STUDENTS' CONJECTURAL OPERATIONS

by

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(Under the Direction of Leslie P. Steffe)

ABSTRACT

In taking a scheme theoretic approach to studying mathematical conjectures, this paper examines the conjectural operations that underlie conjecturing activity. It provides answers to the following two questions: How might conjectural operations engender accommodations in schemes within a cognitive system, and how do students develop a conjectural disposition? In answering these questions, the paper reports on the conjecturing activity of four sixth-grade students, working in the context of fractions problems and using software called `TIMA:Bars`. The researcher also served as the students' teacher during two teaching experiments, one with each of two pairs of students. The teaching experiments were conducted twice per week for one school semester.

Findings include that conjectural operations, themselves, sometimes serve as functional accommodations in schemes. At least one kind of conjecturing operation, generalizing assimilation, can modify the trigger of a scheme and raise student awareness about new constraints to students' ways of operating. Abduction was another kind of conjectural operation and often reorganized existing operations into new patterns for operating. Conjectural operations occurred among all four students and served in actualizing their zones of potential construction.

Not all conjecturing activity was constructive. Several affective and environmental factors contributed to differences between students' success in constructing new ways of operating through conjecturing activity. The role of the teacher in designing appropriate tasks and interpreting students' actions appropriately was particularly important. Differences between the students' initial levels of development were a factor in the particular schemes that they constructed, but such differences did not appear to determine the ways in which conjectural operations were used, nor the constructiveness of their use.

INDEX WORDS: Abduction, Accommodation, Conjecture, Development, Fodor's Paradox, Fractional Scheme, Fractions, Hypothesis, Generalizing Assimilation, Learning, Mathematics, Operations, Scheme Theory, Splitting, Teaching Experiment, TIMA, Zone of Potential Construction, Zone of Proximal Development

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TABLE OF CONTENTS

Page

Chapter 1: Introduction

Research Questions

Cognitive learning is increasingly understood as the gradual refinement of complex structures involving multiple components (Schoenfeld, Smith & Arcavi, 1993; diSessa & Sherin, 1998; Steffe, 2002). Introducing fine-grained structural models allows researchers to delve into the recesses of human thought, taking specific mental operations as basal units. For example, diSessa and Sherin demonstrated that students' concepts could be modeled by "systematic collections of strategies" (1998, p. 1155), called coordination classes, in order to "gain insight on how to improve learning" (p. 1161). In their Fractions Project, Steffe (2002) and Olive (1999) outlined trajectories for fraction learning based on the coordination of cognitive schemes, each of which can be defined as a three-part cognitive structure: a set of triggers that call the scheme, a set of mental operations that are triggered, and a set of expectations for the result of these operations (Glasersfeld, 1998). Both approaches employ general, elemental structures that can be coordinated and developed to produce various concepts, which might otherwise be considered innate. In particular, the second approach presents conceptual learning as accommodating one's schemes in response to cognitive perturbations.

In my study, I examined students' conjectures, or "informed guesses" (NCTM, 2000, p. 57) as occasions for learning, and I adopted a scheme-theoretic approach largely because of the valuable context that Steffe & Olive's study (1990) provides. Moreover, because mathematical schemes are operative structures, I could use scheme theory to study students' underlying conjectural operations: mental operations used to resolve a problematic situation in which the effectiveness of their use is not already established. How might conjectural operations engender accommodations (novel creation, modification, and coordination) of schemes within a cognitive system? My method for approaching this question informed a second question: how do students develop a conjectural disposition?

Before going further, I should clarify what I mean by *operations*, and introduce my general method for studying them, which is further elaborated in Chapter 3. While *actions* refers to students' behaviors (including verbalizations) that a teacher-researcher can observe, operations are the cognitive constructs that those actions may represent. For example, a student's action of folding paper in half may represent the operation of geometric reflection. As a teacher-researcher, I made inferences about students' available operations, operations that may explain students' actions in various situations. I then designed tasks to determine whether inferring a hypothetical operation for a particular student was useful in explaining her actions. By focusing my research on students' operations, instead of their actions alone, I have developed a unique approach to studying conjecture and answering my research questions.

One possible answer to my first research question is that conjecturing activity does not change schemes. In fact, Fodor's paradox suggests that the process of making and testing conjectures can never yield a cognitive system that is richer than the one currently held because all possible conjectures must potentially exist in the system a priori and are simply triggered by experience. Fodor claimed, "there literally isn't such a thing as the notion of learning a conceptual system richer than the one that one already has" (Fodor, 1980, p. 149) if learning consists of the inductive-deductive process of "hypothesis formation and confirmation" (p. 148). To the extent that *hypothesis formation* and *conjecture* can be used as synonyms, it seems that, in relevance to my study, Fodor presented the following two tenets of learning: Learning occurs through conjecturing, and conjecturing is an inductive process. While some research has demonstrated other kinds of learning (Steffe, 1991b; Arzarello et al, 1988), most current research in mathematics education on conjecturing seems to fall in line with Fodor's second tenet, that conjecturing (hypothesis formations) is inductive. My claim is supported by the relatively fruitless results of an extensive literature search for exceptions, which are included in the Chapter 2. But even Fodor did not espouse the first tenet: "What I think [the paradox] shows… is that there must be some notion of learning that is so incredibly different from the one we have imagined" (Fodor, 1980, p. 149). Presently, I attempt to elaborate on such a notion of learning that does involve conjecturing.

Supposing the tenets that Fodor presented, that learning consists of conjecturing and that conjecturing is inductive inference, were accepted, Steffe agreed with Fodor's logical conclusion formulated in the learning paradox, implying that all knowledge is innate—radical innativism (Steffe, 1991b, p. 26). However, Steffe introduced a counterexample to the first tenet (thus undermining the second one): a metamorphic accommodation that transformed a boy's non-numerical counting scheme to a numerical one (Steffe, 1991b). Metamorphic accommodations occur through (possibly subconscious) reflection, in response to the perturbation caused when one's schemes are operating at different levels of abstraction. A detailed account of this research must be relegated to the Chapter 2; for now, suffice it to say that the learning paradox can be avoided by introducing Steffe's model of learning by metamorphic accommodation. Such accommodations will be of importance to my study of conjectures because Steffe has demonstrated that they can be engendered by novel operations in problem solving.

The alternative to the learning paradox suggested here is not as categorical as that of Steffe, who dismissed Fodor's two tenets together. While I do not necessarily agree that all learning is conjectural, I contend only with the second tenet by developing a new model of conjecturing that is not inductive. I pick up a path laid by Glasersfeld who argued against radical innativism by saying that "instead of remembering innate 'true' ideas, the child has the innate tendency to search for 'rhythms, regulations and groupings' and to test constructs for viability in actual experience" (1998, p. 5). As for the psychological mechanisms that allow the child to act on those tendencies, "what has to be assumed innate is no more than the capacity to remember experience, reflect on it, and to make comparisons" (p. 7). Rather than viewing these processes as inductive, Glasersfeld pointed to what he called "the mainspring of creativity" (p. 10)—Peirce's pattern of abduction.

An abduction is a logical inference whereby one links a surprising result to a general rule by supposing that the result follows as a particular case of the general rule (Peirce, 1998, p. 231). It is a sort of reverse deduction because deduction applies a rule to a particular case yielding the result of the particular case as a logical conclusion, and, in abduction, the inferences are reversed. For instance, upon returning home from a trip, one might notice a puddle of water in the middle of the kitchen. He might then assume that there is a leak in the roof because the assumption of this hypothesis would explain, as a matter of consequence, the surprising observation that there is water where there should not be. Peirce attributed the abduction of a general rule (or hypothesis) to an instantaneous and “extremely fallible insight” (p. 227). “It is true that the different elements of the hypothesis were in our minds before; but it is the idea of putting together what we never had before dreamed of putting together which flashes the new suggestion before our contemplation” (p. 227).

Peirce’s formulation still falls into the trap of the learning paradox because it does not explain the creation of new ideas, only the *putting together* of existing ones. Thus, the existing ideas and their potential to be put together would remain innate. However, if we apply the *pattern* of abduction at the operational level, it may help explain how accommodations of schemes (learning) may occur. Moreover, because the pattern involves generating constructs that are untested resolutions to problematic situations, they may be considered conjectural. So abducting—the process of producing, through the pattern of abduction, new operational constructs—is of central interest to my study of conjecturing, although I do not assume, as Peirce (1988) did, that the processes are equivalent. I use the following vignette taken from Fractions Project data to demonstrate how an operational perspective on Peirce’s pattern can be used to explain the production of a “general rule” or conjecture, and thus provide a new alternative to the learning paradox.

Two Cases

A teacher, working with a pair of fifth-graders using computer-based fraction sticks, asked the students to use three-fourths of a whole stick to make parts that would be eighths of the whole stick. Joe, one student in the pair, had been successful in completing similar tasks only with unit fractions, such as one fourth. In the new situation, he divided each of the three fourths into two parts. Figure 1 illustrates the resulting six-eighths stick, as well as the four-fourths stick from which he had pulled three fourths. Assuming his reasoning was similar to his previous reasoning when making new fractions out of one fourth, we can infer from his actions the existence of a *reversible unit fractional composition scheme* that can be characterized in the following way: “To create eighths, I'll cut each fourth into two pieces because two times four is eight.” Assimilating the new situation with this existing scheme seemed unproblematic for Joe until Melissa, the other student of the pair, began counting the parts that Joe had created. This prompted Joe to count the six parts, at which point he experienced a perturbation because his scheme seemed to include the expectation of having created eight parts.

The following transcription includes the actions of the two students, the teacher, and a witness, whose main role was to observe and provide feedback on the teacher’s interactions with the students. Note that the computer program was designed to allow student actions similar to those allowed with string, ruler, tape, scissors and markers; in addition, the program included a built in measuring tool that would display the fractional part of any piece relative to a specified whole.

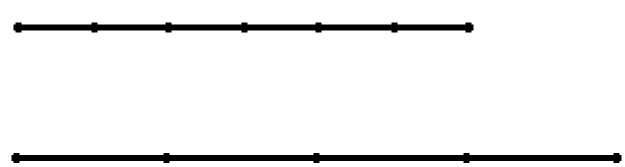

Figure 1. Joe’s construction of eighths from three-fourths of a stick.

Protocol I: *Joe's abduction*.

Teacher: How'd you do that, Joe? You know that's right?

Joe: [Shrugs his shoulders] It is.

Melissa: [Melissa begins counting each of the pieces created within the three-fourths stick.] One, two, three, four, five, six.

Joe: [Joe follows Melissa's counting activity, begins to look worried, and recounts.]

Teacher: How many? Is each little piece one-eighth of the big stick, Melissa?

Joe: No. No. [Joe appears disturbed or even embarrassed, with his head resting on his left hand.]

Witness: Pull a part out Joe and measure it.

Joe: [Pulls out one of the little pieces, measures and sees, with surprise, that the computer displays the measure as "1/8."]

Teacher: You're right Joe. Honest to good correct!

Joe: [shaking excitedly with a big grin]

Teacher: Now you tell us how you thought that out…

Joe: I don't know… like if I put it on two parts and do it like that [pointing to each of the fourths within the six-eighths piece] then it will become six, but this one will add on to it [pointing to the rightmost fourth in the four-fourths stick]. That's what *I* thought.

I claim that Joe's explanation indicates an abduction resulting from conjectural operations that modified his existing scheme. From the start, Joe realized that the three-fourths stick was embedded in a whole stick because he had produced the three-fourths stick by pulling out copies of pieces from the whole. Still, his reversible unit fractional composition scheme did not make use of disembedding three-fourths from the whole in establishing the expected number of parts that the scheme would create. This omission is indicated by his doubt after Melissa's counting activity and may be attributed to the taxing activity of solving the novel task. After finding that he had indeed created eighths, his disembedding of three-fourths from the whole was germane to explaining why he had created six parts instead of eight. By adapting his existing scheme to include this use of the disembedding operation, the surprise that he had made six rather than eight parts became a matter of consequence. In other words, Joe's inclusion of the disembedding operation into the scheme created a conjecture that fit the pattern of abduction and modified the existing scheme. While the disembedding operation was well-established for Joe, its role in the existing scheme was not. Its inclusion modified the existing scheme into the kernel of a new scheme (*a reversible fractional composition scheme*) for dealing with non-unit fractions, such as three-fourths.

The emergence of a new scheme is corroborated in the next protocol of the same episode, in which Joe and Melissa were trying to create twelfths from three-fourths. When Melissa partitioned each of the three fourths into four pieces, Joe claimed she was making sixteenths. However, he remained uncertain, admitting that Melissa's idea "might be right" in creating twelfths, just before she measured to find that each piece was in fact one-sixteenth. He had incorporated his disembedding operation into his existing scheme, resulting in a new scheme that received confirmation in this subsequent situation. Since this new way of operating was relatively permanent, albeit still uncertain, we can infer the emergence of a new scheme.

Glasersfeld referred to abductions as "accommodations… done consciously" (1998, p. 9). In the transcribed protocol, Joe became aware of the relation between the three-fourths part and the whole and his past experience of pulling three fourths from four-fourths, which were crucial elements of his abduction of the disembedding operation. Although Joe did seem aware of the abduction, I only consider the *pattern* of abduction from the *observer's* point of view and do not insist that abducting, nor the broader range of conjecturing activity, be done within a student's awareness. The pattern of reasoning that I developed from my observation of Joe's actions fit that of an abduction because the incorporation of disembedding within his reversible fractional composition scheme resolved the perturbation caused by the disparity between the observed and expected number of parts.

It is important to note that the modification to Joe's scheme occurred through the novel use of an operation. This offers a partial solution to Fodor's paradox (i.e. that conjectures can be created through operating). However, one could argue that this only translates the paradox to the level of operations: people have a limited number of innate operations that can be combined and enacted in order to produce some set of available conjectures, so how can students learn to operate in truly novel ways? To confront such an argument, one needs only to recognize that children construct novel operations through reflective abstraction of experience in their first few years (Piaget, 1977), and there is no reason to believe that this development ceases in adulthood.

In Chapter 2, I will provide a detailed account of reflective abstraction and the production of operations, along with related theoretical constructs such as interiorization and metamorphic accommodation. Using those theoretical constructs, I demonstrate that the concepts on which we operate, which result from operation and contain operations themselves, are by no means static; they continually change through the experience of operating (i.e., operations change through operating). The following example illustrates how concepts (and their associated operations) can change and provides an example of another (non-abductive) sort of conjecture, further refuting the new formulation of Fodor's paradox described above.

In another protocol with Joe and Melissa, Joe was asked to use ninths to make a bar that would be "just a little bit larger than eight-eighths." Joe responded that nine ninths would work, but his tone and subsequent actions indicated that he was not certain; thus, I classify his statement as a conjecture. He tested his conjecture by making a nine-ninths bar and visually comparing it to a previously produced eight-eighths bar. When he found they were the same size, he counted up the parts in each. This action indicates that his conjecture was based on whole number comparisons, where the units were ninths. Note that this protocol actually occurred a few weeks before the one involving Joe's reversible fractional composition scheme.

Protocol II: *Joe's concepts of eighths and ninths*.

Teacher: How many ninths would you need so that it would be just a little bit longer?

Joe: *Nine* ninths.

Teacher: Well try it! …and see if it's right.

Teacher: You think he's right, Melissa?

Melissa: [Nods shyly but affirmatively.]

Teacher: Let's see if you're right, Joe.

Joe: [His expression did not change throughout the activity, nor upon observing the unexpected outcome. He had begun counting the eighths and ninths when the teacher interrupted]

Teacher: What happened? Is that longer?… just longer? …or is it the same?

Joe: Same. [Again, begins to count the eighths and the ninths with pointer that was controlled by the mouse]

Joe had a concept for *longer than* that was based on his experience and operations with whole units. For example, he knew that nine units were longer than eight units. However, he did not have to enact any of the operations embedded in that concept in order to form his conjecture. Rather, his interpretation of the task involving *eighths*, *ninths* and *longer than*, activated his concepts for those three words. Skemp referred to such concepts as schemas (1989, pp. 131-141). Schemas are symbolized schemes in which operations are no longer triggered; when activated, they form a sort of casting net that associates past experiences in operating and connects to (coordinates) other schemas. When multiple schemas are activated, attempts to coordinate them may produce uncertainty and perturbation. For Joe, it is plausible that *eighths* and *ninths* included connections to *eight* and *nine*, respectively, so that their coordination could be reconciled by whole-number comparison operations involved in his schema for *longer than*.

Unlike the conjecture illustrated in the first vignette, Joe did not seem to incorporate any new operation within an existing scheme. Rather, he might have established a new relationship between existing schemas. Skemp's model of resonating schemas points to the idea that such concepts can change in their coordination with other schemas and potential operations. In this case, in response to the perturbation caused by the simultaneous activation of three schemas and his efforts to coordinate them, Joe seemed to generalize his schema and associated operation for *longer than* from strictly whole-number comparisons to fractional ones as well. The result might have changed the trigger for his scheme that calls *longer than* operations (an accommodation of the first part of the scheme), even though his conjecture was refuted by his visual comparison of nine-ninths and eight-eighths.

In a sense, Joe learned from this first attempt at coordinating the three schemas (his conjecture that nine ninths was bigger than eight eighths) that nine is not always bigger than eight! His initial conjecture had not resolved the situation for Joe, as indicated by his counting of the pieces just after he observed that nine ninths was in fact commensurate with eight eighths. And so his perturbation persisted, along with a new realization that something beyond whole number comparisons was involved in comparing ninths and eighths. Based on his actions in subsequent protocols, we could say that the perturbation made Joe explicitly aware that, when more parts are used in partitioning, the parts will be smaller. It is likely that he became more attentive to the differences between compositions of fractional units and compositions of whole units. This hypothesized attentiveness in Joe's reasoning would result in and is corroborated by Joe's operations for conceptualizing the equivalence of $\frac{m}{m}$ and $\frac{n}{n}$ in subsequent protocols. He had constructed records of his reasoning process (in determining that nine ninths was not greater than eight eighths) that constitute a change in his conceptions of one eighth and one ninth (as well as other unit fractions).

Whereas the first protocol exemplifies the abduction of operations in modifying a scheme, the second illustrates how concepts can evolve through operating. Both cases constitute learning within a closed system. Joe's cognitive system is not closed in the sense that his interactions with the world are irrelevant. Clearly, his experiences in operating at least affirm or refute his concepts and ways of operating. If Joe were deprived of sensory experience, we might cast doubt on Joe's potential for growth (though imagined experiences should be considered). Instead, Joe is constantly interacting with his experienced environment and receiving feedback by processing raw perceptual data.

Joe's system *is* closed in the sense that new operations and concepts are not given to him in interaction; they are born in and of the system itself. It is in considering this creative process of learning that we can determine what *is* innate: humans must have a profound ability to form patterns in thought. Limits to their viability are only introduced, against experience, when those patterns are enacted as operations. I believe this ability is what Glasersfeld referred to in suggesting an "innate tendency to search for rhythms, regulations, and groupings" (1998, p. 5). By applying rhythms, regulations and groupings to our experience, operations can change through operating and concepts can change through conceptualizing (assimilating an experience using one or more schemas).

The two examples of conjecturing activity establish that students in middle childhood do make mathematical conjectures, as I have characterized them. The examples also illustrate the nature of conjecturing activity of such students and open the possibility that learning can follow from conjecturing activity. In the current study, I describe the teacher's role in fostering conjecturing activity and specify actual accommodations students have made in their schemes as a result of that activity. Furthermore, by selecting pairs of students operating at qualitatively different stages of development, as described by Steffe and Olive, it was possible to study whether students at different stages made qualitatively different conjectures.

The Zone of Potential Construction

The accommodations that a student makes constitute her zone of actual construction, in proximity to Steffe's *zone of potential construction* [ZPC].

Borrowing from Vygotsky's notion of the *zone of proximal development* [ZPD], Steffe adapted the ZPD to a radical constructivist framework by defining the ZPC as the range "determined by the modifications of a concept a student might make in, or as a result of, interactive communication in a mathematical environment" (1991a, p. 193). This notion differs from ZPD because it involves inferring changes to specific conceptual structures through observed changes in action, rather than a focus on the students' actions in achieving a goal predetermined by a teacher. Considering the scheme-theoretic approach of this study within a radical constructivist framework, "modifications of a concept" may be thought of as accommodations of a scheme, which constitutes learning. Students' "interactive communication in a mathematical environment" is most likely to occur in the classroom, which means that a child's ZPC articulates hypotheses for learning in, or as a result of, activity in a given class period.

By developing zones of actual construction of the students in my teaching experiments, I describe ZPC's for children like them and describe the role of conjecturing and the role of the teacher in actualizing conjecturing potential. My goal was to construct epistemic students: second-order cognitive models that can be used when teaching students cognitively similar to the students in my teaching experiment. The details of this theoretical construct—the epistemic subject—and its use in my study are described in Chapter 2 and Chapter 9, respectively.

As an initial research hypothesis, I expected to find that the six students, operating at three different stages, would have ZPC's that would be more alike within pairs than across pairs. In other words, students operating at the advanced stage would be able to develop novel ways of operating that would be inaccessible to students operating at the two lower stages, and students operating at the middle stage would be able to develop novel ways of operating that would be inaccessible to students operating at the lowest stage. Moreover, I hypothesized that students' ZPC's will be actualized, at least in part, through their conjecturing activity.

Chapter 2: Literature Review

> I am fully aware of the fact that I am merely offering conjectures – but they are conjectures that I have found useful in constructing a model of mental operations.
>
> Ernst von Glasersfeld, 1991

While observing the conjecturing activity of students, I form and test my own conjectures in an attempt to develop a model of students' reasoning. I refer to the latter as hypotheses and emphasize their role in my research. As a central tenet of his research, Ernst von Glasersfeld admitted that researchers of learning are only capable of providing hypotheses about their students' knowledge, thus forming models of students' mental operations; the usefulness of such models can then be tested against further experience when interacting with students. By observing the actions (including verbalizations) of students and reflecting on the teacher's actions in interaction with students, researchers can infer the usefulness of including particular mental operations within their models of each student. In the current study, I infer, from observed actions, the conjectural operations of students in order to build and revise models of their conjecturing.

In this chapter, I will examine previous research on conjecturing that contributes to building models of students' conjecturing. For example, Arzarello et al demonstrated the usefulness of interpreting Peirce's theory of abduction within a theory of student learning. I will lay the theoretical foundation for the operational approach that I employ and explain a new use for abduction within that approach. I use Vygotsky's zone of proximal development in addressing the various roles of interaction in learning. Finally, I consider the affective aspects of learning that are integrated with cognitive activity. Other theoretical constructs will be important to my study, especially the fractions schemes developed by Steffe and Olive, but descriptions of their roles are integrated within my methodology (Chapter 3) and analysis (Chapters 5 and 7).

Conjecture

In his book, *Science and Hypothesis* (1952), Poincaré wrote about developments of the scientific community and attempted to answer the question of how mathematics can be considered infallible without "being reduced to a giant tautology" (p. 1). He described mathematics as the study of relations (p. 20) and recognized that new relations could not be realized through syllogistic (deductive) reasoning. In rejecting the a priori existence of such relations (i.e. rejecting radical innativism) and by recognizing no other alternative, Poincaré deduced that "we can only ascend by mathematical induction" (p. 16). He viewed mathematical induction as the means by which we can move beyond the construction and verification of particular cases to proofs of statements about an infinite number of cases that are useful because of their generality. This mathematical induction is similar to the inductive learning of which Fodor wrote, but differs from it in at least one essential regard.

> Induction applied to the physical sciences is always uncertain, because it is based on the belief in a general order of the universe, an order which is external to us. Mathematical induction—i.e. proof by recurrence—is, on the contrary, necessarily imposed on us, because it is only the affirmation of a property of the mind itself. (Poincaré, 1952, p. 13)

So, Poincaré appealed to psychology in justifying mathematical truth, establishing mathematics as an extension of our psychological operations. On the other hand, he took for granted that we could observe the true nature of the universe through observation of experimental cases, rather than recognizing a symbiotic relationship between our psychological structures (including mathematical objects) and our perceptions of the universe. He claimed that we could generalize from the truthfulness of our physical experiments to probabilistic statements about the universe through analogy and induction, much as we interpolate graphs from a few given points (1952, p. 140). To the degree that mathematical growth can be viewed as quasi-empirical, this kind of inductive learning can be applied to it as well.

Because generalizations are only predictions about unobserved events and are as simple as interpolations, the probability of their truth depends upon the unity and simplicity of the universe (p. 145). Whereas Poincaré argued that the unity of the universe is evident in the cause-effect relations between its observed elements, the complexity of these relations would render the universe anything but simple. Our more detailed observations, then, are accompanied by increasingly complex scientific theories until they are replaced by simple theories that undergo a new cycle of complication (p. 149). Poincaré left his readers with that foregoing and less-than-satisfying explanation for the simplicity of the universe as we know it. I call it unsatisfying because he did not appeal to our psychological and social interactions with the universe. One's universe may well be simple because one perceives it through his mind's simple organization of the universe that she has experienced. It is only when one experiences contradiction that one complicates her theory; Poincaré would have concurred, at least, with that (p. 146).

It is unclear whether Poincaré (1952) viewed mathematics as quasi-empirical. While he wrote about the psychological nature of mathematics, the psychological structures corresponded to strictly formal objects and methods, such as axiomatic systems and mathematical induction. Imre Lakatos, on the other hand, tried to emphasize the neglected, informal methods of mathematics: “None of the ‘creative’ periods and hardly any of the ‘critical’ periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impurities of earthly uncertainty” (1976, p. 2). Lakatos was insistent, throughout *Proofs and Refutations*, on demonstrating the goodness of informal (“impure”) mathematics. His arguments came at a time when he perceived a dominant “formalist school” (p. 1) as providing the paradigm for mathematical development. He argued that such a paradigm ignores the history or “growth” of mathematics. This paradigm, as he demonstrated in a mock dialogue, also undermines the growth of mathematics in the individual.

Lakatos argued, instead, for another view of mathematics, citing the importance of conjecture, counter examples, lemmas, and proofs. Rather than submitting these as finished products, he demonstrated that refuted proofs yield better proofs and that poor conjectures yield better conjectures through the revision process of argued reasoning: “Mathematics grows through incessant improvement of guesses by speculation and criticism” (p. 5); “Naïve conjectures are superseded by improved conjectures in growing out of the method of proofs and refutations” (p. 91).

Throughout his work, Lakatos used “guess” and “conjecture” synonymously. He did, however, distinguish conjecturing from “blind guessing,” by suggesting that conjecturing is an alternative between guessing and machine-like rationalizing (p. 4). It would seem that conjectures (beyond blind guesses) would require some insight into the problem, but, through the voice of a particularly mature student in the mock dialogue, Lakatos rejected insight: “I abhor your pretentious ‘insight.’ I respect conscious guessing because it comes from the best human qualities: courage and modesty” (p. 30).

Lakatos blamed *insight* for the dehumanization of mathematics. *Insight* might imply that one can have knowledge of a platonic structure independent of one’s own cognition. Max Wertheimer (1945), on the other hand, described insight as a focus on “the structure of a problem and the function of statements” (p. 121), which occurs in response to a discord between “actual and expected results” of an action or operation (p. 219). Just like conjectures, insights occur in response to perturbation. Furthermore, the focus on structure and function within a problematic situation may be part of conjectural activity, although some conjectures may be too circumstantial or contextual to call insightful. Viewed in this light, insight is a desirable aspect of informal mathematical development. In fact, Wertheimer believed such a focus on the structure of a problem, beyond the guidance of steps toward its solution, was essential to development: “For real understanding one has to re-create the steps, the structural inner relatedness, the requiredness” (p. 194).

I find it interesting that Wertheimer referred to the “requiredness” of the steps. For me, it alludes to the necessity and infallibility of our mental operations, described in a section of this chapter on *Scheme Theory and Operations*. Furthermore, it seems that Wertheimer’s “structural inner relatedness” does refer to the psychological operations and experiences from which mathematical objects are defined. For example, in describing the insightful approach to finding the area of a parallelogram, he emphasized the importance of going beyond formulas for area to arrive at the area’s *form* in terms of the parallelogram being built up from unit squares (p. 34). He also emphasized the importance of "grouping, reorganization, structurization, [and other] operations" (p. 41) in solving such problems.

If we wish to draw a distinction between insight and conjecture, we may say that conjectures admit uncertainty whereas insight may not. Of course, presumed certainty seems to have been Lakatos' entire concern about insight, but certainty here he only refers to the attention of the learner at the time; he may find fallibility in his insight once he uses it in thought or action. Charles Peirce alluded to a similar relationship between abduction and insight: "The abductive suggestion comes to us like a flash. It is an act of insight, although a very fallible insight" (Peirce, 1998, p. 227). Insights that appear unquestionable, Peirce called perceptual judgments. I find this distinction useful because the judgments we continually make about things, such as the colors of objects, are taken at first as infallible observations. I will address perceptual judgments and abductions in general, later in the chapter.

Regarding questions about the genesis of conjecture, Lakatos demurred, referring to conjecture (in the quote below) as an infinite regression with no beginning. As we shall see, at least two other major authors (Peirce and Polya) took similarly modest approaches to the problem of identifying the origins of conjecture, and I was able to find none who took bolder approaches.

> Our naïve conjecture was not the *first* conjecture ever, 'suggested' by hard, non-conjectural facts: it was preceded by many 'pre-naïve' conjectures and refutations. The logic of conjectures and refutations has no starting point, but the logic of proof and refutations has: it starts with

> the first naïve conjecture followed by a thought-experiment. (Lakatos, 1976, p. 71)

Much like Lakatos, George Polya believed that mathematics is best viewed as an activity requiring a laborious and cyclical process of conjecture and deductive reasoning: “Many a guess has turned out to be wrong but nevertheless useful in leading to a better one” (Polya, 1957, p. 99). Polya also characterized useful conjectures in terms of the activity leading to them. “Guesses of a certain kind deserve to be examined and taken seriously: those which occur to us after we have attentively considered and really understood a problem in which we are genuinely interested” (p. 99). Perhaps Polya was describing the role of insight in intelligent guessing. He also described the kinds of “plausible reasoning” that may support one’s guesses.

Polya cited four broad methods of plausible reasoning: generalization, specialization, analogy, and induction (1954a). He offered examples of reasoning, based on these methods, illustrating how conjectures might gain credibility. The following is an example of such plausible reasoning: “If A is analogous to B [where A and B are conjectures], and B is proven true, then A becomes more credible” (1954b, p. 10). Also, affirming a conjecture in special cases, through observation, renders the conjecture more plausible. Although Polya was not primarily concerned with identifying the source of conjectures, we will see in the next section that other authors refer to some of these methods of plausible reasoning as methods for conjecturing.

Polya did make some speculations on how conjectures arise: “If a naturalist observes a striking regularity which cannot be attributed to mere chance, he or she conjectures that the regularity extends beyond the limits of his actual observations” (p.

49). This statement is reminiscent of earlier writing by Poincaré (described above) about interpolating from points of observation. However, the statement might also be interpreted as referring to inductive learning, and it seems that later authors did interpret his statements that way.

Daniel Chazan and Richard Houde (1989) inferred from Polya's work that "conjectures are the result of plausible reasoning" (p. 3). This inference is quite the reverse of the description of plausible reasoning provided above. This new twist in definition may be useful for attempts to describe students' activities leading to conjecture. To differentiate the use of terms, we might refer to these two directions of reasoning as *pre-conjectural plausible reasoning* and *post-conjectural plausible reasoning*, respectively (Norton, 1999, p. 18). In any case, Chazan and Houde also seem to infer from Polya's work that conjecturing is inductive.

According to Chazan and Houde, "a conjecture in geometry is a statement that may be true or false; at the time of consideration, the conjecturer does not know for sure whether it is true or false, but thinks that it is true" (1989, p. 3). Although I consider conjectures to be based on the use of operations of which the learner does not necessarily need to be aware, Chazan and Houde's definition implies two important aspects of conjecture for my study: a conjecture is an uncertainty (guess), and the conjecturer is concerned about its credibility.

Chazan and Houde further insist that conjectures pertain to whole "sets of objects" that "explicitly mention the intended set of objects" (1989, p. 3). They refer to guesses about particular situations as observations, akin to Peirce's idea of perceptual judgment. Barring conjectures about particular situations, which can be checked by observation, "the only way to determine the truth of a conjecture is through deductive proof" (p. 3). Taking such an absolute positivistic stance, Chazan and Houde do not leave much room for psychological mechanisms that might precede conjecture. If conjectural statements are, in some absolute sense, true or false and if they must pertain to preconceived sets of objects, then conjecturing is reduced to noticing properties shared by some subset of observed objects and attributing those same properties to some extension of the subset.

Ferdinando Arzarello and Federica Olivera stand out as researchers of learning who have moved away from the inductive view in order to describe the psychological processes behind students' activities of conjecturing and proving. Moreover, they do not insist that conjectures take the form of logical statements: "The conjecture in reality is a hypothesis to be checked… It has a logical flavor, but perhaps it is not phrased in a conditional form, nor is it crystallized in a logical form" (Arzarello et al, 1998, p. 84). These authors describe *ascending* and *descending* "modalities of acting," which are roughly equivalent to pre-conjectural and post-conjectural plausible reasoning (p. 84). Much like Lakatos' descriptions of the interplay between conjecturing and proving, they emphasize the "complex switching" back and forth between the modalities (p. 84). In fact, their descriptions of switching from proving (descending) to conjecturing (ascending) fit those of Lakatos; namely, in attempting to prove one conjecture, subsequent problems are experienced, which require new conjectures. But "Lakatos does not analyze the [complementary] conjecturing phase" as the authors do (p. 86).

The authors found fault in both Polya and Lakatos' work, because each work discussed "only half of the story" (p. 86). Whereas they viewed Polya's work as examining the usefulness of four broad methods on the ascending side, Lakatos restricted his attention to the activity of proving. “The analysis of the two sides reveals strong elements of continuity” (p. 86). The authors claim that the missing piece making it possible to switch from the ascending to descending side is Peirce’s idea of abduction. As illustrated in Chapter 1, I too find abduction useful for describing students’ conjectures, and I will elaborate on Peirce’s theories about abduction in the next section. Whereas the authors’ descriptions of the switching back and forth between modalities may be helpful to my study, the detail of their work is not as helpful because they examined only the *logic* of conjecturing and did not consider the psychological operations behind the logical forms.

Analogy, Induction, and Abduction

“The scientist’s procedure to deal with experience is usually called induction” (Polya, 1954a, p. 4). Polya characterized this procedure as one that “begins with observation” of particular instances, which in turn “suggest a general statement” (p. 4). Moreover, he noticed that analogy was always involved in the process. As an example of the inductive procedure, he considered how one might come to formulate what is commonly called Goldbach’s conjecture: All even whole numbers greater than 4 can be written as the sum of two odd primes. I will now examine Polya’s hypothetical trajectory and use it to develop a critique of induction and analogy.

“By some chance, you come across the relation 3+7=10, 3+17=20, 13+17=30 and notice some resemblance between them. It strikes you that the numbers 3, 7, 13, and 17 are odd primes” (p. 4). Here, Polya emphasized the role of analogy in the inductive procedure: “We notice some similarity” (p. 5) between the equations; that is to say that the three equations are analogous. Polya explained that if the learner could identify specific relationships between the respective components of two systems, the systems could be considered analogous (p. 13). In applying this pattern to the case of Goldbach's conjecture, we might say that we are always adding what we have recognized as two primes in order to achieve a sum that is an even number. The analogy would be one drawn between the observed cases: Whereas the specific numbers differ, the pattern of adding primes to achieve various even numbers persists. We could then generalize this analogy in wondering whether we can achieve all even numbers in such a way—hence the conjecture.

Although Polya seemed to be more interested in the probability of logic, it is interesting to note that he recognized the importance for one to break free from logic in order to grow intellectually. For instance, he applauded the genius behind Euler's willingness to apply rules to cases for which they were not intended. Such moves, motivated by analogy, were illogical but served the purpose of generating new theories. Like Lakatos, Polya demurred from an attempt to explain the genesis of conjecture but recognized that extra-logical reasoning was involved. Polya also recognized the role of psychological structures, beyond logical statements, involved in conjecturing, referring to analogy and generalization as "fundamental mental operations" (p. 17).

We will see that Peirce, too, alluded to the importance of mental operations in explaining the advent of conjecture. So, an examination of mental operations appears necessary not only in explaining learning (as presented in Chapter 1), but also in getting beneath the logic-laden surface of mathematics. Mental operations in general are discussed later in this chapter, where it is argued that Polya's *generalization* can be thought of as a special case of reflective abstraction and that the logical limits recognized by Peirce and Lakatos can be circumvented.

Polya also understood the crucial role that operations of analogy play in conjecturing. His ideas are supported by Glasersfeld who, in answering "how hypothetical rules are invented," found that analogy "seems to me a reasonable suggestion" (Glasersfeld, 1998, p. 6). "There may be other ways of intuiting a rule on the strength of a single observation, but I would suggest that the conception of analogy can explain a great many such intuitions" (p. 7). But the *ubiquitous* role analogy played in Polya's view of induction is replaced by *abduction* in Glasersfeld's view. He claimed that, "every inductive inference contains an implicit abduction" (p. 7).

In Peirce's paper on the "Logic of Abduction," Peirce began his outline by introducing the limiting case of abduction: perceptual judgment. As mentioned before, this is roughly equivalent to Chazan and Houde's conception of observation; they are "absolutely beyond criticism" and serve as "the starting point or first premises of all critical and controlled thinking" (Peirce, 1998, p. 227). In other words, perceptual judgments are made before the observer can critically examine them; initially, the observer will accept things as he or she perceives them. However, there is much more to perceptual judgment than meets the eye!

Peirce used ambiguous Gestalt figures, such as that illustrated in Figure 2, to demonstrate that we have already made decisions about objects upon perceiving them. "The very decided preference of our perception for one mode of classing the percept shows that this classification is contained in the perceptual judgment" (p. 228). In the case of Figure 2, there are at least three different ways of perceiving the cube(s) (Do you see the cube in the corner, the two cubes, or the one cube with a missing piece?). One's perceptual preference is immediate and, for the time being, seems beyond question. It's not that one cannot perceive the figure in another way, but that we cannot consider matters involving such *controlled thinking* about the object until we have first perceived it in one definite way. This is, more or less, an elaboration of Aristotle's view that "*nihil est in intellectu quod non prius fuerit in sensu* [nothing is in the intellect that is not first in the senses]" (p. 226).

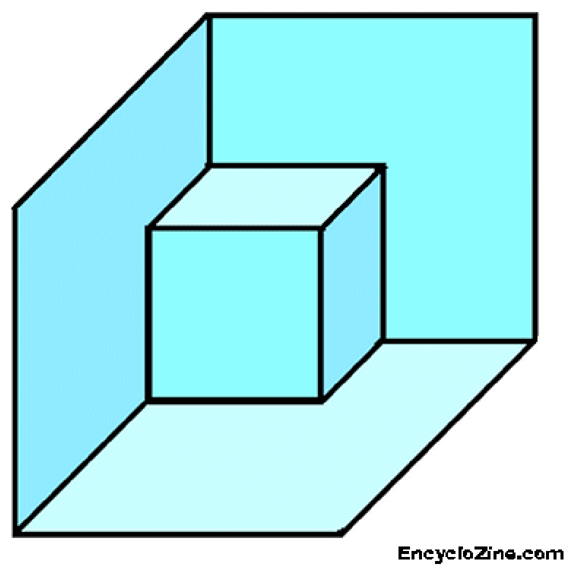

Figure 2. "Cube in a Corner" illusion taken from EncycloZino.

> The first time it is shown to us, it seems as completely beyond the control of rational criticism as any percept is; but after many repetitions of the now familiar experiment, the illusion wears off, becoming first less decided, and ultimately ceasing completely. This shows that those

> phenomena are true connecting links between abductions and perceptions.
> (Peirce, 1998, p. 288)

Peirce referred to abductions as perceptual judgments that are "discretely and consciously performed" (p. 229). This definition demands a short discussion of what it means to be conscious. First of all, in Peirce's view, it is consciousness that opens the possibility for control and rational criticism (including doubt); full consciousness of a situation allows for control and critique of it. Jean Piaget seemed to make a complementary suggestion in his claim that "need creates consciousness" (1955, p. 231). Consciousness corresponds to a degree of control that one has over his mental processes that may be generated by the needs of the learner. In particular, experiencing a problem (a perturbation) appears to have the effect of awakening a learner's consciousness about the particulars of a situation.

In Chapter 1, I discussed the pattern of abduction. Here, I offer one more example of the pattern to elucidate it. Consider again Polya's illustration of Goldbach's conjecture. Polya suggested that the abstraction of the pattern (even numbers as the sum of two odd primes) involves analogy, which may be the case. But it also fits the pattern of abduction, wherein a general rule is adopted or created to explain a surprising situation. In this case, the surprise might involve the recognition of the multitude of prime numbers, being peculiar in their own right. To explain the situation, the student would begin to examine the properties of the equations, combining various whole number concepts and operations in novel ways, starting from just one equation. Then the student could inductively test to see if a particular property holds across all three cases. Eventually, the student might alight on a pattern fitting each of the equations and defining Goldbach's conjecture, which becomes the general rule that explains the surprising situation. This is what Glasersfeld was referring to in saying that every induction includes an implicit abduction.

Glasersfeld's work on conjecture was built mostly from that of Peirce, by demonstrating how abduction can be used as a theoretical construct within scheme theory, bringing abduction closer to the level of mental operations. But Peirce himself viewed abduction as logical inference only.

> Abduction, although it is very little hampered by logical rules, nevertheless is logical inference, asserting its conclusion only problematically or conjecturally it is true, but nevertheless having a perfectly definite logical form. (Peirce, 1998, p. 231)

On the other hand, Peirce did recognize the fruitlessness of applying logical analysis to the genesis of an abduction. We have already noted this in the extreme case of perceptual judgment, but the following passage illustrates that the infinite regression can be applied to abductive inferences as well.

> If we were to subject this perceptual judgment to logical analysis we should find that it terminated in what that analysis would represent as an abductive inference resting on the result of a similar process that a similar logical analysis would represent to be terminated by a similar abductive inference, and so on *ad infinitum*. (Peirce, 1998, p. 227)

Herein, we see the trap in which a strictly logical analysis would place us. As discussed earlier in the chapter, a psychological theory of mental operations may resolve such an endlessly circular (hence useless) logical analysis. The passage above also reiterates the discrete nature of abductions, versus the continuous nature of perception (p. 229).

Although we may need to go beyond logical analysis to explain the origin of an abduction, because they are discrete and conscious, abductions themselves are, in principle, susceptible to logical question and doubt.

In this section we have examined three important mechanisms (analogy, induction, and abduction) involved in conjecturing. Glasersfeld suggested that abductive inference is the ever-present, crucial phase, but I am careful not to dismiss any of the mechanisms as useless or elevate any of them to ubiquity in explaining the formation of conjectures. Polya and Glasersfeld (notwithstanding the latter's emphasis on abduction) each provided examples from their work demonstrating the usefulness of analogy (Polya, 1954a; Glasersfeld, 1998, pp. 6-7). Beyond that, from my own past research (Norton, 2000, p. 293), I have identified conjecturing episodes in which it is difficult to ascribe any particular abduction (including the Protocol II given in Chapter 1). It is crucial that researchers remember Glasersfeld's advice from the beginning of this chapter, that *usefulness* is the criterion for good theoretical constructs.

Scheme Theory and Operations

As demonstrated by Arzarello et al, abductions can be used to explain the transition from the conjecturing modality to the proving modality. But Peirce himself noted that abductions could not explain the novelty of the conjecture: "Self-control is the character which distinguishes [abductive] reasoning from the processes by which perceptual judgments are formed, and self-control of any kind is purely inhibitory. It originates nothing" (Peirce, 1998, p. 233). This means that Peirce would be unable to explain logically how abductions are formed, even though he was able to outline a couple requirements: "Any hypothesis may be admissible, in the absence of any special reasons to the contrary, provided it be capable of experimental verification" (p. 235). In other words, our creative selves are free to construct any novelty that can be tested against experience and does not contradict existing ideas a priori.

Whereas Peirce was concerned with the logic of abduction, Poincaré was concerned with mathematical infallibility and tautology, and Polya and Lakatos were concerned with the growth of mathematics through reasoning. All of them, of course, reached points where they could conduct no further analysis; such points reveal what we take for granted in any theory. However, Polya and Glasersfeld had the foresight to suggest that analogy could be described as a mental operation that might generate new ideas. In the next section, we will see how the mental operation of analogy involves the *reflective abstraction* of a concept to a higher level. We turn first to a scheme theory of operations, developed by Jean Piaget and refined by Glasersfeld, to establish new givens that are fundamental to understanding learning and explaining the origins of conjectures.

Within scheme theory, Glasersfeld confronted the radical innativist idea that the logical structure of all possible conjectures must be present in the learner a priori. That was the trap (later formulated as Fodor's paradox) that Poincaré tried to avoid. Even with his theory of abduction, Peirce would have been resigned to radical innativism, except that he recognized that there must be other unconscious (and therefore extra-logical and extra-linguistic) mechanisms at work. Glasersfeld picked up where Peirce left off by describing those mechanisms in terms of schemes and operational structures, the theoretical constructs that Jean Piaget and Bärbel Inhelder had developed.

Piaget and Inhelder's (1969, p. 44) theoretical development of *operations* reminds us to consider carefully the meaning of Aristotle's credo that nothing is in the intellect that is not first in the senses; the authors dedicated much of their work in *The Psychology of the Child* to providing evidence for the need to consider the actions and operations of the learner, beyond passive perception through the senses. For now, let us be content in defining (mental) operations as actions that are interiorized through abstraction. Whereas actions apply to anything we do in or to our perceptions (possibly in implementing an operation), operations can be applied to concepts. I will elaborate on these ideas more in the next section on reflective abstraction. As an example of the need for actions and operations, Piaget and Inhelder contended that "logico-mathematical concepts presuppose a set of operations that are abstracted not from the objects perceived but from the actions performed on these objects, which is by no means the same" (p. 49). The authors further argued that even perception involves action on the part of the learner. We saw such evidence earlier in considering Gestalt figures.

While Gestaltism, developed in part by Wertheimer and praised by Peirce, highlights the need to consider the operations of the observer in perceiving, this theory diverges from scheme theory when we consider the relationship between perception and operation. Gestaltists claim that the two occur and develop simultaneously. But through their work with children's development of object permanence, Piaget and Inhelder found that "perceptual effect is determined by sensorimotor schemes, rather than explaining them" (p. 33). In other words, our operations determine what we see and how we see; our operations organize and structure our perceptions rather than being given through perception.

Glasersfeld (1998, p. 8) thought about sensorimotor schemes (action schemes) in terms of a three-part structure, as represented in Figure 3. He developed this model from

Piaget and Inhelder's description of schemes. Whereas behaviorists provided only two steps in terms of stimulus and response, but Piaget determined the need for some sort of expected result in order to explain goal directed behavior and the modification of schemes (p. 8). Moreover, the behaviorist's *stimulus* is one that is external to the learner, but the *perceived situation* involves an *assimilation* on the part of the observer.

Perceived Situation → Activity → Expected Result

Figure 3. Glasersfeld's model of a sensorimotor scheme.

. "As in the reflex, every implementation of an action scheme requires the acting subject to recognize a triggering situation. Such a recognition is of course an assimilation, because no two situations in a subject's experience are ever quite the same" (Glasersfeld, 1998, p. 9). Again, this emphasizes that the perceiver is active even in the recognition of a situation. Since the observer's situation that the perceiver assimilates are never identical, when assimilating them he neglects aspects of the situation that do not fit the schemes of perception and action. As such, assimilation is our psychological mechanism for constructing experience from sensory material.

"Assimilation plays a role also in the third part of the scheme. If a scheme is to be considered successful, the actual result of the activity must be such that it can be assimilated to the expected one" (p. 9). When we assimilate a situation using a scheme, we have expectations about the result of acting on the assimilated experience. For example, upon placing a glass on a table that we have never seen before, we might expect that the table will support the glass. If, instead, the glass falls through the table, we would probably be surprised and should want to explain the unexpected result, because we might not be able to assimilate the perceived result of the glass falling into the expected result. This is the realm of conjecture and accommodation, wherein our schemes are modified.

> If, then, a formerly disregarded characteristic of the triggering situation [such as the perception of the table and the goal of putting the glass down] is taken into consideration, this may bring about a modification of the conditions that determine the triggering of the scheme; or it may bring about the formation of a new scheme. Both are instances of accommodation. (Glasersfeld, 1998, p. 9)

Glasersfeld (1995) interpreted Piaget's distinction between the roles of assimilation and accommodation in the following manner: "[assimilation] modifies what is perceived in order to fit it into the organism's conceptual structures" (p. 62); accommodations are the modifications the organism makes to structures when the assimilation "does not yield the expected result" (p. 66). So, whereas we are always assimilating sensory input with existing structures in order to construct our experiential reality, accommodations occur when we experience dissonance in applying our actions or operations to perceived reality (which is already a result of action). Although it is through action and operation that we experience perturbations, accommodations may be made to any part of the scheme.

There is one kind of assimilation that is also a kind of accommodation: a generalizing assimilation. Steffe and Thompson (2000) described these as assimilations of situations into schemes, where the situation involves novel sensory material relative to the scheme (p. 26). The trigger of the scheme may be accommodated in such assimilations because, if using the scheme in the situation produces the expected result, one might modify the trigger to include other situations involving such novel sensory material. Glasersfeld described a third kind of accommodation, below, and we will examine other kinds in the next section of this chapter.

"If the accommodation were done consciously, it would be an abduction, because, at the moment the changes are made, they are hypothetical in the sense that their usefulness has not yet been tested in further experience." (Glasersfeld, 1998, p. 9). Finally, we arrive at Glasersfeld's argument that transforms Peirce's theory of abduction to the operational level, where there is still hope that it will be useful in describing the origins of conjecture. However, Glasersfeld seemed to equate abductions with conjectures and claimed that they are done consciously. In light of his very next line—"Children accommodate their action schemes by means of fortuitous choices quite some time before they begin to reflect on them consciously" (p. 9)—I would argue that it is useful to consider unconscious abductions that, from an observer's point of view, fit the pattern of abduction. Indeed, this is the manner in which I will refer to abductions throughout this paper. Furthermore, I have already mentioned an example of conjecture (Protocol II) that did not seem to fit this pattern; so, I do not accept that all conjectures are abductions.

Glasersfeld found abduction to be pervasive in cognitive development and claimed that it is "the mainspring of human creativity" (p. 10). Certainly, I have found the *pattern* of abduction useful in describing students' conjecture. For example, consider Protocol I in Chapter 1. Still, I have also admitted that considering the pattern of abduction at the operational level only translates Fodor's paradox to the level of operations instead of statements. I will address this new problem in the next section. For now, let us consider how operations might help resolve some of the historic problems faced by Poincaré, Lakatos, and Polya.

First of all, Piaget and Inhelder (1969, p. 50) characterized operations by their reversibility and composibility. I interpret this to mean that the operations that I attribute to students are potentially reversible (composible) from my point of view, although the student may not yet be able to reverse (compose) the operations. As interiorized actions, they act on our conceptions, define relationships between them, and, along with our records of experience, constitute them. It may be said that mathematics is the study of relationships, and so Piaget's reference to the importance of logico-mathematical operations is justified. This point also marks the distinction between the study of mathematics and other sciences. Both may be considered at least quasi-empirical (because mathematical concepts are ultimately abstractions from actions), but, whereas science describes actions on perceptual material, mathematics describes actions on abstracted concepts.

> Now an operation is rigorously additive, for 2+2 make exactly 4 and not a little more or a little less as in the realm of perceptual structures. It seems obvious, therefore, that operations, or intelligence in general, do not derive from the perceptual systems. (Piaget & Inhelder, 1969, p. 50)

Understanding mathematics is such a way demystifies the infallibility of mathematics. For instance, I speculate that we know that 2 plus 2 equals 4 because our operations of uniting 2 and 2, each of which is defined as the result uniting 1 and 1, is precisely the operation defining 4 within an explicitly nested number sequence; or, in other words, we can reverse the uniting operations defining each 2 and compose all three uniting operations as one (1+1+1+1), defining 4. If someone says, “2 plus 2 is 1 because when two raindrops meet two others, the result is one raindrop,” that person is referring to an operation that does not fit our consensual meaning of addition, which involves a uniting operation of a different kind. We might say that the person has abstracted a merging operation, which is a viable operation. But it would not take long in conversation with that person to realize that our operations are not compatible. In such a manner, we can parse out differences between various operations, abstracted from various actions (like those applied to perceptions of raindrops merging) until we can agree on what it means to add. So an operational understanding of mathematics might resolve Poincaré’s problem of explaining mathematical infallibility, elaborate on the *structural inner relatedness* of Wertheimer’s insight, and help to clarify Polya’s and Lakatos’ attempts to humanize the field.

We can extend the argument to logical reasoning in general. Although Ludwig Wittgenstein (1974) did not address specific psychological structures, such as operations, we can interpret one of the results of his *Tractatus* in light of them. “What makes logic a priori is the impossibility of illogical thought” (p. 47). Mental operations are logically necessary because we can carry them out in thought, but Glasersfeld drew an even more direct link between operations and deductive logic: “The certainty of conclusions [through deduction] pertains to mental operations and not results of schemes on the sensorimotor level” (1995, p. 69). Although the application of a scheme to a particular situation is fallible and may yield perturbation, the operations of the scheme are infallible in the sense that Wittgenstein suggested of logic.

To conclude this section, before considering the universal mechanisms of intellectual growth, I would like to mention a pattern of reasoning—*syncretism of reasoning*—that may be particularly relevant to my study. In studying children between ages nine and twelve, Piaget identified a peculiar pattern that appears to precede formal conjecturing. Though I have opened my study to the consideration of informal conjectures that may be attributed to *conjectural operations*, children's development of more formal systems of reasoning should be important to the implications of my results. Moreover, it is interesting that this syncretism of reasoning uses *assimilation* to generate propositions.

> Syncretism of reasoning is the assimilation of two propositions in virtue of the fact that they have a general schema in common, that they both, willy nilly, form part of the same whole. A enters into the same schema as B, therefore A implies B. (Piaget, 1955, p. 155)

The result of syncretic reasoning, then, resembles the *if-then* form that we expect to find in the formal conjectures (hypotheses) of a researcher, although it lacks insight into the relationship between cause and effect. Bärbel Inhelder and Jean Piaget (1958) found that the development of such formal relationship "depends on the establishment of a combinatorial system." This system enables the student to link a set of base associations or correspondences with each other in all possible ways so as to draw from them the relationships of implication (p. 107).

<u>Reflective Abstraction</u>

"The weighty task of constructivism is to explain both the mechanisms of the formation of new concepts and the characteristics these new concepts acquire in the process of becoming logical necessity" (Piaget, 1980, p. 26). In the sections leading up to this one, I have outlined a constructivist model that defines learning as accommodations of schemes and the growth of concepts in general. Realizing that "both interiorized concepts [schemas] and interiorized actions [operations] are needed for intelligent learning" (Olive and Steffe, 2002, p. 118), what is missing from the theory that I have described is a mechanism that would explain how students form concepts and operations. The general operations of reflection and abstraction serve this role in scheme theory by interiorizing perceptions, actions, and concepts. These operations can be applied to any experience and substantiate the innate capacities referred to in Chapter 1, namely "to remember experience, reflect on it, and to make comparisons" (Glasersfeld, 1998, p.7).

Glasersfeld (1995) distinguished in Piaget's work two broad categories of abstraction. *Empirical abstraction* refers to the process by which we form records of our perceptions and are able to remember, recall, and re-present them to ourselves as *figurative material*. These are abstractions that we continually perform in our waking lives, usually (if not always) unawares; they create and modify our concepts. Likewise, sensory-motor experiences from our physical actions can be empirically abstracted as a pattern of action (Glasersfeld, 1995, p. 69), so that they can then be thought of as internalized actions that we can re-present and apply to figurative material, independent of the physical action.

Reflective abstraction can be thought of as reflection on and organization of the products of empirical abstraction—action schemes, internalized actions, and figurative material—or the products of previous reflective abstractions. Sometimes this is done consciously, in the sense that we commonly use the word *reflect*, as in "I will reflect on my thoughts." Other times, reflective abstraction may be used to describe a subconscious coordination of concepts and operations, which may occur as a result of an engendering accommodation. We will consider a special case of this (*metamorphic accommodation*) later in this section. In any case, reflective abstraction generally refers to the (uniquely?) human ability to produce our cognition on two different levels, one (higher) level of cognition using the other (lower) level of cognition as material for operating. As such, through reflective abstraction, humans are also able to "project and reorganize" concepts, schemes, and operations at one level in their cognitive structures to higher levels (p. 70).

As Glasersfeld described in the quote below, reflective abstraction in operating may result in accommodation of the operations involved in resolving an initial perturbation. Still, other times it may happen that a second perturbation is experienced after the learner reflects upon two or more concepts or operations, in which case the perturbation is brought about through reflective abstraction.

> Every time the cognizing subject manages to eliminate a novel perturbation it is possible and sometimes probable that the accommodation that achieved this equilibrium turns out to have introduced a concept or operation that proves incompatible with concepts or operations that were established earlier and proved viable in the elimination of other perturbations. (Glasersfeld, 1995, p. 67)

Whereas Glasersfeld, following Piaget, used the terms *empirical abstraction* and *reflective abstraction*, Richard Skemp (1989, p. 70) distinguished between abstractions from experience (concepts) and abstractions from concepts, which he called secondary concepts. For him, concepts are synonymous with schemas, which "represent regularities abstracted from isolated experiences" (p. 52). We might notice many parallels between the theoretical constructs of Skemp and Piaget. However, as Olive and Steffe pointed out, Skemp never addressed interiorized actions (2002, p. 118), just as Piaget noted of Aristotle.

Notwithstanding Skemp's serious omission of operational change, incorporating Skemp's theory within scheme theory is particularly useful in addressing an issue introduced in Protocol II: How do concepts change when we operate on them? Skemp explained that, "[concepts] grow by assimilating new experiences into existing schemas" (1989, p. 63). Of course this is viable because, if we accept that concepts are abstractions from experience, why should the abstraction cease after any number of experiences? But concepts also grow through interiorization and relationships established between them through reflective abstraction.

To make the relationship between the theories of Skemp and Piaget explicit and incorporate Skemp's theory within the larger scheme theory of learning, I think of schemas as schemes that are symbolized through their interiorization, which occurs in reflective abstraction. For example, Steffe (1991b) described how a child named Tyrone, through operating, could develop a number sequence from his figurative counting scheme. The conceptualization consisted of the child interiorizing his acts of counting to five into a pattern of five, thus establishing a new schema. Tyrone no longer needed to enact the operation of counting by ones five times in order to recognize five. Instead the word *five* could resonate a concept that contained the pattern (the interiorized record of counting) and the potential operation of counting.

We can see that both analogy and generalization involve a kind of reflective abstraction—one in which patterns are abstracted from a way of operating in certain situations and applied to other situations—whereas induction is a kind of empirical abstraction in which we proceed from "*some* to *all* by simple extension" (Piaget, 1980, p. 28). We are now also ready to consider the role of reflective abstraction in metamorphic accommodation: Steffe's reply to Fodor's learning paradox.

"A metamorphic accommodation of a scheme leads to a modification of the scheme that occurs independently but not in any application of the scheme" (Steffe, 1991b, p.38). In Tyrone's case, *five* had been interiorized as an operational pattern on a higher level than the rest of his counting scheme, and this caused a perturbation of dissonance within the organization of his counting scheme. Steffe demonstrated that, during a period of relative inactivity, Tyrone had projected the rest of his counting scheme to the higher level, creating operational patterns up to at least sixteen! Steffe hypothesized that the metamorphosis was a kind of reflective abstraction through which Tyrone's counting scheme was reorganized at the higher level and the perturbation was eliminated.

Social Interaction

"An explanation of a child's development must take into consideration two dimensions: an ontogenetic dimension and a social dimension" (Piaget & Inhelder, 1969, p. 137). In the framework that I have established for my study, the social dimension is a crucial aspect of the ontogenetic dimension; that is, the individual needs to interact with her environment, both physically and socially, in order to develop ontogenetically. Lev Vygotsky presented a different perspective in distinguishing children's learning from their development. In his view, social learning precedes ontological development.

> Learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers. Once these processes are internalized, they become part of the child's independent developmental achievement. (Vygotsky, 1978, p. 90)

Vygotsky's perspective may be useful in describing how some cultural structures, like language, seem to be shared among people, whereas operational structures clearly vary from person to person. According to him, learning a language amounts to awakening the operational structures that support the use of language, but language itself is shared and exists, first, between people in society, before it becomes internalized within them. Other forms of cultural knowledge, such as mathematical concepts, become internalized by the individual through language. So, a child may learn from a teacher by solving a problem in linguistic tandem, and then, once the child has internalized the structure of linguistic interactions within his operational structures, he can solve similar problems on his own. In such a way, "the developmental process lags behind the learning process" (p. 90). In order to engage in problem solving with the teacher in the first place, the child must have already developed suitable operational structures to support the linguistic interactions. The disparity created by the lag introduces a zone that Vygotsky views as essential to learning and development.

> The zone of proximal development of a child is the distance between her actual development level as determined by independent problem solving and her level of potential development as determined through problem solving under the guidance or in collaboration with more capable peers…. It defines those functions that have not yet matured but are in the process of maturation, functions that mature tomorrow but are currently in the embryonic state. (Vygotsky, 1978, p. 86)

The goal for teachers, then, is to promote development by engaging students linguistically with the teacher or other students in solving problems determined by the students' zones of proximal development. Of course, these zones will vary among the students in a classroom, and so teachers might try to identify the overlaps. We will see that such intersubjectivity presents a more complicated dilemma for the perspective of learning that I have proposed. First, let us consider the differences between the two perspectives.

Whereas a child's zone of proximal development is defined in terms of a predetermined set of problems that a teacher may pose, a child's zone of potential construction (of principal interest to my study) refers to mental operations indicated by and explaining her actions. This distinction points to a fundamental epistemological difference between the two perspectives: In Vygotsky's view, problems, concepts, and other forms of knowledge exist in society first before being internalized by the individual; from a scheme theoretic perspective, problems, concepts, knowledge, and even society itself are uniquely constructed by the individual. The former perspective beholds "[children] growing into the intellectual life of those around them" (p. 88). The latter perspective frees the researcher to consider differences between student's conceptions and those around them—difference that may be obscured in language—but it must bear the burden of explaining how concepts can be taken as shared between students. Humberto Maturana took pains to provide a detailed account for the latter perspective and argued for its necessity (Maturana, 1988). I will only mention here that he appeals to the self-organization (*autopoesis*) of the individual mind.

