VILMA MARIA MESA
Conceptions of Function Promoted by Seventh- And Eighth-Grade Textbooks From Eighteen Countries.
(Under the direction of JEREMY KILPATRICK)

This study investigated the conceptions of function enacted by problems and exercises in 35 mathematics textbooks for seventh- and eighth-grade students from 18 countries participating in the Third International Mathematics and Science Study (TIMSS). The notion of conception used was that of Balacheff: a quadruplet consisting of problems, operations, representation systems, and control structures. A coding system was developed that had 10 codes for problems (given by the use of function in the problem), 35 for operations, 9 for representation systems, and 9 for control structures. Von Eye's Configural Frequency Analysis was used to determine types and antitypes of configurations.

Five conceptions of function were identified as promoted in the textbooks: symbolic rule, ordered pair, social data, physical phenomena, and controlling image. The different characteristics of the conceptions suggested that different school practices were associated with each conception. Groups of countries were identified whose textbooks shared similar characteristics. Across countries, the textbooks fell into four clusters according to the predominant conceptions and uses of function: rule oriented, abstract oriented, abstract oriented with applications, and applications oriented. The results suggested that (a) there is no canonical curriculum for teaching function and (b) there are no traditions of organizing mathematics textbook content on function.

Ten items from the TIMSS achievement test were coded and compared with the tasks in the textbook clusters. Performance on the item by students in countries using textbooks promoting the same conception was also examined. The results suggested that (a) the test did not reflect any country's distribution of conceptions and (b) using a textbook belonging to a particular cluster or promoting a certain conception did not provide an advantage to students. Thus, at the micro level of textbook content, there is no evidence that one organization is better than other in terms of student achievement.

Arguments linking student achievement on the TIMSS test to the use of specific textbooks were challenged.

The study illustrated the application of the four-dimensional definition of conception to three questions about textbook content in mathematics. It suggests possible applications to other mathematical notions and other areas of research in mathematics education.

INDEX WORDS: Functions, Conceptions, Textbook Analysis, Curriculum, Middle Grades, International Comparisons, TIMSS, Configural Frequency Analysis, Mathematics Achievement

CONCEPTIONS OF FUNCTION PROMOTED BY SEVENTH-AND EIGHTH-GRADE TEXTBOOKS FROM EIGHTEEN COUNTRIES

by

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To

Alfred Bruno, Ana Sofía, Patricio, for showing me new meanings of love.

In memory of

Sneeha Iyer, Alvaro Mesa, and Mauricio Castro

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And thanks to God, my shepherd; I trust Him, I fear nothing.

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CHAPTER 1

RESEARCH PROBLEM

> "In a riddle whose answer is chess, what is the only prohibited word?"
> I thought a moment and replied, "The word chess."
> "Precisely," said Albert. "*The Garden of Forking Paths* is an enormous riddle, or parable, whose theme is time; this recondite cause prohibits its mention. To omit a word… is perhaps the most emphatic way of stressing it."
>
> Jorge Luis Borges (1996)

> "It was working as a mathematician that I came to learn the right definition of function," he said.
>
> "And which one is it?" I asked with some amusement.
>
> "That's a good question. [Long pause] You know what? Maybe there is more than one right definition; it depends on what you need it for."

About a year ago, I had this dialogue with a mathematician when I was explaining with some difficulty what my dissertation topic was about. We had been chatting about the evolution of function, the differences between Newton's and Leibniz's approaches, Dirichlet's definition, and, finally, the work of Bourbaki. Then he spoke about "the right definition." Our dialogue became, in some ways, paradigmatic of my research. This mathematician saw several right definitions for function, and he realized that their rightness depended on their use. I had been wondering if that could be true for school mathematics.

I was interested not only in distinguishing one definition from another but also in how the curriculum uses those definitions. Are the definitions just tokens presented for the sake of completeness, or do they play a constituent role in building a useful notion of function? In other words, does the curriculum provide contexts in which the definitions of functions are used in meaningful ways? This study was designed to suggest ways to address these questions.

The purpose of this chapter is to give a rationale for studying functions in school mathematics by comparing textbooks from different countries and at the same time to present the evolution of the problem under investigation that ended with the formulation of the research questions. I also provide the theoretical framework that supported the study.

Functions and School Mathematics

The notion of function is a most important one for mathematics. It evolved from being a numerical entity (as represented by Babylonian tables) to become an equation (for Leibniz and Euler), an arbitrary correspondence between numerical intervals (for Dirichlet), and finally a correspondence between any pair of not necessarily numerical sets (a detailed account of its evolution is presented in chapter 2). This last definition, launched at the beginning of the twentieth century by Bourbaki, brought "a coherence and simplicity of viewpoint which did not exist before and led to discoveries … that [made] possible major advances in mathematics (Buck, 1970, p. 237). Thus, for example, something as simple as the addition of natural numbers could now be expressed as a function from $\mathbf{N} \times \mathbf{N}$ into $\mathbf{N}$, that assigned to each ordered pair (a, b) in $\mathbf{N} \times \mathbf{N}$ the natural number $a + b$. That amplification of the definition "made mathematicians realize that the rigorous study of functions [extended] beyond those used in calculus and in analysis" (Even, 1989, p. 47).

In 1908, Felix Klein, interested in the unification of the school mathematics and aware of the importance of the notion of function, advocated the introduction of functional thinking at all school levels (Sierpinska, 1992, p. 32). Klein was "successful in getting Germany to include analytic geometry and calculus in the secondary school curriculum, and other European countries followed suit" (Kilpatrick, 1992, p. 135). The trend, though slower, was also present in the United States (Cooney & Wilson, 1993, p. 137; an expanded account is given in chapter 2).

In the late 1950s, the Sputnik phenomenon marked the beginning of an era of curriculum development projects guided by mathematicians. The new math movement that swept the globe made a stronger commitment to the use of function as a unifying concept for school mathematics. Mathematicians were convinced that teachers who were willing to "introduce the set definitions for relation and function in one or more of their classes" would find that "the results may be rewarding" (May & Van Engen, 1959, p. 110). But they could not foresee that the results, pedagogically speaking, might be anything but rewarding. What had been unifying for mathematics began to create many problems for school mathematics.

The area that has benefited the most from these curricular innovations has been the field of mathematics education itself (Stanic & Kilpatrick, 1992). In the case of functions, the incorporation of the set-theoretical definition into school mathematics stimulated researchers in mathematics education to investigate the connection between the "unifying" definition and the difficulties that students face when attempting to use it (Eisenberg, 1991, p. 141). Research devoted to understanding several aspects related to its teaching and learning include: the meanings given to it by students (Vinner, 1992), by teachers (Norman, 1992), and by prospective teachers (Cooney & Wilson, 1993); the role of representations (Janvier, 1987; Romberg, Fennema, & Carpenter, 1993); the nature of the notion (Freudenthal, 1983; Sfard, 1991) and of the difficulties involved in teaching and learning it (Artigue, 1992; Sierpinska, 1992); and the role of technology (Tall, 1991), among others.

Recent reform movements attempted in many countries justify the teaching of function because of its fundamental character. The rapid development of technology (e.g., graphing calculators) has also made an impact on suggestions for the teaching of functions (see chapter 2). In North America, for example, *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics

(NCTM, 1989) suggested that teachers use more intuitive approaches to the teaching of function that would help students better understand its meaning (see, e.g., Standard 8 on patterns and functions, Standard 9 on algebra, and Standard 10 on statistics for grades 5-8, pp. 98-108). In spite of all these efforts, however, complaints about students' poor understanding of functions at the college level continue to be heard (Carlson, 1998), which suggests that there is discrepancy between expectations as to what should be learned about functions at the school level and what is actually learned. Functions are, indeed, still a difficult topic to teach.

Thus, on the one hand, function is recognized as a fundamental notion for organizing mathematical knowledge and, on the other, efforts to make its definition more understandable appear to be failing. The challenge is still there as to why it is so difficult for students to learn the notion of function and to take advantage of its unifying power.

All new curriculum development projects produce their own textbooks in which the goals, expectations, and philosophy of the project are put into action. In the United States, publishers produce textbooks for students, teachers' editions, batteries of tests, black-line masters for the overhead projector, extra worksheets, and other material considered necessary to put the curriculum into action. In most other countries, the student's textbook is likely to be the only resource available. In any case, though, the student's textbook guides classroom activity and at the same time legitimates the knowledge to be taught (Chevallard, 1985). Neither the teacher nor the student is likely to challenge it. Despite the appealing language of textbooks, they are actually written for teachers (Dörfler & McLone, 1986), who may play a decisive role in choosing the book in those countries in which that decision is not centralized. Other decisions that are available for the teachers include how the topics are organized, the order in which the text is followed, and the choice of exercises that the students are supposed to solve. The initial questions that guided the present study concerned what exactly is available about functions in textbooks and what the implications are for the meaning of *function*.

My experience as a student and as a curriculum developer in Colombia had shown me that there could be differences within a school system in the approach to function taken in textbook materials. In addition, my experience as a graduate student in the United States had indicated that there could also be differences across countries. I decided to undertake an inquiry about differences in the presentation of function in textbooks from different countries.

TIMSS

The issue of differences in presentation of various topics of school mathematics was addressed by the curriculum analysis component of the Third International Mathematics and Science Study. TIMSS, sponsored by the International Association for the Evaluation of Educational Achievement (IEA), was the most important international survey of educational outcomes in the 1990s. The main part of TIMSS involved 48 countries. It had two components: achievement and curriculum. The achievement component tested students in science and mathematics at three moments of schooling: the primary level (grades 3 and 4), the lower secondary level (grades 7 and 8), and the upper secondary level (the last year of schooling). The curriculum component collected curriculum guides, textbooks, and teachers' materials at these grades and in-depth information on curriculum sequencing, in both science and mathematics.

The results of the curriculum analysis presented a mixed picture of the implications for achievement of the emphases given to the topics. The data were analyzed only in terms of the way in which topics were handled in general: space typically devoted to each topic, number of topics per year, presence or absence of topics in curriculum guides and textbooks, number of textbook blocks devoted to particular topics, and so forth. The information published in the reports (Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999; Schmidt, McKnight, & Raizen, 1997; Schmidt, McKnight, Valverde, Houang, & Wiley, 1996) does not allow a description of differences in the organization and presentation of particular curriculum topics that might help explain the students' achievement. I decided to reanalyze the textbooks to describe differences in the presentation of functions in textbooks from different countries. This decision leads to the question of why one would conduct a textbook analysis.

Textbook Analysis

Textbooks have many purposes. They "expound the body of acceptable theory" (Kuhn, 1970, p. 10); they are powerful media for teaching and learning (Tanner, 1988, p. 141); they "determine what is school mathematics (in a similar way to syllabuses and examinations)" (Dörfler & McLone, 1986, p. 93); they are essential for "effective learning in developing nations" (Farrell & Heyneman, 1994, p. 6360), and "together with examinations and assessments, serve an accountability and control function" (Woodward, 1994, p. 6366). Textbooks seem to be an indispensable aid for the beginning teacher, who is "more likely to depend upon formal textbook methods than teachers with several years of experience" (Whipple, 1931, pp. 24-25, cited by Tanner, 1988, p. 116). They provide a source of exercises and assignments even to teachers who do not use them for other purposes.

The empirical study of textbook content has been justified mainly by an interest in predicting the outcome of students' learning as measured by tests. The textbook and test content are matched to see how similar they are, and a large discrepancy is used to explain low student achievement (Freeman et al., 1983). This way of looking at textbooks has been criticized, first, because it assumes that the tests used are valid and reliable (Keitel & Kilpatrick, 1998) and, second, because it does not acknowledge that the teacher, the other students, and the instruction play a critical role in shaping what is finally accomplished in a classroom (Stodolsky, 1989). What students learn from textbooks and the practicality of that learning are mediated by the school context (teacher, peers, instruction, assignments). Thus, the textbook is a source of potential learning. It expresses what has been called the *intended curriculum* (the goals and objectives for mathematics intended for learning at a national or regional level; Travers &

Westbury, 1989, p. 6), which implies that an analysis of textbook content becomes in some ways a hypothetical enterprise: What *would* happen *if*…? becomes the beginning of the inquiry. What would students learn if their mathematics classes were to cover all the textbook sections about functions in the order given? What would students learn if they had to solve all the exercises in the textbook? Would they learn what a function is? Would that learning work well in characterizing *function*?

Thus, I was interested in whether different definitions of function could coexist within school mathematics, as the mathematician of my anecdote suggested, and I decided to focus on how those definitions were made available in textbooks. I considered that the best textbooks to check the presence of those definitions would be those intended for the grades in which function typically begins to appear explicitly in school mathematics. For this reason, I concentrated on seventh- and eighth-grade textbooks.

Theoretical Framework

If the textbook is to be the object of a study, the researcher must determine what aspects of the textbook to focus on. Theoretical and methodological tools are needed to accomplish the task of deciding what to look at in the textbook. I chose the notions of *conceptions* (Balacheff, in press) and *prototypical domains of application* (Biehler, in press) of functions. The works from which these notions come address the meaning of school mathematics concepts. In this section, I discuss the two terms and how they were interpreted in this study to operationalize the research questions.

Conceptions

In this document, following the French tradition, the word *knowing* is used as a noun to distinguish the students' personal constructs from *knowledge,* which refers to intellectual constructs recognized by a social body. Although both terms refer to intangible constructs, the knowings are particular to the individuals that have them. Different situations generate different interactions between the subject (i.e., the cognitive dimension of a person) and the milieu (only those features of the environment that relate to the knowledge at stake), and in consequence lead to different knowings. The different interactions explain the coexistence of multiple knowings by a subject. Contradictory knowings can coexist, either at different times of a subject's history or because different situations enact different knowings. In both cases, what is isomorphic to the observer—probably the teacher—is not for the learner.

To tackle the problem of the existence of these contradictory knowings, (Balacheff, in press) proposed a definition for *conception* as follows:

> A *conception* is a quadruplet (P, R, L, Σ), where P is a set of problems, R is a set of operators, L is a representation system, and Σ is a control structure. P contains those problems for which the given conception provides tools to elaborate a solution. R contains those activities needed to simulate the procedures used by the students to tackle the problems. The representation system is defined as the tools needed to allow the formulation and use of the operators. The control structure can be understood as the metacognitive procedures available to the student by which he or she can check that his or her actions are legitimate and correct.

With these definitions, it is possible to speak of the domain of validity of a knowing as the union of the domains of validity of the related conceptions. As the conceptions correspond to the expression of a subject's knowings enacted by a situation the definition allows the co-existence of more than one, possibly contradictory, knowings in the subject.

This view implies that variations in the set of problems that learners face, together with the operators, the representations, and the metacognitive strategies needed to organize the work, lead to different conceptions of function. For example, the problems that Newton faced, mostly based on physical experiments, in contrast to the problems that Dirichlet faced, analyses of the convergence of Fourier series, required and used a different set of operators, representations, and control structures, which in turn made it possible for Newton and Dirichlet to operate with two different conceptions of function. The interest in establishing the foundations of mathematics at the beginning of the twentieth century and the appearance of set theory led to a different set of problems, operators, representations, and control systems, which resulted in yet another conception of function.

As a way to characterize the issue of more than one definition, I chose to study the possible conceptions enacted by a textbook. As the issue of the set of problems is fundamental—they must be chosen to elicit a particular conception—I chose to analyze textbook exercises with the purpose of describing the possible conceptions these exercises could enact. Consider Table 1, which presents the conceptions in two exercises from two hypothetical textbooks. The two problems yield two different conceptions of function. The need for a physical model for the first situation, together with the need for manipulation of the model, produces a sense of the variation in the period *as consequence* of the change in the length of the pendulum; the reverse interpretation seems illogical (how does the length vary when the period varies?). In the second problem, the relationship is given. The student is asked to make it explicit in a different representation. This second conception seems more powerful mathematically; the relationship stands by itself independently of a possible context from which it originated. The relationship can be established in either way (e.g., abscissa depending on ordinate). Even though from the standpoint of the Bourbaki-Dirichlet definition the two conceptions are equivalent, it is likely that for the student they are separate entities. The differences in the operators and representations used contribute to seeing them as separate.

The issue of the characterization of the problems is crucial for establishing the conceptions. I introduced a second notion, *prototypical domains of application*, to assist in characterizing the set of problems.

Prototypical Domains of Application of Function

Biehler (in press) stresses that a concept may have different meanings in different disciplines, and that those meanings are determined by the differences in practices in each discipline. Regarding the task of teaching mathematics as a social endeavor, he argues

Table 1

Example of a Characterization of Two Conceptions of Function

Component	Conception 1	Conception 2
P: Problems	Determine the relationship between the period of a pendulum and the length of its arm.	Determine the relationship between the abscissa and the coordinate of the given ordered pairs. Seven pairs of ordered pairs, with integer abscissas running from –3 to 3 are given {(–3, –6), (–2, –4), (–1, –2), (0, 0), (1, 2), (2, 4), (3, 6)}.
R: Operators	The student needs to act on a real pendulum. Using a stopwatch, the student needs to take and record several measures of the period as the length of the pendulum varies. Depending on the accuracy of the measures, it might be necessary to use an average for estimating the actual period.	The student needs to find an algebraic expression that takes the abscissa and transforms it into the ordinate of an ordered pair. The expression must work for every ordered pair of the set. Beginning with the pairs with positive abscissas, the student needs to notice that the ordinate is twice the abscissa. The student needs to write an expression for the relationship, using x for the abscissa and y for the ordinate. In a different approach, the student draws the set of points in the Cartesian plane and uses the information to build the expression, either taking into consideration properties of the line depicted or using formulas for slope and intercept.
L: Representation System	Numerical and symbolic.	Symbolic and eventually numerical and graphical.
Σ: Control System	The student may need to take several measures for the same height to control for possible errors.	The expression must work for every pair in the list. Depending on how the problems are embedded in the lesson, the student may find it irrelevant to have more than two ordered pairs to determine the relationship.
Conception	Function as dependency relationship (cause and effect). The number of oscillations per unit time decreases as the length of the arm increases. The student may be able to determine the type of dependence and the expression relating the two quantities.	Function as an abstract entity, with a correspondence determined by a rule that relates two numbers. In this case it is clear that it is a matter of convention which number is called x and which is called y. It is also feasible to begin with the ordinate and by the same procedure find an expression for the abscissa.

that the teaching of a mathematical concept cannot be limited to the meaning given inside the sphere of academic mathematics. Consequently, it is necessary to incorporate the meanings given to the concept in other practices as well:

> As mathematics education, however, has to base its curricular decisions on a broader picture of mathematics than that of academic mathematics, we consider the *reconstruction of meaning,* the development of a *synthesizing meaning landscape* of a mathematical concept, to be an important task for the didactics of mathematics that could serve as a theoretical background for curriculum design and implementation.

For Biehler, three elements are constitutive of the meaning of a mathematical concept: the domains of application of the concept (its use inside and outside mathematics), its relation to other concepts and its role within a conceptual structure (a theory), and the tools and representations available for working with the concept.

Using as an example the concept of function, Biehler (in press) identifies the "prototypical ways of interpreting functions (prototypical domains of application) which summarize essential aspects of the meaning(s) of functions." These are natural laws, causal relations, constructed relations, descriptive relations, and data reductions:

> The relation between the quantity and price of a certain article is a constructed relation: it is imposed by fiat (Davis & Hersh, 1980, p. 70). Using a parabola for describing the curve of a cannon ball has the character of a physical (natural) law. Contrary to this use, a parabola used in curve fitting may just provide a data summary of the curvature in a limited interval. Using functions for describing time dependent processes are different from using functions for expressing causal relations: time is not a "cause" for a certain movement. … In many statistical applications, functions are used to describe structure in a set of data that cannot be interpreted as a natural law.

Biehler notes that the concept of causal relation has been abandoned in mathematics in favor of a

> "functional relation" between two quantities (Sierpinska, 1992). This may be due to philosophical reasons but also to simple pragmatic ones: If we have a 1-1 correspondence, we can invert the cause-effect functional relation to infer the 'causes' from the effects.

The decision to invert the relation is rooted in the academic practice of mathematics; in disciplines such as physics, it might not make sense.

Biehler's characterization of prototypical domains of application of function, that is, its uses, was instrumental for me in initiating a characterization of the problems in a textbook that eventually can be solved by the student. These different uses gave me a stepping stone to use in characterizing the problems needed to define the conceptions that could be elicited by textbook exercises. With these theoretical positions established, I formulated the research questions that guided the study.

Research Questions

The original question about possible definitions of function in school mathematics was narrowed down to consider only those present in textbooks. As practices within a country can be similar, looking at more than one country offers the possibility of eliciting different definitions across countries. I considered that the best textbooks to check the presence of those definitions would be those intended for the grades in which function typically begins to appear explicitly in school mathematics, thus giving me the possibility of studying students' first encounters with the definition. My interest in elaborating the TIMSS findings led me to choose those countries participating in TIMSS whose textbooks I could read. Finally, because what matters is how students perceive a *definition* of function I found that an appropriate theoretical characterization of that perception about function was the students' conception of function. The definitions are fixed, but what students conceive is not. These precisions led me to phrase my first research question in terms of conceptions as follows:

1. What conceptions of function are suggested by the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS?

Because conceptions belong to human subjects, this question is posed to indicate that the application of Balacheff's framework to a textbook leads to conjectures about conceptions that could potentially be held by a student who used the textbook; in this sense it is an abuse of the definition. For Balacheff, the use that I propose "leads to conjectures about the spheres of practice" rather than to conceptions (personal communication, April 26, 2000). Because of the exploratory character of this first question, I chose to assume that the analysis would yield conjectures about conceptions.

The availability of textbooks from several countries, with more than one textbook from some countries, led me to a question about the differences to be found across the textbooks of the countries considered. I posed the following research question:

2. What patterns of conceptions are present in textbooks from different countries?

This question was aimed at disclosing the results of possible influences due to political traditions; it could be possible, for example, that textbooks from Spanish speaking countries are similar because of the links maintained with Spain; similarly, English-speaking countries might have textbooks that are alike as many of them share deep roots with the United Kingdom.

One might discover the possible advantages of one conception over others by observing students solving problems associated with different conceptions. Given the characteristics of the textbooks chosen, this issue was addressed by looking at how students potentially exposed to particular conceptions performed on the function items in the TIMSS test. Thus I formulated the following research question:

3. What is the relation between the conceptions suggested by the textbooks of a country and its students' performance on items related to functions on the TIMSS test for seventh- and eighth-grade students?

As posed, these questions referred to textbooks and their content concerning functions. They did not deal with claims about actual teaching or learning but only with potential implications for both of them.

In Chapter 2, I present a review of the literature related to functions, textbooks, and international comparisons. Chapter 3 contains the description of the methodology used to conduct the study. Chapter 4 presents the results of the analyses, organized by the three research questions. Chapter 5 contains conclusions, implications, and recommendations derived from the study.

CHAPTER 2

REVIEW OF LITERATURE

> At times I found pages where whole sentences were legible; more often, intact bindings, protected by what had once been metal studs…. Ghosts of books, apparently intact on the outside but consumed within; yet sometimes a half page had been saved, an incipit was discernible, a title.
>
> Umberto Eco (1980)

This chapter is divided into three sections. In the first section I discuss the literature on the concept of function grouped according to its history; the history of its teaching; views of function held by students, teachers, and prospective teachers; the nature of the function concept; and the ways function can be known. The second section is devoted to the literature on textbooks. The third section concerns international comparisons of textbooks.

The Concept of Function

Much research has been conducted on the concept of function. I limit the discussion below to very few works, referring the reader to more extensive discussions. I focused on the history of the concept because how working mathematicians shape the concept affects how it is transmitted. I looked at the history of its teaching because how it is taught affects what teachers and students know about it. I looked at the views held by students, teachers, and prospective teachers because those views constitute what it is known about the concept. The confluence of all these elements determines both the ontology and epistemology of the concept, that is, its nature and the ways available to know it. I discuss briefly works that propose explanations of these two fundamental questions about function, What is a function? And how do we get to know it?

History of the Concept

Early versions of the concept of function can be traced to the Babylonians and their use of tables for finding reciprocals, squares, square roots, cubes, and cube roots (Kline, 1972). This numerical treatment of functions, however, did not influence the development of the concept nearly as much as, for example, the algebraic notation developed by Viète or the development of analytic geometry begun by Descartes. Kleiner (1989), in his account of the evolution of the concept of function, uses three images to describe the successive changes that the concept has undergone: *geometrical, algebraic,* and *logical.* The constituent elements of these images are curve, formula, and correspondence (p. 282).

Leibniz introduced the term *function* in 1694 "to denote any quantity connected with a curve, such as the coordinates of a point on the curve, the slope of the curve, the radius of curvature of the curve, and so on" (Eves, 1990, p. 611). Bernoulli in 1718 and Euler in 1748 then described a function as an analytic expression, that is, an equation or formula involving variables and constants (Eves, 1990; Kleiner, 1989). Fourier's work on heat flow published in 1822 marked another important change in the definition. Before then, it had been accepted that "if two analytic expressions agree on an interval, they agree everywhere" (Kleiner, p. 285). Fourier's result showed that it was possible to use a series of sines and cosines to approximate any function on a given interval and thus that it was possible for "two functions given by different analytic expressions [to] agree on an interval without necessarily agreeing outside the interval" (p. 290). Dirichlet undertook the task of revising Fourier's work and in 1829 produced a new definition of function, introducing the concept of correspondence:

> y is a function of a variable x, defined on an interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established. (Luzin, 1998, p. 264)

Dirichlet's function, $D(x)$, where

$$D(x) = \begin{cases} c & \text{when } x \text{ is rational} \\ d & \text{when } x \text{ is irrational} \end{cases},$$

presented with the purpose of showing the necessary condition for the representability of a function by its Fourier series, represented an important break with the previous definitions of function because it was "not given by an analytical expression (or by several such), nor was it a curve drawn freehand; [it] was the first example of a function that is discontinuous everywhere; and [it] illustrated the concept of a function as an arbitrary pairing" (Kleiner, 1989, p. 292). Bourbaki's definition of function, which appeared in 1939, extended the definition to sets (not necessarily numerical) instead of intervals:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation* in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x.
>
> We give the name of *function* to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the *value* of the function at the element x, and the function is said to be *determined* by the given functional relation. Two equivalent functional relations determine the *same* function. (Kleiner, 1989, p. 299)

Bourbaki also gave a definition of function as a set of ordered pairs. In it, the function f is a subset of the Cartesian product $E \times F$ such that if $(a_1, b_1) \in f$, $(a_2, b_2) \in f$, and $a_1 = a_2$, then $b_1 = b_2$. More recent developments related to functions (L_2 functions, distributions, and category theory, see Kleiner, 1989) have not affected school mathematics and are not considered here. What is generally acknowledged is that the concept "pervades much of mathematics and [that] since the early part of the twentieth century, various influential mathematicians have advocated the employment of this concept as the unifying and central principle on the organization of elementary mathematics courses" (Eves, 1990, p. 612). How much of this employment has been accomplished is discussed in the next section.

History of Teaching the Concept

The idea that functions should be a unifying concept for the teaching of school mathematics seems to have appeared in the early 1900s with the work of Felix Klein, "who in 1908 in his *Meran Programme* advocated that functional thinking should pervade all of mathematics and, at school, students should be brought up to functional thinking (*funktionales Denken*)" (Sierpinska, 1992, p. 32). Cooney and Wilson (1993), identified at least three moments in the implementation of functional thinking in school mathematics: from the early 1920s until the 1950s, from the 1950s until the 1990s, and after the 1990s. In the 1920s fostering functional thinking was a priority, justified because it was required to understand and appreciate mathematics and because functions "were prominent in the real world" (p. 137). Barber (1924), in a book intended for high school teachers, suggested the following for the teaching of linear functions in the ninth grade:

> When a problem is met in which the quantities cannot readily be expressed in terms of one unknown, we have found the reason for two unknowns and the linear pair. The solution should be graphical at first, reviewing the knowledge of graphs, and applying it to the new situation. The graph is the bridge between algebra and geometry. It is a good plan to ask the pupil to draw a graph through points where x is 2 more than 3 times y, and then to write an equation expressing the same relation. In preparing to draw the graph, he will make a table of values which meet the conditions. He will sense a certain relationship of table, equation, and graph. This is a part of what is meant by the expression *functional relationship*, which is a very important generalization of mathematics. The teacher will do well to have it in mind as a somewhat remote objective for the child's thinking, but it is not important to use the word *function* in the ninth-grade class. The relationship of the three parts of the pupil's work should be mentioned, but that is all. (pp. 102-103).

