinhardt's framework.

In addition to this lack of explanation, Amy's write-ups frequently contained mathematical errors attributable to language. For example, in a write-up about the medians of triangles, she stated, "When using Pythagorean's Theorem to try and prove that the median triangle is a right triangle it does not work because the two sides added together and then squared does not give the hypothenuse [*sic*]." This statement translated into mathematical notation gives $(a + b)^2 = c$ which is a clear misstatement of the Pythagorean Theorem and if interpreted as stated would give an incorrect solution. Whether or not Amy believed this notation was the true representation for the Pythagorean Theorem cannot be determined from the writing without the parallel representation of the notation to go with the sentence which she did not provide. In addition to having issues with mathematical language, Amy also frequently struggled with grammar. Her writings often contained spelling errors, errors in punctuation, as well as problems with subject-verb agreement.

Amy's work also tended to demonstrate possible gaps in her mathematical understanding. For example, she used the measurement feature of GSP (Version 4.07) to verify some of her claims and she seemed to struggle with the concept of domain. For instance, she stated in her first write-up that the domain of $x^a + y^a = 1$, when a is a positive integer, increases when a is even. That, however, is not the case. The domain remains $-1 \leq x \leq 1$ regardless of the value of the even a. Despite these problems, it must be noted, however, that Amy improved in the last 3 write-ups and drew out some of the mathematics involved in the activities. For example, she discussed reflections across the *x*- and *y*-axes as values in parametric equations are changed.

Part of Amy's shift in how she approached the write-ups was a change in her target audience. During the second interview, Amy expressed a concern about being able to write on a "collegiate level" (October 13) which she described as a strict, formal way of writing that a college professor would expect. During the final interview, she commented that her audience for the first two write-ups was the professor in charge of the course. She strove, therefore, to make her first 2 write-ups as formal as possible to meet what she presumed to be the professor's expectations. When she began to struggle with the mathematics, particularly the geometry, she switched her audience to students and altered her views of what collegiate writing means. She commented during the second interview that her idea of what collegiate writing entailed had relaxed somewhat. She now believed that as long as the writing "flows well, if it makes sense, and . . . [you] use vocabulary you know, then it's fine" (SI, October 13). Amy also stated during the final interview that the switch in audiences had made the writing easier. After the switch, she noted that she thought about how she would teach students about the concept and then she would write from that perspective. According to Amy, letting go of her concerns about the professor and writing for students made the writing less stressful for her.

Despite Amy's implication during the final interview that she had made peace with the writing process, she expressed on the final questionnaire what seemed to be deep-rooted angst about the writing:

> At first when I started doing the write-ups I had a hard time because I was not use to doing work in the manner that I was doing it. However, once I found a system that worked for me I did not have a problem with how I wanted to write the write-up. On every write up I had a hard time trying to describe what I saw. I always knew what I wanted to say, but I had a hard time putting it in words that made sense. I had to start actually using mathematical vocabulary and I had to use it correctly. Another problem I had was grammar and sentence structure. I hate writing and I do not know all of the grammar rules, nor do I completely understand them. Therefore, writing the write-ups were a drag for me. I tried to use less word as possible and I tried to keep my sentences short and simple. (November 17)

Clearly, Amy was uncomfortable and frustrated with expressing in words what she knew mathematically. She emphasized this discomfort when asked in the final interview if she would have preferred to do an oral presentation each week rather than post her write-ups. She stated that she would have preferred the oral presentations and that they "would have been a piece of cake" (November 19) for her because she would have not had to worry about grammar. She lamented, "My problem isn't math—it's writing about math or writing, I think, about anything" (TI, November 19).

Amy was also clear about what she learned from doing the write-ups. When asked in the final interview if she thought completing the write-ups had helped her, she stated that the writing had helped to build her "mathematical vocabulary" (November 19). When asked if she thought doing the write-ups had helped her mathematically, she responded, "No, doing the explorations helped . . . with the mathematics" (TI, November 19). She noted that she was a visual learner and that using the technology had helped her learn the mathematics. She then commented that the "writing . . . just reinforced what I learned" (TI, November 19).

Attitudes and Beliefs.

Like Gwen's, Amy's beliefs about mathematics were substantially influenced by the experiences she was having in graduate school. During the first interview, Amy stated that she had initially believed that doing mathematics was like following a recipe "if you do it the way it's taught" (September 3) but now she was starting to understand that there might be alternate ways to approach problems as well. Because of her graduate work, she was starting to see that doing mathematics was about "being able to understand the concept and being able to apply it" (FI, September 3). She credited this change in beliefs to now "looking at [mathematics] from a teacher's standpoint" (FI, September 3) rather than a student's. Although Amy acknowledged a shift in her views, there was some indication she was still harboring a procedural view of mathematics. During the first interview, she stated that one of the reasons she did not like writing was because it had "too many options" (September 3) but with mathematics "they give you this formula or they give you this concept and here it goes" (September 3).

Amy was very candid about her beliefs about writing in the mathematics classroom. She described her position on the final questionnaire:

> I thought before that writing was not necessary in mathematics, but after this class I think it is very important to do some writing just to make sure concepts are clear in the mind and to make sure one can communicate mathematical ideas and concepts in writing. (November 17)

Amy attributed this belief to the work she had done not only for the project but also in her other graduate studies as well. Despite this change, however, Amy also stated on the final questionnaire that she "would not impose a lot of writing on [her] students" (November 17) but she "would make them write reflections" (November 17).

During the final interview, Amy stated that the post write-up reflection guides were "really helpful" (November 19) to her work. She described that the guides forced her to go back and look at what she had done. She found that process not only helpful with proofing her writing but also with going over her mathematical work. During the interview, she commented that completing the reflections had made her go back and "re-understand" (TI, November 19) her work so that she was able to catch errors and refine both her writing and the mathematics when they were unclear. On the final questionnaire, Amy seemed to expand the use of reflections to include requiring "the students . . . [to] explain in words the mathematical concept discussed" (November 17) in class because she thought it would help to deepen their understanding of the mathematics. She noted in the final interview that because of her graduate work she believed that writing in a mathematics class is beneficial "because you have to think before you write"

(November 19) and it helps “you . . . see in your head what you really don’t understand” (November 19). Amy seemed to face this same awareness during the project when she realized she did not know enough about some of the mathematics to write as exhaustively as she wanted.

Summary.

Of the six participants, Amy seemed to struggle the most with the writing and the mathematics not only in terms of what she produced but also with what she believed about both of them. She came to the program with the belief that doing mathematics was about conforming to formulas and rules. Like Gwen, she seemed to like the cleanliness of this procedural view of mathematics but Amy also contrasted it with the often messy nature of writing which she felt was riddled with “too many options” (FI, September 3). Also like Gwen, however, Amy admitted that she was first perplexed by the notion that writing had a place in the mathematics class but that her graduate studies were helping to change her mind. At the end of the semester, Amy noted that she now believed in using written reflections in the classroom.

Although Amy appeared to experience a shift in her beliefs about writing in the mathematics classroom, she nevertheless struggled with her own writing. Neither the writing nor the mathematics came easily to her and she expressed deep frustration with both at different times. She wanted to write exhaustively about the topics but she frequently did not understand the mathematics well enough to do so. She lamented that she did not have a mathematics book to fall back upon during the semester but she did not indicate that she sought one out. Even in those times when she probably did understand the mathematics, however, her writing was often grammatically incorrect or technically imprecise which made it difficult to read and understand.

Strong in General Writing and Writing Mathematics

Claire

Background.