Such differences in epistemology have spurred arguments over whether the mind exists in the individual or in society (Cobb, 1994; Lerman, 1996; Steffe and Thompson, 2000). Some researchers of mathematics education have tried to put theories into perspective either arguing that scheme theory and Vygotsky's views are complementary (Cobb, 1994; Kieran, 2000) or dichotomous (Lerman, 2000). In my view, the two theories provide complementary benefits and limitations that may render one more useful for one purpose and another more useful in serving another purpose. Although I have primarily relied on the scheme theoretic perspective for my study, both in describing students' individual constructions and interactions between students, zones of proximal development can be identified for my participants and provide a contrasting view of their learning.

The zone of potential construction, as defined in Chapter 1, is based on the researcher's model of the operations of a child in a particular domain. It is important to note that, using such a model, the meaning a child makes of his every interaction, including all linguistic interactions in the domain, is afforded by his existing schemes and operations. If a child is able to solve a problem with the aid of an expert, this only means that he can use his schemes to meaningfully interpret the actions of the expert just as the expert interprets the child's actions as signifying an understanding of the problem. In scheme theory, one would never make the claim that a child comes to share the knowledge of the expert, but that the child and the expert may resolve perturbations caused by perceived differences in their actions until no further differences are perceived. In my final chapter, I will illustrate the dangers of assuming shared knowledge between students acting in the overlap of their zones of proximal development.

Now, the zone of potential construction is also modeled by the researcher in hypothesizing what schemes and operations might become available to the child through a reorganization of schemes and operations within the researcher's existing model of the student. So, it is much more difficult to identify or even appropriately refer to the overlap between students' zones of potential construction. We can resolve this issue by considering yet another researcher construct: the epistemic subject.

While each child in a classroom thinks and behaves differently, teachers can only know their students through the models they build of them. This is what makes it possible for a teacher to infer that two students are thinking in compatible ways about a mathematical problem; it only means that the teacher's models of the two students actions cannot be distinguished or that the teacher may decide that the differences are unimportant in a particular context. While this admits a limitation of teachers' possible understanding of students, it is a useful limitation because it allows the teacher to act effectively within a classroom of thirty minds as if there were only a few minds, at least in consideration of a particular mathematical concept. These general models of thinking are constructions of the teacher that can be thought of as second-order models called "epistemic subjects" (Steffe, 1999, p. 6). They are second-order models because they are neither the mathematical world of the child, nor the first-order world that the teacher ordinarily experiences; they are the teacher's construction of children's mathematical worlds. Epistemic subjects provide the teacher-researcher with an understanding of what children know, how they operate, and what they can learn. My study should yield epistemic subjects that, as such, can be used to understand the mathematical realities, operations and possibilities of children in general.

Affective Aspects

"When behavior is studied in its cognitive aspect, we are concerned with its structures; when behavior is considered in its affective aspect, we are concerned with its energetics (or economics)" (Piaget, 1969, p. 21). Although we have discussed the structures of children's schemes and operations, which are intended to bring the child to a goal state, the establishment of that goal and its relative importance are issues of affect. Suppose that a child experiences a perturbation while using a particular scheme to achieve a goal. The child may respond to the perturbation with frustration, giving up his efforts so that the perturbation simply diminishes. In such a way, we find that frustration and motivation regulate the energy for cognitive activity. "Affective and cognitive [as well as social] aspects of behavior are in fact inseparable… Neither one can function without the other" (p. 114).

Skemp (1989, p. 189) referred to *goal states* when describing several categories of emotion. It is important to note that these goal states do not correspond to goals within the structure of particular schemes. Instead, goal states include aspirations of receiving basic biological needs for survival, such as food and shelter, or social needs such as gaining knowledge and receiving respect. Skemp described how some goal states might have been inherited as products of evolution, while others may be indirectly related to such survival needs. *Anti-goal states* are those that work against our survival (risk death or lower our status).

Pleasure (displeasure) accompanies experiences bringing us toward (away from) a goal state, whereas fear (relief) accompanies experiences bringing us toward (away from) an anti-goal state (p. 193). Confidence and frustration are determined by our construction of our selves and our goal states. If we feel that we are capable of moving toward a goal state, we feel confident in realms of acting designed to bring us to that state; otherwise, we feel frustrated (p. 194). Likewise, security accompanies feelings of capability in avoiding anti-goal states; otherwise, we experience anxiety (p. 195).

Goal states, as Skemp described them, are reminiscent of Maslow's hierarchy, and this may relate well to Piaget's energetics. If students cannot achieve the most basic and important goal states (such as establishing a safe home for themselves), it seems unlikely that they will direct much energy to cognitive tasks at school, which for them may not relate to such goal states at all.

Chapter 3: Methods

The White Rabbit put on his spectacles. “Where shall I begin, please your Majesty?” he asked. “Begin at the beginning,” the King said, gravely, “and go on till you come to the end: then stop.”

Lewis Carroll, *Alice in Wonderland* (1865, p. 151)

The King’s advice is simple to hear and hard to follow because beginning and end are rarely, if ever, absolute. Finding myself in the middle of so much research on learning, I began where all purposeful problem-solving tasks must: I began with questions. My research questions can be summarized by “whence and wherefore conjecture?”

A scheme-theoretic view of learning and cognition focuses my question on the mental operations associated with schemes. So, my approach to answering the question must begin by identifying the available schemes and operations of the participants. In this section, I will describe a type of experiment that targets these constructs and a computer environment that can serve as a medium for students’ representations of those constructs. Next, I will report on some findings of a pilot study that helped me to identify the desired constructs for my participants. I will then characterize the general schemes involved, which, in turn, informed my student selection and interview tasks. Finally, I will describe my method for designing conjecture-rich tasks and analyzing conjectural protocols, which allowed me to form hypotheses about the formation of conjectures and their role in learning.

Teaching Experiments

"Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations that students construct" (Steffe & Thompson, 2000, p. 1). So, teaching experiments provide a unique method for building models of the constructs and learning that are involved in conjecturing activity. I decided to engage in teaching experiments with pairs of students so that student-student interaction would be possible but not too diffuse (i.e. too many possible combinations for student-student and student-teacher interactions) to analyze at the level of individual cognition. My role as teacher-researcher was to build second-order models of the students' conjecturing activity, while attending to and re-constructing second-order models of students' fractional reasoning. While the second-order models built by Steffe and Olive (Steffe, 2002; Olive & Steffe, 2002) provided me with a lens for interpreting children's actions, much as my own mathematics does, I knew I must be open to surprise and find compelling explanations for the actions of students in my own study.

In addition to the teacher-researcher, teaching experiments include a witness and a recording device. The role of the witness is to provide second-order feedback during the flow of teaching protocols and to consult with the teacher-researcher in his attempts to build second-order models of the teaching protocols and plan tasks for subsequent protocols. Video-recordings serve to facilitate the researcher in forming hypotheses about students' activity and to facilitate discussions between the witness and researcher concerning the teaching protocols. As the teaching experiment progresses, tasks are designed, not only to provoke conjecturing activity, but also to test the researcher's hypotheses concerning student activity.

The teaching experiment method for research requires a particular approach to teaching in which the teacher-researcher must "continually establish meaning of the students' language and actions" (Steffe & Thompson, 2000, p. 11) so that the students' actions guide the teacher-researcher in his attempts to "become the students and to think as they do" (p. 13). This approach is important on two levels. First, by continually establishing meaning, the teacher-researcher is developing new hypotheses about students' cognition while remaining open to surprises. Second, by thinking as students do, the teacher-researcher is in a position to understand the students' stages of operating and compare them to his own in order to design tasks to provoke creative activity in the students. On both levels, the teacher-researcher experiences constraints in building viable models and meaningful tasks based on the dichotomy of expected and observed activities of students. This feedback provides the guiding principle for hypothesis testing and the design of new tasks within and between protocols.

Micro Worlds

The computer software, `TIMA:Bars`, consists of many micro worlds (the plurality refers to the potential for students' creation of problematic situations within the program) developed by Steffe & Olive (1996) for use in their teaching experiments related to *The Fractions Project*. I used this program extensively throughout my teaching experiments as a medium for students' activity and for posing tasks. `TIMA:Bars` allows students to enact operations on rectangular bars of varying sizes and shapes, which are created by clicking and dragging the computer mouse. "Making a bar along with possible actions…can [also] be used to engender certain conceptual operations" (Steffe & Olive, 1996, p. 177). The *possible actions* within `TIMA:Bars` are described in Table 1, developed by Barry Biddlecomb (1999, p. 61).

Table 1

Potential Actions in `TIMA:Bars`

MAKE	Allows user to click-and-drag out a bar.
COPY	Copies a selected bar.
PULL PARTS	Copies selected parts from bar.
ROTATE	Rotates a selected bar 90 degrees.
JOIN	Joins two bars together. To be joined, the bars must be the same size along the side to be joined.
REPEAT	Extends a bar by appending copies of the original bar with each click of the mouse. The side of the bar first clicked determines the direction of the extension.
PARTS	Marks a bar into equally sized parts. The marks may be selected horizontal or vertical and run the full width (height) of the bar.
PIECES	Allows the user to make marks on the bar one-by-one. The marks may be selected horizontal or vertical and run the full width (height) of the bar.
BREAK	Fragments a bar into its parts or pieces.
FILL	Fills a bar or a part or a piece of a bar the color of the Color button.
IMAGES	Allows a user to drag a shadowy image of a bar, part or piece for comparison with another bar, part or piece.

UNIT BAR	Designates a selected bar as the unit for purposes of measurement.
MEASURE	Compares the area of a selected bar to that of the unit bar in the form a/b.
COVER	Makes a cover to hide bars.
UNCOVER	Removes a cover by destroying it.
MAT	Allows a user to create a mat by clicking-and-dragging. A mat is a rectangular region that is fixed in place and will appear beneath any other object in the micro world.
COLOR	Clicking on the button selects a color to fill a bar.

I chose to use computer micro worlds because they provide an organized medium for teacher-student and student-student interactions. The software developed by Olive & Steffe (1994) is relatively easy to learn use and includes potential actions that can be used in operating, such as recursive partitioning, which was identified by The Fractions Project as an essential operation in the development of fractions knowledge. `TIMA:Bars` is particularly useful in making recursive partitions because the same bar can be partitioned both vertically and horizontally, making it easy to keep track of each partition. In addition, this environment includes potential actions involving the UNIT BAR, MEASURE, and COVER functions that can support conjecturing activity through their use in posing tasks and testing conjectures.

The designers of `TIMA:Bars` developed actions that could be used to engender mental operations that are essential for fractions learning. However, "children can act in the micro world without having constructed the interiorized mental operations" (Steffe & Olive, 1996, p.120) or they may have constructed the mental operations but use the available actions in a micro world for purposes unrelated to the operations. Therefore, observed actions in a `TIMA:Bars` environment need not reflect students' mental operations, and this underscores the importance of forming and testing hypothesis based on students' actions while remaining open to surprise and revision.

In order to bring forth students' mental operations in their use of `TIMA:Bars`, students' mathematical play is important and was encouraged in the first few episodes. Mathematical play allows students to use their schemes and operations, and this use serves as the goal of the play activity: "[This] is a necessary prelude for students' engagement in independent mathematical activity" (Steffe & Thompson, 2000, p. 25). In fact, students' play in the first few protocols and occasionally throughout the semester informed my task design. From observing the students engage in play activity, I learned about students' interests and comfort zones, and made inferences about their zones of potential construction.

For two of the pairs, the `TIMA:Bars` environment offered too much freedom in that the students' actions and my interpretations of them became overwhelmed with attention to and concern about dimensions of partitioning. Therefore, we backed away from using `TIMA:Bars` and used a companion computer tool, `TIMA:Sticks`, which allows for the construction of very similar micro worlds except that the objects are one-dimensional sticks that can only be partitioned with vertical marks. `TIMA:Sticks` also allows students to partition any part of an existing partition of a stick without breaking the stick. This available function of partitioning recursively was especially beneficial to the two pairs of students because they had not yet constructed recursive partitioning as an interiorized action (i.e. an operation).

Research Setting

During the episodes of their teaching experiment, each pair of students sat to one side of me in front of the computer. We began meeting in a large storage/work room at the end of the sixth grade math hall. The computer rested on a long table that extended lengthwise against one of the long walls. There was also a chalkboard on that long wall, to the left of the computer. Behind the students, there was a camera focused on the computer, and another camera focusing on the students was behind and to the other side of me, where the witness was. The witness was a graduate student in mathematics education who was interested in the study and able to commit time participating in and discussing the teaching experiments with me.

Each student had one mouse of a pair to use that had been spliced so that either student could control the cursor on the screen with his or her mouse. The witness sat behind us taking notes and monitoring the two cameras. One of the cameras was focused on the two students and myself, while the other was zoomed in on the computer screen. The set up varied slightly because we had to use three different rooms throughout the semester: an empty storage room at the end of the sixth-grade math hallway, a back room in the library, and a conference room. However, the general set-up described above was always used.

Fractions Schemes

As a pilot study for the present study, I used videotaped data from The Fractions Project to identify the cognitive constructs used and modified by students in conjecturing activity. In fact, the two examples provided in the first chapter (Protocols I & II) were taken from that pilot study. Through my analysis, I became familiar with children's schemes constructed en route to the development of *rational numbers of arithmetic*: "a scheme in which fractions have become abstracted operations" (Olive, 1999, p. 281). I focused on two students who were operating in a range of development that proved to be conjecture-rich. In particular, one of those students (Joe) appeared to undergo a cognitive metamorphosis that resulted in a new operation, referred to by Confrey (1994) and Olive & Steffe (2002) as splitting. Because of my familiarity with the schemes leading up to and resulting from such a metamorphosis through my work with The Fractions Project, and because of the richness of conjecturing and cognitive change demonstrated by students in the project operating near the development of splitting operations, I decided to continue my study of conjecture by working with students in that same range of development. Figure 4 outlines the network of cognitive structures in the desired range.

The *explicitly nested number sequence (ENS)* is a whole-number scheme of the epistemic subject that allows it to use results of counting as input for further operating. This means that students can view numbers such as 5 as both five 1's and as a composite unit that may be used in further operating. The scheme includes operations of iterating unit items and disembedding (mentally establishing a piece, simultaneously, as a separate item and part of a whole in which it is embedded) a part from a whole (Steffe, 2002, p. 270) that allow the student to solve problems involving the difference of two whole numbers. For example, a student with an ENS can find the difference of 37 and 29 by disembedding 29 from 37 (each of which are records of past counting activity that could be activated again but do not need to be) and counting his acts of iterating by one, from 29 to 37. Partitioning and progressive integration operations may also develop from an ENS. Progressive integration leads to a construction of connected numbers whose development was outlined by Steffe (2002), and partitioning leads to the construction of a critical partitioning scheme called equi-partitioning.

Fractional Schemes Emerging from the Explicitly Nested Number Sequence

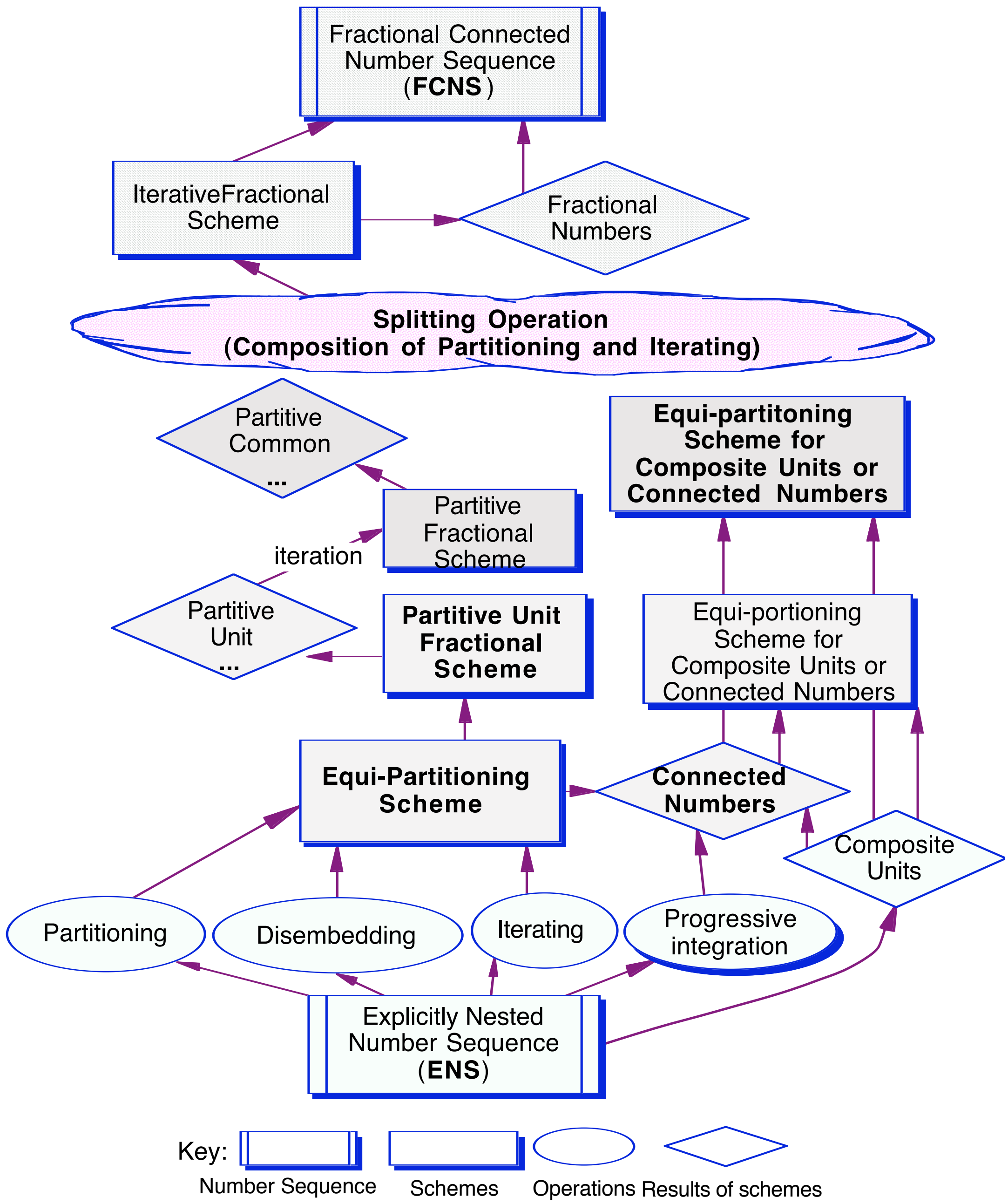

Figure 4. A trajectory for fractions schemes, from (Olive & Steffe, 2002, p. 436).

The *equi-partitioning scheme* is one in which "[students] use their composite units as templates for partitioning [an object] into equal and connected parts" (Steffe, 2002, p. 272). It is not a fractional scheme but a partitioning scheme in which any part may be treated as an iterable unit (as with ENS) to reconstruct the whole. A student attributed with this scheme may also use a *part-whole partitioning scheme* to establish meaning for fractions as some number of parts within the partitioned whole. However, the relationship between part and whole is still a whole number relation, until the student begins to compare the size of the part back to the whole *and* use fractional language to describe the part. Then, we can attribute a *part-whole fractional scheme* (not identified in Figure 4) to the student, recognizing part of the whole as a fractional part of a partitioned whole. *Fractional schemes*, in general, are those that employ fractional language and account for the relative size of the fraction to the designated whole. Students limited to a part-whole fractional scheme rely on a previously partitioned whole and still cannot identify the size of a given fraction by iterating it within an unpartitioned whole. The transition to such a treatment of the composite whole yields a *partitive unit fractional scheme* and the production of unit fractions.

The partitive unit fractional scheme can be used to establish a multiplicative relation between a unit fraction and the whole. Students attributed with this scheme can use fractional language in referring to the *size* of a part, in addition to part-whole references. Although a student without a partitive unit fractional scheme may be able to iterate a unit fraction in activity in order to re-create the whole, the meaning of the unit fraction is still based on additive part-whole operations. With a partitive unit fractional scheme, unit fractions attain their meaning through interiorized operations of iteration, which result in a multiplicative relation between part and whole. The interiorized operations of iteration can also be used to build meaning for non-unit fractions (including improper fractions) as a multiple of the unit fraction, but such constructs are not fully understood as fractions until their size is explicitly compared to the whole. Once students can iterate proper fractions (fractions which can be embedded within the whole) and compare them to the whole, we can attribute to the student a more general *partitive fractional scheme*. Still, many fractions tasks, especially those involving improper fractions, will remain problematic for such students because the student can lose track of the whole in iterating.

The cloud over the next operation in Figure 4—*the splitting operation*—indicates some uncertainty concerning its development. Because the scheme seems to emerge as a global result of activity with fractions and perhaps after extended periods of inactivity with fractions, its construction cannot be attributed to a functional accommodation, which would occur through and in the context of mathematical activity. Instead, Steffe (2002) referred to its construction as a metamorphic accommodation. While I am mostly concerned with conjectures that directly result in functional accommodations, it is possible that a conjecture may engender a metamorphic accommodation (which involves the re-interiorization of concepts on one level to a higher level over a period of prolonged perturbation). The schemes building up to and resulting from the emergence of a splitting operation seem to provide a conjecture-rich realm for students' activity and operating.

Splitting operations enable students to conceive of a whole as a partition into a specified number of pieces and *simultaneously* to conceive of each piece as a fractional part of the whole that can be iterated the specified number of times to re-establish the whole. Because records of the whole are contained in the students' conception of each piece, the whole is not lost in iteration, even when the child iterates beyond the whole to produce improper fractions. This means of *intentionally* producing improper fractions is the most obvious new function provided by an *iterative fractional scheme*. The iterative fractional scheme enables a student to "produce *any* fractional amount through iteration of a unit fraction and establish its multiplicative relation to the fractional whole" (Olive & Steffe, 2002, p. 431). Operating in such a way results in the construction of a *fractional connected number sequence (FCNS)*, which consists of connected numbers, except that the units of the connected numbers are now unit fractions whose whole may be embedded in a larger connected number (improper fraction). Olive & Steffe (2002) determined that Joe had constructed an FCNS by fifth-grade, which was Joe's grade-level in the protocols presented above. Included in my description of those protocols are a few schemes and operations that go beyond those shown in Figure 4.

<u>Student Selection and Pairs</u>

Knowing that I wanted to work with sixth-grade students in the spring of 2003, I began looking for opportunities at local public middle schools in the fall of 2002. In December of that year, I was introduced to an assistant principle from a poor, rural middle school outside of Atlanta. After I had received approval from the county, he introduced me to three sixth-grade teachers: Mrs. Wood, Miss Rose and Mrs. Biltmore. While I had planned on working with one teacher, these three teachers worked closely together and wanted to be involved. All three teachers were experienced and were considered to be good teachers by the administration. They viewed my study as an opportunity for some of their struggling students as well as some of their more successful-but-bored students to receive more individual attention from another teacher. I explained to the classroom teachers that my goal was to study student reasoning rather than to facilitate the construction of particular concepts, but that learning was inextricably tied to my study. I also volunteered to help tutor students on particular skills for as many as fifteen minutes before each episode. After agreeing to work with me, each teacher introduced me to one or two of their regular track math classes (there was an honors math alternative), and I observed each of these classes for one or two days.

The classroom teachers identified students of various stages of ability (but with attention to those who would benefit from individual instruction), and I began data collection in February, 2003, by conducting individual interviews with each student and then working with selected pairs of students at the middle school. Since the students' school year ended that May, I had no more than sixteen weeks to complete the teaching experiment. I was also limited by the number of meetings I could have per week. On the one hand, I needed to meet with students at least two times per week so that lapses between meetings did not break the continuity of tasks. On the other hand, I needed time between sessions in order to complete initial analyses of the previous session and design tasks for the next session. Working with students during their class time, I was able to visit three days per week and schedule two different pairs for episodes on a given school day. This meant that I could reasonably expect to work with three pairs of students.

Within the range of operating described in the "Fraction Schemes" section, I searched for three pairs of students, with the members of each pair operating compatibly. I wanted to select one pair of students who had constructed equi-partitioning schemes but not partitive fractional schemes, one pair with partitive fractional schemes but without splitting operations, and one pair of splitters. The diversity in stages of operating between pairs was designed to allow for greater specificity in patterns of conjecture, while cross-pair analysis would allow for greater generality in findings about these patterns. In choosing these pairs, it was important that members within each pair were compatible affectively as well as cognitively. I relied on the classroom teachers' recommendations in determining affective compatibility. I also left open the possibility that I might have to change pairings once the teaching experiments began, in order to find compatibility between students. Indeed, I ended up switching partners between two of the pairs, in part for that reason. However, my pairings were restricted by student schedules and students' special needs.

I decided to work with *pairs* of students so that my study would include student-student interactions, in addition to student-teacher interactions. Students would need to assimilate each other's actions using established schemes in order to interact meaningfully. Students' responses to each other in action, then, would reveal something about their own constructs. In responding to one another, students might also become more reflective of their own actions, as Joe did after Melissa's counting activity in Protocol I. Such reflection can lead to conjecturing activity and a reorganization of schemes, as it did for Joe. Even the interpretations a student makes of the other's actions may be conjectural. Each of these potential responses contributes to an aspect of classroom interactions that should inform the implications of my study.

Initial Interview Design

In order to identify the six students needed for my study, I conducted individual interviews with eleven students from four sixth-grade classrooms. In order to limit the number of interviews, I first considered the recommendations of the classroom teachers concerning individual students who might be operating at the desired stage and might be interested in participating, as well as recommendations for particular pairs of students who might work well together. These students were taken from class, one at a time, to a workroom for a twenty-minute interview, where I verbally posed a list of tasks. Students were given construction paper, string, scissors, tape, a ruler, and markers to complete the tasks. Students' verbal, written, and motor responses to these tasks were used to determine their available schemes and stages of operating. Once six suitable students were identified and their parents' had signed forms agreeing to let them participate, I discontinued interviews until a replacement was needed (due to personal differences between two students in one of the pairings).

The interview tasks, given in Appendix A, were adapted from Wanda Nabors' dissertation (2000, pp. 298-311). As with Nabors' study, I explicitly intended to design "fractional reasoning tasks that would be difficult to solve using school math rules and algorithms" (2000, i). Each of these tasks, then, would require that the student be able to operate at specific stages in order to complete the task successfully. Below, I describe some of the hypothetical operations identified by Steffe and Olive (Steffe, 2002; Olive, 1999) associated with each task.

1. The first task requires that the student have some sort of equi-partitioning operation available that can be used to create fifths. Then, for part (b), the student would need at least a part-whole concept of fractions that could be used to identify a non-unit fractional part. These may be present in a part-

whole partitioning scheme, but a student limited to such a scheme would not *immediately* recognize the relation between the complement (left-over piece) and the whole. For this, the student would need something like a partitive fractional scheme with which fractions are constituted in relation to their complements and the whole.

2. The second task is much like the first except that it does not introduce fractional language. Students may be more oriented toward using whole number knowledge to solve the task. For instance, a student may use her schema for 6 as a "template for partitioning" as Steffe has hypothesized (2002, p. 272). In order to show that her share is fair, the student may use the cutout piece, placing the piece in each of the successively adjacent positions of the partitioned whole. She might also use folding or marks to demonstrate equal parts. If the student uses a piece to reconstruct the whole through iteration (rather than simply checking its size against the sizes of the other parts), it would indicate that the student had constructed an equi-partitioning scheme.

3. The third task is much like the first except that it deals with pieces that are described by non-unit fractions. In order to interpret the task appropriately, the student must make meaning for two-sevenths. This meaning may be based on iterating one seventh twice, or a student may simply use a part-whole partitioning scheme. Students have to use fractional language to describe the leftover piece. Some students may be able to identify this fraction readily while others may have to count pieces and still others may not be able to use

fractional language at all to describe the leftover piece. These potential responses offer feedback for the researcher to make inferences about students' fractional schemes and their understanding of fractional language. Part (b) is more difficult because students must operate with figurative material rather than perceptual material.

4. This task requires simultaneous operations of partitioning and iterating in order to posit a hypothetical piece of string and conceive of the whole as the result of iterating that piece of string, while recognizing the hypothetical piece as the result of partitioning the whole. This would imply the existence of splitting operations. Part (c) will only be used if the student is successful with part (a). Steffe (personal conversation, 1/29/03) has hypothesized that students who can resolve (a) can also resolve (c) by the same kind of splitting operations, and he suggested that string might lend itself to such activities because it may be perceived as one-dimensional and is easy to manipulate.

5. Solutions to this task may also involve a simultaneous composition of partitioning and iterating. It is possible that a student with a partitive fractional scheme could solve this problem by conceiving three-fourths as one-fourth iterated three times. If the student then partitioned three-fourths to construct three one-fourths as if she were partitioning a whole into three parts, she could iterate once more to create the whole. This would involve using the partitive fractional scheme reversibly in reproducing the whole from a fractional part of it—*a reversible partitive fractional scheme*. Students might also use splitting to partition three-fourths into three of one-fourth.

6. To complete this task the student would have to move beyond a part-whole conception of fractions. Indeed, for many students four-thirds is nonsensical. I would infer an iterative fractional scheme for students who could interpret this task meaningfully and iterate one-third beyond the whole without losing the reference to the whole.
7. Just as Task 5 requires reversibility of a students' partitive fractional scheme, this task requires such operations within a students' iterative fractional scheme—*a reversible iterative fractional scheme*.
8. This task is intended to test whether a student has *recursive partitioning operations* with which to partition each fractional unit in a unit of units produced by a previous partition into smaller fractional units. If the student can name the fractional part without counting each little piece created by recursively partitioning the whole, there is strong indication that the student has constructed a *unit fractional composition scheme* with which to recursively partition and coordinate the sizes of the smaller units in the resulting unit of units of units.
9. If the student lacks a recursive partitioning operation, the student may still be able to act in a situation such as this by partitioning sequentially, as this task demands. However, naming the fractional part (and its complement) that results from the second partition may require the student to count the resulting number of parts in the whole bar, unless they can use a whole number multiplicative relationship in the fractional context.

10. This task may involve the use of a reversible unit fractional composition scheme with which the student could identify one-sixth as half of one-third.

Analysis of the initial interviews for the seven students (the six original students and one replacement) is presented in Chapter 4.

Task Design

Once the teaching experiments with student pairs began, the first session or two with each pair was dedicated to free play, allowing the students to exercise their ways of operating using the actions available in `TIMA:Bars`. Goal-directed activity followed from play through tasks that either the students or I posed, based on experiences in play and students' available operations. Within the context of this goal-directed activity, it was important for me to pose tasks that attempted to provoke conjecturing activity and a general conjecturing disposition. While most of these tasks had to be designed on the spot or between sessions (as described in the analysis section below and with more detail in Chapters 6 and 8), in order to test emerging hypotheses concerning students' conjectures, I began to plan my task design before the first episode based on a couple of hypothetical examples.

As an initial hypothesis, I assumed that tasks fitting both of the following descriptions were likely to provoke conjecturing activity: ones that could be understood by the student in terms of a meaningful goal whose attainment is problematic; ones where students' available operations are sufficient for achieving the goal if these operations (including those embedded in schemas) are used in novel ways. For example, the task posed to Joe and Melissa of creating eighths from three-fourths (Protocol I) fits the description, as demonstrated in the discussion of the first vignette. Questions following students' initial responses to the task can lead to further conjecturing activity. I might ask students how they can test their conjectures or to explain why they believe their conjectures to be true. These questions should be posed to invite the student partner to join in the activity.

In between sessions, I analyzed students' available schemes, their conjecturing patterns, and the effectiveness of the posed tasks. If I hypothesized that a student had constructed a particular scheme through conjecturing activity, I might design tasks just to test for this. In addition, I considered my model of students' available schemes in comparison to my own schemes in order to hypothesize paths for students' reorganization resulting in more powerful schemes. As an initial hypothesis, I claim that students' reorganization can often be achieved through conjecturing activity. As such, I used hypothetical patterns developed from my experience in past sessions in order to hypothesize the kinds of conjecturing activity required to modify schemes and devised tasks to facilitate this. For example, if the characterization of abduction given in Chapter 2 describes one pattern of conjecturing, setting the student up to perceive situations with surprising results (in light of their present ways of operating) may provoke them to search for a general way of operating that resolves such situations.

Analysis

Initial analysis consisted of two major components that lasted the duration of the school's semester: building second-order models of students' operations and examining my role as teacher-researcher interacting with the students. Throughout the teaching experiments, I continued developing my models for students' constructs and conjecturing activity. This was especially important in designing tasks, and it occurred continually during the teaching episodes as I created and tested hypotheses and found ways to provoke student conjectures through posing tasks. To form my models and use them to design tasks, I had to posit ways of operating that would explain students' actions as reasonable interpretations and reactions to posed tasks. The source for these hypothetical ways of operating had to be my own potential ways of operating. Furthermore, my potential ways of operating were used to design tasks that would challenge those of the students in order to provoke conjecturing or to test my second-order models. Each night, after a day of episodes, I would read my notes, watch the tapes, and type a couple of pages describing students' actions. These descriptions would finally include a set of tasks for the next episode, based on the hypothetical mental operations of students.

Second-order models of my role as teacher-researcher resulted from reflections on my own process of hypothesizing, in addition to reflections on my teaching. Since it would be difficult and counterproductive to attempt this in my stream of experience with the students, I relied on the witness and videotaped recordings to perform this analysis between sessions. I would talk with the witness during our travel to and from the school and review the videotapes at home, with the witness' comments and written suggestions in mind. In watching the tapes, I tried to pay special attention to my role as teacher in order to guide my future interactions with the students. This was not easy, because my tendency was to think only about the students' actions and to make inferences about their reasoning. Second-order analysis also informed the questions and tasks that I would pose in subsequent episodes, and resulted in identifying general patterns of students' conjecturing activity and modifications to my hypotheses about the students' cognitive structures.

Once the teaching experiments were complete, I needed to code the data. I had about fifty episodes recorded on about one hundred mini-DV tapes, and it was difficult to get a handle on it all. It was especially difficult because I had spent a summer away from the data, teaching in a summer program for mathematically gifted high school students. I decided to mix the two tapes for each episode (one tape of the students and one of the computer) to a picture-in-picture format and compress them onto a single video stream. This left me with fifty tracks, and within each, I could quickly jump to any given segment. Since the mixing was real-time, I used the time to review the data and re-familiarize myself with the students.

Once the data was compressed, I analyzed all of the episodes for one pair, before moving on to the next pair. This helped me to cut down the volume of data that I needed to consider at one time and to focus on each student's constructs. My goals in this second phase of analysis were to describe the kinds of activity during each segment of the episodes and to establish models of operations and schemes accounting for students' actions. The latter would help me to identify cognitive change resulting from conjecture and the available constructs that students might use during conjectural activity. The former was especially helpful in determining which segments were conjectural. I decided to transcribe initially only those segments that indicated the existence or nonexistence of particular ways of operating and ones that fit my initial (loose) characterization of conjecture described below. I would later transcribe other segments that demonstrated support for, testing of or results from the conjectural activity.

In order to manage the data, I created computer worksheets with brief descriptions of general activity of the students in each episode. Each line in the worksheet [e.g. Appendix B] represents about a minute of video and includes information on who was controlling the mouse for that minute, what schemes might be involved, and whether the segment of video included conjectural activity. I considered a segment of the episode to be at least potentially conjectural if there was evidence of concern or uncertainty about a student's actions in a particular situation. I had to be able to infer that the student had some goal in mind that she could not satisfy through use of well-established constructs. Such segments may indicate perturbations, as in the initial interview with one of the participants (Josh) who paused in the midst of activity and exclaimed, "man, this is confusing!" My initial characterization was sufficiently broad to capture most, if not all, conjecturing activity and was refined through analysis.

The primary goal of my analysis was to determine the nature of conjecturing: How are conjectures formed and what learning do they produce? As indicated in my framework, this goal is attainable by inferring the existence of cognitive structures and changes in them, through observation of students' actions in response to problematic situations. So, I was especially attentive to students' actions and interactions that indicated mental operations that may be employed in problem solving.

Throughout my analysis, I tried to capture my role and the role of the unique micro worlds environments in which the students were working. This is important because the environment (including myself and the student pairs) defines the freedoms and constraints for students' actions. I was only able to observe the actions of students through their interactions with the given medium and their communication with particular people in a particular domain. While the micro worlds constrain action, they also enable action. To speak metaphorically, my analysis is like trying to understand an animal relative to its environment. Though the animal might behave differently in another environment, there are inferences one can make about the lion if she considers the given parameters and is careful about ascribing generality.

Reporting of Analysis

For a few reasons, my final analysis is restricted to the actions of two pairs of students. First of all, I was unable to find pairs of students operating at three different stages of development. Secondly, after reporting on the higher-stage pair, I had already produced a tremendous amount of analysis. Finally, by the time that the two lower-stage pairs switched partners, I had determined which newly formed pair would be most interesting and decided to focus on the students forming that pair.

In Chapters 5 and 7, I describe the nature of conjecturing by drawing inferences about students' ways of operating, through observing students' actions in problem solving. Because the analysis presented in those chapters is so lengthy and difficult to manage, I synthesized them in Chapters 6 and 8, respectively. The reader may choose to read Chapters 6 and 8 first, using Chapters 5 and 7 as an audit trail to support claims made in Chapters 6, 8, and 9.

Chapter 4: Initial Interviews

I conducted the individual interviews with students using the interview tasks described in Chapter 3 and Appendix A. I interviewed a total of eleven students, beginning on February 2nd, 2003 and concluding on February 21st, 2003. In nine of the eleven interviews, I found evidence that the students had constructed, at least, equi-partitioning schemes—the minimal developmental requirement for participation in the teaching experiments. Of the nine students, eight expressed interest in participating but one changed his mind. Among the seven remaining students, I identified two higher-stage (Hillary and Andy), one middle-stage (Will) and four lower-stage (Cory, Matthew, Josh, and Sierra) students. However, the two higher-stage students did not get along, so I formed one middle/higher-stage pair (Hillary & Will) and two lower-stage pairs (Cory & Sierra; Matthew & Josh). The lower-stage pairs were later interchanged due to scheduling conflicts.

In this section, I describe the responses of the seven participants. The descriptions are intended to provide an initial model of the students' fractional knowledge. The protocols within each student's interview are labeled in the following manner: "Protocol [first letter of student's name, or 'T' for myself as the teacher-researcher][sequential numbering of protocol for that student]." Also in the protocols, descriptions within brackets are given in terms of the researcher's concepts.

Sierra

Sierra was a student in Mrs. Wood's first period class. She had long brown, braided hair and dressed casually. She was in the school band and liked to talk about riding dirt bikes at her uncle's house. Mrs. Wood had recommended Sierra for my study because Sierra had expressed interest in participating in it, and Mrs. Wood felt Sierra needed extra help in building fractions concepts. During the interview, Sierra was very reflective, cooperative, and did not seem afraid of being wrong.

She was very good at visualizing and estimating partitions, understanding the importance of creating equal parts. For example, in response to Task 1, she marked off one-fifth of a "candy bar" by drawing four evenly spaced vertical lines, except that the last part was obviously bigger than the others. Dissatisfied with the size of the larger part, Sierra flipped the bar over and repartitioned it into five parts that were nearly equal in size. She was able to identify each of the five parts as one-fifth and could also identify four-fifths by shading four of the five parts. She provided even stronger indication of her ability to estimate equal partitions in response to Task 2, in which I asked her to cut off her share of a candy bar shared between her and six friends. She cut off a piece and demonstrated that she had an equal share by lining it up with the leftover part (using her share as a template) and cutting six more pieces from it; the seven pieces she created were uncannily close in size. Her actions in response to both Task 1 and Task 2 indicate that Sierra had a well-developed equi-partitioning scheme.

After the first task, I asked her how she knew where to make the first partition. She replied, "I have no idea; that's just what they taught me, to start from here [pointing the left side of the bar] and go down [sweeping her hand from left to right over the bar]." It appears that she had some way of anticipating where the other evenly spaced partitions would fall when estimating where to put the first one.

When I asked Sierra, following her response to Task 1b, how she knew that the four shaded parts made four-fifths, she responded, “because there are five spaces in there and like say somebody ate like four of these spaces.” This indicates that she could use a part-whole partitioning scheme and an equi-partitioning scheme in order to work with non-unit fractions. If I could find indication that she compared the four-part bar back to the whole, I would attribute a part-whole *fractional* scheme to her. Task 3 provided Sierra with further opportunity to demonstrate her ability to work with non-unit fractions. In particular, when I covered up all but three-sevenths of a seven-sevenths bar, she was prompt in determining that the hidden part must be four-sevenths by “adding four and three.” So, she was able to use the complement of a partitioned fraction to reproduce the partitioned whole. While her reasoning with non-unit fractions was additive and her responses thus far provided no indication as to whether Sierra had a fractional scheme, they do illustrate the robustness of her equi-partitioning and part-whole relations.

For Task 4a (in which Sierra was asked to make a string so that mine was twice as big), Sierra was unable, at first, to posit any part at all: “I really don’t know how to do that.” Eventually, she marked off with her finger a part of my string that was about three-fourths of the whole. I asked whether there was a way to check whether mine was twice as big as hers, and she replied, “I have no idea.” She expressed even more confusion about Tasks 4c, 5 & 6. So, I decided to come back to Task 4a using a rectangular bar instead of a piece of string.

Protocol S1: *Sierra's response to the revision of Task 4a.*

T: This is my bar and my bar is twice as big as your bar. Could you make your bar?

S: [draws a vertical line about two-thirds of the way from the left end of the bar]

T: Which one is your bar?

S: [shades the larger part of the bar]

T: Is my bar twice as big as your bar?

S: Um… No. Well, I was thinking that this part [the shaded part] was mine, the whole thing was yours, and this [the unshaded part] was just the extra part.

T: So, do you think that my bar is twice as big as your bar?

S: Yes.

T: Okay. How do you know?

S: [thinks for several seconds trying to express herself] I kind of thought that it might start there [pointing to the partition]

T: What might start there?

S: The twice.

T: So what does the word *twice* mean to you?

S: Well, I kind of thought about halving this [marking a line down the middle of the unshaded part] so I would have two equal parts.

Evidently, Sierra thought of *twice* as being two more equal parts. The two equal parts she created by marking half of the unshaded region were scandalously close in size to the sevenths parts she had created in solving Task 2. It seems that she was unable to posit hypothetical parts that were to be iterated so many times to make the whole, and instead used her records of making parts for Task 2 in order to posit two parts that would complete the whole from the desired fraction of the whole bar. Her actions indicate that she was reasoning additively and in the absence of splitting operations. Because her means of checking the desired fraction did not involve iteration of the fraction, I also question whether she had constructed a partitive unit fractional scheme. In order to further test Sierra's meaning of *twice*, I designed a discrete task using previously produced bars.

Protocol S2: *Sierra's iterating in discrete cases.*

T: Here's a bar [placing one of the bars Sierra created in Task 2 in front of Sierra, and placing the other six beside her]. Can you put together some more bars so that you have a bar that's twice as big?

S: [places three of the bars side-by-side] There. That's one bar.

T: How many do you have now?

S: [pauses and looks worried] I couldn't do that, because the 1 is on the bottom, and the 2 is on the top. [Sierra does a calculation on paper, dividing 1 into 2 to yield 2.] So, it would be 2. [She takes away one of the three bars.]

My question to Sierra at the beginning of the protocol engendered a conflict that was resolved in an unexpected way. It seems that Sierra had been assuming that the two more was needed in addition to the original bar, rather than replacing it, and I was surprised that she resolved this issue by referring to a division algorithm. She was subsequently able to identify five times as much as one of the bars by lining up five of the bars. Her actions indicate that she had resolved the conflict and built a new meaning for *n times as much*, but her reasoning may still have been additive and might not translate from discrete to continuous cases. In fact, when I repeated the task from Protocol S1, Sierra acted as she had before, drawing a line of partition about three-fourths of the way across the bar.

There was some indication at the end of the episode that Sierra had a partitive fractional scheme available. Repeating Task 5, I had asked her to make a string such that mine would be three-fourths of hers. She cut off a length of string that was remarkably close to three-fourths of mine, but it could be that her production was based on reasoning similar to that demonstrated in Protocol S1. If she had produced the three-fourths string purposefully, even though she would be confusing our strings, it would be a strong indication that three-fourths was a partitive fraction for her. But after all, she could not explain her production in any way that I could discern as meaningful.

During the next and final task, there was also contra-indication to the claim Sierra had constructed a partitive fractional scheme. I had asked her to make one-fourth of one-half. After I encouraged her to start by marking off one-half, she was able to partition one of the halves into four equal parts and identify one of these parts as the solution. However, when I asked her to name the fractional part, she claimed that it was "one and a half." While the task was designed to test for recursive partitioning (which she also seemed to lack), *one and a half* would not be a viable option for proper fractions, from a student who had constructed partitive fractions. Instead, the name simply described her actions of taking one part from half of a bar. So, it seems that, regarding fraction names, Sierra was confined to a part-whole fractional scheme, at best.

From this initial interview, I have built a model of Sierra's ways of operating that includes an equi-partitioning scheme and a part-whole (partitioning or fractional) scheme. Although she could iterate discrete objects, she seems to lack an interiorized operation for iterating fractions. My model of Sierra would place her in the lower stage, but at least strong enough to participate in the teaching experiment. Throughout the initial interview, she appeared energetic, willing to try out her ideas and unafraid to make mistakes. So, I decided to ask her to participate in my study, initially pairing her with a boy named Cory in one of two lower-stage pairs. Because of schedule conflicts, I had to switch the partners amongst the two lower stage pairs, at which point Sierra began to work with a boy named Josh.

Josh

I interviewed Josh during Miss Rose's first period class on February 10th, eight days after Sierra's interview. He was a football player with short brown hair. He was quiet but not shy, and was very responsive to my questions. Miss Rose had selected him because he was a good student who was struggling in her class, and she thought having Josh work with me would help. His mother was the receptionist at the high school. She wanted Josh to be involved in the teaching experiments because she thought it might have benefits similar to tutoring, and I did tutor Josh for about half an hour before starting the interview.

For Task 1, I asked Josh to mark off one-fifth of a bar. He drew four lines partitioning the bar into five roughly equal parts. When I asked him where one-fifth was, he pointing sequentially to each of the five parts, saying “um…” He did not seem to be indicating the individual parts but the partitioned whole. So, I asked him to show me what he would take to have one-fifth of the bar. At that point, he cut off one of the five parts. Once he had done this, I asked him to show me four-fifths, and he immediately pointed to the four remaining parts: “Just these [four], not this one.”

Josh’s equi-partitioning scheme was even more evident in his response to Task 2. I asked him to cut off his share of a bar that was to be shared between him and six friends. Without making any marks, he cut off a piece. When I asked him how he knew that his share was fair, he began to respond, “because if I equaled all those out,” and placed the cut off piece within the remaining piece six times. He had made a very good estimate, and his verbal response indicated an implicit understanding that his estimate should be one of seven equal parts.

Josh experienced some difficulty with fractional language beginning with the third task. I showed him a bar partitioned into seven parts and asked him how much would be left if I cut off two-sevenths. He knew there would be “five pieces,” but I insisted that he name the fraction of the whole, to which he responded “five-sevenths or fifty-sevenths.” He continued after I asked him which of his answers was right: “Seven-fifths. Oh. Well, no… There’s five left, so maybe seven-fifths.” He was able to sort out the language when I asked him whether the parts were fifths, and in Task 3, he was immediately able to identify the complementary (hidden) part of the seven-part bar (when three parts were visible) as “four-sevenths.” Other than this difficulty with fractional language, Josh appeared to have no problem with the tasks so far. So, I decided to move to the reversible partitive situations set up by Tasks 4a & 4c.

Protocol J1: *Indication of a splitting operation and more difficulty with fractional language.*

T: [measures with a ruler and cuts off one foot of string from a pile of string] This is my piece of string and it's twice as long as your string. I want you to make your string.

J: Can I use the ruler?

T: Yes.

J: [measures and cuts off six inches of string from the pile of string] This is only one time longer. Let's see. [He cuts his string in half and moves it two places from left to right along my string.] Twice as… Man, this is confusing!

T: Okay. So, what are you looking for?

J: Okay. I had a half. So, that's six and that's half of it, but that's only one. Then I had three, that's one, two, three… [placing his three-inch string from left to right across the ruler four times] *four*.

Josh seemed to have an intuitive understanding that his piece should be half of mine, but he also had an expectation that my piece of string should be two more of his piece. He also seemed to think that by cutting his piece in half again, mine would indeed be two more of his. Instead, he found that mine was three more of his (or four total), and he didn't know what else he could do. His actions provide some indication that he had splitting operations and a partitive unit fractional scheme available. I reasoned that I might be able to resolve the issue if I could first help him to resolve the semantics. So, I cut out a few equally sized bars (about the size of the one-seventh bar he had produced for Task 2) and posed some tasks involving discrete bars.

I put one bar in front of Josh and asked him to make a bar that was twice as big. He responded by putting together three of the bars. Through such activity, it became clear that "twice" meant "two more" to Josh. To clarify my meaning of the terms, I showed him two bars and told him they were *twice*—or *double*—one bar because they were two *of* the one bar, and half of the two bars was one bar again. He spontaneously returned to the problem posed in Protocol J1: "That was what I had when I cut that [pointing to my foot-long string]. It would have been right." Now that he had clarified the situation for himself, I felt I would learn a lot about his fractional schemes, reversible reasoning, and splitting operations by continuing with Task 4c.

Protocol J2: *Further indication of Josh's splitting operations.*

T: How about if I told you that my piece of string is *five times* as big as your piece of string? This is my piece of string. It's five times as big as yours, and I'd like you to make yours.

J: [matches up one end of the pile of string with the left end of my string and extends it beyond the other end of my string. He makes four marks on the extended string that, placed at intervals that would equi-partition my string into five parts! Finally, he cuts off four pieces from the extended string, one cut at each mark, and lines them up along my string. He does not make a mark at the right end of my string, and so he does not cut off a fifth piece.] So, I need something just a little bit bigger to equal it up.

T: Okay. Which piece are you saying is yours?

J: Right here [picks up one of the pieces he had just cut off of the extended string, and begins mumbling to himself].

T: So, what's bothering you?

J: I cut mine too short. There only needs to be four, four things to equal yours because yours is five times.

Because Josh did not mark off the right end of the extended string, he did not have a fifth mark from which to make a fifth cut. So, he ended up with four pieces. I do not think that the situation was familiar enough for Josh to realize that this might be problematic. After he lined up the four pieces he had cut off, he recognized that the pieces did not remake my string, as he knew they should. But he explained away the problem of having only four pieces by saying that, since mine was five times his, mine should be four more than his. He may have developed this reasoning from his recent activity with the discrete bars, an intervention I had designed to contend with his notion that *n times* meant *n more*. Now, *n times* seemed to mean *n-1 more*! So, rather than recognizing that he should have a fifth piece, he thought that the pieces were too short.

Once again, I decided to intervene. I drew a line on a sheet of paper and asked Josh to draw one so that mine was five times as big. He drew one about the appropriate length.

Protocol J2: (Cont.)

J: Right there. So yours would be four more.

T: Oh! There'd be four *more*!

J: So, I should add one more [cuts off another piece of string, equal in length to the others].

T: So how do you know that mine is five times yours?

J: Because it's shorter.

T: But, how do you know mine isn't just twice as long or four times as long?

J: Cause I put these down here [lines up the five pieces along side my string], and took these away [removing four of them].

Josh's actions indicate that he might be a splitter and be able to act reversibly with partitive operations, but the novelty of his actions in solving reversible and splitting situations created problems for him. Although he could posit hypothetical pieces satisfying Tasks 4a & 4c, his piece was not embedded in the whole as one of *n* equal pieces. At first, *n* times as much meant *n* more, and, later, it meant that his piece should be *n-1* less than the original whole. Josh had to further develop purposeful use of iteration in naming fractions, as well as work out some semantic difficulties before I would attribute to him a partitive unit fractional scheme. However, splitting operations may have been available to him and partitive fractions were certainly in his zone of potential construction.

Because Josh appeared to be operating in advanced ways without well-developed fractional schemes, it was difficult to assign a stage-level to his cognitive structures. But because the lack of a partitive fractional scheme was a particular distinction of the lower-stage and because there had been contra-indication that Josh had constructed partitive fractions, I decided to pair Josh with a classmate named Matthew, in a lower-stage pair. Due to further scheduling conflicts, I had to switch partners amongst the lower-stage pairs, and Josh began working with Sierra after the first few episodes.

Cory

Cory was a shy, small, funny and very likeable student in Miss Rose's first period class. He wore glasses and was rather animated when I interviewed him on February 11th. He had trouble finding words for his thoughts but was otherwise expressive. We spent several minutes working on tasks similar to Tasks 1 & 2, so that I could assess whether or not he could equi-partition.

I began the interview by asking Cory to mark off one-fifth of a bar. He drew a line down the middle of it, but was unable to justify the result. So, I asked him to mark off one-half of a new bar; again, he drew a line down the middle of it, and this time he explained that he was right because the parts were "equal and cut straight through the middle." At first, he referred to the line itself as half, but was subsequently able to identify each of the two pieces as being one half of the bar. I had hoped that Cory's similar productions of one-half and one-fifth would engender a conflict, but when I asked him about their similarity, he wasn't sure it was a problem.

I continued by asking Cory to mark off one-third of a new bar. He drew a line about one-third of the way across the bar, but then stopped saying, "I need to make a graph." He indicated that he had done these in class and proceeded to draw a pie chart within the larger part of the bar, but he struggled to draw dividing lines that created three parts, accidentally making four parts instead. I explained that the dividing lines can be much more difficult to determine for circles than bars. At that point, he excitedly said, "now I know," drew a line about two-thirds of the way across the bar so that the bar was now divided into three equal parts, and proclaimed, "one-third!" Cory went on to produce fourths by making a horizontal and a vertical line down the middle of the bar. He appropriately identified a one-fourth part of the bar, but when I asked him to make and show me three-fourths of the bar, he shaded three of the four parts and pointed to the unshaded one. His concepts of fractions seemed almost inextricably tied to patterns and pictures.

On his second attempt at creating fifths (which followed his production of three-fourths), he reverted to using only vertical lines and found it problematic that the parts weren't all equal. I think that Cory's initial trouble with partitioning was due to his misunderstandings of what I expected from him. Apparently, he had been assigned partitioning tasks in class, and he seemed accustomed to drawing pie charts to represent fractions rather than representing fractions with manipulatives, such as the construction paper bars used in the interview. Once we had the discussion about the dividing lines, Cory seemed to interpret my expectations differently and was able to equi-partition. He was explicit about the necessity of creating equal parts, and was even able to incorporate partitioning strategies from creating pie charts into the context of fraction bars (creating fourths by partitioning with just two lines—one vertical and one horizontal). His actions and explanations indicate that he had indeed constructed an equi-partitioning scheme.

The next few questions that I asked Cory were designed to determine whether or not he had constructed part-whole fractional and partitive fractional schemes. I showed him a seven-sevenths bar and asked him what fraction would remain if I were to remove two-sevenths. He replied "five" and when I pressed him by asking "five what?" he corrected himself: “Oh, five sevenths.” As the second part of that task (Task 3), I showed him three-sevenths of the seven-sevenths bar and hid the rest. Before I could even ask him a question, Cory said, “Uh… four sevenths.” He was also able to name the seven-sevenths bar as such when I asked him how much the whole bar would be. Although, at first, I had to prompt him to name the fractional sizes rather than simply the number of parts, Cory did seem to be able to readily adopt the language and even name the whole in terms of its fractional parts. So, I think that Cory had constructed at least a part-whole partitioning scheme.

Next, I posed Task 4 to test whether Cory had constructed a partitive operations and reversible reasoning. I placed two equally sized bars in front of Cory, instructing him to make his bar under the conditions specified in Protocol C1.

Protocol C1: *Cory’s lack of reciprocity between twice and half.*

T: This is my bar [pointing to the bar farthest from Cory] and my bar is twice as big as your bar.

C: Twice as… [cuts the bar he was given in half, lengthwise, places it in front of me, and then looks at me]

T: Okay. So mine is twice as big as yours?

C: No… I don’t know. I think I was thinking of halves.

T: Yours is half of mine. Is mine twice yours?

C: [pauses] See. When you asked me that question again, I thought I should cut it like right there [cutting a notch about half way along the side of the one-half bar he had just made].

Cory certainly seemed to understand that his bar should be smaller than mine, but this may be attributed to the fact that I had placed only one bar directly in front of him for him to use. If he were operating reversibly, he should have recognized that my bar was twice as much as his initial construction. Instead, he expressed uncertainty about his initial claim (a conjecture) by saying, “I think I was thinking of halves.” This implies that halving and doubling were not reciprocal operations for him. One might argue that Cory’s problem was one of semantics, but later in the episode, working with discrete bars, he was able to connect bars to make twice, three times, and four times a given bar. In fact, when I repeated my original question, he made a connected bar that was twice as big as my bar from the one-half bar he had made.

Protocol C1: (Cont.)

T: This is my bar; it’s twice as big as yours. And, I want to see yours.

C: …see how bigger it is?

T: Yeah.

C: [tapes together the two halves he had made and tapes another copy of the whole] Not the best chocolate bar, but… [laughs]

T: Okay. So how does this bar [pointing to the two taped whole Cory had made] compare to mine?

C: It’s twice as big.

T: Oh! *Yours* is twice as big as *mine*. How do you know yours is twice as big?

C: Because I have another chocolate bar that's exactly the same as that one and if you connect them, I guess…

So, it seems that Cory did understand *twice* to mean *two of* or *two times*. Still, in working with fractions within a whole, as he had been in the beginning of Protocol C1, his actions were uncertain. Cory's uncertainty indicates that he did not have a partitive unit fractional scheme with which to interpret multiplicative relations between a whole and a part of the whole. At the end of the first half of Protocol C1, he even suggested cutting the one-half part in half again, presumably satisfying his criterion for twice by cutting twice. In order to further examine Cory's reasoning in such situations, I continued with Task 4c.

Protocol C2: *Cory's trouble in positing new wholes.*

T: Here's my bar [points to one bar and places several identical bars in front of Cory] and my bar is five times as big as yours. And, I'd like you to make yours.

C: [picks up one of the identical bars] I think I would just have one.

T: One what?

C: One chocolate bar, since you have… So, I think… No. I'm thinking how many. So, I think I would cut *this* one in half [cuts the bar in half lengthwise]. I think that might be it.

T: Show me that mine is five times as big as yours.

C: Mine is… shorter.

T: How much shorter?

C: Four more times. No. Yeah, four more.

T: Show me that mine is four more than yours.

C: If I had it like this [placing the cut off half next to his half, remaking the whole] it would be four more times, so I think it would be five more times.

Protocol C2 elucidates Cory's concept of *n times as much* and his difficulties of working within the whole with such concepts. After initially claiming that his bar should be one whole, he realized that his bar should be shorter than mine and cut his bar in half. However, *five times as much* still meant that one of us should have five of the other's whole. Cory could not posit this new whole within the given whole and even if he already had such an object available in his perceptual field (as in Protocol C1), he had demonstrated trouble in identifying the object as the desired whole. Cory knew that, since I was to have five times as much as him, I should have four more of something, but that something was my original bar and not his bar. This explains why he said that if he had a whole, mine "would be four more times." This also explains why he continually changed such tasks into ones where his bar was *n* times as much as mine, even though he knew that his bar should be smaller! He made that very mistake in a subsequent task in which I asked him to make a bar such that mine was four times as big as his.

Cory had put together four whole bars in response to the task mentioned above. When I asked him whose bar should be bigger, he expressed surprise, and then cut his bar in half. He realized that my bar wasn't four times bigger than the half that he created, but thought that mine might be "three and a half" times bigger! He tried to correct this by cutting his half in half again. Protocol C3 picks up there.

Protocol C3: *Cory's conjecture concerning iterating a new whole.*

C: [places his part in mine five times while counting aloud] All right. So one, two, three… All right. So that would be five. [Upon my suggestion, Cory and I carefully marked off the number of times his bar went into mine.] Okay. So a little over four.

T: So, is mine four times bigger than yours?

C: Yep.

T: All right. That's good. Now I'm going to ask you a tricky question. Can you make one so that mine is *eight* times bigger than yours?

C: [picks up another whole bar] Eight. I'm going to have to make one smaller. Oh, well why don't I use this one here. [He picks up and cuts the one-fourth piece in half.] It might be too small. [He places the one-eighth piece on the bottom of my whole bar.]

T: You think that's it?

C: Mm-hmm.

T: How do you know?

C: Since both of these connected was four, and if I cut them in half, so that might… So, if I cut it in half, maybe it would be eight… then maybe this will add four more strikes [referring to the marks we had made on the whole in keeping track of the iterations of the one-fourth piece].

T: So how many of those halves would fit in there [pointing to the one-eighth piece]?

C: One, I'm pretty sure.

I refer to Cory's actions in Protocol C3 as conjectural, because thus far in the interview, he seemed unable to consider his part as a whole that could be iterated in the original whole bar. I have pointed out a couple of instances where he seemed to experience a conflict or perturbation in resolving the ambiguity of such situations: While he knew that his bar should be smaller than mine, he was only able to iterate my whole bar. The conjectural operation involved embedding an unknown numerosity of his part within the original whole. Because he actually iterated the part five times within the whole at first, it is doubtful that he had anticipated that taking one-half of one-half resulted in one-fourth, a piece that could be iterated in the whole four times. Moreover, he did not recognize that half of that fourth would fit twice into each of the marks created by the fourths. Still, in tasks where my bar was to be *n* times as much as his, he did understand that the larger *n* was, the smaller his part would have to be.

Cory seems to have constructed equi-partitioning and part-whole partitioning schemes, and he was struggling with concepts surrounding a partitive unit fractional scheme. He was especially close to developing fractional reasoning with one-half, and a generalization of operations with one-half might elevate his reasoning for other unit fractions as well. Beyond that, his conjecture during Protocol C3, could result in reversible reasoning with unit fractions.

Considering his expressive and thoughtful demeanor, his demonstrated propensity for conjecturing, and his stage of development, I felt that Cory would be valuable to my study as the second member of a lower-stage pair, along with Sierra. The two students were in the same class, and although they were not friends (both were actually a little shy in class), I felt they would work well together. However, after a few episodes, scheduling conflicts required me to switch partners among the lower-stage pairs, and Cory began working with a boy named Matthew.