This example, as well as those provided by Cooney and Wilson (1993), seems to show a mismatch between what was advocated—functions as an organizing concept of school mathematics—and the actual expressions of that advocacy. Not only did teachers seem to think that the concept should be taught in the higher grades (Breslich, 1928, p. 54), but the textbook content seem to be geared to skills rehearsal (Cooney & Wilson, 1993, p. 141), very much in the spirit suggested by the example.

The new math movement that swept almost the entire globe at the end of the 1950s saw the implementation of sets as organizing concept of mathematics and of school mathematics; structure and precision were to be emphasized. In 1958, the European Economic Cooperation Administration (EECA) (which became the Organisation for Cooperation and Economic Development, OECD)

> gathered in France a group of representatives from 20 countries with the goal of establishing major guidelines for what became the reform of New Mathematics. Policies necessary for implementing the reform were also discussed…. The purpose was to give unity to mathematics, using sets, relations, functions, and operations as basic concepts, as well as fundamental structures of groups, rings, fields, and vector spaces. The need to adopt modern symbolism was also established. (Ruiz & Barrantes, 1993, p. 1)

Consequently, preparation and translation of textbooks, curriculum changes, and training of teachers were promoted across the globe. May and van Engen (1959), in an article intended for teachers, criticized the definition of *function* given by *Webster's New International Dictionary* (a magnitude so related to another magnitude that to values of the latter there correspond values of the former, very similar to Euler's definition), calling it vague and not useful for the mathematician's or teacher's use: The definition "does not satisfy the requirements for precise statements demanded by the mathematical world. Neither does such a statement satisfy the requirements of good teaching. Vague statements do not facilitate communication between pupil and teacher" (p. 110). They advocated the use of the set theoretical definition because "this is a definite entity; one you can almost put your hands on" (p. 110).

Large curriculum projects were developed and implemented during this era (e.g., the School Mathematics Study Group in the United States, the School Mathematics Project in the United Kingdom, and a number of projects in Latin America; Howson, Keitel, & Kilpatrick, 1981, p. 133; see also Ruiz & Barrantes, 1993, p. 3). However, the diffusion of these projects was problematic:

> In the diffusion … much was lost. The variety of new work and the rigorously deductive methods were reduced or watered down; and topics such as set theory and algebraic structure lost their role as ‘relational’ links and became mere inventories of concepts. (Howson et al., 1981, p. 134)

Strong reactions were heard from mathematicians (e.g., Calandra’s paper, pp. 5-9, and Kline’s paper, pp. 13-16 in Moise et al., 1965; see also Thom, 1985) who believed that the logical, structural, and formal approach was detrimental for the intuitive understanding of mathematical concepts (Thom, 1985, pp. 71-73) and from mathematics educators, who saw the cognitive difficulties that the set theoretical approach posed for both teachers and students (Cooney & Wilson, 1993, p. 144; Eisenberg, 1991, p. 141).

The publication of the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) in the United States reinforced the unifying character of function:

> One of the central themes of mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and build mathematical models to predict the behavior of real-world phenomena that exhibit the observed pattern. The widespread occurrence of regular and chaotic pattern behavior makes the study of patterns and functions important…. In informal ways, students develop an understanding that functions are composed of variables that have a dynamic relationship: changes in one variable result in change in another. (p. 98)

But NCTM advocated a less formal and more intuitive approach to the teaching of function:

> To establish a strong conceptual foundation before the formal notation and language of functions are presented, students in grades 9-12 should continue the informal investigation of functions that they started in grades 5-8. Later, concepts such as domain and range can be formulated and the $f(x)$ notation can be introduced, but care should be taken to treat these as natural extensions to the initial informal experiences. (p. 154)

The *Principles and Standards of School Mathematics* (NCTM, 2000) took a stronger position with respect to the unifying character of functions. It described for each grade group the expectations set with respect to functions within an algebra standard. In Grades

Pre-kindergarten to 2, students should be introduced to patterns—numerical, geometrical, or related to their daily activities—and be able to notice, produce, and continue them (p. 91). The requirements for Grades 3 to 5 included the production of numerical and geometrical patterns and the ability to describe them "mathematically in words or symbols" (p. 159). In Grades 6 to 8, students were to become familiar with linear functions when analyzing constant rate of change (p. 223); students were expected to handle several representations and to study features such as slope and intercept in relation to those representations (p. 224). In Grades 9 to12, the repertoire of functions was broadened to include exponential, polynomial, rational, logarithmic, periodic functions, recursive and explicitly defined functions, and functions in two variables, and to consider arithmetical operations between functions as well as composition and inversion (p. 298). The four aspects that are discussed under the algebra standard have a functional orientation.

The rapid evolution of hand–held technology computing has made it easier to develop undergraduate courses in which function is a central concept (Demana & Waits, 1990; Demarois, McGowen, & Whitkanack, 1997; Gómez, Mesa, Carulla, Gómez, & Valero, 1995; North Carolina School of Science & Mathematics, 1998). The Visual Mathematics curriculum (Center for Educational Technology, 1995) is a technology-intensive curriculum for secondary school mathematics in which the concept of function "allows the organization of algebra curriculum around major ideas rather than technical manipulations" (Yerushalmi, 1997, p. 167). Schwartz (1991) states:

> We took the position that mathematically and pedagogically the primitive and fundamental object of the school subjects of algebra, trigonometry, probability and statistics, pre-calculus and calculus is the function. In fact, we take a stronger position—we maintain that the function is the only pedagogically necessary and desirable object in these subjects. (p. 303)

The central curriculum idea of the project is to consider three dimensions: *mathematical objects* (which include point, linear functions, absolute value, quadratic, power, relational, periodic, and transcendental functions), *mathematical actions* (modeling, transforming, and comparing), and *mathematical big ideas* (representation, conjecture, proof, invariance, symmetry, boundedness, betweenness, continuity, frame of reference, and scale). The project has designed computer programs (e.g., The Algebra Sketchbook, Yerushalmi & Shternberg, 1993; The Function Supposer, Schwartz, 1991) to fit their goals (p. 316).

Gómez et al. (1995) developed their precalculus curriculum by choosing seven *longitudinal* topics—linear, quadratic, cubic, polynomial, rational, radical, and transcendental functions—and treating them through eight *transversal* topics: work within the graphical representation, work within the symbolic representation, relation between manipulations, characteristics of the function, characteristics of the family of functions, systems of equations, inequalities, and applications (p. 14). The resulting matrix is both an organizational and a pedagogical tool; the matrix allows teachers to keep a perspective on what is expected to be accomplished and at the same time helps students contrast and make connections between and across the functions and their properties. The course was designed to take advantage of the availability of the graphing calculator and incorporated almost all of the themes of a course on algebra and trigonometry.

These approaches based on technology offer students tools for dealing with different aspects of functions that are not explicit in the definition (e.g., shape or zeroes). It might be possible, however, that the use of those tools constrains the path towards the development of a logical definition of function, as the relations defined are mostly given by rules. How well students have fared through these changes is the topic of the next section.

Students' Views of Function

The research into students' understanding of function is very extensive, which is not surprising given the importance of the concept in mathematics. Students' poor understanding of the concept has fueled much of this research (Eisenberg, 1991, p. 141).

Research conducted in the 1960s and 1970s highlighted aspects of understanding such as the point-wise view of function, in which students can plot and read points from a graph but are unable to "think of a function as it behaves over intervals (interval-wise) or in a global way" (Janvier, 1978; Marnyanskii, 1965/1975, cited by Even, 1989, p. 17). In the 1980s, the most important work was that of Vinner (1983), which has been considered seminal for much of the research conducted afterward. It provided a picture of what students understood about functions that also seemed to be common to the students in many other countries (Tall, 1986). Vinner coined the expressions *concept image* (what the individual thinks when a term is mentioned) and *concept definition* (how the individual defines a term) to characterize problems of understanding mathematical concepts:

> My basic assumption is that to acquire a concept means to form a concept image for its name. To understand a concept means to have a concept image for it. Knowing by heart a concept definition does not guarantee understanding of the concept. … Very often the concept image is entirely shaped by some examples and it does not fit the concept definition. (Vinner, 1992, p. 199)

In a study carried out with 146 high school students (65 in tenth grade and the rest in eleventh grade) in Jerusalem who were taught Dirichlet's definition of function, Vinner (1983) found that the students believed the following:

- The correspondence should be systematic and established by a rule; arbitrary correspondences are not considered functions;
- The function must have an algebraic expression, formula, or equation; it is a manipulation carried out on the independent variable in order to obtain the dependent variable;
- A function has two representations, a graphical one (either a curve in a Cartesian plane or an arrow diagram) and a symbolic one given by $y = f(x)$;
- A function cannot have more than one rule; a piece-wise function corresponds to more than one function;
- A domain cannot be a singleton; a rule with one exception is not a function; the domain must be constituted by contiguous intervals;

- A correspondence not given by a rule is a function if the mathematical community has so established it;
- The graph of a function is regular and systematic; and
- A function must be a one-to-one correspondence.

Vinner found that even when the students could recall the Dirichlet definition of function correctly, they also held some of the images described above. He called this phenomenon *compartmentalization*; "two items of knowledge which are incompatible with each other exist in your mind and you are not aware of it" (Vinner, 1992, p. 201). Vinner considered that the Dirichlet definition was part of the problem, as did Eisenberg (1991):

> At the definition level the function concept can be introduced in a variety of contexts, through arrow diagrams, tables, algebraic description, as black input-output box, as ordered pairs, et cetera. Of all of these approaches, the pedagogically weakest and non-intuitive one seems to be the approach using ordered pairs. Here, a function is defined as a certain sort of set; one which is made up of ordered pairs in which no two ordered pairs have the same first element and different second elements. This seemingly innocent definition proved to conjure up all kinds of logistic and epistemological problems, which incredibly, were often addressed explicitly in some school curricula. (p. 141)

More recent research, justified in part by the reported separation between the graphical and symbolic representations of function (Janvier, 1987), has dealt with promoting the understanding of the concept with the technology of graphing calculators and computers (Mesa, 1996; Mesa & Gómez, 1996). Ruthven (1990) suggested that the graphing calculator may help in the interpretation of graphs—building precise symbolic descriptions; using salient information such as orientation, position of extreme values, zeroes, and asymptotes; and relating the features of a graph to a symbolic expression (p. 91). Dick (2000) pointed out that the zooming option of the graphing calculator might help students understand holes in the graphs of functions (such as $y = (x^2 - 1) / (x + 1)$), the local linearity of functions (e.g., sin x near the origin), and the behavior of slope fields. Spreadsheets have also been proposed as a tool to help students understand the concept of function. Sutherland (1994) reports the use of spreadsheets for helping students to write expressions for rules defining functions, to find their inverses, and to find equivalent algebraic expressions. She emphasizes how the opportunity that the students have to experiment with a spreadsheet and check the results of their experimentation helps them find algebraically correct expressions.

Promising results obtained using computing technology (Carulla, 1996; Gómez et al., 1995; Schwartz, 1992, Yerushalmi, 1997) have highlighted the importance of the curriculum and the environment in which the technology is used. In many programs, group work, holistic assessment, and explorations with open-ended tasks are promoted, which makes it difficult to know to which change the results should be attributed. Because at the end what matters is students' understanding, however, all these approaches have been welcomed by the mathematics education community. Nevertheless, as with most changes proposed in education, how teachers put them into action may determine their success or failure. How teachers and prospective teachers understand functions is the topic of the next section.

Teachers' and Prospective Teachers' Views of Function

I discuss two studies that have addressed the views held by teachers (Norman, 1992) and prospective teachers (Even, 1989) about functions. Norman (1992), interviewed ten mathematics teachers who were working toward a degree in mathematics education. The purpose of the interview was to ascertain their concept images and concept definitions of function as defined by Vinner (1983). Norman found that teachers

- Preferred to use the graph of the relation (a) to establish whether it was a function or not, or (b) to test characteristics such as continuity or differentiability;
- Tended to think of functional situations as involving only numerical inputs and outputs;

- Had a concept definition aligned with the Dirichlet definition of function but were unable to deal with necessary and sufficient conditions that determine a function; and
- Had difficulty envisioning physical situations that entail functional relationships.

Norman also found that the teachers felt comfortable with and were knowledgeable about their textbook's introduction and development of function but that they had an image *fixation,* a commitment to a single view of function.

Even (1989) gave a questionnaire to 152 prospective teachers in their last year of preparation and then interviewed 10 of them. She found discrepancies between their concept images (both "modern" and "old" images) and their concept definition of function (as an equation). The participants in her study

- Viewed functions mainly as equations;
- Thought that graphs of functions should be "nice" (continuous) and smooth (differentiable);
- Did not accept that correspondences could be arbitrary;
- Rejected the notion of constant function, such as $f(x) = 4$; and
- Thought that the domain and range of a function should be sets of numbers only.

When asked about definitions that they would give to their students, these prospective teachers

- Tended not to use modern terms (e.g., *relation*, *mapping*, and *correspondence*);
- Used the idea of a machine or black box to illustrate a transformation process; and
- Used the vertical line test to characterize functions.

The prospective teachers had difficulty relating symbolic and graphical representations. When asked about general properties of the parameters of an equation defining a function, they tended to base their conclusions on very few examples. They held a point-wise view of functions and were unable to recognize important characteristics of functions in relation to their graphs. They interpreted composition of functions as multiplication; arbitrarily used an "undoing" process to find inverses; did not know the relation between the graphs of a function and its inverse; and had difficulty dealing with trigonometric, exponential, logarithmic, power, and root functions. Even (1989) attributed these results to a lack of "rich relationships that characterize conceptual knowledge" (p. 266). The prospective teachers had the knowledge but were unable to connect the different pieces to make it accessible.

These studies of people's views of function have been criticized conceptually and methodologically. Conceptually, the studies have tacitly accepted that students should learn the Dirichlet-Bourbaki definition of function, implying that it is the "right" definition of function. Markovits, Eylon, and Bruckheimer (1986, pp. 18-19) summarize the disadvantages of this definition in school mathematics from a practical (sets are not used in sciences or applications) and a mathematical (such a definition is not required until the study of analysis or topology) point of view. Methodologically, there are two main criticisms. In the first place, the distinction between concept images and concept definitions is not easily distinguished: "When a student defines a concept, is the definition a concept definition or a verbal description of a concept image? The researcher does not have means to distinguish them!" (Hooper, 1996, p. 7). In the second place, the environment established by the interviewer (or the test) invites the subject to give an answer that he or she thinks the interviewer is seeking. Thus when answering the question "What is a function?" the student returns to what he or she is supposed to know about the definition rather than to express his or her own understanding of the concept. Researchers' codification of students' answers as "fuzzy" is also an indication of deficiencies in both the questioning process and the interpretation framework (Balacheff, in press).

These accounts of the history of function and its teaching and of students', prospective teachers', and teachers' views of the concept of function have helped clarify the nature of and ways of knowing the concept from a pedagogical point of view. A concise description of these developments is given in the next section.

What Is a Function, and How Do Learners Come to Know It?

Sfard (1991) used the concept of function to propose a model for the nature of mathematical concepts and how students acquire them. She ascribed a dual nature to concepts, an operational and dynamic aspect (associated with the ability to carry out procedures) and a structural and static aspect (associated with the ability to see the concepts as objects). She claimed that both aspects are essential constituents of a mathematical concept. To be able to see a concept as a mathematical object, it is necessary to follow a continuum from *interiorization* (the student associates the concept with the procedures) to *condensation* (the student can see the procedures as entities) to *reification* (the student can see the concept as an entity independent of its associated procedures). She noted that historically the concept of function had followed the same path, which could be taken as an indication that in development operational conceptions take precedence over structural ones. That precedence in turn has instructional implications: "New concepts should not be introduced in structural ways [and] a structural conception should not be required as long as the student can do without it" (Sfard, 1992, p. 69).

A similar description of the way in which the concept of function is acquired is given by Breidenbach, Dubinsky, Hawks, and Nichols (1992; see also Dubinsky & Harel, 1992), who defined the terms *prefunction*, *action*, *process*, and *object* to describe the stages of cognitive development of the concept. With a prefunction conception, the individual is not able to display "much of a function concept"; an action conception “implies a repeatable mental or physical manipulation of objects”; a process conception involves a dynamic transformation of quantities, with the individual being able to “think about the transformation as a complete activity” applied to some objects that results in new objects; when the conception of function is that of object, it “is possible to perform actions on it, in general actions that transform it” (Dubinsky & Harel, 1992, p. 85). Note the similarity of the stages of action, process, and object to that of interiorization, condensation, and reification given by Sfard. These authors have shown the importance of procedural work for the act of learning mathematical concepts. At the same time, they show that neither a completely structural approach nor one that is exclusively procedural will help students learn a concept. Instead, it is more desirable to promote an interplay between the two.

These works that attempt to understand students’ cognitive processes can be called *psychological* because their problem is to explore what students have in their minds about functions. Because of the obvious difficulty of knowing what is in other persons’ minds, an alternative is to study the kind of knowledge produced within the situations in which a student acts. The work of Sierpinska (1992) and Freudenthal (1983) (and Balacheff’s, in press, conceptualization described in the previous chapter) represents this alternative.

Sierpinska (1992) identified 16 *epistemological obstacles* (Bachelard, 1938/1983) that appear in the process of understanding the concept of function—some of them rooted in its historical development—together with the acts of understanding that are needed to overcome them. Thus, for example, the view that mathematics is not concerned with practical problems is an epistemological obstacle about the philosophy of mathematics that is overcome by the “identification of changes observed in the surrounding world as a practical problem to solve” (p. 31). These obstacles should be seen as necessary part of the learning process and not as something that should be avoided, because it is through overcoming them that learning occurs. “The only alternative to painful learning [that occurs when acts of understanding are carried out] seems to be no learning at all”

(Sierpinska, 1992, p. 58). Sierpinska's work suggested explanations of the problems when students acquire the concept of function that are not necessarily rooted in students' lack of interest or ability in mathematics but in difficulty with the concept itself and with our perception of the world.

Freudenthal (1983) proposed a similar conceptualization by speaking about the importance of phenomena for the teaching of mathematical concepts:

> *Phenomenology* of a mathematical concept, structure or idea means describing it in relation to the phenomena for which it has been created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is *didactical phenomenology*, a way to show the teacher the places where the learners might step into the learning process of mankind. (p. ix)

In his *Didactical Phenomenology of Mathematical Structures,* Freudenthal (1983) devoted a chapter to the concept of function in which he showed the connections of the concept between different types of phenomena: mathematical, social, physical, and didactical (pp. 491-578). His work showed that the "logical simplicity" of mathematical structures did not imply developmental primacy, an implicit assumption of the new math reform. His work offered an alternative to the study didactic processes as they occur in relation to the mathematics and to the context of its teaching, explaining students' failures in terms of that relation and not in terms of their inability (see also Vergnaud, 1991).

<u>Summary</u>

The evolution of the concept of function has followed an interesting path, changing how people understand mathematics. The rapid changes that occurred in mathematics once the logical definition of the concept was introduced were echoed in school mathematics, generating difficult problems for the mathematics education community, problems that stimulated new lines of research in the field. The research has provided images of students', teachers', and prospective teachers' understanding of function, images shaped by teaching practices, by mathematical discoveries, and by people's cognitive capabilities. The research has also suggested alternative approaches in which less formal presentations are fostered, with technology playing an important role. The textbook, however, remains an object overlooked by researchers on functions. Textbooks synthesize what is known about a concept from multiple perspectives: historical, pedagogical, and mathematical. As documents, they provide valuable information about the potential learning that could occur in the classroom and thus an investigation of textbook content is necessary not only to complement the set of images of function but also to help explain its relation to the difficulty of learning the concept. The issue of research on textbooks is the topic of the next section.

Textbooks

Three yearbooks of the National Society for the Study of Education have been devoted to textbooks: the thirtieth (Whipple, 1931), the eighty-eighth (Jackson & Horoutunian-Gordon, 1989), and the eighty-ninth (Elliot & Woodward, 1990). Compared with other areas, however, and despite the strong association of textbooks with curriculum, the research that has been conducted with textbooks has been limited. Almost a half century ago, Cronbach (1955) proposed ideas for a systematic research program on textbooks, arguing that the research that "has examined the contribution of text materials … has been scattered, inconclusive, and often trivial. Philosophical study of texts has led to equally insubstantial results" (p. 4). In elaborating this program, McMurray (1955, p. 29) identified four different kinds of verbal communication, namely description, prescription, generalization, and theory. McMurray advocated a textbook that would contain a balance of each type of verbal communication; texts that were too prescriptive or descriptive would not help students gain reasoning skills. By making explicit the different types of communication, he hoped to help teachers evaluate the quality of textbooks and researchers analyze their content.

Some years later, the appearance of programmed self-instructional media led Lumsdaine (1963) to declare that textbooks were not an "amenable" object of research, given that the textbook "does not control the behavior of the learner in a way which makes it highly predictable as a vehicle of instruction [and] does not in itself generate a describable and predictable process of learner behavior" (p. 586). He foresaw that programmed self-instructional media would cause the "decline, if not the demise, of the textbook, as now conceived" (p. 586). The decline has not occurred, but developments in electronic communication may pose another threat to the textbook "as now conceived."

The difficulty of "controlling" the textbook as a variable may account for the scarcity of research on textbooks. Most of the research on textbooks conducted in the United States has analyzed textbooks in biology, history, geography, and especially reading. This phenomenon has been observed in France, too. Choppin's 1980 survey (cited by Johnsen, 1993) found that content analyses have dominated French textbook research, usually from a sociological perspective (ideologies, value systems, ways of describing society are questioned), and have centered on primary textbooks more than secondary. With respect to school subjects, the most common have been French as a mother tongue, philosophy, geography, and above all history because "historians and sociologists would probably encounter major problems if faced with the task of analyzing books in mathematics and physics" (Johnsen, 1993, pp. 59-60).

In mathematics, researchers have looked at the content of the textbook from a sociological perspective (Dowling, 1998), at the level of agreement of the curriculum present in the textbook with that of tests (Freeman, Belli, Porter, Floden, Schmidt, & Schwille, 1983) or with the NCTM *Standards* (Chandler & Brosnan, 1994), at the relation between textbook content and instruction (Flanders, 1987; Flanders, 1994; Freeman & Porter, 1988; Freeman & Porter, 1989; Kuhs & Freeman, 1979; Kuhs, Schmidt, Porter, Floden, Freeman, & Scwille, 1979), and at the emphases on certain topics present in elementary school textbooks (Li, 2000; Remillard, 1991; Stigler, Fuson, Ham, & Kim, 1986). The sociological studies—which have principally analyzed bias regarding gender and minorities—have had an immediate impact on publishers and authors, who have been discouraged from using stereotypes (e.g., women doing housework and men doing office work) and have increasingly included minorities doing mathematical work (e.g., portraying mathematicians from several cultures). The studies that link textbooks with achievement have demonstrated the importance of the role of the teacher as a mediator between what is in the textbook (Stodolsky, 1989) and students' performance on tests, corroborating in some ways Lumsdaine's observation about the role of the textbook in instruction.

With respect to teachers' use of the textbook, SIMS, the Second International Mathematics Study (Burstein, 1993; Robitaille & Garden, 1988; Travers & Westbury, 1989), and TIMSS, the Third International Mathematics and Science Study (Beaton et al., 1996), have shown that teachers tend to report high percentages of textbook use in their classroom. Other studies have found that teachers' use of textbooks decreases as they gain experience (Ball & Feiman-Nemser, 1988; Stodolsky, 1989). It also seems that a considerable number of exercises are assigned from textbooks—which might explain the high percentage of textbook use reported by teachers in SIMS and TIMSS.

Another area in which mathematics textbooks have been studied is with respect to the *didactic transposition* (Chevallard, 1985), that is, the process by which the knowledge developed by mathematicians is transformed into teachable knowledge. Balacheff (personal communication, April 1999) points out that this transformed knowledge is a different kind of knowledge that deserves careful analysis. Van Dormolen (1986) characterized aspects of a textbook such as correctness of content (without mistakes, consistency, clarity, and genuineness), global perspectives (cursory and conceptual preparation), and adaptation to student's ability (pp. 160-161). His interest was to discern the match between the textbook content and mathematics. Kang and Kilpatrick (1992) looked at the relationship between didactic transposition, mathematics instruction, and ways of knowing in mathematics as seen in U. S. textbooks.

Following Herbst (1995), these works can be characterized as *external critiques*; in them, the textbook is treated "as a piece of technology inside the educational system," (p. 2) "a technological product, a container, or a funnel of the mathematics to be learned" (p. 3). Those analyses "refer the textbook to its external environment, that being the educational system, the mathematics of the mathematician, or the process of transposition" (p. 3). In contrast, *internal critiques* consider the textbook as an "environment for construction of knowledge" (p. 3); the interactions of the elements inside the textbook (e.g., diagrams, examples, and explanations) are seen "as a product of the conflict between the temporal and spatial nature of texts" (p. 3). Examples of the latter kind of study are Otte (1986, p. 176), who analyzed the relationship between illustrations and explanations, and Herbst (1995), who analyzed the number line as a metaphor for the real numbers in a series of Argentinean mathematics textbooks.

In summary, textbooks are considered a crucial part of schooling, a fundamental curricular agent. The textbook seems to play an important role for novice teachers and seems to be the source of many exercises for students. The extent to which the content of a mathematics textbook matches or fails to match tests or curricular documents such as the NCTM *Standards* has been used to explain students' performance on tests of that content. Research on sociological aspects of textbooks has made textbook writers more aware of the implications of stereotypes. But there is little information on how particular topics are presented in textbooks. One exception is Howson (1995), who offers an interesting but limited account of possible difficulties in the presentations of selected topics in textbooks from eight countries (this work is discussed in the next section, see p. 41). There is also little information about the coherence and the relations among other topics in a textbook. In the case of function, there is little information about issues such as how the topic is introduced, what definitions are given to the student, what examples and exercises students are asked to do, what conceptions are privileged, or what advantages or potential problems might arise with various presentations. These issues are crucial for understanding the difficulties in the teaching of function discussed in the previous section. Although an analysis of a series of textbooks in one country might provide some answers, aspects that may not be obvious become explicit only when comparing textbooks from several countries. In the next section, I discuss cross-national studies that have focused on mathematics and textbooks.

International Comparisons

> Mathematics education has been a central area of comparative international research. Mathematics has always held a privileged position in the school system, in fact, it is one of the few subjects that is taught in most school systems worldwide (Howson & Wilson, 1986)…. This universal status and importance of mathematics, the similarity of mathematics curricula worldwide, and the supposed link between the study of mathematics or science and the development of a nation's economic strength (Walberg, 1983) make studies of international comparison in mathematics education of important interest to researchers, educators, and policy makers. (Robitaille & Nicol, 1994, pp. 405-406)

This quotation captures part of the rationale that has guided international studies in mathematics. Two organizations, the International Association for Educational Achievement (IEA) and the Organization for Cooperation and Economic Development (OECD), have conducted a series of large-scale international comparisons of different elements of school mathematics, focusing on students' achievement. I have limited this discussion to the IEA studies.

The first IEA study of mathematics (First International Mathematics Study, FIMS) was conducted in the 1960s (Husén, 1967); the second (SIMS) in the early 1980s (Burstein, 1993; Robitaille & Garden, 1988; Travers & Westbury, 1989), and the third (TIMSS), which included science, was conducted in the early 1990s (Beaton et al., 1996; Mullis et al., 1997; Mullis et al., 1998). (A summary of findings of the first two studies can be found in Robitaille and Nicol (1994)

The variable *opportunity-to-learn* (OTL) was used in FIMS to indicate the opportunity that students had to learn the mathematics necessary to respond correctly to a given test item. It was measured by asking teachers to rate items according to whether they have taught the related content to students. In FIMS, there was a "positive relationship between students' achievement on an item and the opportunity to learn the content of that item" (Robitaille & Nicol, 1994, p. 408). The results of FIMS were useful for noting the "tremendous variability between countries in many variables that are important to schooling in general, and to the teaching and learning of mathematics in particular" (p. 408). There were criticisms with respect to the operationalization of the OTL variable, as it was "too bound to the form of specific items and more representative of teachers' judgment of items rather than content categories of which the item is an example" and therefore could not be considered as "surrogate for national curriculum" (Schmidt & McKnight, 1995, pp. 344-345).