Claire started the master’s program after graduating from college with a degree in statistics. She stated that she decided to become a teacher after spending a summer interning as a statistician for a government agency. During the first interview, she described the experience:

> I didn’t like the environment. I just couldn’t see myself working in an environment like that long-term. You know, I had a cubicle, I had a project, I was analyzing data, and I was bored out of my mind. I just thought, you know, it’s only been two months and I’m bored already and this is not what’s going to make me happy. (September 1)

She also commented during the interview that she had always thought teaching would make her happy but that she had felt compelled to try something else first. After the internship, she knew she wanted to pursue teaching. When asked why she wanted to teach mathematics, she stated that she “had always loved math” (FI, September 1) and had started her college career as a mathematics major but had a bad experience with her first mathematics course. Because she was enjoying the statistics course she was taking at the same time and because it was similar to the study of mathematics, she switched her major to statistics.

Claire entered the master’s program with a stellar academic record. She was the valedictorian of her high school class and graduated with 33 hours of dual credits. These dual credits enabled Claire to complete her degree in 3 years and she graduated summa cum laude in the spring before entering the master’s program. She described in her first interview what seemed to be a rigorous academic background. Because of dual credit and Advanced Placement courses, Claire had taken a substantial amount of writing-intensive courses and advanced mathematics classes in high school. She did comment, however, that except for two-column proofs in geometry, she had done little writing in her high school mathematics classes. When she started to take mathematics classes during her senior year of college in preparation for the master's program, she reported that she had learned how to write proofs and enjoyed the experience. She also stated that she wrote papers in her statistics classes because there was an emphasis on explaining results to non-statisticians.

In addition to extensive experience with writing in school, Claire also commented that she believed writing was important because her parents stressed the importance of being able to communicate through writing. She commented on the initial questionnaire, however, that although she was very comfortable with writing in high school, she was somewhat apprehensive about it now because she had not done much writing since high school. She stated in the first interview that because she had taken her college writing classes in high school, she had written only one major paper in college and that was for a comparative literature class during her final semester. She stated on the initial questionnaire that "as a result, I have become less comfortable with my writing" (August 19) but she still considered it adequate.

Process.

Because of technical difficulties with her computer, Claire started turning in work slower than most other students in the course. By the second interview, she had completed only 1 assignment but she had told me early on that her computer had crashed because of a virus and that she had to go back and recreate her work. Therefore, the bulk of her work was completed in the second half of the course. She completed 10 of the 11 write-ups before she completed the final questionnaire but had completed the eleventh write-up as well before the final interview. Once her computer problems were fixed, she turned in her work on a consistent basis.

Claire reported that she began each activity by choosing a task from one of two perspectives. If the topic was less familiar to her, she would choose tasks that would not take her "too far outside of [her] comfort zones" (TI, December 15). If she was familiar with the topics, she would choose a task that was interesting and that few people had attempted. During the final interview, she described her next step:

> Once I picked [the task], I tried to think through what I wanted to say about the topic, what I knew about it already, what I wanted to explore, and the big thing for me, and I think it's kind of backwards in the writing process, is I almost have to organize before I write. I want to know "Okay, now I'm going to do a section on this. Now, I'm going to do a section on this. Then I'm going to do a section on this" even if I don't know exactly what I'm going to put in there and so [I] try to get an organization laid out. I guess kind of like an outline. (December 15)

She commented that she felt this step was necessary because if she started "exploring without an outline in mind" (FI, December 15) then she could "go way off where [she] want[ed] to go or not come to some reasonable conclusion" (FI, December 15). She also was aware of her audience. In the final interview, Claire stated that "her ideal audience would be high school students who were looking for this information to try and either learn more about it or to figure [out] how to do something" (December 15). Claire indicated on the final questionnaire, however, that if the write-ups had been written only for the professor, she would not have provided as much background information.

During the second interview, Claire reported that after she had come up with the general outline of her write-up, she started to "flesh out" (October 19) what she wanted her audience to know about each section. She implied that at this stage she did the exploration with the technology and took additional notes as she explored. From there, she would start the write-up by first focusing on the content part of the writing and then adding the background information and summary. She noted in the final interview that she wrote the content part first or the "exploration and findings first . . . so based on what I found, I could then write a goal for what I wanted to express from what I found" (December 15).

Claire commented during the final interview that she found it helpful to use the write-up worksheet as a tool to go back through her notes to look for mistakes or holes to insure the information would be understandable to the reader. She noted that she would go back and "superimpose" (TI, December 15) the headings of the worksheet onto her own set of handwritten notes and then proceed to the write-up. She stated, however, that this was not something she did for every write-up. She relied on this process only for "more lengthy write-ups" (TI, December 15) where she had taken a substantial amount of notes.

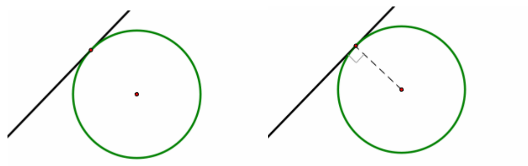
The Write-Ups.

Of the six participants, Claire presented the best work in terms of both the writing and the mathematics. According to the elements of the framework, Claire was strong in both general writing and writing the mathematics. For example, Claire presented write-ups that were tightly organized and well-presented. She made use of sections and headings in each of her 11 write-ups that presented a road-map for her readers to follow and signaled her progression through the presentation. In addition, the representations she used were generally well-explained. Unlike many of her peers, Claire endeavored to explain her graphs and constructions in steps rather than condensing too much information into figures. Therefore, her explanations were often clear and easy to understand as demonstrated by her write-up on tangent circles (see Figure 8). In addition, Claire was also more specific in the animations she created with Geometer's Sketchpad (Version 4.07). She was the only participant to provide an animate button with her linked sketches that allowed readers to animate the sketch without having any prior knowledge of GSP.

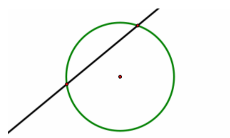
Tangent Circles
By
Claire

A line is said to be **tangent** to a given circle if the line only touches the circle once.

Alternatively, a line is said to be tangent to a given circle if it lies at a right angle with the radius of the circle.



A line is called a **secant** line if it meets a given circle twice.



A circle can be tangent to another circle and be either completely inside that circle, or completely outside of it.

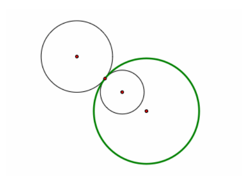


Figure 8. Claire's explanation of tangent and secant.

Although Claire was strong in providing a roadmap in her work as well as in presenting representations, she did struggle in providing a statement of the goal in her write-ups. Of the 11 write-ups, Claire had 3 write-ups that were missing goals and 4 in which the goal was implied by the title or within the document. She did, however, provide introductions with plainly stated goals in 2 of her write-ups. In addition, she implied in the final interview that she had an introduction in mind when she planned out her write-ups; however, several of her writings provided only a presentation of definitions as the introduction.

In addition to presenting her work in an organized manner, Claire also provided substantial justification for the mathematics she presented. Seven of her 11 write-ups provided the strong legitimization that Leinhardt's (1987) framework requires. Claire used "known principles . . . and compelling logic" (p. 227) to substantiate her work in 2 of the write-ups and provided proofs in the other 5. For example, in one write-up, given a point and the slope, she used the point-slope form of the equation of a line to find the appropriate rectangular equation and then again to find and check its corresponding parametric equations. In addition to generally justifying her work, Claire drew out and presented the mathematics in each of her 11 write-ups. As a case in point, she carefully added a presentation on the derivation of the quadratic formula in her write-up on how varying the values of a, b, and c in $y = ax^2 + bx^2 + c$ change the graph of the parabola.

Overall, it was apparent that Claire wanted to provide a clear and understandable presentation of her work. Indeed, she stated in the final interview that she "tried to present more of a finished product" (December 15) than a simple report of what she had done. Therefore, her write-ups tended to be more succinct and organized than those of other students. In addition, her writing was generally grammatically sound except for a tendency to omit periods from the end of sentences. This tended not to be an issue, however, because she used bullet statements in the majority of her writings. As the semester progressed, however, she tended toward the use of longer sentences and mini-paragraphs, and these were generally grammatically sound.