Matthew

Matthew was a good-natured student in Miss Rose's first period class who struggled some in math. During his interview (on the same day as Trent's interview), he was wearing camouflage pants and had a chubby, shy smile. I do not have a recording of my interview with Matthew. So, the analysis given here is wholly reliant on artifacts from Matthew's productions (construction paper cut-outs, drawings, etc.) and notes taken during and immediately following the interview. Although he was a little slow in getting started, he demonstrated a part-whole partitioning scheme for fractions in the first few tasks. He could also equi-partition in sharing tasks and use his part-whole partitioning scheme to identify non-unit fractions and complements.

When asked to make a bar that was twice as much as mine, he created ten-tenths with a bar of the same size because "it has more pieces." I asked him whether he had twice as much, and he began to think about the problem again. He then put two bars together. In the reverse question of creating a bar so that mine was twice as big, he eventually folded his paper in half, without any guidance from me. He then used a trial-and-error approach to finding a bar so that mine was five times as big. At first he used fourths and iterated it in my bar to see that mine was four times as big as his. He then made eighths and checked again. Finally he performed a fold that was in between and was satisfied with the result upon iteration. When I asked him to make a string that was three fourths of mine, he lost the whole entirely and made a row of four pieces of string, three of which were about the size of mine. He seemed more concerned with numbers of parts than making equal parts, and the whole was redefined by his actions with those parts.

I attributed to Matthew a part-whole partitioning scheme and equi-partitioning operations, though he did not use these operations flexibly. He did appear to satisfy requirements for cognitive development in the lower pair. Also, he was willing to act out his ideas, and I imagined he would work well with other students. So I eventually decided to ask Matthew to participate, though at that point I knew that the pairings for which I was looking had to be worked out differently than I had initially planned.

Andy

I interviewed Andy on February 18th, 2003. He was a bright and outwardly confident student from Mrs. Biltmore's second period math class. During the interview, he thought intently about each task and said that he liked difficult questions. From his responses to the first few tasks, it was evident that he had constructed an equi-partitioning scheme, so we quickly moved to more difficult tasks. Protocol A1 was extended from Task 4 of Andy's interview. The task was designed to investigate the splitting operation. I gave Andy several pieces of construction paper referred to as candy bars, all about 12 inches by 8 inches, in addition to a ruler, scissors, markers and tape.

Protocol A1: *The splitting operation.*

T: My bar [touching one of the bars] is twice as big as your bar, and I'd like you to make your bar.

A: [immediately grabs a second sheet and folds it in half careful to match up the edges, then cuts along the folding line, placing the piece on the left half of mine]

T: Is mine twice as big as yours?

A: If yours is twice as big as mine then… if this is it [touching my bar], then you can fit two of these pieces in yours [sliding one of the pieces he had made to the right half of mine].

T: And what is yours compared to mine?

A: One-half.

Andy had no problem with the task, nor in establishing the reciprocal relationship between our bars; he responded immediately, confidently and deliberately in producing the required bar and in making use of his fractional language. He had already demonstrated in previous tasks that he could introduce terminology such as "five-sevenths" meaningfully. Either he had already established that his bar would be half of mine because mine was twice his, or he established it as a result of his production. In the second case, he would also have had to posit a hypothetical bar that, when repeated twice, would constitute my bar. Either way, there is solid indication that Andy had constructed splitting operations, at least for one half, because he seemed to be able to consider partitioning (as in his production of one-half) and iterating (as in "twice") simultaneously. His response to the second part of Task 4 serves as corroboration of the splitting operation.

Protocol A1: (Cont.)

T: This time, here's my bar [a new whole sheet] and my bar is supposed to be five times as big as yours.

A: [raises his eye brows and begins to tap the paper five times sequentially along its length] That's a little bit difficult right there.

T: Well you answered the other ones too easy.

A: [begins making four creases along the length and mutters to himself] …folds. I'm estimating here, so this isn't going to be exactly right I don't think [cuts along first fold and shows me the piece].

T: Okay, so show me that mine is five times as big as yours.

A: That's one, two, three, four... [placing the cut out piece on the leftmost part of my bar and sliding it each time one space to the right] Yeah, I was a little bit off… a little too big.

Andy went on to describe how he could use the ruler to make a better approximation but claimed it was "quite hard because it doesn't have the exact measurements." He explained that he would divide the whole length by 5 to get five equal sections. Based on his concern for exactness in his productions, the task appeared challenging for Andy only because it was difficult to approximate one-fifth of my bar, even with the ruler. Andy seemed to recognize from the start that he needed to make one-fifth, which supports the inference that Andy had established a reciprocal relationship between "n times as big" and "one n^{th}." Such a relation involves splitting the stick into five equal parts to produce a stick such that the given stick is five times as big as the stick produced. Andy's responses to the next task, Task 5, are compatible with the inference that he could split a stick.

Protocol A2: *Reversibility of Andy's fractional schemes.*

T: [cuts off twelve inches of string] This is my piece of string, and my piece of string is three-fourths of your piece of string.

A: [thinks for about ten seconds] That'd mean mine was sixteen inches.

T: Why don't you go ahead and make it and let's see.

A: Can I use the ruler? [marks off one ruler plus four inches of string and asks me to cut, which I do and ask him to explain] Because if you have three-fourths of my string like you said, then divide 12 by 3, which is 4, and that's 3 [of the four-inch pieces]. And then if you go 4 times 4 [inches] with mine [sweeping his hand along the length of the string he made], that's 16. And 12 is three-fourths of mine.

It seemed that Andy solved this problem by determining what one-fourth was and then iterating the composite unit four times. Iterating the composite unit four times indicates that one-fourth was an iterable unit, as well as the composite unit, four. So, I can infer that Andy had constructed, at least, a reversible partitive fractional scheme. We can also see evidence that Andy is coordinating units of units of units, three fours and four fours, indicating a *generalized number sequence* (the number sequence resulting from the interiorization of the explicit number sequence). This coordination, when coupled with iterating the composite unit, four, four times, indicates that he had constructed a *partitive fractional scheme for composite units*—a scheme that can be used within a composite whole to re-establish the whole from any fractional part of it. This is the best confirmation yet that Andy had established splitting operations for composite units; he had to establish one fourth through partitioning before he could iterate, and he seemed to enact both operations simultaneously in satisfying his goal of reproducing the whole. Still, he had problems to work out surrounding such operations.

Protocol A3: *An attempt to compare four-fourths and three-fourths.*

T: Ok great. What fraction is yours of mine? [pointing to each, in turn]

A: One-fourth, maybe?

T: Why do you think one-fourth?

A: 'Cause. Well, that wouldn't work. Couldn't say four fourths; that's a whole. This is hard. I have one more fourth of yours than you have of mine. You have three fourths all together; I have four fourths. I have… I have to go with one fourth because I don't think you can go with four fourths.

T: [To help Andy to check his claim, I asked him to identify one-fourth of my bar, which he was able to do.]

A: [changing his mind] It's *four* fourths.

Andy definitely established my part as three-fourths and his as four-fourths, as well as one fourth more than three-fourths. In this additive comparison, he seemed to establish three-fourths as a three-part whole. But to establish his part as four-thirds of this three-part whole, he would have needed to become explicitly aware that he took four-thirds of three-fourths to produce 16, and so four-fourths is four-thirds of three-fourths.

Such reciprocal reasoning is not a necessary consequence of the partitive fractional scheme for composite units. It appears that Andy had not yet constructed a scheme for producing improper fractions in the case of operating with composite units. Protocol A4 was extracted from Andy's response to Task 6.

Protocol A4: *An attempt to produce a string such that 12 would be four-thirds of the string.*

T: Mine's four-thirds of yours [pointing to a 12-inch piece of string].

A: Four-thirds!

T: And I'd like you to make yours.

A: It might mean that mine is three-fourths of yours [cuts off 8 inches of string from the 16 inch string from the last task]

T: Show me that mine is four-thirds of yours [picking up each piece, in turn].

A: I was thinking if you had four-thirds… It's like an improper fraction and you divide 3 into 4 and you get 1 and one-third.

T: Does that mean that mine is one and one-third of yours?

Andy seemed to have cut off an eight-inch piece of string because it was less than twelve inches by four inches, which is one-third of twelve inches. In other words, he knew that, because my string was one and one-third of his string, he would have to cut off one third to produce his string; so, Andy calculated one-third of 12 (four inches) and cut off that amount. This kind of reasoning parallels that of his solution to Protocol A3, in which he found one-third of the given string (four inches) and added it back on to the given string to recreate the whole (sixteen inches). In that case, however, he was able to posit the missing fourth in the whole as one-third of the given string. In the present protocol, he would have to posit the extra third (beyond the whole) as *one-fourth* of the given string. This would involve the integration of an iterative fractional scheme (for dealing with improper fractions) with his partitive fractional scheme for composite units.

As the protocol continued, he returned to his earlier idea that his might be three-fourths of mine and considered each of three fourths within his eight-inch piece. Once again, he seemed to recognize a reciprocal relationship between our two pieces of string, although this time he appeared less confident about it; it may have been a conjecture influenced by my own reciprocal reasoning in posing four-thirds immediately after posing three-fourths (in the last protocol). He may be in the process of generalizing the relation for unit fractions and "n times as much" to situations involving non-unit fractions, and integrating the splitting operations associated with these relations to his partitive fractional scheme for composite units.

Protocol A4: (First Cont.)

A: [laughs a little and then looks intently at the pieces] I'm stumped [laughs again]. This is 1, 2, and this is three fourths here [pointing to one of three equal parts within his eight inch string, then two of them, then three] and you have four fourths. [sequentially marking off between his thumb and forefinger two inches at a time on the ruler] four-thirds… 4 times 3 is 12, which is a foot. You have four-thirds of mine. Then I would have three-thirds of yours, which would be nine inches.

T: Did you make that?

A: I made eight inches. [cuts off nine inches] I think that's right.

T: Show me that mine is four-thirds of yours.

A: You have four-thirds. Yours would be 12. So I'm thinking *four* thirds would be 4 of 3 [inches?], which is 12. If it's four-thirds of mine… wait a minute. [mumbling to himself with his hand on his face] Four thirds, 12, four-thirds of mine...I think mine would be five-thirds.

Andy seemed to draw an analogy between his reasoning with three-fourths, as in Protocol A3, and reasoning with four-thirds. The explicit analogy, illustrated just as Protocol A4 continued, helped Andy to establish that four-thirds was four of one-third and to determine the size of that one third: 4 times what is 12? Thus, Andy was able to correct his original production by focusing on thirds rather than fourths. But while he knew that my string should have four of his thirds, he referred to his string as being three thirds of mine. As the protocol continues, it becomes apparent that his use of "of" often refers to a sub-collection of parts rather than a reference to a whole that determines the size of a fractional part. For example, he explains that my bar has four of the thirds (three-inch pieces) whereas his has only three of those thirds. This is the kind of regression Piaget noticed in children when they operate in unfamiliar ways. This may also explain why he reasoned that his piece might have to be five thirds, a seemingly absurd claim because he already appeared to recognize that my string was longer; he could take four of his thirds to make mine only if he had more than four thirds to begin with!

Protocol A4: (Second Cont.)

T: What would it mean to be four-thirds of yours?

A: I was thinking at first to be four-thirds, which is 4 times 3 is 12. And if yours is four-thirds of mine… And I thought mine would be three thirds which is 3 times 3 is 9. But if it's four-thirds of *mine*, that sounds like it'd be four *thirds* of mine, not mine being three-thirds of four-thirds. [marking each piece of the nine-inch string into three-inch parts] I was thinking mine might have to be fifteen because yours would be four-thirds of mine and if I added one third to mine it'd be five thirds and five thirds is fifteen inches.

Andy had equated each third with three inches, which he was able to do using his partitive fractional scheme for composite units. His surprise at the start of the protocol (and his impulse to change improper fractions to mixed ones) indicates that he was not prepared to work with improper fractions in the manner I suggested, even though he could produce them through iteration. For these reasons, he transformed his goal to one of finding a bar from which he could pull four three-inch pieces. That action would constitute a meaning for my bar being four thirds of his. In other words, if his bar contained five three-inch pieces, he could explain how my bar would be four thirds of his: it would contain four of the three-inch pieces from the five three-inch pieces in his own bar. I might claim that Andy's solution was the result of abducting because it fit the pattern of adopting a new way of operating in order to explain a surprising situation. However, as I have argued in establishing the framework for my study, I would not be able to examine such patterns of conjecture in depth without first establishing the existence of available operations.

Although the last protocol points to some problems in Andy's use of splitting in the context of composite units, he certainly has constructed unit fractions as iterable units. He can also use them reversibly as he did in producing one-fourth from three-fourths with the intent of iterating that fourth to produce the whole. This indicates the simultaneity of partitioning and iterating that define splitting. In fact, he seemed to use splitting operations in the last task, and the novelty of their use in such a situation may explain the problems that he faced. In particular, he had to consider more than one whole and determine what it might mean to take an improper fraction of one whole in establishing another. Thus, I do attribute splitting operations to Andy and find novel situations for his use of them to be conjecture-rich.

In summary, Andy has splitting operations available, but he seems to be restricted to working within a given string. He could posit a longer string by reasoning additively. For example, Andy demonstrated that he could make sense of four-thirds as a given string, by defining it as one more third than a whole. He had conflated one-third with three inches and had also conflated "four-thirds of" as an operation with four-thirds as the quantity, 12. "Of," in such situations, referred to a number of composite units that could be pulled from a larger number of composite units (hence, five-thirds). He did not see, for example, four-thirds of three-thirds as four-thirds. The activities of Protocol A4, described here, demonstrated the limits of Andy's partitive fractional scheme for composite units.

As evidenced by the transcripts, Andy was adept at verbalizing his thoughts and I felt he would be a wonderful participant as part of the high pair. However, his confident attitude was interpreted as obnoxious behavior by some of his classmates. In particular, the other prospect for the high pair, Hillary, did not get along with him at all, as I learned after the initial interviews, when I tried to pair them. So, I had to choose between the two students. For reasons described in my analysis of Hillary's interview (below), I chose to work with her and only used Andy to work with her partner, Will, when Hillary was absent.

<u>Will</u>

Will was in Andy's math class, and I interviewed him on the same day. He was a football player with nice clothes, dyed spiked hair and a fun personality. He sat at the front of the class next to his class partner Hillary, whom I also interviewed. All three students were performing quite well in the class.

At the beginning of the interview, Will seemed to struggle with equi-partitioning. When I asked him to mark off one-fifth of a rectangular candy bar, he tried to divide it like pizza slices as he had done in class, drawing in the diagonals to create four equal parts and then being unsure on how to continue. After he began acting frustrated in his attempts, I suggested that he divide the bar vertically. This seemed to make sense to him, as did all of the proper fractional names that I used in posing such tasks; he also introduced fraction names himself in response to Task 3. Meaningful use of fractional names in creating fractional parts, such a one-fifth, is a sufficient but not necessary indication of equi-partitioning. However, Will exhibited further complications in using his equi-partitioning scheme and fractions concepts in his response to the following task.

I asked Will to share a candy bar between him and five friends. He drew five lines lengthwise across the bar, creating five equal pieces and one large leftover piece. After he drew the second, third and fourth lines, he counted the number of equal pieces he had created. The following dialogue begins after Will cut off one of the smaller equal pieces, and cut off and discarded the larger leftover piece.

Protocol W1: *Will's struggle with equi-partitioning.*

T: Okay, so you're sharing this with five friends, right?

W: Um-hmm.

T: Where's your piece?

W: Right here [touches the small piece that he had cut off].

T: Where's the pieces for your five friends?

W: [touches the other four equal-sized pieces that were still connected] Right here. Well that's four so…

T: Okay, so now you pretend like there were five of you, instead of six of you. So, it was you and four friends, but five total. Okay. What about that piece? [pointing to the larger discarded piece]

W: [picks up the discarded piece and shakes it while showing it to the teacher] That's just extra, left over.

T: Okay, so there'd be extra candy bar left over?

W: Um-hmm.

T: Okay, what if we wanted to use the whole candy bar and you were going to… So you were going to share this with your four other friends; there's five of you, but you're going to use the whole candy bar.

Will then began cutting the discarded piece (which he was still holding) into five equal parts lengthwise. This time he was able to partition the piece efficiently and precisely. This stands in contrast to his initial attempt: Whereas in the initial attempt he was only concerned with creating a specified number of equally sized pieces (as indicated by his counting acts in creating them and his discarding of the left over piece), the second time he had been explicitly instructed to use the whole candy bar and was able to apply equi-partitioning, including the instruction as a goal and acting appropriately to satisfy that goal. So, it seems that Will did have an effective equi-partitioning scheme available, but did not recognize the importance of preserving the whole and, instead, altered it to make the task easier. As the protocol continues, it also becomes clear that Will has trouble reconciling his classroom knowledge in the new context.

Protocol W2: *An attempt to name the equal shares.*

T: Where's your piece?

W: [places one of the old pieces and one of the new pieces in front of me] This is mine right here, and that's theirs [touching the remaining pieces].

T: So, if I was going to get one of these, I would get maybe this [scanning one of the uncut old pieces] and what else?

W: Um-hmm. [picking up one of the new cut pieces]

T: …and one of those. So, everybody would get like you – one of the big ones and one of the small ones. And how much is this [holding Will's share]?

W: That is two… fifths out of all of it.

T: Okay, so the whole candy bar is cut into how many pieces?

W: [sub-vocally counts the pieces while pointing with his right index finger] ten.

T: …and I've got two of them. So I got two out of ten.

W: …so you'd get two tenths of it.

T: So, if I've got two tenths, can you change that into another fraction?

W: You might be able to get it into two-fifths.

T: So, maybe two-tenths is the same as two-fifths? So, if I have two-tenths, like I've got two of these [holding the old and new piece that Will had set aside as his share], right? Then I might have two-fifths of the whole candy bar, too?

W: Um-hmm.

Will recognized the importance of giving equal shares in distributing one of each size to each person. I thought that, in distributing five sets of two, Will might also recognize the commensurability of two-tenths (although, there were not two equal tenths in each share) and one-fifth, but he became confused with the numbers. While he was familiar with fractional names and part-whole relations, he would lose track of the whole and resort to playing with the numbers themselves without much regard for part-whole relations. I *speculate* that part of this might be attributed to issues of affect; Will appears to be very comfortable with guessing and could have been very successful in his math class by finding patterns in numbers without considering context. In some ways, this could make him a powerful mathematician, except that he is lacking the critical reflection necessary to question his own ideas. His equating two-fifths with two-tenths, in which “two fifths” may have referred to the five sets of two, is one such example; others follow.

Using the four-fourths bar that he had produced in a previous task, Will was able to recognize the commensurability of two-fourths and one-half. Still, in returning to the case of two-tenths, Will continued to struggle. While drawing models on paper, he was able to produce guesses such as “two-sixths” but could not establish any equivalence or commensurability with two-tenths. As the protocol continued, it became clear that Will was preoccupied with the numbers and not so much concerned with their meaning in the context of his drawings.

Protocol W3: *Naming an equivalent fraction.*

W: The 2 has to be in there, don’t it? Cause there’d have to be something like two eighths or something like that… or could it like three sixths or…

T: Well let’s see. Let me give you an example: you said that two-fourths is the same as one-half. [writes on the board “2/4=1/2”] Right?

W: Uh-huh.

T: …and I want to know if something similar goes on with two-tenths [writing “2/10=” under the other equation]. If you have two-tenths, is there something that you can say that that’s equal to?

W: One-fifth?

T: What makes you say that?

W: Well, from there it went down to 1 and then half of that because 2 is half and then if you went down it would be 1 and fifth.

This time Will was very explicit in his strict attention to number. He had invented a rule that 2 must somehow be involved in a fraction equivalent to two-tenths. When I asked him to consider an equivalent fraction, rather than directing him to consider quantities, my actions led him to notice a pattern in the numbers. He seemed to be reasoning only with his whole number knowledge, and the situation had lost any contextual meaning regarding the pieces he had made. We might further argue that Will did not understand the relation between fraction names and fractional quantities of wholes (which implies the paucity of a partitive fractional scheme), but then he seemed able to relate them in Task 4a.

Protocol W4: *Indication of a partitive unit fractional scheme.*

T: [Showing Will a rectangular piece of construction paper (a bar)] This is my candy bar, and I want you to show me what your candy bar would look like if mine is twice as big as yours.

W: [draws a line down the middle (widthwise) of a second, congruent bar using a ruler. He places it on top of mine, muttering]. It's half of that bar. It's going to be once. [He then divides the right half in half with a new line and cuts off the marked quarter of his whole bar. He places the piece at the right edge of my whole bar and places it again one space over.] I guess it's just going to be one half [begins to cut off the other quarter from his whole bar, leaving half of the whole].

T: What did you say about that last one?

W: It's not going to be it because I was going to try to put this in there two times [stopping his action of cutting and repeating his action of placing the quarter piece in my whole bar twice].

T: Oh. Okay.

W: [finishes cutting and places the half in my whole bar twice, just as he had done with the quarter piece] That's going to be it right there.

T: That's it?

W: Um-hmm.

T: So mine's twice as big as that one?

W: Mmm… [looks a little confused and slides his piece adjacent to my whole bar before sliding it back on top again]… You got two of my pieces.

Once again, Will's confusion at the end might be attributed to his difficulty in keeping track of the whole. When he slid his piece adjacent to my whole bar, he may have been considering them together as a whole. However, he was able to resolve the situation because he had made meaning for "twice as big": it meant that I had two copies of his piece within mine. He had tested this initially with a one fourth piece. Initially, he seemed to think that my bar would be twice his if he took a half twice (i.e. a half of a half). The fact that he didn't immediately recognize a half as reciprocal to "twice as big" may give us reason to doubt that he had constructed splitting operations. However, there was also indication that he did at least have a partitive unit fractional scheme.

Both one-half and one-fourth were iterable units for Will (at least within the whole); he demonstrated this in testing his first and second solution. Since he used iteration in satisfying a goal, we have good reason to infer the existence of a scheme that incorporates operations of iterating and can anticipate iterating a hypothetical piece. The next protocol tested whether he could operate similarly with larger values.

Protocol W5: *Producing a bar such that a given bar is five times as big.*

T: Here's mine [pointing to my whole bar]. This time I'd like you to make one so that mine's five times as big as yours.

W: Five times as big… [Will immediately grabs the one-fourth piece and begins iterating it within my whole bar. He first keeps track of the spaces using his finger, then scissors, then a marker. After iterating the piece four times, he utters] That's four.

T: So how many times bigger is mine than yours?

W: This one right here is four, so I've got to make it where it's five.

T: Okay.

W: [Will repeated his act of iterating the one-fourth piece four times] …cut half that up. [He then cut the one-fourth piece in half lengthwise, and began iterating one of the halves within my whole bar using a marker to keep track as he had done before. He stopped when he got to the fourth iteration.] …too small. I've got to make one bigger.

T: [laughing] First it was too big, then too small…

W: [Will laughed a little, then iterated another one fourth piece in the whole four times, but without using a marker. He then cut off about one third of the one fourth piece and iterated it into the whole five times using the marker.] Five.

T: So what fraction is yours out of mine?

W: [looking down at the piece, he immediately responds…] One fifth.

Will's responses to this last task, Task 4c, offer stronger indication that Will had constructed a partitive unit fractional scheme. He iterated each of the three different sized pieces in purposefully checking them against his goal of creating a piece so that mine was five times as big as his. Moreover, he made appropriate adjustments when he found that a piece went in too few or too many times, seemingly recognizing a relationship between the size of the part and the number of parts in a fixed whole. On the other hand, I would argue that Will had not yet constructed splitting operations, because he had trouble in each of the last two tasks positing an appropriate hypothetical piece. In particular, for Task 4c, he used the one-fourth piece first even though he had already found that it went into the whole four times and not five times. His subsequent and immediate naming of the final piece, "one fifth," offers another indication that he had constructed a partitive unit fractional scheme.

Splitting operations and a partitive fractional scheme did appear to be within Will's zone of potential construction. Although he could not posit the precise size of the piece he wanted to make, he was able to posit a hypothetical piece with the goal that it would form the whole when repeated so many times. For lack of a precise hypothetical image, Will instead relied on tangible pieces that he had available. After checking, he knew that he needed a smaller piece but relied on halving to create the fifth part in the whole, without realizing that this would actually create four more parts in the whole. This may also point to a lack of recursive partitioning and fraction composition, as did his trouble in equating two-tenths and one-fifth.

Although he faced some complications in equi-partitioning, I felt that many of these complications could be attributed to his classroom experience and his unfamiliarity of my expectations for him in the new setting. Once he accepted the premise of using the whole bar as a goal, he was able to meet this goal. There is also plenty of evidence to indicate that Will had constructed a partitive unit fractional scheme, and that he could make meaning of goals involving partitive and splitting operations.

I felt that Will would work well with other students, and he seemed to enjoy the activities. So, given my assessment of his reasoning, I decided to ask Will to participate as one of the students in the middle pair. However, pairs did not work out as I had anticipated, largely due to the problems that I mentioned existing between Hillary and Andy. Thus, Will worked in the high pair with his classroom partner, Hillary.

<u>Hillary</u>

Hillary was a bright and pleasant girl who, along with Will and Andy, was in Mrs. Biltmore's second period class; I interviewed her three days after the other two. She had long, curly brown hair and glasses. She was a little quiet but was thoughtful and verbalized her ideas well when prompted. Unfortunately, when I videotaped her interview, the sound did not record. So, the following analysis relied upon my notes taken that day, artifacts of her productions, and observations of the silent video.

Hillary had no problem solving sharing tasks. In Task 1, she was able to estimate one-fifth of a whole bar by making sweeps at five intervals and then drawing the first dividing line; she did not need to draw in the other parts in order to recognize the small part as one-fifth of the whole. Furthermore, having drawn three-fifths within the whole bar (one-fifth from the left end and two one-fifth pieces from the right end), she was able to recognize the unpartitioned two-fifths part in the middle as two fifths of the whole. This suggests that she may have had a partitive fractional scheme available in order to make sense of two-fifths without having to refer to the individual units making it up.

When I asked Hillary to make a bar such that mine would be twice as big (Task 4a), she cut out a sort of picture-framed piece that was *similar* to the whole. She may have perceived this as an additional requirement, as a result of schoolwork with similarity and dilations (although the class had not been working with similarity in recent weeks). She later demonstrated that she could easily produce the required piece with a single cut down the middle of a whole bar, so it appears that she had assumed the additional requirement as a goal in her initial production. While it was easy for her to justify that her simpler production worked (because it fit twice into the whole), she struggled some in justifying her initial, more complicated production. Eventually (after I questioned her about the size of the left over frame), she was able to cut up the frame to show that it filled half of the whole, arguing that her initial production must then be the other half. This demonstrated to me that she had a strong understanding of quantity and the contextual meaning of one-half and twice as much.

Hillary smiled, seeming very pleased with her justification that her initial production was indeed one-half of mine, thus arguing that my bar was twice hers. She behaved similarly in response to Task 4c: creating a part so that my bar would be five times as much as hers. Rather than producing the part with a single cut, as she later demonstrated she could easily do, she cut out a piece from the bottom of the whole that was similar to the whole again, this time using only three cuts (there was no need for her to cut across the bottom). Her accuracy was uncanny; I measured her production after the interview and found the area of the whole to be 4.72 times as big as the area of the cut out piece! Still, Hillary had trouble justifying her production. In fact, after placing her piece within the whole, she recanted and cut off a smaller piece from her original cut out; but when she traced copies of that new piece within the whole five times, she found the new piece to be too small.

Hillary's trouble in Tasks 4a and 4c seemed to stem from her concern that the initially produced pieces could not be iterated adjacently within the whole without exceeding either the length or width of the whole. Given her trouble in justifying through iteration, it is even more amazing that Hillary was able to produce the pieces so accurately; it indicates that she could split, but that her splitting operations were not always based on the iteration of a hypothetical part. Rather, the operations may have been based on a more fluid sense of quantity (such as a dilation) that could not be represented easily with the construction paper. In any case, she did appear to have the splitting operations required to posit a hypothetical piece that, when iterated so many times, would reproduce the whole.

For more complicated reversible tasks, such as Task 5, Hillary confused our parts and made three-fourths of my string when she was supposed to make a string such that *mine* would be three-fourths of *hers*. This serves as contra-indication that Hillary had constructed an iterative fractional scheme or even a reversible partitive fractional scheme. Still, if her goal was to create a string that was three-fourths of mine, she was able to accomplish her goal without first partitioning my string. In other words, she could anticipate both partitioning my string into four parts and iterating one-fourth three times. Her ability to anticipate the result of both operations together indicates that the two operations might be simultaneous for her (i.e. she could split), or at least they had been interiorized to a level where they could be treated as abstract entities that she could anticipate composing.

In justifying her construction, Hillary did in fact mark the partitions of my four-fourths string and her three-fourths string. When I restated the task that I had intended for her, I used the three-fourths string as my string and disposed of the four-fourths string. Hillary then lined up one end of my string with one end of a pile of string and cut off a piece from that pile of string that was nearly identical to the disposed string. Had she been able to do this without the previous experience of creating three fourths and without the aid of the marks on my string, it provided solid indication that she had constructed a reversible partitive fractional scheme. As things were, she must have been able to recognize the three pieces marked in my string as fourths rather than thirds and posit a missing fourth, unless, of course, she had simply remembered the size of the discarded string and reproduced it. Her deliberate alignment of the three-fourths string with the pile of string makes me believe the first scenario was the case. As such, her actions indicate that she could preserve the relationship between three fourths and a whole in the absence of the whole. Once again, we could attribute this to splitting and her reversible partitive fractional scheme—establishing an extra fourth from a piece that already contained three iterations of that fourth.

At the end of the episode, I asked Hillary to make one fourth of one half. She had little trouble making the bar: she cut a whole piece of paper in half and then cut off one fourth of that half. However, she could not name the resulting part as one eighth without iterating it within the whole. After placing the part in one half of the whole four times, she knew that it would go into the other half four times and then called the part "one eighth." While Hillary could act out partitioning recursively, I claim that she could not recursively partition because she seemed to be unable to coordinate whatever records she had of her production. It was only in iterating the final piece in the whole, that she could determine its fractional part of the whole.

An iterative fractional scheme was within Hillary's zone of potential construction. Since she was also very thoughtful, reflective and expressive in her activity, I felt that Hillary would make a wonderful participant. So, I asked her to participate as a member of the high pair. I had planned on pairing her with Andy until I learned, after the initial interviews, that she did not get along with him. When I was forced to choose, I chose Hillary as the primary participant because I was confident she would work well with anyone else that I had interviewed, especially her class partner Will who was operating at a similar stage.

As things turned out, I had six students with whom to work, but only one high stage and one middle stage student. I decided to pair those two students and form two lower stage pairs from the others. It was beneficial that Hillary and Will worked together in class and seemed to get along well. I paired Sierra with Cory because they were also from the same class and I felt they both had mild personalities that might be drowned out by more boisterous partners. That left Matthew and Josh whom, again, were from the same class and knew each other from playing football.

I worked out a schedule in which I would visit the school for the first two periods for three days per week. I worked with Hillary and Will two days per week and always during their second period math class; I worked with Sierra and Cory during first period math class one day and second period another day, under an agreement with their second period English teacher; and I worked with Matthew and Josh during their first period math class two days per week. After a few protocols, it became apparent that Sierra did not like leaving her English class. I thought of switching her with Josh because Cory and Josh seemed more advanced than the other two, but Josh could not afford to miss his second period science class. So, I switched Sierra with Matthew, who was in Cory's second period English class as well. Such logistical decisions played as much of a role in the final pairings as did my initial cognitive design for pairs.

Chapter 5: Hillary and Will

Synopsis

Several of Hillary and Will's conjectures are highlighted within this chapter and synthesized in Chapter 6. The main purpose of this chapter is to build models of the students' ways of operating, which changed significantly as a result of conjecturing activity. I provide examples from my data to indicate that Will and Hillary could operate in the `TIMA:Bars` environments using particular schemes and operations. In the first few teaching episodes, Will's actions indicated that he had constructed a part-whole fractional scheme and a partitive unit fractional scheme, but not a (more general) partitive fractional scheme. During the teaching experiment, he constructed several procedural schemes in response to tasks and in assimilating Hillary's actions, but these were not truly fractions schemes. His conjecturing activity did engender a partitive fractional scheme for composite units and, eventually, a partitive fractional scheme.

Hillary's actions in the first few teaching episodes indicated that she had constructed a partitive fractional scheme and a splitting operation; a reversible partitive fractional scheme was emerging. By way of a few conjectures and simpler schemes, she constructed a commensurate fractional scheme during the teaching experiment. Her conjecturing activity engendered several other schemes, eventually including an iterative fractional scheme. I also note her agreeable disposition throughout the teaching experiment in working with Will and its effect on her development.

Introduction

Through my analysis of their initial interviews, I determined that Hillary and Will were advanced (especially Hillary) relative to other students that I interviewed. Whereas Hillary had constructed a partitive fractional scheme and splitting operations, Will had constructed a partitive *unit* fractional scheme; I place a partitive fractional scheme and splitting operations within Will's zone of potential construction.

Through analysis of the first several episodes, I will elaborate on my models of the students' fractional knowledge. These first episodes provided students with the opportunity to familiarize themselves with the available computer tools, and they provided me with the opportunity to get to know the students as mathematical operators and to build a foundational model for their fractional knowledge. Throughout the protocols, "O" precedes the comments and actions of the witness, "T" precedes those of the researcher, and, likewise, the first letter of each student's name is used. Also in the protocols, the comments within brackets are the researcher's descriptions of student action and interaction, task elements, etc.

24 February, 2003 Teaching Episode

The focus of this first episode was for the students to develop familiarity in using `TIMA:Bars`. Will and Hillary took turns asking about and trying out various tools available in the program, including MAKE, CUTS, PARTS, SHADE, BREAK, PULLOUT, ERASE, and JOIN. They traded control of the mouses (which were spliced so that either student could control the cursor on the screen) back and forth about every minute with little need for supervision from me on the issue of sharing. When the students asked about the role of the MEASURE and UNIT BAR tools, I encouraged them to predict the fractional names of a couple of bars. This activity was helpful in providing an opportunity for the students to become familiar with the tools, but revealed little about the students' fractional knowledge because the activity required only part-whole reasoning.

Will's conjecture about constructing an improper fraction. I decided to challenge the students to produce a fraction that would exceed the whole. In response to that challenge, Will operated conjecturally, and both students interpreted the results of their actions (following the conjecture) using their available fractions schemes. Protocol 1 documents Will's conjecture and refers to objects depicted in Figure 5.

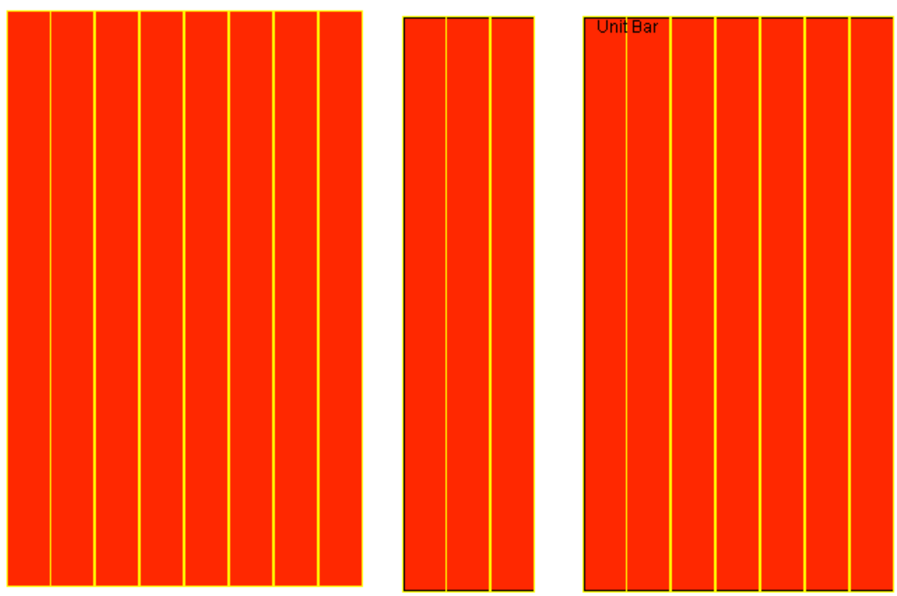

Figure 5. Hillary's production of eight sevenths.

Protocol 1: *Will's conjectural production of eight-sevenths.*

T: I want ya'll to get one so that eight-sevenths comes out in the box [referring to the measure box in the upper left corner of the screen].

W&H: [Both students looked intently at the screen for about twenty seconds.]

H: [pulls out all seven sevenths with some instruction on how to use PULLOUT]

T: So, what do you think is going to happen if you measure that? Is it going to be eight-sevenths?

W: [after looking at the screen for about five more seconds] Couldn't you add another one on the end? …When you took that away you'd have eight sevenths or… [mumbling in ambiguous reference to the bars].

Will, whose thoughts in the initial interview were often consumed with attempts at associating the numerator and denominator of a fraction with the numbers of parts visibly produced, was concerned with producing eight parts. However, it is not clear that he saw these parts as sevenths. Rather, he might accept the existence of a bar with eight parts in proximity to a bar with seven parts as constituting eight-sevenths. It is not even clear that the bar marked "unit bar" held any distinction for him. Yet, Hillary continued by joining another seventh part to her pullout of seven sevenths, under the guidance of Will who informed her that she could pull the extra seventh from a three-sevenths piece that was still visible. After the students had produced the eight-sevenths, the screen looked like the image in Figure 5.

As indicated by his prolonged pauses, Will's verbalization at the end of the protocol was uncertain and conjectural. He conjectured, **Conjecture W1**, that eight-sevenths could be produced from seven-sevenths by adjoining one more part. His conjecture was based on the conjectural operation of adding one more to seven to produce eight. Still, after Hillary acted on his suggestion, he and Hillary seemed to interpret the result in terms of part-whole fractions.

Protocol 1: (First Cont.)

T: What do you think is going to happen [if we measure the bar in question]?

H: It's going to be eight-sevenths.

T: Let's try it.

H: No, it's going to be eight-eighths [continues on to measure the bar].

T: Hold on; don't do it yet. Eight-sevenths or eight-eighths?

W: I say eight-sevenths.

T: Why do you think eight-sevenths?

W: Well, because first it was seven-sevenths and then when you got… Well! That would make it. Well, never mind, I think it will be eight-eighths.

T: Hillary?

H: I think it will be eight-eighths because you add another bar and it's not going to be seven any more.

W: [after Hillary said "…you add another bar," Will began saying] Eight. It will be eight total.

After Hillary added one more one-seventh bar to the seven-sevenths bar, Will construed the new bar as an eight-eighths bar rather than an eight-sevenths bar. That is, a unit fractional part had fractional meaning only in relation to the whole of which it was a part, and so the one-seventh bar could not be said to be an iterative fraction. Rather, it was still a partitive unit fraction (unit fractions determined through iteration in the whole) or a part-whole fraction. For the time being, Will seemed to ignore the bar labeled "unit bar," as Hillary had in saying, "It's not going to be seven anymore." The students did not act this way when dealing with proper fractions, presumably because they could still make sense of such fractions within the original whole. This is not surprising in Will's case, because I had noted in his initial interview (Protocol W5) that he lacked splitting operations, which Steffe (2002) showed are necessary for a meaningful construction of eight-sevenths. Hillary, on the other hand, had demonstrated a powerful use of partitive unit fractions in the initial interview and seemed to have already constructed splitting operations. While her response "eight-eighths" casts doubt on the existence of an iterative fractional scheme, the confusion of both students in the protocol above might be attributed to their lack of familiarity with the functioning of the computer program, particularly with the role of the unit bar. In any case, Hillary's assertion that the bar would measure eight-eighths indicates that she had established a new whole, and Will followed suit.

Protocol 1: (Second Cont.)

T: Okay, let's try it. Measure…

H: [measures the bar to show "8/7" in the measure box] Eight-sevenths.

W: [simultaneously] Eight-sevenths! So it *was* it!

T: Why?

W: Well, because at first it was sevens…

H: Because we added on.

W: …we added one more.

T: So it's eight-sevenths of something. What is it eight-sevenths of?

W: Bars.

T: What bars?

W: Well, we added on that one [pointing to the leftmost part of the eight-sevenths bar, which Hillary had joined to her pulled out seven-sevenths bar].

Once the students realized that the bar did indeed measure eighth-sevenths, they both attributed this surprising result to the unusual circumstance that they had added on another bar: "Because we added on"; "We added one more." This is the logical form of an abduction, and it may have served in supporting Will's initial conjecture.

Hillary and Will's fractional schemes. Although I did not identify any more conjectures in the remainder of the episode, the students' actions regarding UNIT BAR and MEASURE provide some indication of their partitive unit fractional schemes and their interpretation of the roles of the computer tools. An understanding of the students' fractional schemes (beyond the initial interviews) and the meaning they ascribed to the computer tools is vital to analyzing Conjecture W1 and subsequent conjectures. I provide this section, describing the students' actions following Conjecture W1, in order to build that understanding.

Recognizing that the students' shifted the unit bar from a seven-sevenths bar to an eight-sevenths bar, the witness intervened after Protocol 1 in order to ask the students what would happen if we changed the unit bar. I picked up on the idea and clarified his question by designating the old eight-sevenths bar as the unit bar and asking the students whether that would change anything. Hillary thought that the measure box would display eight-eighths this time. Will measured to reveal "1" in the measure box, and Hillary said "one whole!"

Protocol 2: *Will's equivalence of eight-eighths and one whole.*

T: Why wasn't it eight-eighths? What's eight-eighths?

W: Cause that unit is not just on one… [pointing across the "unit bar" label, which covered the top of two of the parts]. Oh! Eight-eighths *equals* one whole!

Will began by blaming the surprising fractional name on the position of the "unit bar" label at first. He may have been looking for peculiar details in the situation (as Glasersfeld suggested as a possible reaction to perturbation) and simply found the label odd. He may also have thought that "unit bar" was supposed to refer to only one of the parts within the bar, rather than the entire connected bar. In any case, it is clear that the students needed to become more familiar with the actions in `TIMA:Bars`. It is not clear how Will viewed the role of the unit bar, and I could have been more direct on the matter, but decided to let them work it out.

Will's second explanation was more insightful. He recognized that the output from MEASURE ("1") was consistent with the bar that he measured because eight-eighths and one whole were equivalent for him. This explanation offers further indication that Will had constructed partitive unit fractions, although my prompt—"What's eight-

eighths?"—may have elicited his response. Will's surprise toward the end of Protocol 1 may indicate that it was a novel perceptual judgment. This would cast doubt that he had previously constructed the relationship between eight-eighths and one whole, as he would have done upon establishing a partitive unit fractional scheme because the scheme contains the operations for rebuilding the whole from one eighth by iterating eight times. As the episode continued, Hillary demonstrated that she had already established a similar equivalence, but then struggled when I asked her to predict what fractional name the seven-part bar would have.

Protocol 2: (Cont.)

H: It's going to be eight-sevenths.

T: Let's see.

W: [picks up his mouse to measure]

H: [before Will measures] Seven-sevenths. It's going to be a whole.

W: [measures the bar to reveal "7/8"]

T: Seven-eighths! Why is that one seven-eighths?

W: [counts the pieces in the bar] There's seven. Where's the eighth one at though?

Once again, Will's primary concern was identifying whole numbers of the numerator and denominator with the number of visible parts. It also seems that he had not established a unit bar as an item with which to compare the bar being measured. This assertion is based on the fact that the bar labeled "unit bar" had eight parts in it, but Will didn't use it in finding where the eighth one was. Hillary also neglected the unit bar in claiming that the seven-part bar would be a whole. Hillary demonstrated that she could reason as Will had, when she established the seven-part bar as a new whole. In previous situations involving proper fractions, the students had pulled a specified number of parts from the unit bar creating a part-whole fraction. In Protocol 2, the students began with the seven-part bar, leaving Will (who had not yet constructed a general partitive fractional scheme) to wonder, "where's the eighth one?"

Before we picked up with more problems this type, I decided to demonstrate how to use REPEAT in making bars. The REPEAT action can be used to make a copy of a bar or a piece of a bar while joining them together as a connected but partitioned whole. I felt that this was an important action the student's could use to iterate a piece because both students had demonstrated that they could iterate unit fractions. The students also spent time playing with two tools they had not yet explored: MAT and COVER. Following their play activity, they tried to make "3/2" appear in the measure box by making a copy of the unit bar, partitioning the copy into three parts and pulling two of them. Their three attempts initially yielded "2/3," and in the last two attempts, they incidentally produced a bar measuring "3/2" (with the two-part bar now labeled as the unit bar), but they could not explain why measuring the three-part bar yielded "3/2."

After my initial analysis of the episode, I decided that the students should pose problems to each other in the next episode, but they should focus on proper fractions until they established a more coherent meaning for the role of the unit bar. I felt that I might be able to help by explicitly using "out of" language when describing fractions. For example, instead of referring to a bar as two-thirds, I might say that the bar is two-thirds *out of* the unit bar.

26 February, 2003 Teaching Episode

Early in this episode, Will and Hillary began playing a game of asking each other to make proper fractions. In doing so, they used PULLOUT, MEASURE, and UNIT BAR appropriately. It also gave them an opportunity to use their part-whole fractional schemes with `TIMA:Bars`.

Hillary's progress toward constructing commensurate fractions. TIMA:Bars always records the fractional name of a bar in reduced terms. So, when Will asked Hillary to make six-ninths, the task presented an opportunity for conjecture. In fact, Hillary did respond conjecturally, and her conjecture was instrumental in her eventual construction of a *commensurate fractional scheme* with which to construct commensurate fractions (fractions having the same size relative to the whole) by uniting or composing units in one fraction to produce those of the other. The activities described in this section build toward and from that conjecture, but because of the critical role MEASURE and UNIT BAR play in this activity, I include Protocol 3, describing the students' meaning of those computer tools.

In response to Will's challenge, Hillary produced six-ninths by partitioning a copy of the unit bar into nine parts horizontally and pulling out six of them. She did this easily except that she had trouble pulling out the six parts (it was difficult to draw a box around the six parts in order to pull them). I anticipated measuring six-ninths would cause additional trouble for the students because the computer would show the fraction in reduced form. So, I asked the students to predict what MEASURE would display. Both students thought that the bar would measure six-ninths, but, before we measured Hillary's production, I wanted to make sure the students understood the role of UNIT BAR and MEASURE.

Protocol 3: *Explaining the role of the unit bar.*

T: So, you know what the unit bar means then?

W&H: Mm-hmm.

T: Okay, so the unit bar… If I measure the unit bar it's always going to be what?

W&H: One.

T: [measures unit bar and screen displays "1"] One… and if I measure this thing [pointing with the cursor to be nine-ninths bar], it's also going to be one [measures the nine-ninths bar and screen displays "1" again]. Why is it going to be one?

W: Cause it's one whole.

T: Yeah, but how does it compare to the unit bar?

W: They're the same size.

I had never before been so explicit about the role of the unit bar. For the first time, Will began to focus on sizes of the bars, and both students realized the importance of making copies of the unit bar. At this point, both students were fairly certain that the six-ninths bar would measure as such, but then they measured the bar. I asked the students to explain why "2/3" came up, and they thought for about twenty seconds before Will responded, "Maybe it wasn't the same size as the unit bar; maybe it was bigger."

However, we measured the nine-ninths bar to display “1,” and Will seemed to withdraw his argument. Fifteen seconds later, Hillary responded.

Protocol 4: *Understanding the common measure of six-ninths and two-thirds.*

H: Is it because we have three pieces left in there?

T: Okay. There’s three pieces left in there…

W&H: [long pause, looking at the screen]

T: She took out six out of nine parts, and so that makes sense that she said it should be six-ninths. But the computer says two-thirds. Why would the computer say two-thirds when we think six-ninths?

W: [measures the nine-ninths bar and the unit bar and the six-ninths bar again] Is six-ninths the same as two-thirds?

T: That’s a good question. Is six-ninths the same as two-thirds, Hillary?

H: [focusing on the screen, excitedly jerks her head up] Is it because it’s like the same size as the unit bar [pointing with the cursor to the unit bar] and it’s two-thirds of the unit bar?

T: [to Will who had leaned in to write on the chalkboard] You want to use paper. [To Hillary…] Show me on here [pointing to the computer] what you mean while Will does his calculation.

The students began trying to explain the unusual circumstance of the six-ninths bar’s measure by looking for peculiarities. Will thought that the nine-ninths bar might not be the same size as the unit bar. This was a reasonable thought because he had just observed that the only other unexpected numbers we had seen could be attributed to differences between the sizes of the unit bar and the whole used to produce the fraction. Hillary then seemed to reason as Will had in previous protocols, trying to relate the “3” in “2/3” to some observable part: She found the leftover three parts from six-ninths and wondered whether they were to blame. Finally, Will wondered whether the expected fraction and the fraction displayed by MEASURE might be the same.

I am not sure how much insight I can attribute to Will’s claim. He may have visualized two-thirds of the unit bar and compared its size to the six-ninths bar. But I believe that Will relied more heavily on another idea: He knew that the bar was six-ninths by using his part-whole fractional scheme; so, when the computer measured “2/3” it may have led to Will’s establishing a second identity for the same fraction. In any case, Will’s question about a connection between the two fractions may have oriented Hillary to the sizes of them, and she was able to explain the connection with insight into the relative sizes of the fractions. When she said “it’s two-thirds of the unit bar,” this indicated that she had constructed a partitive fractional scheme with which to understand two-thirds as a size relative to the unit bar. This was also a conjecture whose nature became more apparent as the episode continued. I encouraged Hillary to make two-thirds from another copy of the unit bar in order to help her explain her idea.

Protocol 4: (First Cont.)

H: [a few seconds after putting a copy of the unit bar into three parts] Oh! That [pointing to the six-ninths bar] is the whole size of those two put together

> [pointing to two of the three thirds she had just created] and that [pointing to the last part] is the third part!

Even before she produced the three-thirds bar, Hillary was able to consider the size of the two-thirds bar as potentially the same as the size of the six-ninths bar; she indicated as much just after Will had asked whether two-thirds was the same as six-ninths. In order for her to reason in such a way, she would have to anticipate the partitioning of one of the whole bars in her perceptual field. Her focus on size throughout the protocols leads me to believe that she was using a partitive fractional scheme. She seemed to be using the scheme in a novel way, for when she actually followed through on the production of three-thirds, she conveyed surprise ("Oh!"), and the image that she produced appeared to test and verify the novelty. It is interesting that she also indicated the third part of the whole, just as she had noticed the three left over parts from six-ninths of the whole. Her focus on complements offers further indication that she was using partitive fractional operations, which make use of the complement in re-establishing the whole from a fraction of it.

Being explicitly aware of the complement may have served as a means for Hillary to keep track of the whole from which two-thirds was pulled. I hypothesize that she used her disembedding operations along with the iteration operation available within her partitive fractional scheme in order to consider fractional parts without losing their relationship to the unit bar. This hypothesis is affirmed in my analysis of Hillary's actions just before Protocol 7. She had behaved similarly in producing a whole from three-fourths during her initial interview. I do not know whether there was a novelty during the initial interview, but there is strong indication (her initial confusion and subsequent surprise) to suggest that the use of those operations in Protocol 4 was novel, uncertain, and therefore conjectural. I refer to her conjecture—positing two-thirds of the unit bar as having the same size as six-ninths—as **Conjecture H1**.

Will's conjectural use of parts-whole comparison. I mentioned that, in the middle of Protocol 4, Will had begun writing (drawing a picture) in order to explain the equivalence of six-ninths and two-thirds. His picture is represented in Figure 6 (although Will had drawn his picture on paper without the aid of TIMA:Bars), and after inspecting it, I realized that he was not reasoning at the same stage as Hillary. Still, his drawing represented novel actions in response to a perturbation, and his question "Is six-ninths the same as two-thirds"—indicated uncertainty about the use of operations underlying those actions. I argue that his question and subsequent actions indicate a conjecture: a conjectural comparison of two fractions.

Will had drawn a nine-ninths bar and shaded six of the ninths. Below it, he had drawn a three-thirds bar that was considerably smaller, and he shaded two of the thirds. He could have completed each drawing using only equi-partitioning and part-whole partitioning schemes. Then he could apply his impoverished fractional comparison to the drawings, which would give erroneous results because there was no criterion that the two wholes be the same size. In fact, in partitioning the first bar, he was only able to fit six partitions into the bar initially and had to extend the size of the bar by three more equal-sized partitions. This, along with making a smaller whole to make two-thirds, indicates that Will did not understand the importance of conserving the size of the whole. Even when he finished his drawing and explained, "I was seeing if they were the same size [referring to the sizes of the two shaded fractions]," he did not notice a problem with the sizes of the wholes. Furthermore, when I asked why the two whole bars were different sizes, he explained that it was "because there are nine parts in that one and only three in the other one."

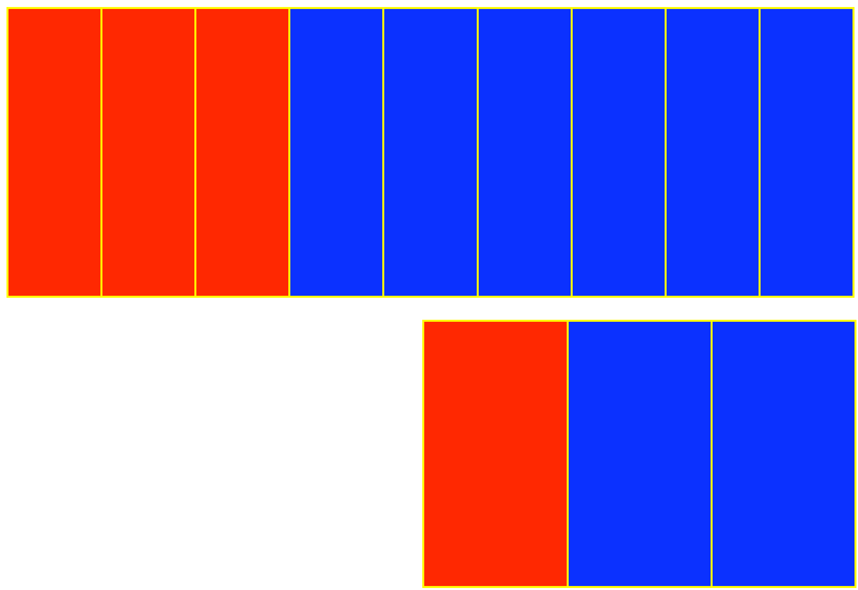

Figure 6. Will's picture for explaining the equivalence of six-ninths and two-thirds.

I call Will's fractional comparison *impoverished* because it seemed to satisfy only the goal of making fractions to figuratively compare, but not determining the relative sizes of the fractions to each other and a *fixed* unit bar. For lack of another way of operating, he had invented a means for making a fractional comparison, but the size of the fractional whole was not relevant. His means entailed a generalization of his whole number comparison scheme, applying it to the connected numbers formed by shading 2 out of 3 and 6 out of 9, in order to compare their extents. I refer to his conjecture that he could compare the fractions in such a way as **Conjecture W2**.

Comparison of Hillary and Will's emerging structures for partitive and commensurate fractions. As the episode continued, just after Protocol 4, Hillary pulled out two of the three thirds and dragged them together over to the six-ninths bar saying, "See. It's the same size!" After a brief interlude during which I showed the children how they might use IMAGE to compare the sizes of two-thirds and six-ninths, Hillary repeated her demonstration, which seemed to work better. Will seemed to follow this demonstration in saying, "that's the same thing!" He even went on to measure the two-thirds bar. The mere fact that the two-thirds bar actually measured "2/3" seemed to add credence to his belief. This may indicate that Will's concept of two-thirds in terms of his part-whole fractional scheme had not previously been reconciled with the computer's MEASURE tool.

Protocol 4: (Second Cont.)

W: Two-thirds of the unit bar is the same as six-ninths. …See. Look. I measured it [the two-thirds bar].

T: [writing 2/3=6/9]. So two-thirds is the same as six-ninths?

H: I didn't get it at first.

W: I didn't either.

Both students got it now, but what they got were two different things. While Will noticed the common measures and observed the equal sizes of the six-ninths bar and the two-thirds bar, Hillary seemed to have constructed an explicit relationship between the relative sizes of two-thirds of a bar and six-ninths of the same bar. It was an instantiation of a novel way of operating that established the kernel for a new scheme, which I call her *complementary fractional comparison scheme*. In using such a scheme, she could keep track of the wholes involved in comparing the sizes of two fractions by making explicit reference to the complements of each fraction, although I have only noticed the scheme used in cases where the complement of the simpler fraction is a unit fraction. I name it because it becomes a relatively permanent structure in my model of her fractional reasoning, as it is instantiated in subsequent protocols. This scheme could be used in much the same way as a commensurate fractional scheme, except that there was not necessarily a numerical relation between the parts, simultaneously understood in terms of the sizes of the fractions.

The remainder of the teaching episode included many corroborations of Hillary's new way of operating. It also provided opportunity for elaborations of both students' partitive schemes (fractional or unit fractional). We erased all but the unit bar and Will made a few copies of it. I asked him to make two-fourths, which he did by partitioning vertically and pulling two of the parts. The students were very confident that MEASURE would display "2/4" (Will was 95 percent sure and Hillary was 100 percent sure.), but it read "1/2" instead.

Protocol 5: *Hillary and Will's common measure of two-fourths and one-half.*

H: It's one half of that! [excitedly pointing to the middle of the unit bar and smiling]

W: Yep! Watch. [Will dragged the two-fourths bar over the left half of the unit bar.]

I have already noted Hillary's attention to the parts, their complements, and their sizes relative to the unit bar. In the protocol described above, Hillary used the middle of the three marks to identify one-half of the bar and establish an identity between one-half and two-fourths. I am interested in finding out whether a commensurate fractional scheme could emerge by operating with her complementary fractional comparison scheme and whether such learning could be engendered by a particular conjecture.

Smiling, Hillary seemed completely satisfied by her explanation even before Will dragged the two-fourths bar over to visually compare with the left side of the unit bar. Because Will also responded immediately, it is possible that he reasoned as Hillary seemed to reason, but it is also important to note that he had established the equivalence of one-half and two-fourths through his actions in Protocol W3. It is unclear whether, for Will, two-fourths was yet a determined fractional size relative to a fixed whole, because he did visually compare to test whether two-fourths was the same size as one-half. Hillary did not have to visually compare the bars because she was able to anticipate the production of one-half and compare this image to the two-fourths bar on the screen.

The next protocol provides a test for the questions raised above, especially regarding the students' uniting operations and partitive schemes. Warning the students that I was giving them another tricky problem like the last one, I asked Hillary to make three-ninths.

Protocol 6: *Hillary's prediction about three-ninths*.

> H: [after making three-ninths in a manner similar to Will's production of two-fourths] I think it's going to be two-thirds. No! One-third!
>
> W: Put it up there [suggesting that Hillary drag the three pulled parts over the unit bar, which she subsequently did].
>
> T: What do you think, Will?
>
> W: [begins dragging the three-ninths piece three times across the unit bar].
>
> H: It's one-third. [smiling] One hundred percent sure… cause three of them can fit inside that.
>
> W: [seemingly a little frustrated that his iterations, done manually with the three-ninths bar, did not fill the unit bar exactly] I think it's one-third.

Next, I encouraged the children to make one-third to check their assertions, but I should have reminded Will how to use the REPEAT tool because it would better fit his actions in attempting to iterate manually. Will's action opens the possibility that one-third was an iterable unit for him (at least within the whole), which would corroborate that he had constructed a partitive *unit* fractional scheme. However, the action of iterating a unit fraction would have to be more than an *internalized* action for Will in order to ascribe a partitive unit fractional scheme to him. It would have to be *interiorized* so that, as an interiorized operation, it could be used to make material for further operating, just as the interiorization of iterating 1 (through his ENS) makes 5 an object on which he can operate.

Even though Will carried out the iterative actions, Hillary was the only one who expressed that she was convinced that three-ninths and one-third were commensurate. Also, Hillary's initial confusion that the bar would measure two-thirds may be further indication that she was focusing on the complement of a fraction in establishing the whole from it (just as she seemed to do in the case of six-ninths). This time Hillary seemed to recognize that the three ninths constituted one third, but it was easier to do so because the simplified fraction was a unit fraction. Will's frustration at his inaccurate attempt to act out the iterations and his expressed uncertainty ("I think it's one-third") indicate that he expected to be able to iterate the part three times to produce the whole, but cast doubt on whether he had *interiorized* the iteration of three-ninths as he would in coordinating a unit of units. Instead, his iterative actions seemed to constitute a test of Hillary's claim that the part was one-third.

As the protocol continued, Hillary ended up making a three-thirds bar with horizontal partitions, whereas the three-ninths bar had been produced using vertical partitions. This made it difficult to compare one-third with three-ninths until Hillary realized she could cut up the one-third piece: "Yes you can. You can cut it up into pieces and it will all fit into the same as that [the three-ninths bar]." She then cut up the one-third part into three pieces, estimating very accurately using CUTS. Next, she lined up the three pieces on top of the three-ninths bar. The description of these actions highlight Hillary's focus on size and its conservation, and demonstrates the fluidity of her splitting operations (as did her responses to initial interview Tasks 4a and 4c).

In the next part of the episode, I encouraged Hillary to give a similar tricky problem to Will. She asked him to make eleven-twelfths. Will partitioned a copy of the unit bar into twelve parts and accidentally pulled ten of them instead of twelve. He measured the ten-twelfths bar anyway, and the screen displayed "5/6." Will admitted that he thought it would say "10/12," but Hillary explained "Mm-hmm… because you can cut that [pointing to the unit bar] into six pieces and you have one piece left over, and all that [pointing to the ten twelfths bar] can be cut into six pieces and it will be five pieces out of six of that [sweeping her finger from the top to the bottom of the unit bar]."

Once again, Hillary explicitly mentioned the complement of a fraction in explaining its commensurability with another fraction: "… and you have one piece left over." Although her explanation was a little confused when she referred to cutting the ten-twelfths bar in six pieces (instead of five), it is clear that she imagined five sixths being embedded in the unit bar and that the ten-twelfths bar would fit into those five sixths, with one sixth left over. She did not seem to unite each pair of twelfths into one sixth as she would with a commensurate fractional scheme, but rather anticipated the production and size of five-sixths using her partitive fractional scheme while using her disembedding operation to keep track of its complement and the unit bar. This affirms my previous hypothesis (from Conjecture H1, during Protocol 4) that she had coordinated those operations in forming her complementary fractional comparison scheme.