SIMS included a more intensive curriculum analysis than FIMS. Curriculum was analyzed through a three-level framework that included the educational system, the school and classroom, and the student. At the first level, the goals at a national or regional level for mathematics to be learned were called the *intended curriculum*. The interpretation of the curriculum by teachers in the classroom was called the *implemented curriculum*. Finally, what students learned as determined by their achievement on tests was called the *attained curriculum*. The study also took into consideration *curricular contexts* (institutional settings, school and classroom conditions and processes, and student behavior) and *curricular antecedents* (educational system features and conditions; community, school and teacher characteristics; and student background characteristics), to provide a comprehensive analysis of the three levels of the curriculum (Travers & Westbury, 1989, p. 6-9). One major finding dealt with an apparent decline in the study of geometry and increase in the study of algebra since FIMS. The new math reform movement that spread worldwide during the 1960s and 1970s apparently explained this trend. Differences across countries in tracking practices were found, but they were not enough to explain differences in achievement (Robitaille & Nicol, 1994, p. 410).

One important lesson that was drawn from these two studies concerns the difficulty entailed in comparing achievement results across countries. Countries with similar intended curricula showed different patterns of achievement. This result was also found in TIMSS. Neither FIMS nor SIMS included an analysis of textbooks, although SIMS reported a high rate of teachers' use of textbooks (Robitaille & Garden, 1988, p. 53-61).

TIMSS included an ambitious curriculum analysis project that sought information about curricular and textbook organization in mathematics and science in the 48 participating countries. Initial results of that project are reported in Schmidt, McKnight, Valverde, Houang, and Wiley (1996) and in Schmidt, McKnight, and Raizen (1997). The SIMS model was modified to include the *potentially implemented curriculum*, an intermediate level between the school system and the classroom, that included textbooks and other organized curriculum materials and that attempted to acknowledge, among other factors, the role of the teacher in mediating the implementation of the curriculum (Schmidt et al., 1996, p. 174).

Two listings of topics, the science and mathematics curriculum frameworks, were developed to give coherence to the TIMSS curriculum analysis (Martin & Kelly, 1997, pp. 5-7). The mathematics framework has 10 main topics with 24 subtopics. Some of the main topics contain two levels of specificity. For example, Topic 6 (Functions, Relations, and Equations) is divided into Topic 1.6.1 (Patterns, Relations, and Functions) and Topic 1.6.2 (Equations and Formulas). Topic 1.6.1 is then divided into 11 subtopics: Number Patterns; Relations and Their Properties; Functions and Their Properties; Representation of Relations and Functions; Families of Functions—Graphs and Properties; Operations on Functions, Related Functions—Inverse, Derivative, etc.; Relationship to Functions and Equations; Interpretation of Function Graphs; Functions of Several Variables; and Recursion. The available reports, however, do not consider the eleven subdivisions.

In the TIMSS curriculum analysis the researchers found a “pervasive variation” (Schmidt et al., 1997, p. 165) among countries in terms of the topics that were included in the textbooks and in the number of topics that were studied in a given year. Some countries had few topics each year (and in consequence teachers might devote considerable time to each), and others had a large number of topics each year, some of which were repeated across grades. Each textbook was divided into blocks (smaller segments within a lesson that could be narrative, informative graphic blocks related or nor related to instructional narrative, exercises and question sets, suggested activities, worked mathematical examples, or other; Schmidt et al., 1996, p. 200). The proportion of a textbook (the percentage of blocks into which the book was partitioned) “devoted to a particular topic was used as an indicator of the emphasis on that topic within a particular grade” (Schmidt et al., 1996, p. 113). Schmidt et al. acknowledge that these data suggest “possible rough bounds on emphasis” because teachers decide what use to make of textbooks and whether to cover all the information in them. Unfortunately, the report does not include information as to textbook use for the subtopic of Patterns, Relations, and Representations, because this topic was not “commonly intended and emphasized” (p. 115) at the eighth-grade level.

Large-scale studies have been criticized because they handle the results of students’ achievement as a “horse race” in which the public is told who won (had the highest score), which assumes that such a comparison is possible (Rotberg, 1998, p. 1030). In all these studies, there has been a tacit assumption that the tests were fair to all the students, that student populations and instructional practices were homogeneous within countries, that there is a canonical curriculum in mathematics, and that all countries are happy with their mathematical instruction (Atkin & Black, 1997, Keitel & Kilpatrick, 1998). The curriculum model (intended, implemented, and attained) has been seen as disrespectful to teachers, whose intentions are not considered and who are assumed to blindly follow “plans drawn by others” (Kilpatrick & Davis, 1993, p. 206). The impact that these studies have on policy has been strongly questioned, as the implication that successful neighbors are to be imitated might not be the solution to any country’s instructional problems in mathematics (Atkin & Black, 1997, p. 22; Keitel, 2000).

Smaller-scale international studies have offered more detailed accounts of particular topics in school mathematics and have been pursued mostly to complement the results observed in the large-scale studies. I report on the results of six of the former.

Schutter and Spreckelmeyer (1959) found that the American curriculum as seen in arithmetic textbooks lagged approximately two years behind curricula from European countries (Austria, Belgium, Bulgaria, Czechoslovakia, Denmark, England, France, East and West Germany, Greece, Holland, Hungary, Ireland, Italy, Poland, Rumania, Russia, Spain, Sweden, Switzerland, and Yugoslavia). Their analysis of the content of the examples and problems posed to the students at each comparable grade (determined by pupils' age) showed that European textbooks were

> carefully planned and [presented] a well-structured curriculum, with explanations tailored to fit the experience and background of the children who [studied] the texts. At every stage, students [were] encouraged to use both the information that [was] available and their own reasoning abilities to verify their work. Appeals to flexibility of mind and to creative thinking [were] made to a greater degree than in American textbooks. (p. 32)

They recommended a revision of arithmetic programs (pp. 34-35) so that their materials would devote more time to the study of arithmetic in the elementary school by including more challenging work, explicit application of fundamental laws of number operation (e.g., distributive property), explicit connections among arithmetical ideas, and more emphasis on early and gradual development of geometrical concepts and by deleting informational arithmetic (e.g., insurance, business, and budget). This study was important in that it used arithmetical problems as indicators of the content and the possible connections enacted within arithmetical topics. One major weakness, however, is that no clear guides with respect to the method of selection of the tasks are provided ("samples that were typical of their source texts," p. 2), nor are the criteria for analyzing them, which to some extent suggests that the report sought evidence to support the new math movement. Max Beberman, in the introduction to the report, praises the recommendations as "all children (and this includes American children) have a genuine need for working with abstractions…. The most successful mathematics curriculum will be the one which caters to this need for playing with ideas" (p. iii).

Stevenson and Bartsch (1992) analyzed surface (e.g., number of pages and length of chapters) and content characteristics of elementary and secondary mathematics textbooks from Japan and the United States. They selected the most popular series used in each country (it was difficult to accomplish this at the secondary level in the United States because of the lack of comprehensive statistics, p. 116). By determining the number of concepts and the place where they appeared for the first time in the textbooks, they could establish similarities and differences between both curricula in terms of content and timing. At both levels, elementary and secondary, Stevenson and Bartsch found that

> Japanese textbooks tend to be tersely written, while the American textbooks contain information that is not necessary for developing the concepts under consideration… [They] appear to be written so that understanding the content of the lesson is less dependent upon what happens in mathematics class. (p. 125)

The American textbooks also used a step-by-step approach (p. 109). Although the content of the curricula was similar, Japanese textbooks introduced concepts and skills earlier than American textbooks, which implies that Japanese children have more time to practice concepts and skills than American children and indicates that Japanese children are expected to master them faster than the American children (pp. 132-122). This study suggests that the "canonical curriculum" (Howson & Wilson, 1986, cited by (Kilpatrick, 1992, p. 139) expresses itself differently within countries. Thus, a study of these differences is worth pursuing because they may relate to achievement.

Stigler, Fuson, Ham, and Kim (1986) analyzed the difficulty of different types of addition and subtraction word problems by comparing four American text series with one series from the Soviet Union (Grades 1 to 3). They used a two-dimensional framework (semantic structure of the story—change, combine, compare, and equalize—and position of the unknown in the equation representing the story) to characterize the difficulty of the word problems present in the textbooks. They found that (a) the Soviet textbooks included more two-step problems than the American textbooks; (b) the number of one- and two-step problems decreased across grades in the Soviet series, whereas it increased in the American series; and (c) the Soviet textbooks tended to include a similar amount of word problems of all the types whereas the American textbooks tended to favor three of the simplest ones (pp. 163-165). This study is important because it highlights the importance of incorporating research results from psychology ("variation [and] repetition [are] crucial for learning," p. 169) and from mathematics education (problem characterization) into textbook construction, and because it shows the feasibility of conducting empirical studies with textbooks problems.

Howson (1995), as part of TIMSS, did a qualitative analysis of the content of eighth-grade textbooks from eight countries (England, France, Japan, the Netherlands, Norway, Spain, Switzerland, and the United States), analyzing six topics in particular (place value and decimals, fractions and proportionality, geometry, linear equations, measurement, and data analysis). His analysis included both surface characteristics (such as length and number of lessons and availability of review sections) and content (motivation and organization). The U.S. textbook, for example, "contained over three hundred units—each requiring one to three periods of class time! There is material here for three grades' work" (p. 28), but the Japanese textbook did not contain "work on arithmetic and indeed no review at all. This omission looks somewhat singular unless students in Japan have retentive powers not possessed by their peers elsewhere" (p. 51). He discusses differences in motivations for introducing some topics, such as multiplication of whole numbers or geometry, and some problems inherent in them (e.g., the model of the witch's cauldron for negative numbers from the Netherlands, p. 66). He also found that "the English, French, and U.S. texts go further than most in offering a wider range of learning situations, should the teacher wish to take advantage of these" (p. 87). The most important implication of Howson's study is that it is not possible to assume that "if a topic is introduced in several countries then it is always treated in the same manner and with the same degree of emphasis" (p. 66), which suggests that the different pedagogical resources available in each country to teach particular topics are worth knowing. One limitation is that Howson seems to ignore other aspects that might influence how textbooks are written (e.g., extreme societal pressures to excel in mathematics in Japan require children to attend after school programs in which students review and practice what is being learned; thus, the textbooks do not need to provide review sections that are common in textbooks from other countries).

Li (1999) analyzed the algebra content and the problems in nine eighth-grade textbooks from China, Hong Kong, Singapore, and the United States. Two features of content presentation and organization were analyzed: the inclusion and organization of units (content instruction, content review, cooperative learning, problem solving, technology, more practice, content extension, tests, other) and the instructional approaches (explanations, worked examples, illustrations, and to-be-solved—TBS—problems). Three features of TBS problems were analyzed: mathematics (the same as in the unit where the problems are, different, or mixed), context (illustrated or purely mathematical), and performance requirements (type of response—explanation required or not—and cognitive demands—conceptual understanding, routine procedures, complex procedures, problem solving, other; see pp. 98-107).

Li found differences within the American textbooks in terms of content covered and similarities in their tendency to split the content into various small units, put less emphasis on content instruction but more on student practice, and provide more problem-solving activities. The Asian textbooks tended to offer larger chapters that presented the algebra content from a pure mathematics perspective. The U.S. nonalgebra textbooks placed "less emphasis and lower requirement on algebra content than [did] the textbooks from East Asia" (p. 124). The U.S. textbooks tended have a greater variety of TBS problems than the East Asian textbooks, which included a single type of TBS problem that emphasized the performance of routine procedures in a purely mathematical context (pp. 125-126). Li also contrasted eighth-grade students' performance on five TIMSS items and concluded "textbooks' variations in mathematics requirements can provide partial explanations of differential student performance across educational systems" (p. iv).

Methodologically, Li's approach is innovative, as it combines both content and problem analyses. The quantitative measures used in Li's study, though, were limited to such measures as space devoted to a topic, number of chapters, and number and nature of section titles. These measures act as proxies for both quality of content and organization. Even though they can be appropriate at a macro level, they can mask the role that the actual content and organization play in presenting the mathematics. If a common section called “graphing systems of linear equations” were to appear in five textbooks, that commonality would not guarantee that all five books would treat the topic in the same way or that the conceptions enacted would be the same. Also, less instructional content in a textbook does not necessarily imply a poorer treatment of a topic.

Li provides extensive descriptions of the several features he considered, an effort that is valuable for disclosing patterns of content organization (e.g., content explanation may come before or after a worked example; worked examples may have illustrative contexts or verbal explanations; TBS sets are opportunities for practice; the content is introduced either with a problem that has a real-world like context or with an explanation of the mathematical knowledge, pp. 157-171). Li’s implication, however, that the Asian textbooks account for much of the higher students’ performance on the TIMSS algebra related items can be challenged. In a situation in which the outcomes of two countries are so different (in this case, students’ performance on the TIMSS test for the United States and the Asian countries) it is not very difficult to attribute that a difference to particular aspects in which their corresponding textbooks differ. Perhaps there are other countries whose textbooks look similar to the East Asian textbooks and whose students do not perform as well as the Asian students. The researcher needs to look for both confirmatory and disconfirmatory cases before making a generalization of this kind.

In a study involving the problems following the sections on addition and subtraction of integers in five U.S. and four Chinese textbooks, Li (2000) applied a three-dimensional framework (see Table 2) to analyze problem requirements. All the dimensions provided measures of complexity of the problem. The mathematical dimension referred to the number of operations required for solving the problem. The contextual dimension described the type of information provided in the problem. The performance dimension accounted for the type of response and the cognitive demands of the problem. Li found that the textbooks from the two countries were similar with respect to the first two dimensions. In both, the majority of problems required a single computation procedure and used a purely mathematical context (p. 238). U.S. textbooks, however, offered more problems that required conceptual understanding and explanations or solutions than the Chinese textbooks. The Chinese textbooks offered more problems requiring procedural practice than the U.S. textbooks. Li interpreted this result as a demonstration of the influence of the NCTM *Standards* on the U.S. textbooks (pp. 238-239).

Table 2

Dimensions of Problem Requirement.

Mathematical		
		Single computation procedure required
		Multiple computation procedures required
Contextual		
		Purely mathematical context in numerical or word form
		Illustrative context with pictorial representation or story
Performance		
	Response type	
		Numerical answer only
		Numerical expression only
		Explanation or solution required
	Cognitive requirement	
		Procedural practice
		Conceptual understanding
		Problem solving
		Special requirements

Note: Adapted from Li, 2000, p. 237.

Li's (2000) study is important in that it shows the viability of using problem analysis to describe salient features of content demands and cognitive requirements of the textbooks. One limitation is that the elements of the framework used to analyze problems are taken as separate entities and not in relation to each other. Thus, it might be possible that a cognitive requirement such as "problem solving" is tied to the response type "solution required" when an "illustrative context" is provided and that in those cases a "multiple computation" is required. As the categories seem interdependent, the issue of how the categories appear separately is less important than the issue of how and why they appear together.

Large- and small-scale international studies show that there are important differences in content presentation across countries and that these differences might have an impact on students' achievement on tests. Large-scale studies have been useful in showing general trends, which in some cases have been further explored with small-scale studies. The information and suggestions that have been produced as a result have nurtured the field in many productive ways by suggesting better methodologies and more interesting problems to study. Nevertheless, I find interesting the selection of countries against which U. S. textbooks are compared. During the new math movement, the United States focused on European and Soviet textbooks; and during the formation of the European Union and the Asian miracle, Japan, China, Taiwan, Singapore, and the European countries were studied. All efforts to find explanations for the intriguing phenomena involving textbook content and use from the natural competitors of the United States are welcome, but the dismissal of the analysis of textbooks from other countries—in fact, from any developing country—seriously limits the generalizability of the findings of these studies and their implications for the U.S. curriculum.

I find it intriguing that since the 1950s studies have consistently reported that the content presented in U.S. mathematics textbooks is fragmented. The issue of why this characteristic has not changed in 50 years is less important than the issue of why researchers still conduct studies that report the same "finding" repeatedly. This lack of progress reflects the minimal advancement in our techniques for analyzing textbooks and the limited status that the textbook has as "a piece of technology inside the educational system" (Herbst, 1995, p. 2), instead of being "an environment for construction of knowledge" (p. 3). With this study, I wanted to contribute to both aspects of the analysis of textbooks.

CHAPTER 3

METHOD

> The world was so recent that many things lacked names, and in order to indicate them it was necessary to point.
>
> Gabriel García-Márquez (1969)

This chapter is organized into five sections. The first section describes the procedure used to select the textbooks; the second describes the process of designing the coding system, with some examples; the third discusses the reorganization of the categories of the elements of the conceptions, with a brief description of the Configural Frequency Analysis program, CFA (von Eye, 1990, 2000), used to identify and characterize them; the fourth presents the procedures used to connect the information from the TIMSS achievement test with the conceptions; and the last discusses my own biases when conducting this research.

Textbook Sampling

Access to the TIMSS textbook archives at Michigan State University was made possible through the auspices of William Schmidt, director of the TIMSS curriculum analysis project. The archive contains textbooks and curriculum guides in mathematics and science from 48 countries. The first criterion for selecting a textbook was that I could read it. I therefore selected all the English, French, German, Portuguese, and Spanish mathematics textbooks in the archive. These textbooks came from twenty countries: Argentina, Australia, Austria, Canada, Colombia, the Dominican Republic, England, France, Germany, Hong Kong, Ireland, Mexico, New Zealand, Portugal, Scotland, Singapore, South Africa, Spain, Switzerland, and the United States.

The second criterion was that the textbooks were intended for seventh grade or higher. That eliminated the one textbook from France. It was included in the original coding but was dropped later when I discovered that it was intended for fourth grade.

The third criterion was that the textbook contained sections devoted to functions. I looked in the table of contents for words or phrases such as *functions, linear functions, graphing in two coordinates, graphing in the Cartesian plane, tables, patterns,* and *relations,* and I also looked under the entry corresponding to *function* in the index (only the textbooks of Canada and the United States provided an index). This criterion eliminated the textbooks from two countries, the Dominican Republic and Germany. Because some countries had more than one textbook, I numbered the textbooks in each country alphabetically by author. A list of the 35 titles and authors of the selected textbooks, together with the grades for which they were intended, is provided in Appendix A.

In each textbook, all pages of those sections marked as being related to functions were photographed with a digital camera, which allowed me to handle the pages as files in a computer, thus facilitating the analysis. The procedure also saved the considerable amount of time that photocopying would have taken. The Statistical Package for the Social Sciences (SPSS, 1997) software was the basic tool used to record the data from the textbooks. Table 3 lists the textbooks with their intended grades, the number of pages analyzed, and the number of exercises initially selected for the analysis.

Table 3

Intended Grades, Number of Pages, and Exercises Analyzed in Each Textbook

Textbook	Grade	No. of pages	No. of exercises	Textbook	Grade	No. of pages	No. of exercises
Argentina1	8	22	18	Scotland2	10	24	86
Australia1	8	29	26	Singapore1	8	20	41
Australia2	9	24	89	SouthAfrica1	10	45	90
Australia3	7	26	14	SouthAfrica2	7	15	22
Austria1	8	6	14	SouthAfrica3	9	14	40
Canada1	8	6	25	Spain1	8	19	77
Colombia1	8	30	87	Switzerland1	9	18	38
Colombia2	8	22	27	Switzerland2	7	8	36

Textbook	Grade	No. of pages	No. of exercises	Textbook	Grade	No. of pages	No. of exercises
England1	8	17	31	Switzerland3	9	15	51
England2	8	17	23	Switzerland4	7	9	22
HongKong1	8	12	12	UnitedStates1	7	12	174
Ireland1	8	13	53	UnitedStates2	8	12	207
Ireland2	9	20	72	UnitedStates3	7	11	160
Mexico1	8	42	36	UnitedStates4	8	12	195
NewZealand1	12	12	116	UnitedStates5	7	7	94
NewZealand2	12	26	41	UnitedStates6	8	12	98
Portugal1	8	26	56	UnitedStates7	9	30	345
Scotland1	11	11	48	Total		644	2564

Except for the textbooks from the United States, all the pages photographed were consecutive, and for only two textbooks, Australia3 and Switzerland2, were the pages under a single heading. The textbooks with the largest number of exercises tended to be from the United States. The U.S. textbooks numbered the exercises differently from those in the textbooks from other countries. For example, in the U.S. textbooks, each missing entry in a table was counted as an exercise. Figure 1 presents an exercise from an Australian book that deals with tables and four similar exercises from an American textbook. The exercises ask the student to find a functional relationship between the entries in a table; the previous examples and exercises have dealt with similar situations. Whereas for the Australian textbook there is only 1 exercise and six tables, for the American textbook there are 34 exercises for eight tables. I did not attempt to adjust the number of exercises from the United States; I decided to use the textbook's own definition of an exercise in the analyses.

I chose not to analyze problems that did not deal explicitly with functions or relations even if they were included in a section with such a title. Such problems included arithmetic problems, exercises in simplifying algebraic expressions, and exercises dealing with geometric properties of figures that appeared under headings like "Review." That reduced the sample by 303 exercises (73% were from textbooks from the United States, 16% from Australia3, and 11% from the Colombian textbooks). Unreadable pages in the original or in the photograph were discarded; these corresponded to 8 exercises

62. In each of the following tables [six tables are provided]
(i) Find the relationship between the *y*-coordinate and the *x*-coordinate;
(ii) Copy and complete the table

x *y*	1 3	2 6	3	4
	A(1,3)	*B*(2,6)	*C*(3,)	*D*(4,)

Source: Australia1, p. 407, Exercise 62.

Complete each table [eight tables are provided]

Hikers	Canoeing
1	1
2	2
3	3
4	**5.**
6.	5
7.	6
7	**8.**

Source: UnitedStates1, p. 40, Exercises 5 to 8.

Figure 1. Two ways of numbering exercises.

(from Switzerland1, Switzerland3, Portugal1, and SouthAfrica3). The final number of exercises was 2253. In this report, the problems and exercises from the textbooks are all referred to as *tasks.*

Coding System

Balacheff (in press) defined a conception as a quadruplet (see p. 8 of chapter 1). I used the labels P for the tasks, O for the operations required to solve the tasks, R for the representations used, and Σ for the control structures—controls, for short—for the means available for the student to determine that he or she has an answer and that it is correct. In English, the word *control* portrays the idea of "power or authority to guide or manage" (*Merriam Webster's College Dictionary*); that is not its intended meaning in this study. Instead, *control* refers to possibilities available for the student for legitimating solutions and verifying answers. In this report, I refer to P, O, R, and Σ as the elements of the quadruplet or the elements of the conception.

The development of the categories for coding each element of the quadruplet was a lengthy process that I accomplished in four steps. First, I selected one task from the first section of each textbook to analyze in depth. I worked each task, following as much as possible the textbook presentation that preceded the exercise section and developing categories for each element of the quadruplet. Second, I used the resulting categories to code the remaining tasks in all the first sections of each textbook, looking for new categories and refining the properties of each. I followed a *constant comparative* method (Glaser & Strauss, 1967) in which I described the salient features of the categories for an element and at the same time looked for possible breaks or mismatches that could lead to the creation of a new category. This second step involved 518 tasks and resulted in 133 categories. Because there were so many categories, the third step consisted in merging categories within common groups, thus yielding a smaller, more manageable number of categories for each element. The final step was to test the coding system by having other raters use it to code tasks. The details are provided in the following sections. I have included examples taken from the textbooks. Appendix B contains the original of those examples from books in languages other than English for which I have provided a translation together with the versions of one example in English that is abbreviated below.

Development of Categories

Because every problem section contained more than one task, I needed a procedure for selecting the first tasks to be analyzed to create the categories. I wanted to maximize the differences among tasks to obtain a variety of possibilities. I selected those tasks having the most questions because I assumed that longer tasks would provide more different types of questions than single-question tasks. In the cases of tasks with the same number of questions, I chose the first even-numbered task. Next, I read the text content that preceded the task and worked it using as a guide what was suggested by the text. I produced a narrative response, based on the solution and on the preceding text content, to each of the following four questions:

A. What is the use given to function in the task?

B. What does the student need to do to solve the problem?

C. Which representations are necessary to solve the problem?

D. How does the student know that he or she has gotten an answer and that the answer is correct?

The following is an example of the narratives produced:

> Colombia 1, p. 140, Tasks 5-9.
>
> For Exercises 5 to 9 represent in the Cartesian plane the relationship whose solution is the given set.
>
> 5. $R = \{(x, y) \mid x, y > 0 \wedge x, y \in \mathbf{R}\}$
>
> 6. $Q = \{(x, y) \mid y = -x \wedge x \in \mathbf{Z}\}$
>
> 7. $S = \{(x, y) \mid y = x \wedge x \in \mathbf{N}\}$
>
> 8. $T = \{(x, 0) \mid x \in \mathbf{R}\}$
>
> 9. $H = \{(0, y) \mid y \in \mathbf{R}\}$
>
> The task does not deal with functions but with solution sets of relations as ordered pairs, described symbolically and by how they are represented in the Cartesian plane. It uses symbolic and graphical representations. There are no similar examples solved previously: the closest one gives a relation in words and the problem there is to graph the relation and to write the solution set in symbols. In that example there is only one point plotted in the Cartesian plane. It seems that to solve the exercise the student does not need to plot many points, at least not many specific points, although it is likely that the student may do that. Note that the emphasis seems to be on making the students recognize the differences in systems of numbers chosen for each set. If a student chooses for set *R* points such as (1, 1), (1,2) and so forth, he or she will be corrected—probably by the teacher—because **R** represents all the real numbers; that is, non-integer numbers are to be considered too. Then he or she needs to shade the whole first quadrant. The set *Q* is made up of the points with integer coordinates that lie on a line with slope –1 that passes through the origin. The student will probably need to write ordered pairs in the form $(x, -x)$ before plotting. Set *S* is made up of the points with integer coordinates that lie on the line with slope 1 passing through the origin and in the first quadrant; it does not consider the point (0, 0). The sets *T* and *H* correspond to the *x*- and *y*-axes, respectively. The student will basically need to pick numbers, probably integer numbers, check whether the number satisfies an initial condition (is integral, real, positive, natural); find another number that could be paired with it (Is it positive? Is it the same? Is it its additive inverse? Is it zero?); and then plot the pair in the Cartesian plane. By applying this procedure a

> number of times—how many is not clear from the task but from previous work it seems that four or five times would be enough—a representation in the Cartesian plane will be obtained. Controls: in order to know if a plotted set is correct, the student might need to choose an arbitrary point in the plane that is not represented in the set and test whether the point meets the conditions given by the set definition. The work that has been done before does not guide the student in this way, though. In fact, one of the examples consists in locating four ordered pairs in the Cartesian plane (only one of these points has a non-integer coordinate, $\sqrt{2}$). The other one was described above; so it is not clear that the student knows how he or she can verify that the answer was correct. The presentation is devoted to giving precise definitions related to the Cartesian plane (unequivocal correspondence between $\mathbf{R} \times \mathbf{R}$ and the Cartesian plane; definitions for quadrants, abscissa, coordinate, and domain of the relation). It is likely that the student will decide to retest the points obtained to check that they meet the conditions proposed. (First analysis, p. 3)

Biehler's (in press) definition of prototypical uses of a concept was fundamental for characterizing the answers to Question A. He proposed the following uses: *natural law* (e.g., a parabola as a representation of the curve of a cannon ball), *constructed relations* (e.g., to express a price depending on a quantity), *descriptive* (e.g., functions involving time-dependent processes), and *data reduction* (e.g., functions in statistics). After working several of the tasks, I found that this classification did not include tasks lacking a real context: namely, when the function was treated as a set of ordered pairs (as in the previous example) or as a rule, when a pattern (with numbers or figures) was sought, or when there was a proportion involved. I treated each of these as a category.

Among the tasks that enacted relations that could be classified as constructed using Biehler's characterization were cases in which the content of the task used geometrical definitions or principles (e.g., similarity). That suggested an additional category. The following is an example from England2:

> The slide projector puts a picture on the screen. The size of the picture changes as you move the projector. The picture gets bigger and bigger as you move the projector further away. When the projector is 300 cm from the screen, the picture is 120 cm high. Here are figures for other distances [a table with six values for distance and height is given].

1. Draw two axes on graph paper. Mark the across axis from 0 to 500 and the up axis from 0 to 200. Label the across axis ‘Distance from screen in cm’. Label the other axis correctly. Use the figures in the table [given] to plot points.

   (a) What do you notice about the points you have plotted?