When asked on the final questionnaire what she had learned from doing the write-ups, Claire stated that she had "learned the importance of assumptions" (December 13) and to "consider what assumptions you are making about your audience" (December 13). She implied that it is necessary to include background information so that the reader can follow the explanation. She also stated that it was "difficult to trim the amount of relevant information about a topic into a manageable yet coherent subsection" (Final Questionnaire, December 13) but she believed the "assignments gave [her] practice in deciding what is important to teach, how much background information to give, and what tools to use to best present the information" (Final Questionnaire, December 13). She also stated in the final interview that the fact the write-ups were posted on the Internet helped her mathematically by making her more conscientious in checking her work for mathematical accuracy. She stated that she did not "want to miss some important case that [she] just didn't consider" (TI, December 15).

Attitudes and Beliefs.

Claire entered the master's program with clearly defined beliefs about the teaching and learning of mathematics. She implied a few times during the semester that she believed teachers should promote a conceptual understanding of mathematics. On the initial questionnaire, she described her beliefs about whether or not doing mathematics is like following a recipe:

> I personally disagree with the idea that matematics [*sic*] is like following a recipe. I view doing math as a puzzle that requires the student to use techniques that they already know to solve problems that are unfamiliar to them. I do acknowledge that some teachers teach mathematics algorithmicly [*sic*]. Students of such teachers are more likely to view math as "following a recipe" since they are not asked to solve novel problems, only problems that they have seen before. (August 19)

During the first interview, Claire described the type of mathematics teaching that she believed does not promote mathematical learning:

> You sit on your stool, next to your overhead projector, and you copy the example problem from the textbook onto the board and then you assign them problems 1 through 10 to do in class and then you collect that and then you assign them problems 10 through 30 to do for homework and then you collect that and that's the same routine every day. No group work. No discussion. No room for exploring anything, just "I'm going to show you what the book says to do. Then I want you to repeat it 30 times and then supposedly you'll learn it." (September 1)

In contrast, Claire indicated that she felt the best way to teach students mathematics is to help them "learn how to use the different skills and combine them appropriately" (FI, September 1) so they can successfully respond to new and different situations in the same manner that good cooks do not follow a recipe.

By the end of the semester, Claire appeared to have somewhat refined her beliefs about the teaching and learning of mathematics. She described her thoughts on the final questionnaire:

> To keep with the analogy, if the recipe says it will make a cake, it is important that the students understand the ingredients, how the ingredients work together, why the recipe makes a cake and not a cookie, and how to modify the recipe if they want to make cupcakes. We are doing our students a disservice if we do not teach them to question and ask why in mathematics. (December 13)

Unlike her response on the initial questionnaire, this response includes the requirement of justification which she endeavored to provide in her own write-ups.

In addition to having strong beliefs about mathematics when she entered the master's program, Claire also brought similar beliefs about writing in mathematics. She stated on the initial questionnaire that she believed "being able to communicate about mathematics and being able to communicate ideas in general is an essential part of any good education" (August 19). During the first interview, she specifically noted that in "unmotivated remedial type math" (September 1) writing could be useful to help students learn how to "communicate effectively and specifically to communicate about math" (September 1). She also commented that she had been reading about the use of writing in mathematics classes in her other graduate classes and was getting excited by the idea:

> I think it's such a great idea to have students get in the mindset of communicating and to write out their thought process so they can reference it later . . . and so that the teacher can look at it and have some sort of concrete look into a student's head. (September 1)

Therefore, it was obvious from the beginning of the semester that Claire was a staunch advocate of writing in the mathematics classroom because of its benefits for student and teacher alike.

Claire also stated in the final interview at the end of the semester that her position about writing in mathematics had not changed. She noted that she was still “very pro-writing in mathematics” (December 15). She did indicate, however, on the final questionnaire and in the final interview that one of the benefits of writing in a mathematics class is that it helps students learn “self-questioning” (TI, December 15). She described her own experience with writing for the project:

> I started asking myself those questions as I was writing, “Do I understand everything? What if something equaled zero? How would that affect it?” Internalizing those questions, I think is the most valuable part of the writing experience. (TI, December 15)

Claire also indicated that the post write-up reflection guides had the same effect on her. She stated on the final questionnaire that “it was good to write about why I chose the format and content and it was also helpful in making me think about my own reasoning and writing style” (December 13). She noted in the final interview, however, that she thought she got more benefit from completing the first two reflections than the later ones because on them she started to ask herself “those questions” (December 15) as she was “going through the process” (December 15) so reflecting became a matter of simply writing down her responses on the guides. The implication is that Claire had internalized the process and filling out the guides had become something of an annoyance. Nevertheless, she commented on the final questionnaire that such internalization had influenced her writing process. She stated that “after completing the first couple final write ups, I began to think more carefully about my formatting and the content I chose to include while I was writing” (December 13).

During the final interview, Claire was clear that she wanted to use writing in her own classroom someday regardless of the subject or grade level because she felt "it is crucial that students learn to express themselves in writing even if they're going to have a career that has absolutely nothing to do with mathematics" (December 15). She stated that she believed it was particularly useful in the mathematics class to ascertain whether or not students conceptually understood the material. She noted that she was entertaining the idea of using journals with prompts such as "Explain to your friend Sally why the quantity $(a+b)^2$ isn't $a^2 + b^2$" (TI, December 15).

Summary.

Of the six participants, Claire was the strongest communicator in both her writing and her mathematical skills and she was acutely aware that her designated audience was high school students. Although she generally did not provide a clearly stated goal in her work, she nevertheless presented well-organized write-ups comprised of clear representations and sound mathematics. Over the course of the project, Claire also expressed strong beliefs in promoting a conceptual understanding of mathematics in the classroom. These beliefs seemed to evolve from notions of helping students build the proper tools to solve new and different problems to helping them learn how to justify their methods and solutions.

Claire also deeply believed in being able to communicate through writing—a belief that had been fostered by her parents. She demonstrated that belief by investing a substantial amount of time and effort on the research project as shown by her detailed notes, write-ups, and reflections. For Claire, the write-ups had helped to hone her skills at crafting explanations and in developing self-questioning techniques. She stated in the final interview that the internalization of questions was "the most valuable part of the writing experience" (December 15). She also implied that completing the post write-up reflection guides had provided her with a similar experience. She noted that after she completed the first two guides, she had started to internalize the questions and was able to keep them in mind as she prepared the remaining write-ups.

Mathematical Language and the Integration of Mathematics and Words

Two areas of mathematical writing in which all participants showed signs of struggle were the use of mathematical language and the integration of mathematics and words. Five of the six participants struggled with the use of imprecise or non-technical language in their write-ups. For example, in one of her write-ups, Grace described an ellipse as “tall up and down” or “long left to right” rather than using the terms *vertical* or *horizontal*. In other examples, Kim described an inverted parabola as a “negative” graph, Gwen wrote about the number of “humps” in the graph of a parametric equation, and Grace referred to areas of triangles as congruent. Although a reader with some mathematical knowledge could reasonably infer what the writers intended by using these words and phrases, they are nevertheless informal at best and imprecise at worst. For example, the word *congruent* is used in reference to two geometric figures that have the same size and the same shape. The concept of area is not included in that definition.

The use of mathematical language also evoked different beliefs and attitudes in three of the participants. Amy commented on several occasions that she struggled with the use of mathematical language. She stated during the second interview that “a lot of why I can’t communicate mathematically [is] sometimes I don’t know the language” (October 13). She acknowledged that her problem was “using the right math terminology” (SI, October 13). She believed, however, that creating the write-ups had helped her build her mathematical vocabulary. Kim, in contrast, wanted to avoid the use of technical language. She stated in the second interview that she thought her write-ups were “mathematically written” (October 15) but they were not like textbooks. She thought “textbooks are just too mathematical language” (SI, October 15) and she implied she wanted to “use just normal conversational language” (SI, October 15). In the final interview, she noted that she “wanted to make sure that [her] words were universal” (December 16). In contrast, Claire stated during the second interview that she had recently learned about the mathematics register in one of her classes and implied that it had helped to make her aware of the language she was using in her write-ups. She implied that she needed to make sure that the readers understood the technical language she was using so they could understand her explanations.