In examining the next protocol, it becomes clearer that Hillary's size comparisons were estimations rather than being constituted in a numerical relationship between two-twelfths and ten-twelfths (the latter being five times the former) or an equivalence relation between two-twelfths and one-sixth. In that protocol (Protocol 7), Will went on to produce the eleven-twelfths bar, as Hillary had intended for him to do initially. The dialogue of Protocol 7 illustrates a limitation to the way of operating that Hillary had constructed as a result of Conjecture H1.

Protocol 7: *Hillary's estimate for eleven-twelfths*.

H: [just as Will started his production] It's going to be a whole.

W: [after finishing his production of eleven-twelfths] What's half of eleven-twelfths. Nothing goes into eleven.

T: I heard you say before, Hillary, that you thought it was going to be a whole. Do you still think it's going to be a whole?

H: Hmm. No. I think it's going to be five-sixths.

W: I think it's going to be something sixths or eleven-twelfths. [Will measured to reveal "11/12."] Eleven-twelfths.

T: Why didn't this one do a tricky one, Hillary?

W: Is this one the same as that unit bar… cause we made copies of it so it must be. [Will measured the twelve-twelfths bar, revealing "1"]. Yep. Could it be the way you divide it make it different? [Will divided another copy of the unit bar horizontally, pulled eleven parts and measured "11/12."] Eleven-twelfths!

Hillary seemed to be approximating the size of the eleven-twelfths bar in terms of fractions that she could imagine producing. "One whole" and "five-sixths" were viable candidates, not because of an equivalence established by numerical relations or units coordination, but because she anticipated that they would be about the same size as eleven-twelfths. While such size estimations may form the figurative basis for a commensurate fractional scheme, her reliance on estimations presents a limitation to her complementary fractional comparison scheme. She demonstrated the same limitation when she initially posed the problem: Hillary did not anticipate the production of a fraction commensurate to a given fraction until the given fraction was produced on the screen. To do so would mean that not only could she imagine two equally sized fractions of a whole, but she could engage in the operations of transforming, say, two-thirds into four-sixths or ten-twelfths into five-sixths, which involves recursive partitioning and reasoning with three levels of units. She did want to pose a tricky problem for Will, and she seemed to guess that eleven-twelfths might result in something like the previous case with six-ninths.

Will, on the other hand, was entirely interested in numerical relationships, but these were not insightful relationships referring to the sizes of the fraction bars that he produced, and they were certainly not established through units coordination. He knew that he could follow a pattern in finding the solution. In two of the last three protocols, the equivalent fraction could be found by taking half of the numerator and half of the denominator. When his answer ("something sixths") was refuted by MEASURE, Will began looking for peculiarities, as he had before. In fact, the first peculiarity that he mentioned—about the twelve-twelfths bar being the same size as the unit bar—was the same as one he had mentioned before. He tested the second one, discovering that MEASURE was unaffected by the direction of partitioning.

On his next (and last) turn, Will posed thirteen-fourteenths for Hillary, which she easily produced. Will said that it would measure "13/14." Hillary thought for a while before exclaiming, "I think it's going to be three-fourths." The bell rang for the end of the period before she could explain. She had started to say, "four of these [fourteenths from the thirteen-fourteenths bar] could be split up into that [the unit bar]." She then divided the unit bar into three equal parts with her finger. The students then left, so it is difficult to determine what Hillary meant here, but she may still be focused on approximating the size of one fraction in terms of another that she could anticipate producing. Although there is little evidence to substantiate additional claims, it appears that she thinks that each third of the unit bar could be filled with four of the fourteenths and that the three thirds would exhaust all thirteen of the fourteenths. If that is so, I'm not sure what to make of it at this point, but it may represent a diversion from her previous focus on approximating the size of the fraction with a simpler fraction and reconstituting the whole from its complement. Perhaps her failure to do so in the case of eleven-twelfths generated a perturbation so that she began conjecturing new possibilities for action. Such possibilities may include uniting smaller fractions into units within a larger fraction, as in a commensurate fractional scheme, except that she still seemed to be estimating.

For his part, Will may have thought the fraction would measure "13/14" because this problem, like the last one, involved a numerator that could not be divided in half evenly. In other words, he may have revised his rule for finding commensurate fractions after observing a break in the pattern of halving with eleven-twelfths. Will does not seem to consider the relative sizes of the bars involved when he uses such rules. Rather, the rules serve as a procedure—a sequence of steps—generalized from records of past experiences (possibly through analogy, as Polya suggested).

3 March, 2003 Teaching Episode (Andy and Will)

At the end of the last episode, Hillary and Will appeared to be acting meaningfully with the actions of MEASURE and UNIT BAR so that they could begin to reason more freely and meaningfully in representing and testing their mathematical ideas. Unfortunately, Hillary was absent for the next episode, but her absence did provide an opportunity for Will to catch up with Hillary on estimating fractional sizes, coordinating units, and working toward the construction of an iterative fractional scheme (it is unclear whether he had even constructed a partitive fractional scheme). Andy took Hillary's place for the day, and he had yet to become familiar with the potential for operating in the micro worlds. Because, as a substitute partner for the other participants, Andy is not included in many of the teaching episodes, I do not analyze his language and action. So, I focus only on the schemes and conjectures of Will; Andy's actions are considered only in as much as they substantiate interactions with Will.

Will's procedural schemes. After I introduced an example (two-fourths), the students began finding new ways to make "1/2" appear in the measure box. They went on to produce three-sixths, recognizing that it should also be one-half.

After Andy suggested eighths as another partition that would lead to a fraction commensurate with one-half, Will produced eight-eighths and pulled four eighths. Will then suggested that Andy could make six-twelfths "because six is half of twelve." This was an instantiation of what I refer to as Will's *procedural scheme for producing fractions commensurate with one-half*. Will had acted similarly at the end of the teaching episode on February 26th, in which I attributed his actions to a generalized procedure.

Because this procedure has persisted and served Will in constructive problem solving, I now refer to it as a scheme.

We continued the episode with the students posing tricky fractions to one another. Will posed thirteen-sixteenths to Andy, presumably because he thought it might simplify, as three-sixths had. When he observed that the fraction actually measured "13/16," Will at first attributed this fact to the observation that the thirteen-sixteenths bar had been produced from a copy of the unit bar. When I reminded him that we had used a copy of the unit bar in the case of two-fourths, he stated, "Thirteen-sixteenths is not like a half." He was continually inventing rules like this to explain surprising situations in which his existing numerical rules did not produce the desired results. His rules can be considered abductions, but they were not insightful in the context of fractions because they did not focus on fractional quantities, and there were no conjectural fractional operations.

Will's use of iteration using composite units. For the next problem, Andy posed six-eighteenths, which Will produced using a copy of the unit bar. Before producing the fraction and dragging it over the unit bar, I asked Will what the fraction would measure, and he admitted that he did not know. His procedure of relating the numerator and denominator seemed restricted to identifying pairs in which the denominator was double the numerator. After putting the copy into eighteen parts and pulling six eighteenths, Will carefully lined up this fraction with the leftmost third of the (unpartitioned) unit bar and then moved it over one space to the middle third.

Protocol 8: *Will's iteration of a six-eighteenths bar in establishing it as one-third.*

W: [after moving the six-eighteenths bar to the middle of the unit bar] Two-thirds.

T: What did you pull? What did Andy ask you to make?

W: I pulled six eighteenths.

T: Six eighteenths, and why do you think its going to be two-thirds?

W: Because it's two-thirds of that right there [pointing to the unit bar].

T: Show me that it's going to be two-thirds.

W: [demonstrates that the six-eighteenths bar fits inside of the unit bar three times] One, two, three.

A: I think it's going to be one-third.

T: What does that mean? So, if it goes in there three times, what should it make it be?

W: It could be two-thirds. Oh!

T: What does two-thirds mean?

W: Oh, it could be one-third.

T: Why do you think one-third?

W: Cause this [dragging the six-eighteenths bar into the unit bar again] is one-third of that.

T: How do you know it's one-third of that?

W: [pointing to the six-eighteenths bar] Cause it's one thir… [laughs!] Because this goes into that three times.

Will was able to verbalize a partitive conception of one-third at the end of the protocol, but it did not seem to be available through most of the protocol. Otherwise, he might have experienced a conflict between his "two-thirds" answer and the situation. So, it seems that one-third was an iterable unit for Will, but he did not seem to be able to carry out the mental operation of iterating figurative material in this situation. Instead, he relied on the action of putting the six-eighteenths bar in the unit bar three times. Later, he admitted that he had thought the fraction was two-thirds because, once he had repeated it in the unit bar twice, he saw the remaining third and stopped. Having recorded his two iterations of the fraction, which he then recognized as one-third, he answered "two-thirds."

Will had not been able to (mentally) iterate composite fractions in past protocols either. The action of manually iterating a composite fraction was a novelty that may have consumed him so that he was unable to reconstitute the fraction as a third and, instead, simply recorded his actions of repeating twice and his visualization of the remaining one-third part. Even so, Will was eventually able to carry out the action of iterating in order to satisfy his goal of constituting the composite fraction as a new fraction.

Protocol 8 marks the first time that Will used fractional language to refer to the size of a fraction relative to the whole. Although he had mistakenly called the fraction "two-thirds," this is the strongest indication yet that Will had constructed a partitive unit fractional scheme, because he used iteration to determine the size of a given fraction and even attempted to apply it to composite units. Will's actions in this situation involved the novel use of his iteration operation in resolving the perturbation caused by the unknown commensurate fraction for the given fraction. I refer to his use of analogy in trying to determine the fractional size of a composite fraction as he would an unpartitioned unit as **Conjecture W3a**. We will see another similar conjecture involving the iteration of a composite unit, which I will call Conjecture W3b.

Another example. Will was still unable to anticipate the production of commensurate fractions other than those in which the denominator was twice the numerator. Intending to give Andy another commensurate fractions problem, Will posed five-elevenths to Andy. Given the analysis of the protocol described above, this should not be a surprise because Will needed to carry out of action of repeating a composite fraction to reconstitute it as a simpler fraction, except in the cases where he could use his procedural scheme for producing fractions commensurate with one-half. However, Andy (a splitter) also had trouble determining whether the fraction could be simplified. The following transcription begins after Andy had produced the objects illustrated in Figure 7. When Andy guessed the measure of the five-elevenths bar would be "5/11," I asked Will what measuring would show.

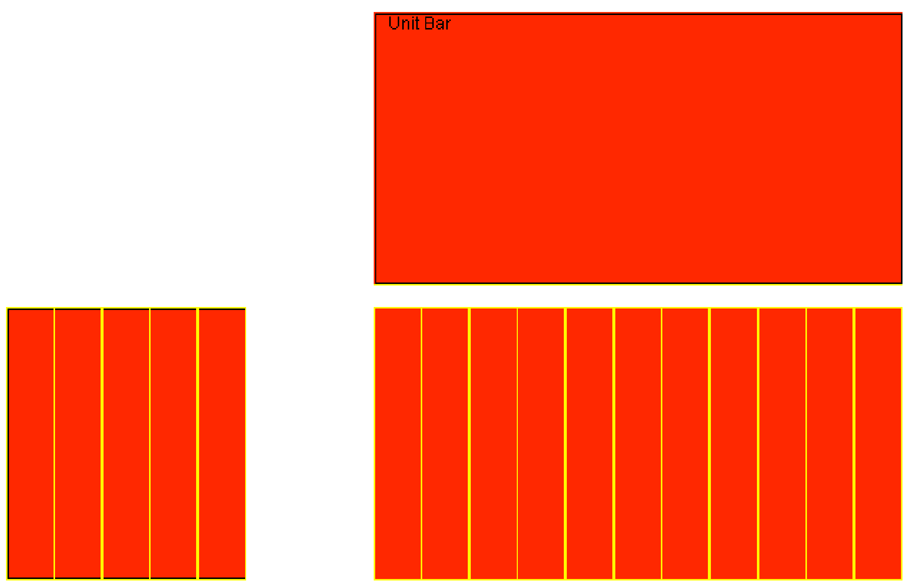

Figure 7. Andy's production of five-elevenths.

Protocol 9: *Will's confusion in working with composite units*.

W: Uh, I think it's going to be two… [looking intently at the screen] Let's see. That's five… Two and… 'Cause see that's five [touching the left side of the eleven-elevenths bar] and then there's five right there [touching the right side] and then there's one left over [pointing to the right-most eleventh].

T: Yeah.

W: So it's going to be two, four… Ten-elevenths!

T: Why do you think ten-elevenths?

W: 'Cause five and five and there's one left over.

Will was trying to iterate a fractional part in order to segment the fractional whole. His action corroborates my inference that he had constructed a partitive unit fractional scheme. His goal was to produce the five-elevenths bar as one out of so many parts that exhausted the eleven-elevenths bar—the goal of a partitive unit fractional scheme. As with the last protocol, he became confused about the result of the scheme because he was using it conjecturally. Ordinarily, working with unit fractions, his partitive unit fractional scheme would yield the number of iterations of the fraction within the whole, which would be used reciprocally to name the unit fraction. In the two most recent cases, Will used the scheme with composite units, which indicates that his use of the scheme constituted a generalizing assimilation. However, he experienced unexpected constraints in the use of the scheme, and the adjustments that he made were conjectural.

In Protocol 8, Will only iterated twice (possibly because at that point he had achieved his goal of identifying the fraction as one-third) and recorded the “2” in the name of the fraction he was iterating, either because he was focused on the complement or because he was counting the iterations. In the present case, Will experienced new constraints; the fractional part did not exhaust the whole. This generated a perturbation as exemplified by his stutters: “Two... Let’s see… Two and….” He could not name the fraction as “1/[two and…].” Instead, once again he iterated the five-elevenths bar twice, this time recording the number of elevenths. Failing to exhaust the eleven-elevenths bar, he produced “ten-elevenths,” which satisfied Will’s goal of naming a new fraction.

His production of ten-elevenths was a conjecture (**Conjecture W3b**) produced as a result of the constraints that he met in the use of his partitive unit fractional scheme. Because he did not ordinarily consider non-unit fractions as measures of size, it did not even strike him as strange that he was equating five-elevenths with ten-elevenths, until I asked him to produce ten-elevenths later in the episode.

In Protocol 8, involving the reconstitution of six-eighteenths, Will had been successful in unitizing six-eighteenths and iterating it to determine that it was one-third (although, while iterating the piece by dragging it, the partitions of the six-eighteenths were invisible). So, it was reasonable to try operating similarly in the latest protocol involving five-elevenths. The immediacy of his approach to the latest situation indicates that his partitive unit fractional scheme had already changed as a result of his experiences in the previous protocol, at least in its trigger: He did not hesitate to apply the scheme to another situation involving a composite fraction.

After Protocol 9, Will produced ten-elevenths and was immediately able to recognize five-elevenths as half of it. The immediacy of his recognition corroborates that he had been using a partitive unit fractional scheme, and indicates that he had constructed an experiential unit of units of units (ten-twelfths as two of five-twelfths). Now that there was no left over piece, Will could quickly name the five-elevenths bar as one-half of the ten-elevenths bar because the five parts went into the ten parts twice.

5 March, 2003 Teaching Episode

The episode began with Hillary making two-thirds by pulling two parts from a copy of the unit bar that she had partitioned vertically into three parts. When I asked Will to make two-thirds a different way, he did the same thing, except that he partitioned the copy horizontally. We had a discussion about the two productions and both students said that the productions were the same. However, at first, only Hillary seemed to recognize that they had the same areas and that one could be cut up to neatly fill the other.

Will's part-whole fractional scheme. When I asked the students to make two-thirds without partitioning the copy into three parts, Will partitioned a copy of the unit bar into six parts, pulled three parts, and then pulled two of those. Protocol 10 picks up from there and demonstrates Will's reliance on whole number concepts.

Protocol 10: *Will's attempt to produce two-thirds from six parts*.

T: Don't measure yet. I want you to explain what you did.

W: I made… I put it into six parts. I don't know why I put it into six parts, but I just did. I took three out of that, and then I took two out of the three, and maybe it will come out to be two-thirds of that [pointing at the screen].

T: Okay. It will be two-thirds of what?

W: That [pointing to the three-part bar]

T: Will it be two-thirds of the unit bar?

W: [pause] Nuh-uh. [begins to drag his “two-thirds” bar (two-sixths) into the unit bar]

T: Okay. So we want something that will be two-thirds of the unit bar. How much will [the two-sixths bar] be out of the unit bar?

W: [begins placing the two-sixths bar in the bottom third, then the middle third, then the top third of the unit bar].

H: [as Will was beginning to drag the two-sixths bar to the middle third of the unit bar] Two… [Will had just reached the middle third.] thirds.

W: Yep. Uh… One-third.

H: One-third.

T: Why did you say two-thirds, Hillary?

H: I got mixed up [smiling].

T: I know. You fixed it, but could you explain how someone could get mixed up on this? [pause] How could somebody get mixed up and think it might be two-thirds?

H: Cause you are taking two out of that.

Even after the discussion on area, Will’s actions provided no indication for a partitive fractional concept of two-thirds. Rather than producing two-thirds as a fractional size relative to the given whole, to make two-thirds, Will needed to take two out of three parts. The imposed constraint that he use the six-part bar led to the necessity that he redefine the whole. To construct the desired two-to-three relation using his part-whole fractional scheme, he took three parts from the six-sixths bar and pulled two of those three parts. His goal had been satisfied by making two-thirds of something, where the *something* was the three-part bar. He held little regard for the unique role of the bar labeled “unit bar” and his actions serve as contra-indication of a partitive fractional scheme, commensurate fractional scheme, and recursive partitioning operations. It was unclear (even to Will) why he chose to use six parts. It may be due to numerical relations between 2, 3 and 6: 2 times 3 is 6, 2 and 3 both divide 6, and half of 6 is 3.

Once he had made his “two-thirds” bar (two-sixths) and was prompted to consider its size in relation to the unit bar, Will could iterate through action. This is not an indication that two-sixths or two-thirds was an iterable unit for Will, but, rather, that his records of experience in such action could later become interiorized as the operation of iteration. It is important to note that when the students drag a composite fraction in the unit bar as Will did here, the partitions in the composite fraction are no longer visible, so it is easier to perceive the object as a unit. Thus, Will’s actions in Protocol 10, for example, may be more akin to iterating one-third than to iterating two-sixths, and this corroborates only his partitive unit fractional scheme.

<u>Hillary’s conjecture concerning commensurate fractions and Will’s assimilation of her actions</u>. Later in the episode, Hillary also attempted to make two-thirds from six parts, but made one-half instead. Each student recognized Hillary’s production (three-sixths) as one-half, but by different means. Whereas Will supported his claim by saying that “3 is half of 6,” Hillary referred to the relative sizes of the fractional stick in question

and the whole (six-sixths) stick. Recently, however, Hillary had also begun focusing on the numbers of parts in fractions, possibly as a result of interacting with Will who seemed to rely solely on the numbers of parts in his productions to justify their measure. In fact, as Hillary renewed her attempts to produce two-thirds without using three parts, Will suggested that she try using twelve parts, presumably because he recognized that 2 and 3 each divide 12.

Hillary's actions in attempting to use twelve parts to produce two-thirds are recorded in Protocol 11, and her resulting productions are illustrated in Figure 8.

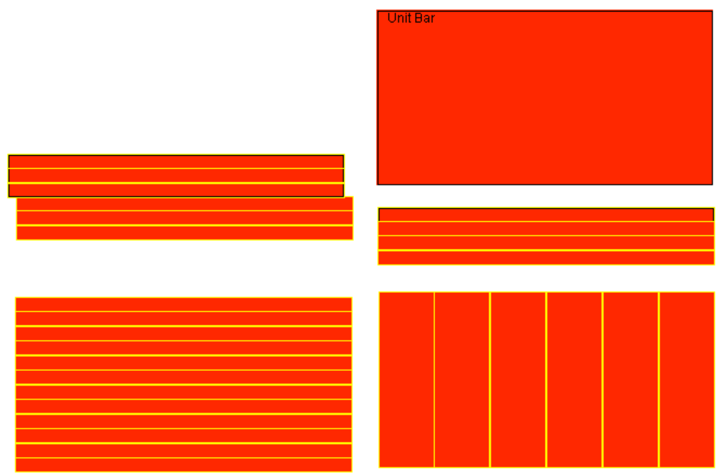

Figure 8. Hillary's production of four-twelfths.

Protocol 11: *Hillary's production of two-thirds from twelve parts.*

H: [pulls four-twelfths and places it in the bottom third of the unit bar, then the middle third, then the top third]

W: [following Hillary's actions] One… two… Three!

H: That's two-thirds!

T: You think you've got two-thirds?

H: Nods.

T: Okay. What makes you think you've got two-thirds?

H: You can put… Oh, never mind, that's not two-thirds.

T: What is it?

W: I think it would be *one*-third.

H: [simultaneously] It's one-third.

T: Oh! It's *one*-third. If you've got *one*-third, can you think of a way to get *two*-thirds?

H: Mm-hmm. [begins counting eight parts in the twelve-twelfths bar]

Hillary's pulling of four parts may represent a divergence from her previous ways of operating in similar situations. Had she used her complementary fractional comparison scheme, she might have anticipated the production of two-thirds and compared its size to the twelve-twelfths bar. This would entail a novel use of the scheme because, previously, she had never used it to construct a commensurate fraction; she had only used it in post-hoc explanations of commensurate fractions. If she did estimate two-thirds of the twelve-part bar, it's possible that she would pull about eight parts from the twelve, and her action of pulling four parts could only be explained by some confusion between the desired fraction and its complement.

Considering Hillary's recent focus on number and recent experiences of uniting composite fractions (such as six-eighteenths), it seems more plausible that her pulling of four parts involved a novel use of uniting operations and units coordination. She knew that three units of four is twelve units, so she united four parts into a one composite unit in an attempt to make two-thirds, as indicated by her iterating the four-twelfths bar three times in the unit bar. Her attempt constituted a test of **Conjecture H2a**, that she could produce two-thirds by coordinating units of four parts. Hillary was beginning to understand the importance of numerical relationships in these situations. The fact that she thought the fraction was two-thirds indicates that she confused the composite fraction and its complement upon iterating. If she were able to resolve the use of her iteration operation with composite units, she would be able to construct a partitive fractional scheme for composite units, as well as a commensurate fractional scheme.

As she began to justify her production of two-thirds, Hillary realized that she had in fact produced one-third; she knew that one-third was the fraction you could put into the unit bar three times. She also knew that two-thirds was one-third iterated twice. After Hillary placed the four-twelfths bar in the unit bar three times, Will knew that she had made one-third. He seemed to have a concept of one-third that was similar to Hillary's, but he did not seem to understand two-thirds as she did. These facts highlight Will's limitations with non-unit fractions: They were not partitive fractions, but part-whole fractions. As the protocol continued, Will reverted to his part-whole fractional scheme as he had in past protocols involving non-unit fractions.

Protocol 11: (First Cont.)

> W: I was going to say she could take two out of that [pointing to a three-twelfths bar that Hillary had pulled (but not named) by mistake] and put it into the unit bar and see if it would make two-thirds.

T: Oh. What would happen though?

W: It would be more than *one*-third.

T: You mean like more on the bottom.

W: Mm-hmm.

T: So it'd really be less…

Once more, the stark contrast between Will's reasoning with unit fractions and non-unit fractions was apparent as he suggested that he might create two-thirds by pulling two of the three twelfths. But something had changed in his reasoning with composite fractions: After using his part-whole fractional scheme to generate his suggestion, Will understood that the suggestion was flawed because he considered the size of the resulting fraction in relation to the unit bar. Although his language was a little ambiguous, he understood that two-twelfths would go into the unit bar more than three times, and so it could not be two-thirds or even one-third. Will's confusion continued as Hillary attempted to justify her production of two-thirds from eight-twelfths.

Protocol 11: (Second Cont.)

H: [finishes pulling out eight-twelfths and starts to move it to the bottom two-thirds of the unit bar] *That's* two-thirds. I had four and…

W: Nope.

H: … four was one-third of that. So you add double the number, so I got eight, so that's two-thirds of [the unit bar].

T: [to Will] What do you think about her explanation?

W: I don't think so because if you put [eight-twelfths] again into [the unit bar], it would be over it. [begins counting eight parts from the bottom of the twelve-twelfths bar and whispers] She took out eight.

H: If you put [four-twelfths] right there [in the top third of the unit bar] in that one-third.

T: [to Will] Look what she just did.

W: She just made another whole. Yep. I think it might be two-thirds.

T: So what was confusing you at first, Will?

W: What was confusing me at first was that if she put [eight-twelfths] in there again, it would go over.

T: So if you put two-thirds in there twice, it's okay if it goes over?

H: Yeah. The other half wouldn't matter.

T: [to Will] So what should be left over if you put two-thirds in [the unit bar]?

W: Just four little parts like [the four-twelfths].

T: …which is how much out of the whole?

W: Uh. One-third.

From her explanation ("Four was one-third of that. So you add double the number, so I got eight, so that's two-thirds."), it is clear that Hillary knew that two-thirds was two of one-third. Later in the protocol, she even demonstrated her understanding that one-third was half of two-thirds when she said, "the other half wouldn't matter," indicating a reversible partitive fractional scheme. Her explanation also affirms that she used units coordination, uniting four twelfths as one third and iterating it. Lining up the eight-twelfths bar with the unit bar may have helped her to reconcile the production with her complementary fractional comparison scheme, but it is important to note that she did this *after* claiming that she had produced two-thirds. We might refer to this way of operating as the basis for a partitive fractional scheme for composite units, which may grow from her complementary fractional comparison scheme as she continues coordinating units of units with the latter scheme.

When Will expressed doubt about Hillary's production (and possibly her explanation), Hillary demonstrated that the complement of the eight-twelfths bar was four-twelfths. This indicates that she could use unit fractional composition (composing four parts in one-third) with her complementary fractional comparison scheme, possibly establishing the kernel for a *partitive unit fractional scheme for composite units*—a scheme that uses the operations of a partitive unit fractional scheme on composite units with coordinating the resulting units of units. If she were able to combine the operations of those schemes into one scheme with which to purposefully construct commensurate fractions, she would also have a commensurate fractional scheme. This is a scheme with which students can create units of units within a fractional part and the whole in order to generate another name for the fraction.

Hillary's action of putting the last four-twelfths bar into the unit bar to complete it made sense to Will and convinced him that Hillary's production of eight twelfths was indeed two-thirds. This is an example in which Will was able to assimilate Hillary's actions and make local adjustments to his fractional schemes and concepts. It may be that Will could understand two-thirds as the complement of one-third (as Hillary could), but his hesitation in naming the four-twelfths bar as one-third (at the end of the segment) opens the possibility that he may instead have considered the complement of eight-twelfths as four-twelfths. This could be done in action rather than as an operational necessity, because of the explicit nature of Hillary's construction. The results of Hillary's activity are illustrated in Figure 9, except that I exaggerated the misalignment that was visible on the screen.

Figure 9. Hillary's reconstruction of the unit bar.

Will realized that if a fraction went into the unit bar three times, it would be one-third. His iteration of composite units had helped him to resolve (correctly or otherwise) problematic situations in which he was supposed to find simpler fractions for fractions such as six-eighteenths (Protocol 8) and five-elevenths (Protocol 9). He seemed to be in the process of constructing a partitive unit fractional scheme for composite units. As indicated by his admission that he had thought two-thirds repeated twice should not go over the unit bar, Will was initially trying to make sense of the two-thirds bar by treating it as he would treat a composite one-third bar and using the iteration operation of a partitive unit fractional scheme. But then Hillary's depiction of the unit bar's reconstruction oriented him differently: “She just made another whole.” I hypothesize that, after he agreed that Hillary had produced two-thirds, Will could visually unite the three composites of four twelfths in order to establish the part as two composites and the whole as three composites, producing an experiential unit of units of units. This claim is corroborated by Will’s statements at the end of Protocol 11, when I tried to orient Will to consider the complement of two-thirds. His initial identification of the complement as “four little parts” and his subsequent claim that this was one-third affirms that he had created at least an experiential unit of units.

Will seemed to be engaged in progressive uniting operations in the production of an experiential structure. As the protocol continued, I pressed Will to explain his production. During my interaction with Will, Hillary created another novelty.

Protocol 11: (Third Cont.)

T: Why would eight out of twelve give us two-thirds?

W: Well, 3 goes into 12, no 2 goes… no… I was going to say something goes into 12 three times. That’s the reason why I told her to go to 12, because the last number right there [pointing to the “3” in the “2/3” displayed in the measure box] could end up a 3 because something like that goes into 12…. Eight-twelfths is the same as two-thirds.

T: How do you know?

H: [Hillary, who had been in silent reflection for about a minute (but still appearing to follow my discussion with Will), turns from looking at the screen and smiles] Um…

T: [writes 8/12=2/3 on the chalk board and repeats the question for Will, ignoring Hillary for the moment]

W: 4 times 2 is 8 and then 4 times 3 is 12.

When I asked Will to explain the commensurability of eight-twelfths and two-thirds, he used a procedure for determining the equivalence of fractional numerals, involving whole number operations of multiplication and division. This may have been a generalization of his procedural scheme for producing fractions commensurate with one-half. I will refer to it as his *procedural scheme for producing equivalent fractions*, but note that it seems to occur without any consideration of commensurability (in terms of fractional sizes) and was not yet used to *produce* new fractions other than those commensurate with one-half. The generalization of the original scheme involved generating post hoc explanations for the common measures of non-unit fractions. This modification may have been the conjectural result of an abduction explaining the surprising measure, but the conjecture and scheme were not based on fractional operations. Will had apparently not used uniting or iteration operations at all. The fact that he did not immediately know what “something” went into 12 three times would be unlikely if he had created an experiential unit of units of units as I had hypothesized. When he did establish the relation that “4 times 2 is 8 and then 4 times 3 is 12,” it seemed independent of his previous actions. This serves as contra-indication that Will could coordinate a unit of units of units and produce commensurate fractions. Instead, it seems that he had only created a unit of units (four parts as one-third).

Will's partitive unit fractional scheme established fractions as the reciprocal of the number of iterations needed to reproduce the unit bar. Because two-thirds was a non-unit fraction, he did not know whether it would be established by two or three iterations. With this in mind, Will suggested that Hillary use twelve-twelfths in their attempts to produce two-thirds. Whereas his explanation in the latest segment focuses on finding a number divisible by 3, in previous segments he had expressed interest in finding a fraction that would go into the whole twice. In the end, he found a number divisible by 2 and 3: "Well, 3 goes into 12, no 2 goes… no… I was going to say something goes into 12 three times. That's the reason why I told her to go to 12, because the last number could end up a 3 because something like that goes into 12." Will had reasoned similarly in choosing six-sixths in the previous protocol, but had begun to consider the role of denominator ("last number") of non-unit fractions more.

<u>Operation of a commensurate fractional scheme.</u> Toward the end of Protocol 11, Hillary had begun to excitedly express an idea, but I was focused on Will's explanation at the time. I don't know whether her initial idea was related to the six-sixths bar, but the witness interjected with a question about it, and Hillary appeared equally excited to consider that question.

Protocol 12: *Hillary's explicit use of commensurate fraction operations.*

O: Is it even possible to do it using sixths [referring to the six-sixths bar still displayed on the screen]?

H: Yeah there is!

W: Let's see. There's two-thirds right here [pointing to the eight-twelfths bar] and you took out eight. You could add some. [After several seconds of looking at the screen…] *Is* there a way to do it?

H: Uh-huh. What I've been thinking about is six can… There's two parts, every two parts is one-third [pointing her thumb and index finger to the three pairs of sixths in the six-sixths bar] and if you put two parts together [pointing across four sixths] that's going to make two-thirds [pointing to the unit bar].

W: That's what I was thinking about.

T: Okay. You do it, Will.

W: [pulls four sixths from the six-sixths bar and measures it to reveal "2/3"]

I think that Hillary had finally constructed the operations of a commensurate fractional scheme. Her expression that "every two parts is one-third," without pulling any of the sixths, demonstrates that she could perceive six-sixths as a unit of units of sixths that could be partitioned to form a unit of three units, each containing two sixths. Her sporadically heightened level of excitement throughout the last several minutes of the episode indicates she was beginning to operate in distinctly novel ways. I refer to her explicitly stated conjecture as **Conjecture H2b**. Her use of the four-sixths and six-sixths bars in referencing the units of units also served as a test of her conjecture.

Will's predisposition to numerical relations continued to interfere with his fractional reasoning in this last segment. In suggesting that Hillary "add some" to the six-sixths bar, he seemed to have invented a rule that eight parts, regardless of their size relative to the twelve-twelfths bar, would constitute two-thirds. However, he was able to make meaning of Hillary's expressions and complete her suggested production. This indicates that while Hillary could coordinate three levels of units to create new fractions, Will could only act on experiential units of units of units once they were established. In this case, Hillary created two out of the three units of sixths, and, once she verbalized this, Will could use the parts on the screen to follow her instruction.

The episode concluded as Will and Hillary tried to produce two-thirds with different numbers of parts. Will suggested that they use sixteenths, and, "in a way" Hillary thought this might work. Will was still not focused on size; he initially thought that eight-sixteenths might be two-thirds before realizing that it would be one-half. Similar to his initial approach in trying to produce two-thirds using twelfths, he then tried half of the eight sixteenths. Although he had been able to interpret Hillary's previous comments regarding the production of four-sixths as two-thirds, Will could not act creatively with a commensurate fractional scheme of his own and instead resorted to using his whole-number procedures again. Hillary, on the other hand, began using discreet partitioning, trying to "count into parts where they will all be even, like seven, seven and seven." These actions may represent another instantiation of her commensurate fractional operations.

12 March, 2003 Teaching Episode: Hillary's Schemes for Working with Improper Fractions

Will was out of school with strep throat for this teaching episode. I used the opportunity to work with Hillary alone in order to test whether she had constructed a reversible partitive fractional scheme or an iterative fractional scheme with which to meaningfully interpret improper fractional language. I began by asking Hillary to "make" a bar so that the bar on the screen was three-fourths of hers. She used MAKE to draw the bar because of my language. However, she drew a very good estimate in area and length. Protocol 13 begins with her explanation of that initial production, which served as indication for a reversible partitive fractional scheme.

Protocol 13: *Hillary's reversible partitive fractional scheme*

T: Can you show me how mine is three-fourths of yours?

H: [nods affirmatively]

T: How do you know?

H: Um… 'cause that's like three pieces in here [sweeping the cursor across my bar and then tracing an imaginary segment where her bar extended beyond mine to the right (the bars were lined up of the left side)], and there's the fourth piece.

T: So this would be in four pieces [pointing to her bar], and this [pointing to my bar] would be three of those?

H: [nods again]

Hillary's ability to accurately reproduce the fourth piece from an unpartitioned three-fourths bar indicates that she could use her partitive fractional scheme reversibly. This way of operating did not appear to be novel in Protocol 13 because she responded immediately and with a precise explanation for her actions. It should not surprise us if Hillary had constructed a reversible partitive fractional scheme because she could split, meaning that iteration and partitioning were not only inverse operations, but were interiorized as a single operation. As the episode continued, I asked Hillary to produce the desired whole exactly, rather than drawing an estimate for it.

Protocol 13: (Cont.)

T: [after drawing a new bar…] There's my bar, and I would like you to do something so that mine is three-fourths of yours.

H: [makes a copy of the unit bar, dials parts to 4, pauses, then dials it to 3 and partitions the unit bar into three parts. She then pulled out one of the three parts in the unit bar and joined it to the copy]

T: Did you get it?

H: [nods affirmatively]

T: How do you know you got it?

H: That's [pointing to the left side of her bar] the same size as all three of those [pointing to my bar], and if you add a fourth… another square to that [tracing a fourth piece (equal in size to one third of my bar) with her finger] it would be the same size as that [tracing her bar].

T: So mine is three-fourths of yours?

H: [nods again]

Hillary was learning to carry out her existing splitting operations in the `TIMA:Bars` environment in satisfying the goal of producing a bar so that the given bar would be three-fourths of it. If she had not already constructed a reversible partitive fractional scheme, she was able to use operations of splitting and disembedding in satisfying a reversible partitioning goal, with no help from me. She may have acted with certainty because her available ways of operating could give meaning to the situation and suggest its resolution. I note her activity as an important example of the power Hillary had gained through Conjecture H1: estimating a fractional size of a bar while keeping track of the whole by explicitly referring to the fractional complement. It could be argued, however, based on her response to Task 5 of her initial interview, that Hillary had already possessed a reversible partitive fractional scheme or that she recalled records of that similar experience.

Hillary did leave some room for doubt after she had finished, claiming that she was seventy percent sure she had produced the desired bar. When I asked her how she could be more sure, she tried joining another third of the unit bar to the unit bar (except that `TIMA:Bars` does not allow other bars to be joined to the unit bar, so she just lined them up). This raised her confidence to ninety percent. It also provided evidence that she had been operating in ways compatible with her complementary fractional comparison scheme, because rebuilding the whole using the complement served as a test for the production of her bar. This is interesting in itself, but I cannot identify Hillary's actions in Protocol 13 as definitively conjectural because Hillary seemed to be rather certain of her results and had acted similarly in the initial interview. The novelty was in acting reversibly using `TIMA:Bars`.

Hillary's conjectures concerning improper fractions. Although Hillary did seem to reason reversibly using her partitive fractional scheme, there is reason to doubt that she had constructed the operation that underpins an iterative fractional scheme. As the episode continued, I asked Hillary what her bar would measure when mine was labeled as the unit bar. After several seconds in silent thought, she responded, "a whole?" When I challenged her to consider the size of a whole, she began to focus on the fourth part in her bar and claimed that her bar would be one-fourth. We will see in the episode on March 20th that this sort of regression (from her partitive fractional scheme in conflating her bar with the extra part) was common to Hillary and Will when dealing with improper fractions. Although Hillary was eventually able to name her bar "four-thirds," it was only through our interaction as I directed her to consider iterating one-third that she was able to do so, and it did not seem to be the result of an established scheme for operating with improper fractions. She could name improper fractions through iterating but could not yet consider them as she could common fractions such as three-fourths, with her partitive fractional scheme. This conclusion will be further corroborated by her actions in dealing with improper fractions in the episode on March 20th. First, let us consider one more protocol in this episode.

There had been no indication thus far that Hillary had constructed an iterative fractional scheme, but she could make sense of and even produce fractions such as eight-sevenths or four-thirds by "adding on" (Protocol 1) or iteration. She had also demonstrated that she could operate reversibly with her partitive fractional scheme. In order to test whether her reversibility would extend to improper fractions and possibly provoke conjectural activity, I decided to challenge her with a reversible task involving an improper fraction.

Protocol 14: *Attempts to produce the whole from an improper fraction.*

T: [after making a bar on a cleared screen and labeling it the unit bar] This is my bar again. This time, my bar is five-fourths of your bar, and I want you to make your bar.

H: [makes a copy of my bar, partitions the original into five parts and the copy into four parts] I think that's it.

T: Okay. So, mine's five-fourths of yours, right? So, how should mine compare to yours?

H: You should have one extra bar.

T: Let's see if it does.

H: [lines up the two bars] Yeah it does. Mine are just bigger.

T: Is that okay?

H: [nods affirmatively].

T: So, if mine's five-fourths of yours, should mine be bigger, smaller or the same size.

H: Smaller.

T: So, if mine is five-fourths of yours, mine should be smaller than yours?

H: Your squares [the pieces] should.

Hillary knew that a five-fourths bar should have one more part than the whole, but she did not recognize the whole embedded in the five-fourths bar as she would in using an iterative fractional scheme. A student with an iterative fractional scheme might still have trouble with the task because the task involves operating reversibly with respect to the iterative fractional scheme—partitioning the given fraction with the goal of removing parts to reproduce the whole. Hillary could use partitioning in such a way, but because she did not recognize the embedded whole, she acted on a new whole (a second copy of the five-fourths bar) and created partitions that numbered one less than the given fraction.

She was still operating reversibly as she would with her partitive fractional scheme, but there was a novelty: She posited a whole and used partitioning in order to *reduce* the number of parts from her bar and satisfy the five-to-four relationship between my bar and hers. Her uncertainty ("I think that's it") indicates the conjectural nature of her claim (**Conjecture H3a**) that the five-part bar was five-fourths of the four-part bar she made. In operating the way she did, Hillary neglected any requirements about the sizes of the pieces and instead focused on the fact that my bar had one more part. As a post hoc explanation, she even used her partitive fractional scheme to point out the necessity of my parts being smaller than hers (because my bar was the same size as hers and had more parts), and used this to justify the sensibility of her production. It was only later when she focused on the sizes of our bars (at my prompting), that she found fault with her production.

Once Hillary realized that my bar should be bigger than hers, she tried again to produce the desired bar. However, she conflated the situation with one in which the bar she would create would be five-fourths of the given bar.

Protocol 14: (Cont.)

H: [clears the parts on the unit bar, partitions it into four parts (instead of five), pulls one fourth from it, and lines it up with the unit bar]

T: Okay. I see what you did, and it's sneaky. Now, go ahead and explain.

H: [lines up the copy (still in four parts) with the left side of the five-fourths bar she had just made] That's four [pointing to the unit bar within the five-fourths bar]. That's like the whole. So, you just add one more and it would be four… I mean five-fifths… five-fourths.

T: So this whole thing [pointing to the five-fourths bar] is how much of this one [pointing to the four-part bar]?

H: Five-fourths.

In Task 5 of her initial interview, in which she was asked to make a bar so that mine was three-fourths of hers, Hillary conflated the two bars and made three-fourths of my bar. At that time she seemed to have no means for keeping track of the whole when an experiential whole was absent. After the first couple of teaching episodes, when working with proper fractions, Hillary had learned a means of keeping track of the whole by explicitly referring to the complement of the fraction. This was not possible with improper fractions. In her first attempt at producing a bar so that mine was five-fourths of hers, she had made a copy of my bar to use as the whole. In her second attempt, Hillary again conflated the roles of our two bars. She used the *given* bar (my bar) as the whole because, this way, she would not have to identify the whole within the five-fourths. However, once she produced the five-fourths bar from my bar, she could identify the whole within it because she was working from an experiential whole.

I refer to Hillary's conflation of our bars as **Conjecture H3b** because she was trying to determine how her bar might be embedded in mine, and she was uncertain how this might be done. She needed a referent whole from which to construct my bar as five-fourths and conjectured that producing five-fourths *from* my bar might result in the same construction. Once she partitioned her bar into four parts, she knew that it was four-fourths and she also knew (from her experiences in Protocol 1) that she could "add one more" fourth to make five-fourths. Even then, her reference to the five-fourths bar was momentarily ambiguous: "It would be four… I mean five-fifths… five-fourths." She could conceive of the parts as both fourths and fifths depending on which whole she was considering. So, the fractional sizes of the unit parts were dependent on the production of an experiential whole.

Hillary seemed to resolve the ambiguity by positing the give bar (my bar) as the whole. She must have understood five-fourths as five of some part of which the whole had four, in order to produce five-fourths as she did.. But to construct an iterative fractional scheme, that part would have to contain records of the whole so that she could understand five-fourths as five of one-fourth of the whole. Because she could split, I hypothesize that this way of operating was within Hillary's zone of potential construction if she had not *already* constructed an iterative fractional scheme. Either way, Conjectures H3a and H3b might also engender the construction of a *reversible iterative fractional scheme*.

As the episode continued, Hillary posed a problem to me. She drew a new bar and said, "Mine is eight-sixths of yours." I worked through the problem out loud referring to my bar as having six of her eight pieces. When I made my bar the unit bar and measured her bar, her bar measured "4/3." When I asked Hillary why that measure appeared, she responded, "You can connect these two, connect these two, and connect these two [pointing in turn to the three adjacent pairs of sixths in my bar]. So that's three parts, and… there's four parts [pointing out the four pairs in her bar]." Her actions and explanation indicate that she had assimilated my language and action using her commensurate fractional scheme. Her explanation also corroborates my hypothesis that she could conceive of fractions like four-thirds as four of one-third of the whole—an indication of iterative fractional operations. In fact, by the end of the May 2nd teaching episode, there is strong indication that Hillary had constructed an iterative fractional scheme, affirming my hypothesis that the scheme had been in her zone of potential construction.

As the episode concluded, Hillary demonstrated that she could also use her commensurate fractional scheme to construe a nine-thirds bar as "three ones" before actually measuring the fraction. In fact, she had produced the three-thirds whole given a bar that was to be nine-thirds of her bar. She did this by dividing the given bar into nine parts and pulling three of them. Although her actions appeared unproblematic, I interpret them as a regeneration and generalization of her assimilated experience in observing my actions in the case of eight-sixths. This argument is supported by her subsequent actions on March 20th, which indicate that she was yet to construct an iterative fractional scheme for meaningfully producing improper fractions. She could produce fraction bars like four-thirds, but she did not seem to compare them back to the unit bar and sometimes conflated them with the unit bar; except, in the case of four-thirds, she hesitated after the third ("that's three parts") iteration. This may indicate an increased awareness of the unit bar within an improper fraction.

20 March, 2003 Teaching Episode

Will's conjectural production of an improper fraction. The students' classroom teacher asked me to spend some time reviewing the conversion of improper fractions to mixed numbers with the students. I decided to incorporate this into the teaching episode, in order to test the students' understanding of improper fractions. I began by asking Will to produce seven-fourths. He made two copies of the unit bar and partitioned one of them into seven parts, but then realized, "Oh! I need to cut it into four parts, not seven, because we're making seven-fourths [switching his index and middle fingers back and forth]." Hillary, who had been sitting quietly, smiled and admitted that she had decided to let Will figure that out for himself. Will went on to produce seven-fourths by partitioning the two copies of the unit bar into four parts each, pulling seven parts from among them, and joining them together. Hillary was confused because she thought that the unit bar would also have to be partitioned into four parts for the measure of the seven-part bar to be seven-fourths.

The last time that Will attempted to produce an improper fraction in the teaching experiment was during Protocol 11, in which he attempted (unsuccessfully) to produce four-thirds. Of course, we know that he had been working with improper fractions in class. Since Protocol 13, and perhaps through classroom experience, Will seemed to have enriched his concept of fractions like seven-fourths: He understood that seven-fourths should be seven of a part that comes from four equal parts making up a given bar. It is unclear whether Will necessarily conceived of the given bar as the unit bar, and he did not appear to compare the seven-part bar back to it. Hillary, on the other hand, had already demonstrated that she could meaningfully produce such fractions. Her confusion seemed to be related to the functioning of the computer in using the measure tool, as described above.

Because I had never before observed Will successfully produce an improper fraction, I refer to his production as conjectural: **Conjecture W4a**. He had initially attempted its production by partitioning a copy of the unit bar into seven parts, before using four parts. When he finished the production, he was not certain he had made seven-fourths, but could explain why he thought it was seven-fourths: "'Cause I took seven parts out of four out of all of these [pointing to one of the four-fourths bars]." When I asked Will how much each of the seven parts was out of the unit bar, he did not immediately know, and instead looked at the screen for a few seconds before responding, "one… fourth." This indicated that he may not have been using a fraction scheme in his production, but was simply partitioning two bars into four parts each and taking seven parts. "Seven-fourths" would then mean seven parts compared to four parts. This is indicated by his uncertainties and, in turn, indicates that he had been reasoning with ratios of whole units.

The episode continued as I asked the students what the mixed number for seven-fourths would be. Will knew right away that it would be one and three-fourths. However, when I asked him to show this with the bars, he pulled one-fourth and three-fourths and joined them, thinking aloud, "I'm going to try measuring that because I took one, and then I took three out." His prompt recognition of the mixed number and his response in making it corroborate my claim that Will had been using whole number algorithms and had not treated the units as fourths but as *whole* units. Before measuring, he realized that this would be one whole because it was the same size as the unit bar, but he did not seem to experience a conflict between the two expected measures of the bar, thinking it could be both measures at the same time.

Hillary seemed initially confused but, after listening to Will's explanation, agreed that he had produced one and three-fourths. Once we actually measured the production as "1," I reminded the students that we wanted one and three-fourths. At this point Will seemed to realize what "one" referred to, "Oh! I get it now [completing the production of one and three fourths by pulling another three-fourths from the four-fourths bar and lining it up below his previous production]." He explained his initial confusion and subsequent insight saying, "'cause I took one part out and then three other ones and then when I saw it, it was a whole entire thing – a unit bar."

Will's insight was based on a perceptual judgment (recognizing the whole in his perceptual field), and, as such, was not conjectural. But the students' actions in the segment highlight important ideas about their reasoning. First, Will still seemed focused on numbers of parts and was only considering relative sizes of the parts in post hoc explanations. This indicates that he had constructed fractions concepts with which to make meaning of relative sizes, but had no fractional schemes constructed for meaningfully producing improper fractions. Second, it is unclear whether Hillary had constructed an iterative fractional scheme either. Although she seemed uncomfortable, at first, with Will's initial production, she agreed with his explanation. This may be an example of what Piaget (1955) called "syncretism of reasoning" (pp. 140-170) referred to in Chapter 2. Hillary's assimilation of Will's explanation and her conception of one and three-fourths called the same schema—her concept of one as a singular unit, even if the unit was a fourth—so that one implied the other without her deductively examining the details of the situation further. Syncretism of reasoning commonly occurs in children between the ages of nine and eleven (Will and Hillary were not yet twelve) and precedes formal deductive reasoning (p. 140).

There are affective issues to consider here as well. Hillary's disposition in working with Will had been very agreeable. She looked for ways to validate Will's reasoning, even when she was unable to reconcile it with her initial reasoning concerning a situation.

Hillary's conjectural progress toward an iterative fractional scheme. From the students' classroom experience, they were proficient in applying an algorithm to convert improper fractional numerals to mixed numbers. So, after Will had successfully produced ten-thirds, Hillary calculated that it should be three and one-third, and I asked her to show that. Surprisingly, even after we had resolved the situation with one and three-fourths, Hillary lined up a three-thirds bar with a one-third bar (as displayed in the lower-left corner of Figure 10) and claimed that was three and one third!

Protocol 15: *Hillary's novel use of uniting operations*.

H: Three and one-third?

T: Where's three and one-third?

H: This is three [sweeping the cursor over the three-thirds bar] and that's one-third [pointing with the cursor to the one-third bar].

T: Is it the same as ten-thirds [pointing at the ten-thirds bar and then looking at Hillary]?

H: [looks at the screen]

W: [shakes his head *no*] Mm-mm.

T: Doesn't look the same… This one [pointing to the ten-thirds bar] looks a lot bigger!

W: I think it's three and one-thirds but it's not the same as three-tenths or ten-thirds or whatever.

H: We're going to take three of these bars [circling the cursor around three thirds at a time, down the ten-thirds bar].

T: Do you want to fill them in to show?

H: [fills in three of the one-third bars at a time, with each group of three a different color] Three and three...

W: [pointing at the groups] That's three, three and three and we've got one left over.

T: Is that three and one-third? You said three, three times.

W: It could be nine and one-third.

H: [after a few seconds pause looking at the screen] Three whole unit bars and one-third out of it.

T: Did you hear what she said? Does that make sense?

W: Yeah.

T: So, explain in your own words, Will, what she means in terms of this.

W: Those three right there are unit bars. There's three of them, so there's three whole unit bars. See [dragging the unit bar over each] they're the same size as the unit bar.

T: So this right here is not three and one-third [pointing to the lower left corner of the screen displayed in Figure 10], is it?

W: [pauses and shakes his head "no"] Uh-uh.

T: What is it?

H: [pauses for a couple of seconds] One and one-third?

W: [Joins the three-thirds and one-third bars and measures "4/3"] Four-thirds is the same as one and one-third.

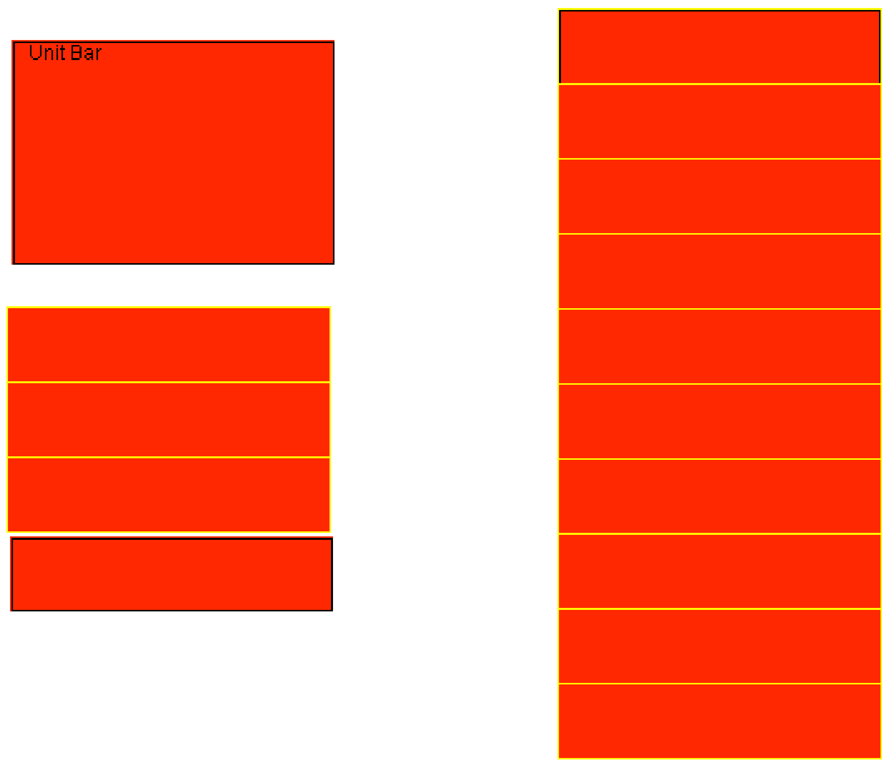

Figure 10: Hillary's production on three and one-third.

I have argued that Hillary had constructed a commensurate fractional scheme. In fact, at the end of the March 12th episode, she was even able to use this scheme to interpret an eight-sixths bar, arguing that eight-sixths had the same measure as four-thirds by uniting every two sixths into one third. However, she had not previously united composite wholes within an improper fraction, and her commensurate fractional scheme would not be of immediate use in converting improper fractions to mixed numbers because in such conversions there is no n-to-1 relationship that can be used to form equal units of units within the whole fraction (for example, in the present case, there is a third left over). Instead, she had to conceive of the whole within the fraction—a crucial step in constructing an iterative fractional scheme and a modification to her commensurate fractional scheme.

Hillary and Will used their conversion algorithm treating the fractional units ambiguously as units of one and units of thirds (as they had with seven-fourths) until I questioned them about the sizes of the bars: “Is it the same as ten-thirds?” Even then, only Hillary deduced that the bars should be the same size. Will seemed to see no problem in the different sizes associated with the “ten-thirds” and “three and one-third” fractions, even though he had established that one could be converted into the other. Although he could make meaning of Hillary’s subsequent actions and explanations, his meaning did not seem to be based on uniting fractional wholes within the bar, and he was not constructing the parts as fractions either. Rather, he first claimed that the “three, three and three” might make “nine and one third,” and later used the unit bar to show that the whole fit into ten-thirds three times. He had not recognized the copies of the unit bar within ten-thirds until Hillary had colored them.

Hillary eventually united the three three-thirds parts as composite fractional wholes using SHADE and dragging the unit bar to help her enact the novelty. Since this was a novel use of her uniting operation and units coordination, I refer to it as a conjectural operation. The uncertainty of its use in such situations is indicated by her tentative assertion about her initial production: "One and one-third?" It seems that she had conjecturally made a functional accommodation of her commensurate fractional scheme to coordinate composite fractional wholes within improper fractions. Such coordination might also contribute to the construction of an iterative fractional scheme or even a reversible iterative fractional scheme if she could use it to compare improper fractions back to the whole and posit a whole within an unpartitioned improper fraction. For example, if I were to ask her to find a bar such that mine was five-fourths of it, she might be able to partition my bar into five equal parts, and unite four of them to reestablish the intended whole. So, this episode includes a potentially powerful conjecture, which I refer to as **Conjecture H4**: identifying the fractional wholes formed within an improper fraction, m/n, by uniting each numerical composite of *n* parts as 1.

Will's conception of and new procedure for improper fractions. We continued the episode as Will asked Hillary to produce seven-sixths. Initially, she partitioned the unit bar into seven parts, but quickly changed this to sixths, with no intervention from Will or myself. She made a copy of the unit bar, partitioned it into six parts, pulled one more, and joined to make seven-sixths. It is interesting that she still insisted on partitioning the unit bar as she had done on March 20th. Whereas it indicates an awareness of the identity of the copy and the unit bar, it would hardly seem necessary to partition the unit bar. Perhaps she thought that it was necessary for the functioning of the MEASURE tool, or she may have needed it as an experiential referent for the partitioned whole, with which she could compare the number of parts in the seven-sixths bar. The latter necessity would indicate that comparing improper fractions back to the whole was indeed novel for her, as she now appears to be operating as she would with an iterative fractional scheme.

Will agreed that Hillary had produced seven-sixths and immediately knew that it would convert to one and one-sixth. He produced the mixed number by making a new copy of the unit bar, partitioning it into six parts (saying, "that's one"), and pulling another sixth: "That's going to be one and one-sixths." After Hillary agreed that he was right, Will measured each of the two bars in turn to reveal "1" and "1/6."

Will's production constituted a new way of acting for him. The question remains whether he had assembled a novel way of operating based on his interpretation of Hillary's activity during Protocol 15. He had explicitly established that "one" referred to one unit bar and, of course, he knew how to produce one-sixth. He had also conceptualized "one and one-sixth" as "one" and "one-sixth." What remains uncertain is whether he could unite composite wholes within a given improper fraction, with the goal of reconstituting the fraction as a mixed number.

Will's actions later in the episode indicate that he could not conceive of seven-sixths as seven iterations of one-sixth of the whole, as Hillary could. Hillary posed sixteen-fifths to Will. In response, he made a copy of the unit bar, partitioned it into sixteen parts, pulled five parts, and joined them to make twenty-one sixteenths.

Protocol 16: *Will's ambiguous sense of improper fractions.*

W: [finishing his production] That's going to be sixteen-fifths.

T: All right, let's check. What do you think Hillary?

H: It's not.

W: [measures to reveal "21/16"] Twenty-one sixteenths?! Okay…

T: [laughing] Why did it come out to be twenty-one sixteenths?

H: It's because he used sixteen parts instead of five.

T: Do you get what she's saying?

W: Yeah. That's where I messed up; I cut the unit wrong.

T: Yeah. Ya'll both switch it sometimes, so you have to be careful. So, what should you cut it into?

W: Fifths.

T: Why fifths?

W: 'Cause it's sixteen fifths.

Will's actions in Protocol 16 indicate that he still lacked a fraction scheme for producing improper fractions. He seemed to be using contextual procedures, invented in the social context of assimilating Hillary's actions, to relate the numerators and denominators in such fractions. In past protocols, he used procedures successfully, but it was evident that his actions did not represent schemes for meaningful fractional operation. Like the other procedures that he had invented, the social context of observing and interacting with Hillary in solving problems provided occasion for him to assimilate Hillary's actions and to coordinate old operations in new ways.

In this case, Will's assimilation of Hillary's actions in producing seven-sixths resulted in a procedure that simply concatenated the number of parts indicated by the numerator and denominator. He had partitioned a copy of the unit bar just as Hillary had, except that he used sixteen parts whereas Hillary would have used five; he then pulled a number of parts to join just as Hillary had except that Will used the number in the denominator instead of building up to the number in the numerator. So, in assimilating Hillary's action, he did not distinguish the unique roles of numerator and denominator as one would in using a partitive fractional scheme. Will's lack of a partitive fractional scheme accounts for the disparity between his actions and Hillary's. Hillary knew that Will's production was not appropriate and reminded Will that "sixteen-fifths" referred to the numerosity of fifths making up the fraction, and Will understood this "cause it's sixteen fifths." Similar to Conjecture W4a, Will's procedure amounted to a conjecture (**Conjecture W4b**) that he could produce sixteen-fifths by adjoining five more parts to a sixteen-part bar.

Once the students had successfully produced sixteen-fifths, I asked Hillary to find the mixed number. This time, the unit bar was not partitioned, and she dragged the unit bar into the sixteen-fifths bar three times (top, middle, and bottom) before I suggested that she use FILL to keep track, as she had done with ten-thirds. When she was done filling the three composite units, she claimed, "That's three and one-fifth." After a moment of reflection, Will seemed to understand, saying, "there's three out of all of those, there's one left over, and there's five in each." In that Will interpreted Hillary's language and action appropriately, transforming sixteen-fifths to three and one-fifth seemed to be in his zone of potential construction. But it is important to keep in mind that his understanding occurred in a social context of interpreting Hillary's language and actions. Hillary seemed to take the five-fifths bar as a given, which, coupled with her transformation of sixteen-fifths to three and one-fifth, indicates that she was working at the level of a unit of units of units. There is no indication that Will could operate independently with such units.

I have claimed that Hillary had been operating conjecturally in Protocol 15. Protocol 16 provides further indication that her use of uniting operations and units coordinating in such situations was novel. In the latter protocol, because the unit bar was unpartitioned, there was no visual cue for Hillary to determine the numerosity of the groups for shading. Instead, she dragged the unit bar within the improper fraction, establishing three wholes within the fraction. This activity and her subsequent shading of the groups of five may have served as a test for Hillary's way of operating.

<u>24 March, 2003 Teaching Episode</u>

This episode was dedicated to producing various fractions by pulling parts from a twelve-twelfths bar. The students began by pulling eight parts and measuring to find that it was two-thirds. Finding commensurate fractions had not been their goal, but they were able to give post hoc explanations for the measures, much as they had in previous episodes. For the next fraction, while trying to pull six parts, Will pulled out seven-twelfths and thought that it would measure four-sixths.

Protocol 17: *Will's numerical procedures and a modification to Hillary's commensurate fractional scheme*.

T: What do you think this one's going to be?

W: I pulled out seven, so I think… I say four-sixths.

H: I think two-thirds.

T: [to Hillary] Why do you think Will said four-sixths? Will, you try to think about why she was thinking two-thirds.

H: ‘Cause if you cut that [pointing to the seven-twelfths bar] in the middle, you are going to have four.

T: Go ahead and do that [suggesting that she use CUTS].

H: [cuts the bottom seven-twelfths and the top five-twelfths in half, but the top cut wasn’t visible. She then drags one of the 3.5-twelfths pieces (which had no visible partitions) in the unit bar four times (with some overlap)]. Four, and two of these will fit into that piece [seven-twelfths]. But he said four-sixths…

T: Will, do you know why she said two-thirds?

W: Well, it’s pretty much like last time. We did a little bit over half [dragging the seven-twelfths bar into the unit bar, then lining it up beside the unit bar].