   (b) Use your ruler to draw the graph through the points.

   (c) Use the graph to find the height of the picture when the projector is 350cm from the screen.

   (d) How far is the projector from the screen when the picture is 50cm high? (p. 15, Task G1)

The category *geometrical* was used in such cases. Biehler’s “descriptive relation” was renamed *cause and effect* and was used to characterize the cases in which the task dealt with physical phenomena not dependent on time. In the end, I had 9 categories: cause and effect, constructed, data reduction, proportion, geometrical, pattern, rule, set of ordered pairs, and time. I kept a record of all the different instances of uses within each category. These lists of examples of uses were crucial in fully characterizing the categories for uses of function (see Table C1 in Appendix C).

A similar process was followed to develop the categories for operations and controls. I compared manually the narratives of all the problems looking for common words and comparable activities and processes. When a common activity or process appeared, a short name was assigned and written on an index card, with an abbreviation and a brief description. The continuous comparison of the narratives and the classification of the instances allowed me to refine the descriptions, constructing terms and sentences that encompassed groups of operations and of controls. Thus, for example, the operation *locate points in a graph* was initially described as follows:

> Begin with an ordered pair; the first component is located on the *x*-axis, and a mark is drawn at that point; a perpendicular line through that mark is traced; the second component is located on the *y*-axis, and a mark is drawn at that point; a horizontal line is traced through the mark; the point of intersection is the point sought.

The control *use check points* was described as follows:

> There are sentences giving the expected answers; there are warnings as to what is not a result; the answers to subsequent tasks contradict the answer obtained; there is another person performing the same activity.

The majority of the descriptions for the operations were taken from the textbooks themselves because many described thoroughly the processes that the students were expected to perform in the tasks. As for representations, I began with those given by Balacheff—*symbolic* and *graphical*—and added those presented in the textbooks. The new categories included *table*, *picture*, *arrow diagram*, and *number line*. There were also cases in which none of these representations was used and in which both the task and the solution required natural language, as in the following example from Scotland2:

> In which of the following can you say that one quantity is inversely proportional to the other?
>
> a. The time taken to deliver a batch of leaflets, and the number of people delivering them.
> b. Company sales, and the money spent on advertising.
> c. The distance walked at steady speed, and the time taken.
> d. The distance walked in a certain time, and the speed.
> e. The number of people on a job, and the time taken to do it. (p. 144, Task 1A)

This representation was called *verbal*. The last type of representation corresponded to cases in which there were equations that involved natural language, as in the following example from Spain1:

> There are 8 liters of a gas at a pressure of 1 atmosphere. The temperature is constant. Under these circumstances it is known that:
>
> Pressure x Volume = Constant
>
> **P** x **V** = Constant
>
> Fill the table:

P	1	2	4	8
V	8			

> (p. 99, Task 6)

This representation was called *semi-symbolic*. At this point I had 30 operations, 8 representations, and 8 controls, but each of these had subcategories (e.g., the symbolic representation could be a list of ordered pairs, a formula, an equation, or a set), which meant that the actual numbers of codes was larger. There were, in all, 87 codes for operations, 25 for representations, and 12 for controls.

Analysis of First Sections and Category Reduction

I then coded all the tasks in the first sections of exercises of each textbook, a total of 518 tasks. Each task received one code for use of function and one or more codes for each of the other three elements, as it was possible that a task required more than one operation, representation, or control. The codes were assigned considering both what the student was asked to do and what he or she needed to do to work the task. The following example, from Switzerland3 illustrates the need for this distinction:

> Produce graphs and descriptions with the functions defined by the following equations [only one is shown; for the complete version, see Appendix B]:
>
> a) $y = \begin{cases} \frac{1}{2}x + 1 & x \le 0 \\ 2x + 1 & x > 0 \end{cases}$
>
> (p. 72, Task 23)

Although the task explicitly requires only that the student produce a graph and its description, the student needs to determine the domain of the relation and find the image of particular elements of that domain (e.g., the image of 0 to find the *y*-intercept and of another couple of elements to fully determine the lines). Two additional operations, therefore, were assigned to the task besides those explicitly required.

This process helped to refine the coding system by providing better descriptions for each category and enlarging the set of examples for the uses of function. I reorganized the operations; one category disappeared, and two subcategories became categories (*rehearse notation* was dropped; *name point on axis* was a subcategory of *locate point in* *graph*; and *use proportionality within entries* was a subcategory of *find relation between two (sets of) numbers*). After refining the system, I had 9 codes for prototypical uses of function (uses, for simplicity), 31 for operations, 8 for representations, and 8 for controls. For practical purposes, I then dropped all the subcategories.

Testing of the Coding Procedure

With this set of codes, I produced a document that other people could use to code tasks. I randomly selected 11 tasks from the pool of 518 tasks (using the random function in SPSS) and created seven groups of 4 tasks and one group of 2 tasks. Eight tasks were assigned to three groups and 3 tasks to two groups to guarantee that each task was coded by at least two people. There were tasks from all the languages (English, Spanish, Portuguese, and German). I sent an electronic message (see Appendix D) to 30 colleagues asking for their collaboration in using the coding procedure and providing feedback about it. Nine people agreed to test the coding system: five university faculty members (one at the University of Michigan and four at the University of Los Andes), each of whom was knowledgeable about the research problem and the process I was using, and four graduate students in mathematics education (three at the University of Georgia and one at the Royal Danish School of Educational Studies). Three of the students were second-year doctoral students, and the other was a fifth-year doctoral student.

I sent each person two Acrobat PDF (Adobe Systems, 1987) files, one containing the tasks to be coded and other containing the coding procedure. Of the nine colleagues, I got feedback from seven. The tasks written in German were not analyzed because the coders did not understand the tasks. From the 11 tasks selected, only 8 were coded.

Because one purpose of the test was to fine-tune the categories, the coders were allowed to create new codes if the ones provided were not enough. This was done with the purpose of exhausting the possible definitions of the categories. Therefore, in addition to assigning the codes I had developed, the coders proposed four new categories.

Calculation would be any operation that refers to the process of operating on numbers (e.g., subtraction). *Continuity* would be a control that allows the student to assume continuity in finding a value associated with a function (e.g., when the student is to find a pre-image of a function defined by a table or a set of ordered pairs, and the image is not one of the values given). A representation would be *numerical* if it were used to describe manipulation with numbers in any number system. Finally, *implicit/explicit* would be a variant case of the uses for a relation. Of these four, only the first three were incorporated into the final coding system. The last one was ignored because it would have created two subcategories, *implicit* and *explicit*, for each of the uses proposed, and those characteristics not only would have been difficult to distinguish but also would have added more complexity to an already complex system.

Although initially intended to be a test of interrater agreement, this test did not fit that purpose. In the first place, the raters used their own understanding and knowledge coming from their different backgrounds to solve the tasks, which implied that they did not always use the solution that was fostered in the lesson from which the task was taken. In the second place, the coders were allowed to modify the coding system but could not share the modifications with each other. The coding was a solitary task. I did not have access to his or her solutions, which made it more difficult for me to discover the rationale behind their code assignment. However, the test was useful for designing a different reliability test.

First, I produced more examples of code assignments. Then I asked two doctoral students in higher education from the University of Michigan and one in mathematics education from Michigan State University to participate in an individual one-hour interview. I chose 5 of the 8 tasks (one in Spanish, the others in English) and gave the coders the whole chapter from which the task was taken and a revised coding procedure. Each coder was given time to select a task, check the chapter in which it was embedded if he or she wanted to, solve the task aloud, and assign the codes to their solutions. I encouraged the coders to make explicit their rationale in each case by asking why a code could be assigned or why not. I did not attempt to negotiate their assignment with them, but I did provide explanations about what I meant by some of the codes. The interviews were audiotaped. After each interview, I listened to the tape, went over the solutions and my notes, adjusted the coding system, and revised the coding of the five tasks.

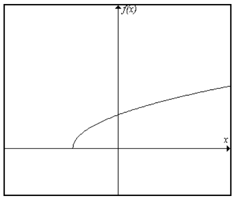
The agreement between the codes assigned by the interviewees and my coding was 80% after the first interview, 85% after the second, and 100% after the last. The procedure helped me produce better descriptions for almost all the codes. Using the new codes, I recoded the 518 tasks. About 20% of the codes were modified.

I am aware that the reliability and validity of coding are crucial for the study. The fine-tuning test and the reliability test helped me to provide better examples for each code and explanations of its meaning, which gave me confidence in the validity of the results. A third test that I carried out was to select six tasks at random and recode them 2 weeks after the modified coding system was completed. In this test, the agreement between the codes that I assigned at different times was 100%, which means that my own coding was consistent.

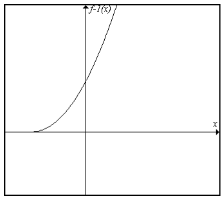
Final Coding Procedure

I used the modified coding system to code the tasks in the remaining sections. A total of 2304 tasks were coded (this figure included the French textbook that was later dropped because it did not met the grade requirement). As I was coding these tasks, I found that I needed to add five new codes: *graph* for uses of function; *change form*, *find composite*, and *operation between functions has characteristic*, for operations; and *use calculator or a computer*, for controls. These codes were needed for tasks in several final sections of upper-grade textbooks from New Zealand and the United States (twelfth grade and ninth grade, respectively). The following task from NewZealand1 illustrates the code *graph* in its use of function:

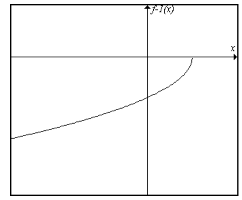
> Here is the graph of a function f. Which of the graphs (a)-(e) is the graph of the inverse function f^{-1}?



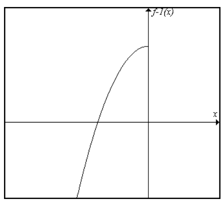
a)



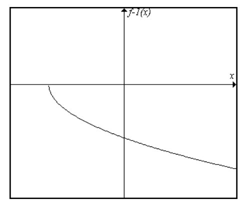
b)



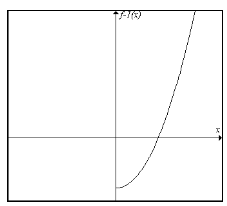
c)



d)



e)



(p. 29, Task 8)

Appendix C presents the complete coding system—with examples of uses of the codes—which consists of 10 codes for uses of function, 36 codes for operations, 9 for representations, and 9 for controls.

Examples of Coding

The following tasks illustrate the coding. A comma separates each element of the conception; a hyphen separates codes within an element.

A Task From the Geometrical Use of Function

> Is there proportionality between the length of the edge of a cube and (a) the sum of the lengths of its edges? (b) Its surface area? (c) Its volume? (Mexico1, p. 195, Task 3)

The student needs to determine the type of relationship that exists between the two given variables; he or she may need to use drawings and test particular numerical cases to determine whether the proportionality is direct or not. The task is coded GR, DTR, V-N-P, UAR.

GR	*Geometrical relation.* Used to code content that refers to geometric figures and their characteristics.
DTR	*Determine type of relationship.* The student needs to determine whether the relation between two sets of numbers is direct, indirect, linear, or nonlinear, or whether there is no relation.
V-N-P	*Verbal.* The task uses a description of a situation using natural language (e.g., a pound of apples costs 30 cents) or requires the student to interpret a situation with natural language. *Numerical.* The task does not require any symbols; instead, it requires numbers. *Pictorial.* The task uses drawings of machines, maps, geometrical shapes and figures, photos, or pictograms (frequency diagram where the *y*-axis is not present), pies (only one variable is sketched) or any other kind of drawing.
UAR	*Use alternative (given or not given) representations.* The student can use other representations (e.g., results in a table vs. results with a formula or a graph, a set of ordered pairs as an arrow diagram). These can be explicitly given in the statement of the task or can be result of something the student was asked to do.

A Task From the Pattern Use of Function

> Each of the following set of points represents a linear pattern in the Cartesian plane. By plotting each set of points and using a rule, find the coordinates of the next two points in the pattern.
>
> {(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2)}
>
> (Australia1, pp. 255-256, Task 3. There are 13 more items like this one that follow; see the full text in Appendix D)

The task was classified as a pattern use because the setting allows the student to find the next two points by following the pattern given by the abscissa and ordinate of each ordered pair: beginning with –3 and increasing by ones for the abscissas and beginning with –6 and increasing by twos for the ordinates. The student needs to plot the points in a

Cartesian plane; he or she will thereby know that the answer is correct if all the points are on a line. The task was coded PR, LPCP-RPCP, G-S, LFLUR.

PR	*Pattern relation*. Used to code content in which given a sequence the question is to find the general term (or an expression for the *n*th element) of the sequence.
LPCP-RPCP	*Locate points in graph*. The student needs to locate points in a graph; a graph can be any of the types defined in the section about representations. Whenever a Cartesian plane is involved the code must be applied if both elements of the ordered pair are known and need to be located. If that is not the case (e.g., the time at which the temperature is 50°C), then use the operation FIP. LPCP always requires NPOX when a Cartesian plane is involved. *Read points from graph.* The student needs to read the coordinates of a point or a set of points from a graph. A graph can be of any of the types defined in the section about representations. Whenever a Cartesian plane is involved the code must be applied if both elements of the ordered pair have to be determined (e.g., the coordinates of the maximum value of a relation). If that is not the case, then use the operation FIP (e.g., the time at which the temperature is 50°C). RPCP always requires NPOX when a Cartesian plane is involved.
G-S	*Graph in two axes*. It can be a Cartesian plane, a frequency diagram, a histogram, a broken line (time series), or a scatterplot. *Symbolic*. The task uses expressions with only symbols. This includes arithmetical notation, sets (e.g., $\{x \mid x > 0, x \in \mathbf{N}\}$), ordered pairs, equations (e.g., $f(x) = x + 1$; $y = x + 1$, $f(2) = x + 1$), mappings ($f\colon x \to x + 1$), or intervals.
LFLUR	*Look for likely or unlikely results*. The student can use indicators in the statement of the task (e.g., the student obtains a number too big or too small for a given scale in a Cartesian plane, or he or she is getting decimals or negative numbers when whole or positive numbers are expected, or a set of points in a Cartesian plane are not aligned on a line) or use previous knowledge (e.g., the sides of a square have the same length).

A Task From The Constructed Relation Use

> Mai's parents allow her to watch 12 hours of TV each week during vacations.
> Find three solutions to her new equation: $y = 12 - (1/2)\,x$.
>
> (UnitedStates6, p. 153, Task 17. *Note: x* stands for the number of half-hour TV shows)

The text preceding the task used the context of television viewing as a vehicle for developing ideas about graphing linear equations. The standard procedure given in the text required that the student select three values for *x* and find the corresponding value for *y*. The student might possibly use a table or plot points in a Cartesian plane, but that might be unlikely because in this exercise the instructions "make a table and graph" are not explicitly given. The section is about linear equations, which may help as a control for the correctness of the answer if the points plotted do not lie on a straight line. The task was coded CR, FIP, S-G-T, DC-UAR.

CR	*Constructed relationship.* Used to code content that refers to 'real life' situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables originates equally valid—for the context—relationships.
FIP	*Find element of the range or of the domain of a relationship*. The student needs to find in the range of the relation a value (or element) associated with a given element of the domain, or find a domain element associated with a range element, or both. This includes finding one more ordered pair of the relationship, in which the student might need to choose an element of the domain and find its corresponding value in the range through the relation. It includes algebraic manipulations that involve for x in $f(x) = k$, where k is a given value, or finding $f(m)$ where m is an algebraic expression, finding the solution of $f(x) = f^{-1}(x)$, or finding asymptotes. This code is also used when the student needs to find the function that results from the operation of two given functions; the process can be made through operating component by component in a table or by operating on the expressions that define the relation. This includes finding, for example, the image when $x = 0$ and the pre-image when $y = 0$, with all the algebraic manipulation that may be required. There is no restriction on the representation used for the pair.
S-G-T	*(S-G see above.)* *Tabular.* The task uses a table. The table can be given, asked for, or a requisite for the process of keeping track of the entries.
DC-UAR	*(UAR see above.)* *Double check*. The student either repeats the process used to obtain the answer (e.g., relocates points in the Cartesian plane) or reverses the process to get something that is given in the statement of the task (undoes the sequence of operations).

Data Analysis

I constructed an SPSS file for handling the data, with each entry corresponding to a task. I created variables for country, language, intended grade, textbook number, textbook section, task number in the textbook, and use of function. These variables required 7 columns in the file. To handle the other elements of the quadruplet, I created 26 more columns—10 for operations, 8 for representations, and 8 for controls—to allow for multiple codes for these elements. In the end I used 7 columns for operations (i.e., there were tasks that received as many as seven operation codes), 5 for representations, and 5 for controls. Each different combination of operations, of representations, and of controls received a code. I obtained 396 different combinations of operations, 70 of representations, and 74 of controls. This large number of combinations created difficulty in data handling and interpretation. I reorganized the existing combinations into new categories according to several criteria to facilitate the analysis and interpretation. The process was different for each element. The following four sections describe these reorganizations.

Uses of Function

The ten coding categories were combined into five categories by similarity of the relations between function elements as follows. The uses that referred to physical phenomena, cause-and-effect relations, and time relations were grouped into a new category called *physical* to capture the character of these relations. Because they relate to human activity, data-reduction relations and constructed relations were grouped into a new category called *social*. Geometrical relations, graph-defined relations, and pattern relations were grouped into a new category called *figural,* to highlight the crucial role of images and patterns for defining functions with these relations. Rule and direct proportion/proportion relation were grouped together into the category *rule*. *Set of ordered pairs* was left as a separate category.

Operations

The basic criterion for reorganizing the categories of operations was the frequency of assignment. I determined the number of countries for which those operations were the most frequently assigned, as determined by a configural frequency analysis (CFA, see p. 66). The configural frequency analysis marks as *types* those cells in a contingency table that show a frequency that is larger than would be expected by chance. If a cell frequency is less than would be expected by chance, the cell is called an *antitype*. In this case, and as an initial way to disclose patterns of code assignment, I was interested in tagging those operations that were types, taking into consideration the composition of the sample. I considered only the data from seventh- and eighth- grade textbooks, which covered 24 textbooks from 15 countries and 32 operations (the codes *find composite*, *give period*, *operation has given characteristic*, and *trace regression line* were not assigned to the tasks of the seventh- and eighth- grade textbooks). I divided the operations into three groups according to the frequency of the countries in which CFA yielded a type. In each case, I was able to characterize the operations in the group. Table 4 presents the three groups of operations together with the number of countries for which those operations were labeled as types.

Table 4

Operations Grouped by Frequency of Countries in Which They Were Types

Group	Number of countries
Operations requiring manipulation of the relation	
Locate point in a graph (LPCP)	9
Find images and pre-images (FIP)	7
Read point from a graph (RPCP)	7
Operations attempting to characterize the relation	
Find relation between two sets of numbers (FR2N)	6
Compare without calculation (CWC)	5
Determine domain and range (DDR)	4
Fill table (FT)	2
Describe shape in graph (DSCP)	2
Relation is function (RIF)	2
Operations involving numerical and concrete activities	
Perform a computation (CALC)	1
Carry out experiment (COE)	1
Measure (M)	1
List elements of the relation (LER)	1
Give definition (GD)	1

To use the three operations marked as types by 7 to 9 countries, an explicit relation must be known. The operations also give particular instances of the relation (a value or a point); general features may be obtained by a repeated application of these operations. I called this group of operations *manipulate* (in the sense of "utilize skillfully," *Merriam Webster's College Dictionary*); they *do* something *with* the relation. In contrast, the operations that were labeled as types for 2 to 6 countries do not need an explicit definition for the relation and attempt to make its features explicit. I called this group *appreciate* (in the sense of "grasp the nature, worth quality, or significance of"; also, "judge with heightened perception or understanding: be fully aware of," *Merriam Webster's College Dictionary*); they *tell* something *about* the relation. The group of operations that were labeled as types by one country only may or may not need an explicit definition of the relation and are to some extent peripheral to the relation itself. I called this group *calculate* because they *do* something *for* the relation. I characterized the combinations as having operations in the manipulation group only, in the appreciation group only, in both the appreciation and manipulation group, in all three groups, or in no group (other). These five categories were used to characterize all the combinations of operations.

Representations

Because there were 70 combinations of representations, I explored several alternatives for reorganizing them. There were some tasks that used only one representation, but the majority used at least two, which made it difficult group them. A compromise was needed to balance the need for diversity and the need to highlight particular characteristics of the combinations. I chose to emphasize the use of the symbolic representation and created three groups. The first group, called *symbolic*, contained those combinations that used either a symbolic or a semi-symbolic representation only. The second group, called *symbolic and other*, contained those combinations that used the symbolic representation in conjunction with any other representation (graph, table, picture, number line, arrow diagram, verbal, or numerical); the last group, *other*, contained those combinations that did not use a symbolic representation. With this classification, it was not possible to make claims about the use of representations other than symbolic ones, but the classification was good enough to characterize the conceptions, which was the main purpose of the study.

Controls

The case of the controls was similar to that of the representations in that there were 74 combinations, but in this case it was possible to regroup the combinations according to the nature of the activities involved. I defined three groups of controls. The first group encompassed activities that rely on the solution *process* only. These were *double check*, *compare with previous examples or exercises*, and *use checkpoints.* The second group referred to activities requiring use of the mathematical *content* that was at stake. These were *more than one point* (or the vertical line test), *continuity*, and *use alternative representations*. The final group, *use a computer or a (graphing) calculator, look for likely or unlikely results*, and *use given information*, encompassed activities that seem to be related to the *didactical contract* (Brousseau, 1997). Use a computer or a (graphing) calculator was defined as a control because the student might be using the instrument to look for familiar results—established by the didactical contract (see also Mesa & Herbst, 1997).

To suitably group the 74 combinations, I characterized each combination according to the three types of controls. A given combination could have controls of one type only (process, content, or contract) or a combination of two or three types. From these seven possibilities, I chose to highlight those combinations in which the content was important. With this in mind, I created three categories: *content and other* contained all the combinations that had at least one control of the content type; *process-contract*, which contained the combinations with controls of these two types only; and *process*, which contained the combinations with controls of the process type only.

Configural Frequency Analysis

Because all the variables in this study were categorical, I used cross tabulations to provide the basic input for subsequent analyses. Configural Frequency Analysis (CFA) is a data analysis technique developed by von Eye (1990). The analysis is related to data-mining processes (DuMouchel, 1999) in which the researcher is interested in the most frequent configurations of events that occur in large databases. Examples are groups of products most commonly bought in supermarkets, groups of words that tend to go together, and groups of adverse effects reported by types of drugs (p. 178). The CFA provides several statistical tests that allow the researcher to determine whether the differences between observed and expected configuration frequencies are statistically significant. The program provides a Bonferroni adjustment to protect the test-wise α (von Eye, 2000, p. 3).

I applied CFA to a four-way table (uses of function, operations, representations, and controls) with 225 ($5 \times 5 \times 3 \times 3$) possible configurations for the sample of tasks from the seventh- and eighth-grade textbooks with $\alpha = .05$ as the level of significance. I used the resulting types and antitypes to characterize the conceptions enacted by this sample of tasks.

Textbooks and the TIMSS Achievement Test

To examine the performance of students who might have used the textbooks, I selected the items related to functions from the released set of TIMSS test items for the seventh and eighth grade (IEA, 1997b). I chose from the TIMSS categories of algebra; geometry; data representation, analysis, and probability; and proportionality (p. vii) all the items that were similar to tasks presented in the textbooks. Those items that asked the student to describe a relation, to interpret a Cartesian graph of a relation, to analyze a pattern by finding an element, or to deal with proportionality were marked as possible items to be considered. I selected the 10 items that fit these descriptions and coded them using the procedure developed for the study. The items and their codes are reproduced in Appendix E.

Because Mexico and Argentina did not have data from the TIMSS achievement test (Mexico chose not to make the results public; Argentina did not administer the test), I did not include their textbooks in the analysis of achievement. The achievement test analysis included 13 countries and 22 textbooks.

Each country's performance on each of the 10 items was obtained from the TIMSS almanac (IEA, 1997a), which provided the percent of correct answers and the standard error for each sample of students. This information was used to build 95% confidence intervals for the percent of correct answers for each item for each country to establish the extent to which observed differences across countries were statistically significant. The significance test used the Dunn-Bonferroni adjustment for multiple comparisons based on the number of countries involved, as suggested by Gonzalez (1997, pp. 151-152), which yielded a critical value of 2.96. Similar intervals were constructed for the average percent of correct answers for the ten items as a group.

There were two difficulties in linking the student data with the textbook data. First, the TIMSS textbook analysis had used the official national textbooks when they were available or else the most widely used commercial textbook in each country (Schmidt et al., 1996, p. 9). That decision implied that for countries in which more than one textbook was used, the textbook used by some of the participating students was not present in the sample. Also, even though the teacher questionnaire contained a question about the textbook used by the students who took the test, that information was not analyzed in TIMSS because it was too diverse (L. Cogan, personal communication, March 6, 2000). Thus it was impossible to determine the textbook used by a specific student taking the test in those countries in which more than one text was used. Austria, Hong Kong, Singapore, and Spain were the only countries in the sample that had, at the time of the test, a nationally centralized decision process about textbooks and used only one book. Any inferences about the connection between conceptions present in textbooks and patterns of performance in the TIMSS test, therefore, required considerable caution.

Sources of Bias

Researchers' views, knowledge, and beliefs affect the way in which they conduct research studies from selecting a research topic to interpreting and presenting their results. I was not an exception. My preparation as mathematician and engineer put me in contact with two different approaches to function, the abstract and the logical, that have shaped my own conceptions of it. The mathematics curriculum that I had as a school student practically banned geometry, and as an undergraduate I had only one course on non-Euclidean geometry. I believe that this meager exposure to geometrical thinking affected two aspects of the study. In the first place, I looked for functions only in chapters that were devoted to algebra; it did not occur to me that I might find a geometrical treatment of functions in chapters devoted to geometry or in geometry textbooks. This bias affected the sampling process. Second, when I was solving the tasks in each textbook, the first solutions that I thought of used algebraic approaches supported by graphs in the Cartesian plane when possible. My confrontation with the textbook content showed me that I needed to re-solve the tasks, giving preference to what was presented in the textbooks. My bias against geometric approaches could have affected the definition of the categories, despite my efforts to control its influence.

I tend to be sympathetic toward statistical analyses. For that reason, I made a strong commitment to find statistical evidence that the patterns and differences that I was observing were not due merely to chance. This approach might have affected the results of the study: I could have dismissed interesting patterns that did not reach the .05 level of significance, which might be detrimental for an exploratory study such as this one.

Because I was raised in a developing country that has been negatively affected by the policies of economically and geopolitically powerful countries, I tend to look with sympathetic eyes at countries that are in a situation similar to mine and tend to be harsh toward countries that dominate the world's destiny. This bias might have influenced my interpretations of the results of this study, as it was difficult for me to be fair to the textbooks on their own terms.

CHAPTER 4

RESULTS

After Columbus one should not be surprised if one
does not solve the problem one has set out to solve
Imre Lakatos (1976)

The data collected were used to address the three research questions:

1. What conceptions of function are suggested by the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS?
2. What patterns of conceptions are present in textbooks from different countries?
3. What is the relation between the conceptions present in the textbooks of a country and its students' performance on items related to functions on the TIMSS test for seventh- and eighth-grade students?

The chapter is divided into four sections. The first presents an overview of the characteristics of the tasks and the results of the Configural Frequency Analysis (CFA) program (von Eye, 2000) for the sample of tasks from the seventh- and eighth-grade textbooks. The second presents data on how the conceptions distribute within textbooks from each country. The third presents a comparison between the conceptions present in the textbooks of each country and the performance on selected items of the TIMSS achievement test. The last presents some results that were not directly related to the research questions but that provided valuable information about the textbooks.

Conceptions in Seventh- and Eighth-Grade Textbooks

Below I present the frequencies and percentages of occurrence of each element of the conception corresponding to the sample of tasks from the seventh- and eighth-grade textbooks. Then I use the results of the CFA program to examine how the elements defining a conception interrelate.

Elements of the Quadruplet

The purpose of looking at the elements of the quadruplet separately is twofold. I wanted first to illustrate tasks in each category and second to give the reader a sense of how the categories were distributed across categories. This descriptive information helps in understanding the data. The results in this section are based on a sample of 1319 tasks from 24 seventh- and eighth-grade textbooks from 15 countries.