All six participants also demonstrated problems with integrating mathematics into sentences. For example, Grace used the phrase “the sum of AB+AC” in one of her write-ups to indicate the addition of two segments. Clearly, the phrase is redundant. It translates as “the sum of AB added to AC.” The most common integration error seen in the write-ups, however, was the omission of punctuation from statements involving both mathematics and words. For example, Claire stated the condition “Let a>b” without providing a period at the end. In another example, Lisa described a parabolic equation (see Figure 9) and omitted the periods after the equations. Such a lack of punctuation within the writing produces run-on sentences that can make the writing seem unfocused and rambling.

Graph the parabola $y = 2x^2 + 3x - 4$

h and k can be calculated using the above formulas giving the vertex form of the equation as

$$y = 2(x +.75)^2 - 5.125$$

Figure 9. Lisa’s description of a quadratic equation without punctuation.

Overall Summary

In this study, the six participants offered varying responses to the writings they created for the class and project. The diversity of responses was frequently based on the type of writing they did, the target audience they selected, and their beliefs about and attitudes toward mathematics and writing in mathematics. The participants' responses also tended to be tied to the level of self-awareness they experienced while engaged in writing and exploring the mathematics. Although their responses were varied in some areas, all participants attempted to explain the results of their explorations and all struggled to some degree with providing effective explanations. Their write-ups indicated that they struggled with providing introductions with clear statements of goals, adequately described graphs, the proper use of technical language, and the integration of mathematical notation, graphs, and figures into sentences. Overall, such issues tended to interfere with their ability to provide an effective explanation of the mathematics. Despite these issues, however, all participants reported that they gained some value from one or more of the three writings they were asked to do and all reported that they would consider using writing in their own classrooms to some degree.

CHAPTER 5

DISCUSSION

In this study, I sought to explore the experiences of both preservice and inservice teachers in a graduate mathematics education course as they engaged in the problem solving processes of writing and doing mathematics. Throughout the study, the participants investigated mathematics with technology and prepared write-ups of their results for Internet publication. During and after this process of exploration and writing, I asked the participants to take notes about their work and to reflect in writing on what they had done. Through the use of initial and final questionnaires, three interviews of each participant, field notes based on weekly class meetings, and the analysis of all writings, I gained insight into how to answer the questions that directed the study:

- How do secondary teachers of mathematics respond to writing to communicate mathematics?
- How do secondary teachers of mathematics respond to reflective writing?
- What do secondary teachers of mathematics believe about the use of writing in the mathematics classroom?

In this study, I defined responses to include both affective and product-oriented components. In other words, I focused on how the participants felt, what they believed, and the quality of the work they produced for Internet publication. In so doing, I focused on their responses to both the private and public components of writing and doing mathematics in which the participants individually made sense of the assigned problems and then prepared their work for Internet publication.

The six participants in this study brought a myriad of responses to the writing they were asked to do both for the class and the project. How they responded was often predicated on the type of writing they were asked to do, the audience for whom they wrote, and the beliefs they held about mathematics and writing in mathematics. Despite these differences, all six participants endeavored to explain to some extent what they had explored or discovered but all struggled in their explanations in some manner. Issues of providing clear goals, adequate explanations of graphs, the proper use of mathematical language, and the integration of mathematics into sentences sometimes interfered with their abilities to effectively communicate the mathematics. Nevertheless, all six found some benefit to the writing and agreed to some degree that they wanted to use a form of writing in their own classrooms.

In the sections that follow, I examine participant responses according to the three types of writing they performed. In the section “Writing to Communicate Mathematics,” I address participant responses to both the public write-ups they completed for the class as well as the informal, private notes they took while exploring the mathematics with or without the aid of the write-up worksheet. In the section “Writing to Reflect,” I examine how participants responded to the post write-up reflection guide. In the section “Attitudes and Beliefs,” I discuss what the participants believed about the use of writing in the mathematics classroom.

Writing to Communicate Mathematics

Process and Preparation

Most of the participant responses regarding the process of writing tended to fall along a spectrum clearly defined by two polar ends in which the participants regarded the writing as either deeply connected to the exploration of the mathematics or far removed from it. The term "writing" is used here to refer to both the notetaking process aided by the use of the write-up worksheet as well as the preparation of the formal write-ups. Gwen and Claire tended to operate at the end of the spectrum where the writing process and the investigation of the mathematics were viewed as strongly connected.

Both Claire and Gwen indicated that completing the write-ups had helped them learn self-questioning skills which in turn helped to propel them deeper into the exploration of the mathematics. For Claire, "internalizing questions" (TI, December 15) such as "What if something equaled zero?" was "the most valuable part of the writing experience" (TI, December 15). Although Gwen commented that she intentionally worked on being more inquisitive because of suggestions made by the professor, she also noted she realized that "as you're writing up [your results], you sometimes realize that there's things that you didn't think to ask when you were just investigating" (TI, December 16). These comments indicate that the writing served as an intrapersonal means of communication for both Claire and Gwen. In essence, preparing the write-ups generated an awareness for Gwen and Claire of the thinking that Sfard (2001) described as a "a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our response" (pp. 4-5). Using writing in this manner to generate and explore their own questions helped both Gwen and Claire "to dig a little deeper" (Gwen, Post Write-Up Reflection Guide #2) into the mathematics. For these two participants, the self-questioning prompted by the writing was a metacognitive behavior which Pugalee (2001) defined as the "monitoring of one's mental activities" (p. 237) that can occur during the problem solving process. Essentially, Gwen and Claire were aware they asked questions about the mathematics as they wrote, worked to refine the process, and found value in the questioning.

Although writing about the mathematics was a “planned and conscious” (Sierpinska, 1998, p. 45) endeavor for Gwen and Claire, Amy and Kim seemed to operate at the opposite end of the spectrum and seemed unaware of any connection between the processes of writing and the exploration of the mathematics. Unlike Gwen and Claire, Amy and Kim tended not to use those “habits of mind” that Cuoco, Goldenberg, and Mark (1996) imply are characteristic of mathematical thinking in which the process of doing mathematics is as valuable as the end result. Specifically, they were not willing to be “tinkerers” (p. 379) with the mathematics by “taking ideas apart and putting them back together” (p. 379) through the use of questioning. Kim did, however, use questioning on three sets of her notes but she did not make reference to the questions during the project as Gwen and Claire did. Therefore, the questions did not seem to play a conscious role in how she processed the mathematics or prepared her write-ups.

Amy, in contrast, seemed to believe that she did not know the mathematics well enough to tinker with it at all. In effect, she seemed to possess an instrumental understanding of the mathematics. From the beginning, she lamented that the course did not provide her with a book thus implying that there was little she could do to learn the math. Essentially, she was “dependent on outside guidance for learning” (Skemp, 1987, p. 163). This unwillingness to tinker is also demonstrated in how she used the write-up worksheet as reported in chapter 4. She seemed to view the write-up worksheet not as a tool of exploration to assist in notetaking but rather as a way to organize her final product. When she realized she did not know the mathematics well enough to organize her work, she abandoned the write-up worksheet as a tool of preparation and simply used it to report what she had done after the write-up was completed. Therefore, she reported that she did not find the worksheet useful.