H: Two-fourths.

W: I changed my mind. I think it’s going to be… I took seven out… [drags the cursor across each twelfth in the twelve-twelfths (unit) bar] I think it’s just going to be what I pulled out, seven-twelfths.

T: I’d like y’all to explain to each other.

W: I think it’s going to be seven-twelfths, first of all, because I pulled out seven out of twelve [drags the seven-twelfths bar back into the twelve-twelfths bar].

H: …’cause seven is an odd number…

T: Do you still think two-fourths might be it?

H: [nods affirmatively]

T: Okay. Can you explain why two-fourths?

H: It's just like you cut that in half and you cut that part in half too, and you're going to have four pieces.

W: [After indicating that he understood Hillary's explanation, Will cut a copy of the seven-twelfths bar into two parts and begins to count parts in a copy of the twelve-twelfths bar.]

H: I think it's seven-twelfths… because you can't split seven in half; it's not an even number.

Will seemed to make an analogical guess when he said that the seven-part bar would measure "four-sixths." Given his reference to the previous case, when eight twelfths measured "two-thirds" ("it's pretty much like last time"), he may have used his procedural scheme for producing equivalent fractions to produce four-sixths as a similar measure. Will had begun focusing on the sizes of bars relative to the unit bar. For the first time in the teaching experiment, Will initiated the activity of explicitly comparing a non-unit fraction to the whole in order to approximate its size; dragging the bar in question over the unit bar, he said, "we did a little bit over half." He must have been able to figuratively posit the size of half of the unit bar and two-thirds of the unit bar and compare them to the size of the bar in question, thus establishing an ability to figuratively compare non-unit fractions. If his activity were to modify his concept of non-unit fractions, he would eventually have to reconcile a conception of approximate fractional size with his conception of non-unit fractions as ratios. In so doing, he might construct a partitive fractional scheme. If he could also develop an ability to coordinate units of units of units, he might construct a commensurate fractional scheme from his current procedural scheme for producing equivalent fractions.

Hillary initially approximated that the fraction would measure two-thirds, but when she considered Will's suggestion, she could make some sense of it by considering four apparently equally sized parts. Perhaps she reasoned similarly with three such parts in offering her own suggestion. At least with the four parts, her actions represented the conjectural use of partitioning and units coordination. Her conjecture (**Conjecture H5**) was to explain Will's "four-sixths" response by creating four equal parts within the seven-twelfths bar, but resulted in a conjectural production of two-fourths. This was, again, a socially based conjecture. The crucial difference in her conjecture and Will's conjectural procedures is that she had the fractional operations available to evaluate her conjecture.

Recall that Hillary's complementary fractional comparison scheme involved estimations of size, and her commensurate fractional scheme grew from the former scheme by uniting parts within a composite fraction and its complement. Hillary partitioned seven-twelfths and its complement into two parts each with the goal of creating four approximately equally sized parts. Once she had done this, she was able to assimilate the resulting bar using her commensurate fractional scheme and recognize it as two-fourths. Having dragged the 3.5-fourths piece within the unit bar four times, Hillary was convinced that she had created equal parts. But after observing Will make the cuts, she realized a problem with cutting seven-twelfths in half.

When Hillary was cutting, she was operating within the framework of assimilating Will's production using her commensurate fractional scheme. Prior to this task, it worked, so she was simply repeating what had worked. She was not expecting there to be differences. When Will operated, and produced seven-twelfths, Hillary's goal now was to explain why Will did not get a fraction other than seven-twelfths. She was still learning to reflect on her operating. Being free from the activity of cutting allowed her to notice that the groups of fourths would be uneven and consist of broken parts: "You can't split seven in half; it's an odd number." Through her conjecturing activity, she had introduced a new constraint to her production of commensurate fractions: She now understood the importance of divisibility in the numbers of parts in establishing composite units within a partitioned whole. Will could not coordinate units as Hillary had, and his reasoning that the bar would measure seven-twelfths probably was based on his whole number division operation (7 is not divisible by 2).

Toward the end of the episode, the students produced a ten-twelfths bar. I asked them to predict its measure, and the students' responses are recorded in Protocol 18. Hillary's responses, in particular, indicate a modification to her commensurate fractional scheme as a result of the new constraint introduced as a result of her test of Conjecture H5.

Protocol 18: *A newly observed aspect of Hillary's commensurate fractional scheme.*

W: I think it's going to be five-sixths… because it's half of ten-twelfths.

H: [after slowly moving her cursor down through the twelve twelfths] Six-sevenths.

O: Can you explain to Will?

H: Okay. There's two bars left over, so… you can't have three or more. You have to have two [sliding the cursor under every other twelfth starting from the top of the twelve-twelfths bar. Two, two, two, two…

T: Do you want to use fill to keep track of them?

W: Where'd you get the seven?

H: [Fills the twelve-twelfths bar different colors by two's and counts the pairs] Five-sixths.

T: So why'd you say six-sevenths before?

H: It's confusing [pointing her cursor across the parts].

Will was able to use his procedural scheme for producing equivalent fractions once more to quickly solve the problem of simplifying the fraction. His language, "it's half of ten-twelfths," indicates both his method and the need for reconciliation between his numerical computation and the sizes of fraction bars: His numerical computation was based on ratio's with which "half" meant half of each number in the ratio but he also used the term to refer to half as much. Meanwhile, Hillary demonstrated what might have been a novel way of operating for her: In order to decide how to coordinate the units of parts, she considered the complement of ten-twelfths and concluded that 2 was the *only* possibility because "you can't have 3 or more." This way of operating might be attributed to a functional accommodation of her commensurate fractional scheme resulting from Conjecture H5. In particular, she had learned the importance of finding equal composites of whole parts within a fraction *and* its complement. Her initial claim that the bar was six-sevenths may be explained by her pairings of the numerous parts involved, but it is interesting that even then she chose a fraction whose complement was a unit fraction; she knew that the two parts in the complement of ten-twelfths should be united in the complement of the simplified fraction.

<u>26 March, 2003 Teaching Episode</u>

This episode was unusually short (about eighteen minutes) because Will and Hillary had to complete a classroom assignment before I could meet with them. I decided to challenge the students to make the unit bar given a bar that was a proper fraction of it. I asked the students to close their eyes as I partitioned the unit bar into twelve parts, pulled out nine of them and covered the unit bar. I chose nine-twelfths to begin because it was a fraction that the students had not thought to make in the previous episode when using twelve-twelfths to pull out various fractions.

When the students measured and “3/4” appeared, they thought for several seconds about why this happened. Hillary then began smiling and explained this happened “because there’s three parts in each one.” Will then shaded the three groups of three saying, “there’s three in each one of these; there’s three bars, and there’s nine in all,” but then he stopped and looked puzzled, apparently confused about where the 4 came from. Hillary responded by pulling a three-twelfths bar from the nine-twelfths bar and joining the two bars to produce the whole. After Hillary had completed her production and uncovered the unit bar to check, Will still seemed confused.

Protocol 19: *Novelties in Hillary’s use of partitive fractional and commensurate fractional operations.*

T: Can you explain why she was right?

W: [both hands at his mouth, looking at the screen] I'm trying to figure out… Because you shade all of them and you have three, it equals nine. But where do you get the three-fourths from?

T: [after several seconds, to Hillary] You can help him if you want.

H: There's three parts right there, there's three parts right there, there's three parts right there, and that part was left over [pointing in turn to the four groups of three and then turning to look at Will]… nine times three is twelve… or nine *plus* three is twelve [smiles].

W: [begins shading the twelve-twelfths bar by three's and talking mostly to himself] There's three parts right there, three parts right there, three right there… That equals up to nine, and you add that [pointing to the last three parts] and you get twelve. [turns to the teacher] I get what she's saying.

Protocol 19 illustrates the growth in Hillary's ways of operating as a result of Conjecture H5. Hillary demonstrated the power of her commensurate fractional scheme in coordinating the units of three to establish nine-twelfths as three-fourths in a three-to-one relationship, even in the absence of the whole. She could do this partly because her commensurate fractional scheme contained operations for keeping track of the whole. In particular, she was able to consider the complement of three-fourths: "and that part is left over… nine plus three is twelve." The manner in which she was able to maintain the whole provides further indication that her commensurate fractional scheme had grown from her complementary fractional comparison scheme. Her actions also indicate reversibility in her partitive fractional scheme for composite units, which seemed to operate in coordination with her commensurate fractional scheme: establishing the whole by adjoining the three-part complement as the fourth fourth. Such reversible reasoning and coordination of schemes was novel for Hillary. I attribute the novelty to Conjecture H5, which introduced the constraint of producing common composite units in both the given fraction bar and its complement.

Will could not reason as Hillary had. He was once again trying to relate the numbers of parts in the fraction to the numerator and denominator displayed in the measure box. He had been successful operating in such ways before, and, in fact, we will see a profound example of success in Protocol 20. However, such whole-number operations, with little attention to fractional sizes, may have circumvented efforts that otherwise might have led to the construction of a commensurate fractional scheme. He did not yet coordinate units at three levels, absent of experiential units. Will could relate nine-twelfths to its measure, only after Hillary had produced the fourth group of three and he had shaded the groups (at the end of Protocol 19). In the absence of the entire twelve parts making up the whole, Will was not even able to use his procedural scheme for producing equivalent fractions.

As the episode continued, Hillary pulled six sixteenths, and challenged Will to reproduce the covered whole. Will measured the six-sixteenths bar and found that it was "three-eighths of the unit bar." He looked at the screen and thought for about a minute before explaining what he was thinking.

Protocol 20: *Will's novel use of his procedural scheme for producing equivalent fractions*.

W: I was counting [the number of parts in the six-sixteenths bar] and then I was going to look [at "3/8" in the measure box] to see if that number had anything to do with… [pulls one of the sixteenths from the six-sixteenths bar and repeats it to make another six-sixteenths bar, then joins the two] That's it right there.

H: [shakes her head left and right, almost imperceptibly, but Will looks over at her.]

T: Okay. So, tell us how you did that.

W: Well, first I was counting how many there were overall, and there were six in there. And then I noticed the 3 [in measure] and that was half of it [the six parts]. So, I just figured if I added on to twelve, maybe… 'cause 8 can… no… I think I should have made sixteen instead of twelve.

Will completed the production and explained that he needed sixteen parts because 3 is half of 6 and 8 is half of 16. He also volunteered that this was the toughest problem he had done so far. Because Will had used his ratio reasoning in a new way to resolve a problematic situation, I refer to his operations as conjectural (**Conjecture W5**). He conjectured that if six parts constituted three-eighths of the whole, then there should be sixteen parts in the whole because 3 is to 6 as 8 is to 16. Whereas before he had used his procedural scheme for producing equivalent fractions in inventing post-hoc explanations, in Protocol 20 he was able to determine the total number of parts in the whole. In doing this, it was important that he distinguish between the roles of the numerator and denominator. His initial attempt of making twelve parts indicates these distinctions had been problematic: He doubled the number of parts in the visible bar rather than the denominator of its fractional measure.

31 March, 2003 Teaching Episode

Will's procedural schemes. The students returned to the game they had been playing in the last episode. Will began by producing nine-thirteenths, covering the whole, and challenging Hillary to reproduce the whole from that fractional part, which she measured as "9/13." After about ten seconds, Hillary pulled one thirteenth from the nine-thirteenths bar, repeated it four times, and joined the parts to make the whole. When Hillary had completed her production, she checked the number of parts, determining that it had thirteen parts. This determination seemed to satisfy Hillary's goal, as she indicated that she was certain with no need for further checking. Next, Hillary posed three-sixteenths to Will, who had to reproduce the whole. Will measured the visible three-part bar (the original unit bar was covered) as "3/16," pulled one part, joined it to the others, and repeated the resulting four-part bar four times. Finally, he checked the number of parts in the production to find "16."

Protocol 21: *Will's new procedure for reproducing wholes from composite fractions*.

T: You got it?

W: [nods affirmatively]

T: Let's measure and check.

W: [measures "1"]

T: …and then you can uncover. There's lots of ways to check. You are getting more sure every time aren't you?

W: [nods affirmatively, uncovers the original unit bar, and drags it over his production, apparently to compare sizes]

T: All right. Good job!

O: How did you know how to do it that way?

W: 'Cause I got it off of Hillary's idea.

O: [laughs] What was Hillary's idea?

W: To, uh… Well, when we measured it, I did… I put down nine-thirteenths. So, she measured it, and it was nine-thirteenths. So, she just added four more [pointing to the last four parts in his production of sixteen-sixteenths].

When Hillary solved the problem that Will had posed, her goal seemed to be to make thirteen parts because she knew that nine-thirteenths was nine of one-thirteenth and that thirteen of these one-thirteenth units would recreate the whole. This sort of reasoning would involve a reversible partitive fractional scheme. In observing Hillary and subsequently solving a similar problem, Will had constructed a procedure by analogy. In other words, Will had used his whole number operations in assimilating general steps to Hillary's solution: Hillary had *nine* thirteenths and added four more to make *thirteen* thirteenths; Will had been given *three* sixteenths and knew he had to add on until he reached *sixteen* sixteenths. Will had been very explicit about his analogy in explaining "Hillary's idea." It may have been conjectural if he had been aware of his assimilation of Hillary's actions with the goal of generalizing them. But Will's actions indicated little uncertainty. It was more like a perceptual judgment because it was constructed without question.

It is interesting that Will converted this goal of adding on parts to make sixteen into a goal of making sixteen through a multiplicative relationship, as indicated by his actions of making four-sixteenths and repeating it. This may indicate that he could treat the sixteenths as he would treat units within his explicitly nested number sequence, anticipating the production of a unit of units of units.

I hypothesize that Will's actions were based on a procedural scheme that could emulate the operations of a reversible partitive fractional scheme: his *procedural scheme for reversing ratios.* The goal of this scheme was to reproduce the whole from a measured fractional part by creating a numerosity of parts equivalent to the denominator of the measured fraction. We will see further indication of the scheme and affirmation of my hypothesis throughout the rest of the teaching episode. I refer to it as a scheme because it represents a general way of acting for Will that is consistent throughout this episode. But we will see in future episodes that his procedural schemes were not as permanent as Hillary's conceptual schemes.

The constitutive characteristic of such a procedural scheme is that it is constructed in the context of assimilating the language and action of another student, using operations of a scheme different than the one used by the operating student. In this case, Will used the operation of his adding scheme for whole numbers in interpreting Hillary's language and action that were produced using her reversible partitive fractional scheme. Will could generalize some of the contextual details, such as the specific numbers in the fractional measure, but we will see that his procedure depended on other contextual details, such as starting from a *partitioned* fraction bar.

As the episode progressed, the students engaged in posing and solving more problems like that of Protocol 21. I encouraged the students to pose fractions to each other that might be tricky, without specifying what that might mean except that the measures of those fractions might say something different from what one might think. On his next turn, Will posed six-twelfths to Hillary and rotated the bar ninety degrees. His choice of using six-twelfths as a tricky problem indicates that Will was able to use his procedural scheme for producing equivalent fractions with the goal of creating a commensurate fraction, in addition to that of simply justifying equivalence (a restriction of his scheme before Protocol 20). The inference that Will had acted purposefully is supported by his explanation after Hillary had reproduced the whole: “Well, I just did half of it. So, instead of ‘6/12,’ it should come out to be ‘1/2.’”

Hillary posed three-elevenths for her next problem, and Will produced the whole by repeating one of the parts until he had eleven in all. Then, Will posed ten-twentieths to Hillary. These occurrences respectively provide further indication that Will had constructed general procedures for reproducing the whole from a proper, irreducible fraction (by considering its measure) and that he could purposefully produce commensurate fractions: his procedural scheme for reversing ratios and his procedural scheme for producing equivalent fractions, respectively.

In the latter problem mentioned above, Hillary measured “1/2.” At which point, Will, smiling nervously, exclaimed, “I have got to figure out how *I* would solve this.” Meanwhile, Hillary proceeded to pull out a copy of the ten-twentieths bar and joined the two visible bars to reproduce the whole. Hillary's actions may indicate the power of her reversible partitive fractional scheme and her ability to treat ten-twelfths as one half, using her commensurate fractional scheme, whereas Will's expressed uncertainty indicates a limitation to the procedural scheme for reversing ratios: It could not yet be applied to reducible fractions because the measured fraction in such cases does not specify the number of parts in the whole. This provides further indication that he had been acting with such a procedural scheme, rather than acting with partitive operations as Hillary seemed to do.

Modification of Will's procedural schemes. Later in the episode, Hillary posed seven twenty-firsts to Will. When Will measured the given bar as "1/3," he reacted with surprise, furrowing his brow and then turning to Hillary with a puzzled look. But then he counted the number of parts in the seven-part bar, made two copies of it, and joined all three bars.

Protocol 22: *A modification to Will's procedural scheme for producing equivalent fractions*.

W: That's the bar.

H: [nods affirmatively]

T: *That's* the bar?

W: Mm-hmm.

T: You pretty sure?

W: Mm-hmm.

T: What percent [sure are you]?

W: Ninety-nine.

T: All right. How can you check?

W: Uncover it. [uncovers the unit bar and begins to drag it over the bar that he produced] Yep.

Will had successfully used his numerical procedure for reversing ratios three times during the episode. He may have tried to apply it again in Protocol 22, but the procedure would lead him to make a total of three parts (3 was the denominator of the measure), when he already had seven parts. This conflict may account for Will's puzzlement, and a similar conflict may account for his nervous smile in Hillary's last problem of "1/2." However, I hypothesize that Will had another way of operating available as a result of his success in Protocol 20: a reversibility in his procedural scheme for producing equivalent fractions, which enabled him to reproduce wholes through a multiplicative relation defined by the number of parts in the given fraction and the numerator of its measure. The relationship in this case was seven-to-one, and was established after Will counted the number of parts in the given fraction.

Using the procedural scheme, Will's goal would then be to create twenty-one parts because 7 times 3 is 21. If indeed he was using the procedural scheme, Protocol 22 would serve as a test for the conjectural modification to the procedural scheme (using the relations with the procedure to reproduce the whole) made in Protocol 20. Alternatively, it is possible that Will had constructed a *reversible partitive unit fractional scheme for composite units*, in which case his goal would be to iterate the seven-part bar three times rather than to create twenty-one parts. Because Will counted the parts and chose to use

COPY rather than REPEAT, it seems that Will had been using the procedural scheme, corroborating my hypothesis.

Further limitations to Will's procedural schemes became apparent toward the conclusion of the episode. I made an unpartitioned three-fourths bar (with the unit bar covered), and asked the students to reproduce the whole. Hillary measured the bar as "3/4," and the students sat looking intently at the screen for about twenty seconds before Will made a suggestion.

Protocol 23: *A limitation to Will's procedural scheme for reversing ratios*.

W: [to Hillary] Try to copy one… copy that [pointing to the unpartitioned three-fourths bar].

H: It would be too much.

T: Do you know what I did that was sneaky?

W: [nods affirmatively and smiles] I think the other one has parts on it.

T: Yeah. I erased the parts.

H: [dials PARTS to 4 and then 3, partitions the given bar into three parts, pulls one and joins it to make a whole. When she finishes, she turns to look at the teacher, smiles and turns to Will, still smiling.]

W: [Meanwhile, Will appears to silently mouth the words "one fourth."]

T: How'd she do that, Will?

W: I was trying to figure out what she was doing at the beginning, but she took out one fourth of the three fourths and added another one to make one whole. So, I think that's it.

If Will had been attempting to use his procedural scheme for reversing ratios in Protocol 23, he would have tried to add one more part to the given fraction because it measured “3/4” and 3 plus one more would be 4 (the whole). Of course, in assimilating the situation using his procedural scheme he would have to supply the parts. Will was aware that this was an aspect of the situation that made it tricky: “I think the other one [the hidden unit bar] has parts on it.” More importantly, Will did indeed suggest that he and Hillary should add another copy to the given fraction! If he were using a reversible partitive fractional scheme (or a fractional scheme at all) he would be unlikely to offer such a suggestion because he could recognize that two copies of three-fourths would go beyond the whole. In fact, Hillary recognized this fact right away: “It would be too much.”

Hillary’s actions in Protocol 23 provide the best indication yet that she had constructed a reversible partitive fractional scheme, that she could split, and that an iterative fractional scheme was within her zone of potential construction. Will could make sense of her actions once she had partitioned the given bar into three parts. At that point, his procedural scheme for reversing ratios could be applied without a problem. His statements at the end of Protocol 23 serve as indication of this.

<u>14 April, 2003 Teaching Episode: Relative Permanence of the Students’ Schemes.</u>

This was the first episode in two weeks, following the students’ weeklong spring break. Although the students’ actions in the episode often do not represent new ways of operating, I mention them to highlight the relative permanence of the student’s schemes and operations. At the beginning of the episode, I asked the students to remake a hidden unit bar given an unpartitioned bar that was two-thirds of it. Hillary immediately grabbed the mouse and dialed PARTS to “3.” She hesitated a moment before dialing PARTS to “2,” but then immediately partitioned the given bar into two parts, pulled one of them and joined it to the others to make the whole. Despite the prolonged break from working with fraction bars or even with classroom mathematics, Hillary’s reversible partitive fractional scheme was still available for solving such tasks.

Next, Hillary made a wiped (unpartitioned) three-fifteenths bar for Will, and he was supposed to reproduce the covered whole. After measuring the given bar to be “1/5,” he partitioned it into two parts, pulled one of those parts, repeated it until he had three additional parts (for a total of five parts), and joined them. I use this protocol as a test for the flexibility and relative permanence of Will’s procedural schemes.

Protocol 24: *A test of Will’s procedural scheme for reversing ratios*.

W: I think that’s it.

T: Okay. How did you think about that?

W: Well, I measured and it said one-fifth, and so I pulled out one and I added [holding out three fingers on his left hand] about… I think it was three more? Three or four more, and that equaled five, and that equals one whole.

T: Okay. Let me ask this. The first thing you did was you put the piece into two parts. Why did you do that?

W: Just so I could add… So I won’t have to… So when I cut them down lower, I won’t have to add as many on.

T: Okay… Because you thought you would run out of room or something?

W: No. Because, see, if you cut it down like more [pointing to the bar he had just made] then it's going to be smaller than that, and since she covered up a whole lot [pointing to the cover over the whole], I figured she cut it down as bigger pieces.

T: If you're right, what's it going to say with MEASURE?

W: [thinks for a few seconds] One whole. [measures "1/2"]

T: Do you know why it says one-half instead of one whole?

W: Because I cut it into halves?

T: Yeah. May be.

O: Can you fix it?

T: Yeah. Can you fix it? Can you fix it and make it equal to the whole now?

W: Add a whole 'nother one. [repeats the bar that he produced, doubling its size] Or maybe I could wipe those [WIPES clear the partitions] That's it.

T: Okay. Let's measure it and see.

W: [measures "1"]

T: Oh! Okay. Good. So, you fixed it.

W: [nods affirmatively]

If Will had been acting using his procedural scheme for reversing ratios, he might have made four more parts and joined them to the given part to make a total of five parts. Indeed, he did make four more parts by partitioning the given part in two and joining on three more parts. But he had presumed that the one-fifths bar was composed of smaller parts: "If you cut it down more, it's going to be smaller." Because the size of the cover Hillary had used was large relative to the given bar, he surmised that the given bar could not be cut into very small pieces. He figured that it might have been partitioned into two parts before Hillary wiped it because such a partition would produce the largest possible parts. All the while, Will planned to join four more copies of this part to remake the whole. So, it seems that Will had been operating with his procedural scheme for reversing ratios, only modifying it to take into account his presumption that the one-fifth bar was composed of smaller parts.

Will had modified his procedural scheme to account for his observations and interpretations of Hillary's actions, forming a new procedure. When Hillary was given a wiped two-thirds bar, she partitioned it into two parts and joined three copies to make the whole. Will's actions with one-fifth were completely analogous. Will's previously constructed numerical procedure, which had proven adequate for resolving such situations before spring break, had been unnecessarily abandoned, bringing into question whether such numerical procedures (procedural schemes) were permanent enough to call schemes. Moreover, it seems that Will's propensity for inventing new procedures was impeding his construction of a partitive fractional scheme.

Will's modification was conjectural ("I think that's it"), but it did nothing to increase his power in operating with fractions. His conjecture (**Conjecture W7**) that he should partition the given fraction into two parts before producing a total of five parts merely supplanted his old procedure. But the old procedure was not operationally flexible enough to survive.

At the end of the episode, the difference between Hillary and Will's ways of operating was yet again apparent. I posed a wiped five-sixths bar to Will and asked him to remake the whole with Hillary's help. Will measured the bar and paused for several seconds appearing to be stuck, so Hillary suggested that he partition the given bar into five parts. Will followed her suggestion and proceeded to add one more sixth to the five-sixths bar, but then continued repeating that sixth until he had produced twelve-sixths. Hillary looked perplexed by this, and Will explained, “I was going to add five more to that 5 [pointing to the numerator of 5/6 in the measure box] and make it 10, and then six more to that 6 and make it 12, so that way it would be ten-twelfths.”

Will seemed to assume that he needed to create more parts to establish the equivalent fraction that he assumed I had used to make the wiped five-sixths bar. This may explain why he did not use his procedural scheme for reversing ratios, which he had used effectively at least three times in the episode on March 31st. Hillary, on the other hand, had constructed a reversible fractional scheme that was as effective in the new situations as it had been in situations from the previous episode. So, at least in that regard, Hillary’s operative schemes were more permanent than Will’s procedural schemes.

Of course, Hillary had to continually modify her schemes as well, but the modifications still included the necessary relationships established by previous operations. For example, consider Protocol 25 (below), which illustrates a problem that Hillary struggled to solve in the middle of the episode. Will had posed a wiped fourteen-sixteenths bar to her, which she measured as “7/8.” As she began to partition the given bar, she noticed that PARTS had been dialed to “16.” So Hillary assumed, as Will had previously, that the given fraction was composed of smaller parts.

Protocol 25: *A limitation to Hillary's reversible partitive fractional scheme, and a new conjecture coordinating two schemes*.

H: [partitions the given bar into sixteen parts and begins shading by threes, stopping when she reaches four groups of three]

T: What were you doing there, Hillary?

H: I was trying to figure out something.

T: Why were you shading by threes?

H: I was seeing what it would end up to be [begins shading by two's instead, until she reaches five groups of two. She then wiped the bar and repartitioned it into fifteen parts].

O: How many pieces are in that bar?

H: Fifteen

W: [looking at Hillary who was still looking at the screen] You see, that [pointing to "7/8" in the measure box] is simplified.

T: I want to get a view for what you are thinking, Hillary. I can't tell what you are thinking, but there is a lot going on. First, you split it into sixteen parts, right?

H: [nods]

T: Why did you do that?

H: Because it was already on there [pointing with the cursor to the PARTS dial].

T: Oh! It was on there [smiling]. Then you said 'okay, I'm not going to use sixteen' and you split into fifteen instead. Why did you split into fifteen?

H: Because… when you try three it gets in four pieces, and when you try two, it gets into eight pieces.

T: So, you are trying to find something that will be in how many pieces?

H: Seven.

T: Oh! You are trying to find something that will be in seven pieces. I see.

H: [wipes the bar, repartitions it into fourteen pieces, shades by twos making seven groups of two, and joins on two more of the fourteen parts to remake the whole] That's it.

T: Explain to Will what you did.

H: I put it into fourteen pieces, because I just thought of 7, and then I doubled it.

W: If you double 7, you get 14; if you double 8, you get 16. And, I took fourteen out of sixteen pieces. So, seven-eighths is half of fourteen-sixteenths.

Hillary's assumption that Will had used sixteen parts transformed her goal from that of a reversible partitive fractional scheme, which she used successfully within this episode, to first finding something with which to make seven composite units. Once she was able to establish these composite units, she immediately knew that she needed to adjoin one more of the composite units in order to produce the whole, because the whole would be eight eighths, and she only had seven of them. So, even though she did not immediately use her reversible partitive fractional scheme, she still recognized the necessary relations established by its operations. By engendering a more complicated goal, the situation provided an opportunity for Hillary to coordinate operations of her reversible partitive fractional scheme with her commensurate fractional scheme.

What Hillary was lacking in the relationship between the two schemes was the relationship between the numerator of the measure (namely, 7) and the number of parts in the partition. Of course, because she did not know the number of units in each composite unit, she had no occasion to know the total number of parts in the partition. Instead, she could have surmised that it would have to be a multiple of 7. Eventually, she did recognize that necessity: "I just thought of 7, and then I doubled it." Having resolved this, it was certainly within her zone of potential construction to coordinate her reversible partitive fractional scheme and commensurate fractional scheme into two new schemes, namely a *reversible fractional composition scheme* and a *reversible commensurate fractional scheme*. The first scheme would enable her to produce a specified unit fraction within an unpartitioned proper fraction. The second would enable her to produce various composite fractions from a simplified one by distributing units of units within a given fraction bar.

Because Hillary's actions in establishing seven units of units within the fraction seemed to represent genuine problem solving and a novel way of operating, I refer to them as conjectural. I label the associated conjecture—Hillary's assertion that she could produce seven composite units in the given bar and use an eighth one to reproduce the whole—as **Conjecture H6**.

18 April, 2003 Teaching Episode

Will's ambiguous fraction language. In recent episodes, Will had demonstrated an ambiguous use of terms like *half* and *double*. For example, he would say that ten-twelfths had the same measure as five-sixths because five-sixths is "half" of ten-twelfths. I wanted him to confront the ambiguity in his use of such terms in describing the sizes of fractions. So, I began the present episode by asking Will which was bigger, five-sixths or ten-twelfths. Will produced five-sixths and claimed that ten-twelfths was "one time bigger."

He went on to make ten-twelfths by joining two copies of five-sixths and pulling all but two of the parts. It seems that he was experiencing a conflict: While his partitive unit fractional scheme would lead him to join the two copies in order to double a fraction, his part-whole fractional scheme led him to pull all but two of the resulting parts because ten-twelfths involves pulling ten parts and leaving two from the twelve (although he ended up with eight-sixths). His activity might indicate that, in joining the two copies of five-sixths, he thought that he had created twelve parts, never minding that the six parts referred to by five-sixths were contained in the unit bar and not in the five-sixths bar. His reasoning throughout the activity was more germane to ratios, with which the whole is not a fixed length and *ten out of twelve* is like doubling *five out of six*.

Once Will actually made five-sixths and ten-twelfths, he lined them up to determine that they were equal (Hillary had recognized the equality from the start, presumably using her commensurate fractional scheme). Still, he used ambiguous language: "Five-sixths is half of ten-twelfths and that's the reason it's probably the same—like how big it is." After pulling five of the ten parts in ten-twelfths to produce half of ten-twelfths as I had requested, Will iterated the five-part bar within the unit bar two times, saying, "that goes in there two times with two left over." He had acted in much the same way in previous episodes, in which I attributed his actions to a partitive unit fractional scheme. When Hillary explained that the fraction would measure five-twelfths, Will could make sense of this by referring to the number of parts in the fraction and the number of parts in the unit bar from which ten-twelfths had been pulled: "If you get the higher number out of all of those, you are going to get twelve." This explanation sounds similar to the kinds of rules Will had invented before in his post-hoc explanations, and I doubt that he viewed the five-twelfths bar as five of the twelve parts making up the unit bar because he had not pulled the five parts directly from the unit bar.

Another numerical procedure. Toward the end of the episode, I asked the students to reproduce a hidden whole from a wiped two-fifths of it. Hillary quickly completed the production by partitioning the bar into two parts and repeating to five parts. Will immediately claimed, “that’s your unit bar.” The immediacy of his response combined with his explanation—“When I measured, it was two-fifths, and she copied and made four, then added another one to make a whole”—surprised me, and led me to think that Will had generalized the complicated procedure while recognizing two-fifths as two of one-fifths. However, when I asked him to produce the whole from a wiped three-sevenths bar, Will partitioned the bar into two parts and repeated to make seven parts. His actions further corroborate my claim that Will did not recognize fractions such as three-sevenths as three of one-seventh, especially when the parts were not visible! He had generalized Hillary’s scheme to account for various denominators, but could not meaningfully account for variation in the numerator. As such, his actions provide further indication that Will had not constructed a partitive fractional scheme. Will did modify the procedure in consideration of the numerator once Hillary helped him to correct his production, but it still appeared to be independent of a general fraction scheme: “I should have cut it into three parts because the first number [in the fraction] was 3.” Such post-hoc explanations and procedures are characteristic of Will’s behavior throughout the teaching experiment. What they lack is the necessity or *requiredness* of operating mentioned in Chapter 2 and of which Wertheimer wrote.

30 April, 2004 Teaching Episode: A distinction between the students' zones of potential construction.

School personnel needed to use the room we had been using in the back of the library, so we met at a back table in the main room of the library. The set up was nearly identical, but there was more noise and more visual distraction with other students moving about the room. This episode was intended to test Will's meaning of fractions and to orient him to considering the sizes of fractions. In past protocols, it appears that Will had been considering only the numbers of parts as related to the numerator and denominator of the fraction and often lost sight of the unit bar and its role.

I began the episode by asking Will to make a fraction that was just a little bigger than one-half. He responded by partitioning a copy of the unit bar into two parts, pulling out one of them, partitioning that half into two parts, pulling one of those quarter parts and joining it to the half part to make three-fourths of the unit bar. However, he thought that he had made one-half "because [Will] didn't cut any more out than just half." I interpret his remarks as an explanation that he had taken half of the unit bar and then half of the resulting part (which was already a half), never using more than two parts. In other words, his meaning for half (at least for the moment) was restricted to a relation between the number of parts used and his actions of partitioning, and not related to the size of the fraction in relation to the unit bar; the fraction he made was visibly larger than half of the unit bar.

Further indication that Will still lacked a partitive fractional scheme was provided by his actions concerning Hillary's production of a bar that was even closer to one half of the unit bar. Hillary had pulled one-half of the unit bar, partitioned that half into three parts, pulled one of them, partitioned it into two parts, pulled one of them, and joined it to the original one-half part of the unit bar; an illustration of Hillary's final and intermediate productions is provided in Figure 11. The final fraction looked like the one shown in the lower left hand corner of Figure 11, except that when she joined the three-sixths part to the one-twelfth part, the partitions disappeared. Protocol 26 illustrates a conjecture that Hillary formed based on her production.

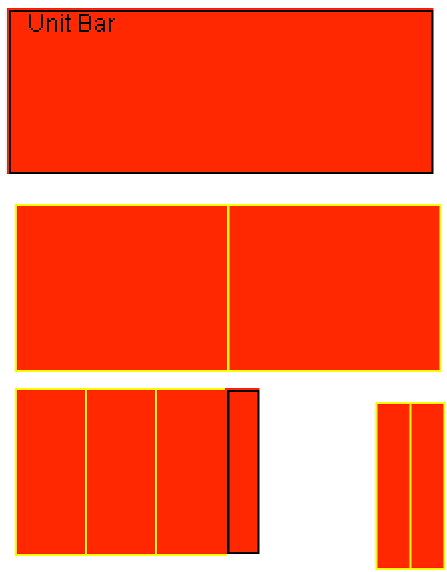

Figure 11. Hillary's production of a bar that was a little bigger than one-half.

Protocol 26: *Hillary's partitive conjecture.*

T: Did ya'll keep track of all of that to figure out what fraction she did?

H: [drags the bar in question (at the bottom of Figure 11) into the unit bar, pulls one of the two smallest parts (in the lower right of Figure 11), and begins moving it from right to left within the bar in question, as if iterating it]

T: Hillary, why don't you explain to Will what you are doing; I want to make sure that ya'll are working as a team.

W: I get why she's putting them in there [pointing to the bar in question], but why is she taking it out of there [pointing to the two smallest parts].

T: [after noticing that Hillary was having trouble keeping track of how many times the smallest part went into the bar in question] If you want, you can use REPEAT… if that helps.

H: [uses REPEAT to iterate the smallest part seven times within the bar in question, filling the latter. Then, she pauses.]

W: I think it's something with sevenths.

H: [continues iterating to fill the unit bar]

W: I think it's going to be one-sevenths.

T: One-sevenths? All right. See if you can see what Hillary is doing, too.

W: [having looked away, begins paying attention to Hillary's actions again, just as she completed filling the unit bar with small parts] That would make the whole.

T: Okay. So, she made a whole, right? Does that help her figure out the fraction?

W: I'm trying to figure out why she made the whole.

T: All right, Hillary, why did you make the whole? I want to make sure you and Will are talking.

H: [She had been focusing intently on counting the number of parts in the whole, and then comparing the bar in question to the twelve-twelfths bar that filled the whole. When she finally answered, she seemed a little put out by the distraction.]

Because I'm trying to figure out how many parts are in there [sweeping the cursor back and forth across the twelve-twelfths bar].

T: So, you made the whole with some part. Will didn't know why you were choosing that part to begin with. Why did you choose one of these [pointing to one of the twelve parts] to start with?

H: Because it's small.

T: Oh. She started with the smallest one, it looks like, Will. [to Hillary] And you made the whole by repeating. Is that going to help you, Will? Can you use what she did to figure out what fraction this [pointing to the bar in question] is out of the unit bar?

W: I could but, if I was her, I would have stopped at seven.

T: So, if you stopped at seven, that would tell you what?

W: That it would probably be one-sevenths.

T: And what did it tell you Hillary?

H: One-sevenths.

At this point, I asked the students to produce one-seventh of the unit bar to compare to the bar in question. Will produced a one-sevenths bar and realized that the bar in question could not be one-seventh because the bar in question was bigger and "because it won't take up seven spaces; it will only take up a little bit over two." He admitted that, "if you count all of the parts, there's seven of them; that's the reason I said one-seventh, but, then she kept on going." I asked both students to think about the problem and try to figure out what measure the bar in question would have. Will changed his mind to one-twelfth, after which Hillary claimed that it was seven-twelfths, and then Will immediately agreed, “because that’s seven right there [pointing to the bar in question, which was now embedded in the unit bar] and there’s twelve left over.”

I have already argued that Hillary had a partitive fractional scheme and Will did not. Will’s claims that the bar in question was one-seventh and later that it was one-twelfth corroborates the latter argument. When he finally recognized that the bar in question was, in fact, seven-twelfths (after Hillary had made the claim), he seemed to be using his part-whole fractional scheme, trying to explain that it was seven out of twelve parts. It seems that, in dealing with non-unit fractions, Will conflates the part and the whole. He knew that seven parts fit within the bar in question and thought that Hillary should have stopped when she had established this. On the other hand, he did experience conflict when he tried to reconcile his assertion that the bar was one-seventh with his partitive unit fractional scheme. Will’s partitive unit fractional scheme determined the measure of a unit fraction as the reciprocal of the number of times it would fit into the whole, but the bar in question “won’t take up seven spaces.” Experiencing such conflicts may be instrumental to engendering a partitive fractional scheme for Will.

Hillary did have a partitive fractional scheme but deferred to Will (at the end of Protocol 26) either because she had been unsuccessful in her attempts to establish the measure of the bar by iterating the one-twelfth part in it and the unit bar, or because I had interrupted her train of thought. Will’s argument sounded reasonable, and we have seen previous examples of Hillary agreeing with Will simply because she could understand his reasoning, even when it was very different than her own approach. In any case, Hillary had been using her partitive fractional scheme to solve a novel problem—one that seemed to consume her energy and resulted in uncertainty. So, I refer to her actions as conjectural. Her conjecture (**Conjecture H7**) was that she could use the smallest piece as a common partition of the unit bar and the bar in question.

Hillary's goal had been to establish the measure of a bar by using a common partition of it and the unit bar. In her partitive fractional scheme, the common partition of the part and whole is established by the measure of the fraction; the multiplicities of the common partition in producing the part and the whole are given. In the present case, the part and whole were given, but the common partition and the multiplicities were not. So Hillary had to use her partitive fractional scheme in a novel way to solve the problem: hypothetically composing part and whole of smaller pieces So, her conjectural operations employed fraction composition with her partitive fractional scheme, much as they had for Conjecture H6. However, once she arrived at her conclusion that the bar would measure seven-twelfths, she could use her part-whole fractional scheme (even as Will had) to affirm the novel use of her operations.

2 May, 2003 Teaching Episode

We established two teams, with me (the teacher) and Hillary on one team and Paul (the witness) and Will on the other, so that we could differentiate between those problems appropriate for Hillary and those appropriate for Will. Hillary and Will's primary roles were problem solving while the observer and teacher's roles were primarily problem posing.

Hillary and I began by making one-seventeenth of a unit bar, covering the unit bar and asking Will to reproduce the whole. Will measured "1/17" and repeated until he had seventeen parts, exclaiming, "there it is." His actions corroborate that he had constructed a partitive unit fractional scheme with which to recognize one-seventeenth as one of seventeen equal parts making up the whole.

Hillary's iterative fractional scheme. With guidance from Paul, Will was able to make a wiped five-thirds bar (as Paul had suggested), and he challenged Hillary to remake the whole. Hillary appeared to have little trouble in remaking the whole. She dialed PARTS to "3," quickly changed to "5" and partitioned the bar vertically. After looking at the cover (covering the original unit bar), she wiped the five-thirds bar and decided to partition it horizontally, presumably because the shape of the result of her eventual production would resemble the shape of the cover (and thus the original unit bar) better if she were to partition it horizontally instead of vertically. In fact, the shape of her eventual production did fit the shape of the original unit bar, and Hillary said that she had changed the direction of her partitioning for that reason.

Because her actions were so deliberate in establishing the whole from an improper fraction, it seemed that Hillary had constructed an iterative fractional scheme. Her subsequent actions in the present episode substantiate that the pattern for operating was relatively permanent. It will be important for me to find further instances to substantiate this claim and for me to identify how this scheme was constructed, conjecturally or otherwise. Hillary's actions at the end of this teaching episode address the former concern, affirming that she had constructed the scheme. As for the latter concern, I noted from my analysis of Protocol 14 that Hillary had not constructed an iterative fractional scheme by March 12th. Although she could purposefully produce improper fractions and had constructed a reversible partitive fractional scheme, she could not identify the whole within an unmarked improper fraction and compare the improper fraction to it. I hypothesize that the critical period of operational change can be traced back to Hillary's conjectural operations on Protocol 15 (Conjecture H4 on March 20th), during which she united unit wholes within improper fractions with the goal of converting improper fractions to mixed numbers. So, I do not view her actions as conjectural, but as the result of an engendering accommodation that can be attributed to Hillary's conjectural use of uniting operations in Conjecture H4.

Will's partitive conjecture. As the episode continued, Hillary made a wiped five-fourteenths bar and challenged Will to reproduce the hidden whole. Will's subsequent actions indicate that he had not yet constructed a reversible partitive fractional scheme. Indeed, I question whether he had constructed a partitive fractional scheme at all. But Will was operating conjecturally in a manner that might lead to the construction of both schemes. Protocol 27 picks up after Will had measured the fraction as "5/14," thought about he problem for several seconds, and began verbalizing his thoughts to his partner, Paul.

Protocol 27: *A conjecture that may engender a partitive fractional scheme.*

W: That one equals five [pointing to the five-fourteenths bar and then the "5/14" in the measure box], and if you did another one, it would probably equal 10. Well, if you add another one, it would be *fifteen*-fourteenths.

O: Right.

W: [partitions the five-fourteenths bar into five parts horizontally, pauses, wipes the partitions, and then partitions it into five parts vertically instead]

O: I think you are on the right track.

W: I need to make fourteen of those [sweeping the cursor over the five fourteenths]

O: Make fourteen of what?

W: Take one of these boxes out and repeat it.

O: Okay.

W: [pulls one of the fourteenths and places it below the right end of the five-fourteenths bar]

O: So, what's the fraction there?

W: One-sixth? [repeats the pulled fourteenth part nine times so that he has a five-fourteenths bar and a nine-fourteenths bar lined up]

O: So how many total do you have?

W: That's fourteen right there.

Will's actions in Protocol 27 were novel. Although he previously had been able to use his procedural scheme for reversing ratios in order to reproduce the whole from a partitioned, non-unit, proper fraction, he had never been able to recreate the partitions of an unpartitioned (wiped), non-unit fraction. I can think of two explanations for the novelty of his actions: Either he had constructed a new procedure from observing Hillary's actions in previous episodes, or his experiences over the past couple of episodes engendered partitive fractions. I am inclined to agree with the former, after considering the following observation made by Will at the end of the April 18^{th} teaching episode: "I should have cut it into three parts because the first number [in the fraction] was 3." However, the past couple of episodes had been designed to focus Will's attention on the relative sizes of non-unit fractions and so to elicit partitive fractions; the episodes may have been met with success. I hypothesize that Will had not yet constructed a partitive fractional scheme with which to act consistently in treating fraction bars as partitive fractions.

Will's actions of partitioning the unpartitioned fraction conjecturally transformed the unfamiliar situation into one in which he could use his procedural scheme for reversing ratios in order to reproduce the whole. His conjecture, **Conjecture W6**, was that, since the given bar measured "5/14," it must contain five parts from a fourteen part whole. As was the case in past uses of the procedural scheme, Will embedded the given fraction within the whole by adding to the given bar to make the whole. He knew that he needed to add nine parts, but he had trouble keeping track of what the parts were. Will referred to the part he was repeating as one-sixth (at an intermediate stage when he had six parts), which indicates that he had not understood the given five-fourteenths fraction as a partitive fraction made up of five fourteenths.

As the episode continued, Will seemed unhappy with the non-rectangular shape he had created, so he continued by using cuts. Even then, he was only able to get two five-fourteenths bars lined up with a four-fourteenths bar in the middle, as displayed in Figure 12. So, he filled in the gap by copying one of the fourteenths and joining it to the four-fourteenths bar, and claimed, "That's your whole." So, it seems that Will's goal of completing the rectangle superceded his intention to create fourteen parts, further indicating that his actions in producing fourteen parts from the given bar were conjectural and not operationally necessary. He was surprised to find that he had created fifteen parts when he counted them later; the test for his conjecture had not yielded complete success.

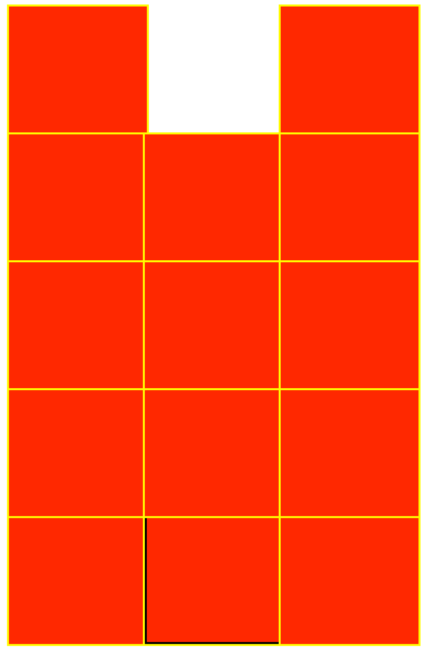

Figure 12. Will's production of fourteen-fourteenths from five-fourteenths.

Will's actions in solving a similar problem at the end of the episode provide further indication that Will had constructed a new way of operating that may not have been based on partitive fractions. When Hillary challenged him to reproduce a hidden whole from a wiped three twenty-eighths bar, he initially partitioned the bar into *two* parts. But after wiping it, he repeated it three times, uttering "three, six, nine." He continued repeating "to see how close I can get the numbers to 28," stopping after he had reached twenty-seven. It is unclear whether he considered his production of "twenty-seven" as twenty-seven twenty-eighths, and we ran out of time before he could complete his production. His actions in solving this final problem of the episode seem to be related to a partitive unit fractional scheme for composite units, except that he counted by threes with the goal of reproducing the whole from a wiped fraction, rather than counting by ones with the goal of naming a given partitioned fraction.

Hillary's actions late in the episode provided further indication that she too continued operating in the manner that had been novel earlier in the episode. Will had challenged her to reproduce the whole from a seven-fifths bar, and she was able to immediately satisfy the goal by partitioning the seven-fifths bar into seven parts and pulling five of them. Before she could pull out the five parts from the seven parts, I asked her to tell me what one of the seven parts would measure, and she replied, "one-fifth." This is a strong indication that she had been using an iterative fractional scheme.

7 May, 2003 Teaching Episode: Results of Conjecture W6

Will's actions in the middle of this episode provide for strong implications about his partitive fractional reasoning. In my analysis of Protocol 27 from the last episode, I hypothesized that Will had not yet constructed a partitive fractional scheme. It seemed that he did recognize fractions such as five-fourteenths as five of something, but that the *something* was not *necessarily* one-fourteenth of the unit bar. His goal in reversible tasks such as reproducing the whole from five-fourteenths of it had been restricted to creating fourteen of the something. Now, there is indication that Will could view fractions such as two-fifths as partitive fractions if this goal were isolated from larger goals, such as reproducing the whole from two-fifths. Protocol 28 begins with Will's response to a task requiring him to make one-fifth from a wiped two-fifths bar. The unit bar was hidden, and Will had just measured the given bar to be "2/5."

Protocol 28: *Will's construction of two-fifths as a partitive fraction.*

T: Can you use that [the wiped two-fifths] to make one-fifth?

W: [dials PARTS to 2, partitions the given bar, pulls one out, releases the mouse and turns to look at me]

T: You sure?

W: [nods]

T: How do you know?

W: Well, it's two fifths and if you cut it in half, it will take one off.

T: Oh, okay. Good.

Protocol 28 continued after Will had followed my instructions to reset the screen to the way it had been before the protocol began. I then asked him to make the whole from the wiped two-fifths bar. Will repeated the bar once, partitioned the result horizontally, pulled out two of the parts and lined them up on the right as illustrated in Figure 13. When the witness asked him how much he had, Will thought for a moment and replied, "six-fifths." He completed his production by removing the rightmost sixth, eventually lining up the five parts side-by-side.

Something had changed in Will's ways of operating. Whereas in the last episode five-fourteenths only meant five of something (apparently unrelated to the size of the five-fourteenths fraction bar), now Will was able to identify that two-fifths was two of one-fifth, and he could identify the one-fifth within two-fifths by removing the other half of two-fifths. Moreover, he did not lose track of the fact that he was operating with fifths when he went on to produce the whole from two-fifths. He began much as he had with three twenty-eighths in the last episode, repeating the two-fifths fraction to get as close to five as he could. He also seemed to insist that the original two-fifths bar be embedded in unit bar that he was producing, as he had with four-fourteenths and three twenty-eighths. However, with the knowledge that he was operating with fifths, he was now counting by fractions rather than whole numbers, as indicated by his claim that he had created *six-fifths*! He also accepted the restriction that he could not create additional fifths beyond five-fifths in order to complete a rectangular shape (unlike his actions with five-fourteenths in the last episode).

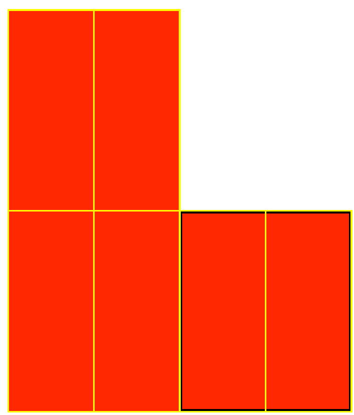

Figure 13. Will's production of six-fifths in attempting to produce the whole.

Rather than considering Will's actions as representative of a new conjecture, I consider them as another test of Conjecture W6—a conjecture that may have engendered partitive fractions. If he had indeed begun to construct partitive fractions and a partitive fractional scheme, he could also transform his procedural scheme for reversing ratios into a genuine reversible partitive fractional scheme.

9 May, 2003 & 15 May, 2003 Teaching Episodes

The final two episodes were analyzed to determine the students' available operations at the end of the teaching experiment. The episodes provide indication that Hillary had constructed at least a kernel for a reversible unit fractional composition scheme (like the scheme Joe used in Protocol I), which included recursive partitioning operations. The episodes provide indication that Will may have splitting operations available, but there is contra-indication that he had constructed a partitive fractional scheme.

In the episode on May 9th, Hillary was asked to make and determine the measure of one-third of one-fourth. She produced the desired fraction by partitioning a copy of the unit bar into four parts vertically, pulling one of these, and partitioning it into three parts horizontally. After pulling one of the small parts, she determined that it was one-twelfth of the unit bar by iterating the three-part bar in the unit bar four times. In this way, she was able to recursively partition the unit bar into twelve parts and compose the two partitions to determine the size of the fraction in question. Later in the episode she appeared incapable of using recursive partitioning in such a way when given non-unit fractions (e.g. two-thirds of two-sevenths). Instead, she acted much as she had during Protocol 26 on April 30th. Hillary's actions in these in the cases described here indicate that her fractional composition scheme was restricted to unit fractions of unit fractions. However, with such recursive partitions, she could also operate reversibly, as indicated during the episode on May 15th. During that episode, with the goal of creating a part such that the unit bar would be twelve times as big, Hillary partitioned the unit bar into three parts vertically and four parts vertically and pulled one part, confident that it was one-twelfth of the unit bar. Furthermore, when I asked her if she could make one-twelfth from one-sixth, she partitioned a one-sixth piece into two parts and pulled one.

In the same two episodes, I had also given Will tasks in which he was to make a part such that the unit bar was *n* times as big. His responses were varied and sometimes ambiguous, but he demonstrated in each case that he could posit a hypothetical piece to check through iteration. In the first such task, I asked Will to find a bar such that the unit bar was twice as big as it. He responded by taking half of the unit bar and then taking half of that half. However, he was disappointed to find that the unit bar was four times bigger. He acted analogously when I asked him to find a bar such that the unit bar was five times as big. Finally, during the final episode, he was able to produce a bar so that the unit bar was twelve times as big, by partitioning the unit bar into twelve parts. These cases indicate that splitting was at least in Will's zone of potential construction.

A partitive fractional scheme was also in Will's zone of potential construction. However, in each of the last two episodes, there is contra-indication that he had constructed the scheme. On May 9th, Will claimed that five-fifths was five times as big as two-fifths. With both bars already made, he iterated the latter into the former counting, "two, four, and another one makes five." He had only iterated the two-fifths bar twice within the five-fifths bar, counting by two's. During the last episode, I asked Will to produce one-sixth of a unit bar and then to produce a bar that was five times as big. He had no problem producing the one-sixth bar, by partitioning the unit bar into six parts and pulling one. He easily produced the bar that was five times as big as one-sixth by repeating it four more times. Still, he named the result as a whole rather than five-sixths.

Chapter 6: Synthesis of Hillary and Will's Conjectures

Whence and wherefore conjecture? In taking an operational approach to answering this question, I needed to examine students' conjectural activity and consider how conjectural operations might engender accommodations of schemes. In the case study of Hillary and Will in Chapter 5, I identified conjectural operations and the conjectures that those operations formed. So, in the present chapter, I summarize and synthesize the conjectures that I identified for Hillary and Will. I use the analysis presented in Chapter 5 to specify those accommodations that the students made in their schemes as a result of their conjectural operations, and demonstrate how the students' zones of potential construction were actualized through conjectural activity.

Will

Conjecture W1. Will had constructed a partitive unit fractional scheme before February 24th. During the teaching episode on that date, he was challenged to produce an eight-sevenths bar, which is beyond the scope of the partitive unit fractional scheme. So, he resorted to using his whole number knowledge in a novel way in acting on a previously produced seven-sevenths bar to produce an eight-sevenths bar (Protocol 1). He treated each seventh as a unit and suggested adding one more of those units to make eight in all. I have argued that, in acting this way, he was not treating the units as fractional units (sevenths). Instead, he used the *one more than* operation that he had available for whole numbers in order to produce a relation between the seven units that he started with and the eight units that would result from his action. So, his conjectural operation of *adding one more* in the fractional situation was based on a generalizing assimilation using his number sequence.

By conjecturally assimilating the situation as he did, Will was able to establish (what an observer might refer to as) an improper fraction as a ratio of whole numbers, which constituted his conjecture. But with ratios, the whole is not a fixed quantity and, indeed, Will lost the whole in action. Through the rest of the teaching experiment, he used this sort of ratio reasoning to produce fractions in various situations in which he found his partitive unit fractional scheme insufficient. For example, in attempting to produce two-thirds from six parts (Protocol 10), he pulled three parts from a six-sixths bar and then pulled two of those three parts. In doing so, he produced a two-to-three ratio, and apparently was unaware that the two-to-three ratio was not two-thirds of the six-sixths bar. Also, beginning in Protocol 18, Will insistently referred to five-sixths as half of ten-twelfths. When interpreted as ratio reasoning, his actions were rational (e.g. five hits out of six at-bats is half as many hits and at-bats as ten out of twelve).

Because Will used his ratio reasoning so flexibly, he seemed to experience no need to change his pattern of reasoning through most of the teaching experiment. In fact, there is contra-indication of a general partitive fractional scheme even at the end of teaching experiment, although Conjecture W7 may have engendered the scheme. And, during the teaching experiment, there is no indication that he had begun constructing an iterative fractional scheme with which to meaningfully produce improper fractions, in spite of tasks that I posed involving fractions like eight-sevenths and my attempts to induce conflict between his and Hillary's results of operating.

Conjecture W2. Will lacked partitive fractional and commensurate fractional schemes. So, when he was challenged to compare the measures of a two-thirds bar and a six-ninths bar (during Protocol 4), he drew a picture (illustrated in Figure 6) and compared the extensions of the fractional parts drawn within each whole bar. He did not seem to attend to the relative sizes of fractional wholes until I questioned him on it, at which point he reasoned that the nine-part whole should be longer than the three-part whole because it had more parts. As in Conjecture W1, his reasoning was compatible with ratio reasoning rather than fractional reasoning. He had produced the two fractions as part-whole ratios in the connected numbers, three and nine, and he conjecturally used his whole number comparison operation, applying it to the connected numbers.

Will had conjectured that he could compare the two fractions as ratios of parts in connected numbers, and his use of his whole number comparison operation was a generalizing assimilation. The comparison was unsuccessful in establishing the commensurability of the two fractions, and, at the end of the teaching experiment, Will still had not constructed a commensurate fractional scheme. So, this may be a second example of Will's ratio reasoning circumventing a need to change his pattern of reasoning.

Conjectures W3a and W3b. In Protocols 8 and 9 (March 10^{th}), Will formed two conjectures in identifying simpler fractional names for given proper fractions. Conjecture W3a was based on a generalizing assimilation involving his partitive unit fractional scheme, and Conjecture W3b occurred as a result of the assimilation. In his generalizing assimilation, Will was aware that he was acting in a new situation as he would in situations involving unit fractions. He conjectured that he could determine the fractional size of a six-eighteenths bar by iterating it within the unit bar just as he would with a partitive unit fraction. Polya might refer to Will's generalizing assimilation as an analogy: Will ignored some of the contextual details of the situation, such as the partitions of the six-eighteenths bar, in order to operate as he would in another situation.

Conjecture W3a was successfully tested and modified the trigger of his partitive unit fractional scheme so that he would begin using it to operate on composite fractions, iterating them within the whole to determine their fractional measure as he had with unit fractions. Conjecture W3b was formed in response to an unexpected constraint introduced by Will's use of his partitive unit fractional scheme on a composite fraction that did not exhaust the whole. Asked to predict what a five-elevenths bar would measure (Protocol 9), Will immediately began to iterate the bar within the unit bar (corroborating the generalizing assimilation of Conjecture W3a), but there was one part left over. So, Will iterated again, this time attending to the units of units created by his iterations, and naming the bar ten-elevenths.

Will demonstrated in subsequent episodes that he could use his partitive unit fractional scheme to act in a manner consistent with the functioning of a partitive unit fractional scheme for composite units. But having constructed such a scheme would imply that Will could coordinate a unit of units of units resulting from his iterations of a composite unit. Whereas he created an experiential unit of units of units through Conjecture W3b (treating ten-elevenths as two of five-elevenths), Protocol 11 includes contra-indication that he could coordinate units at three levels. So, Conjectures W3a and W3b resulted in a generalization of Will's partitive unit fractional scheme in that he contextually used a composite unit as iterable. For this reason, the partitive unit fractional scheme for composite units seemed to be in his zone of potential construction. Will's conjectures also raised his awareness of the operations of his partitive unit fractional scheme in that he began using fractional language to refer to the size of a fraction relative to the unit bar and explicitly justified the size of fractions like one-third, "because it goes into the unit bar three times."

Conjectures W4a and W4b. I have mentioned that Will treated non-unit fractions as ratios. Acting on Conjecture W1, he was even able to use an eight-to-seven ratio of parts in order to conjecturally establish eight-sevenths. Whereas in Conjecture W1 Will had been working from a seven-sevenths bar, in acting on Conjecture W4a Will began with an unpartitioned unit bar with the goal of producing seven-fourths. He conjectured that, by partitioning copies of it into four parts, pulling a total of seven parts among them, and joining the parts, he could produce seven-fourths. His meaning for seven-fourths was based on an assimilation of Hillary's previous actions using a procedure that was based on ratio reasoning. As such, his assimilation was a conjectural operation that did not result in fractional meaning (he did not immediately know that each part was one-fourth of the unit bar), much less an improper fraction.

Ordinarily, Will could use ratios of parts within the unit bar to produce fraction bars, which he could then give a part-whole fractional meaning based on the whole he created (which sometimes differed from the unit bar). Conjecture W4a resulted in an ability to produce fraction bars beyond the unit bar, but he could not give them meaning using his part-whole fractional scheme. Will had simply assimilated Hillary's productions of fractions like seven-fourths using his partitioning operation and ratio reasoning, learning to reproduce such fractions by partitioning the unit bar into a number of parts specified by one number in the fraction and joining together a number of parts specified by the other number in the fractions.

Conjecture W4b (during Protocol 16) illustrated that even Will's *production* of bars such as seven-fourths and sixteen-fifths was still ambiguous. He was not acting out of operational necessity, but with procedures constructed from his assimilations in the social context of observing and interacting with Hillary. In forming Conjecture W4b, for example, he had assimilated Hillary's actions of working with improper fractions using his partitioning operations and ratio reasoning again, but in a different way. His conjecture was that he could produce sixteen-fifths by adjoining five more parts to a sixteen-part bar and resulted in a new procedure. This time, he used the number of parts in the *numerator* of the fraction to determine the number of parts to use in his partition, and he added on five more parts to it because he could not add on to the number of parts in the partition (sixteen) to produce the other number in the fraction (five) as he had in the procedure from Conjecture W4a.