Uses

The element uses had five categories: rule, set of ordered pairs, physical, social, and figural. The following are examples of tasks with each of these five uses.

Rule: Trace the graph of the following functions in the same coordinate system: $y = -3x + 2$; $y = 2x + 2$; $y = -x + 2$; $y = 2$; $y = 2x + 2$; $y = 3x + 2$. (Mexico1, p. 219, Task 1)

Set of ordered pairs: State the domain and range of $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$. Write down the couples of R^{-1}. (Ireland2, p. 62, Task 1)

Social: Represent graphically the following function: In certain city the cost of a taxi fare is given by:

- Initial charge: $150.00.
- Cost of trip: $5.00 for every 100 m.
- Time spent waiting is not considered. (Colombia2, p. 248, Task 4)

Physical: Make a table of six values using the relation. Then draw a graph.
a. Phil runs 9 km/h.
b. Gayle cycles 16 km/h.
c. The train travels 90 km/h. (Canada1, p. 311, Task 4)

Figural: A formula to produce consecutive odd numbers is $2n - 1$ where $n \geq 1$.
a) Draw up a table to produce ordered pairs which satisfy this formula for $1 \leq n \leq 8$.
b) Graph this information. (SouthAfrica3, p. 177, Task 6)

The categories are listed above in descending order of frequency across all the textbooks. Table 5 presents the frequencies and percentages of uses of function in the tasks. The most frequent uses were rule and set of ordered pairs. Only one third of the uses corresponded to those involving concrete contexts: namely, social, physical, and figural. Social uses were almost twice as frequent as physical uses, which suggests that at these grade levels physical phenomena in which functions can be defined do not play a very important role.

Table 5

Frequency and Percentage of Tasks by Uses of Function

Uses	Frequency	%
Rule	556	42
Set of Ordered Pairs	319	24
Social	227	17
Physical	136	10
Figural	81	6
Total	1319	

Rule was by far the most frequent use (it grouped situations in which either a proportion or a transformation of an input to obtain an output was involved and there was no context included). One possible reason for this result might be didactical: Because the notion of correspondence is so fundamental to the definition of function, and because the seventh and eighth grades mark the transition period from arithmetic to algebra, transformation of numbers by means of basic operations seem to fit the double purpose of defining valid functions, with a notion of correspondence as transformation or constrained variation, while at the same time linking known operations with the new idea of correspondence. In this way the burden of considering apparently unrealistic cases in which the correspondence can be arbitrary is overcome. This purpose might also explain why the rule use was more frequent than the set-of-ordered-pairs use, in which such arbitrary correspondences take place but may seem unrealistic to a student who has been moving slowly from a concrete to an abstract stage of reasoning.

The set-of-ordered-pairs use had a surprisingly high frequency, which might be due to the interest of authors in keeping their textbooks updated mathematically: If the most sophisticated definition is available, why not present it? It could be possible too that in some cases the influence of the new math movement might have been operating. Textbooks with copyrights from the 1970s or early 1980s might have shown this tendency. I expanded on this issue in the section devoted to the patterns of textbook conceptions (see p. 96).

The figural use of function accounted for only 6% of the tasks. This category included geometrical, pattern, and graph relations. One possible reason for the few instances of this use might be linked to the separation between geometry, arithmetic, and algebra in school mathematics curricula. The textbooks tended to contain separate chapters for geometry, and it might be possible that within those chapters, functions did not get much attention. In addition, the low frequency of geometrical uses could be a result of the new math movement, which almost eliminated geometry from school mathematics in several countries (Ruiz & Barrantes, 1993).

Operations

The operations element had five categories: manipulate only; manipulate and appreciate; manipulate, appreciate, and calculate; appreciate only, and others. Below I provided examples of three tasks illustrating three of these categories.

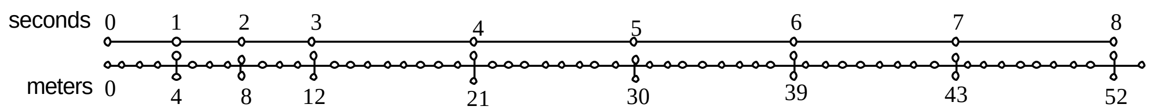
In the following task, from England1, the needed operations belonged to the manipulation group:

> A girl walks for 3 seconds at 4 m/s and then runs for 5 seconds at 9 m/s.
>
> a. Copy and complete this table.

Time in seconds	0	1	2	3	4	5	6	7	8
Distance in meters	0	4			21				

> b. Draw the distance-time graph. (p. 17, Task F2)

In a previous exercise the student has been told how to use a number line to represent the distance that a person walks at different speeds. With that technique, the need to calculate is overridden. The student merely needs to read off the number line the distance walked after marking the appropriate number of line segments of sizes 4 and 9:



For Part b, the student needs to locate the pairs from the table in a Cartesian plane.

Operations in the manipulation group do not require very elaborate activities. The relation is given, and the instructions are very precise as to what the student has to do with the functions. This information leaves little room for other activities in which the student might need to draw information from other sources or consider the relations from a different perspective.

An example of a task with operations in the appreciation category is the following, taken from UnitedStates1:

> Explain what happens to x in each function:
>
> **1**. $f(x) = x + 4$ **2**. $f(x) = (1/2)x + 9$ **3**. $f(x) = x^2 + x$ **4**. $f(x) = (x/7) + 5$. (p. 560, Tasks 1 to 4)

The process of describing the transformation to be applied to the variable x implicitly characterizes the function that defines the transformation; the student does not need to perform calculations, or to find particular values of the functions or its graph.

The following task, taken from Singapore1, requires manipulate, appreciate, and calculate operations:

> Draw the graph of each of the following equations on the same graph paper: (i) $y = -x + 6$ (ii) $y = x - 2$ (iii) $y = -x + 10$ (iv) $y = x + 2$. What figure is formed by these four lines? Write down the co-ordinates of the vertices of this figure. (p. 197, Task 5)

In this task, the student needs to determine points that belong to the lines and use them to trace the lines, operations that belong to the manipulation group. He or she also needs to describe the resulting figure, a rhombus, an operation from the appreciation group. He or she needs to verify that the figure is indeed a rhombus, most likely by measuring lengths and angles, an operation from the calculate group. Finally, he or she needs to establish the coordinates of the vertices, probably by estimation with recalculation with the equations, operations from the manipulation group again.

In this sample, a large proportion of the tasks (38%) used combinations of the operations belonging to the manipulation group only (find images and pre-images, and locate and read points from a graph). Table 6 presents the frequencies and percentages of combinations of operations. The large number of operations in the manipulation group can be linked to the fact that the Cartesian plane is often introduced in Grade 7 or 8. Thus, finding values of numbers through a relation to form pairs that will later be located in a Cartesian plane and reading points from it become standard tasks that students need to master.

Table 6

Frequencies of Combinations of Operations in the Tasks

Combination of Operations	Frequency	%
Manipulate only	496	38
Manipulate-Appreciate	321	24
Manipulate-Appreciate-Calculate	243	18
Appreciate only	241	18
Others	18	1
Total	1319	

The proportion of tasks requiring only operations that tell something about the function—namely, from the appreciation group (find the relations between two sets of numbers, determine domain and range, describe shape of the graph of the relation, compare without calculations, determine if a relation is function, and fill a table)—was about half that from the manipulation group. This difference means there were relatively few instances in which the function was considered the object of an operation. This result might be suggesting that in these grades, when functions are introduced, the textbooks need to fulfill the demand of familiarizing the students with the notion by giving them opportunities to interact *with* the notion. The result may indicate a tendency by textbook authors to familiarize students with the tool (as in a concrete experience) before moving to a further abstraction in which the function is considered as an object.

About one fourth of the tasks required a combination of operations from the appreciation and manipulation groups (e.g., a request for, say, domain and range might be followed by a second question in which the student needed to sketch a graph for the functions given). In comparison, relatively few tasks required operations from the calculation group (carry out experiment, measure, calculate, list the elements of the relation, and give definition) in combination with operations from the other two groups.

Only 1% of the tasks did not make use of any of the operations included in these groups. At Grades 7 and 8 more sophisticated operations with functions seem to play a secondary role because the notion is just being introduced. Operations such as find an inverse or a composite or produce a proof were not typical when these textbooks dealt with functions (for a list of these operations, refer to table C2 in Appendix C).

Thus, even though a considerable proportion of tasks required very few simple operations (38% from the manipulation group only), a sizable proportion (60%) required more elaborated combinations of operations with different levels of complexity.

Representations

The representations element had three categories: symbolic only, symbolic with another representation, and nonsymbolic. About a fifth of the tasks required only a symbolic representation, and almost half required it in combination with other representations: graphical (23%), graphical and numerical (8%), numerical (7%), verbal (6%), and combinations of all these (4%). Thus the symbolic representation was overwhelmingly prominent in this sample of tasks (see Table 7). A possible explanation of this result has to do with the fact that the curriculum of the grades considered in the study may be designed to provide a transition from arithmetic to algebra. Symbolization is, in a way, a prerequisite for carrying out the transition, and equations and functions provide a context for symbol use. The high frequency of the combination of symbolic and graphic representations might also be due to the fact that the Cartesian plane is being introduced in these grades, which requires both representations.

Table 7

Frequencies of Combinations of Representations

Combination of Representations	Frequency	%
Symbolic only	254	19
Symbolic with another representation	628	48
Nonsymbolic	438	33
Total	1319	

Controls

The element of controls had three categories: process only, process and contract, and content and others. In 55% of the cases, the only means available for the student to check that an answer was obtained and that it was correct was based on the process of solution. About 30% of the tasks used controls based on the didactical contract and on the process of solution, and only 17% required controls based on the content of the task—alone or in combination with the other two types of controls. These results indicate that the tasks related to functions in these textbooks provided relatively few opportunities for the students to use the content to verify the correctness of their answers. In addition, the results show that overall there were few opportunities for the students to learn, apply, and enlarge their metacognitive strategies, to move from the stage in which they do what they are told to do to a stage in which they control what can be done. Table 8 presents the frequencies and percentages of combinations of controls.

Tables 5 to 8 suggest that a large number of tasks in the sample portrayed functions as rules, used a reduced set of operations, with mainly a symbolic representation, and with controls based on the solution process of the task. Because a conception was defined as a combination of four elements, however, an analysis of the configurations of elements was needed to corroborate this result.

Table 8

Frequencies of Combinations of Controls

Combination of Controls	Frequency	%
Process	726	55
Process-Contract	364	28
Content and others	230	17
Total	1319	

Configurations of Elements

The CFA for the configurations of the four elements of the quadruplet yielded 24 types and 4 antitypes (see Appendix F). To determine what these 28 configurations were portraying, I grouped them by use. The configurations within each use followed patterns that helped me to characterize the conceptions associated with each particular use. For example, there were six configurations with a rule use; three of them were types (3111, 3321, and 3322) and three antitypes (3131, 3331, 3332). The third digit of these configurations corresponds to the combinations of representations. In the antitype configurations, nonsymbolic representations were always assigned. I interpreted this fact as an indication that tasks in which the function is used as a rule do not require nonsymbolic representations; in other words, the cases in which they do require nonsymbolic representations are so rare—the program flags them as antitypes—that the opposite characteristic should be the norm. Thus, nonsymbolic representations do not appear by themselves when dealing with functions defined by rules.

Thus I defined five conceptions: *symbolic rule, ordered pair, social data, physical phenomena*, and *controlling image*. In the following five sections, I describe the characteristics of each conception.

Symbolic Rule

In the tasks that belong to a symbolic-rule conception the use of function is as a rule. If manipulation operations are used, the representation selected is symbolic. If manipulation and appreciate operations are used, the symbolic representation is not the only representation required. In any case, the task does not require controls based on the content.

Thus, within this conception two kinds of tasks appear. In both kinds, the task provides an equation that defines the function without a reference to a particular real situation. However, in one of them, say Task A, the student only needs to apply the transformation given in order to obtain particular values of the function, using the symbolic representation only, whereas in the other, Task B, the student applies the transformation, tells something about the function, and uses another representation that supports the symbolic one. In both tasks, the student may verify his or her solutions by repeating the calculation or by contrasting the results with hints given by the setting of the tasks. Function machine tasks are of the first kind, whereas tasks in which the Cartesian plane is being introduced are of the second. The following is an example of Task A, taken from Portugal1, which can be considered paradigmatic of the tasks enacting these conceptions:

> Consider a function h defined as $h\ (x) = 2x + 1$.
> 1. Find $h\ (-1)$, $h\ (0)$, and $h\ (1)$
> 2. Find x such that $h(x) = 11$. (p. 67, Task 4)

In this task, there is an input x that is transformed by certain procedure—multiply by 2 and then add 1—to obtain an output; there is no reference to an external context. The student has to obtain particular values of the function (at –1, 0, and 1) and a number, a value for x, such that the function is 11 when the transformation is applied to that number. The only representation used is symbolic. The student has to repeat the procedure that was given in the preceding text (substitute the values into the equation), which, at the same time, acts as the indication that an answer was obtained. It is unlikely that the student will determine by himself or herself whether the answer is correct. If he or she gets an indication in this sense, the path to follow would be to repeat the process.

The characteristics of tasks in this conception were suggested in the previous section: rule use, symbolic representation, and process controls were more frequent than other categories. The operations were almost equally divided between manipulate operations only and manipulate and appreciate operations. This result, which might have been expected, seemed to result from a combination of several factors: The need to give meaning to the notion of correspondence while at the same time linking the work in arithmetic to the work in algebra leads to an emphasis on procedures that combines familiar activities (performing numeric transformations) with unfamiliar activities (assigning values to variables or getting used to the Cartesian plane). Because it might be easier for students to accept the notion of a controlled assignment of values (controlled in the sense that the student knows what he or she is doing) than an arbitrary assignment (as in the case of relations defined with Venn diagrams), the possibilities for the assignment are reduced to equations involving arithmetic operations and powers (i.e., polynomial expressions). In addition, the lack of context is useful in that it reduces the burden of interpreting the situation. Thus these tasks were abundant (20% of the tasks), simple (because they used simple operations and mainly one representation), fulfilled a familiarization purpose (needed to advance in the abstraction process), and did not require sophisticated methods to legitimate the answer; the process and the task setting acted as *the* control structure.

Ordered Pair

In the tasks that belong to this conception, the use of function is as a set of ordered pairs. They require manipulate or appreciate operations or a combination of the three types and they use any of the possible representations and controls.

In this conception the tasks offered the most alternatives for the elements of the configuration. That is not surprising, given that the set theoretical definition of function is the most flexible, mathematically speaking. The tasks may be solved using only the symbolic representation or using a combination of a set and other representations (e.g., an arrow diagram, a number line, or a Cartesian plane). Because one common operation deals with determining whether a given relation is a function, the controls maybe based on the conditions defining a function, that is, a content type of control. The following example taken from Ireland1 illustrates one such task:

> The domain of the relation $R = \{(x, y) \mid y = x^2\}$ is $\{1, 2, 3, 4\}$. What is the range or R? List the couples of R^{-1}. (p. 182, Task 6)

In this task, operations of all three types are needed. The student needs to use the relation to find the values of the relation at each point of the domain, which in turn will determine the range; these correspond to manipulate and appreciate operations. The listing of the elements of the inverse relation (exchange the ordered pairs) corresponds to a calculate operation. In this example, the representation used is symbolic, and the controls available are the procedures themselves; the student might repeat the calculations if there were a hint that there is a mistake in the solution.

Fourteen percent of the tasks fell into this group. This is a large percent, which could be explained by the effects of the new math movement, as the majority of the textbooks in the sample were produced during the late 1970s and the 1980s. That the set theoretical use of function was associated with all the categories of the quadruplet could be related to the overarching character of the definition. The use of all types of operations serves the purpose of showing that the function can be something that is usable and something that one can discuss; the use of several representations serves the purpose of showing that arbitrary correspondences are possible, something that cannot be shown when the rule of assignment is explicitly given (with an algebraic expression, for example); and the use of several controls also serves the purpose of calling attention to the processes associated with the definition and to the conditions by which the function exists.

Social Data

In the tasks that belong to the social-data conception, the use of function is social (as constructed relations or data reduction relations). The task requires appreciation operations alone or in combination with manipulation operations. It uses nonsymbolic representations mainly, although the symbolic representation can be combined with other representations. Controls are based either on the content or on the process plus the didactical contract.

The tasks that belonged to a social-data conception in this sample did not necessarily involve symbolic representations. The relations tended to be defined through tables, graphs, or words, and the task might not ask for a symbolic expression. The presence of real contexts in many of the tasks might explain why both manipulate and appreciate operations were often needed: The task might offer the student the possibility of operating with the function so as to become more familiar with it, but interpreting the results requires an examination of the function as an object. The context acts as a means of controlling, either by limiting the reasonableness of an answer (most of the situations deal with positive numbers only) or by asking for an interpolation of values, which invariably assumes the continuity of the relations depicted. The following task taken from UnitedStates4 illustrates one of such tasks:

> Make a table for [the] relation: A car gets 26 miles to a gallon of fuel. Show the relationship of the number of miles driven to the number of gallons of fuel used. (p. 457, Task 17)

The task is numerical; the preceding text contains a similar example that uses integral positive numbers and describes the relation without symbols; these uses are thus determined by the contract. The described relation—miles driven per gallons used—acts as a control for the calculation of the entries in the table (if the number of miles driven does not increase as the number of gallons increases, then there is a mistake).

These tasks satisfied the purpose of providing meaning to the correspondence that defines a function as a dependence between two variables that relate to the student's world, fulfilling in this way a motivational objective too. The tasks tended to be used for motivation, which might explain the reduced emphasis on symbolic representation. The text preceding these tasks might present a definition of function as a set of ordered pairs and give definitions for domain and range; in some cases, these names were not used again in the tasks, or if they were, they appeared without a reference to an external context helping in this way to create a separation of the practices in using the definition:

> A **relation** is a group of ordered pairs. A relation can be shown in a table or a graph. [A graph and a table are provided showing six integral values for *number of gallons* (x) and *cost in dollars* (y) of ethanol fuel.]
>
> The **domain** of a relation is the set of all the values of x. The **range** of a relation is the set of all the values of y. (UnitedStates4, p. 456)

Observe that in these definitions the variables are referred to as x and y and not as *number of gallons* and *cost in dollars*. The tasks that followed the definition and that dealt with ordered pairs used a symbolic representation, whereas the tasks with a context required nonsymbolic representations. Only 7% of tasks belonged to this conception.

Physical Phenomena

In the tasks that belong to this conception the use of function is physical (cause-effect or time relationships). The task requires manipulate operations alone or in combination with appreciate and calculate operations, or it requires operations outside these types. The task does not use symbolic representation. The controls are based either on the content in combination with other types or on the process only.

The tasks belonging to a physical-phenomena conception (4% of the total) shared common characteristics with those belonging to a social-data conception. Beside a difference in the use of function, important differences are that for the physical-phenomena conception, the controls were based on the content or the process rather than on the contract, and the operations used were not necessarily within the three main groups (e.g., determine the type of relation between the variables and use proportionality to find values in a table). The tasks belonging to this conception required the students to collect data from experiments (e.g., timing a pendulum, England2, p. 16), which might explain why the process and the content were so frequently used as controls: Students' unexpected results for the task could be attributed to the data collection processes or taken as falsifiers of conjectures posed (in the case of the pendulum the conjecture that the smaller the angle the longer the period, for example, would be falsified by the unexpected result that the period is the same for every initial angle). The following example taken from Mexico1 illustrates a task belonging to this conception:

> The following table shows the distance traveled by a car after the brake is pressed over a dry road; for example, a car driving at 40 kilometers per hour will need 18.6 meters to reach a complete stop. Is there proportionality between the speed and the stopping distance?

Speed (in km/h)	Stop Distance (in m)
40	18.6
50	26.5
60	35.7
70	46
80	57.5
90	70.7
110	101
130	135.6

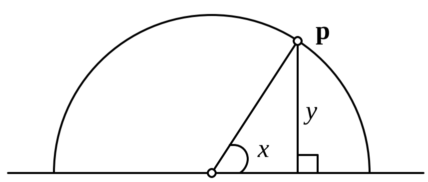
> (p. 197, Task 5)

This task was contained in chapter on direct proportion, and the student had at hand two strategies to test the proportionality: Find the ratios of corresponding entries and produce graphs. The tabular presentation suggests the ratio approach. The student might need to repeat the procedure for almost all the entries because the ratios (distance/speed) of the smaller numbers seem to group around 0.5. By finding all the ratios, the student shows that there is no (direct) proportionality. In this case, the process is fundamental to the solution and for establishing the correctness of the answers.

Controlling Image

In the tasks that belong to this conception, the use of function is figural (geometrical, graph defined, or pattern relations). The tasks use operations of all types, but they do not use manipulate or appreciate operations alone. Nonsymbolic representations are used almost exclusively, and the task uses any of the types of controls. Three percent of the tasks belong to this category.

The few instances in which the symbolic representation is used in combination with other representation correspond to cases in which the symbols are not manipulated; they act as labels as in expressions like $A = b \times$ h for the area of a rectangle with base b and height h. The main difference between the controlling-image conception and the social-data and physical-phenomena conceptions, however, is that the tasks belonging to the controlling-image conception do not require operations from one type only. This difference may indicate a greater complexity in these tasks, which is supported by the fact that all types of controls were available. The following task from Switzerland2 is an example:

> Angular Height
> Draw a semicircle with radius 10cm. Draw several angles x $(0 \leq x \leq 180^{o})$ with
>
> 
>
>
> origin at the center of the circle and one side lying on the horizontal radius.
>
> For each angle x determine the height y. Draw an approximate graph. Describe the behavior of the curve. (p. 131, Task 18B)

This task, which the student has to solve without using trigonometric relations, uses x and y as labels for the angle and the height, respectively. The student is not asked to find a relation between the two variables of angle and height. To solve the task the student has to collect data by measuring several angles and then measuring the corresponding heights. The set of values obtained has to be plotted, and a description of the graph, which will be continuous because of how the angles vary, must be obtained. Thus, despite the function not being explicitly exposed, the student uses the relation (to plot the points), discusses it (by describing the graph), and does something for it (collecting the data). The picture given in the task helps to illustrate that for an angle of 0° the height is 0 cm and that for an angle of 90° it is 10 cm, which will act as control for the values obtained for the height (it must be between 0 and 10).

Summary of Findings on Conceptions

Table 9 gives the percentages of tasks belonging to each conception. The proportion of tasks for the symbolic-rule and ordered-pair conceptions—in which there is no context involved—is almost twice the percentage of tasks for the conceptions that involve a context. One explanation may be that it is easier to set up a larger number of tasks when there is no context. Real applications require a lot more work. It is difficult to construct tasks that satisfy academic purposes and at the same time resemble the real situation from which they are derived. Similarly, tasks involving physical phenomena may be more difficult to set up than tasks involving social phenomena because the former may involve situations that are less familiar to the students or more difficult for teachers to explain. The low percentage of tasks in the geometry conception might be a consequence of the separation of subjects that is common between arithmetic, geometry, and algebra. Because function tends to be considered an algebra topic, it is less likely that geometric situations or patterns involving numerical sequences would be treated under a functional perspective.

In the tasks belonging the conceptions in which the use of function was not rule or set of ordered pairs—function without a context—the role of the symbolic representation was minor. Tasks belonging those contextual conceptions may fulfill a motivational purpose; the need to handle the context imposes other demands (as part of a modeling

Table 9

Percentages of Tasks Within Each Conception

Conception	%
Symbolic rule	20
Ordered pair	14
Social data	7
Physical phenomena	4
Controlling image	3

process, for example) that relegate the issue of symbolic manipulation to a secondary status.

The tasks in the symbolic-rule conception were the only ones that did not take advantage of the content as control. Three activities were included under the content category of controls (*more than one point—the vertical line test,* MT1P, *continuity,* CON, and *use alternative representations,* UAR). UAR could not be assigned to the tasks because within this conception the only representation used was symbolic. MT1P was not assigned because in these tasks the relation, given symbolically, always represented a function. Thus any correspondence interpreted as a dependence relation always defined a function, and therefore the necessary condition of unique assignment became superfluous. The student does not need to provide a method to test that the relation satisfies the condition within the symbolic representation. CON was not assigned because the tasks tended to ask for discrete values of the relations, even for continuous relations. It could be the case that textbook authors for these grades consider the treatment of continuity in symbolic expressions for functions to be beyond students' level of understanding.

With respect to operations, the physical-phenomena and controlling-image conceptions were the only ones that required operations from all the sets. A possible explanation is related to the level of complexity that these tasks manifest because of the introduction of a more specialized context as compared to the contexts present in the tasks belonging to the social-data conceptions.

The classification into five conceptions reveals the most frequent interrelations of the four elements defining a conception. The requirement that four elements of the quadruplet be analyzed at a time reduced the number of cases in each cell of the associated contingency table, which in turn reduced the possibility of classifying a larger fraction of the tasks. That only 48% of the tasks were accounted for in this classification indicates that there are other interrelations that, as a group, did not constitute extreme cases. To elicit these interrelations, it would be necessary to have a much larger sample of tasks.

Altogether, these results imply the existence of a separation of practices enacted by these tasks that is strongly associated with the uses given to function within each task and that results in different potential conceptions of functions that the students might learn. This separation has implications for the ways in which students generate their own conceptions of function.

Patterns of Conceptions Across Textbooks and Countries

Because the tasks came from different textbooks within the countries, a natural question concerned the similarities between the conceptions of function across countries: Were there patterns of conceptions? Because there were cases in which there was more than one textbook in a country, the results are discussed at the textbook level.

The question of patterns of conceptions was addressed by studying the distribution of tasks eliciting each conception at the textbook level. I did not perform a CFA for each textbook, because the extremely small ratio of sample size to the number of cells in a five-way table (of book by use by operation by representations by controls) would have made the test too powerful and, consequently, the results unsuitable for interpretation (von Eye, 2000, p. 9). Thus, no statistical analysis was carried out beyond the calculation of frequencies and percentages of tasks eliciting particular conceptions.

The section is divided into two parts. In the first part, I describe how the tasks were distributed across textbooks according to conception. Because not all tasks fell into a particular conception, a further classification considering the most frequent use of function was added to create clusters of textbooks. The results of this process are discussed in the second part.

Conceptions Within Textbooks

Table 10 shows the proportion of tasks from each textbook that contributed to the characterization of each conception. The table is alphabetical by country. The Canadian textbook, for example, had 25 tasks. Of these, 4% (1) had a configuration within a rule use of function that was labeled as an antitype by the CFA program; thus this task enacted a conception of function that was not common to the symbolic-rule conception of function, as the tasks required nonsymbolic representations when the function was used as a rule. Another 4% (1) had a configuration within the social use of function that was labeled as a type by the CFA program; thus this task elicited a conception of function that required appreciation operations either alone or in combination with manipulation operations, without exclusive use of symbolic representation, and with controls based on the content or on the didactical contract. Eight percent of tasks enacted a physical-phenomena conception, and another 8% a controlling-image conception. In all, 24% of the tasks of the Canadian textbook could be classified according to the five conceptions found. The remaining 76% of the tasks had configurations that, when considered in the pooled sample, did not have a more than expected frequency of occurrence, in other words, the interrelation of the four elements of the conception in those tasks could be expected by chance at the pooled-sample level.

At the textbook level, one might well have expected different interrelations to play out. The reduced sample size did not permit a statistical analysis like the one carried out with the pooled sample. Even without such an analysis, however, the proportions in the table showing each textbook's contribution to the characterization of each conception reveal interesting features of the textbooks.

Table 10

Percentage of Tasks in Each Textbook Eliciting Each Conception

Textbook	No. of tasks	Symbolic rule	Ordered pair	Social data	Physical phenomena	Controlling image
Argentina1	18		17	17		
Australia1	27	3	11			4
Australia3	26	4	15			31
Austria1	14			7	57	
Canada1	25	(4)		4	8	8
Colombia1	57	5	47			
Colombia2	28	7	4	4	4	4
England1	31				65	
England2	23				57	
HongKong1	12			25		
Ireland1	73	3	37			
Mexico1	32	6		3	3	6
Portugal1	52	12	6	17	2	
Singapore1	41	27	27			
SouthAfrica2	22	(36)				
Spain1	77	39		3		
Switzerland2	34	6	3	3	15	3
Switzerland4	22	18		(14)		
UnitedStates1	174	16	3	29		
UnitedStates2	123	27	21			17
UnitedStates3	88	5	51			
UnitedStates4	180	29	16	6		1
UnitedStates5	38	45		5		3
UnitedStates6	99	53		(1)		

Note: Entries in parentheses indicate the proportion of tasks with configurations labeled as antitypes.