Amy’s unwillingness to tinker with the mathematics through the informal notetaking process can be contrasted to Gwen’s approach. Like Amy, Gwen struggled with the mathematics, and she was acutely aware that she did not have the mathematical background that the majority of her peers in the class possessed, including Amy. Most importantly, however, Gwen realized, unlike Amy, that she needed to push herself to engage in the exploration of mathematics as the creative process that Cuoco et al. (1996) envisioned. Amy, on the other hand, seemed to want to bypass the exploration and directly arrive at the conclusion and the finished product. Without a textbook to guide her, she rebelled against the cultivation of the type of mathematical thinking that the writing could support. In effect, she refused to become a “flexible thinker with a broad repertoire of techniques and perspectives for dealing with novel problems and situations” (Schoenfeld, 1992, p. 335). In contrast to Amy, Gwen found the write-up worksheet useful as a way to “focus on . . . the different aspects of what [she] was doing” (TI, December 16). Without the worksheet, she commented that she “probably just would have written notes on paper and then ciphered through it” (TI, December 16).

In the middle of the spectrum, Grace and Lisa offered mixed responses to the notetaking process and the preparation of the write-ups. On eight of her ten sets of notes, Lisa indicated questions that arose for her during the exploration process and implied in the second interview that the process of exploring the mathematics and taking notes sometimes created questions in her mind about the mathematics. Although Lisa was aware she was asking questions, she did not seem aware of any connections among the questioning, the notetaking, and the investigation of the mathematics. Essentially, self-questioning did not appear to be a metacognitive behavior for Lisa. She was aware she was creating questions for herself but she did not indicate that she monitored or regulated the process nor did she express an appreciation for the questioning process. For Lisa, the notetaking aided by the write-up worksheet had simply helped her to organize her work which she found useful. In contrast, Grace asked few or no questions on her notes and did not find the write-up worksheet useful at all. She did imply, however, that both writing and mathematics are somewhat related because “math is all about reasoning and logic” (TI, December 7) and the writing was logical in the sense that it helped to organize her thoughts.

According to Bruner (1966), writing and mathematics can be thought of as “devices ordering thoughts about things and thoughts about thoughts” (p. 112). Indeed, all six participants referred to the topic of organization at some point during the project. Grace stated that preparing the write-ups helped her “organize her thoughts a lot more” (TI, December 7). Lisa believed that organizing her thoughts into a written document was the hardest part of the activity for her because she wanted to provide a clear and coherent explanation to her student audience. Claire expressed the desire to have an outline in place before she explored so that her investigations would stay on point. Gwen commented that completing the write-up worksheet had helped to organize her notes and thus her write-up. When specifically asked if taking notes had helped her organize her write-ups, Kim stated that “having the notes in front of me always helped me organize the write-up and where I really wanted to go” (TI, December 16). She also stated that she kept the headings from the write-up worksheet in mind as she took notes and that it provided a map for her to follow in preparing the write-ups. Amy was the only participant who did not mention that the writing in any form had helped her organize her work. On the contrary, as stated earlier, Amy implied that because she did not know the mathematics she could not plan out or organize her write-ups in advance.

Although all participants referred to issues of organization in regards to their work, only two were concerned with their work being published on the Internet. By requiring that students post their write-ups on the Internet, the professor was attempting to draw out and make public their thinking (Franke, Kazemi, & Battey, 2007). Huang, Normandia, and Greer (2005) implied in their study that when students are required to publicly communicate what they have learned, the experience tends to push them into a deeper understanding of the material. Of the six participants in this study, however, only two commented that the publishing of their work on the Internet had an influence on them or their work. Kim commented that the Internet publication made the writing "a little stressful" (Final Questionnaire, December 15) because it was "for everyone in the world to see" (Final Questionnaire, December 15) but she did not indicate that it forced her to do the work more carefully. Claire, in contrast, indicated that the Internet publication had made her want to be sure that her work was mathematically sound and complete. She stated that she did not "want to miss some important case that [she] just didn't consider" (TI, December 15). Essentially, Claire was the only participant to move through Polya's four phases of problem solving and to engage in the fifth phase that I described in chapter 2. In other words, she took seriously that her work would be published on the Internet and it caused her to look deeply at the mathematics and to carefully consider her explanations.

Even though Internet publication did not seem to influence the majority of the participants, most of them had an audience in mind other than the professor when they prepared their write-ups. Essentially, the majority considered that they were being asked to convey information through writing to a specific audience rather than solely for evaluation by the professor (Cohen & Riel, 1989). For Lisa and Claire, the audience was clearly high school students. Gwen confessed that she had not really thought about audience until she was asked about it in the second interview but she, too, considered high school or college students as her target audience. Kim's audience shifted throughout the semester but overall, she stated that she thought she was explaining the information to someone who wanted to know something about the mathematics. Amy, however, experienced a fundamental shift in audience. She reported that she wrote for the professor in the first two write-ups and that that target audience put a considerable amount of stress on her because she felt that any writing for the professor had to be more formal. When she switched her target audience to students, she felt she could be more informal in her explanations. In essence, this position is a reversal of the positions held by Lisa, Gwen, Kim, and Claire who all reported that they would have explained less if the write-ups had been written only for the professor. Grace, in contrast, wrote for her fellow teachers and reported that whether or not it was written only for the professor did not really matter because she always strove to do her best work regardless of who would read it. Therefore, with the exception of Amy and Grace, the fact that the other participants wrote for someone other than the professor increased the depth of explanation that they provided.

As indicated by their choice of audience, all participants approached their write-ups from the perspective of a teacher. Flores and Brittain (2003) stated that “writing serves as a tool to organize the thoughts of prospective teachers about issues related to the teaching of mathematics” (p. 114). For Grace, Lisa, and Kim, the write-ups were resources they could use in their careers. As a case in point, Lisa used two of her write-ups in group work in another class and in her tutoring job. Kim also used one of her write-ups in her own classroom. In addition, Claire, Lisa, and Gwen reported that they felt completing the write-ups had given them necessary practice as future teachers in being able to explain concepts. Emily, Gwen, Claire, and Grace also viewed writing as a way to assess what the students know about the mathematics they are studying.

Quality of Product

The write-ups or products that the students prepared for Internet publication were the center of this study. My goal was to examine each writing to ascertain how well the participants wrote to communicate mathematics overall. As indicated in chapter 4, analysis of the write-ups revealed issues in the use of mathematical language, the integration of mathematics into sentences, and the quality of explanation assessed according to Leinhardt's framework.

The use of mathematical language in an explanation can present a dilemma for mathematics teachers in terms of how technically precise they should be when presenting mathematical information. Essentially, they feel pulls from what can appear as two opposing directions—from the need to use an informal language that students know and from the need to cultivate the proper use of mathematical language. Adler (1997) described this tension as one of the "dilemmas of mediation" (p. 235) in which teachers bear the burden of "shaping informal, expressive and sometimes incomplete and confusing language, while aiming towards the abstract and formal language of mathematics" (p. 236). As indicated in chapter 4, three of the participants in the study specifically responded to this tension in their write-ups between the use of informal language and the use of precise mathematical language. Claire wanted to emphasize the use of technically correct language, Amy realized she struggled with it, and Kim wanted to avoid it. The majority of the participants, however, were informal at best in their writings or at worst imprecise, using words and phrases such as "tall up and down" or "congruent area."

In addition to requiring the use of precise mathematical language, effective mathematical writing also requires the proper integration of mathematics into sentences. Higham (1993) stated that "the mathematical writer needs to be aware of a number of matters specific to mathematical writing . . . such as choice of proper notation . . . [and] how to punctuate mathematical expressions" (p. 12). He noted that "mathematical expressions are part of the sentence and so should be punctuated" (p. 24). In this study, all participants struggled with integrating mathematics into their writing. Omission of punctuation marks, such as periods, was a common occurrence in the write-ups and helped to create writings that were sometimes difficult to read. The write-ups also contained other integration errors such as the proper use of mathematical notation in sentences as demonstrated by the redundant phrase "the sum of AB+AC" written by Grace.