<u>Conjecture W5</u>. During Protocol 20, Will had used ratio reasoning and his whole number multiplication operation in a novel way to reproduce the unit bar from a six-part bar making up three-eighths of the unit bar. His reasoning involved a recognition that 3 is to 6 as 8 is to 16. Given the procedures that Will was prone to construct from his interactions with Hillary, it is likely that he had learned to posit a unit bar with an indeterminate number of parts by reflecting on his records of experience in observing Hillary's reconstruction of the unit bar from nine-twelfths of it. His conjecture consisted of independently positing such a unit bar in the new situation and constructing a numerical relationship between the number of parts given and the numerator and

denominator of their measure. In fact, he initially confused the proportional relationship by producing a twelve-part bar (6 times 2, rather than 8 times 2). Will's conjectural operations included a novel use of his disembedding operation in positing a unit bar with an indeterminate number of parts, in addition to his subsequent use of ratio reasoning and multiplication in determining that number of parts.

In formulating his conjecture, Will had to sort out the roles of the numerator and denominator of the fractional measure in setting up a proportion of ratios: 3 is to the given number of parts (6) as 8 is to the number of parts in the whole. Due to the successful test of his conjecture in reproducing the unit bar using sixteen parts, Will could use his disembedding operation in subsequent situations to confidently posit a unit bar in relation to a given ratio. Although he still did not consider a given ratio as a size relative to the unit bar, as he would with a partitive fractional scheme, he had established a meaningful numerical relation between the bars. This made it possible for Will to construct his procedural scheme for reversing ratios (Protocol 21) with which to interpret fractional measures. His ratio reasoning made it possible for him to create equivalent ratios (in the middle of the March 31st teaching episode) and to reproduce the whole from a seven-part bar that measured one-third (Protocol 22, March 31st).

Although his procedural schemes enabled Will to act as one would with reversible partitive fractional and commensurate fractional schemes, we will see in Conjecture W6 that his structures were not as permanent or operationally flexible as those of Hillary.

Conjecture W6. As indicated by his actions in Protocol 23, Will's procedural scheme for reversing ratios was limited to working with partitioned fraction bars. So, he could not produce the unit bar from an unpartitioned three-fourths bar. Hillary's reversible partitive fractional scheme, however, was operationally flexible and could be used in that situation and a similar one in the next episode (April 14th) involving an unpartitioned two-thirds bar. She partitioned the fraction into two parts and joined on one more part to produce the unit bar. In the social context of observing Hillary's actions, Will constructed a new procedure by conjecturally using his partitioning and iterating operations. When Will was asked to produce the unit bar from a one-fifth bar (Protocol 24), he acted analogously to Hillary's actions, by partitioning the given bar into two parts and joining on parts until he had five of them in all.

Will's actions served as a test of his conjecture that the procedure would produce the whole just as Hillary's actions had in case of two-thirds. Will's test failed and left him no more powerful than he had been before. He had abandoned one procedural scheme and constructed a new procedure in its place, but neither procedure contained operational necessities from which he could build. Although he could recall his procedural schemes from one episode to another, they were not as permanent as Hillary's fractional schemes because they lacked flexibility, which necessitated Wills abandonment of them when he experienced new challenges.

Conjecture W7. Although Conjecture W6 did nothing to increase Will's operational power, it's failure, as well as subsequent failures of similar conjectural procedures in the April 14th teaching episode, did seem to re-establish the relevance of his procedural scheme for reversing ratios. He began using that scheme again after April 14th and, in particular, during Protocol 27. Furthermore, Will's test of Conjecture W6 seemed to heighten his awareness about the role of the numerator in observing Hillary's actions of reproducing the unit bar from a fourteen-part bar that measured seven-eighths

(Protocol 25): “If you double 7 you get 14.” His attention to the role of the numerator and his confidence in the usefulness of his numerical scheme for reversing ratios were indicted in Conjecture W7, during the May 2nd teaching episode.

When asked to reproduce the unit bar from an unpartitioned bar that measured five-fourteenths of the unit bar, Will conjecturally applied partitioning operations to the given bar. He had conjectured that since the given bar measured “5/14,” it must contain five parts from a fourteen part whole. This conjectural operation produced a five-part bar from which he could produce the unit bar using his procedural scheme for reversing ratios (adding on nine more parts to make fourteen). But his reference to the parts was ambiguous; the parts were not necessarily each one-fourteenth of the unit bar.

Will later demonstrated that he could treat an unpartitioned three twenty-eighths bar as a unit of three. And, during Protocol 27 in the next episode, he was able to produce one-fifth from an unpartitioned two-fifths bar. He was even able to keep track of the unit measure when he produced and named six-fifths. So, sometime between the two episodes (May 2nd & May 7th) Will had resolved the ambiguity in naming the partitive fractions that he had begun to produce in Conjecture W7. I suggest that his conjecture and subsequent productions in testing it engendered partitive fractions. Although there was still contra-indication to his production of a partitive fractional scheme at the end of the teaching experiment, he was in the process of constructing that scheme and a reversible partitive fractional scheme.

Hillary

Conjecture H1. In forming conjecture H1 during the February 26th teaching episode, a six-ninths bar was visible on the screen but a two-thirds bar and the unit bar were not. The students were surprised to find that the six-ninths bar measured two-thirds, and Will's question—"Is two-thirds the same as six-ninths?"—oriented Hillary to estimate the size of the two-thirds bar using her partitive fractional scheme. In mentally positing a two-thirds bar, she conjecturally used her disembedding operation to explicitly refer to its complement (one-third) in order to maintain the unit bar from which six-ninths was produced. She could thus compare the two fractions and conjecture that two-thirds of the unit bar was the same size as six-ninths of the unit bar. Her conjecture was a sort of abduction whereby she explained the surprising measure of six-ninths by establishing its size as that of a two-thirds bar; because two-thirds and six-ninths of the unit bar were the same size, their common measure was a matter of course.

Hillary's partitive fractional operations and her conjectural use of disembedding operations formed the operations of a new scheme—a complementary fraction comparison scheme—that was eventually reconstructed as a commensurate fractional scheme. The new scheme differed from a commensurate fractional scheme in that it relied entirely on size estimations of fractions and their complements and did not yet include the coordination of units of units. However, it was used successfully in Protocols 5 and 6 to simplify given fractions. Limitations to the scheme can be found in Protocol 7 in which Hillary estimated eleven-twelfths as five-sixths.

Conjecture H1 may have also contributed to Hillary's construction of a reversible partitive fractional scheme. She was able to use her partitive fractional scheme reversibly (reproducing the unit bar from an unpartitioned proper fraction bar) because she could maintain the unit bar by explicitly referring to the complement of the fraction bar. Considering her initial interview (Task 5), it is possible that she had constructed the reversible scheme before the teaching experiment began, but Protocol 13 (March 12th) at least marks the first indication that she could use the scheme with `TIMA:Bars`.

Conjecture H2a. Even before Protocol 11, I noted that Hillary had begun attending to the numbers of parts in fractions, perhaps due to her interactions with Will who predominantly considered number over fractional size in working with fraction bars. In considering number and working from a partitioned unit bar (rather than a partitioned fraction bar), Hillary's complementary fractional comparison scheme was not called to produce two-thirds from twelve parts. Instead, she conjecturally used her uniting operation, iteration, and units coordination in conjecturing (Conjecture H2a) that she could produce thirds from the twelve parts by constructing three units of four parts in the twelve-twelfths bar.

Because using units coordination to produce a non-unit fraction was novel for Hillary, she had to act out the iterations of a four-part bar in the twelve-part bar and conflated the goal of producing two-thirds with one of producing a bar that would fit into the unit bar three times. She was able to correct herself by using two units of four parts each, and her actions in testing and correcting her conjecture may have engendered novel uses of units coordination. In particular, her actions indicated that a partitive fractional scheme for composite units and a commensurate fractional scheme were within her zone of potential construction. The latter scheme seemed to be actualized by Conjecture H2b.

Conjecture H2b. This conjecture was based on the same operations used in Conjecture H2a except that she coordinated the units of units in both the fraction and its complement, indicating that she was reconciling units coordination with her complementary fractional comparison scheme. She made the units coordination explicit in Protocol 12, uniting three units of two in the unit bar and using two of these units in the fraction bar in order to identify it as two-thirds. In explicitly referencing the units of units in the fraction bars, Hillary tested her conjecture that she could form the units of units in order to produce two-thirds, and I hypothesize that she constructed commensurate fractional operations as a functional accommodation of her complementary fractional composition scheme. Her actions in Protocols 15, 17, and 18 indicate that she had constructed a relatively permanent way of operating—a commensurate fractional scheme.

Conjectures H3a and H3b. In attempting to produce the unit bar from an unpartitioned five-fourths bar (Protocol 14), Hillary understood the necessity of the five-fourths bar having five parts and the necessity of the unit bar having one part fewer than the five-fourths bar (perhaps using the iteration operation of her partitive fractional scheme). But Hillary was unable to identify the unit bar within the five-fourths bar as she might with an iterative fractional scheme. So, she made a generalizing assimilation using her partitive fractional scheme to conjecturally (Conjecture H3a) posit a copy of the five-fourths bar, partitioned into four parts, as the desired whole. Then she partitioned the original five-fourths bar into five parts. In using her partitive fractional scheme, Hillary had to neglect the necessity of creating equally sized parts in the two bars. She recognized this necessity only after completing her production and examining it. Because her generalizing assimilation failed, the trigger of her partitive fractional scheme was not modified, as indicated by her next attempt. But the test of her generalizing assimilation did seem to raise Hillary's awareness of the criterion for producing equal parts among the two bars.

In Conjecture H3b, Hillary again posited a whole to use as a reference for the five-fourths bar. This time, she conjectured that she could use the original five-fourths bar as the unit bar and produce five-fourths of *it*, conflating the two bars. This involved another generalizing assimilation using her partitive fractional scheme. While Hillary acted with the operational necessity of producing equally sized parts in the unit bar and the five-fourths bar, her conflation of the bars was a necessary product of the partitive fractional scheme.

In each case, Hillary needed to produce an experiential whole because she had no record of the whole within the improper fraction. Her actions using these posited wholes indicate that she could iterate unit fractions beyond the whole to produce improper fractions, but she had not been comparing the improper fractions that she produced back to the whole from which she produced them. In my analysis of Protocol 14, I hypothesized that Conjectures H3a and H3b might engender Hillary's construction of an iterative fractional scheme and a reversible iterative fractional scheme. Her raised awareness of the embedded whole, due to her conjectural activity, could focus her attention on comparing improper fraction bars back to the unit bar after producing them, thus embedding the unit bar in the improper fraction bars. In fact, Hillary's actions in producing four-thirds at the end of the March 12th teaching episode and her actions in inventing Conjecture H4 (Protocol 15, March 20th) do indicate increased attention to the unit bar embedded in her productions of improper fractions.

Conjecture H4. After calculating that ten-thirds was the same as three and one-third, Hillary pulled three-thirds and one-third from a ten-thirds bar, casually claiming that she had produced three and one-third (Protocol 15, March 20th). This indicated an ambiguous concept for the 3 in *three and one-third*. When I prompted Hillary to compare her production with that of ten-thirds, she experienced a perturbation that provoked her to consider the relative sizes of the unit bar and the partitioned ten-thirds bar (both of which were visible on the screen, as illustrated in Figure 10). She then conjectured that the 3 referred to the three composites of three parts (each equal in size to the unit bar) that she could identify in the ten-thirds bar. This was a peculiar aspect of the situation, in part, because there were *exactly three* of these of these composite whole in the ten-thirds bar. In order to coordinate the peculiar composite wholes, her conjecture relied on her commensurate fractional scheme, conjecturally using units coordination to establish three such wholes within the improper fraction. She had used this scheme on improper fractions before in order to simplify them, but she had not used it to convert them to mixed numbers.

Hillary's conjecture served as a functional accommodation of her commensurate fractional scheme because she began using the operations (units coordination) of the scheme differently in order to satisfy the new goal of converting improper fractions to mixed numbers. The relative permanence of her modification is indicated by her actions in converting sixteen-fifths to a mixed number later in the teaching episode. There is also indication in the episode that Hillary had begun comparing back to the whole the improper fractions that she produced—the critical aspect of an iterative fractional scheme that she lacked before. By the end of the May 2nd teaching episode there was strong indication that Hillary had indeed constructed that scheme.

I had hypothesized (in my analysis of the May 2nd teaching episode) that the critical period of operational change in Hillary's construction of an iterative fractional scheme could be traced back to Conjecture H4 on March 20th. However, the only operation that she lacked in constructing the scheme during Protocol 14 on March 12th was that of comparing improper fractions back to the whole. During Protocol 14, Hillary formed Conjectures H3a and H3b, which seemed to raise her awareness of the need to make such comparisons. While Conjecture H4 certainly involved a novel use of Hillary's commensurate fractional scheme, the missing aspect of an iterative fractional scheme was engendered by Conjectures H3a and H3b. Indeed, Hillary did construct an iterative fractional scheme by the end of the May 2nd teaching episode.

Conjecture H5. I have noted that many of Will's conjectures were socially based, assimilating Hillary's actions using his whole number operations in order to construct procedures for operating with fractions. Some of Hillary's conjectures were socially based too, except that she had the fractional operations available to evaluate and refine her conjectures. For example, Conjecture H5 was Hillary's attempt to explain Will's assertion that a seven-twelfths bar would measure four-sixths, by conjecturally uniting two composite units within the fraction and two more within its complement, thus creating four units. But when she had produced the units, she refined her conjecture, claiming that she the fraction would be two-fourths.

Hillary had made a generalizing assimilation using her commensurate fractional scheme, using it to work with cut parts and unequal units. Because she had been working within the framework of assimilating Will's assertion, she did not experience a problem with her use of the scheme until she was free of the activity and reflected back on it. At that point, she noted that "you can't split seven in half; it's an odd number," introducing a new constraint to her scheme and indicating insight as to why her generalizing assimilation had failed. The new constraint served as a functional accommodation of her commensurate fractional scheme that became evident during Protocols 18 and 19: Hillary began using divisibility in the numbers of parts in the fraction and its complement as a criterion for producing units of units. This modification also contributed to Hillary's construction of a reversible partitive fractional scheme for composite units (Protocol 19).

Conjecture H6. This conjecture provides a stark contrast to Conjecture W6 because Hillary's conjecture demonstrated operational flexibility. Like Will, when Hillary measured a given unpartitioned proper fraction bar (seven-eighths) and was asked to produce the unit bar, she assumed that the fraction was composed of smaller parts. This was due to the fact that she had observed "16" in PARTS. Unlike Will, Hillary did not abandon her existing scheme for producing the unit bar from a proper fraction bar—her reversible partitive fractional scheme—but used it in the new situation of working from a simplified fraction.

Hillary used her reversible partitive fractional scheme in coordination with her commensurate fractional scheme to conjecturally compose seven units of two in the fraction and eight units of two in the unit bar that she eventually produced. Her conjecture was that she could coordinate seven units of smaller units in the given fraction bar, all the while knowing that she would need to join one more unit of units to produce the unit bar. She used trial and error to determine that she should form units of two.

Hillary had used her commensurate fractional scheme to posit seven units of two within the unpartitioned fraction bar so that she could use her reversible partitive fractional scheme to produce the unit bar from the seven units. In doing so, she recognized the numerical relationship between the numerator of the fraction, 7, and the number of parts she created, 14: "I put it into fourteen pieces, because I just thought of 7, and then I doubled it." She could now operate as she would with a reversible commensurate fractional scheme, which produces composite fractions from a simplified one. She had also constructed the operations of a reversible fractional composition scheme, which produces specified unit fractions within a partitioned proper fraction by partitioning each unit in the partitioned fraction into a determined number of parts. Both schemes were now in her zone of potential construction.

Conjecture H7. During the April 30th teaching episode, given the unit bar and the fraction bar illustrated in the bottom left corner of Figure 11, Hillary attempted to name the size of the fraction bar without using MEASURE. Using her partitive fractional scheme, Hillary could transform the goal of the situation to one of finding a common partition (co-partition) of the fraction bar and the unit bar. In Conjecture H6, she had posited parts within a fraction based on its measure, but in the absence of the new fractions' measure, Hillary relied on the conjectural use of an experiential part (the one outlined at the bottom of Figure 11). She conjectured that she could use the part as a co-partition of the fraction bar and the unit bar in order to determine the fractional size of the fraction bar. This involved a conjectural use of her partitive fractional scheme and iteration operation, iterating the potential co-partition into both the fraction bar and the unit bar.

Hillary's conjecture might have engendered a functional accommodation of her partitive fractional scheme, but, because the conjecture occurred so late in the teaching experiment, there is no indication of its subsequent use.

Chapter 7: Josh and Sierra

Synopsis

Several of Josh and Sierra's conjectures are highlighted within this chapter and synthesized in Chapter 8. The main purpose of this chapter is to build models of the students' ways of operating, which changed significantly as a result of conjecturing activity. I provide examples from my data to indicate that Josh and Sierra could operate in the `TIMA` environments using particular schemes and operations. In the first few teaching episodes, Josh's actions indicated that he had constructed a part-whole fractional scheme, but not a partitive unit fractional scheme. This level of development placed him in a lower-stage pair, but he did seem to have a splitting operation available and used it constructively in conjecturing. His conjecturing activity engendered several schemes, including a partitive unit fractional scheme, a partitive unit fractional scheme for composite units, a partitive fractional scheme, a partitive fractional scheme for composite units, and a commensurate fractional scheme.

Sierra's actions in the first few teaching episodes indicated that she had constructed a part-whole partitioning scheme, but not a part-whole fractional scheme. During the teaching experiment, her conjecturing activity engendered a part-whole fractional scheme and some fractional comparison schemes. I also note her propensity for forming abductions throughout the teaching experiment.

Introduction

Josh and Sierra began working together on March 18th, after working with other partners of lower-stage pairs for the first four or five teaching episodes. Because those first few episodes were devoted to play while the students developed familiarity with the computer environments, they do not provide much meaningful data on conjecturing. In fact, after two episodes of observing continued confusion on the part of the students in working with the two-dimensional bars in `TIMA:Bars`, I decided to start using `TIMA:Sticks` with each of the lower-stage pairs. This change required the students to make additional adjustments in learning to use the computer tools. Still, the data from the initial pairs is used in determining the schemes that the students had available for use in those environments. So, I begin the present chapter by analyzing the data from the initial pairs and focus on Sierra and Josh within the initial pairs. This chapter will then proceed with analysis of the lower-stage pair that Sierra and Josh formed.

<u>Sierra and Cory</u>

Sierra and Cory were classmates in Mrs. Wood's first period math class, but they did not interact much in class. Both students appeared quiet and shy in class, but they had responded energetically in their initial interviews. The students' responses during their initial interviews indicated that each of them had constructed equi-partitioning and part-whole partitioning schemes, but neither of them seemed to have constructed a partitive unit fractional scheme. Analysis of the five episodes in which these students worked together is devoted to characterizing Sierra's schemes.

I introduced the students to the teaching experiment on February 21st, during their first period math class. We met in the same storage/work room in which I had worked with Hillary and Will during the first several teaching episodes. The set up was also the same: The computer sat on a long table below a chalkboard, and two cameras were set up to capture our physical actions and the actions represented on the computer screen. The students began playing a game in `TIMA:Bars`, challenging each other to make various numbers of parts within a whole by partitioning the whole vertically and horizontally. During such playful activity the students used PARTS, SHADE, and BREAK. But they were yet to use the UNIT BAR and MEASURE tools. So, I decided to introduce those tools during the next episode in order to direct their activity toward fractional measures of a whole.

The second episode took place on February 26th and provided rich data for examining the students' fractional schemes or lack thereof. After demonstrating the use of UNIT BAR and MEASURE in measuring fractions, I asked the students to make various fractions and predict their measures. The students were free to choose whatever fraction bars they wanted to make, and eventually challenged each other to make specified fraction bars.

Cory began by partitioning a copy of the unit bar into three parts and breaking it into three pieces. Sierra knew that each of the pieces would measure "1/3" because each was one of three equally sized parts. Her prediction indicates that she had constructed a part-whole partitioning scheme. But it does not necessarily indicate a fractional scheme with which Sierra would compare the *size* of the part to the whole (beyond simply naming the part based on it being one of three equal parts). In fact, there is indication within the episode to suggest that she had not yet constructed any fractional scheme.

Sierra had challenged Cory to make a three-fifths bar, and he attempted to do so by partitioning a copy of the unit bar into three parts horizontally and five parts vertically. Cory was not yet satisfied with his production even after Sierra reassured him: "Cory, you're right. That's three-fifths." Cory continued by breaking up the fifteen-part bar, joining various numbers of pieces and measuring them. Sierra seemed to attribute Cory's lack of success to his action of breaking up the partitions. In her own attempts to create three-fifths, she replicated his partitions, but measured the result before she broke the bar and "1" appeared in the measure box. The students' making of three-fifths by using horizontal and vertical parts is contraindication that they had constructed a fractional scheme because at the very least, "three-fifths" refers to three parts in five parts.

Other indications for the lack of a fractional scheme include her subsequent identification of a partitioned two-fifteenths bar as "one-half." Cory had produced the two-fifteenths bar by joining two of the fifteen pieces that he had created. It seemed that Sierra's concept of one-half in that case was based on the partition marking the fraction bar into two parts. She seemed to pay no attention to the unit bar or to the relative sizes of the fraction bars.

While her trouble in the cases stated so far may be attributed to Sierra's unfamiliarity with the new environment and, in particular, the role of the unit bar, the strongest contraindication of a fractional scheme occurred at the end of the episode. Cory had challenged Sierra to make one-tenth of the unit bar. She partitioned a copy of the unit bar into ten parts and pulled one, but measured the ten-part bar rather than the one-part bar. Although she went on to measure the one-tenth bar, she had been surprised to see that "1" appeared in the measure box after she measured the ten-part bar. Her initial action and subsequent surprise indicate that Sierra had been considering the partitioning into ten parts as creating one-tenth, rather than considering the size of a part relative to the whole, as she would with a fractional scheme.

Sierra's actions in the next episode (February 28th) also corroborate my argument that she had not yet constructed a fractional scheme. The students had begun the episode by making and lining up bars such as two-halves, three-thirds, four-fourths, all the way up to a ten-tenths bar. When I asked Sierra to use the bars to make three-sevenths, she dragged the three-thirds by over to the side and began to dial parts to "7". When I asked her to make the three-sevenths bar without using PARTS, she did not act and sat quietly in thought while Cory attempted to make the desired bar by joining the three-thirds and four-fourths bars. When Cory joined the two bars, the partitions were cleared away, and Cory explained that he had wanted to pull three of the joined parts, at which point Sierra suddenly exclaimed that she had an idea. She pulled three parts from the seven-sevenths bar!

I attribute Sierra's final solution to a possibly novel use of her disembedding operation as indicated by her pulling of three parts out of the seven parts. She constructed the seven-sevenths bar as a seven-part whole from which she could disembed three out of seven parts. I refer to this construal as a perceptual judgment because she acted with certainty. Sierra's part-whole partitioning scheme, of which the disembedding operation was a part, would be sufficient for deciding that the three parts should be pulled from seven parts given that it was her goal to make three-sevenths. Her initial attempt of using the three-thirds bar indicates that she was primarily concerned with the whole numbers involved in the fractional numeral "3/7," and not with a fractional part of the unit bar. For this reason, I retain my hypothesis that she had not yet constructed a fractional scheme.

Some of Sierra's actions in the episode indicate that she *had* constructed a part-whole fractional scheme. First of all, Sierra was able to make a one-half bar from a four-

fourths bar by pulling two parts from the four-fourths bar. She was also able to justify that a given piece was one-ninth (as was previously measured) by referring to the commensurability of a nine-ninths bar and the unit bar; the piece in question had been lined up under the left side of the unit bar when she said, "The bars [nine-ninths and the unit bar] are the same size so, automatically, you know it's going to be one-ninth." She then made nine copies of the piece, lined them up and joined them under the unit bar. Finally, she was able to make one-third from ninths by pulling three parts from the nine-ninths bar.

These cases indicate that Sierra was able to estimate the relative unit fractional sizes of bars, or at least compare composite fractions to the unit fractions that were visible on the screen. Making such size comparisons using fractional language as she did indicate the use of a part-whole fractional scheme. But when she attempted to justify the common measures of three-ninths and one-third, she said, "that's one [pointing to the three-ninths bar] and there's three parts," apparently conflating the number of parts in the one-third (three-ninths) bar with it being one of three parts in the whole. If she had constructed a partitive unit fractional scheme, I would expect her to iterate the part three times within the unit bar, because such actions would be essential to her understanding of one-third. So, I maintain that Sierra had not constructed a partitive unit fractional scheme, although a part-whole fractional scheme may be emerging from her actions in `TIMA:Bars`.

In the next episode, held on March 3rd, there is indication that Sierra's apparent success with comparing and estimating fractional parts was based on something other than a fractional scheme. During that episode, I introduced the students to

`TIMA:Sticks`. Beginning with a set up of n/n-sticks very similar to the set up of long skinny bars in the previous episode, the students took turns secretly pulling one part from one of the sticks, and the other student had to determine the part's fractional size. I demonstrated the game by pulling a one-sixths bar while the students closed their eyes. Upon opening their eyes and examining the piece I had pulled, both students guessed that the piece was one-sixth, and Sierra explained that she knew "because it looks like one of the sixths bar." For the next mystery piece (one-eighth), she counted down the rows of n/n-sticks in determining it's fractional size (the sticks were lined up in order from top to bottom, starting with an unpartitioned unit stick and ending with the ten-tenths stick). As the students continued the activity, I decided to have them cover the n/n-sticks after pulling the mystery piece (although the ruler was still visible). With this restriction in place, Sierra did not even guess the fractional size of the piece. Her responses to these cases indicate that she was simply visually comparing the piece in question to the parts in the n/n-sticks rather than estimating the fractional size of the piece relative to the (unpartitioned) ruler.

Sierra and Cory worked together for the last time on March 10th. I had to change the pairings because Sierra did not want to leave her second period class to participate in the teaching experiment. In fact, she was supposed to work with Cory again on March 14th, but declined because she did not want to leave class (English). Even in the March 10th episode, which was held during first period, she seemed disengaged. I hoped that working with a new partner would not only relieve scheduling conflicts, but also raise her level of interest. In any case, Sierra's responses to tasks posed in this last episode with Cory affirm that she had not yet constructed a fractional scheme.

I began the episode by making a copy of the ruler, partitioning it into two equal parts, and then partitioning one of those parts into two equal parts. When I asked the students what one of the smaller parts would measure, Sierra guessed that it would measure "half." Even after indicating (by sweeping the cursor across the unpartitioned half of the copy) what half of the ruler would look like, she maintained that her guess was viable. So, after Cory had repeated one of the parts from the first partition to demonstrate that it was one-half, I asked Sierra to demonstrate that the smaller part was one-half. She pulled out the piece in question, lined it up with the ruler, and repeated it once. Then, she pulled out one of the parts from the first partition (one-half) and lined it up beside the repeated pieces (two-fourths), thus regenerating the image with which we had started. She seemed to be reasoning that the piece in question was a half because it was half of something, which was, in turn, half of the ruler. She subsequently seemed bothered that her demonstration required two steps (whereas Cory's had only required one) and changed her guess to "one and a half." Presumably the "one" represented the larger piece that she appended to the two "halves."

The actions described above do not fit those that I would expect to observe in working with a student who had constructed a partitive unit fractional scheme. Instead, I would expect such a student to iterate the part in question four times to reproduce the ruler and determine that the part was one-fourth. Both Sierra's initial guess and her revised one indicate that her concepts for the fraction names she used were, at best, restricted to those of a part-whole fractional scheme. Using such a scheme, she would be able to name a fractional part if the whole were partitioned evenly. Because in this case the whole had uneven parts, Sierra treated the two smaller parts as the partitioned whole and named one of them as one-half; the bigger part was another whole.

In justifying that the piece in question was actually one-fourth, Cory repeated it four times along the ruler, creating a partitioned copy of the ruler. Cory's actions indicated that one-fourth was an iterable unit for him. Sierra indicated that she understood Cory's justification. It is unclear whether she was able to follow his actions in iterating the piece or whether she understood only after his actions produced four partitions in the ruler. In the latter case, Sierra could assimilate the situation with her part-whole partitioning scheme. In the former case, she could assimilate his actions of iteration only with her equi-partitioning scheme, because she had not constructed a partitive unit fractional scheme nor iterative unit fractions. Her equi-partitioning scheme included the tacit understanding that any part could be repeated so many times to reproduce the ruler, and Cory's actions had indeed reproduced the ruler from one part.

Sierra's action in the final segment of the teaching episode indicate a novel use of partitioning and a conjecture. Cory asked Sierra to determine the fractional name of a four-tenths stick that he had made while she had her eyes closed. In response, Sierra dragged the stick to the left side of the ruler and repeated it twice. She paused for a moment between the repetitions, apparently considering the implications of going beyond the ruler, but she resolved this situation by repeating the second time and cutting off the two extra parts. She then counted the ten parts and said, resolutely, "ten." The novelty of her actions in using REPEAT in such a way was indicated by her stated resolution. She had lost the original fraction stick (four-tenths) in the ten-part stick. Moreover, her goal in repeating the stick seemed to be restricted to recreating the partitions of the ruler from which it was pulled, making it possible for her to use her part-whole partitioning scheme to determine the fractional name of the stick, even if she lacked a part-whole fractional scheme with which to compare it back to the ruler.

The novelty of Sierra's actions involved her use of REPEAT with a partitioned fraction stick in order to determine its fractional name. She did not seem to use REPEAT to iterate the fraction and determine how many times it fit into the whole, as she might in using a partitive unit fractional scheme. Rather, I hypothesize that she used the tool in a non-iterative way to reproduce partitions in the ruler from those in the stick she was repeating. As such, she had used the partitioning operation of her equi-partitioning scheme in a novel way to partition the ruler. Her conjecture (**Conjecture S1**) was that she could evenly partition the ruler by repeating the partitioned fraction stick beyond it. Conjecture S1 may have engendered a functional accommodation of Sierra's equi-partitioning scheme because she had modified her tacit understanding that she could use a unit stick to reproduce the ruler. Since she had tested her conjecture, she could now use a composite stick to reproduce the parts in the ruler.

In sum, I hypothesize that Sierra had begun using her equi-partitioning scheme to reproduce partitions in the ruler from a given part. She had at least constructed a part-whole partitioning scheme with which to name fractions and may have been constructing a part-whole fractional scheme. My only reservation in attributing the latter scheme to her is that it is unclear whether she compared parts back to the whole of which they were a part. I have cited one case in which she changed the whole in naming a one-fourth part as one-half. At least in that case "one-half" may have meant nothing more than *one of two parts*.

<u>Josh and Matthew</u>

Josh and Matthew worked together for the first time during an episode on February 28th, which was Matthew's first episode of the teaching experiment. Josh had participated in one other episode with Andy on February 24th. In that earlier episode, Josh and Andy had practiced using various tools and features of the `TIMA:Bars` software, including MAKE, COPY, REPEAT, PARTS, COVER/UNCOVER, JOIN, and BREAK. While exploring various ways to partition, the students started playing a game of partitioning copies of a whole into twenty-two parts in various ways. After they had partitioned three copies of the whole in three different ways (using only horizontal partitions, using only vertical partitions, and using a combination of two vertical and eleven horizontal partitions), I asked the students which method of partitioning created the largest parts. Josh claimed that the horizontal-only partition would create the largest parts "because there is more space" between the partitions.

Although my question suggested that one of the parts should be larger than the others, I would expect a student using a partitive unit fractional scheme to understand the necessity of having produced equal parts after partitioning a fixed unit bar into *n* parts in various ways. I would expect this because the partitive unit fractional scheme establishes fractional size through iteration of the part in the whole. When I asked Josh to name one of the parts created from the vertical partitioning, he said that it was "one out of twenty-two." His language indicates that he was not reasoning with a partitive unit fractional scheme at all, but rather with a part-whole partitioning scheme. In contrast, Andy, who had constructed a partitive unit fractional scheme, knew that a part from one of the partitioned wholes should be the same size as a part from another partitioned whole because "they're all being divided up into twenty-two."

In Josh and Matthews' first episode working with each other, Josh's actions indicated ambiguity in his non-unit fractional concepts, at least in the context of using `TIMA:Bars`. The students had been practicing the use of various computer tools, such as JOIN and BREAK when Matthew challenged Josh to make four-fifths of a given bar. Perhaps because of his experience from the last episode with Andy in creating twenty-two parts using vertical and horizontal partitions, Josh partitioned the bar into four parts vertically and five parts horizontally. His production was similar to Cory and Sierra's production of three-fifths in one of their first episodes. It indicates the lack of a (general) partitive fractional scheme and a kind of homonym for fraction names that can be manifested in the two-dimensional environment of `TIMA:Bars` (i.e. four-fifths might mean partitions of four and five, as well as four our of five parts). Even after Matthew completed his action for producing four-fifths—by partitioning a whole into five parts and shading four of them—Josh claimed that either production could represent four-fifths.

We used Josh's production (a twenty-twentieths bar) to explore some situations related (from my frame of reference) to commensurate fractions. Josh had some success in meaningfully assimilating those situations, but his ambiguities concerning partitive fractions (proper fractions determined through iteration of parts in the whole) persisted. Whereas he was able to shade in four-fifths of the twenty-twentieths bar that he had produced, he could not discern one-fourth of it until Matthew helped him, presumably because he had trouble orienting his perceptions to the vertical partitions (fourths) after focusing on the horizontal partitions (fifths). After Matthew demonstrated how to shade two-fourths (by shading the left two columns), Josh exclaimed, “he made one-half.” When asked for another name for the fraction, he responded, “one whole because there’s two [shaded parts] and two [unshaded parts] and if you put them together, you get the whole.” But, then he also claimed that it might be “four-fifths.” It seems that Josh’s concepts for fractions in the `TIMA:Bars` context were dominated by visual patterns that he could recognize in the partitions, as they would be using a part-whole partitioning scheme. As the episode continued, he also demonstrated that he could anticipate the relative sizes of fractions, at least for unit fractions, as he would in using a part-whole fractional scheme.

The students had made a set of unit fraction bars—one half, one-third, one-fourth, and one-sixth—when I asked Josh how one-fifth would compare to the others. Josh responded that it would be “just a tad bigger than one-sixth.” This would indicate that Josh had constructed a partitive unit fractional scheme, unless he was able to determine the relative size of one-fifth by generalizing a pattern from the unit fraction bars in his perceptual field. I am unable to determine whether there was a new development here, but from his previous actions in the episode, it seemed that he had not yet constructed a partitive unit fractional scheme.

For the next episode, held on March 5th, I decided to introduce the students to `TIMA:Sticks`. I thought that the one-dimensional environment of the new program would help avoid some of the ambiguity concerning the production of non-unit fractions. The students spent the first several minutes of the episode playing with the available tools, such as PARTS, BREAK, JOIN, PULL PARTS, COVER, and COPY. Then, I introduced the role of the ruler. Basically, it serves the same role as UNIT BAR (in `TIMA:Bars`); a stick is designated as the ruler when is it copied into a box labeled "ruler," and all other sticks are measured relative to that stick. In addition to the familiarity the students gained with `TIMA:Sticks`, the episode provided an opportunity for me to learn more about Josh's conception of unit fractions.

In the middle of the episode, Matthew had accidentally partitioned the left half of a two-halves stick into two parts, and I asked the students to consider what one of the resulting parts (the leftmost fourth) would measure. Josh responded that it would be "one third… if it was even." When Matthew said that he thought it would be one-third anyway, Josh agreed. But when they pulled out the part and measured it to be "1/4," Josh reiterated his initial qualification as an explanation for the surprising result: "because it's not even." This response fit the pattern of abduction, but his expressed confidence in reiterating the reason indicates that his response was a based on a perceptual judgment. Still, his agreement with Matthew that the part could be one-third indicates that *one-third* was not determined through iteration in the ruler and served as contra-indication that he had constructed a partitive unit fractional scheme. So did his actions surrounding Matthew's subsequent production of a three-thirds stick.

The three-thirds bar was broken into three equal parts when I dragged one aside and asked Josh what it would measure. He knew that it would measure "1/3," but when I began to drag aside a second third (the middle one), he said it would measure "one-half, because you had two of them together." Whereas Josh demonstrated an ability to consider the relative sizes of fractions, he established fractions in terms of what visual pattern at the moment.

Josh could name and determine fractional parts within a partitioned whole, and so he may have constructed a part-whole fractional scheme. I have mentioned two cases in which a fraction stick displayed a fractional name that was surprising to him, and these surprises may have engendered change in his fraction concepts, at least for unit fractions. In particular, Josh might be able to use his splitting operation (identified in the initial interview) to transform concepts based on partitioning to ones based on iteration and construct a partitive unit fractional scheme. There is indication at the end of the episode that he was beginning to do just that.

The stick that Matthew had accidentally made earlier in the episode (the two-halves stick with the left half partitioned into two parts) was still visible on the computer screen. We revisited the measure of one of the smaller parts, which had been pulled out of the stick. Protocol 1 picks up there.

Protocol 1: *Josh's abductive conjecture about an unevenly partitioned stick.*

T: If I measure this [pointing to the pulled piece]?

M: One-fourth.

T: How do you know that's one-fourth?

M: Because we already measured it.

T: Okay…

J: Let's see. Because… them two look the same [pointing to the two fourths]; you could put one more [partition] *in* there [pointing to the middle of the right half].

Although Josh seemed to form an abduction about the same stick earlier in the episode, it did not seem as insightful or operational as the present one. Using his part-whole fractional scheme (or, at least a part-whole partitioning scheme integrated with fractional language) in Protocol 1, Josh knew that one-fourth meant that the piece in question should be one out of four equal pieces making up the ruler, but there were only three visible parts. The new abduction, then, was to use partitioning in a novel way (segmenting), creating the fourth equal part in the whole from the three unequal parts, which would explain the surprising measure (one-fourth) of the piece. He conjectured (**Conjecture J1**) that he could produce the desired part-whole fraction by partitioning the larger part into two parts.

If he were indeed a splitter, partitioning and iterating would be inverse operations for him so that one-fourth might become an iterable unit. In other words, Josh's abduction of his partitioning operation within the context of conceptualizing a unit fraction could yield partitive unit fractions. In fact, at the end of the episode and for the first time in my observations of him, Josh was able to estimate the fractional size of a given piece in the absence of a partitioned whole.

While the students closed their eyes, I pulled a one-fourth piece from a four-fourths stick and covered everything except for the one-fourth piece and the ruler. When the students opened their eyes, Josh looked at the piece and the ruler for a moment and said, "that's one-fourth of it." It seems that Josh had mentally repeated the piece within the ruler to segment the ruler into four parts, thus determining that the piece was one out of four parts in the ruler. This would indicate a novel use of his part-whole fractional scheme. Alternatively, he may have treated the piece as a fractional quantity of the ruler that could be iterated four times to reproduce the length of the ruler. This would indicate a partitive unit fractional scheme. Whether or not Will had yet constructed a partitive unit fractional scheme, the relevance of Conjecture J1 is evident when we consider Josh's actions in subsequent episodes.

Conjecture J1 marked a point of transition after which Josh appeared to use partitioning to determine the fractional sizes of given pieces. He could anticipate using a given piece as a template for partitioning (or segmenting) a whole into "even" pieces, thus determining the size of the piece. Similar partitioning activity occurred several times in the episode following Conjecture J1, on March 7th. In that episode, the students were creating unit fractions from n/n-sticks, and I encouraged them to use REPEAT to recreate n/n-sticks from their unit fractional pieces.

In order to test the students' concepts of one-half, the witness made an arbitrary mark about one-third of the way across a copy of the ruler and asked the students whether he had marked one-half. Josh looked at the marked stick for a moment and replied, "That's a third of the stick. You can put another one in there [pointing to the middle of the larger part of the stick]." He did something similar after Matthew had accidentally created an unpartitioned whole stick with a one-third stick appended to it: "He made fourths." When asked whether the stick made fourths *of the ruler*, Josh realized that it didn't because "it is longer than the ruler." In both cases, Josh seemed to use a single visible partition to mentally create the other partitions. While he was particularly attentive to making sure the marks were "even," he sometimes neglected to consider the ruler, just as he had during the March 5th episode when he named a one-third stick "one-half." Of course, this may be a consequence of his unfamiliarity with the computer program in which he had only been using MEASURE and the RULER for two episodes.

It seems that Josh was in the process of constructing a partitive unit fractional scheme and conceptualizing unit fractions as iterable units. After Matthew had made a five-fifths stick, pulled one-fifth and covered the five-fifths stick, I asked Josh to justify that the piece was indeed one-fifth of the ruler. He dragged the piece to the leftmost part of the ruler and said, "If we mark it right here it would be…" He then dragged the piece over within the ruler until he reached the rightmost part of the ruler, counting "one, two, three, four, five." These actions demonstrate how his novel use of partitioning (segmenting the ruler with marks) might translate to a novel use of iterating. I have provided indication in the initial interview that Josh was a splitter, and suggested in my analysis of this teaching episode that he was capable of considering partitioning and iterating simultaneously. But Josh's actions at the end of the episode indicate that his fraction concepts were not yet based on iteration.

The students had returned to the game from the end of the previous episode (March 5th), posing mystery unit fractions with all but the mystery unit fraction and the ruler covered. On the two occasions in the game that I posed problems, I created a one-seventh piece and a one-twelfth piece as the students hid their eyes. Josh guessed one-fifth and one-tenth, respectively. He checked each guess by making a copy of the ruler, partitioning it into the anticipated number of parts, and comparing the size of the first part to the piece in question. He persisted with this method of checking even after observing Matthew using REPEAT to check that a piece was by seeing if n repetitions of the piece reproduced the ruler. Josh's preference for the more time-consuming method of partitioning indicates that he was not operating with partitive unit fractions, but was probably operating based on his part-whole fractional scheme.

The next episode, held on March 12th, was the last in which Josh and Matthew worked together. We began the episode by continuing the game that we were playing at the end of the last two episodes, except that Matthew was the one posing the mystery fractions to Josh. Matthew began by pulling two parts from a thirty-thirtieths bar and covering the thirty-thirtieths bar as Josh hid his eyes. When Josh opened his eyes, he made a copy of the ruler, lined up the two-part piece with it and began making copies of the piece. When he indicated that he was planning on joining the copies, I suggested that he might use REPEAT because it would join the pieces for him as he repeated them. Josh then used REPEAT to reproduce the ruler, thus determining that thirty of the small parts would fit in it.

This was the first time that Josh used copies of a piece rather than partitioning of the whole using PARTS in order to determine the fractional size of a piece. In fact, Josh seemed wholly immersed in determining the number of times the small parts would fit into the ruler, for when he had finished repeating and I asked him the name of the fraction, he replied, “30.” He subsequently revised his claim a few times: “2… two out of the whole bar… 2 over 30.”

It may be that this way of operating had been available to Josh all along, but that partitioning with PARTS had been more meaningful, when he could use it, in determining the size of a given fraction. Josh was unable to determine what number to use in PARTS in order to partition the ruler as he had in the past because the given fraction was not a unit fraction and the parts were unusually small. Instead, Josh relied on making copies of the piece, similar to actions he had observed Matthew perform in the previous episode (except that Matthew used REPEAT). In using COPY instead of PARTS, his goal had been transformed, and so he was not immediately able to use fractional language as he had before. Copying or repeating the piece had become a method for counting (by two's in this case) the number of unit within a ruler, producing a connected number that was a multiple of the two-unit piece.

When Matthew posed a similar problem—three-nineteenths—Josh solved it similarly, lining up the three-part piece with a copy of the ruler and using REPEAT until he had almost reached the end of the copy with eighteen parts. He assumed that one more of the small parts would complete the copy of the ruler, and so claimed that it was 19. Once again, his goal of determining the number of parts in the partitioning of the ruler superseded the initial fraction goal. When I explicitly asked him to name the given fraction, he said, "three out of nineteen." I had already noted, in my analysis of Josh's initial interview, that he had trouble naming non-unit fractions. But he did not even try to name the fraction when using REPEAT, until I prompted him. So, it seems that Josh had not yet reconciled iteration in the context of determining a given fraction with partitioning in that same context, and it is unclear whether any fractions were yet iterable units for Josh.

<u>18 March, 2003 Teaching Episode</u>

In the previous teaching episodes, with other partners, Josh and Sierra had each used partitioning conjecturally to determine fractional names of pieces. Indeed, the two students had acted similarly in several situations, and both seemed to be on the verge of constructing a partitive unit fractional scheme. However, as a splitter, Josh was operationally more advanced than Sierra. Being a splitter implies that Josh's novel use of partitioning in Conjecture J1 should also yield a novel use of iterating. Each student seemed to have constructed at least a part-whole partitioning scheme. Josh, at least, had constructed a part-whole fractional scheme, and he could also use iterative actions in estimating a fractional part of the whole. So, I hypothesize that Josh will construct a partitive unit fractional scheme from the conjectural operations of Conjecture J1, whereas that scheme may not yet be in Sierra's zone of potential construction. It's not that she would need to construct splitting operations first, but that she had yet to establish multiplicative relationships between part and whole as Josh had demonstrated he that had done in his initial interview.

This was the first episode in which Josh and Sierra worked together. The students' classroom teachers had asked me to spend a few minutes working with the students on changing improper and mixed numbers. So, after allowing time for the students to familiarize themselves with all of the available actions in `TIMA:Sticks`, I decided to try to integrate the classroom topic with part of the teaching episode, and so I asked the students to try to produce three-halves. In hindsight, working with improper fractions in `TIMA:Sticks` this early in the teaching experiment was a bad idea. The students appeared frustrated at several points in the episode, were given little opportunity to interact, and my attempts to broach the subject of improper fractions yielded a funneling effect that often guided the students' responses. So, I present here only those segments of the episode in which the students were able to operate independently and conjecturally.

Josh's conjectures about improper fractions. Not only did Josh lack an iterative fractional scheme, but he seemed to lack a partitive fractional scheme as well. So, early in the episode, when I challenged the students to produce three-halves of the ruler, Josh (and Sierra) had no appropriate scheme available. Josh partitioned a copy of the ruler into three parts, broke it up into three pieces and dragged one piece below the others. Protocol 2 begins as Josh completed this activity and released the mouse.

Protocol 2: *Josh's conjectural operations in attempting to produce an improper fraction*.

T: All right. Where's the three-halves of the ruler?

J: [picks up the mouse again and sweeps a path across the three pieces]

T: So, tell me what you meant by this.

J: I made one-third.

T: Oh, okay. You made one-third. So, this [pointing to the bottom piece] is one-third.

J: [nods affirmatively]

T: So, why did you do that in trying to do three-halves? What got you to do that? Is there something that made you think of making three parts when I asked you to make three-halves?

J: [drags a second piece down below, beside the first one that he had dragged] That's *two* halves. [He then lines up the third piece beside the other two.] That's just a whole.

T: Okay. So, show me *one* half, then.

J: [drags the leftmost piece back up above the others]

T: Is that one half of the ruler?

J: Yeah. [Josh drags the three pieces down to the ruler, one at a time, lining them up along the length of the ruler and joining them.]

T: What do you think, Sierra?

S: [silence]

T: What do you think, Josh? Is it half of the ruler?

J: Uh-huh… Well… No! It wouldn't be because you can't make it half with three [sweeping the cursor back and forth across the middle piece].

S: It would be one-third of that ruler.

I have hypothesized that Josh had constructed a part-whole fractional scheme and was in the process of constructing a partitive *unit* fractional scheme. Using either scheme, he was constrained to working within the copy of the ruler, always using the larger number in the fraction to partition. If Josh were using a partitive unit fractional scheme, he would also be constrained to treating the copy as a *fixed* whole. But he considered each of the three parts as a half, even as he understood them as being one-third of the ruler. By dragging one of these pieces below the others, he could use his part-whole fractional scheme to conceive of each piece as a half of the two pieces on top. I have noted that Josh had acted similarly in naming a fractional part relative to another fractional part (rather than the ruler) during the March 5th teaching episode. But this time, he acted with the goal of producing the relation, and, once he had completed his action and reflected on it, he assimilated the situation using his part-whole fractional scheme in order to recognize each piece as one-third of the ruler.

Josh's uncertainty about the fractional size of the part indicates that he had operated conjecturally in producing it. He wavered between treating the pieces as halves (e.g. "that's two halves [as he dragged two pieces together]") and thirds (e.g. "that's just a whole [as he dragged the three pieces together]"), until he finally realized that "you can't make it half with three" because the pieces would not divide evenly into two parts. Whereas Sierra's conclusion that the pieces were thirds may have been based on her part-whole partitioning scheme (once the three pieces were aligned), Josh's actions indicated a fractional conception because he could consider the sizes of parts separated from the whole. It seems that his conjectural activity had raised his awareness of the potential ambiguity in considering a part separated from the whole, unlike the March 5th situation in which he had simply made a perceptual judgment.

Josh conjecturally partitioned a copy of the ruler into three parts to produce a (two-part) whole of which his (one-third) part would be one-half. I claim that Josh's actions represented a conjectural use of splitting operations because the three parts contained records of the two-part whole, which is why he was able to conjecture that they were three halves. His operations generated conflict because he was trying to use splitting in two different ways. On the one hand, he used it to establish one of the three pieces as being one-half of the other two. On the other hand, use of splitting, when considering the ruler, established each piece as one-third of the ruler. The novelty was in considering the relative sizes of the pieces within the copy of the ruler. He conjectured (**Conjecture J2a**) that he needed three of something that could be considered a half, and, in testing the conjecture, was able to satisfy the condition that one piece was half of the other two.

However, his recognition of the piece as one-third of the whole refuted the conjecture and introduced the constraint of considering fractions relative to the unit bar.

The new constraint should contribute to Josh's construction of a partitive unit fractional scheme because he had explicitly established the importance of considering the number of iterations of a fractional part in the ruler in determining the part's fractional name. In forming his conjecture, Josh seemed to consider three-halves as three of one-half. So, I hypothesize that Conjecture J2a and the problematic situation of considering improper fractions as iterations of unit fractions may engender partitive fractions and a (general) partitive fractional scheme, beyond a partitive *unit* fractional scheme.

My argument that Sierra was using a part-whole partitioning scheme in Protocol 2 is corroborated by her actions following the protocol, when Josh completed his second attempt at producing three-halves. He had broken a copy of the ruler into six parts and joined three of them, apparently trying to satisfy the goal of making three-halves by producing one-half using three parts (below, I analyze this as conjectural). Upon examining the result of Josh's attempt (three joined pieces from six making up the ruler), Sierra claimed that it was "one-third." This indicates that "one-third" referred to the three parts rather than the size of the three joined pieces relative to the ruler. After she joined together all six pieces as a six-sixths stick and lined it up with the ruler, and after I pulled out three of the six parts, she constituted them as "half." So, it seems that fractions concepts for Sierra so far were based on part-whole partitioning operations and numerical relations, such as *three parts are half of six parts*.

I refer to Josh's constitution of three-halves as a half made with three parts and his production of this by pulling three of six parts of a given whole as **Conjecture J2b**).

It is difficult to analyze by itself, in part because it was closely related to Conjecture J2a. On the other hand, Josh's actions in independently producing the three-part stick were, in themselves, interesting. He construed "three-halves" as three parts in one-half, and anticipated that he would need six parts in the whole. This indicates Josh had disembedded the three-part half from the whole and iterated it to produce six parts in the whole, conjecturally treating one-half (and a composite unit!) as a partitive fraction.

Josh and Sierra's abductive conjectures concerning surprising measures. Even after the students discussed and agreed that the three-part stick was one-half, Josh still thought that it might measure three-halves and Sierra had "no idea." After we measured the stick, the students found that it measured one-half, and were in the position of having to explain this. Each student invented explanations for surprising measures a few times within the episode, and each time the explanations took the form of an abduction. Protocol 3 begins with the students' explanations after they measured the three-part stick as one-half.

Protocol 3: *Sierra and Josh's abductions explaining the measure of the three-part stick.*

T: What do you think it's going to measure?

S: I have no idea.

J: Three-halves?

S: [measures the three-part stick and "1/2" appears]

T: *One* half. Why is that only *one* half?

J: Because it's joined.

T: So do you think it's going to be different when we break them?

J: [breaks the three-part stick into three pieces and measures one of them "1/6"]

T: When they're broken, it just measures one of them. Why would that be one-sixth of the ruler?

S: I think it's because there are six little things in there [indicating the six-sixths stick that was still visible above the ruler and below the three pieces].

T: [joining the three-pieces back together] Can you explain why, when I have three of them, it's half of the ruler?

S: Because there's six of them, and 3 plus 3 is 6, so it's half.

Josh explained his surprising observation—that he had created *one* half and not *three* halves—by assuming that the computer treated joined pieces as one of something and treated broken pieces with respect to their number. His explanation fits the abductive pattern of making sense of a surprising observation by assuming a general rule to explain the observation. However, his test of the abduction failed. The abduction was certainly a conjecture, but not a fractional conjecture. Rather, it was a local conjecture, bound within the context of interpreting the functions of the MEASURE and JOIN tools in the computer environment. So, I doubt that it would have general implications for his fractional knowledge. For that reason, I do not distinguish the conjecture with a label, but it does indicate that Josh was not yet familiar with the role of MEASURE. The students' unfamiliarity with the computer tools may distract them, and I need to continue to consider how such distractions may explain the actions that I analyze.

Sierra's final explanation—"because there's six of them and 3 plus 3 is 6"—did involve her fractional knowledge. She had already noted, based on her part-whole partitioning scheme, that six pieces make up the ruler, and so one of them would be one-sixth. Her final explanation addressed the measures of three of the pieces, joined. She used a whole number comparison between the number of pieces in the part and the number of pieces in the whole. It is especially interesting that she did not use multiplicative reasoning, such as *3 times 2 is 6* or *3 is half of 6*. She was still reasoning additively, further indicating that Sierra had not yet constructed a partitive unit fractional scheme with which she might have mentally iterated the three parts twice within the ruler. This supports part of my hypothesis from the beginning of the March 18th teaching episode: a partitive unit fractional scheme had apparently not been in Sierra's zone of potential construction.

The explanation was a novel abduction with which Sierra was able to perceive the three-part stick as being embedded in the six-part stick, and perceive the complement as being three more parts, thus forming two equal parts in the ruler. I label this abduction as **Conjecture S2** and use it as one example of Sierra's propensity for inventing abductive explanations. The conjecture may have resulted in a way of operating similar to Will's procedural scheme for producing fractions commensurate with one-half, except that Sierra used her way of operating to explain commensurability rather than to produce it. This is indicated by her actions during Protocol 11 on March 28th, in which Sierra tried to use three parts to produce a fraction stick commensurate with one-half.

The principal difference between Josh's actions in the teaching episode and those of Sierra was Josh's use of multiplicative reasoning and Sierra's apparent lack of it. Whereas Josh focused on the multiplicity of halves and thirds in the ruler, Sierra focused on the additive relationships between parts produced within a copy of the ruler (e.g. the sum of the two composites of three parts in Protocol 3). Both students were able to unite units, such as uniting three-sixths as one-half, but Sierra seemed to lack the ability to iterate units.

I have noted in previous episodes that both students had seemed to be on the verge of constructing partitive unit fractional schemes. I also hypothesized (March 18th) that, because he was a splitter, Josh would more readily construct iteration from his use of partitioning and that a partitive unit fractional scheme was in his zone of potential construction. It appears now that this was the case and that Josh was indeed constructing a partitive unit fractional scheme. Moreover, he was able to iterate composite units, so that a general partitive fractional scheme now seems to be within his zone of proximal development.

21 March, 2003 Teaching Episode

In my analysis of the episodes in which Josh worked with Matthew, I have noted that Josh had learned to use REPEAT in order to recreate the partitions of a copy of the ruler from which unknown fractions were pulled. For example, when Matthew pulled a partitioned three-nineteenths stick from a nineteen-nineteenths stick and hid the latter, Josh repeated the former six times and knew that Matthew had used nineteen parts in the copy of the ruler. In the present teaching episode, Josh continued to use REPEAT to determine unknown unit and composite fractions. He also demonstrated an uncanny ability to visually estimate fractional sizes, which suggests that he may have constructed a partitive unit fractional scheme. However, his difficulties with fractional language persisted.

Sierra, on the other hand, had no difficulty with fractional language, but her visual estimates for given fractions were not nearly as accurate as Josh's. She could use REPEAT to reproduce partitions, but I suggest that REPEAT was still strictly a partitioning tool for her. The subsections in this section provide contrasting descriptions of the students' actions. They are intended to modify or support the various claims made in the previous teaching episode and clarify the students' present ways of operating.

Josh's estimates and use of REPEAT. As Josh hid his eyes, Sierra made a copy of the ruler, partitioned it into seventeen parts and pulled one of them. After she hid the seventeen-part stick, Josh opened his eyes and was asked to determine the fraction of the ruler that Sierra had made. He made a copy of the fractional piece, dragged it to the left end of the ruler, and repeated it seventeen times while sub vocally counting. Protocol 4 picks up there.

Protocol 4: *Josh's use of REPEAT in determining the size of a unit fraction.*

J: She made seventeen little mark things [waving his forefinger vertically to indicate the marks created by his repetitions of the piece].

T: Yep. So what was her fraction out of the ruler?

J: One out of seventeen. Uh… Yeah. One out of seventeen.

From his comment about the "little mark things," just after completing the repetitions of the unit fraction, it seems that Josh had been using REPEAT as a partitioning tool, rather than one for iterating the unit fraction with the goal of determining how many times it went into the ruler. This is a subtle but important distinction that indicates he was not yet using a partitive unit fractional scheme in such situations. In fact, his naming of "one out of seventeen" indicates a part-whole partitioning scheme that was employed to name the fractions, as an after-thought. I have hypothesized that Josh was in the process of constructing a partitive unit fractional scheme and was doing so by using his splitting operations to transform partitioning actions into iterative actions.

Josh demonstrated that he could use REPEAT with composite fractions as well. In Protocol 5, Sierra had posed a four-nineteenths stick for Josh to figure out. Just a few seconds after he opened his eyes and looked at the fraction, he made a remarkably accurate guess.

Protocol 5: *Josh's fractional estimate for a composite fraction*.

J: I think it might be twenty.

T: So, what would the whole fraction be?

J: Four out of twenty or four-twentieths.

T: You think four-twentieths? Okay. Let's check.

J: [drags the four-nineteenths stick to the left side of the ruler and repeats it until he has extended just beyond the ruler, with twenty parts. He then looked confused and indicated that he wanted to get rid of the extra part.]

T: [reminds Josh how to use CUTS]

J: [cuts off the extra part and begins counting the nineteen parts extending along the length of the ruler] You'd have… I think nineteen.

S: Mm-hmm.

T: What was her fraction, then?

J: Uh, nineteen out of four… I meant four out of nineteen.

Josh's initial estimate indicates that unit fractions were iterable for him, even if the previous situation (Protocol 4) did not trigger a partitive unit fractional scheme. He was able to look at the four parts in the given stick and imagine the number of parts making up the ruler. When prompted, he was also able to name the fractional part, although he experienced fractional language difficulty at the end of the protocol. I hypothesize that these struggles in naming fractions run parallel to his struggles in moving from a part-whole conception of fractions to a partitive conception of them. Terms like "nineteen out of four" may indicate his recognition that he had made nineteen parts from four parts.

Sierra's estimates and use of REPEAT. In his first problem for Sierra, Josh produced a one-tenth piece, leaving no other visible sticks except for the unmarked ruler. Upon opening her eyes, Sierra's task was to determine the fractional size of the piece.

Protocol 6: *Sierra's fractional estimate and use of REPEAT with a unit fraction.*

T: Can you guess what it is before you even do anything?

S: I'm thinking sixteenths.

T: Sixteenths. Okay.

S: [drags the piece to the left side of the ruler and repeats it to the length of the ruler, ten parts] Ten.

Sierra's estimate for the given fraction was not nearly as accurate as Josh's estimate in Protocol 5; the actual size of the fraction given to Sierra was sixty percent larger than her estimate of it, whereas Josh was off by little over five percent. Still, she was able to use REPEAT to determine the number of parts making up the ruler. Later in the episode, she also demonstrated that she could use REPEAT with a composite fraction (seven-eighteenths), as Josh had, in order to determine how many parts would be in the ruler. So, the only notable difference between the students' actions was in their fractional estimates. Josh's actions in the previous episode and during Protocol 5 indicate that he had constructed iterable unit fractions, and was in the process of constructing a partitive unit fractional scheme.

24 March, 2003 Teaching Episode

The students were not interacting very much in the last episode. So, many of my actions in the present episode were intended to encourage communication between the students and facilitate meaning-making of each other's actions. My efforts seemed to pay off, and the students' interactions indicated significant differences between their available operations.

Josh's struggles in constructing a partitive unit fractional scheme. From the beginning of the episode, we find more support for the claim that Josh could iterate unit fractions. Having produced a five-fifths bar (in order to pose a problem for Sierra), he had accidentally cut off about half of the rightmost fifth. Protocol 7 begins after Josh dragged the piece that he had cut off below the rest of the stick. The protocol not only shows that Josh could iterate but reveals more about his struggles in constructing a partitive unit fractional scheme.

Protocol 7: *Josh's iteration of an unknown fraction.*

J: I cut half of it.

T: Yeah. Now, I tell you that this one is going to be real tricky. What do you think it's going to say?

J: That's what I'm going to see.

T: Okay.

J: [drags the piece across the remaining parts of the stick, moving the piece twice within each uncut fifth]

T: What do you think it's going to say, Sierra?

S: I don't know.

J: Nine out of… Uh. One out of nine, I think.

Once again, we see Josh struggle with the fractional language. I have suggested that this may be due him reasoning that he was making nine parts out of one part, through his iterations. I have only observed him using such language in these last two episodes, in which Josh seemed to be struggling to construct a partitive unit fractional scheme (although he had used language such as "seven-fifths" for five-sevenths in the initial interview). Josh's actions in Protocol 7 also indicate that his use of CUTS and BREAK instead of PULL PARTS, may have contributed to additional struggles.

When using PULL PARTS, unlike CUTS or BREAK, the pieces pulled are only copies, and the original parts from which they are pulled remain in the original stick. When Josh used CUTS to create the piece in Protocol 7, the original stick, which would have contained about ten of the pieces, was reduced to one that would contain only nine of the pieces. Josh did not consider this reduction in the size of the original stick when determining the fractional measure of the piece. This indicates that he had not established the unique role of the ruler in using iteration to determine the fractional size of the piece. He had acted similarly in the March 18th teaching episode when he attempted to make three-halves by breaking a stick into three pieces; having dragged one of the pieces away, he referred to it as one-half, but it was one-half of the reduced stick and not the original.

Although he was struggling with a couple of aspects of naming the iterable fractions, he did seem to be using the iteration of a fractional piece to determine its measure. In that way, Protocol 7 serves as indication for Josh's continued progress towards a partitive unit fractional scheme. Examining the remainder of the teaching episode, I examine the students' use of REPEAT, with a particular focus on whether iteration and partitive fractions seem to be involved.