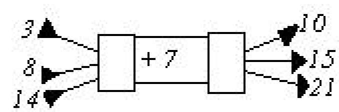
All but five textbooks contained tasks that promoted a symbolic-rule conception, and about half contained tasks that promoted an ordered-pair conception. Tasks in the symbolic-rule conception use functions as a rule, require manipulation of the relation with a symbolic representation or manipulation and appreciation with additional representations, and base the controls on the process of solution. The ordered-pair conception uses function as a set of ordered pairs and combinations of all the representations, operations, and controls. As was discussed earlier these conceptions were the most commonly promoted. Li (1999), in his analysis of to-be-solved problems in eighth-grade textbooks from East Asian (China, Hong Kong, Singapore) and the United States found that the configuration “same mathematics content as introduced in the chapter, pure mathematical context, no explanation required, application of routine procedures" was the most frequent in each of the textbooks analyzed (pp. 180-183). This result is consistent with the results in this study for Singapore1, UnitedStates2, UnitedStates4, and UnitedStates6. In these textbooks, the proportion of tasks promoting either a symbolic-rule or an ordered-pair conception was high. The discrepancy in the textbook from Hong Kong appears to have occurred because the sections analyzed were different in each study. Functions are not treated as an algebra topic in the eight-grade textbook from Hong Kong ; they are treated through applications in statistics.

Five countries provided more than one textbook at the seventh and eighth grades: Australia, Colombia, England, Switzerland, and United States. The textbooks from Australia and from England had very similar distributions of tasks in each conception; this similarity was probably due to the fact that in each country the authors of those textbooks were the same. Each author or group of authors has a particular agenda (in most cases shaped by curriculum guides) that they tend to follow as they write a series of textbooks. The textbooks from Colombia and from Switzerland were intended for the same grade but had different authors; and in each case one of the textbooks had tasks in all the conceptions, whereas the other has tasks in only two of them. The pairs UnitedStates1–UnitedStates2, UnitedStates3–UnitedStates4, and UnitedStates5–UnitedStates6, which were three different series of seventh- and eighth-grade textbooks, were each written by a different group of authors, and they showed different distributions of conceptions.

In the case of the United States, the textbooks also showed differences across grades within the series. In every case, there was an increase in tasks belonging to the symbolic-rule conception from the seventh grade to the eighth grade. This increase suggests an interest in fostering rigor as the students advance in their study of function.

Table 10 also shows the contribution of the textbooks to antitype configurations. An antitype indicates a configuration that was less frequent than would be expected by chance. Antitypes were used to corroborate the descriptions of the conceptions: Their characteristics were so rare that they helped to highlight (or make evident) characteristics of the most frequent configurations. For a textbook, an antitype exemplifies configurations that are different in critical aspects from those used to characterize the corresponding conception. In the case of the South African and Canadian textbooks, the tasks in the antitype use a rule, but they do not use a symbolic representation. The following is an example of this situation taken from SouthAfrica2; the Canadian tasks are comparable:

> Examine [the] following number machine. When you are sure that you know how the machine works copy the table into your book and complete it by filling in the values of the outputs (*y*).



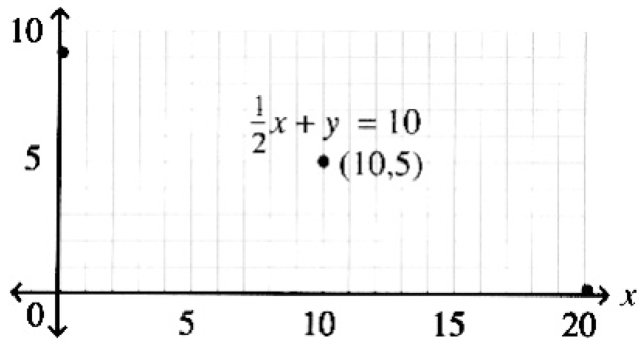
input (*x*)	1	2	3	4
output (*y*)				

> (p. 290, Task 1)

In this kind of task, the function machine exemplifies the transformation that affects the numbers that enter the machine. It performs an arithmetic operation that is also made visible in the label (+7) on the machine. The student needs to apply the transformation to each number of the table, after checking that the label (+7) does correspond to the transformation of adding 7 to the numbers that enter the machine. Note that the symbols *x* and *y* do not play a role as variables; they act merely as names for input and output. This task was labeled as numerical for its representation. The student might redo the computations to make sure that the transformed values are correct.

In the case of the Swiss and U.S. textbooks, the use of function in the antitype tasks is social, but the tasks use manipulate operations only and the controls are based on the process only, as in the following example from UnitedStates6:

Mai is allowed to watch a total of 10 hours of TV during the school week. Some shows are a half hour long and some are an hour long.



Find the three points graphed above. Which ordered-pairs solutions to the equation do these points represent? (p. 152, Task 3)

This task seems to serve the purpose of giving the students practice in reading points from the Cartesian plane. The word *find* in the task is to be taken literally (in the original there are three red dots where the black dots are), and once the points, actually dots, are found, the student needs to write the corresponding ordered pairs. The graph does provide a checkpoint through one of the points, which is already labeled (10, 5). The choice of the numbers in this ordered pair seem to be intended to show that the 10 corresponds to the 10 on the *x*-axis and goes first in the ordered pair, and the 5 must be located on the *y*-axis and goes second in the ordered pair. The other points in the graph are located on the axes, which would facilitate the discrimination as to which number goes in which place. The dot that corresponds to (0, 9) is not a solution to the equation. It is treated as if it were, however, which suggests that there was a printing error and that the authors meant to highlight the pair (0, 10). The use of the process indicates that if an answer was obtained and a mistake detected, redoing the process would help as a check. Thus there is only one operation needed, read points from the graph, which is a manipulate operation, and only one way to control the solution, through the process.

Only 14 tasks were labeled as antitypes (about 1%), which I considered to have a negligible effect on the analysis of percentages and frequencies that follows. The characteristics of the antitypes, however, were crucial in characterizing the conceptions enacted by the tasks in these textbooks, as was discussed previously.

Completing the Picture: Tasks Not Eliciting a Conception

For each textbook, the set of tasks that did not lead to distinctive conceptions and that therefore were not considered in Table 10 were classified by use of function, and the most frequent use was used to characterize the textbook. By looking at similarities in both the conceptions enacted and the predominant use of function in the remaining tasks, I determined four clusters of countries. Those textbooks whose tasks belonged to a symbolic-rule conception and for which the majority of the other tasks had a rule use composed the first cluster, called *rule oriented*. Textbooks with tasks belonging to a symbolic-rule or ordered-pair conception and for which the majority of the other tasks had a rule use or a set-of-ordered-pairs use formed the second cluster, called *abstract oriented*. Textbooks whose tasks belonged to the controlling-image, social-data, or physical-phenomena conception and to either or both of the symbolic-rule or ordered-pair conception composed the third cluster called *abstract oriented with applications*. Textbooks whose tasks belonged to the controlling-image, social-data, or physical-phenomena conception and for which most of the other tasks had either a figural, social, or physical use composed the fourth cluster, called *applications oriented.* Table 11 presents these clusters, together with the intended grade and the percentage of the tasks contributing to each conception or use of function.

More than half of the textbooks were abstract oriented with applications, one sixth were abstract oriented, one sixth were applications oriented, and the rest were rule oriented. The textbooks tended to offer more tasks in which both abstractions and applications were used than situations in which the function was treated as an abstract entity only. This tendency might well be connected to the grades for which these textbooks were intended, grades that are seen as transitional between the primary grades, in which more work is done with concrete objects and situations, and the secondary grades, in which more abstract work is expected. The contribution of these textbooks to developing an abstract practice of functions, however, cannot be neglected: 20 textbooks (83%) had tasks belonging to a symbolic-rule conception or used function as a rule, and 13 (57%) had tasks belonging to an ordered-pair conception or used function as a set of ordered pairs. Although many books offered a variety of situations in which different practices of functions occurred, they still emphasized abstract uses of function.

Table 11

Clusters of Textbooks With Percentage of Tasks by Conceptions and Use of Function

Cluster	Textbook	Grade	SR-R	OP-SOP	SD-S	PP-P	CI-F
Rule Oriented	SouthAfrica2	7	58				
	Switzerland4	7	54				
	United States6	8	69				
Abstract Oriented	Colombia1	8	5	80			
	Ireland1	8	3	96			
	Singapore1	8	59	27			
	UnitedStates3	7	27	51			
Abstract Oriented with applications	Argentina1	8	30	17	17		
	Australia1	8	33	11			4
	Australia3	7	4	15	31		31
	Austria1	8	14		21	57	
	Colombia2	8	46	4	4	4	4
	Mexico1	8	47		3	3	6
	Portugal1	8	12	6	42	2	
	Spain1	8	78		3		
	Switzerland2	7	6	3	35	3	3
	UnitedStates1	7	55	3	29		
	UnitedStates2	8	44	21			17
	UnitedStates4	8	54	16	6		1
	UnitedStates5	7	45		31		3
Applications Oriented	Canada1	8			40	8	8
	England1	8				100	
	England2	8				79	
	Hong Kong1	8			100		

Note. SR-R = Symbolic-rule conception or rule use; OP-SOP = Ordered-pair conception or set-of-ordered-pairs use; SD-S = Social-data conception or social use; PP-P = Physical-phenomena conception or physical use; CI-F = Controlling-image conception or figural use.

All but one textbook from the Spanish- and Portuguese-speaking countries belonged to the abstract-with-applications cluster. The similarity in languages and culture may have given educators in these countries common experiences. They may have taken similar approaches to teaching mathematics, which may have been reflected in how these textbooks posed tasks about functions. The conjecture about the influence of common experiences, however, did not apply to textbooks written in English. Textbooks from three countries in which English is the language of instruction (England, Hong Kong, and Canada) were in the applications-oriented cluster; textbooks from Australia and the United States were in the abstract-oriented-with-applications cluster, and textbooks from Singapore and the United States were in the abstract-oriented cluster. Thus in the case of the countries that share some common background—being former English colonies—the distribution of conceptions and uses of function within textbooks was not uniform.

All but one of the textbooks from continental Europe belonged to the abstract-oriented-with-applications cluster. There may be a cross-country influence that made these countries' textbooks similar with respect to function use. If Spanish-speaking countries tend to be influenced by Spain, that could explain why all but one of the textbooks written in Spanish belonged to this cluster too. In the case of the Australian textbooks, the influence is not clear (but one could ask why the South African textbook is not in this cluster, given South Africa's links to continental Europe).

Searching for possible explanations for the cluster organization of textbooks, I looked at the copyright dates of the textbooks. The textbooks analyzed were all copyrighted between 1972 and 1993. One might expect that textbooks with copyright dates in the 1970s and 1980s would tend to be abstract-oriented because of the influence of the new math movement, which advocated a formal approach to the teaching of mathematics. In the abstract-oriented cluster, however, two textbooks were copyrighted in 1987 (Ireland1 and Singapore1) and two in 1992 and 1993 (UnitedStates3and Colombia1). Thus the expectation that older textbooks are abstract oriented cannot be supported. In this cluster, more than 76% of the tasks in the textbooks belonged to the symbolic-rule and the ordered-pair conceptions and used function as a rule or a set of ordered pairs. Perhaps in some countries there was an interest in maintaining certain content to cater to teachers who were used to this kind of approach. It could be, however, a consequence of constraints that did not allow a rapid production of new textbooks for some countries (e.g., small readership, competing priorities in the case of state publishing, protectionist practices, strict copyright laws, or shortage of paper for printing; Farrel & Heynemann, 1994, pp. 6362-6365).

In Colombia, Switzerland, and the United States, the country's textbooks were spread over two or three clusters. In countries in which decisions about textbook use are not centralized (Colombia and the United States), publishers tend to offer a variety of alternatives so that teachers can select or suggest a textbook based on their experience. In the case of Switzerland, the difference in approaches in the two textbooks can be attributed to ways of dealing with tracking practices. Both textbooks were intended for the medium/high track. Switzerland4, however, incorporated content for two grades. The first half of the textbook was intended for seventh graders; the second half, for eighth graders (W. Durandi, personal communication, June 22, 2000). The section analyzed in this study corresponded to the introduction of functions, whose tasks were similar to the those from the South African textbook (see p. 92) and belonged to the first part of the textbook.

Conceptions and Achievement

To address the question of the relation between conceptions promoted by textbooks and student performance, the data from students' performance on selected items of the TIMSS achievement test for each of the participating countries were compared with the conceptions fostered by the tasks in the textbooks of that country.

The codes for the ten items selected from the released set of TIMSS achievement test, organized by use of function, are shown in Table 12. The distribution of uses of function in these items does not resemble the distribution found in the tasks analyzed in this study (see Figure 2). In only two items, J-18 and L-14, is function seen as a rule, in four the use of function is figural, three are physical, and only one is social.

Table 12

Items Related to Function on the TIMSS Achievement Test

Item	Description	Use of function	Operations	Representations	Controls
J-18	Number missing from table	Rule	Manipulate-appreciate-calculate	Nonsymbolic	Process-content
L-14	Missing values in proportionality table	Rule	Manipulate	Nonsymbolic	Process (A)
I-04	Number sequence	Figural	Manipulate-appreciate-calculate	Symbolic-others	Process-contract
I-08	Point on a line	Figural	Manipulate-appreciate-calculate	Nonsymbolic	Process-contract-content
J-16	Likely coordinates of P	Figural	Manipulate-calculate	Nonsymbolic	Process
S-01	Sequence of triangles	Figural	Manipulate-appreciate-calculate	Symbolic-other	Process
V-02	Price of renting office space	Social	Manipulate-appreciate-calculate	Nonsymbolic	Process
L-11	Total distance traveled by ball	Physical	Manipulate-calculate	Nonsymbolic	Process (T)
O-01	Speed of car from graph	Physical	Manipulate	Nonsymbolic	Process
R-08	Distance car will travel	Physical	Manipulate	Nonsymbolic	Process-content (T)

Note. (A) indicates that the item was similar to tasks in the textbooks with a configuration labeled as an antitype. (T) indicates that the item was similar to tasks in the textbooks with a configuration labeled as a type.

Figure 2 also shows the distribution of uses of function in the tasks in three other textbooks that were selected because the distribution of tasks in those textbooks were similar to the TIMSS items in the proportion of physical or figural uses of function. Even in the case of these countries, however, there are striking differences with respect to the overall distribution of uses. The data imply that the textbooks in this study had a very different emphasis on conceptions of function than did the function items in the test.

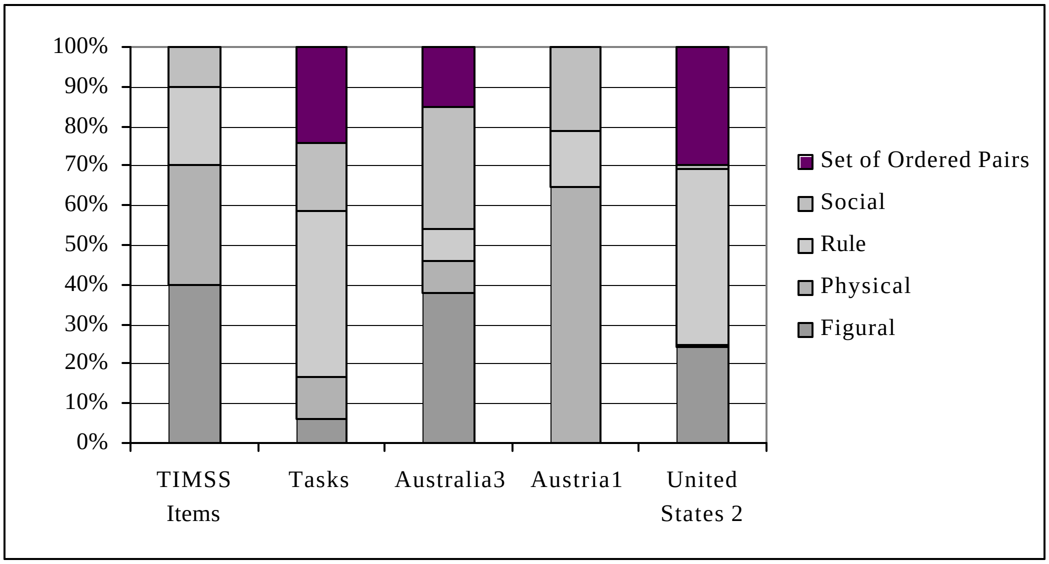


Figure 2. Distribution of uses of function in the TIMSS function items, in the tasks in the sample of 24 textbooks and in tasks from three textbooks.

The test items do not mirror the textbook tasks' uses of representations either. Almost all the items require nonsymbolic representations (numerical, graphical, and tabular), and when a symbolic representation is required it is not in combination with a rule use of function. The distributions of operations and of controls do resemble the distributions observed for the tasks of the study. The test situation, however, was different from a situation in which the student is solving a textbook task, because the test provided multiple-choice items. If the student reaches a solution that is not among the choices, he or she immediately knows that there is a mistake. The choices act as checkpoints. In solving a textbook task or an open-ended test question, the student does not necessarily have these checkpoints available. Also, when choices were provided, the student might use the choices provided in the item and test their reasonableness instead of following a certain process (e.g., in Item J-16, the student might have plotted the points given in the choices instead of looking for the coordinates of the point). There were seven multiple-choice items, and they were given the code use-checkpoints.

Thus the TIMSS items, as a set, do not share the same characteristics as those depicted by the tasks in the textbooks. Only two items, L-11 and R-08 (a physical conception), had a type configuration. England1, England2, and Switzerland2 had tasks with the same configuration as Item L-11, and Austria1, Canada1, England1, England2, and Portugal, had tasks with the same configuration as Item R-08. I conjectured that the students from those countries with textbooks containing tasks with the same type configurations would perform better on items enacting those configurations than would the students from other countries.

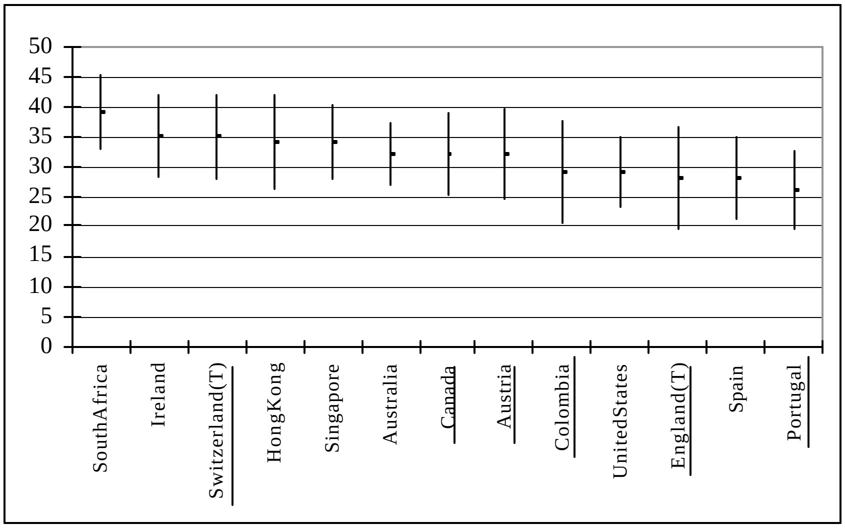
To determine the differences in achievement across countries, I constructed 95% confidence intervals around the mean percentage of correct answers by item and by country to get an estimate of the true percentage of correct responses for the two items L-11 and R-08. The distribution across countries of performance on the items with 95% confidence intervals is presented in Figures 3 and 4 for Grades 7 and 8 separately. In the figures, the countries are ordered by performance on the item, and the names of those countries with textbooks having the physical phenomenon conception are underlined. A (T) is used to mark those countries in which at least one textbook contained tasks having a type configuration similar to that of the item.

Item L-11

> A rubber ball rebounds to half the height it drops. If the ball is dropped from a rooftop 18m above the ground, what is the total distance traveled by the time it hits the ground the third time?
>
> A. 31.5m
> B. 40.5m
> C. 45m
> D. 63m
>
> (IEA, 1997b, p. 37)

On Item L-11, the performance of the English students at both levels was not statistically different than the performance of other students (see Figure 3). The Swiss eighth graders outperformed those from Portugal, the United States, and Colombia, but the performance of the Swiss seventh graders was not significantly better that that of students in those same countries. The textbooks from England and Switzerland had tasks

Grade 7



Grade 8

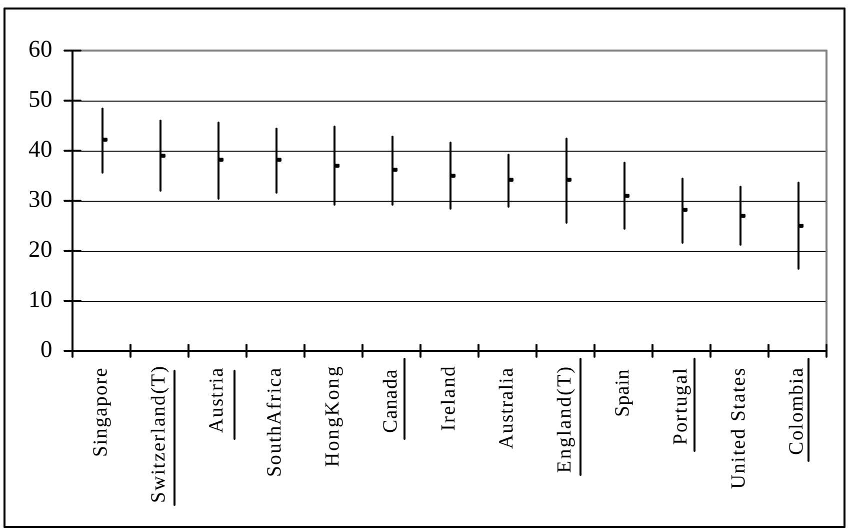
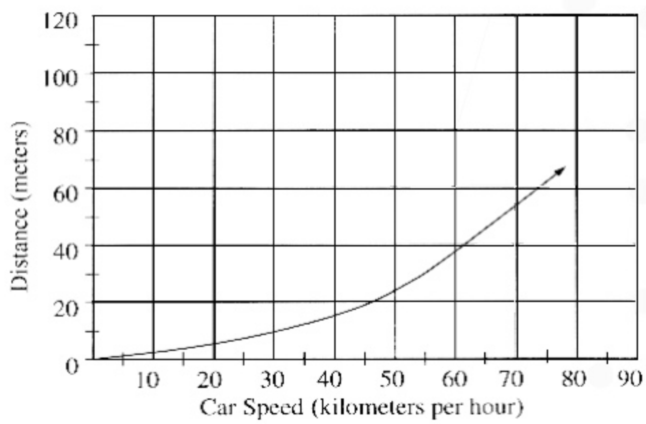


Figure 3. Percentage of correct answers on Item L-11 at Grades 7 and 8 by country, with 95% confidence intervals. Countries with textbooks whose tasks belong to the physical phenomenon conception are underlined. A (T) beside a country name indicates that at least one textbook from that country had tasks with a type configuration similar to that of the item.

with the same configuration of use of function, operations, representations, and controls as Item L-11, with the Swiss textbooks intended for the seventh grade and the English textbooks intended for the eighth grade. At neither of these grades did the students from Switzerland or England those countries outperform the students from all other countries. Of the six countries having books with tasks that belonged to a physical phenomenon conception, Portugal and Colombia showed the lowest performance. Thus, as a group the students from countries with textbooks having the physical phenomenon conception did not perform better on this item than the students in countries with textbooks eliciting other conceptions. Across grades there are not many differences.

Item R-08

The graph shows the distance traveled before coming to a stop after the brakes are applied for a typical car traveling at different speeds.



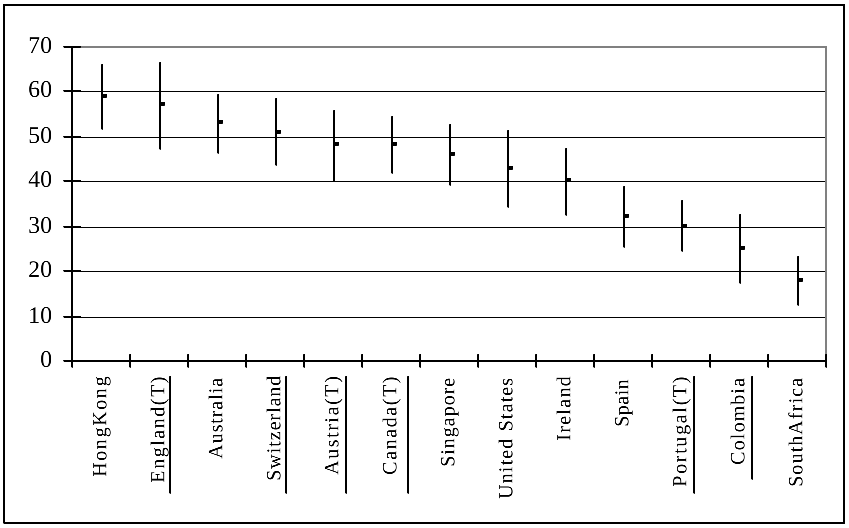
A car is traveling 80 km per hour. About how far will the car travel after the breaks are applied?

A. 60 m
B. 70 m
C. 85 m
D. 100 m

(IEA, 1997b, p. 101)

On Item R-08, the students in England, Canada, and Austria, countries with textbooks that had tasks with the same configuration as the item, performed statistically better than four countries, Portugal, South Africa, Colombia, and Spain at both grades (see Figure 4). Of the six countries that had textbooks with tasks that belonged to the physical conception, only two, Portugal and Colombia, exhibited low performance at both grades. Thus for this item there might be some benefit for some students whose textbooks had tasks belonging to a physical-phenomenon conception. On this item, there was an improvement in performance across grades for some countries.

Grade 7



Grade 8

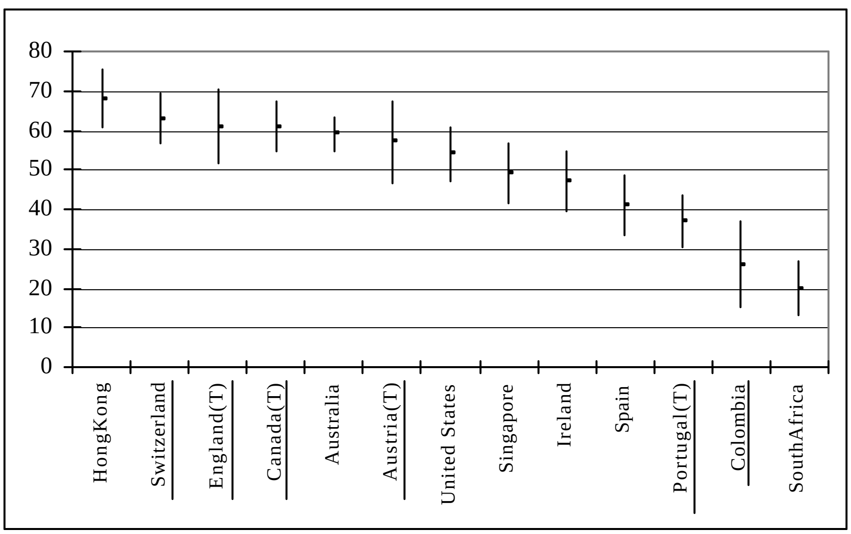


Figure 4. Percentage of correct answers on Item R-08 at Grades 7 and 8 by country, with 95% confidence intervals. Countries with textbooks whose tasks belong to the physical phenomenon conception are underlined. A (T) beside a country name indicates that at least one textbook from that country had tasks with a type configuration similar to that of the item.

In summary, for Items L-11 and R-08, which had a configuration in a physical-phenomena conception of function, students in countries with textbooks containing tasks that belong to the same configuration performed better, about the same, and worse than students from the other countries, which might or might not have had textbooks with tasks belonging to the same conception of function. Thus it is not possible to conclude that countries that have textbooks with tasks promoting a physical conception have an advantage on test items eliciting the same conception.