To gauge how well the participants communicated the mathematics, their writings were analyzed according to Leinhardt's features of an oral expert explanation. These features, however, were rearranged to specifically create a framework through which to analyze writing about mathematics (see Figure 1 on p. 35). This restructuring of Leinhardt's framework allowed me to examine both how the participants wrote in general and how well they specifically addressed the mathematics. All participants struggled to some extent with the components of the general writing. Specifically, they all had issues with directly stating the goal of their write-ups in a well-defined introduction. With the exception of Claire, the remaining five participants also struggled with providing adequate linkages between the representations they used in their write-ups. Indeed, in general, most of the participants struggled with providing adequate explanations for the representations they used. They tended to condense a significant amount of information into their geometric sketches and assumed that the reader would draw the correct inferences. Despite these difficulties, however, all participants endeavored to provide some type of map for their work by providing signal monitors of their progression as they moved through their explanation of the mathematics.

In writing about the mathematics, Gwen, Amy, and Kim struggled with providing the level of legitimization dictated by Leinhardt's framework in which mathematical work is justified by "known principles, cross-checks of representations . . . [or] compelling logic"(1987, p. 227). They rarely provided proofs or attained the depth of mathematical explanation that Grace, Lisa, and Claire achieved. However, it must be noted that Grace, Lisa, and Claire were taking a geometry class at the same time they were participating in the project; therefore, they had immediate access to proofs which were helpful for some of the activities in the class. Nevertheless, they tended to provide richer explanations that drew out more mathematics in general than Gwen, Amy, or Kim provided. In addition to showcasing problems with legitimization, the write-ups also tended to expose possible errors in the mathematical understanding of some participants such as the use of measurement to justify geometric claims.

Writing to Reflect

Flores and Brittain (2003) suggested that writing can "allow[s] teachers to look back at their thoughts and reflect on their growth" (p. 114). With this objective in mind, I asked the participants to complete a post write-up reflective guide after each write-up. There were mixed responses to this reflective piece of the project. In general, Grace found little use for the reflections but did state she liked to reflect on how to use the activities in the classroom. Like Grace, Kim tried to find something positive about completing the reflections but implied that she resented having to explain herself time and time again. Gwen, Claire, and Amy, however, found the reflections beneficial to their work. Gwen and Claire both reported that they eventually internalized the questions of the guide which in turn influenced future explorations and write-ups. In other words, both reported that they were able to think about their experiences as they were engaged in the activities and reacted accordingly. Essentially, the post write-up reflection guide was a tool that promoted metacognitive behavior in both Gwen and Claire. For Amy, completing the guides was the most helpful part of the project because it was beneficial to her to go back and "re-understand" (TI, November 19) her work. In addition, Lisa stated that she also liked the fact that the reflections made her go back over her work to insure that she was satisfied with the write-ups. Therefore, completing the post write-up reflection guides seemed to help Amy and Lisa engage in the looking back phase of Polya's (1945/2004) problem solving process.

Attitudes and Beliefs

Using a broad definition of beliefs as "teacher conceptions [or] mental structures" (Thompson, 1992) with Philipp's (2007) definition of attitudes as "manners of acting, feeling, or thinking" (p. 259), I address in this section the participants' views on doing mathematics and on writing in mathematics in general. Claire came to this project with the strongest beliefs that writing should be incorporated into the teaching and learning of mathematics; therefore, she seemed to put the most effort and work into the project and to gain the most benefit from the experience. Claire was also emphatic that she did not subscribe to what is often regarded as a traditional style of teaching mathematics in which mathematics is taught as a list of rules and procedures (Hiebert, 2003). Instead, she subscribed to the belief that mathematics should be taught and learned conceptually in a classroom where students are encouraged to justify their answers. In essence, she believed that mathematics teachers should promote relational understanding in their classrooms in which students know "not only what method worked but why" (Skemp, 1987, p. 158). For Claire, writing was a method that could assist students in learning how to justify their answers and a way for teachers to ascertain whether students understood the mathematics conceptually. Essentially, Claire tended to place value on mathematics as a process. Like Claire, both Grace and Lisa intimated that they held the belief that mathematics was something more than rules and procedures. However, they both seemed somewhat reserved in their attitudes towards writing in mathematics and its use in the mathematics classroom. They both indicated that they found the preparation of the write-ups helpful and thought writing could be useful in the mathematics classroom, but neither was sure how often she would use it.

Gwen and Amy started the program with similar views about mathematics. Both implied during the interviews that they had initially taken comfort in their beliefs that mathematics was "very black and white" (Gwen, FI, September 1) with its rules and procedures (Hiebert, 2003). In addition, both indicated that they did not initially understand how writing could be used in a mathematics classroom. By the end of the semester, however, both reported that their beliefs were shifting somewhat because of their graduate studies. Both spoke of promoting conceptual understanding in the mathematics classroom but Gwen was more enthusiastic about the use of writing as a method to do so.

For Gwen, this shift in beliefs was not a major transition because she wrote well, she enjoyed it, and she believed in what she called the power of writing. Essentially, Gwen's "manner of acting, feeling, [and] thinking" (Philipp, 2007, p. 25) indicated a positive attitude toward writing in mathematics. It seemed a small step for Gwen to connect the writing process to the building of conceptual understanding. For Amy, writing had always been difficult. She did not like to do it, and the most value she could find for writing in mathematics was writing reflections. By the end of the semester, however, she seemed to have expanded her idea of using reflections to include having students "explain in words the mathematical concept . . . discussed" (Final Questionnaire, November 17) in class.

Kim also had beliefs similar to those held by Amy. She, too, tended toward a traditional view of mathematics. She implied that she believed basic mathematics, such as geometry and algebra, did not require conceptual understanding. In addition, she expressed concerns about implementing writing in her classroom because she felt students might “lose the mathematics” (TI, December 16) by focusing on the writing. She implied that writing could be useful as a review tool but not much else. Mewborn and Cross (2007) indicated in their work that how teachers view mathematics can influence the structure and content of their lessons. Arguably in this study, Kim and Amy’s predominantly procedural view of mathematics interacted to some extent with their reluctance to acknowledge writing as a process connected to doing mathematics that could be useful in the mathematics classroom.

Summary of Themes

There were several themes in how the participants responded to the writings. For example, the participants offered mixed views about the writing they did on the post write-up reflection guides. Some found it helpful. Others did not. However, all participants referred to the topic of organization in regards to their write-ups and five of them felt that the writing had helped them to organize their thoughts. In addition, having to publish their write-ups on the Internet seemed to only push one participant to a greater consideration of her work but four participants implied that writing for an audience other than the professor had helped them explain their work in greater detail. All participants also tended to write from the perspective of a teacher and several of them noted that they felt writing could be a useful method to assess the mathematical understanding of their students.

However, what the participants believed about the use of writing in the mathematics classroom tended to be related to how connected they viewed the writing to be to the exploration of the mathematics. For the two participants who tended to view the writing and the exploration of the mathematics as intertwined, using writing in the mathematics classroom was imperative. For the two participants who saw some connection between the writing and the mathematics, writing was useful but not necessary in the mathematics classroom. The two participants, who saw the writing and the exploration of the mathematics as loosely connected, if at all, expressed a reluctance to use writing in their lessons. These associations also tended to correspond to the beliefs the participants held about the nature of mathematics. For example, the four participants who saw some connection between the writing and doing mathematics believed in promoting a conceptual understanding of mathematics in the mathematics classroom. The two participants who saw little or no connection between the writing and doing mathematics expressed views about mathematics that tended to be procedural and instrumental in nature.

Flores and Brittain (2003) suggested that “teachers probably will not use [writing] . . . unless they have had the experience themselves of writing in relation to mathematics” (p. 112). However, this study highlights the notion that the quality of the experience teachers have with the writing may also be a factor in what they glean from the activity. This quality of experience in turn appears to be related to the level of awareness teachers bring to the writing activity as well as the level of awareness that the writing invokes. Of all the participants, two seemed most open to the experiences of the project. These two participants were able to use the notetaking process, the write-ups, and the reflections to promote metacognitive behavior that helped them both with the exploration of the mathematics and the development of their writing skills. Essentially, they were aware of what they were doing and learning throughout the project and seemed fully engaged in the overall process. This heightened level of awareness distinguished them from the rest of the participants and made them the ones most likely to consistently use writing in their own classrooms.