Changes in Josh and Sierra's uses of REPEAT. The first case occurred after Josh had posed a partitioned four-fifteenths stick while Sierra closed her eyes, challenging her to determine it's fractional size in relation to the (unpartitioned) ruler. Sierra dragged the fraction to the left side of the ruler and repeated it until the result extended just beyond the ruler, with sixteen parts. She then cut off the rightmost part (extending beyond the ruler) and counted the remaining parts. When she had finished, the following dialogue began.

Protocol 8: *Sierra's rule-based use of REPEAT.*

S: Okay. I got fif… sixteen, but I don't know…

T: What do you mean?

S: There's sixteen over there, but when I cut off…

T: Uh-huh.

S: Okay. All of the others, including the one I cut off, is sixteen. Not including the one I cut, it's fifteen.

T: Okay. But what was his fraction… the fraction that he started with?

S: Four-sixteenths?

Although Sierra's actions in repeating the fraction were similar to her actions in previous episodes, she now seemed unsure of how to use the results of her actions in determining the fractional name. She did not know whether to use the total number of parts in the name or just the number that made up the ruler. This indicates that her present and previous actions in naming fractions using REPEAT had been ambiguous: She could follow the pattern of action but had constructed no permanent meaning for the results of her actions. She did not seem to be using iteration in any fashion that might lead to partitive unit fractions, if she were using iteration at all. Rather, I maintain my hypothesis from the end of the March 10th teaching episode that Conjecture H1 enacted a functional accommodation of her equi-partioning scheme. The modified scheme included integration operations but its goal was still simply to produce partitions in the ruler.

Sierra's confusion about whether or not to count the cut-off piece may have resulted from her observation of Josh's trouble in Protocol 7. In that case, Josh learned that he should have counted the cut-off piece. Sierra's lack of distinction between the situations represented in Protocols 7 and 8 indicates that her meaning for the situations in naming fractions had been procedural, and not based on iteration.

Next, it was Sierra's turn to pose a similar problem for Josh. She made a four-eighteenths stick. When Josh opened his eyes, he made two copies of the stick and repeated one of them four times along the ruler, much as he had in past episodes. However, this time, instead of repeating the stick again and cutting off extra parts, he cut off two parts from the other copy and joined those parts to the end of the sixteen-part stick that he had created using REPEAT. He was then able to name the fraction as four-eighteenths.

Josh's novel action of joining on two more parts rather than repeating a fourth time is an interesting departure from his previous approach to similar problems. First of all, the novelty indicates flexibility in his understanding of the situation. Rather than using a fixed procedure, he had demonstrated a meaningful approach. Beyond that, the novel action was one of joining pieces rather than repeating parts. This indicates that his use of REPEAT may have represented an intention to determine the number of times each piece would go into the ruler, and not just a goal to recreate partitions in the ruler. The difference (discussed in my analysis of the previous episode) was subtle, but may indicate that he was actually iterating parts within the whole. Such use of an iterating operation is a distinctive aspect of partitive schemes over part-whole schemes.

Students' conjectures about commensurability. Once Josh had completed the production of four-eighteenths, as described above, he measured the fraction as "2/9." The students then tried to explain why this unexpected measure appeared. Protocol 9 documents several of their conjectural attempts to explain this.

Protocol 9: *Josh's conjectural iteration of a composite fraction*.

S: 2 times 9 is 18 and there's eighteen [parts] over there.

T: Ah. Okay. So maybe that will help you.

J: [Meanwhile, Josh was dragging the fraction stick across the top of the eighteen-eighteenths stick making marks as illustrated in Figure 14.]

T: [to Sierra] Do you know what he's doing?

S: [shakes her head, negatively]

T: Josh, what are you doing? Sierra's not sure. I'm not either. [after a few seconds of silence] It looks like you had an idea…

J: Remember last time? We had like, say, one of these bars equaled up to three things [dragging the four-eighteenths fraction between each pair of marks that he had just made]. You remember?

T: Oh! Okay. So, you were hoping that it might equal up to something.

J: Yeah. [continues dragging the fraction in the other direction until it reaches the right side of the rightmost mark, as illustrated in Figure 14] It wouldn't work.

T: It doesn't work? Why doesn't it work?

J: Because I would have two left over [pointing to the two rightmost parts in the eighteen-eighteenths stick].

Sierra's initial explanation was an abductive relation of the numerator and denominator to the eighteen visible parts in the whole. Such a relation might help her establish an invented rule or procedure, but it was not an operative conjecture about fractional measures. Josh, on the other hand, had begun to iterate a composite fraction within the whole, using marks to keep track of his iterations. This appears to be a novel use of iteration. Indeed, I have claimed that he had only recently begun to treat unit fractions as iterable units, and now he was acting out the iterations of a non-unit fraction! Josh seemed to unite four parts and iterate them in order to affirm their measure just as he would iterate unit fractions with a partitive unit fractional scheme. The iteration of such a unit in justifying its measure was conjectural. His **Conjecture J3** was that the four-sixteenths stick would fit evenly into the ruler and establish it as a simpler fraction.

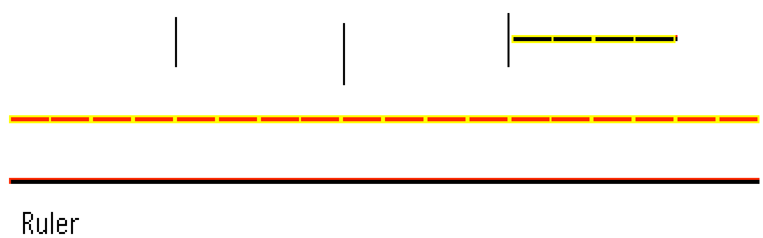

Figure 14. Josh's marks for iterating a composite fraction.

I hypothesize that Josh's conjecture could contribute to his construction of a partitive unit fractional scheme for composite units, as well as a partitive unit fractional scheme and would engender commensurate fractions if he could coordinate the units of units. Units coordination seemed to be occurring during Protocol 9, as indicated by Josh's disappointment that two parts were left over after his fourth iteration. He seemed to be determining the measure of the fraction by figuring out how many times the composite fraction would evenly fit into the ruler. If he could then name the fraction as the reciprocal of the number of iterations, he would demonstrate the essence of a partitive unit fractional scheme for composite units. Moreover, Josh's actions in Protocol 10 indicate the construction of commensurate fractional operations.

Multiplicative reasoning with composite fractions. There were no more significant conjectures in the teaching episode, but we might learn more about the students and the results of their previous conjectures by considering a couple of short segments toward the end of the teaching episode. The students had resumed the game they had been playing earlier. When Josh posed five-nineteenths to Sierra, she was able to use REPEAT appropriately and unambiguously this time: She repeated the fraction four times, cut off the extra part and counted only the remaining nineteen parts. At first, she seemed to have forgotten how many parts she had started with, but she either remembered after a few moments or somehow figured it out, naming the fraction five-nineteenths. I decided to test whether she could use some sort of multiplicative reasoning in her process by asking her if there was a way she could figure out how many parts were in the whole without counting. She thought in silence for about ten seconds before admitting, "I don't know."

While Sierra's language and action indicate that she had corrected her procedure for acting since the previous, similar situation, they also corroborate my previous arguments that she had not been using iteration of the five-part unit, nor iteration and multiplicative reasoning with fractional situations in general. Josh, on the other hand, seemed to be using multiplicative reasoning more and more as his iteration operation became more manifest. For example, after Sierra had named the five-nineteenths stick, Josh commented that, had it been five-twentieths, it would measure something different. I encouraged him to make five-twentieths, which he did, but neither student knew exactly what it would measure. Protocol 10 picks up after Josh measured the fraction to be one-fourth and as Josh began trying to explain this. I encouraged the students to share ideas with each other.

Protocol 10: *Josh's operations for constructing commensurate fractions*.

J: [lines up the five-twentieths stick with the left side of a twenty-twentieths stick] There's one. Okay. Then, if I put that in there [moving the stick over the next five parts in the twenty-twentieths stick] it's two. And then that one's three, four. It's one-fourth out of the whole bar.

T: Don't tell *me*; tell *her*.

J: [to Sierra] There's a fifth. [dragging the five-twentieths stick across the twenty-twentieths stick again] And then you put that one right there, and that's another fifth. So, that'll make that ten. Put that one right there, it'd be fifteen. Put that right there, it'd be twenty. That's one fourth… There's four of these little things right here [pointing to the places he had dragged the five-twentieths stick] going into that.

S: [nods] Yeah.

Josh's explanation fit the pattern of abduction, but the novelty had occurred in Conjecture J3. He used his uniting and iterating operations to justify that the five-part stick was actually one-fourth of the whole. His actions included double-counting (five is one; ten is two, etc), which indicates that he might be coordinating the units of units within the ruler. His actions also demonstrated his continuing struggles with fractional language, referring to the five-part stick as "a fifth." Considering this, along with his inability to predict the measure of five-twentieths, I am not yet willing to attribute to Josh a partitive unit fractional scheme for composite units, nor a commensurate fractional scheme. But the essential operations for the schemes had been constructed and the schemes themselves were within his zone of potential construction. Sierra claimed to, and probably did, understand Josh's explanation. She may have assimilated his actions using her part-whole partitioning scheme and one-fourth concept, if not a part-whole fractional scheme, once Josh had identified the four equal parts in the ruler.

We are now beginning to see the construction of operations and schemes that had been latently available to Josh, as a splitter. He seemed to be on the verge of constructing four significant schemes: a partitive unit fractional scheme, a partitive fractional scheme, a partitive unit fractional scheme for composite units, and a commensurate fractional scheme. I have provided contra-indication of the latter two schemes. While I expect to find consistent use of a partitive unit fractional scheme in subsequent episodes, his construction of the more general scheme was still in question.

When I asked Josh and Sierra (just before the dialogue described in Protocol 10) which would be bigger, five-nineteenths or five-twentieths, they each claimed that five-twentieths would be bigger and agreed this was due to the "higher number," as Josh put it. A student with a partitive fractional scheme might assimilate the fraction with that scheme and make a comparison between the number of parts in each (five) and the relative sizes of the parts. After producing the two fraction sticks, Josh focused on the relative sizes of their parts in order to explain the surprising result of his comparison:

"Because you have to put twenty of those things in there, and it makes it smaller." His conclusion does indicate a partitive unit fractional scheme because he seemed to understand the reciprocal relationship between the number of parts in a partition of the whole and the size of each part, but there was no indication of a (general) partitive fractional scheme.

The students' final exchanges in the episode further contrast their ways of operating. Josh had just correctly determined that a fraction that Sierra had posed was nine-fifteenths of the ruler, but measured it as "3/5." Sierra quickly exclaimed, "three times five is fifteen," whereas Josh noted that "if you have five of those little [parts], it would go in [the ruler] three times." Each student had formed an abductive explanation and both explanations were somewhat inadequate for insightfully resolving the situation. Josh had iterated a composite unit (five parts), but it was not the appropriate one (three parts). But the students' attempts represented differences in their operations available for use in assimilation and abduction. Sierra's abduction was much like one that she had employed before, in explaining why four-eighteenths measured two-ninths. It was based one whole number multiplicative reasoning. Josh's abduction was based on fractional multiplicative reasoning, and represents his recent focus on iteration. Josh's subsequent production of the five-part stick that he had described elicited further differences. Sierra thought that it would measure five-ninths, indicating her reliance on part-whole schemes (the five parts were pulled from a nine-part stick). Josh thought it would be "one out of fifteen" indicating his continued struggles in conceptualizing and naming non-unit fractions. He was, however, able to use iteration to explain that the stick was in fact one-third, after he had measured it, as the episode ended.

28 March, 2003 Teaching Episode

I decided to test whether the students could create commensurate fractions. By asking them to make a specified unit fraction in as many ways as they could, I imagined that the students might begin to use their uniting, partitioning, and iterating operations in novel ways. Since I have already argued that a commensurate fractional scheme was within Josh's zone of potential construction, such activity might engender his construction of that scheme. Sierra's potential for learning through the activity was less clear. Indeed, she responded unexpectedly to the activity.

Sierra and Josh's operations in producing fractions commensurate with one-half. I began the teaching episode by asking the students to produce sticks, in as many ways as possible, that would measure "1/2." Once the students had completed a simple production of one-half by breaking a copy of the ruler into two parts, Josh led the first attempt at an alternate production. He broke in half one of the halves from the first production and measured one of the resulting parts. He should have been able to recall (from a similar production during his March 5th teaching episode with Matthew) that the piece would measure "1/4", but his goals and his explanations since his previous production had changed.

Protocol 11: *Novel uses of partitioning*.

J: [just after measuring "1/4"] One-fourth… Now, if that wasn't part of the *ruler*, if it was part of the line, it would probably be one-half… if that wasn't there [dragging the unbroken half away]. See, if that was just a bar [circling the cursor around the two remaining pieces (fourths)].

T: Okay, but we're trying to find out if it's half of the ruler, right? It's always comparing to the ruler. So, is the piece you clicked on one-half of the ruler?

J: Uh-Uh. It's one-fourth.

Josh's actions described in Protocol 11 are reminiscent of his actions in two previous segments. The first was one with Matthew on March 5^{th}, during which Josh formulated his partitioning conjecture (Conjecture J1). Although the production was similar, Josh's use and interpretation of the production were more in line with his reasoning in forming Conjecture J2 during Protocol 2. During that protocol, Josh had ignored the ruler in treating two-thirds as two-halves. In the present case, Josh was treating two-fourths as two-halves by ignoring the other half of the ruler. This time, however, Josh was explicitly aware of the new whole that he had used and the relative sizes of the pieces to each of the original whole (the ruler) and the new whole. His flexibility in considering two different measures depending on two different wholes indicates a development in partitive reasoning that he seemed to lack before. Otherwise, his attempt at producing one-half did not involve a particularly novel way of acting.

Sierra's first attempts were not distinctly novel either, but they do help to elaborate on her previous uses of partitioning and her existing fractional concepts.

Protocol 11: (Cont.)

T: Sierra, can you think of a way to use a different number of parts, besides two, to make a half?

J: [makes a new copy of the ruler and erases the two one-fourth pieces, leaving only the one-half piece and the new copy of the ruler on the screen]

S: [partitions the new copy into three parts, pulls out the middle part and measures "1/3"]

T: Okay. Now it says a third.

S: Mm-hmm.

T: Does that surprise y'all?

S: [shakes her head "no"]

T: Can you make a half, though, using three parts?

S: [excitedly grabs the mouse] Umm… [partitions the pulled third into two parts, breaks it and measures one of them as "1/6].

T: Do you know why it said one-sixth?

S: I think so.

T: Can you explain to Josh?

S: Okay. Umm… It's like two goes into each one and so there's six of them.

J: So, it's like two boxes in there [pointing to the rightmost third in the three-thirds stick]. One, two, three, four, five six [pointing twice within each third].

T: That's a good job guys! All right. But I want y'all to try to make a *half* with a different number of parts, besides two.

S: Is there a way?

J: [begins making a new bar]

Sierra's initial action of pulling and measuring the middle third indicates that her concept of one-half was based on a place in the middle of the stick. She may not have been focused on sizes of fractions at all, but considered only the marks of partitioning. This is in synch with my previous observation (from her initial interview) that she had used a geometric operation of reflection in considering one-half of a stick. After measuring "1/3," she actually partitioned that middle third into two parts, creating a mark in the middle of it.

The novelty of Sierra's actions occurred when she was challenged to explain the measure of "1/6." She was able to mentally partition each of the thirds in the three-thirds stick into two parts. This action fit the operations of a *unit fractional composition scheme* with which she might distribute a partition of a unit fraction across all of the units making up the whole. Josh had performed a similar operation in forming Conjecture J1. Sierra's **Conjecture S3** was that the one-sixth measure could be explained by producing six parts in the whole, each the size of the one she measured, and that she could use the one-third part that she had already partitioned (into sixths) as a template in the other one-third parts. As such, her conjecture was an abduction whose uncertainty was indicated by her language: "I think so." Obviously, it might lead to a unit fractional composition scheme, if she could coordinate the units of units to anticipate the fractional sizes of the smallest parts.

Of course, Josh had been able to assimilate Sierra's actions using his operation for composing units of units, but Josh's novelty occurred when he resumed his attempts at the end of the protocol: He made a copy of the ruler, broke it into ten parts, joined five of them, and measured the result as "1/2". His choice of using ten parts and BREAK indicate that he had been using whole number reasoning, such as *five is half of ten*. Had he been considering fractional size, he would have cut or pulled parts so that he could find half of the ten-part stick. His use of ten is significant because it was a familiar number in his whole number reasoning. This was the first time either student had used whole number, multiplicative reasoning with the goal of producing a commensurate fraction. Although his action was novel, there was no indication of a fractional operation being used, nor was there uncertainty in his use of whole number reasoning.

In previous episodes, the students had used whole number reasoning only to explain the numerators and denominators in surprising measures. It is interesting that Sierra, who had used whole number reasoning in the latter context, could not generate a fraction at all ("Is there a way?"). This might suggest that Josh's fractional operations were significant in his application of whole number reasoning, but I can't say how other than to note that he was capable of iterating a unit of five units. Anyway, Sierra was able to assimilate Josh's actions and independently complete similar productions of one-half, such as six-twelfths. In those cases, Sierra did appear to use multiplicative reasoning, saying things like, "six is half of twelve." It seems that this assimilation was based the way of operating that Sierra used in Conjecture S2, which now seems as powerful as the procedural scheme for producing fractions commensurate with one-half that Will had constructed.

<u>Sierra's conjectural use of units coordination</u>. Moving on, I asked the students to produce fractions that would measure one-third. Sierra began by producing the canonical example using three parts. Josh then produced and cut off the leftmost two parts of a six-sixths stick. I was more impressed with this production until Josh admitted that he had

remembered the measure of two-sixths from a previous task. Protocol 12 begins with Josh's explanation and continues with a powerful conjecture by Sierra.

Protocol 12: *Producing a fraction commensurate with one-third.*

T: What's the name of that one, Josh?

J: Six out of two.

T: Six out of two?

J: Two out of six.

T: So, two out of six is the same as…

J: Uh… It's one-third.

O: How did you decide that that's what you wanted to do?

J: 'Cause somebody did that last time. [Josh points out to Sierra, who had picked up the mouse, that the stick she was about to partition was not a copy of the ruler.]

S: [Sierra makes a new copy of the ruler and partitions it into nine parts, and immediately drags the one-third stick (still visible on the screen from her first production) above the left side of the nine-ninths stick. She then cuts off three parts, where the one-third stick ended, smiling all the while.]

T: That's good, Sierra. I like the way you did that. Can you explain what she did, Josh?

J: Mm-hmm. She took one third and measured it with three of them.

T: Ah! Okay. So, she knew she was going to get a third that way. And what's the name of what she made?

J: Nine. Nine-thirds? *Three*-thirds… Three-ninths!

O: [to Sierra] How did you know to make ninths?

S: They're all going to be even, so I just took a stick and measured it. Like all the lines are going to be matched up. So, they're going to have to be even with the little pieces.

T: So, how did you know that nine parts was going to allow you to make them even?

S: 'Cause 3 times 3 is 9.

First of all, I note that Josh's language difficulties are still prevalent in this most recent protocol. It will be noteworthy, as well, when he is able to resolve these fractional, linguistic issues once and for all. At that point, I will be better able to judge the root of them. Unambiguous use of fraction language appears to be lagging far behind Josh's operational development.

Sierra was consistent in her use of fractional language, but seemed to be using her part-whole partitioning scheme alone in naming fractions. It is unclear whether she had yet constructed even a part-whole fractional scheme because she did not appear to compare parts back to the whole. Nonetheless, Sierra's actions in Protocol 12 are powerful because they indicate that she had independently begun to consider fractional parts relative to one another. The operations driving her actions were certainly conjectural, and her excitement at the novelty in acting them out showed in her smile.

Sierra's conjecture consisted of two phases. First, she had to decide what number of parts to use. Second, she had to decide how many of those parts she needed in order to make one-third. She used nine parts because 3 times 3 is 9 and because she and Josh had been using such whole number multiplication knowledge in their previous productions of one-half and one-third. Even with that knowledge, she did not know how many of the ninths would be needed to constitute one-third. From previous productions, Sierra had noticed the various numbers of partitions within the various fractions commensurate with one-half or one-third. Even with these two observations together, Sierra would not be able to connect them and independently produce three-ninths without the crucial operational potential introduced by Conjecture S3.

Conjecture S3 had opened the possibility for Sierra to coordinate partitions within a part being distributed across a whole: units coordinating of connected numbers. In the conjecture under present consideration, **Conjecture S4**, Sierra's novel way of operating involved assimilating the observations described above by coordinating three units of three in the connected number, 9. She conjectured that she could partition the whole into three equal parts by using nine parts. Whole number units coordination (available within her explicitly nested number sequence) established the necessary relationship between the number of parts in the whole and the potential for using some of those parts to constitute one-third. Her action of comparing the existing one-third stick to the nine-part stick helped her to test or determine that three parts from nine established one-third. Her actions also provide strong indication for a part-whole fractional scheme. I hypothesize that Sierra had constructed this scheme as the result of a functional accommodation of her part-whole partitioning scheme, an accommodation occurring as a result of Conjecture S4. Corroboration of this hypothesis can be found toward the end of my analysis of the March 31st teaching episode.

As the students resumed making fractions commensurate with one-third, Josh acted much as he had been, but Sierra never again used one-third to measure off how much she needed to cut. Instead, she and Josh seemed to have invented whole number multiplication rules for determining the number of parts to pull, as they had for one-half. Josh, however, was the only student to justify his productions by referring to iterations of the composite fraction. For example, after pulling ten parts from a thirty-thirtieths stick, Josh argued that the fraction was one-third because it would go into the whole three times: "ten, twenty, thirty." His actions throughout the episode not only indicate the kind of iteration involved in a partitive unit fractional scheme, but that of a partitive unit fractional scheme for composite units, which includes coordinating the units of units making up the fractional parts of the whole. The schemes seemed to be well-established for him at this point. There is no solid indication that Sierra could iterate fractions at all.

Testing the boundaries. To test the boundaries of Josh's schemes and the flexibility of Sierra's new way of operating, I decided to complicate the tasks by asking the students to make fractions commensurate with two-thirds. Josh began by partitioning a copy of the ruler into three parts, cutting off one of them, and measuring the remaining two parts. These actions indicate only a part-whole fractional scheme.

Sierra attempted the next fraction, with Josh's two-thirds bar still visible at the top of the screen. She made a copy of the ruler and thought for a few moments before partitioning it into six parts. I presume that she chose 6 because 2 times 3 is 6, and she thought this was a plausible choice because she was trying to produce two-thirds. Anyway, she pulled three of the six parts, but realized this would not work even before measuring it, and she erased it. Perhaps she recognized it as one-half of the whole. Next, she pulled two parts and actually measured these as "1/3," even though she had recently produced and measured an identical stick. At that point, Josh began pointing to the two-thirds stick that he had made and mumbled something unintelligible. His pointing actions seemed to direct Sierra's attention to the size of the two-thirds stick because she then made a cut on the six-sixths stick that was below the right end of the two-thirds stick; the two sticks had been nearly aligned on the left ends. Sierra admitted that she had used the two-thirds stick to visualize the place to cut on the six-sixths stick.

Once again, Sierra had used the length of one stick to mark off the length of another. Her early trials indicate that she had not constructed a general *fractional composition scheme*, nor a partitive fractional scheme. Instead she seemed to use a part-whole fractional scheme and whole number multiplication (choosing 6 because it is a multiple of 2) in the novel manner she had learned as a result of Conjecture S4. She was not able to coordinate the units of units to determine the number of sticks needed in the non-unit fraction, as she would with a commensurate fractional scheme or a fractional composition scheme. But she seems to have constructed a part-whole fractional scheme, which had become quite useful in resolving fractional composition and commensurate fractional situations. We will see its limitations in Protocol 13.

Protocol 13: *Josh's conjecture in producing two-thirds from nine parts*.

J: [partitions a copy of the ruler into nine parts, pulls out three parts, and measures them as "1/3"]

T: One-third. What did you think that was going to say, Sierra? Did you think he had it?

S: Uh… I don't know.

J: There's one-third [dragging the three-part stick over the leftmost three parts in the nine-ninths stick] and then you have another one-third [dragging the three-part stick over the middle three parts in the nine-ninths stick].

S: Oh! He could cut seven out of it…

J: I'm going to pull *six* out of it [pulls the leftmost six parts].

T: So, six or seven. [to Josh] Don't measure yet. [to Sierra] So, why did you think seven?

S: Because it leaves two left over, and with the last one there was two left over.

T: Oh! Okay. So, with the last one there were two left over? This one [pointing to the four-sixths stick that was still visible on the screen] you mean?

S: Yeah.

T: Okay. So, with this one if you had two left over, maybe it would work. That's what she was thinking. What did you think, Josh?

J: 'Cause, see, I had that one equal to one-third and if I put one more it would be six. So, it may be one-hal… I mean, it may be *two*-thirds.

Josh probably chose nine parts for reasons similar to Sierra's selection of six parts in the previous production (6 and 9 are multiples of 3). The fact that he pulled and measured only three of those parts indicates that, while he may have constructed a partitive unit fractional scheme, he had not yet constructed the more general partitive fractional scheme. The latter scheme may progress as a result of his actions later in the protocol.

Josh iterated the three-part stick twice within the whole with the goal of producing two-thirds. He had been initially uncertain about how to make two-thirds, and his use of iteration in the situation was conjectural (**Conjecture J4**) as indicated by his confusion in naming the fraction. He conjectured that if three parts made one-third, six parts would make two-thirds.

Having iterated the composite fraction, he could then identify the number of ninths necessary to constitute two-thirds. This indicates that his conjecture involved the use of the operations involved in his partitive unit fractional scheme for composite units. In particular, he used uniting, iterating, partitioning, and disembedding operations to construct a new concept of two-thirds. Through his conjectural operation of iteration, two-thirds could be conceived of as two of one-third, where three-parts were united as one-third embedded in the three-thirds making up the whole. I hypothesize that his conjecture would yield a partitive concept of two-thirds, which might, in turn, engender a partitive fractional scheme.

Sierra’s claim that two-thirds could be created from seven of the parts was based on a generalized procedure abstracted from her recent experiences in making two-thirds. She noticed that she had cut off two parts in producing four-sixths, and thought the same procedure might work again. Such claims do not involve iterative, multiplicative, or fractional reasoning at all; they are additive and based on whole number reasoning. Sierra seemed to turn to this kind of reasoning and these sorts of procedures when her available fractional operations may have been insufficient for assimilating the situation.

31 March, 2003 Teaching Experiment

Early in the episode there was corroboration for the models I had built for the students' reasoning and operations so far. After the students had produced one-fourth with various numbers of parts, I had challenged the students to produce two-fourths. Having pulled two parts from a four-fourths stick, Josh predicted that the stick would measure one-half and Sierra agreed, pointing to the center partition in the four-fourths stick and noting that the whole was "exactly in two parts." She went on to attempt to produce two-fourths from an eight-part stick by pulling three and then four parts, measuring each. Next, Josh pulled two parts from either end of the eight-part stick and joined them.

Whereas Sierra's justification for the measure of the students' initial attempt corroborated my claim that her fractional reasoning included concepts of geometric reflection about partitions, her later decision to use eight parts corroborates that she was using whole number multiplication in her productions in the previous teaching episode: She might choose eight parts to make two-fourths because 2 times 4 is 8. Josh's construction of partitive fractions is corroborated by his attempts to make two-fourths by making two copies of one-fourth.

Josh's construction of partitive fractions was further corroborated, just after the students' attempts at producing two-fourths, when I challenged him to make three-fourths. He partitioned a copy of the ruler into sixteen parts, dragged one of the previously produced one-fourth sticks to the right side of the sixteen-part stick, and cut off the rightmost four parts from the latter. He immediately measured the remaining twelve parts as "3/4." Rather than representing a new conjecture (his confidence and certainty were manifested in his decision to use twelve parts and not four), Josh's actions corroborate my hypothesis from my analysis of Protocol 13 (March 28th) that Josh was progressing in his construction of a new way of operating based on Conjecture J4. The result might be a (general) *partitive fractional scheme for composite units*. After all, he could use a composite unit to recognize the complement of a unit of composite units as the desired fraction. Sierra's struggles (which immediately followed) to assimilate and emulate Josh's actions are described in Protocol 14.

Protocol 14: *Josh's partitive actions and Sierra's attempts to assimilate them.*

T: [to Josh] Can you explain that?

J: Because sixteen… You can put four of these little line things in there [dragging the same four-part stick he had used before across each of the four groups of four parts making up the sixteen-part stick], and it will equal up to one-fourth.

T: Do you see what he did on that one, Sierra? Do you think you can do one like that?

S: [makes three-fourths by pulling three parts from a four-part stick]

T: [presuming that Sierra could complete the previous production using only her part-whole fractional scheme] All right. I want you to do one more, Sierra. See if you can come up with a different way to make three-fourths.

S: [partitions a copy of the ruler into twelve parts, then pulls and measures a three-part stick and then a four-part stick]

It is apparent throughout Protocol 14 that the students were operating in distinctly different ways. Josh had been able to iterate a composite unit within the partitioned whole in order to justify its measure. Sierra seemed to rely on whole number reasoning, choosing twelve parts because 3 times 4 is twelve, and choosing to pull three or four parts to make three-fourths. She appeared unable to assimilate Josh's actions using her own fractional operations and instead had focused on the numerator and denominator of the fraction.

New constructs. The segments of the teaching episode described thus far only corroborate existing models of the students' schemes, operations, and concepts. As Josh began the next production of three-fourths, I noticed new constructions. Josh made the fraction by partitioning a copy of the ruler into eight parts and pulling six of those parts. It was the first time he had produced a fraction commensurate with a non-unit fraction, without first establishing a fraction commensurate with one of its units. For example, in his previous production of three-fourths, Josh had used four-sixteenths to mark off one-fourth of the sixteen-part whole. Regrettably, I did not ask him to explain how he was able to complete his most recent production without intermediate productions. It may be that he made use of the previous three-fourths productions that were still in his visual field or that he had formed some kind of algorithm using whole number reasoning, like *3 times 2 is 6* and *4 times 2 is 8*. Otherwise, he might have mentally partitioned each of the four-fourths, in which three-fourths was embedded, into two parts. Because I did not ask, I have no solid indication that the latter was the case. If it were, it would certainly be conjectural, involving a novel use of fractional composition and recursive partitioning.

Sierra attempted to emulate Josh's actions by partitioning a copy of the ruler into nine parts and pulling seven of them. Although she was unable to explain why she did this, I suspect that she chose nine parts because it was a multiple of 3, and she chose seven parts because it was two parts fewer than the nine parts making up the whole. Recall that she had used a similar procedure for producing two-thirds in the last episode. Josh knew that nine parts would not work and Sierra elaborated, noting that "4 won't go into it but 3 will." Her explanation was an abduction based on her experiences in the two most recent episodes. In these episodes, she had been successful at producing fractions commensurate with m/n by using m·n parts. After observing that her most recent production measured "7/9" and not three-fourths, Sierra explained the surprise based on the distinction that I have described between her unsuccessful production and previous successful ones. Having eliminated the surprise, she did not question her procedure for using two fewer parts than the number of parts in the whole.

Since Josh had used twelve parts, Sierra continued her attempts to produce three-fourths by using twenty-four parts (24 is divisible by 3 and 4). She pulled five connected parts and carefully dragged them five times in the twenty-four-part whole: once in each group of five-parts starting on the left side of the whole, and once more extending just beyond the right side of the whole. She then erased the five-part stick and pulled six parts from the whole. She dragged the six-part stick across the whole too, but in an erratic and imprecise manner. Finally, she lined up the six-part stick with the left side of the whole, used it to measure and cut off the leftmost six-parts in the twenty-four-part whole, and measured the remaining eighteen-part stick as "3/4."

This was the first time I had observed Sierra dragging a fraction stick across a whole. I cannot be certain that she was keeping track of the number of times the five-part stick went into the whole, but she seemed to be, at least, determining whether it would go into the whole evenly. Her actions resembled those that she acted out during Protocol 12, in which I identified Conjecture S4. In forming that conjecture, she had used units coordination to produce a connected number, 9: “They’re all going to be even, so I just took a stick and measured it. Like all the lines are going to be matched up. So, they’re going to have to be even with the little pieces.” She could establish partitions within a partition and recognized the importance of the smaller units evenly dividing up the whole into larger units of units. She also seemed to have constructed a part-whole fractional scheme as a result of that way of operating, particularly in comparing a fraction back to the whole and considering its complement. I had hypothesized (toward the end o the March 28th teaching episode) that Sierra’s operations in forming Conjecture S4 would serve to accommodate her part-whole partitioning scheme into a part-whole fractional scheme.

Sierra’s action of dragging the six-part stick was too inaccurate to be useful in determining how many times the stick would go into the whole. Her action may have been half-hearted because she had already recognized a telling multiplicative relationship between six and twenty-four: 6 goes into 24 four times. That relationship provided enough reason to carry on with the rest of her procedure, cutting off six parts from the whole and measuring the remaining parts. I refer to these final actions as procedural because she had acted similarly in producing two-thirds and three-fourths from various numbers of parts. But in such cases, she never seemed to be acting independently, based on her fractional operations. Rather, she had assimilated Josh's actions with a procedure for acting.

My hypothesis (early in my analysis of the March 18th teaching episode) that Sierra had not been iterating as Josh might in using a partitive unit fractional scheme is corroborated by Sierra's actions toward the end of the episode. Josh had challenged her to produce two-fourths using twelve parts. She pulled three of them and then four of them, each time dragging the parts carefully across the length of the whole, as if to mark the whole into four and three sections, respectively. She had just completed the action with the four-part stick when the observer asked her, "What fraction is that?" Sierra responded by counting parts and saying, "I think it's four-twelfths." Had she been keeping track of her iterations within the whole, it is unlikely that she would have responded in such a way instead of answering that the fraction was one-third. This indicates that Sierra had been using the dragging action to determine the evenness of her fraction composition, rather than to act out iterations. Regardless, I maintain that she might construct iteration as a fractional operation through reflective abstraction upon her dragging actions.

14 April, 2003 Teaching Episode

The students had been on spring break for a week, so this was our first meeting in two weeks. Much of the teaching episode was spent on tasks that were designed to determine what reasoning they had retained (the permanence of previous constructions) and to elicit partitive fractions. The tasks designed to elicit partitive fractions were so simple that there was no notable conjecturing activity until the end of the episode. So, mostly, I use this account of the episode to refine my models of the students' operations surrounding the construction of partitive fractions. In particular, the episode includes indication that Sierra had constructed part-whole fractions but not partitive fractions. Until she could iterate unit fractions, I would not attribute to her a partitive unit fractional scheme. The episode also includes indication that Josh had constructed a partitive fractional scheme with which to estimate the relative sizes of non-unit fractions. However, Protocol 15 describes Josh's difficulty with some aspects of comparing partitive fractions. I begin here with support for my general claims and then examine the students' interesting actions in that protocol.

At the beginning of the episode, Josh and Sierra chose numbers: 22 and 27, respectively. I assigned to each of them the fraction created by the reciprocal of their number and asked them whose fraction would be larger. Both students knew that one-twenty-seventh would be smaller, and Sierra justified this by saying, "his lines would be separated more so it would make his lines longer, and mine would be pushed together." She seemed to have constructed an inverse relation between the number of parts used in a whole and the size of each part. Her explicit reference to such a relation indicates a part-whole fractional conception of unit fractions that surpassed the part-whole partitioning conception upon which she typically relied. She could have established the relation by reflecting on her previous experiences in producing fractions; operationally, the conception would require a partitioning operation, but not necessarily an iterating operation, which would be essential to her construction of a partitive unit fractional scheme.

There were at least two segments in the episode during which Josh appeared to be using a partitive unit fractional scheme. The first occurred in response to the tasks designed to elicit partitive fractions. Sierra had just made eight twenty-sevenths, and I asked Josh to produce a fraction using twenty-seconds that would be just a little bigger than Sierra's fraction. He immediately decided to use eight of the twenty-seconds. The immediacy of his decision to use the same number of parts Sierra had used indicates that he understood the context in which eight units of one size might be greater than eight units of another size. The second occurrence, toward the end of the episode, was in response to Sierra's challenge of determining how many fiftieths she had used in making a wiped fraction. Josh had closed his eyes as Sierra pulled twenty fiftieths. No fractions were visible on the screen when Josh opened his eyes, except for the fraction in question and the ruler. Once again, Josh immediately responded, claiming, "I think she pulled twenty of them." Combined, the occurrences demonstrate Josh's ability to consider the sizes of non-unit fractions relative to each other and the whole. His use of fractional language encompassed my only lingering concern in attributing a partitive fractional scheme to Josh, but from this episode forward, I never observed him reverse the numerator and denominator in a fraction as he had in previous episodes.

The interesting protocol mentioned above occurred toward the end of the partitive fraction activity, during which the students alternated turns in determining how many twenty-seconds or twenty-sevenths they would need to pull in order to make a fraction just longer than the preceding fraction. Protocol 15 describes the students' responses to my final question of the activity.

Protocol 15: *Comparing fractions just less than the whole.*

T: If we kept playing this game all the way through, what's the last one that you would have before you had the whole thing, Josh?

J: I'd have twenty-two… Well, no [long pause].

T: You know what I mean? What could you make that's just a little smaller than the whole stick, using your pieces?

J: Twenty-one out of twenty-two.

T: Sierra, what would you have?

S: Twenty-six twenty-sevenths.

T: Now, whose would be longer, twenty-one twenty-seconds or twenty-six twenty-sevenths?

S: Twenty-one twenty-seconds.

J: Mm-hmm.

T: I want ya'll to talk about it together and tell me why you think that's true.

J: 'Cause these are a little bit longer than those [pointing to the unit fractions].

In determining which fraction was bigger, the students only considered the sizes of the unit fractions of which the non-unit fractions were composed; after all, this consideration had been sufficient in solving the task given at the beginning of the episode. So, I encouraged the students to make each non-unit fraction stick and compare their sizes. They produced the two fraction sticks by cutting off one piece from each partitioned copy of the ruler, and lining up their left sides. Once they had done this, the dialogue resumed.

Protocol 15: (Cont.)

J: Hers is a tad bit longer.

T: Can ya'll figure out why?

S: [to Josh] You have less than I do. [After a long pause, Sierra drags one twenty-second and one twenty-seventh (from previous productions) to the ends of the respective fractions the students were comparing, thus completing each whole.]

J: You have like a half of one longer than me. [removing the pieces Sierra had dragged] She has twenty-six, and I only have twenty-one, but mine are a little bit bigger than hers, but she has like half of one bigger than mine.

For the first time, Josh seemed to consider the dichotomy between the numbers in the numerator and the numbers in the denominator in determining the relative sizes of two fractions. Josh's struggles to resolve this dichotomy might explain his use of the conjunctive "but" twice in the same sentence. This dichotomy would be difficult to resolve, even with a partitive fractional scheme, unless he considered the complements of the two fractions, as Sierra seemed to do.

Sierra had rebuilt the two wholes by adjoining the two unit fractions that were the respective complements of the non-unit fractions in question. She was capable of determining the complements of the two non-unit fractions by using her part-whole fractional scheme and disembedding operation, even without first producing the two fractions to use as perceptual material. Once she determined the unit fraction complements, her part-whole fractional conceptions of them would be sufficient to compare them and thus determine which non-unit fraction was bigger. However, Sierra did not consider the unit fraction complements until after the students had produced the non-unit fractions, and, even then, she did not seem to have the described operations organized in a scheme for resolving the situation.

I refer to Sierra's actions of adjoining the two unit fractions as conjectural because they represented a novel use of her disembedding operation with consequences that will become clearer in the next episode. In comparing the lengths of the twenty-one twenty-seconds stick and the twenty-six twenty-sevenths stick, and in searching for a reason why one was longer than the other, Sierra seemed to notice that their right sides would be made even by adjoining the two unit fractions. She conjectured (**Conjecture S5**) that she could reproduce the whole, in which the two non-unit fractions were embedded, by adjoining their complements and that the unit fractional complements could be used to justify the difference in the sizes of the non-unit fractions. She was not able to explicitly justify her action or determine how it might be useful, but the action itself helped to resolve the peculiarity of the fractions' misalignment.

16 April, 2003 Teaching Episode

The present teaching episode began much as the previous one did and revealed even more about the differences between the students' fractional schemes and operations. Later in the episode, we will see how each student used operations to form a powerful conjecture. Given the differences between the students existing schemes and operations at this point, the students' respective conjectures will be interesting to compare in Chapter 9.

Elaborations on available structures. At the beginning of the episode, I asked the students to think quietly for several seconds about which was larger, two-fourths or one-half. Within five seconds, Josh claimed to know, but I asked Sierra to respond first. She claimed that one-half was bigger, after which Josh responded, “they’re both a half.” He explained that if you took two one-fourth pieces and put them together, they would form one-half. His response indicates that he had understood two-fourths as a partitive fraction and corroborates my hypotheses (in my analysis of Conjectures J2a and J4) that he had been constructing a partitive fractional scheme, although it is difficult to attribute the development of partitive fractions to a single conjecture or experience.

In a previous episode (March 31st), both students had established the commensurability of two-fourths and one-half, but Sierra appeared unable to reason about the sizes of non-unit fractions in the absence of perceptual material. Instead, she might have assumed that one-half was bigger because two-fourths was divided into more pieces. In fact, she used such reasoning to justify a subsequent claim that one-fourth was bigger than one-fifth. So, Sierra had constructed part-whole fractions, but still appeared to lack an iteration operation for unit fractions that could be used to establish partitive unit fractions.

Conjecture S5 revisited. I asked the students to consider which would be bigger, twenty-four twenty-fifths or thirty-five thirty-sixths. This question was similar to one the students had considered in Protocol 15. Even though the students explicitly recognized the similarity between the two problems, they agreed that twenty-four twenty-fifths would be bigger. I asked the students to make the two fractions to compare them, but, because they had previously claimed that one twenty-fifth would be bigger than one thirty-sixth, I asked them to check this first. Josh produced the unit fractions from partitioned wholes and compared them, affirming the students’ previous claims. Then, Sierra proceeded to make the non-unit fractions by cutting off one piece from each of the partitioned wholes. When she had completed the production and lined them up to compare, both students realized that thirty-five thirty-sixths was bigger. The dialogue described in Protocol 16 began just after each student expressed his or her realization.

Protocol 16: *Sierra's conjectural use of complements*.

T: Why do ya'll think that is?

S: Because the twenty-fifth one… the things are going to be bigger. So, that's going to take away more, but, if they're smaller, that's going to take away less, so it's not going to be… [pauses]

T: Can you show me what you mean by "take away more" and "take away less?"

S: 'Cause if you take [cuts off the rightmost three parts in the twenty-four twenty-fifths stick] and you cut three, and then you cut three off of this [the thirty-five thirty-sixths stick], then that's just going to be less. But there's going to be more of the little ones.

T: [after Sierra explained her idea again, to Josh] Does that make sense, Josh?

J: Mm-hmm.

T: Why don't you explain in your own words.

J: These… There's not as many of these [twenty-fifths], but they're longer than these [thirty-sixths]. But there's like, say, three of these [thirty-sixths] equals one of those [twenty-fifths], and you've got more of these [thirty-sixths].

Sierra had formed a new way of operating, using her concepts of complements, her disembedding operation, and her part-whole fractional concepts. In cutting off the unit fractions from each partitioned whole, she had become explicitly aware of the implications of her actions in Conjecture S5. Her conjecture from the previous episode seemed to make her operationally aware of the complement embedded in the whole when cutting off pieces. Being aware of the complementary parts of the whole when cutting allowed her to establish a complementary relation between the sizes of the pieces. If the cut off piece were smaller, the complement would be bigger, and vice versa.

From his statements at the end of the protocol, it is evident that Josh had been unable to assimilate Sierra's actions in terms of complements. I hypothesize that this was so because he had not performed the cutting action and may not have constructed the kind of operational understanding of such an action that Sierra had constructed. He was still struggling with the dichotomy between the number of parts and their respective sizes, as noted in the previous episode. His assimilation of Sierra's actions seemed to rely on uniting and composition operations, attempting to form units of the larger parts from units of the smaller parts.

<u>A new use of Josh's composition and units coordinating</u>. Later in the episode, the students began playing a game in which one student would pull a number of parts from a partitioned whole, while the other hid his eyes. The problem poser would then wipe the fraction stick formed from the pulled parts and hide the partitioned whole. In this case, when Josh opened his eyes, he was told that Sierra had partitioned the whole into sixty-three parts, and he was supposed to determine the number of parts in the wiped fraction stick (that stick and the ruler were the only sticks visible on the screen).

Sierra had used fourteen of the sixty-three parts in challenging Josh. Upon opening his eyes, Josh responded to the challenge by lining up the fraction stick above the left side of the ruler. He then sat looking at the screen for ten seconds, before the following dialogue began.

Protocol 17: *Josh's conjectural use of fractional composition*.

J: I'm going to say fifteen.

T: How'd you do figure that?

J: I used 25 'cause… If that was 25 [pointing to the fraction in question], it would give you 50 right there [pointing to a mark that would be one iteration to the right of the fraction, and about half way across the ruler], and it would be too big [moving his finger across the right half of the ruler]. That [pointing to the fraction again] would be too small for 25. So, I used 15 for my number.

Josh posited hypothetical parts within the fraction and tested them against the whole through iteration and units coordination, double-counting the iterations and the hypothetical parts. Moreover, he was able to adjust the hypothetical number of parts in the composite unit to account for the number of times the fraction would iterate within the whole. This substantiates a novel and fascinating use of Josh's composition operation. He used fractional composition to double-count a hypothetical composite unit (keeping track of the number of times the fraction iterated into the whole while counting the number of hypothetical parts within the iterations) and to appropriately adjust the hypothetical composition based on his iterations and double-counting. So, it seems that Josh's use of iteration and units coordination were integrated quite flexibly, corroborating my claims that he had constructed the operations of a partitive fractional scheme for composite units and a commensurate fractional scheme. Josh's actions in Protocol 17 also indicate that Josh had constructed operations similar to those of a reversible unit fractional scheme: Instead of positing the number of parts needed within a given unit fraction in order to generate a specified number of parts in the whole, he could estimate the number of parts needed without first establishing the size of the unit fraction through iteration.

However, there is no indication that Josh's actions were conjectural. His initial response, "fifteen," was an estimate, and the pause that preceded his estimate can be attributed to his calculating of the estimate, operating as I have described. Rather, his powerful use of iteration and units coordination (and the associated schemes mentioned above) corroborate my hypotheses about the power of Conjectures J3 and J4.

25 April, 2003 Teaching Episode

Sierra's complementary fractional comparison scheme. At the beginning of the episode, I posed a task that provided occasion to test the consequences of Conjecture S5. The task was similar to the one described in Protocol 16; I asked the students to determine, with an empty screen, which was larger, seven-eighths or four-fifths. Both students thought that four-fifths would be larger, indicating that Sierra was still unable to act on figurative material and continued to rely on perceptual material instead. Indeed, once Josh had made the two fractions (by cutting off unit fractions from partitioned wholes) and the students had observed that seven-eighths was longer, Sierra was able to argue as she had before: "because his [eighths] are littler, and that's less to take away, and then mine are bigger and that's more to take away." The persistence of her reasoning across three episodes and eleven days indicates that Sierra had constructed a scheme for comparing the lengths of complements of unit fractions. Her scheme was still dependent on perceptual material and involved her disembedding operation in order to conceptualize the additive and complementary relationship between complements of a whole. I refer to it as Sierra's *unit fractional complement comparison scheme*.

Even though Josh had completed the production this time and even though he had done so by cutting off the unit fractions as Sierra had in Protocol 16, he still did not seem to assimilate Sierra's described actions. When I asked him to explain Sierra's reasoning, he made an argument very similar to the one he had made during the protocol held nine days before: "Mine are littler than hers and hers are bigger than mine, but I have more than her…" I had hypothesized that Josh had not assimilated the situation using complements in Protocol 16 because he had not completed the production, cutting off the unit fractions. In the present case, he had done this and still seemed to ignore Sierra's references to pieces that were *less to take away* or *more to take away*. So, my hypothesis is refuted, and I am left to assume that Josh had no scheme compatible with Sierra's unit fractional complement comparison scheme, although Josh's demonstrated ability to recognize complements embedded in a whole indicate that the scheme was in his zone of potential construction. I have also noted instances in which Josh seemed to lose track of the whole after cutting off a piece, and his ambiguity in referencing the whole may account for the disparity between Josh's potential structure and the actual one.

<u>Josh's splitting conjecture and Sierra's assimilation of it</u>. I have provided evidence indicating that Josh had constructed a partitive unit fractional scheme for composite units (if not the more general scheme), and that he had been able to posit hypothetical units within a unit that he was iterating (Protocol 17). Figure 15 and

Protocol 18 provide indication that Josh could conjecturally use a similar way of operating in order to determine fractional sizes of unknown non-unit fraction sticks.

I had been posing problems to the students, challenging them to determine the sizes of various *unpartitioned* fractions in the absence of any figurative material except for the fraction stick in question and the ruler. I began with a unit fraction, which Josh was able to determine to be one-sixth by repeating the fraction six times along the top of the ruler. Next, I posed two-fifths and asked Sierra to take control of the mouse. She made a copy of the fraction and repeated it three times along the ruler, but seemed unable to proceed after observing that the three parts went beyond the ruler. She released the mouse, and Josh immediately picked it up and cut off the piece that extended beyond the ruler. He then iterated it just beyond the ruler, cut off that piece and continued, recursively, until he had created the image that is represented by Figure 15. I describe the rest of the students' actions concerning the image in Protocol 18.

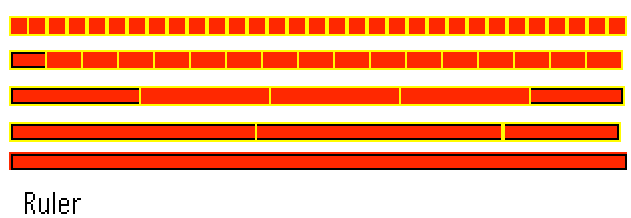

Figure 15. The results of Josh's recursive actions.

Protocol 18: *Using the results of Josh's recursive process.*

J: [after counting the small parts on the top stick illustrated in Figure 15] Maybe thirty-one bars.

S: [drags the original fraction above the left side of the thirty-one part stick]

T: And how much is this fraction [pointing to the original fraction]?

J: Um… [leans in, apparently counting parts]

S: [cuts off the thirty-one part stick at a mark that is at the right end of the original fraction]

J: [uses the mouse to count the number of parts cut off] Twelve.

T: So, what's the fraction?

J: Twelve out of thirty-one?

Josh recognized that Sierra's repetitions of the fraction did not fit evenly within the whole, as they would have to in order for him to immediately use any of his partitive fractional schemes. But in Protocol 17, Josh had demonstrated an ability to hypothetically posit units with the fractional unit that he was iterating. Moreover, he was a splitter, meaning that he could compose partitioning and iterating simultaneously. So, Josh could anticipate cutting off the piece of the six-fifths stick (the result of Sierra's repeating action) extending beyond the ruler and positing it as a sub-partition of the original fraction, which could then be iterated. That piece would be an ideal choice to posit because it represented the peculiarity of the situation (that Sierra's repetitions did not fit evenly in the whole).

So, Josh conjectured (**Conjecture J5**) that he could compose the fraction stick and ruler from iterations of the extended piece. He needed to act out the iteration because he was not sure how many times it would fit into the original fraction or the ruler. My guess is that he thought it would go into the fraction stick twice, but was not certain. In fact, upon repeating the piece, he found that it would not fit into the ruler evenly either, but it did not extend quite as far beyond the whole as the original fraction had. Recursively applying his novel ways of operating, Josh produced better and better partitions of the whole until he produced a thirty-one part stick, approximately the same size as the ruler. But in positing the pieces, his initial goal of finding the size of the original fraction was transformed to one of finding a piece that would fit into the ruler evenly and finding out how many times the piece fit.

Once Josh had partitioned the ruler, Sierra was able to assimilate the situation using composition operations as she had during Protocol 12 (Conjecture S4) of the March 28th teaching episode. She cut off a number of parts in the thirty-one part stick that appeared to be the same size as the fraction in question. From that point, the students' part-whole fractional schemes were sufficient for determining the size of the fraction.

20 May, 2003 Teaching Episode

There had been no teaching episodes with Josh and Sierra in nearly a month. I decided to schedule this final one to test for the relative permanence of the students' structures and to determine whether they acted in any manner incompatible with my models of them. Through most of the episode, the students acted predictably: Josh appeared to be operating with partitive fractions, whereas Sierra relied heavily on part-whole fractions. The students' behavior became interesting as I observed Josh's response to a challenge that Sierra posed. Sierra had produced a wiped two-ninths stick and challenged Josh to determine its fractional size with no perceptual material available except for the wiped fraction and the unpartitioned ruler. Protocol 19 documents Josh's response.

Protocol 19: *Josh's recursive fraction composition*.

J: [makes a few copies of the fraction and lines one of them along the top of the left side of the ruler]

O: Do you have an initial guess?

J: Is it over ten [referring to the number of parts Sierra had used in her production]?

T: It's not over ten.

J: I think nine something [lines up four copies of the fraction along the top of the ruler, extending from the left side of the ruler almost to the right end].

T: What's your specific fraction?

J: Two-ninths, maybe.

Josh resumed action by dragging the copies of the two-ninths stick away and using one of them to repeat four times along the top of the ruler, extending from the left end of the ruler to just beyond the right end. He cut off the extra piece and repeated it from the left of the ruler until he had almost reached the right end. Unfortunately, the left ends were not perfectly aligned, so he decided to repeat once more and cut off the extra piece. He acted recursively until he had made a twenty-three part stick that was approximately the same size as the ruler. He guessed that the fraction in question was five twenty-thirds, but figured that it must simplify because he knew that Sierra had used ten parts or fewer. He proceeded to iterate the five part stick within the twenty-three part stick, but became frustrated: "I know that there's five of these [twenty-thirds] in one of those [the original fraction], but it's not going to be five because it would be over ten."

Josh's initial (and correct!) guess of two-ninths also indicates that he had constructed a partitive fractional scheme. His actions following Protocol 19 resemble those of Conjecture J5 and indicate the operations of his partitive fractional scheme, which enabled him to guess accurately. He determined that Sierra had used nine parts because, in mentally iterating the fraction, he realized that it would extend beyond the whole on the fifth iteration, and this implied for him that there were a large number of partitions. Once he had determined that she used ninths, he could estimate how many ninths would fit into the fraction. This estimate is a strong indication for partitive fractions in operating with ninths. Because ninths are fairly arbitrary units, my claim that Josh had constructed a partitive fractional scheme is substantiated. This affirms my hypothesis—made during my analysis of Conjectures J2a and J4—that Josh would construct such a scheme, although it is difficult to determine which, if either, conjecture engendered the construction.

Chapter 8: Synthesis of Josh and Sierra's Conjectures

In the case study of Josh and Sierra in Chapter 7, I identified conjectural operations and the conjectures that those operations formed. In the present chapter, I synthesize the conjectures that I identified for Sierra and Josh. I use the analysis presented in Chapter 7 to specify those accommodations that the students made to their schemes as a result of their conjectural operations, and demonstrate how the students' zones of potential construction were actualized through conjectural activity.

Sierra

Conjecture S1. During the last teaching episode in which Sierra worked with Cory (March 10th), Sierra was asked to determine the fraction name for a four-part stick that Cory made while Sierra had her eyes closed. At the time, she had an equi-partitioning scheme, a part-whole partitioning scheme, and possibly a part-whole fractional scheme available, but no partitive schemes. The four-part stick and the unpartitioned ruler were the only visible sticks on the screen when Sierra opened her eyes. She repeated the four-part stick twice and cut off the two parts that extended beyond the length of the ruler, conjecturing that she could reproduce the partitions in the ruler to determine the number of parts Cory had used in the ruler during his production of the four-part stick.

I have hypothesized that Sierra's conjecture involved a novel use of her equi-partitioning scheme. In fact, she may have used her equi-partitioning scheme to assimilate Cory's immediately preceding actions of reproducing the partitions in the ruler by repeating a one-fourth part. If her conjectural operations for Conjecture S1 had involved equi-partitioning, they might enact a functional accommodation of her equi-partitioning scheme, resulting in a scheme that could be used to partition the ruler by progressively integrating parts from it. Indeed, her subsequent actions in Protocols 6 and 8 indicate that she had used the integration of parts (both unit parts and composite parts) to produce partitions in the ruler. In both protocols, she named the number of parts in the ruler rather than the fraction name that she was asked to determine. This indicates that she had not been using iteration in order to determine the fraction name, but was acting with the goal of determining the number of parts in the ruler, corroborating my hypothesis.

Conjecture S2. In attempting to produce a three-halves stick (Protocol 3, March 18th), Josh pulled a three-part stick from a six-sixths stick and measured it as one-half. Sierra had been unable to predict this surprising measure but offered this post-hoc explanation: “because there’s six of them, and 3 plus 3 is 6; so it’s half.” Her explanation was a conjecture that fit the pattern of abduction, explaining the surprising measure of the three-part stick by constructing a general rule about the number of parts in it, its complement, and the ruler.

Operationally, the abductive pattern was one in which Sierra had a goal of constituting the three-part stick as one-half, but had no available scheme to do so. If she were to have a part-whole fractional scheme available to use in the situation, for example, she would have anticipated the measure of the three-part stick by comparing its size to that of the ruler. Instead, she abductively (and conjecturally) used her disembedding operation and whole number addition operation. She used disembedding to project the three-part stick into the six-part ruler and establish its complement as another three-part stick. She used addition to relate the number of parts in the three sticks to her whole number notion of taking one-half of something (taking one share from two equal shares).

On March 28th, between Protocols 11 and 12, Sierra was able to use disembedding and her whole number multiplication operation to *produce* sticks commensurate with one-half, such as six-twelfths. Although she observed Josh produce five-tenths first, Conjecture S2 may have enabled Sierra to assimilate his actions meaningfully in order to purposefully and appropriately act on her own with a procedural scheme for producing fractions commensurate with one-half. Moreover, Conjecture S2 serves as an example of Sierra's propensity for inventing abductive explanations throughout the teaching experiment.

Conjecture S3. During Protocol 11 (March 28th), in attempting to produce one-half of the ruler using three parts, Sierra pulled out one of the three parts (the middle one) and broke it into two pieces. Upon measuring one of those two pieces, Sierra was asked to explain why the piece measured one-sixth, instead of one-half as she had predicted. She explained the surprising measure by conjecturing that it was due to the fact that "two goes into each one and so there's six of them." This was an abduction that explained the surprising measure of the one-sixth piece by identifying six equally sized pieces in the ruler.

During Protocol 3 (March 18th) just before Conjecture S2, Josh had measured another one-sixth piece. Sierra was able to explain that measure by referring to the six parts that were visible in the ruler: "I think it's because there are six little things in there." This was an abduction too, but it did not involve a fractional operation. In forming Conjecture S3, on the other hand, Sierra had to mentally distribute the partition of one of the one-third parts in the ruler across the other two with the goal of producing the six parts. This involved the conjectural operation of units coordinating, and Sierra's actions were similar to those of a unit fractional composition scheme.

In order to construct the unit fractional composition scheme, Sierra would have to be able to anticipate the fractional sizes of the smallest parts, rather than simply explaining the one-sixth measure by producing six parts from three. It is unclear whether she could so that because her goal in forming Conjecture S3 seemed to be to produce six parts rather than a one-sixth part. Conjecture S3 did open the possibility for Sierra to consider partitions within a part being distributed across the whole, which was instrumental in her formation of Conjecture S4.

Conjecture S4. During Protocol 12 (March 28th), Josh and Sierra were producing fractions commensurate with one-third. Sierra produced a one-third stick using three parts; then, Josh produced a two-sixths stick using six parts. With the previously produced sticks still visible on the screen, Sierra partitioned a copy of the ruler into nine parts and used the one-third stick to measure off one-third of the nine-part stick. She explained that she used nine parts because "3 times 3 is 9" and that the lines partitioning the ruler into three parts would be "even with the little pieces" produced by using nine parts. So, her conjecture was that she could evenly partition the whole into thirds by using nine parts.

Sierra's conjecture relied on the conjectural operation of units coordination in the production of a connected number, 9. This operation established the necessary relationship between the number of parts in the ruler and the potential of using some of those parts to constitute one-third of it: The parts in the ruler had to divide evenly into three composite parts that evenly partitioned the ruler. Although one might argue that Sierra had simply used her whole number multiplication operation to guess that nine parts would work, such a simple production would not explain Sierra's immediate use of the one-third part in measuring off the desired fraction, nor would it account for Sierra's own explanation about the lines of partitioning (using three parts and nine parts) being even with each other.

Sierra's use of the one-third stick to measure off three parts from the nine-part stick also marks the first time that I had observed Sierra independently comparing a fraction stick back to the ruler. This was a strong indication of a part-whole fractional scheme. I hypothesized Sierra had constructed that scheme as a functional accommodation of her part-whole partitioning scheme, as a result of Conjecture S4. In fact, I had never observed her comparing fractions back to the ruler before Protocol 12 but observed her making such comparisons again later in the same teaching episode, leading up to Protocol 13.

Conjecture S5. I had asked Josh and Sierra to predict which stick would be longer, a twenty-one twenty-seconds stick or a twenty-six twenty-sevenths stick (Protocol 15, April 14th). The students agreed that twenty-one twenty-seconds would be longer because it would have bigger parts. To test this, the students produced the sticks in question by cutting off one part each from a twenty-two twenty-seconds stick and a twenty-seven twenty-sevenths stick. Observing that the twenty-one twenty-seconds stick was actually the *shorter* one, Sierra sat in quiet thought for several seconds before dragging the cut-off unit fractional sticks back over to their respective complements. In doing so, Sierra rebuilt the whole sticks, conjecturally establishing an inverse relationship between the size of the cut-off unit fractional stick and the size of its complement in order to explain the surprising sizes of the sticks in question (the complements of the unit fractional sticks), relative to one another.

During Protocol 15, Sierra had not been able to explicitly verbalize the inverse relationship that she was conjecturally forming. In fact, during Protocol 16 (April 16th), Sierra was still unable to predict the relative sizes of two sticks that were complements of the unit fractional sticks (twenty-four twenty-fifths and thirty five-thirty sixths). But once the sticks were constructed, Sierra was able to explain the relationship: The twenty-four twenty-fifths stick is smaller "because the parts are going to be bigger, so that's going to take away more; but if the parts are smaller, that's going to take away less." She had become aware of her conjectural operation of disembedding, involved in reproducing the whole sticks from the sticks in question by adjoining the unit fractional sticks.