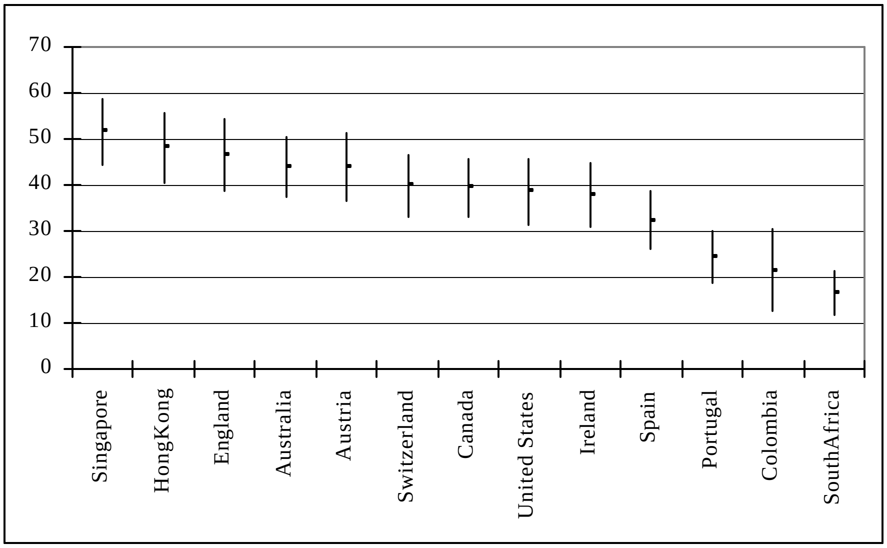
The Items As a Group

As information based on only one item can be inconclusive, I repeated the analysis using the entire set of function items. I found the average percent correct for the 10 function items for the students in grades seven and eight, using 95% confidence intervals (see Figure 5) in order to determine whether there were patterns of achievement that could be related to the textbook clusters. I did not find a clear pattern regarding the cluster organization of textbooks and students' achievement on the items. There was a slight improvement from seventh to eighth grade, but at both grades the students from Spain, Portugal, Colombia, and South Africa obtained the lowest scores—and were outperformed by at least two other countries—and the students from the other countries performed about the same (there are not many significant differences in their scores). Thus, it is not possible to claim from these data that exposure to a textbook with a certain orientation improves students' performance on these items. It is clear that other factors must be operating beyond just the use of a certain textbook.

That some countries within a given cluster performed better and some worse than others can be a consequence of two different factors. In the first place, the existence of more than one textbook for the same grade in some countries (Switzerland and Colombia, for example) implies that there were students who might not have used the textbook during their school year, and thus they might not have been exposed to those conceptions. Because the performance score for a country reflects the attainment of all students in the country, it is difficult to interpret the results without examining particular segments of the student population. Such information was not available for this study.

In the second place, and even in the cases in which the decisions regarding textbook use was centralized (Austria, Hong Kong, Singapore, and Spain), it might be that only a very small fraction of the tasks in a textbook were done by the students. The decision as to which tasks were used in instruction depended on the teacher and his or her

Grade 7



Grade 8

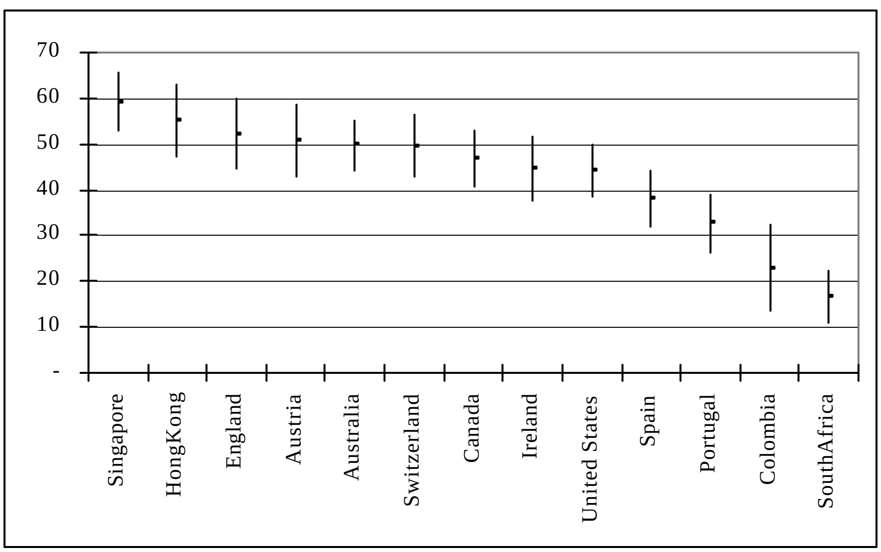


Figure 5. Average percentage of correct answers on ten function items at Grades 7 and 8 by country, with 95% confidence intervals.

teaching situation. This observation implies that the everyday activity within a classroom creates another set of practices about function that do not resemble the ones described here. A subsequent study of the practices of function as revealed by the interaction between teachers, students, the textbook, and other resources in real classrooms might explore the relation between textbook conceptions and conceptions enacted in the classroom.

Additional Observations

As part of the process of answering the questions, I made several observations that were indirectly connected to the research questions.

A Limited Set of Examples for Physical Situations

Only 136 tasks in the textbooks presented a functional relation that was related to some physical situation, either a cause-and-effect situation or a situation involving time as a continuous variable (see Table 5, p. 72). There were 16 different cause-and-effect relations (e.g., density of water versus its temperature, Hooke's law, or Ohm's law) and 13 different time relations (e.g., speed versus distance, speed versus time, or time versus distance) with several contexts (e.g., people, cars, or trains in the time relations). Thus, each possible example in which function was used in physical phenomena occurred in four to five tasks, which implies that these examples were more or less standard across the countries.

In contrast, there were about 145 different situations that defined social uses of function (e.g., number of items versus their price, percentages of discounts of items, recipes, scales in maps, height, weight, see Table C1) for 227 tasks. These numbers mean that each situation could be used in one or possibly two tasks at most. Thus, social environments apparently offer a more prolific source of problems for illustrating functions than the physical world does, and the limited number of relations derived from the physical world may allow their exploration from different perspectives in these tasks.

Number of Sections Not Devoted to Tasks

The number of sections in the textbooks that did not contain tasks ranged from 0 for the English textbooks to 14 for the Spanish textbook, with an average of 5. The English textbooks were work booklets without explanations for the students. The other textbooks contained text that explained or illustrated related content. Not all the textbooks contained task sections immediately after these explanatory sections (e.g., Switzerland2 had a very concise basics section—*Grundlagen*—at the beginning of the task section and an elaborated presentation in the second part of the book without any task following it). These apparently simple differences suggest different strategies across countries for organizing content about functions, and these strategies seem different from those described by Li (1999, p. 157).

Surface Characteristics

That textbooks come in different sizes and shapes has been reported in several other studies (Howson, 1995; Li, 1999; Schutter & Spreckelmeyer, 1959), and the textbooks in this study illustrated the same variation. I found that almost every seventh- and eighth-grade textbook contained pictures, drawings, and photographs apparently related to the content but without any information necessary for solving the task (e.g., the picture of a car in a task about distance traveled per unit of time or the photograph of a plant in a task about length of the leaves of plants). Only four textbooks—Colombia1 (which has side boxes labeled “think!” with logic puzzles), Ireland1, Singapore1, Spain1, and Switzerland2—did not contain such illustrations. Thus despite these exceptions, it seems that there is a strong tendency toward making the content presentation appealing to the students in these grades.

CHAPTER 5

CONCLUSIONS

> "Yes, I suppose you'd be over when that was done," Alice said thoughtfully: "but don't you think it would be rather hard?
> "I haven't tried it yet," the Knight said, gravely; "so I can't tell for certain—but I'm afraid it *would* be a little hard."
>
> Lewis Carroll (1964)

The concept of function has dramatically changed the landscape of mathematics, and mathematicians consider it a key concept. Its introduction into school mathematics, however, has proved to be a challenge. Many efforts to help students understand the concept have not been successful. These two issues—that function is a key concept for mathematics and that students seem to have considerable difficulty understanding it—have spawned much research. Researchers in mathematics education have studied various aspects of the process of teaching and learning functions at different levels—by students, teachers, and prospective teachers—and have also investigated the nature of and the ways of thinking about functions. My experience as curriculum developer in Colombia has shown me the crucial role that textbooks play in teaching and learning any topic. They contain particular views about the mathematics students should learn. Across countries there are important differences and similarities in how textbooks treat functions that are worth investigating. One attempt to disclose these similarities and differences was pursued by the curriculum analysis component of the Third International Mathematics and Science Study, TIMSS. TIMSS did not address textbook content for particular topics in school mathematics. It did, however, assemble a valuable collection of documents that allow in-depth studies of how topics are organized in the mathematics curricula of some 48 countries.

The present study was designed to answer the following research questions:

1. What conceptions of function are suggested by the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS?
2. What patterns of conceptions are present in textbooks from different countries?
3. What is the relation between the conceptions suggested by the textbooks of a country and its students' performance on items related to functions on the TIMSS test for the seventh- and eighth-grade students?

From the TIMSS curriculum database, I selected 35 textbooks for seventh grade or higher that were written in English, French, German, Portuguese, or Spanish and that had specific sections devoted to functions. Balacheff's (in press) definition of conception (a quadruplet consisting of problems, P, operations, O, and representations, R, needed to solve those problems, and controls, Σ, required to legitimate the solution and to determine that it is correct) and Biehler's (in press) characterization of the prototypical uses of function were used to construct a system for coding exercises in the sections on functions in the textbooks.

The development of the coding system was a four-step process. I wanted to characterize the elements that define a conception as it would be elicited by students who solved all the problems—or tasks—on functions in each textbook. In the first step, I solved one task in the first section of each textbook according to the process suggested by the book, and I wrote a narrative answering four questions (What use is given to function in the task? What does the student need to do to solve the problem? What representations are necessary to solve the problem? How does the student know that he or she has an answer and that it is correct?). The narratives were used to formulate an initial set of codes for the quadruplet of prototypical uses, P; operations, O; representations, R; and controls, C. In the second step, I applied these codes to all tasks in the first sections in each textbook. The purpose was to find new categories and refine the characterization of the existing ones. The coding of 518 tasks resulted in a set of 133 categories, which, in the third step, were reorganized to obtain a more manageable set. In the fourth step, the system of 9 codes for uses of function, 31 for operations, 8 for representations, and 8 for controls was given to seven people from three countries (Colombia, Denmark, and the United States) who used it to code up to three tasks. As a result, improved characterizations were produced, and new codes were added. The revised system was tested with in-depth interviews with three people who used the system and provided feedback about it. The final coding system consisted of 10 codes for uses of function, 36 for operations, 9 for representations, and 9 for controls. I applied the coding procedure to each task in every section on functions in the selected textbooks. In all, 2304 tasks were coded. Each task received one code for use of function and a combination of codes for each of the other three elements of the conception.

In the data analysis the categories for each element of a conception were grouped to facilitate the analysis. The grouping characterized the combinations of codes assigned to operations, representations, and controls.

The frequencies of observed quadruplets were tested statistically with the Configural Frequency Analysis program (von Eye, 2000) to identify those quadruplets that were more frequent (types) and less frequent (antitypes) than would be expected by chance. The classification of types and antitypes by use of function resulted in conceptions that the tasks in seventh- and eighth-grade textbooks could potentially enact in a student, which addressed Question 1. To address Question 2, I used the classification of conceptions in a textbook together with the uses of function in that textbook to identify common patterns. To address Question 3, I located ten items from the TIMSS achievement test for seventh and eighth graders and compared students' achievement in countries whose textbooks had tasks enacting the same conceptions as the items with students' achievement in the countries whose textbooks did not have such tasks.

The Conceptions of Function

Five conceptions are promoted by the tasks of the textbooks analyzed: symbolic rule, ordered pair, social data, physical phenomena, and controlling image.

Symbolic Rule

The most common conception of function across textbooks was the symbolic-rule, which was enacted by 20% of the tasks. It presents functions as rules. It requires the students either to manipulate the function with a symbolic representation or to appreciate some aspect of the function using symbolic and other representations. It relies on the solution process to legitimate the solution and to verify its correctness. Tasks enacting a symbolic-rule conception seem to fulfill a familiarization purpose, serving as a link between the arithmetic of previous grades and the algebra of secondary school. Thus what they ask from the students tends not to be very demanding. The correspondence that defines the function is never counterintuitive, because the rule gives it a structure. The student is never confronted with the problem of determining whether a given symbolic expression is a function because the rule guarantees that it is. If the task makes use of the vertical line test, there is invariably an alternative representation (e.g., a graph or an arrow diagram). Tasks enacting this conception do not address the issue of continuity. Most of the expressions used are linear polynomials, but some are quadratic and cubic. The issue of discontinuous functions is never at stake.

Ordered Pair

The next most frequent conception in the textbooks, enacted in 14% of the tasks, was the ordered pair. Tasks enacting this conception use function as a set of ordered pairs and vary the most widely across the other three dimensions of the conception. This variety shows the power of the set theoretical definition of function: representations can be symbolic (a set), arrow or number line diagrams, graphical, or verbal; the student can be asked to manipulate, appreciate, or calculate; the student can legitimate a solution through the solution process, clues given by the didactical contract (Brousseau, 1997), or the content at stake. The raison d'être of these tasks is to determine whether or not a relation is a function. Nonsymbolic representations are invariably used to address that issue. The tasks address both mathematical and nonmathematical situations (see Table C1 in Appendix C), and the correspondences illustrate arbitrary or counterintuitive assignments that are not present in the other conceptions.

Social Data

The tasks enacting a social-data conception, 7% of the tasks, use functions as constructed relations (dependence relations that apply to real-life situations that are not causal or time related) or as data-reduction relations (those involving statistical data). The tasks require the student to appreciate the relation only or in combination with a manipulation. They do not necessarily require symbolic representations, and the controls are based either on the content of the task or on the process of solution and the didactical contract. The use of contexts related to the student's world satisfies a motivational purpose and at the same time provides an interpretation of an arbitrary correspondence between sets as a bi-directional dependence relation. Such tasks can be seen as helping the student toward formalizing the arbitrariness of the correspondence. The context also helps the student to legitimate a solution because the answers obtained must make sense within the particular context.

Physical Dependence

The 4% of the tasks enacting a physical-dependence conception used function as a time or a cause-and-effect relation. The task might require manipulations only, manipulations, appreciations, or calculations, or none of these operations. It does not use symbolic representations. The controls are based either on the content in combination with other types of control or on the process only.

An interesting feature of these tasks is the absence of symbolic representations. The tasks address content that technically belongs to physics and not to mathematics. Perhaps to make these physical situations more manageable for students and teachers, textbook authors downplay a formal treatment in favor of showing how the relation works. For that purpose, tables, graphs, and diagrams play a dominant role. In this conception, the relation defining a function is unidirectional (because the use of function can be cause-and-effect dependence or a time dependence).

Controlling Image

The use of function in the 3% of tasks having a controlling-image conception is geometrical, graph, or pattern. The operations may be of any type except that they are never just manipulations or appreciations alone. The representations are almost exclusively nonsymbolic, and all types of controls may be present. The tasks enacting this conception seem to require a more set of operations from the student and to use the symbols as labels rather than to represent variables. In these tasks, the context provided is mathematical expressed through a geometric figure, a graph, or a pattern (figural or numerical). The context plays a constitutive role for the task and helps to legitimate the solution processes.

The Function of Conceptions

The five conceptions accounted for 48% of the tasks in the textbooks. The conceptions require various actions, representations, and controls from students, which suggests that the practices associated with each conception are different, which in turn explains why these different conceptions can coexist simultaneously without being contradictory. The symbolic-rule conception is the only one in which symbolic representations are dominant and in which the tasks proposed do not allow questions about the necessity for a relation to be a function or about the possibly pathological nature of the assignment. The other conceptions involve more representations, and except for the ordered pair, the relation is never counterintuitive. Context plays an important role for those controls based on the content (e.g., by using several representations) and the didactical contract (by establishing the plausibility of a solution). Context is also associated with the use of nonsymbolic representations.

The use of a context in tasks on function raises an important issue. It is helpful to provide a context so that students can relate to their experience the mathematics to be learned. However, the presence of a context determines a particular conception of function, one in which the correspondence is not counterintuitive, symbols are tokens, and the controls not based on the procedures of solution are shaped by the context. The use of a context may act as an obstacle for students in making the transition to a more sophisticated conception of function just as it did for mathematicians when Dirichlet proposed the idea of an arbitrary assignment (see p. 16). The obstacle might arise not only for students but also for teachers (who might prefer to use a graph of a relation to determine whether it is a function; Norman, 1992) and for prospective teachers (who may not believe that the correspondence should be arbitrary or that the domain and range may not be sets of numbers).

The symbolic-rule conception was present in the most textbooks (71%,) and the physical-phenomena and controlling-image conceptions in the fewest. This difference could be due to textbooks authors' interest in including topics such as probability and statistics, which would require the elimination of other topics. The most likely topics to be eliminated appeared to be those for which the teachers might feel less prepared: the new math topics introduced in the 1960s and those topics that cross subjects (such as physics or biology), both of which would require stronger preparation. The preparation of teachers is an important issue because countries need to satisfy the increasing demand for more prepared teachers as the years of compulsory education increase and the ratio of pupils to teachers decreases.

If the symbolic-rule conception is the most common one that seventh and eighth graders encounter in their textbooks, it is not surprising that they believe that the correspondence defining a function should be systematic; that the function must have an algebraic expression, formula or equation; and that a function is a manipulation carried out on the independent variable in order to obtain the dependent variable (Vinner, 1983). Nor it is surprising that experienced teachers think that functional situations involve only numerical variables and have difficulty envisioning physical situations that entail functional relationships (Norman, 1992), or that prospective teachers view functions mainly as equations and tend to use the idea of a machine or black box to illustrate a transformation process (Even, 1989). Rather than showing a defect in students', teachers', and prospective teachers' understanding of functions, these results suggest why those views arise in the first place. It is not a problem of inadequate definition of function—the participants in these other studies were taught Dirichlet's definition of function—but rather an indication of how that definition had been made operational for them. Also if all these conceptions coexist, it is likely that they will emerge when the appropriate situation enacts them, which may account for the apparently contradictory and compartmentalized images and definitions that students, teachers, and prospective teachers exhibit in research studies. Of course, the symbolic-rule conception, then, may act as an obstacle to a conception that would admit arbitrary assignments or arbitrary sets (ones that are not necessarily numerical), multiple representations, or controls not based on procedures only. It is apparent that at these grades such properties of function are not relevant.

Limited as they are, these conceptions may play an important role in making function accessible to students. They may help students construct a more flexible conception of function. As Sierpinska (1992) notes, the absence of obstacles implies that learning does not occur. One problem with the ordered-pair conception may be that because it is so transparent, comprehensive, and general, the student may not distinguish which features of a function are relevant to know and when those features can be used. When the correspondence between the argument of a function and its value is counterintuitive, the student might not be able to interpret the relation. At least with the other conceptions the student knows what to do and what to expect. Because these conceptions are more constrained, they may offer the student a more secure ground for learning. In particular with a rule to use or a context in which operate, the student can explore what a function is in a variety of ways.

Patterns of Conceptions Within and Across Countries

I established four clusters of textbooks that captured the patterns to be seen within and across countries: rule oriented, abstract oriented, abstract oriented with applications, and applications oriented. In the rule-oriented textbooks, more than 50% of the tasks in each textbook enacted a symbolic-rule conception of function or at least used function as a rule. In the abstract-oriented cluster, more than 78% of the tasks within each textbook elicited an ordered-pair or a symbolic-rule conception of function. The abstract-oriented-with-applications cluster contained about half of the textbooks, and beside the symbolic-rule conception, their tasks elicited at least one of the contextual conceptions and in some cases an ordered-pair conception. The textbooks in the applications-oriented cluster did not have tasks that elicited the symbolic-rule or ordered-pair conception. It appears that textbook authors think it desirable for students to be exposed to tasks that elicit different conceptions of function. On the one hand, some or all of these conceptions may present obstacles to the development of a more flexible conception on the part of the students. On the other hand, multiple views—even conflicting ones—should be welcome if they can be used as springboards toward more flexibility.

The purpose of the research question on patterns was to disclose curricular influences across countries. One conjecture was that textbooks from former colonies of Spain might be similar in their approaches to function to those of the textbook from Spain and similarly that the approaches in textbooks from former colonies of England might be similar to those from England. Neither of these conjectures was supported by the data. The four clusters in which the textbooks were organized contained textbooks from different regions; moreover, in several cases textbooks from the same country were in separate clusters. This finding indicates that when it comes to functions, there may be no such thing as a canonical curriculum in school mathematics. It seems to be false—and this result is also supported by the TIMSS curriculum analysis—that mathematical content is expressed in the same way across the globe. The TIMSS analysis found that the only commonly intended and emphasized topic in the eighth-grade textbooks from 42 countries was that of equations-related algebra (Schmidt et al., 1997, p. 115). The emphases were different across countries, with Spain devoting more than 35% of the textbook blocks to the topic, and South Africa devoting about 1% of the textbook blocks (p. 117, Figure 7.3). I found the same sort of differences at a micro level, when analyzing a particular topic. The absence of a canonical curriculum raises the obvious question of the pertinence of international comparisons based on achievement testing. It has become increasingly clear that any comparison of achievement destines many countries to a failure. It seems impossible to build a test that will reveal what students know about mathematics while accounting for curriculum differences in a rational way (Keitel, 2000; Keitel & Kilpatrick, 1998). If students in one country are exposed to certain set of conceptions, and students of another country to another set, how can a test be built to show what both groups of students know, and not what both do not know, which is what is actually happening?

The failure to find a clear-cut pattern organizing the textbooks of these countries leads to the question of why this lack of pattern occurs. In the countries in which there is a national curriculum but a decentralized decision about textbook use (e.g., Colombia), it is possible to produce textbooks that are essentially different. Tracking may be one reason that textbooks within a country are different (e.g., Switzerland), but it may be that other undisclosed factors are operating (textbooks from the United States do not all belong to the same cluster, even when they are in the same series, and there are no tracking practices in Colombia). Thus, the issue seems to be more than just the similarity between the textbooks; other factors are determining what is in the textbooks and in turn what students can learn from them. The issue of within-country variability is certainly one that deserves attention, especially for the United States, which may be considered as a collection of 50 countries (as an aside, the number of textbooks and documents that the US provided for the TIMSS analysis is comparable to the number of documents provided by the other 47 countries). A more detailed analysis, either in the United States or in other countries in which the variability is present might help identify those factors influencing variability by specifying different expressions of those factors in different countries.

Conceptions and Performance

I faced many difficulties in addressing the nature of the relation between conceptions suggested in textbooks and students' achievement in the TIMSS test. In the first place, the items that I selected from the test, as a group, enacted a set of conceptions shared by few tasks from these textbooks. The figural and physical uses were overrepresented, and the rule and social uses were underrepresented. The use of function as a set of ordered pairs was not present on the test. No item used only the symbolic representation; in fact, only two items used it in combination with other representations. The distribution of operations and controls, in contrast, resembled that of the tasks in the textbooks. Thus, the test items did not seem to be testing the same conceptions of function that were enacted by the tasks in the textbooks from the participating countries. In the second place, for countries with more than one textbook, it was not possible to trace which students used which book, and for the countries in which only one textbook was used—Austria, Hong Kong, Singapore, and Spain—there was no indication that potential exposure to textbook tasks enacting the same conceptions as those enacted by the items led to better performance. There were items in which the students from these countries outperformed, performed about the same as, or performed worse than the students from the other countries. In the third place, the students' exposure to any particular conception was always potential. There was no guarantee that the students, even in countries having only one textbook, would work all the tasks or that the conception enacted would be exactly that promoted by the tasks. Teachers, peers, the knowledge at stake, and even the culture of school and society play a role in shaping those conceptions.

Any subsequent study that would carefully consider the advantages of being exposed to one conception or another should begin by establishing the conditions in which each conception is made available to students so as to determine whether or not the situations are comparable. Simply borrowing a textbook used by students in countries labeled "high achieving" in TIMSS is disrespectful both to the donor system and to the recipient system. Such a textbook "implant" fails to take into consideration the particular histories, traditions, societal, cultural, or other powerful reasons that affect each country's educational system. For example, after the U.S. Secretary of Education, Richard Riley, visited Singapore, whose seventh and eighth graders did very well on the TIMSS achievement test, newspaper accounts indicated that some schools in several U.S. states had adopted Singapore mathematics textbooks (Dizon, 2000; Quek, 2000). Students in the country of Brunei follow a mathematics curriculum almost identical to that of Singapore, and many Bruneian secondary students tend to use mathematics textbooks written and published in Singapore. Because these same students do not perform nearly as well as Singaporean students on the O-level and A-level examinations set by the Cambridge Examination Board, it seems clear that textbooks alone do not account for the superior performance of Singaporean students (M. A. Clements, personal communication, May 4, 2000).

There are no simple remedies for the difficulties many students have on tests of mathematics achievement, but researchers may not have been asking the right questions either. Students from some countries perform consistently better on international tests, and students from other countries perform consistently worse. It has been suggested that the potential exposure to textbooks with certain characteristics may have benefits for students' achievement from particular countries (e.g., for Asian countries, see Li, 2000). Is that the case for all the countries whose textbooks share those certain characteristics? In the present study, I found that it is not the case, at least for the specific topic of functions. It has been suggested, instead, that extra-curricular practices may have a greater impact on student achievement on tests than the textbook does. Besides private lessons taken by Singaporean students, their school system is very selective because it "exports low-achievers and only imports high-achievers" (Keitel, 2000, p. 18). In Japan about 70% of the lower secondary students attend the Juku, an after-school program where “besides working on math drills and other exercises, students [learn] to solve problems and take practice exams when admission tests approach” (Japan Information Network, 1998). This activity may have a strong influence on student performance.

I hinted in this report that conclusions about achievement based on single curricular elements are problematic. The data that are available from international studies allow one to build cases supporting contradictory results. The claim that the U.S. textbooks are substantially different from the textbooks from other countries seems to be based on global characteristics of the textbooks rather than on how the textbooks introduce and develop particular topics. The tasks in the U.S. textbooks in this sample promoted conceptions of function similar to those of the tasks in 11 textbooks from other countries (see Table 11, p. 95), and no clear pattern of achievement can be derived from them.

Other Results

I found that relatively few situations were used in these textbooks to deal with functions in the physical world; that textbooks may be only work booklets or may present extended explanations after the problem sections; and that most textbooks use pictures or photographs to enhance their presentation. These results suggest that there is much variability across textbooks with respect to *how* the information is presented, but that such variation is very limited when attending to *what* knowledge is presented. Analyses of superficial aspects of textbooks have suggested that there are important differences across countries. Analyses of what the particular content is eliciting, however, show that there are no big differences, which leads me to conjecture that an analysis of the titles of the sections of chapters devoted to a particular topic (Li, 1999) can also be misleading. It is only through a microscopic analysis that actual differences or similarities can be uncovered. There is a great need to move beyond the superficial analysis of textbooks toward more content-centered analysis.

Methodological Observations

The framework for identifying conceptions present in textbooks that I developed for the present study revealed that some textbooks had tasks that enact a variety of conceptions of function that might be simultaneously available for students. The process of developing the coding system and coding the tasks put me in contact with the many different ways of presenting mathematical ideas to students, ideas that were captured by the framework because I solved the tasks after a conscientious reading of the textbooks.

In this study, I did not define in advance the categories of the constructs used to analyze the textbooks; rather, I built their meaning from what the textbooks contained. By building up the categories and their meaning from the textbook content, I could reveal particular features that might otherwise be overlooked when imposing a predefined set of categories. Difficulties arise when the textbooks are booklets or workbooks. In these cases, the authors' intentions have to be extrapolated from the tasks or from accompanying documents. Fortunately, the TIMSS database has supporting documents that allows one to trace those intentions.

It is useful to have the authors' intentions documented, but when faced with solving a task on functions, it is one's own conception of function that takes precedence and that is reflected in the solution of the task (N. Balacheff, personal communication, December 12, 1999). To overcome this difficulty, it is critical to produce as many alternative solutions as possible, and then with the textbook in hand, estimate the plausibility of those approaches. Textbooks eventually privilege certain approaches over others (as do teachers, and eventually students).

I aimed at category exhaustion, and for that reason I also looked at textbooks for grades 9, 10, and 12. It turned out that some operations—not many—in textbooks for these grades were not present in the seventh- and eighth-grade textbooks. I suspect that had I begun with a larger sample of textbooks from the higher grades, I might well have found a somewhat different set of categories. Thus, the framework developed for the present study is based on what the textbooks in the sample offered the students at Grades

7 and 8. An application of the framework to a different group of textbooks would likely yield a different set of uses, operations, representations, and controls.