What seemed essential to the engagement of these two participants, however, were not their beliefs or attitudes about mathematics but rather their beliefs and attitudes about writing. This connection was particularly highlighted in the main difference between these two participants. One came to the project with strong beliefs in promoting a conceptual understanding of mathematics, but the other readily admitted that she started her master's program believing that mathematics was a “black and white” subject in which writing had no place. Because the second participant enjoyed writing and believed in its power to support her learning, however, she was willing to explore connections between the process of writing and the exploration of the mathematics. In so doing, her procedural views of mathematics tended to give way to new conceptual views. Arguably, the converse could occur as well—that if teachers become aware of a connection between the process of writing and doing mathematics then their beliefs about and attitudes toward writing in mathematics could shift toward the positive. No participant in this project exhibited this awareness without an initial belief in or a positive attitude about writing in mathematics.

In addition, this study suggests that a single type of writing in one classroom or one lesson may not be adequate to engage teachers in the writing process. For the two participants most engaged in the study, completing the write-ups and the post write-up reflection guides engendered the metacognitive behavior of self-questioning that refined both their writing and mathematical skills. One also spoke favorably of the write-up worksheet as a way to organize her work. Although the rest of the participants did not exhibit the level of awareness that these two participants demonstrated, they nevertheless found different writings useful. Of the three types of writing included in the study, one participant found the reflections the most useful and two indicated that the headings of the write-up worksheet helped them organize their write-ups. In addition, two participants spoke most favorably about the write-ups. Essentially, by using three types of writings in the project, I cast a wider net of engagement for the participants, and each participant was able to find a form of writing with which she was comfortable and to which she responded best.

How teachers affectively responded to writing in mathematics, however, was only half of their overall response. The other half was how well they communicated the mathematics through the write-ups they prepared for class. Those participants who were clear that they wrote to students endeavored to produce explanations that provided adequate background knowledge and basic information. Despite these endeavors, all participants struggled with aspects of providing an expert explanation as specified by Leinhardt's (1987) framework.

Aspects of this study, however, indicate that the framework should be modified to specifically address the demands that writing to communicate mathematics entails. For example, in the category of general writing, I observed in this study that it was not enough simply for the participants to provide linkages between representations but they also needed to adequately explain the representations as well. In addition, I found that there were three issues that were a unique blend of the two categories of general writing and writing about mathematics. First, the readability of the write-ups was an issue in some cases because of the number of major grammatical errors that the writings contained. Major grammatical errors in this sense were a common use of sentence fragments and run-on sentences. There was also an aggregation of minor errors, such as spelling, that rendered some write-ups difficult to read. The lack of proper integration of notation and figures with words also frequently created the run-on sentences because of the omission of punctuation. Finally, the lack of formal mathematical language played a role in how mathematically unsophisticated the write-ups sounded and the use of incorrect terms indicated gaps in the mathematical understanding of some participants. Based on these results, I modified Leinhardt's (1987) framework to better reflect what writing an expert explanation in mathematics entails (see Figure 10). Just as Leinhardt assumed in her framework, I assume in this modification that the mathematics offered in the explanation is correct. As the results in this study indicate, if a written explanation contains mathematical errors then it logically follows that the writer has not effectively communicated the mathematics.

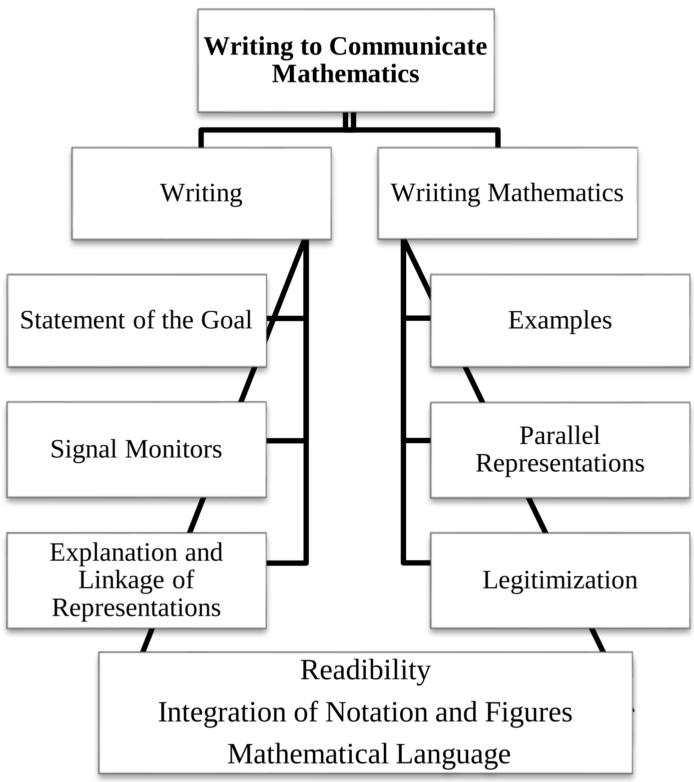


Figure 10. Modified Framework for writing to communicate mathematics using the features of an expert explanation described by Leinhardt. Adapted from "Development of an Expert Explanation: An Analysis of a Sequence of Subtraction Lessons," by G. Leinhardt, 1987, *Cognition and Instruction, 4,* pp. 226-227. Copyright 1987 by Lawrence Erlbaum Associates.

Implications

This study has two major implications. First, if mathematics educators want to help preservice and inservice teachers become promoters of writing in mathematics then they must seek out and use teaching and learning methods that heighten student awareness of the connections between the processes of writing and doing mathematics. And, if writing in general is to stand a better chance of being promoted in the mathematics classroom, mathematics educators must find ways to promote in preservice and inservice teachers the level of awareness of the connection between writing and doing mathematics that both Gwen and Claire exhibited. Cooney (1998) stated that the consideration of teacher beliefs "enables us to create activities that encourage teachers to wonder, to doubt, to consider what might be, to reflect, and most important, to be adaptive" (p. 332). Mewborn and Cross (2007) implied "helping teachers . . . become aware of their beliefs" (p. 262) increases the chances that those beliefs can be altered in ways that may lead to eventually improving students' opportunities to learn mathematics. Although teacher beliefs are often difficult to change (Pajares, 1992), seeking out those activities that provide for rich experiences with both the mathematics and the writing may help in promoting the level of awareness in teachers that Gwen and Claire showed during this study. Essentially, mathematics educators need to seek out those activities that use writing to support metacognitive behavior in exploring mathematics.

One way in which that awareness can be raised is to help teachers see how much informal writing they can do during the notetaking phase of exploring the mathematics. For three of the participants in this study, taking notes during the exploration phase of the mathematics seemed to be an integral part of their private processes but all three were somewhat equivocal about regularly using any type of writing in their own classrooms. They did not seem to regard this notetaking process as a form of writing and appeared to view writing as the generation of a formal document. This observation leads to the supposition that if they had been made aware of the writing in their notes then their views about what it means to write in mathematics might have been altered. The lesson here may be that students do more informal writing than they think to process mathematics.

Second, mathematics educators cannot assume that teachers know how to write effective mathematical explanations. As students, teachers may need direct instruction in what it means to target an audience, state the goal in a well-defined introduction, link and explain representations, and to properly integrate mathematical notation and figures with words. They may also need to be encouraged to view mathematical language as mathematical content. Ball and Sleep (2007) indicated that “mathematical language is both mathematical content to be learned and [a] medium for learning mathematical content” (p. 13). Therefore, the use of proper mathematical language should be promoted in a mathematics classroom with the same attention to detail as other mathematical content. In addition, teacher educators could use the Modified Framework (see Figure 10, p. 124) in the classroom to help preservice and inservice teachers learn how to construct and assess writings that effectively communicate mathematics.