Because Sierra had established a general rule for determining the longer stick in such situations, her explicit statement was an abduction that explained her initial surprise about the sizes of the stick. Her use of disembedding fit the pattern of abduction at the operational level, using it unawares, at first, to build a link between her insights into the reason for the size difference between the unit fractions and her confusion about the relative sizes of the sticks in question.

Josh

Conjecture J1. I asked Josh to consider the fractional size of a part that Matthew produced when he accidentally partitioned the left half of a two-halves stick into two parts (March 5th). Josh responded that the part would be "one-third, if [the parts] were even." After observing that the stick actually measured one-fourth, Josh formed an abductive conjecture to explain the measure: "[the two fourths] look the same, so you could put one more [partition] in [the other half]" (Protocol 1). His abduction was to explain the surprising measure by imagining one more partition in the (copy of the) ruler that would produce four *even* (equal) parts.

Assimilating the fractional name with his part-whole fractional scheme, Josh understood the necessity of producing four equal parts in the ruler. But in past experience, he had only produced four equal parts in the ruler using his equi-partitioning scheme, and, in Protocol 1, the ruler had already been partitioned into three uneven parts. So, his abduction was formed through a novel use of partitioning, with which he partitioned the large part into two parts that would be equal in size to the other two parts.

Because Josh was a splitter, partitioning and iteration were inverse operations for him. So, in using partitioning as Josh did, he could simultaneously understand that the parts that he produced through partitioning could be iterated in the ruler four times. Such a conception of the parts would substantiate an iterable one-fourth unit. During the ensuing teaching episodes and for the first time, I observed Josh estimating the fractional sizes of pieces (one-fourth, one-third, and one-fifth, respectively) without a partitioned copy of the ruler. Josh seemed to mentally iterate the pieces within the ruler to determine the fractional sizes of the pieces, treating them as iterable units and partitive unit fractions. Alternatively, he may have been using the pieces as templates to produce equal partitions in the ruler. Either way, I hypothesized at the beginning of the March 18th teaching episode, that Josh's novel use of partitioning would engender a novel use of iterating: iterable unit fractions, and a partitive unit fractional scheme. Josh's actions in Protocols 2, 3, 6, 7, and 9 indicate that he had begun constructing partitive unit fractional operations, corroborating my hypothesis. Furthermore, by the April 14th teaching episode, he appeared to be operating consistently with such a scheme.

Conjecture J2a. Attempting to produce three-halves during Protocol 2, Josh partitioned a copy of the ruler into three parts, broke it into three pieces, and immediately dragged one of the pieces below the other two. He conjectured (Conjecture J2a) that his actions had produced three-halves because, using his part-whole fractional scheme, he could conceive of each piece as being one-half of the two pieces that were left together on top of the third piece. His uncertainty about the production was indicated by his wavering between referring to the pieces as being one-half and one-third.

I have hypothesized that Josh's actions and Conjecture J2a were based on a novel use of splitting because he had used partitioning and iteration simultaneously in his production of the pieces. He had used *partitioning* to produce the three pieces, but with the goal of establishing one of them as a half that he could *iterate* three times to make three-halves. Moreover, he was simultaneously able to recognize the piece as being one-third of the ruler and one-half of the other two pieces. While such recognition refuted his conjecture and introduced the constraint of considering fractions relative to the unit bar, it also indicated that Josh had used a splitting operation because the piece contained records of two different wholes.

I further hypothesized that Conjecture J2a, and the problematic situation of considering improper fractions as iterations of unit fractions, would engender partitive fractions and a (general) partitive fractional scheme, beyond a partitive *unit* fractional scheme. But Josh did not seem to construct any non-unit partitive fractions before Conjecture J4. So, I revise my hypothesis and claim that Conjecture J2a contributed to

Josh's continuing development of partitive unit fractions and his construction of a partitive unit fractional scheme. First of all, the new constraint (mentioned above) might contribute to Josh's conception of the importance of explicitly considering the number of iterations of a fractional part in the ruler in determining the part's fractional name. Indeed, Josh's actions over the next two teaching episodes (March 21st and March 24th) indicate that Josh had constructed such a concept, and during Protocol 11 (March 28th), Josh explicitly referenced the sizes of a fraction stick relative to two different wholes. Secondly, Josh's actions in forming and testing Conjecture 2b indicate that Josh had constructed one-half as a partitive unit fraction.

Conjecture J2b. In his second attempt at producing three-halves, Josh broke a copy of the ruler into six parts and joined three of them, apparently conjecturing that he could satisfy the goal of making three-halves by producing one-half using three parts. Conjecture J2b also included Josh's anticipation that the whole would need six parts if three parts were to be like a half. This indicates that Josh could conjecturally treat one-half as a partitive unit fraction, but the 3 in three-halves now seemed to refer to the number of parts making up a unit fraction rather than the number of iterations of one-half or the number of parts that he needed to produce in the ruler.

Once again Josh had used both partitioning and iteration in his production. He partitioned the ruler into six parts with the understanding that three of those parts could be iterated in the ruler twice. His use of the two operations together indicates another conjectural use of splitting. The conjecture resulted in Josh's construction of a fraction commensurate with one-half.

<u>Conjecture J3</u>. Having produced a four-eighteenths stick, Josh and Sierra were surprised to find that it measured two-ninths. I challenged them to explain this during Protocol 9. Josh conjectured that the four-eighteenths stick would fit evenly into the ruler, establishing it as a simpler fraction. He dragged the four-eighteenths stick across the ruler, making marks to keep track of the iterations (as illustrated in Figure 14). When I asked him what he was doing, he referred to a previous experience (that of Conjecture J2b and Protocol 3) in which he had noticed a three-to-one relationship between the parts in a stick and the numerator of its measure: “Remember last time? One of these bars equaled up to three things.”

Josh formed an abduction, attributing the surprising measure of the four-eighteenths stick to the generalization of a peculiar aspect of a previous experience. In the previous experience, he had used iteration conjecturally to establish a three-part stick as being one-half of the six-part ruler. This introduced the possibility of fraction sticks having simplified measures. In forming Conjecture 3, Josh attributed the surprising measure to a similar simplification of the expected fractional measure and again used iteration conjecturally. This time, the fraction stick and its measure were more complicated and he was unable to mentally iterate the four-eighteenths stick within the ruler.

Josh’s abductive operations fit those of a partitive unit fractional scheme for composite units, except that he experienced the constraint of having two of the smaller units left over after producing four units of units. His realization of the constraint indicates that Josh had indeed been involved in units coordination. So, I hypothesized that Josh’s conjecture might contribute to his construction of a partitive unit fractional scheme for composite units, as well as a partitive unit fractional scheme, and might engender commensurate fractions. In fact, apart from his trouble with fractional language, Josh demonstrated the operations of a commensurate fractional scheme during the next protocol (Protocol 10). And, between Protocols 12 and 13 (March 28^{th}), Josh acted successfully with the operations of a partitive unit fractional scheme for composite units, including a coordination of the units of units in the ruler.

Conjecture J4. Toward the end of the March 28^{th} teaching episode, I challenged Josh and Sierra to produce a stick that would measure two-thirds, without using three parts in the ruler. During Protocol 13, Josh responded by producing a three-ninths stick and measuring it as one-third. He then began to drag the three-ninths stick across the ruler, reasoning aloud: “There’s one-third, and then you have another one-third… I’m going to pull *six* out of it… because if I put one more it would be six. So, it may be *two*-thirds.” He conjectured that if three parts made one-third, then six parts would make two-thirds.

At the end of my synthesis of Conjecture J3, I mentioned that Josh could coordinate units of units in the ruler in order to produce commensurate fractions. But in iterating a composite unit in the ruler and coordinating the units, Josh had always referred intermediate productions in terms of whole units, rather than fractional units. For example, during Protocol 10 (March 24), he counted his iterations of a five-part stick, “that’d make that ten… it’d be fifteen… it’d be twenty,” in order to justify that the stick was one-fourth of the ruler. The novelty of Conjecture J4 was that he used units coordination to identify a non-unit fraction produced by his iteration of a composite unit fraction.

Conjecture J4 involved a conjectural use of Josh's units coordinating operations available within his partitive unit fractional scheme, which had already been emerging partitive unit fractional scheme for composite units. The conjecture resulted in a functional accommodation in that scheme, constructing a partitive unit fractional scheme for composite units that included the units coordination of fractional units with the ruler. I hypothesized that the conjecture would engender a partitive fractional scheme. After all, Josh already seemed to have a partitive conception of two-thirds. Furthermore, his partitive unit fractional scheme for composite units already contained the operations of a (general) partitive fractional scheme for composite units. Indeed, over the next three episodes (March 31st, April 14th and April 16th) Josh's actions provided several indications for both new schemes, and his difficulty with fractional language dissipated!

<u>Conjecture J5</u>. During the teaching episode on April 25th, I challenged Sierra to determine the fractional size of an unpartitioned two-fifths stick. After repeating the stick beyond the ruler (producing six-fifths), she did not know what to do. Josh picked up where she left off by cutting off the extra piece, except that, in using CUTS, the piece was not exactly one-fifth. So, when Josh iterated the piece in the ruler, it did not fit evenly either. He continued this recursive process of cutting and iterating until he had produced the thirty-one part stick illustrated at the top of Figure 15, closely approximating the length of the ruler. Protocol 18 records his subsequent actions, beginning with his act of counting parts and saying resolutely (though with uncertainty), "maybe thirty-one bars." Once I reminded him to consider the size of the fraction, he lined it up with the thirty-one part stick and determined that it was "twelve out of thirty-one."

Josh conjectured that he could use the pieces extending beyond the ruler as co-partitions of the fraction stick and the ruler in order to determine the size of the fraction stick. Each time he attempted to use one of these extended pieces, he found a new one to use in its place. In his actions of cutting and repeating, Josh lost the original goal, supplanting it with one of evenly partitioning the ruler. His conjecture and actions were products of his conjectural operations, which included iteration and the manner of operating demonstrated in Protocol 17. He posited hypothetical pieces to use in iteration within the ruler. The extended pieces were viable options because they represented the peculiarity of the situation—the pieces that wouldn't fit.

Josh acted similarly in Protocol 19 (May 20th), during which his actions also affirmed that Josh had constructed a partitive fractional scheme. I attribute that construction to Conjecture J5, but note that his actions of finding co-partitions seemed to help him in making estimates of partitive fractions.

Chapter 9: Conclusions

Originally, I intended to work with pairs of students at three different stages of fractional development. Based on the assumption that I would be working with such pairs, I hypothesized that the students' zones of potential construction would be more alike within pairs than across pairs, and that this hypothesis could be affirmed by considering students' actual constructions in the teaching experiments. Having found only one higher-stage student (Hillary) and one middle-stage student (Will), the three pairs that I formed included one heterogeneous pair (Hillary and Will) and two lower-stage pairs, of which I have only reported on one (Josh and Sierra). Whereas that lower stage pair was homogenous in terms of the students' construction of the schemes illustrated in Figure 4 in Chapter 3, one of the students (Josh) seemed to have splitting operations available. So, it will be at least as interesting to consider zones of potential construction within pairs as it will be to consider them across pairs.

The first few episodes of the teaching experiment provide indication for the schemes that the four students had constructed. Hillary had constructed a partitive fractional scheme and a splitting operation. Will had constructed a partitive unit fractional scheme and splitting operations were in his zone of potential construction. Josh could split but had not yet constructed a partitive unit fractional scheme. So, I placed him in the lower-stage pair with Sierra. Like Josh, Sierra's only fractional scheme, with which she could name fractional parts, was a part-whole scheme. The only significant difference between my models Josh and Sierra's fractional schemes was that Josh's part-whole scheme seemed to be a part-whole fractional scheme, whereas Sierra's was a part-whole partitioning scheme. The distinction between the schemes is that a student with a part-whole fractional scheme compares fractional parts back to the whole. So, although I knew that Hillary and Will formed a heterogeneous pair, Josh and Sierra seemed quite compatible at first, in terms of their constructed fractional schemes.

If we consider the available operations of the students at the beginning of the teaching experiments, apart from established fractional schemes, Josh and Hillary were very similar. Both of them were splitters and had demonstrated that they could produce experiential units of units, reasoning multiplicatively with constructed parts. So, comparing the constructions of these students to each other and to those of Sierra, in particular (because she had not even developed fractional iteration), will test whether students of similar *operational* development have similar zones of potential construction. It may be especially interesting to compare Josh and Sierra's constructions because they differed significantly in terms of available operations, but not in terms of fractional schemes. Because Hillary and Will were at different stages in terms of their fractional schemes (higher-stage and middle-stage, respectively), a comparison of their constructions in the teaching experiment will also test my original hypothesis: Students at different stages in terms of their available fractions schemes will demonstrate more pronounced differences in their zones of potential construction. It may be particularly interesting to compare Josh and Will's constructions because Josh was more advanced operationally (he was already a splitter at the beginning of the teaching experiment), but Will was more advanced in terms of constructed schemes (he had already constructed a partitive unit fractional scheme).

Comparing Actual Constructions

I provide this section to summarize the students' actual constructions that can be attributed to conjecturing activity. Then I can compare the power of their conjectures relative to their operational development, initial stages in terms of schemes, and zones of potential construction. I focus on comparisons between Hillary and Will, Will and Josh, Josh and Sierra, and Hillary and Josh for the reasons provided at the end of the previous section. These comparisons serve to test the hypotheses mentioned there. They also answer questions about whether students actualize their zones of potential construction through conjecturing activity and whether there are qualitative differences between the conjecturing activity of students operating at different stages.

Hillary. From the beginning of the teaching experiment, I had hypothesized that a reversible partitive fractional scheme and an iterative fractional scheme were within Hillary's zone of potential construction. As I learned more about her ability to coordinate units of units, I also hypothesized that a commensurate fractional scheme was within her zone of potential construction. Whereas she may have constructed the first scheme before the teaching experiment, the third scheme was constructed by way of a complementary fractional comparison scheme that Hillary formed as a functional accommodation in her partitive fractional scheme. In forming Conjecture H1, she had abducted a new use of disembedding operations within her partitive fractional scheme. Conjecture H2b was based on a way of operating that served as a functional accommodation in the intermediate scheme (her complementary fractional comparison scheme) in order to form a commensurate fractional scheme.

Although it is difficult to attribute engendered schemes to any particular event, Hillary's generalizing assimilations using her partitive fractional scheme (Conjectures H3a and H3b) seemed to engender the construction of an iterative fractional scheme. By the end of the teaching experiment and possibly engendered by Conjectures H5 and H6 (respectively), Hillary had also constructed a reversible partitive fractional scheme for composite units and a reversible commensurate fractional scheme.

It seems that through most of the teaching experiment, we had been acting in situations involving schemes and operations that were within Hillary's zone of potential construction. After all, by the end of the teaching experiment, she had constructed each of the schemes that I had placed in that zone. Moreover, *she had constructed two of those three schemes (and many others!) through conjecturing activity.*

Hillary and Will. Will's conjectures were not generally as powerful or constructive as those of Hillary because they often resulted in constructions that were not as operationally flexible or permanent as hers. I have referred to these constructions as procedural schemes. Conjectures W1 and W2, for instance, resulted in procedures that were based on generalizing assimilations of Will's whole number operations and ratio reasoning. I will elaborate on the limitations of such procedures and procedural schemes in the implications section.

To the extent that Will was able to assimilate aspects of problematic situations using his fractional schemes, he was successful in making accommodations in those schemes, resulting in more powerful fractional schemes. Conjectures W3a and W3b, for instance, seemed to engender an accommodation in his partitive unit fractional scheme, which might have engendered the construction of a partitive unit fractional scheme for composite units.

Conjectures W4a, W4b, W5, and W6 (Chapter 6, pp. 236-239) were Will's responses to tasks that involved improper fractions, commensurate fractions, and reversible partitive fractions. Because Will had not yet constructed a partitive fractional scheme, these tasks that I posed involved ways of operating that were beyond his zone of potential construction. Because Will had demonstrated a pronounced ability to construct procedures that emulated Hillary's fractional schemes, I did not know, at that time, that the tasks were inappropriate for Will. In response to such situations, Will constructed procedures using his whole number operations in making generalizing assimilations of Hillary's actions. *My misunderstanding of Will's actions and his propensity for constructing procedures created a cycle of stagnation in Will's learning that only abated toward the end of the teaching experiment, when I realized that Will's actions were contextually based.* After that time, I began using tasks explicitly designed to engender partitive fractions, and Will's operations in forming Conjecture W7 seemed to do just that.

So, the initial differences between Hillary and Will's available operations (Will was not yet a splitter) and stages of fractional scheme construction (Will had not yet constructed a partitive fractional scheme) contributed to profound differences in their actual constructions. *This indicated that their zones of potential construction were quite different due to disparities in their operations or fractional schemes.* In fact, I have argued that much of the teaching experiment was composed of interactions between me and the students (task design and interpretations of students' actions) that were within

Hillary's zone of potential construction, but outside of Will's zone of potential construction.

Will and Josh. We might say that Josh had powerful but latent fractional operations because he had not yet coordinated his ways of operating into schemes (not even a partitive unit fractional scheme) for resolving problematic fractional situations. From this perspective, it should have come as no surprise that Josh constructed several powerful fractional schemes during the teaching experiment. Will, on the other hand, had demonstrated a propensity for constructing procedural schemes, which often circumvented his need to construct fractional schemes. I have suggested that this propensity was due, in part, to the fact that the problematic situations of the teaching experiment with Hillary and Will involved ways of operating that were outside of Will's zone of potential construction. This was not the case for Josh in the teaching experiment.

Not only did the tasks in the teaching experiment involve schemes that were within Josh's zone of potential construction, but he had many operations available to use in abducting and conjecturing in general. Conjecture J1 involved the abduction (conjectural evocation with the goal of explaining a surprising situation) of Josh's partitioning operation, Conjecture J3 involved the abduction of his iterating operation, and Conjectures J2a and J2b involved a novel use of splitting. Those conjectural operations engendered a partitive unit fractional scheme, a partitive fractional scheme for composite units, and a commensurate fractional scheme. So, before the end of March, Josh was already operating more powerfully than Will. As an accommodation engendered by Conjecture J4, Josh also constructed a partitive fractional scheme before Will did (if Will did at all).

Throughout most of the teaching experiment, Josh experienced difficulty in naming fractions. For example, in Protocol 5, he referred to a four-nineteenths stick as, "19 out of 4… I meant 4 out of 19." In Protocol 7, he was similarly confused after his iteration of a unit fraction: "9 out of… Uh. 1 out of 9, I think." Because his difficulty was most pronounced when he was attempting to iterate fractions within the unit bar, I hypothesized that Josh's struggles with fractional language ran parallel to his struggles in moving from a part-whole conception of fractions to a partitive conception of them. In fact, *his struggles with language only dissipated after Conjecture J4, the conjecture that engendered his partitive fractional scheme*.

Josh had powerful operations from the start of the teaching experiment but no fractional schemes other than his part-whole fractional scheme. His construction of fractional language did seem to run parallel to his construction of fractional schemes and this construction was rapid. However, it would be precarious to attribute Josh's relative success to his operational superiority over Will because Will was working with a different partner on different tasks. If Josh had been working with Hillary, it is possible that he would have experienced the same kind of stagnation in his constructions. *I can say that Will's superiority over Josh in terms of initial fractional schemes was insufficient for actualizing superior constructions*.

Josh and Sierra. I have mentioned that Sierra demonstrated a propensity for forming abductions in the teaching experiment. Whereas many of them did not involve fractional operations and were not constructive, three of her five conjectures presented in Chapter 8 (Conjectures S2, S3, and S5, Chapter 6, pp. 327-329, 330-331) were constructive fractional abductions. I have suggested that Sierra's abductions were prevalent among her conjectures because she was often unable to act independently in the teaching experiment and was left in the position of explaining Josh's actions.

At the beginning of the teaching experiment, Josh seemed to have constructed a part-whole fractional scheme. Moreover, because Josh was a splitter, he was able to conceive of situations involving partitioning as situations that also involved iteration. Sierra's only scheme for naming fractions was a part-whole partitioning scheme. So, whenever Josh acted using iteration and partitioning operations, Sierra was able to assimilate those actions using her partitioning operation. Indeed, in forming two of the three abductive conjectures mentioned above (Conjectures S2 and S3), Sierra had abducted her partitioning operation to explain Josh's actions. Conjecture S1 involved Sierra's conjectural use of partitioning to assimilate the actions of her partner at that time (Cory), and two of those first three conjectures (Conjectures S1 and S3) resulted in accommodations in her partitioning schemes. Throughout the teaching experiment, she had been unable to construct schemes involving iteration.

As a result of Conjecture S4, Sierra did construct a part-whole fractional scheme as a functional accommodation in her part-whole partitioning scheme, but this scheme does not involve iteration either. Her final conjecture (Conjecture S5) was an abduction that occurred independently of Josh's actions and involved her disembedding operation. Partitioning and disembedding operations were fractional operations that Sierra seemed to have developed before her initial interview. So, her conjectures involved only the coordination of existing operations in new ways to satisfy new goals. These coordinations constructed new schemes through functional accommodation. There is no indication that any of her conjectures *engendered* the *development* of new schemes or operations. And, I hypothesize that the construction of new fractional operations, such as iteration, could only result from engendered development rather than functional accommodation. Their construction might rely upon the third kind of reflective abstraction in which concepts at one level of abstraction are projected and reorganized at a higher level. Steffe (1991b) referred to this as metamorphic accommodation and provided the example given in Chapter 1.

Sierra was able to use conjectural operations to construct new fractional schemes, but her schemes were limited to ones that did not involve iteration or splitting. Whereas Josh's initial schemes were not very different from Sierra's, his available operations were clearly superior. I attribute his success, relative to Sierra, to such operations. *This affirms the hypothesis that students' zones of potential construction are determined, at least in part, by their available operations*.

Hillary and Josh. I have argued that Hillary and Josh were very similar operationally (as splitters) but that Hillary had constructed many more fractional schemes. In their respective teaching experiments, both students were engaged in tasks that involved ways of operating that were within their respective zones of potential construction. Based on their actual constructions, it appears that there was a difference between their zones. But Josh made remarkable progress and it seemed that his zone of potential construction at the end of the teaching experiment was very similar to that of Hillary at the beginning. In fact, he had constructed every fraction scheme with which Hillary had begun and more. So, it seems that the differences between Hillary and Josh's zones of potential construction were temporal and shrinking.

<u>Summary</u>. Considering the pair-wise comparisons, *it seems that splitting operations contribute more than fractional schemes in determining students' zones of potential construction*, and perhaps I should have defined the three stages of development in terms of operations rather than fractional schemes. Hillary and Josh's actual constructions were remarkably similar considering the initial differences in their fractional schemes, and this was not due to any lack of constructive activity on Hillary's part. In comparing the construction of students who were operationally different, the student that was more operationally advanced actualized a greater zone of potential construction each time, even when, in the case of Josh and Will, the other student began with more fractional schemes.

I have hypothesized that the students' zones of potential construction would be actualized through their conjectural activity. *To varying degrees, each student did actualize her or his zone of potential construction through conjecturing*. This was most remarkable with Hillary and Josh. Their partners' constructions (especially Will's) suffered because some of the tasks were not within their zones of potential construction, and they coped with this by constructing procedures and inventing abductive explanations. This observation informs another important research question.

In designing my study, I asked whether students at different stages (as determined by the fractional schemes outlined in Figure 4, p. 60) would make *qualitatively* different conjectures. Considering the conjectures of Hillary and Josh, this does not seem to be the case; both students used their available operations in forming generalizing assimilations and abductions, resulting in functional and engendering accommodations. Rather, students' zones of potential construction relative to the tasks posed seem to determine qualitative aspects of their conjectures. *Problematic situations that were beyond a students' zone of potential construction provoked conjectures that involved inadequate ways of operating*. Even when a student was successful in resolving such problematic situations (as Will often was), the result was not a permanent and flexible way of operating.

<u>Characterization of Conjecture</u>

In Chapter 2, I characterized *conjecture* as a student's spontaneous construction that the student is motivated to test because its usefulness is uncertain. This characterization is more open than that of Chazan and Houde's (1989); most notably, I do not require that the conjecture be explicitly stated. Considering the conjectures presented in Chapters 6 and 8, it was useful to omit such a requirement because the students were rarely able to succinctly verbalize their conjectures, and yet they were able to test and use them in constructing new schemes. Such conjectures remind us that, while Glasersfeld (1998) was correct to say that conscious accommodations are abductions, *not all conjectures should be considered as conscious accommodations, nor should they all be considered as abductions* (as I argue later in this section).

Whereas many of the conjectural operations presented in Chapters 6 and 8 led to the construction of new ways of operating, not all of them were accommodations, nor were all of them conscious acts. Consider, for example, Conjecture H6, in which Hillary conjecturally coordinated her reversible partitive fractional and commensurate fractional schemes in order to conjecturally compose seven units of two in an unpartitioned seven-eighths bar and to join one more unit of two to produce the unit bar. I am sure that she was aware of her attempts to posit parts within the fraction bar because she said as much in Protocol 26, but was she aware that she produced seven units of two in the fraction bar? And, was she aware that her conjecture would engender a new way of operating? Finally, because her conjectural operations only engendered the eventual construction of new schemes, can they be considered as accommodations? I cannot answer these questions confidently based on my observations of Hillary's actions, but I can confidently say that she was conjecturing because she was operating uncertainly in new ways to resolve and to test her solution to a problematic situation.

Glasersfeld's assertion about consciousness is important because conjectures emerge in novel situations and, by their nature, are uncertain. The extent to which the conjecturer is aware of the uncertainty is the driving force behind the conjecturer's critique and test of the conjecture (Peirce, 1998). Furthermore, *the motivation for conjecturing appears to be generated by the need to achieve a goal state without an established means for doing so*. In other words, the need to eliminate a perturbation provokes conjectural activity, and "need creates consciousness" (Piaget, 1955, 231). I prefer to use the term *awareness*, rather than *consciousness*, because consciousness has so many philosophical implications. The conjectures presented in Chapters 6 and 8 certainly illustrate a raised level of awareness when students are engaged in conjecturing activity. For example, considering Conjecture H6 again, it seems unlikely that Hillary would be aware that she needed to find something that would divide into seven pieces if she had simply assimilated the situation using a reversible commensurate fractional scheme (which she had not yet constructed). Later in this chapter, we will examine cases in which conjectures resulted in students' raised awareness about aspects of a situation that served to modify schemes.

Glasersfeld referred to abduction as "the mainspring of human creativity" (1998, p. 10) and implied that all conjectures are abductions. There is a way in which we could claim that all conjectural operations fit the abductive pattern. Conjectural operations allow students to act as if they know how to reach a cognitive goal that would be beyond their ability to reach otherwise. We might consider Conjecture W1 to be an abduction because Will's conjectural use of the *adding one more* operation in a fractional situation made it possible for him to produce an eight-sevenths bar from a seven-sevenths bar. That is, a production that at first seemed confusing to Will became a matter of course once he applied the conjectural operation of adding one more part to the given seven-sevenths bar. However, I reserve the use of the term *abduction* for conjectures where a student is trying to explain a surprising observation.

In my view, *conjectural operations are abductive if and only if they produce a result that explains the surprise*. In explaining a surprise, a student would be aware of the conjecture, its role in explaining, and possibly even her novel way of operating. Perhaps this is why Glasersfeld referred to abductions as "accommodations done consciously" (1998, p. 9). As an example of such an accommodation, consider Conjecture H1 in which Hillary's conjectural use of her disembedding operation fit the pattern of abduction. She used disembedding to produce the complement of a two-thirds bar (thus, maintaining the unit bar) to explain the surprising measure of a six-ninths bar. She seemed explicitly aware of the complements of fractions, both in forming Conjecture H1 and in subsequent applications of the resulting scheme, her complementary fractional comparison scheme.

Conjectures are uncertainties that occur in response to perturbation. Indeed, uncertainty itself is a kind of perturbation. So, we might ask whether provoking students to experience perturbations is sufficient to foster conjecturing activity. We might also ask whether all conjecturing activity fosters learning and development. The answer to both questions would be *no*. *There are other responses to perturbation besides conjecturing, and, even when conjecturing activity does occur in response to a perturbation, some conjecturing activity is unconstructive*.

Students between the ages of nine and eleven years (the students in my study were eleven) are prone to syncretism in reasoning (Piaget, 1955). As an example, on March 20th, Hillary and Will had been attempting to produce the bars representing the mixed number for seven-fourths. Will pulled a one-fourth bar and a three-fourths bar from the seven-fourths bar and claimed that he had produced one and three-fourths. At first, Hillary appeared uncomfortable with Will's production, but then agreed with his subsequent explanation. Hillary's assimilation of Will's explanation and her conception of one and three-fourths called the same schema so that one implied the other without her deductively examining the details of the situation further.

Part of the reason for Hillary's agreement in the previous example was that Hillary had developed a particularly agreeable disposition in working with Will. Conjecture H5, for instance, was Hillary's vain attempt to justify Will's claim that seven-twelfths would measure as four-sixths. When she considered Will's idea, she was able to make some sense of it by considering four apparently equally sized parts. Hillary tended to agree with Will simply because she could understand his reasoning, even when it was very different from her own. Such social and affective dispositions introduce important implications for fostering conjecture, which I will discuss in the final (implications) section of this chapter.

Social and affective dispositions aside, there are other ways that perturbations might be eliminated. If a situation provokes a student to operate in a way that is radically different than her present ways of operating, the student might not attend to some aspects of the situation that she otherwise would and use schemes that are more elementary than others that she has available. I refer to this as regression. For example, during the March 12th teaching episode, I asked Hillary to tell me how much the old unit bar would measure if her three-fourths bar were to be used as the new unit bar. Hillary began to focus on the fourth part in her bar and claimed that her bar would be one-fourth. Whereas she lacked an iterative fractional scheme with which to meaningfully interpret the task, she could have used her partitive fractional scheme to recognize that the old unit bar could not be one-fourth of her bar. But instead, she had relied on a part-whole fractional scheme to name one-fourth. This sort of regression was common to both Hillary and Will when dealing with complicated tasks involving improper fractions, until Hillary had begun to produce improper fractions.

Even when the students did respond to perturbation with conjecturing activity, their conjectures were not always constructive. In particular, attempts to induce ways of operating that are outside of a student's zone of potential construction were often met with the construction of procedures and procedural schemes. Such responses were especially common among Will's conjectures. I will describe these responses and elaborate on potential factors provoking them in the implications section.

Whence Conjecture?

Analogy. Polya (1954a) suggested that conjectures are constructed through analogy, "identifying specific relationships between the respective components of two systems" (p. 13). Glasersfeld (1998) agreed that this was plausible, but did not go so far as to say that analogy was the root of conjecture. He said that analogies involve intuiting a rule based on a single observation (p. 7). I begin with an example of a conjecture that fit Polya and Glasersfeld's characterizations, so that I can examine their role in the conjecturing of the students in my study.

In Conjecture W6, having observed Hillary's production of the unit bar from an unpartitioned two-thirds bar, Will attempted to produce the unit bar from a two-fifths bar by partitioning it into two parts and joining on three more of those parts. His actions fit Polya's assertion about analogy concerning systems, interpreted here as experienced situations. Will was specifically aware of the relations that he was forming between a new situation and one in which he had observed Hillary acting. He used the relations that he identified in order to construct a procedure for acting in the new situation. This pattern is also closely aligned with Glasersfeld's assertion about analogy: Will had intuited a rule for acting based on a single observation.

Conjectures like Conjecture W6 often resulted in procedures that only allowed the student to act *as if* he had constructed particular fractional schemes and concepts. *This may be symptomatic of analogies in general; the only operational novelties involved are those of assimilation*. In the case of Conjecture W6, there was a sequence of assimilations of Hillary's actions that served as a sequence of steps for acting in similar situations. Whereas Will's assimilations in the context of attempting to understand Hillary's actions served to raise his awareness about his actions in trying to emulate Hillary's actions, he did not experience any new constraints in operating as he might in a generalizing assimilation.

Generalizing assimilation as conjecture. A generalizing assimilation is the assimilation of a novel situation through which the trigger of the scheme is accommodated (generalized) to include similar situations. Steffe and Thompson (2000) have suggested that teachers might encourage generalizing assimilations by varying the contextual details of problems that involve familiar ways of operating. In the next section, we will examine the potential effects of generalizing assimilations in terms of accommodations in schemes. For now, I only mention that *generalizing assimilation can introduce new constraints to a student's way of operating, which can both modify the trigger of the scheme used in assimilation and engender operational change*. Conjecture W3a, Will's assimilation of a situation involving composite units within his partitive unit fractional scheme, is an example of a generalizing assimilation that introduced a new constraint—that composite units sometimes do not exhaust the whole. Other conjectures that can be considered as generalizing assimilations are Conjectures W1, W2, H3a, H3b, and H5 (Chapter 6, pp. 232-234, 243-244, 246-247). *In the other two cases with Will (Conjecture W1 and W2), the generalizing assimilations resulted in procedures that did not contribute to Will's fractional knowledge, further supporting my claim that not all conjectural activity is constructive*.

From the examples mentioned above, we see that generalizing assimilations were a common conjectural response to perturbation for Hillary and Will. I claim that this was due to the fact that Hillary and Will had constructed more fractional schemes with which to assimilate various situations. Moreover, in working with Hillary, Will was inclined to assimilate her actions using whole number schemes when his fractional schemes were inadequate. If we were to consider generalizing assimilations involving operations (without specialized schemes), many of Josh's conjectures would fit the description of generalizing assimilations. For example, Conjectures J2a and J2b might be considered as generalizing assimilations using his splitting operation. In Conjecture J2a, Josh used splitting to produce a three-thirds bar that he could also treat as three-halves. Attempting to produce three-halves introduced a problematic situation in which Josh had not used splitting before, and so his assimilation of the situation using splitting might be considered a generalizing assimilation. However, operations, by themselves, do not have as much structure as schemes. They do not provide a pattern for acting in a situation to achieve a goal; they only provide for action in transforming one object into another. So, Josh's conjecture cannot be described as a generalizing assimilation that used splitting to take into account the various aspects of the situation. It would be just as appropriate to consider Conjecture J2a as an abduction, but I have already argued against such a loose use of terms.

Glasersfeld (1998) alluded to analogy as a potential response to perturbation when he wrote of "formerly disregarded aspects of the triggering situation being taken into consideration." I refer to generalizing assimilation in that regard. The key difference between assimilation and generalizing assimilation is that assimilation never takes into consideration those aspects of the situation that do not trigger the scheme; the student does not notice them. Consider again Conjecture H5 (already mentioned in describing Hillary's agreeability). The situation of trying to predict the measure of a seven-twelfths bar did not trigger Hillary's commensurate fractional scheme, presumably because she could not produce equal composite units from the seven twelfths. But when Will suggested that the bar might measure four-sixths, Hillary made a generalizing assimilation using her commensurate fractional scheme by considering two equal composite units within the seven-twelfths part and two more in its complement. *This was an assimilation in which she was aware of novel aspects of the situation, and being aware of the novel aspects involved in her use of the scheme gave her more control to use the scheme flexibly*.

<u>Abduction, perceptual judgment, peculiarities, abducted rules</u>. Glasersfeld (1998) suggested that one plausible response to a perturbation is to re-examine the situation that provoked the perturbation and identify peculiar details about it. Indeed, such activity accounts for much of the students' activity in explaining surprising observations—i.e., in forming abductions. As with syncretism of reasoning and some of the generalizing assimilations mentioned above, many of these abductions seemed to do little in terms of constructing new schemes. For example, Will blamed surprising measures on the position of the "unit bar" label (Protocol 2) and the direction of partitioning (Protocol 7). I have noted that Will, in particular, was continually inventing rules like this to explain surprising situations in which his existing schemes and procedures did not produce the desired results. His rules can be considered abductions, but they were not insightful in the context of fractions.

Other times, the students' abductions helped them to gain insight into "the structure of a problem and the function of statements" (Lakatos, 1976, p. 121). This was the case when Hillary (Protocol 1) explained why a bar had measured eight-sevenths instead of eight-eighths as she had predicted: "because we added on." Of course, such insights do not have to be conjectural, but may be perceptual judgments in which a student notices something he had not noticed before and attributes, with certainty, its role in the situation. During the March 20th teaching episode with Hillary and Will, for example, Will eventually realized why he needed four one-fourth parts to produce the *1* in *one and three-fourths*: “It was a whole entire thing—a unit bar.” Also, just before forming Conjecture J1, Josh realized why a part that was a half of one-half of the ruler did not measure one-third as he had expected: “Because [the parts are] not even.” These are insights based on perceptual judgments, which, unlike abductions or other conjectures, are unquestioned at the time they are made.

Sierra had a particular affinity for forming abductions, perhaps because she was operationally lagging behind Josh, whose actions she attempted to explain after he had completed them. For example, after Josh had produced four-eighteenths and measured it as two-ninths, Sierra explained (Protocol 9) that this was so because, “2 times 9 is 18, and there’s eighteen parts in there.” This is another example of an unconstructive abduction; I will further examine Sierra’s propensity for them in comparing her actual constructions to those of Josh.

Sierra was able to use abductions constructively too, particularly when her abductions involved fractional operations. In fact, this was the case for each of the students. In Chapter 2, I referred to the use of operations in following the pattern of abduction as abducting. I now return to that terminology and examine a few examples of how abducting generated conjectures.

In forming Conjecture S2, Sierra abducted her disembedding and number addition operations to constitute a three-part stick as one-half because she had no available scheme for producing one-half from three parts. By coordinating the two abducted operations, she was able to link the goal of producing one-half with the situation of having three parts. She formed a similar abduction in Conjecture S3. In Conjecture S5, she used disembedding to link an established relationship (between number of parts used in the ruler and the sizes of the unit fractions produced) to an unestablished one (between the sizes of the complements of the unit fractions). This time the link involved the use of only one operation to translate an unfamiliar relationship into a familiar one. Conjecture J1 was an abduction that depended on Josh's recognition of a peculiarity. His notice of the peculiarity was a product of the perturbation that he had experienced (in the sense that Glasersfeld suggested), and the peculiarity called Josh's iteration operation, which he used abductively. Other examples of abduction include Conjectures H1, H4 and J3 (pp. 240-242, 244-246, 335-336). *Will alone appeared not to engage in abducting, perhaps due to his propensity for generalizing procedures from his observations of Hillary's actions*.

In saying that naïve conjectures are preceded by pre-naïve conjectures and refutations, Lakatos (1976) may have been referring to conjecturing as *the* means of learning and development: We build from previous operations and schemes by conjecturing. I have not made such a contention. Indeed, I have interpreted Lakatos' remarks as referring to the origins of conjecture. *He seemed to suggest that conjectures come from conjectures. I have provided examples to demonstrate that this is not always the case, citing generalizing assimilation, abducting, and the use of existing operations in new ways or new situations to conjecturally satisfy a goal. However, I have also cited examples in which one conjecture followed from another*. For instance, I have mentioned that Conjecture S3 opened the possibility for Sierra to consider partitions within a part being distributed across the whole, which was instrumental in her formation of Conjecture S4.

Considering Lakatos' remarks further, conjectures can be formed and refined through a series of miniature conjectures that serve to transform the goal of a situation. In Conjecture H1, Hillary and Will formed these miniature conjectures in interaction. Both Hillary and Will experienced perturbations at the surprising measure of a six-ninths bar. Will began with a miniature conjecture that took form in his question, "Is two-thirds the same as six-ninths?" In assimilating Will's question, Hillary's partitive fractional scheme was evoked to estimate the size of a two-thirds bar. This transformed her goal of explaining the surprising measure to one of comparing the two bars, which she was able to do by conjecturally using her disembedding operation.

In Conjecture W7, Will used his partitioning operation to conjecturally transform an unfamiliar situation to a familiar one. His original goal had been to reproduce the unit bar from an unpartitioned five-fourteenths bar. He had a scheme available to produce the unit bar if the fraction bar were partitioned, so he formed a new goal of producing the appropriate partitions in the fraction bar and conjecturally partitioned it into five parts. But *in transforming a goal through a series of miniature conjectures, a student can loose track of the larger goal*. This was the case in Conjecture J5 when Josh became so involved in recursively positing and iterating pieces that he never named the fraction he had originally intended to name and was, instead, satisfied with evenly partitioning the ruler. Such examples illustrate what Polya (1957) must have meant when he said, "Many a guess has turned out to be wrong but nevertheless useful in leading to a better one" (p. 99).

Wherefore Conjecture?

Poincaré (1952) claimed, "we can only ascend through mathematical induction" (p. 16). I am now prepared to refute that claim, at least as it might be applied to the construction of schemes, by citing specific constructions resulting from students' conjecturing activity. In this section, I consider students' constructions resulting from two broad categories of conjecture: generalizing assimilations and abducting. I then provide examples of constructions resulting from conjecturing in general. *These examples also refute Fodor's (1980) tenet that conjecturing is an inductive process, by presenting "a notion of learning incredibly different" from that of induction (mathematical or otherwise) and deduction* (p. 149).

Results of generalizing assimilations. In the previous section, I mentioned some possible results of generalizing assimilations. Obviously, a student's successful generalizing (conjectural) assimilation of a new situation into an existing scheme should modify the trigger of the scheme. This is, in part, because in acting conjecturally, a student's awareness about their activity is raised. *Not only does this make the student more aware that she is acting in a new situation, but it also raises her awareness about the details of the situation and how she is acting, often introducing new constraints to her ways of operating*. Below, I offer examples of these three possible results from generalizing assimilations (modification in the triggers of schemes, raised awareness of situations and actions, and the introduction of new constraints to operating).

In Conjecture W3a, Will was aware that he was operating in a new situation as he would in situations involving unit fractions. This raised awareness may have contributed to the modification of the scheme's trigger (to include situations involving composite units) because, when he was challenged with a similar situation in forming Conjecture W3b, he acted immediately using the scheme. Furthermore, when he acted on Conjecture W3b, he explicitly attended to the units of units that he was producing through his iterations of five-elevenths and experienced the constraint of being unable to exhaust the ruler. These two conjectures might engender Will's eventual construction of a partitive unit fractional scheme for composite units. I have already mentioned Hillary's raised awareness due to her generalizing assimilation of Conjecture H5. She became aware of a new constraint because her assimilation failed to produce the desired result, but this new constraint served as a functional accommodation in her commensurate fractional scheme.

Considering the new constraints that students experienced in operating reinforces my claim in Chapter 1 that operations change through operating. Indeed, the operations of Will's partitive unit fractional scheme had changed because he conjecturally used it in a new situation and experienced a new constraint. Recognizing the constraint provoked Will to produce experiential units of units using the scheme, which he eventually abstracted as a units coordination operation within his partitive unit fractional scheme.

<u>Results of abducting and other conjectures</u>. Conjectures H1, H4, S3, J1, and J3 (pp. 240-242, 244-246, 328-329, 331-333, 335-336) were abductions that yielded new ways of operating. *My analysis of these conjectures has demonstrated that abducted operations often engender the construction of new schemes*. Consider as examples, Conjectures J1 and J3. Other examples that fit neither abducting not generalizing assimilation follow.

Conjecture J1 involved Josh's abduction of his partitioning operation to conjecturally partition an unevenly partitioned fraction in order to make the partitions even and produce one-fourth. Because Josh was a splitter, his actions of partitioning translated to actions of iterating and engendered his construction of a partitive unit fractional scheme. Conjecture J3 involved Josh's abduction of iterating operations to explain the measure of a four-eighteenths stick and also engendered a scheme, a partitive unit fractional scheme for composite units.

Many of the conjectural operations presented in Chapter 6 and 8 do not fit the pattern of abducting, nor are they generalizing assimilations. Some of them served as functional accommodations in schemes, as in Conjectures H2b, S1, and S4 (pp. 242-243, 326-327, 329-330). Others engendered accommodations in schemes, as in Conjectures H2a, H6, and J4 (pp. 242, 247-248, 336-337). *All of them affirm that conjectural operations can result in the construction of schemes*. I will use a couple of the examples already presented in this section to summarize *how* conjectural operations do this, thus providing an answer to one of my main research questions: "How might conjectural operations engender accommodations in schemes?"

Sometimes, conjectural operations played an even more direct role than *engendering* accommodations in schemes. *Not only do conjectural operations engender accommodations, but in being acted upon, they often serve as the accommodation itself*. I have described how Hillary's generalizing assimilation in Conjecture H5 introduced a new constraint in her ways of operating that constituted a functional accommodation in a scheme. I have argued that Will's generalizing assimilation in Conjecture W3a raised his awareness about his experience of a situation, modifying the trigger of his partitive unit fractional scheme. When conjectural operations do *engender* accommodations, this appears to occur through the reflective abstraction of novel actions used in the situation.

Such was the case in Conjecture W3b, through which Will eventually constructed units coordination as part of his partitive unit fractional scheme, possibly engendering a partitive unit fractional scheme for composite units.

Modalities of conjecturing and testing. I conclude this section by answering some questions implied by the work of Arzarello et al (1998). Their work described an ascending modality of forming a conjecture and a descending modality of testing (proving) it, and they suggested that abduction was the means of switching from the ascending modality to the descending modality. I complicated this question by considering students' conjectural operations rather than their explicitly stated conjectures. After considering the conjectural operations of the students in my study, the question of how students switch seems much more clear. I would say now that *the descending modality begins the moment that a student begins acting out her conjectural operations. This may even precede any verbalization on the part of the student*. Indeed, students' verbalized conjectures may be part of their actions in testing the conjecture. For example, Sierra began testing Conjecture S5 long before she could describe it, and her attempts to do so (Protocol 16) seemed to increase her confidence in the conjecture.

The ascending modality is only the conjectural evocation or coordination of the operations involved. This may be the reason Peirce (1998) said that, "the different elements of the hypothesis were in our minds before; but it is the idea of putting together what we never dreamed of putting together that flashes the new suggestion" (227). This does not imply that all possible conjectural operations are determined, as an operational translation of Fodor's paradox would suggest; I have already argued that operations change through operating, particularly through conjecturally operating.

The question that remains is why, in conjecturing, students call particular schemes or operations instead of others that are available to them. Skemp (1989) offered the following suggestion concerning the calling of schemas: Schemas form a sort of casting net that associates past experiences in operating and connects to (coordinates) other schemas (pp. 131-141). Perhaps students' initial assimilations of problematic situations using schemes similarly connect to other schemes and operations. Those schemes and operations, then, might be used in conjecturally operating. If additional aspects of a situation are recognized once the student had experienced the perturbation of being unable to resolve the problematic situation, assimilations of those aspects may resonate with additional schemes and operations, which then may be used in conjecturally operating. In fact, *in each conjecture that we have examined, we can find connections between the students' initial assimilation of the problematic situation and the conjectural scheme or operation used*. For example, in Conjecture W1, I have argued that Josh assimilated eight-sevenths as a ratio of whole numbers, and then he used the conjectural operation of *adding one more*, which was a whole number operation that he had used in previous situations to make 8 from 7.

<u>Implications</u>

In considering the implications of my study on conjecturing, it is important to keep in mind that this study was conducted in the context of working with fractions, using dynamic software. `TIMA:Bars` and `TIMA:Sticks` were designed to provide occasions for cleaner and quicker actions than students could otherwise perform with drawings or construction paper. Still, students' perceptions of the fractional situations, as well as the available means for them to act on their conjectures, included perceptions of the software. This first concern may have been minimized by my focus on students' operations. While the schemes and operations that students used may have been triggered by contextual details and limitations in working with the computer programs, I was using the students' specific actions to make inferences about changes to the constraints in their ways of operating. As a second concern, conjecturing activity in working with fractions may differ from other mathematical conjecturing activity and, of course, conjecturing in general. This second concern is minimized to the extent that students' fractional operations generalize to operations in other situations, just as fractional operations are, themselves, constructive generalizations of whole number operations.

It is also important to consider that I worked with four students who, as individuals, represented only themselves. But my models of them serve as epistemic students that teachers may use to make inferences about the conjectural operations of their students, if teachers can interpret their students' actions to be similar to those described here. Comparing and contrasting the epistemic students that I have created and described should inform decisions that teachers make in attempting to promote learning and development through conjecturing activity.

Fostering constructive conjecturing. In order to study students' conjectural operations as a teacher-researcher, I needed to encourage and foster conjecturing. Indeed, my second major research question involved determining how students develop conjecturing dispositions. In characterizing and describing students' conjectures in this chapter, I have noted several factors contributing to or inhibiting constructive conjecturing activity. I summarize those factors here while elaborating on others in order to inform future research, as well as classroom teaching.

I have mentioned a few social factors contributing to conjecturing. In particular, Hillary's disposition of agreeability in working with Will often resigned her simply to concur with his assertions, halting her potentially constructive conjectural activity. Other times when Will made conjectural assertions, she abandoned her activity in working toward a goal to begin conjecturing as to why Will's assertion was viable, as in Conjecture H5. Still other times his assertions seemed to trigger her assimilation of the situation using a different scheme, as in Conjecture H1. All but the first were potentially constructive responses to her interactions with Will.

Perhaps the most constructive contribution provided by Hillary's interactions with Will was that she had begun focusing on the numbers of parts in fractions, as he did. At the beginning of the teaching experiment, she had been focused on the relative sizes of fraction bars, often ignoring the parts within the bars. Will's focus was quite the opposite, and, in assimilating his arguments, Hillary began to use her whole number operations more, coordinating them with her fractional operations. This seemed to be a key factor in her construction of a commensurate fractional scheme.

So, interaction with peers and even Hillary's agreeable disposition were particularly positive factors in generating constructive conjectural activity. I suggest that teachers encourage students to verbalize their conjectures so that other students might build from those assertions. But it is equally important for teachers to encourage students to be skeptical about such assertions, attempting to explain why they are viable or not.

At the end of Chapter 2, I mentioned several potential affective issues that might affect conjecturing, including confidence and frustration (Skemp, 1989). Will and Josh exhibited frustration when they experienced difficulty in acting on their conjectures, during Protocols 6 and 19 of their respective teaching experiments. Their frustration halted their activity, and they were resigned to guess at an expected result of their actions. For example, in Will's case, he was trying to iterate a three-ninths bar within the unit bar by dragging the three-ninths bar to determine its measure. When he experienced frustration in trying to drag it without overlaps, he simply agreed with Hillary's previous assertion that it was one-third. Frustration also stifled activity during much of the March 18th teaching episode with Josh and Sierra. I have attributed this to the nature of the tasks posed in that teaching episode: Many of the tasks involved the production of improper fractions and were well outside of the students' zones of potential construction. From these examples, I draw two conclusions about how to avoid student stagnation in conjecturing activity: *Design tasks that are within the students' zones of potential construction, and provide intermittent opportunities for students to break from task-based activity and engage in mathematical play with the available actions in their mathematical environment*.

In working with pairs of students, I was able to provide individual attention that may have minimized differences among the students' levels of confidence. Still, I can make a few comments about Will, the only student to exhibit a high level of confidence through most of the teaching experiment. Will was willing to act conjecturally, even in situations that were well outside of his zone of potential construction. This was probably a contributing factor to his construction of procedures for acting, and it was certainly a factor in my overestimations of his ways of operating. So, *while confidence may foster conjecturing activity, the activity is not always constructive activity*. In fact, Will seemed to form more conjectures than the other students but demonstrated the least progress in terms of new ways of operating.

Conjecturing is about achieving goals by uncertain means. Skemp (1989) claimed that pleasure accompanies experiences that bring students toward a goal state. To the extent that cognitive goals can be considered as miniature goal states, pleasure should accompany constructive conjecturing. In fact, Piaget (1969) claimed that *affect* provides the *energetics* of all cognitive activity and is inseparable from it. Furthermore, I have noted throughout my analysis that expressions of excitement accompanied students' conjectures. *Although fun may not be the reason teachers encourage conjecturing, student enjoyment during mathematical activity serves as a good indicator that the activity is constructive*.

Finally, it is important to consider the ages of the students when trying to foster conjecturing activity. Whereas I have demonstrated sixth-grade students' ability to operate conjecturally, I expect that conjecturing activity among younger students would be qualitatively different. I have provided examples in Chapter 1 that fourth-grade students are capable of operating conjecturally too, but developmental issues, like syncretism of reasoning, may be more of a factor among them.

<u>ZPD and ZPC</u>. I have hypothesized that Will's propensity for constructing procedural schemes interfered with his construction of a partitive fractional scheme. Because he had constructed a partitive unit fractional scheme by the beginning of the teaching experiment, it seems that the more general scheme should have been in his zone of potential construction. But until the end of the teaching experiment, the tasks that I posed did not engender that scheme. *I claim that the tasks that I posed were within Will's*

zone of proximal development [ZPD], but they involved ways of operating that were outside of his zone of potential construction [ZPC]. I attribute Will's propensity for constructing procedural schemes to this claim. And, I note that tasks at the end of the teaching experiment took into account Will's ways of operating, involved new ways of operating that *were* within his ZPC, and engendered a partitive fractional scheme.

As the teacher-researcher, I designed tasks accounting for Hillary's ways of operating and potential reorganizations of her schemes and operations. In other words, she and I were working within her ZPC. Will's actions indicated that he could act as Hillary did, and I inferred that Will had constructed the same operations that Hillary had constructed. I argue that, by definition, Will and I were working within his ZPD, and, at the time, because of the inferences I had made, I thought that we were working within his ZPC. Because ZPD emphasizes an *expert's* mathematics and the student's *actions*, teachers using this zone may create the same kind of cycle of stagnation that I did in working with Will. The problem is that, in describing learning, ZPD does not account for students' operations, assimilations, and accommodations.

"For real understanding," Wertheimer (1945) claimed, "one has to re-create…the structural inner relatedness, the requiredness" (p. 194) of a situation. From a scheme-theoretic perspective, operations provide the structural inner relatedness and requiredness of problem solving situations, but Will seemed unable to reorganize his existing operations to construct ways of operating that indicated understanding of the tasks that I was posing. His actions in the teaching experiment indicate that he was often able to act as Hillary did, but lacked the inner relatedness that would allow him to operate flexibly, modifying his operations to account for variances in the tasks and problematic situations.

His actions were based on procedures and procedural schemes that served as a sequence of steps that he could follow.

For example, in forming Conjecture W4a, Will was able to produce a seven-fourths bar, but he did not understand that the parts in the bar were each one-fourth of the ruler. Such procedures were invented in the social context of assimilating Hillary's actions with his whole number knowledge and ratio reasoning, relating the numerators and denominators in such fractions to the number of parts needed in the unit bar and the fraction bar. He could not assimilate Hillary's actions using his fractional operations because he had not yet constructed a partitive fractional scheme. Limitations in his understanding of the situation were indicated by similar attempts at producing improper fractions with some variation in the context, as was the case in his actions surrounding Conjecture W4b.

Based on Will's actions in forming Conjecture W4a and producing a seven-fourths bar, I might conclude that Will had learned to produce improper fractions through his interactions with Hillary and that he would soon develop iterative fractions. After all, Will's actions and that conclusion fit Vytgotsky's (1978) description of a "social learning that precedes ontological development," one *lagging behind* the other (p. 90). Indeed, I did draw such a conclusion in working with Will, and it was to Will's detriment. Hillary had been working with Will at a "level of potential development as determined through problem solving in collaboration with more capable peers," otherwise known as Will's ZPD (p. 86). But what he learned was not a flexible way of operating, and the development that was supposed to lag behind (the construction of iterative fractions) did not develop during the teaching experiment.

Will could not develop iterative fractions from the procedures that he had learned in working with Hillary because (as I noted in my analysis of Protocol 17) he lacked the fractional operations to insightfully evaluate them. In my analysis of Protocol 19, I noted that Will's ability to resolve perturbations by acting with procedures, and his inability to identify reasons for their limitations, circumvented the need for operational changes that might have otherwise served in constructing a commensurate fractional scheme. In other words, *his procedures resulted from his assimilations of Hillary's actions as the students worked on tasks that were within Will's ZPD, but involved ways of operating that were not within his ZPC*. To illustrate the difference between students' constructions caused by the disparity between their ZPD and ZPC, I contrast Will's actions with those of Hillary during Conjectures W5 and W6.

As a result of Conjecture W5, Will constructed a procedural scheme for reversing ratios. In many situations, he used the procedural scheme to act just as Hillary would with her reversible partitive fractional scheme. This indicates that the latter scheme was at least in Will's ZPD. In fact, Will could use his procedural scheme to generalize some contextual details, such as the specific numbers in the fractional measure, but his actions in forming Conjecture W6 indicate that his scheme depended on other contextual details, such as starting from a *partitioned* fraction bar that does not have a simplified measure. In forming Conjecture W6, Will abandoned his procedural scheme for a new one, bringing into question whether his procedural schemes were permanent enough to call schemes at all. And, there was still contra-indication at the end of the teaching experiment that Will had constructed the reversible partitive fractional scheme.

Similar examples can be found in my analysis of Sierra's actions too (e.g. Protocol 13). *Both Sierra and Will had a motivation to assimilate their partners' actions using operations that were not insightfully related to the situation. This was due to the power of their partners' operations and my failures as a teacher. While the tasks that I posed were usually in the ZPD for each student in each pair, they involved ways of operating that were often in the ZPC for only one student in each pair.*

<u>Implications for further research</u>. Because I was focused on provoking conjecture, rather than promoting learning, it took several teaching episodes before I noticed that Will was not building new operational schemes from old one, and instead was constructing procedures that seemed to operate independently of each other. This problem would be compounded in a classroom of twenty-five students. I would like to conduct future research with whole classes, attempting to design tasks provoking conjecturing activity. I could then consider learning within the social dynamic of the classroom, while attempting to identify students operating as Will had versus those students operating as Hillary had. Teaching experiments could be conducted with individual students outside of the classroom to test my hypotheses about their conjectural operations and operational change. Building from the models of conjecturing and findings of this study, such future studies would have more direct implications for classroom instruction, specifically on how teachers can foster conjecturing activity in the classroom that would result in operational development.

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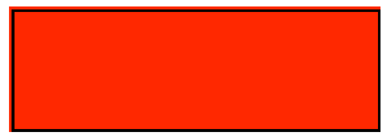
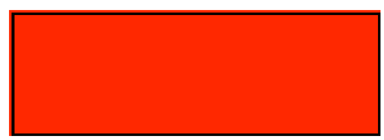
Appendix A: Interviews Tasks

1. a. Mark off 1/5 of the bar.



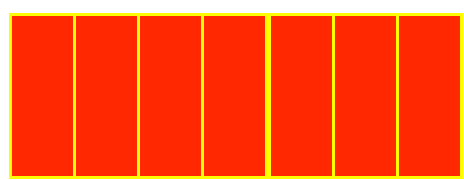
   b. Mark off 4/5 of the bar.

2. a. Suppose that you share this candy bar with 6 other people. Cut off your share.



   b. Do you think that your part is one of 7 equal parts? Show me that it is one of 7 equal parts.

3. a. If you cut off 2/7 of this candy bar, how much will be left over?



   b. (Hiding all but 3/7 of the bar). What fraction of the bar is hidden?

4. a. This is my piece of string. It is twice as long as your piece. Make your piece.

   b. How does the length of your string compare to the length of mine?

   c. (Repeat a. with “five times as long”)

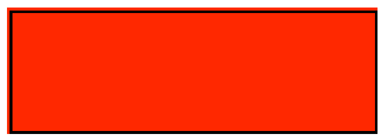
   d. (Repeat b.)

5. This string is 3/4 as long as the string I want you to make. Make your string.

6. Here is a piece of string. Make a string that is 4/3 as long as this string.

7. This string is 5/4 as long as the string I want you to make. Make your string.

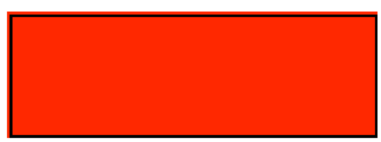
8. Show me 1/4 of 1/2 of this bar.



   What fractional part of the bar is filled?

9. a. Pretend this is a rectangular cake and that I take half of it. Cut off my half.

   b. Now you take 1/3 of the leftover part. Cut off your share.

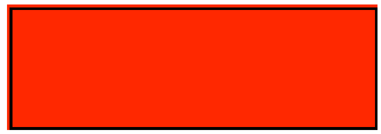


   c. What fraction of the whole do you have?

   d. What fraction of the whole is left over?

10. a. Cut this bar into thirds.

b. Now cut one third so that it is one sixth of the whole.



c. How can you tell whether it is one sixth of the whole?

Appendix B: Example from Spreadsheet

4/21/2003	**hw**	**students give problems for t**	
Time	**Mouse**	**Activity**	**Schemes**
0:00:00		make 7/8 using 16 parts	
0:01:15	w	pulls 8 from 16 and 7 from 8	part-whole
0:03:15	w	iterates in unit bar. W: 2 and 1/16	pufs for cu's
0:05:00	w	measures 7/16 "bc I took 7 parts out of 16"	pw
0:05:30		what is it 7/8 of?	
0:07:00	w	"well, if you reduce 16 you get 8"	
0:08:00		`	
0:08:30	w	makes 7/8 and compares to 16/16 to draw 14	size
0:10:15		what fraction is half of 14/16? W: 7/8	
0:11:00	w	shades 14/16 by 4's, then 2's to find half	
0:12:30	w	pulls 7 W: 7/14	
0:13:30		students discuss and count 14. H had thought there were 16 and then agreed w/ 7/14!	
0:15:30	wh	pose for t: large #'s, h/v, recursive part/pull, wipe, rotate, big cover	
0:21:00	t	measures 1/8 and iterates to 8/8	
0:22:50	hw	pose for t: 32x32 h/v, recursive pulling until 1 part, rotate	
0:26:00	t	measures 1/1024, "1024=32x32" repeats 32 v, measures 1/32 and repeats 32 h	
0:31:15	hw	W: "we need to get a number that can be reduced (like 14/16) but with a high number"	
0:32:30	hw	W thinks 32 h and 20 something v won't be able to multiply	
0:34:00	hw	restarts with smaller numbers bc of slow CPU. 18h/16v. Pulls some and measures 1/16	
0:36:00	hw	W: "cut that in half" to make harder. H pulls differently and stilll measures 1/36	
0:37:30	hw	pulls, measures 1/32. pulls 2 of those, measures 1/96. pulls 2 of 3, measures 1/144	
0:38:30	hw	want to get a non-unit fraction so I can't just add more	
0:39:30	t	measures 11/36. W wants to save this one (thinks it's a good one)	
0:40:00		End	