There is, however, more work to be done. More research is needed in search of methods that will help teachers build an awareness of and appreciation for writing as a thinking skill that has merit to promote mathematical understanding at both the informal and formal stages of writing. Researchers need to examine the influence of various methods to help teachers as students make the connections to the mathematics they explore and explain through writing. In addition, the Modified Framework needs to be tested to ascertain the influence it may have in helping teachers to become effective communicators of mathematics through writing and the extent to which it fosters an appreciation for the overlap between the writing process and doing mathematics.

This study also highlighted the need for case studies in which two teachers are compared and contrasted as they respond to writing in mathematics. For example, in this study, Gwen and Amy were different in terms of educational backgrounds and beliefs about writing but initially similar in their beliefs about the nature of mathematics. Examining in-depth where their responses to writing in mathematics converged and diverged in light of their backgrounds and beliefs could help researchers find and highlight those factors critical to helping teachers build a positive attitude toward writing in mathematics.

Conclusion: Overcoming the Divide

Despite the research done noting the value of writing in the mathematics classroom, many mathematics teachers remain reluctant to use it as Weiss, Banilower, McMahon and Smith (2001) reported in their work. At the heart of this reluctance may be the fundamental misunderstanding that the processes of writing and of doing mathematics are distinctly unrelated. However, research suggests that both writing and doing mathematics can be characterized as problem solving processes in which students engage to craft products they often share with others. Therefore, both processes can engage students at a private level in which they make sense of the given problem and at a public level in which they formally communicate the results of their work. Specifically, the results of this study indicate that when the writing process is used at a private level to support metacognitive behaviors essential to problem solving, the distinctions between the writing and doing the mathematics can blur under the general heading of critical thinking.

Cuoco, Goldenberg, and Mark (1996) urge a refocusing of the study of mathematics not necessarily on content but on those habits of mind that manage the content. Schoenfeld (1992) advises that the focus of mathematics teaching should be on teaching students how to think mathematically. But arguably those habits of mind and methods of thinking mathematically can transcend the boundaries of mathematics. In writing as well, students need to learn how to experiment with ideas, conjecture about different methods and solutions, and how to tinker with different ways of viewing and experiencing the world. Indeed, both the mathematics teacher and writing teacher should heed Schoenfeld's (1992) call to develop "flexible thinkers" (p. 335) who realize that their respective endeavors are not simple procedural pursuits but rather "complicated intellectual [processes]" (Flower & Hayes, 2009, p. 467) that are uniquely the students' own. Such ownership helps to break down the dyad of teacher and student and empowers the student to freely take on the title of writer and mathematician.

In the article "Who Owns Writing?" Douglas Hesse (2009) wondered if the word *writing* conveys a less than desirable message—that writing is only "the physical act of graphemic production" (p. 1253) and nothing more. Perhaps, the same argument can be made about the use of the word *mathematics*—that to most people it means only numbers and nothing more. However, if more teachers are to find the motivation to use writing in their mathematics classroom, these superficial notions of what it means to write and to do mathematics must be transcended and merged into a single idea of what it means to think critically and creatively. In so doing, mathematics teachers would be freed up to use writing as a form of critical thinking that can help students learn how to think mathematically.

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APPENDIX A

INITIAL QUESTIONNAIRE

Name:______________________________ Date:_______________________________

Directions: Please respond with as much detail as possible in the space provided. Feel free, however, to make your answers as long as you want. Additional pages are more than welcome.

1. Please state your major and minor in college.

2. If you are an inservice teacher, please state the number of years you have taught. In addition, please indicate the subjects you have taught.

3. Some people think doing mathematics is like following a recipe. Do you agree or disagree? Please explain your answer.

4. Were you comfortable with writing in your high school and college classes (other than mathematics) or was it a struggle for you? Please describe your experience.

5. As a student, have you ever used writing in any of your mathematics classes either in high school or in college? If so, please describe the type of writing you have done in these classes – e.g., lists, proofs, journals, etc.

6. How do you feel about writing in a mathematics class? Do you have any concerns about doing the write-ups required for this class?

7. How do you feel as a teacher about having students write in your mathematics classes? Is it an activity you would consider using? Why or why not?

APPENDIX B

FINAL QUESTIONNAIRE

Name:______________________________ Date:_______________________________

Directions: Please respond with as much detail as possible in the space below the question. Feel free to make your answers as long as you want. Additional pages are more than welcome.

1. Describe your experience with the write-ups in this class. Describe what you have learned from writing up the assignments.

2. If the write-ups had been written only for the professor and not posted on the Internet, would that have changed how you approached the write-ups? Please explain your answer.

3. Did completing the post write-up guides assist you in your work? If so, please describe.

4. Now that you are nearing the completion of this course, how do you feel as a student about writing in a mathematics class? Have your thoughts and feelings changed? Please describe.

5. How do you feel as a teacher about having students write in your mathematics classes? Is it an activity you would consider using? Why or why not?

6. Some people think doing mathematics is like following a recipe. Do you agree or disagree? Please explain your answer.

APPENDIX C

WRITE-UP WORKSHEET

Write-Up #:___________ Name_________________________

Write-Up Worksheet

Goal:

Exploration:

Findings:

Conclusion:

APPENDIX D

Name: Date:

POST WRITE-UP REFLECTION GUIDE

Describe your experience in exploring the mathematics in this lesson.

Describe your experience in writing up this activity.

What did you learn that you could use in future mathematical explorations?

Describe any ideas you have about how any part of this activity can be used in a high school mathematics class.

APPENDIX E

INTERVIEW GUIDE

Interview Questions:
1. Tell me about your academic experiences in school.
Possible Probes:
- Describe the high school you attended – overall size, rural vs. urban, class size, etc.
- Describe your academic performance in high school.
- Describe your academic performance in college.
- What were your favorite classes in high school and why?
- What were you favorite classes in college and why?

2. (For Inservice Teachers) Describe your teaching job.
Possible Probes:
- Describe the school where you teach.
- What subjects do you teach?
- How big are your classes?
- Describe a typical lesson in your class.

3. Describe your experiences with writing in school.
Possible Probes:
- Describe the type of writing you did in high school and in what classes.
- How did you feel about the writing at the time?
- Describe the type of writing, if any, that you enjoyed or did not mind doing.
- Describe the type of writing that you found difficult to do or did not enjoy.
- Tell me about your experiences with writing in your college classes.

4. Tell me about your experiences with writing in mathematics.
Possible Probes:
- Describe the type of writing, if any, that you did in your high school mathematics classes.
- As a high school student, how did you feel about writing in mathematics?
- Describe the type of writing, if any, that you do or have done in your college mathematics classes.
- As a college student, how have you felt about writing in mathematics?

5. As a future (or current) mathematics teacher, tell me how you feel about using writing in your mathematics classroom.
Possible Probes:
- If you have any concerns about using writing in your classroom, please describe them.
- What types of writing, if any, would you consider using? Why or why not?

- Describe a situation in which you think the process of writing might help a high school student in mathematics.

6. Describe your experience in exploring the mathematics and writing up the activity.
Possible Probes

- What did you learn mathematically?
- Describe your experience with exploring the mathematics.
- Are you seeing improvement in how you are learning the mathematics? Please describe.
- Describe your experience with writing up the activity.
- Are you seeing improvement in your writing? Please describe.
- Have you noticed any changes in how you feel about mathematics? If so, please explain.
- Have noticed any changes in how you feel about writing in mathematics? If so, please explain.
- If you were to use this activity in your classroom, would you change it? If so, explain how and why.

7. Possible questions based on participants' written work such as….

- On your worksheet (or questionnaire) you stated that "…." Please explain.
- On your write-up, you indicated that…. Please explain.

** Additional questions may be based upon participant responses.