The framework is very powerful because, as I defined it, a conception requires a particular configuration of four elements. It may have been too powerful for the data available in the present study. Such a specialized tool makes it difficult to detect patterns when the sample is small (DuMouchel, 1999, von Eye, 2000). In other words, the quadruplet may have restricted the possibility of finding patterns. It might prove interesting to study the patterns that occur with pairs or triples of the elements. Thus, for example, an analysis of the combinations of uses and representations might illustrate the extent to which uses of function as rules require other types of representations, and an analysis of operations versus controls might illustrate the groups of operations associated with a particular type of control.

The formulation and shaping of the categories of controls was an especially difficult step in developing the framework. The reason is that textbooks rarely provide clues as to why topics are treated the way they are treated, or why one procedure is more effective than another. Very few textbooks provide explicit indications for the students to control their activities (e.g., U. S. textbooks) or problems that are solved in more than one way (e.g., Mexico1), or recommendations as to what the answers should look like (e.g. the English textbooks). Thus the metacognitive strategies associated with controls are difficult to discern from the tasks in textbooks. It may be that these strategies are typically left for the teacher to illustrate.

An issue related to the coding process has to do with the selection of the task as a unit of analysis. By making that choice, I was able to perceive the specific characteristics of the conception for a task. I could not, however, account for what a given group of tasks together might accomplish. An analysis of task sequences would require an approach similar, for example, to that used by Stigler et al. (1986), who counted the changes in complexity across groups of ten consecutive arithmetic problems in U.S. and Soviet textbooks. It might have been that by coding task by task, I lost sight of phenomena such as controls that depend on how the tasks were linked. Certainly the flow of the set of problems was lost by the approach I took.

Regarding the reliability of the final coding system I developed, because my principal goal was to make Balacheff's framework operational, I aimed at getting the broadest possible coverage; I wanted to obtain a comprehensive framework. When I decided to explain the coding procedure to other people to see whether they could reliably use it, I began to realize how important it was to be explicit about the meaning of the categories. I wanted the framework to be usable by other people, people who might not even be geographically close. I found that the coding was feasible and at the same time that repeated applications and uses by more people would clearly result in an improved framework, one that would account not only for the most salient features of the elements but also for the most important features. It would be interesting to test the applicability of the framework to a different set of textbooks. What new uses might appear? What new operations, representations, and controls might there be? Does this framework really give a comprehensive picture of what is asked about functions in seventh- and eighth-grade mathematics textbooks around the world?

I used both qualitative and quantitative methods to organize and analyze the data collected. Qualitative methods helped in the development of the categories for the coding system (through constant comparison) and quantitative methods allowed me to test that the patterns I observed were not due merely to chance (with the Configural Analysis Program). This work illustrates a successful combination of two apparently different orientations in research methods that helped to explore and analyze textbook mathematical content and that may eventually help to predict their impact on classroom processes. The work also illustrates the potential of working with large data sets and the feasibility of such enterprise.

The categories developed needed to be reorganized in order to facilitate the analysis. Because of the nature of the coding system, the regrouping made in this particular study highlighted some aspects of the data but at the same time hid others. It is important, though, that the coding was already done so that other reorganizations and analyses (e.g., the different operations) could be carried out that would show other interesting patterns in these data.

Implications of the Study

By conducting this study I wanted to explore the possibility of simultaneously analyzing textbooks both as "environments for construction of knowledge" and as pieces of technology inside educational systems (Herbst, 1995, p. 3). One product of this study, the coding system, together with the analysis of the configurations (of conceptions of function) present in textbook tasks, aimed at performing the first type of analysis. The comparison across different educational systems outlined possible paths for the second type of analysis. In the following sections I elaborate on several alternatives that could make use of similar analyses, considering the implications of the study for research, for international comparisons, for textbook authoring, for curriculum development, and for teaching.

Implications for Research

The framework developed in the present study provides a powerful tool that allows researchers to study several aspects related to the teaching and learning of functions in particular and mathematics in general. As shown in the study, the framework contributes to the study of the relationship between the four components that define a conception. This characteristic of the definition, which constitutes its strength, requires a large number of cases to obtain statistically significant results. In this study, I addressed the issue by pooling the tasks from all the textbooks. One possibility would have been to drop one of the components. Being aware of the need to consider the four components simultaneously, however, is an important requirement in developing a better understanding of how the interrelationships of these aspects are manifested.

Quantitative empirical research in mathematics education is less common nowadays than in the past, in part because of the technical difficulties involved in collecting, organizing, and analyzing the complex information inherent in the phenomena analyzed. In their attempts to simplify the enterprise, researchers tend to sacrifice in-depth analyses and look for superficial characteristics. The development of more powerful tools—faster and larger computers, advancements in software development for data organization, and techniques associated with data mining (e.g, CFA)—offer the possibility to deal with complex interrelations, which would in turn help in the building of stronger theoretical frameworks to nurture the development of mathematics education as a scientific discipline. The assumption that phenomena associated with teaching and learning mathematics are such that they cannot stand rigorous statistical analyses or that they are not suitable for prediction might be acting as a vicious circle: Because we believe that we should not analyze the phenomena with these techniques, then we do not use them or use them superficially; consequently, we do not test their potential, or we get superficial analyses that assure us that the approaches should not have been used in the first place. That vicious circle needs to be broken.

Follow-up research derived from this study might take any of several paths. Much discussion and investigation are needed of how these conceptions interrelate, in other words, how students live with them, and how the conceptions evolve towards more flexibility. Immediate questions are how to trace the presence of these conceptions. What should the problems look like so that important aspects of function are at stake? What combinations of operations, representations, and controls should be available to the students, so that they can effectively put those aspects into action? In other words, it is the problem of how to pose a question for which *function* is the solution, a problem envisioned by Balacheff (in press).

Because little information is available as to how teachers use textbooks in their teaching, and in particular for teaching functions, the framework can provide a tool researchers could use to study how teachers treat textbook content as suggested by

Stodolsky (1989). For example, when teachers go over homework based on textbook exercises, do they propose different uses of functions? Do they use multiple representations? Do they make explicit the operations involved? Do they model metacognitive activities for the students, illustrating how to legitimate the solution process or verify that a solution is correct? And how are these elements combined? There may be interesting contrasts between what is in the textbooks and teachers' actions and decisions with respect to how exercises are solved.

The *Principles and Standards of School Mathematics* (NCTM, 2000) offers new alternatives to the teaching of key areas of school mathematics, whose implications in terms of students' conceptions have yet to be explored. One example is, How is the transition made in the document from contextual conceptions of function towards abstract ones? What obstacles are associated with particular conceptions of function and what acts of understanding (Sierpinska, 1992) are offered to overcome them? Answers to these questions would give us an honest perspective on the significance of these approaches that would ease the transitions to projects based on these documents.

The enterprise of refining and enlarging the framework by analyzing a different set of textbooks will tell what conceptions are potentially available, shaping our knowledge about functions. A possible study would code textbooks from different time periods. Changes in the distribution of the categories could provide valuable information about the evolution of the potential conceptions in school mathematics. A further investigation that analyzed the data for the upper grades might reveal new conceptions that appear or old ones that disappear as students advance in school mathematics. Such studies would help in building a more robust framework that at the same time would enlarge our pedagogical subject matter knowledge (Shulman, 1986) about functions.

The method that I followed could be used to detect conceptions of other key concepts of school mathematics (e.g., multiplication or area in the elementary grades, or probability in secondary school mathematics) promoted in textbooks as a way to exemplify the domains of validity of those concepts. These conceptions could be traced a students advance through the school years. The method could also be used to study how exposure to functional thinking in earlier grades could provide a foundation for students' understanding of functions when formal definitions are presented.

It would be interesting to study whether teachers tend to privilege a particular conception of function and the reasons for that preference. It could be that textbooks they have used as students have played an important role, but also it could be that their own experience as teachers has determined their selection of a particular conception. An interesting avenue would be to explore whether teachers are aware of all these different, apparently competing, conceptions, or whether there are others that emerge when they solve particular sets of problems. Such analyses would have an impact on the ways in which functions are taught in methods courses for teachers because their results would help teachers evaluate their content knowledge about functions.

This study also illustrates that it is possible to conduct external and internal critiques of textbooks (Herbst, 1995). My analysis looked at functions as presented in textbooks—an internal analysis—but it also aimed at explaining the reasons that some of the differences existed—an external analysis—as textbooks play an important role in the schooling system. This study, however, showed that characteristics of textbooks alone do not account for the differences observed, which implies that a future study should consider the collection of other complementary information about issues such as textbook production in each particular system, authors' statements about the textbooks, or actual textbook use by students and teachers.

Finally, since the time in which the TIMSS curriculum materials were collected a number of new projects (some of them influenced by the NCTM *Standards*, others relying on technology) have appeared, not only in the United States, but in other countries as well. I conjecture that a similar analysis with more recent textbooks that have organized the middle-school mathematics curriculum around functions (e.g., Demarois, McGowen, & Whitkanack, 1997; Gómez et al., 1996; Lappan, Fey, Fitzgerald, Friel, &

Phillips, 1998; Yerushalmi, 1997) would show a different set of potential conceptions as a result of the evolution in how problems abut functions are posed in these projects.

Implications for International Comparisons

Information on how other countries' textbooks treat functions is valuable for understanding the complexity of ways of approaching functions used in the United States. For example, in other countries, physical situations are more often emphasized; an inquiry into the reasons for promoting that emphasis might prove useful for understand why we use the tasks and approaches we do.

Keitel (2000) warns us about the perils of *globalization* as it

> corresponds with ideological shifts to neo-liberalism, economic rationalism, where educators are replaced by business people, and education becomes object of a deregulated training market where it is just treated as tradable good with an exchange value. (p. 3)

If globalization of economies is an imperative, then it is mandatory for the United States to set an example by being knowledgeable not only about its competitors' advancements but also about those of other countries that contribute to the world economy. That might help to provoke a shift towards *internationalization*, which "keeps its invitational characteristics and allows to maintain the autonomy of all partners." As diversity increases, understanding other people's cultures is a necessary condition for avoiding judgments based on decontextualized aspects of school mathematics (e.g., test achievement or a curriculum "that is a mile wide and an inch deep").

Many lessons can be drawn by looking at countries that have different conditions from those in the United States or its competitors because such analyses highlight hidden assumptions that may be taken for granted by the dominant countries (e.g., that after-school time is spent in the same ways by all school children). As I was carrying out the test of the coding system, I took the opportunity to conduct the test with researchers with different backgrounds and learned how such an interchange could contribute to disclosing their own assumptions about school mathematics. Although language can be a stronger barrier than distance, it is up to us as community committed to learning to overcome language barriers.

Implications for Curriculum Development

An informed decision about which conceptions should be promoted must guide the design of any curriculum project that uses functions as an organizing concept. As new projects emerge, it is fundamental to establish their potential in terms of what can be learned. The framework that I developed helps in establishing those potential conceptions a priori, so that developers might foresee and predict difficulties that they could address before making a big investment in textbook production and during the process of dissemination of their textbooks.

Implications for Textbook Authoring

As several other studies have demonstrated, there is more to learning a mathematical concept than simply remembering the “bare bones” definition of the concept. How the definition is made operational as reflected in the exercises that students solve can shape students’ conceptions of function in significant ways. Thus textbook authors should be aware not only of how the elements of a conception play out in their exposition but also of how they are enacted in the exercises proposed. The scarcity of controls available to the students is probably one of the most pressing problems to address. Questions such as the following should be presented, illustrated, and reinforced in textbooks: Did I solve the problem that I had at the beginning? Could I think of a different way to solve this problem? How can I be sure that there are not other solutions or answers? Although teachers might present these questions themselves, if textbooks are to be used as a reference, they should contain them too. Students who for various reasons cannot attend a regular lesson should have also the opportunity to encounter these questions.

Implications for Teaching

Teachers could benefit from adopting a framework such as the one developed in this study to analyze textbook content. Nowadays teachers have a proactive role in the decisions concerning textbook use, and in consequence a tool that would help them to disclose what different textbooks are offering to them and to their students in terms of conceptions would be very useful. The use of the framework would allow them to make better decisions regarding textbook adoption.

In the classroom, the framework would be helpful before, during, and after teachers teach a lesson or group of lessons on functions. The framework may help teachers design tasks that would elicit particular conceptions about functions in their students. Also, knowing that some conceptions are inevitable and desirable obstacles for promoting learning about functions might help them to organize their teaching sequences accordingly.

While teaching functions, teachers might use the framework to probe and guide students' thinking about functions. It would be possible to, for example, ask whether more operations or representations could be used and why, or whether the answers obtained are legitimate or not.

Particular problems could be used to assess students' conceptions of function (Have they changed? Are there new conceptions?) and to evaluate teacher's organization of the activities they use to teach functions.

Coda

Balacheff's (in press) and Biehler's (in press) theoretical developments have been proposed for studying people's cognitive activities, in particular, the meaning that students and teachers give to mathematics concepts in relation to the practice of teaching and learning mathematics. I borrowed these developments to study an apparently static element of the curriculum, the textbook. Even though this application might seem inappropriate, as it is difficult to envision the actual cognitive activities that a textbook by itself could promote, the process of making those definitions operational illustrates their potential for systematic application. I believe that soon mathematics educators will begin systematically corroborating the many theoretical and not-so-theoretical developments that the discipline is producing today. That corroboration will build and sustain our knowledge about the complex phenomenon of teaching and learning mathematics.

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APPENDIX A

TEXTBOOKS USED IN THE STUDY

Below is a list of the 35 textbooks that were used in the study classified by country and the grade for which they were intended. When more than one textbook for a country was used, they are listed alphabetically by author.

Country	Grade	Textbook
Argentina1	8	Sadovsky, P., Melguizo, M. P., & Waldman, C. (1989). *Matemáticas 2*. Buenos Aires: Santillana.
Australia1	8	Lynch, B. J., Parr, R. E., & Keating, H. M. (1980/1988). *Maths 8*. Melbourne: Longman Cheshire.
Australia2	9	Lynch, B. J., Parr, R. E., & Keating, H. M. (1979/1991). *Maths 9*. Melbourne: Longman Cheshire.
Australia3	7	Lynch, B. J., Parr, R. E., & Keating, H. M. (1981/1991). *Maths 7*. Melbourne: Longman Cheshire.
Austria1	8	Albrecht, R., Gutschi, H. P., Langgner, D., & Wiltsche, H. (1991). *Lebendige Mathematik, Band 4*. Vienna: Hoelder-Pichler-Tempsky.
Canada1	8	Connely, R. D., Lesage, J., Martin, J. D., O'Shea, T., Charp, J. N. C., Beattie, R. H., Bilous, F., Bober, W. C., Drost, D. R., Hope, J. A., Lee, R., & Tossell, S. (1988). *Journeys in Math 8*. Scarborough, Ontario: Ginn Publishing Canada.
Colombia1	8	Londoño, N., Guarín, H., & Bedoya, H. (1993). *Dimensión matemática 8*. Bogotá: Norma.
Colombia2	8	Villegas, M. (1991). *Matemática 2000*. Bogota: Editorial Voluntad.
England1	7/8	School Mathematics Project. (1984/1991). *Speed 1: Graphs*. Cambridge: Cambridge University Press.
England2	7/8	School Mathematics Project. (1984/1991). *Graphs 2: Graphs*. Cambridge: Cambridge University Press.
Hong Kong1	8	Chan, L. K. F., Leung, C. T., & Wise, S. R. (1988/1992). *Mathematics for Hong Kong*. Hong Kong: Canotta.
Ireland1	8	Morris, O. D. (1987/1992). *Text & Tests, 1*. Dublin: Celtic Press.
Ireland2	9	Morris, O. D. (1988/1991). *Text & Tests, 2*. Dublin: Celtic Press.
Mexico1	8	Alarcón, J., Lucio, M. G., Parra, B. M., Rivaud, J. J., Waldegg, G., & Rojo, A. (1991/1994). *Matemáticas 2*. Mexico, DF: Fondo de Cultura Económica.
New Zealand1	12	Barton, D., Johnson, W., & Laird, S. (1989/1992). *Delta Mathematics*. Auckland: Longman Paul.
New Zealand2	12	Barton, D. (1986/1991). *Sigma Mathematics*. Auckland: Longman Paul.
Portugal1	8	Ferreira-Neves, M. A., & Carvalho-Brito, M. L. (1992). *Matematica 8*. Lisbon: Porto Editora.

Country	Grade	Textbook
Scotland1	11	Howat, R. D., Mullan, E. C. K., Nisbet, K., & Brown, D. (1989/1992). *Mathematics in action 5S*. Glasgow: Thomas Nelson and Sons.
Scotland2	10	Howat, R. D., Mullan, E. C. K., Nisbet, K., & Brown, D. (1988/1990). *Mathematics in action 4A*. Musselburgh: Blackie-Chambers.
Singapore1	8	Seng, T., Keong, L. Yee, L. (1987). *New Syllabus: Mathematics 2*. Singapore: Shing Lee.
South Africa1	10	Laridon, P. E. J. M., Burgess, A. G., Jawurek, A. M., Kitto, A. L., Pike, M. J., Rhodes-Houghton, R. H., & Rooyen, R. P. v. (1988/1993). *Classroom Mathematics Standard 8*. Isando: Lexicon.
South Africa2	7	Laridon, P. E. J. M., Brown, M., Jawurek, A., Kitto, A., Stafford, H., Strauss, J., Strimling, L., & Wilson, H. (1990/1991). *Classroom Mathematics Standard 5*. Isando: Lexicon.
South Africa3	9	Laridon, P. E. J. M., Burgess, A. G., Kitto, A., Pike, M. R., Strauss, J., Strimling, L., & Wilson, H. (1986/1993). *Classroom Mathematics Standard 7*. Isando: Lexicon.
Spain1	8	Gil, J., García, P., Vázquez, C., & Mascaró, J. (1984). *Matemáticas 8*. Madrid: Santillana.
Switzerland1	9	Hohl, W., Egli, B., Möckli, H., & Rick, H. R. (1987/1992). *Arithmetik und Algebra 3*. Zürich: Lehrmittelverlag des Kanton Zürich.
Switzerland2	7	Holzherr, E., & Ineichen, R. (1972/1986). *Arithmetik und Algebra 1*. Zürich: Sabe.
Switzerland3	9	Holzherr, E., & Ineichen, R. (1973/1988). *Arithmetik und Algebra 3*. (Vol. sekundarschulen and polygimnasien). Zürich: Sabe.
Switzerland4	7	Deller, H., Gebauer, P., & Zinn, J. (1992). *Algebra 1*. Zürich: Orel Füssli.
United States1	7	Bolster, L., boyer, C., Hamada, R., Leiva, M., Linduist, M. M., Robitaille, D., Swafford, J., van de Walle, J. (1991). *Exploring Mathematics Grade 7*. Glenview, IL: Scott Foresman.
United States2	8	Bolster, L., boyer, C., Hamada, R., Leiva, M., Linduist, M. M., Robitaille, D., Swafford, J., van de Walle, J. (1991). *Exploring Mathematics Grade 8*. Glenview, IL: Scott Foresman.
United States3	7	Fennell, F., Ferrini-Mundy, J., Ginsburg, H, Murphy, S., Tate, W. Cavanagh, M., Altieri, M. B., Sammons, K., Long, D., Sherman, C., & Vogeli, B. (1992). *Mathematics 7*. Morristown, NJ: Silver, Burdett & Ginn.
United States4	8	Fennell, F., Ferrini-Mundy, J., Ginsburg, H, Murphy, S., Tate, W. Cavanagh, M., Altieri, M. B., Sammons, K., Long, D., Sherman, C., & Vogeli, B. (1992). *Mathematics 7*. Morristown, NJ: Silver, Burdett & Ginn.
United States5	7	Eicholz, R., O'daffer, P., Fleenor,C., Charles, R., Young, S. & Bernett, C. (1993). *Mathematics grade 7*. Reading, MA: Addison-Wesley.
United States6	8	Eicholz, R., O'daffer, P., Fleenor,C., Charles, R., Young, S. & Bernett, C. (1993). *Mathematics grade 8*. Reading, MA: Addison-Wesley.
United States7	9	Brown, R. G., Dolciani, M. P., Sogenfrey, R. H., & Cole, W. L. (1990/1994). *Algebra structure and method, Book 1*. Boston: Houghton Mifflin.

APPENDIX B

ORIGINAL TEXTS OF SELECTED TASKS

This appendix contains the original texts of tasks cited in the report that I translated into English and the complete text of a task from Australia cited in the report. They are ordered by country, within a country by textbook, and within a textbook by the page from which the task was taken.

Australia1, pp. 255-256, Task 3:

> Each of the following set of points represent a linear pattern in the Cartesian plane. By plotting each set of points and using a rule, find the coordinates of the first two points in the pattern.

a. {(-3, -8), (-2, -4), (-1, -2), (0, 0), (1, 2)}
b. {(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, -1)}
c. {(-3, 9), (-2, 6), (-1, 3), (0, 0), (1, -3)}
d. {(-3, -15), (-2, -10), (-1, -5), (0, 0), (1, 5)}
e. {(-3, -2), (-2, -1), (-1, 0), (0, 1), (1, 2)}
f. {(-3, -5), (-2, -4), (-1, -3), (0, -2), (1, -1)}
g. {(-3, -6), (-2, -5), (-1, -4), (0, -3), (1, -2)}
h. {(-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3)}
i. {(-3,4), (-2, 3), (-1, 2), (0, 1), (1, 0)}
j. {(-3, 0), (-2, -1), (-1, -2), (0, -3), (1, -4)}
k. {(-3, -7), (-2, -5), (-1, -3), (0, -1), (1, 1)}
l. {(-3, -7), (-2, -4), (-1, -1), (0, 2), (1, 5)}
m. {(-3, 9), (-2, 7), (-1, 5), (0, 3), (1, 1)}
n. {(-3, 8), (-2, 5), (-1, 2), (0, -1), (1, -4)}

Austria1, p. 189, Task 999

a. Vervollständige die Wertetabellte (Zuordnungstabelle).

Geschwindigkeit *x* in km/h	60	30	20	15	12	10	5
Fahrzeit *y* in Stunden für dieselbe Strecke	1						

b. Stelle die Zuordnung in Fig. 169 graphisch dar!

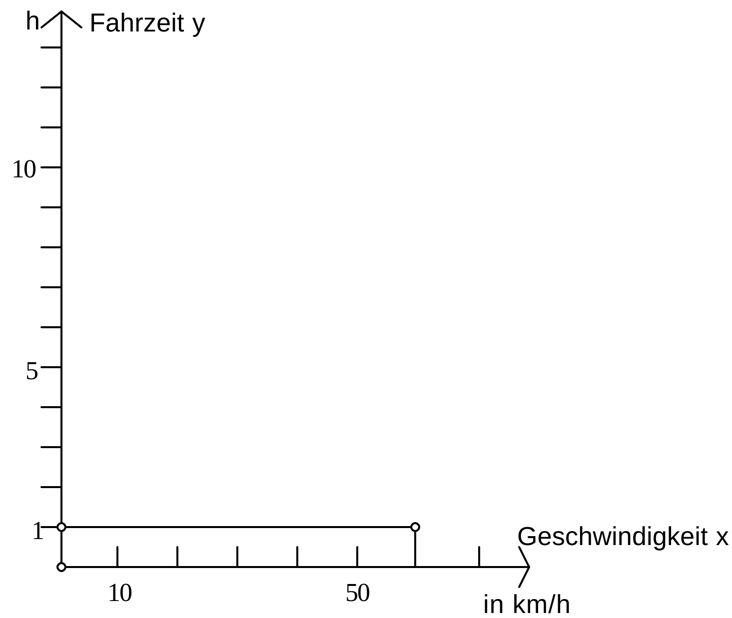


Fig. 169

Colombia1, p. 140, Tasks 5-9.

En los ejercicios 5 a 9, representar, en el plano cartesiano, la relación cuyo conjunto solución es el indicado:

5. $R = \{(x, y) \mid x, y > 0 \wedge x, y \in \mathbf{R}\}$

6. $Q = \{(x, y) \mid y = -x, \wedge x \in \mathbf{Z}\}$

7. $S = \{(x, y) \mid y = x \wedge x \in \mathbf{N}\}$

8. $T = \{(x, 0) \mid x \in \mathbf{R}\}$

9. $H = \{(0, y) \mid y \in \mathbf{R}\}$

Colombia2, p. 248, Task 4

Representar gráficamente la siguiente función:

En cierta ciudad el precio de la carrera de un taxi se cobra de acuerdo con la siguiente tarifa:

- Banderazo: $150.00
- Costo de recorrido: $5.00 por cada 100 m.
- No se tiene en cuenta el tiempo de espera.

Mexico1, p. 195, Task 3: r

¿Hay proporcionalidad entre la longitud de la arista de un cubo y:

a. la suma de las longitudes de las aristas?
b. el area total de la superficie del cubo?
c. el volumen del cubo?

Mexico1, p. 197, Task 5

La siguiente tabla muestra la distancia de frenado de un vehículo en suelo seco en función de la velocidad; por ejemplo, un vehículo que va a 40 kilómetros por hora necesita 18.6 metros para frenar. ¿Hay proporcionalidad entre la velocidad y la distancia de frenado?

Velocidad (en km/h)	Distancia de frenado (en m)
40	18.6
50	26.5
60	35.7
70	46
80	57.5
90	70.7
110	101
130	135.6

Mexico1, p. 219, Task 1

Dibuja las gráficas de las siguientes funciones en un mismo sistema de ejes coordenados.

$y = -3x + 2$; $y = 2x + 2$; $y = -x + 2$; $y = 2$; $y = 2x + 2$; $y = 3x + 2$

Portugal1, p. 67, Task 4

Considere a funçao h assim definida
$h(x) = 2x + 1$

1. Calcule: $h(-1)$, $h(0)$ e $h(1)$

2. Determine x de modo que $h(x) = 11$.

Spain1, p. 99, Task 6

Se tienen 8 litros de un gas a la presión de 1 atmosfera. La temperatura se mantiene constante y se sabe que en estas condiciones se verifica:

$$\text{Presión} \times \text{Volumen} = \text{Constante}$$

P x **V** = Constante

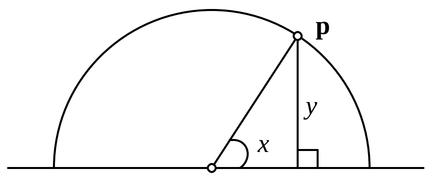
Completa la tabla:

P	1	2	4	8
V	8			

Switzerland2, p. 131,Task 18B

Winkelhöhe

Zeichne einen Halbkreis mit den Radius 10 cm. Zeichne verschiedene Winkel x ($0 \le x \le 180^o$) mit dem Scheitel im Kreismittelpunkt und einem Schenkel als waagrechten Strahl nach rechts.



Bestimme für jeden Winkel x die zugehörige Höhe y. Zeichne einen möglichst genauenGraph. Diskutiere den Verlauf der Kurve.

Switzerland3, p. 72, Task 23

Zeichne und beschreibe die Graphen mit den folgenden Funktionsgleichungen.

a) $y = \begin{cases} \frac{1}{2}x+1 & x \le 0 \\ 2x+1 & x > 0 \end{cases}$

b) $y = \begin{cases} x+2 & x \le -1 \\ 1 & -1 \le x < 1 \\ \frac{3}{2}x - \frac{1}{2} & 1 \le x \end{cases}$

c) $y = \begin{cases} -x-3 & x < 0 \\ x+3 & x \ge \end{cases}$

d) $y = \begin{cases} |x| - x & x < 0 \\ 1 & x = 0 \\ 2-x & x > 0 \end{cases}$

APPENDIX C

CODING PROCEDURE

This appendix presents the final document used to code the tasks in the sample. It is divided into four sections, one for each element defining a conception. In the case of the prototypical uses of function, the examples span those situations that were found in the sample of textbooks. In the other cases, only few characteristic examples were added to illustrate the application of a code.

General Instructions

Each task is given four types of codes, one for each element defining a conception: a unique code for prototypical uses of function, one or more codes for operations, one or more codes for representations, and one or more codes for controls. Each of these is discussed in the following sections. Assign the codes only after solving each exercise and taking into consideration the content of the chapter in which the exercise was embedded.

P Codes: Prototypical use of function

This code refers to the content that is being addressed by the task. Table C1 presents the available codes with a description and examples of contents that have received those codes.

Table C1

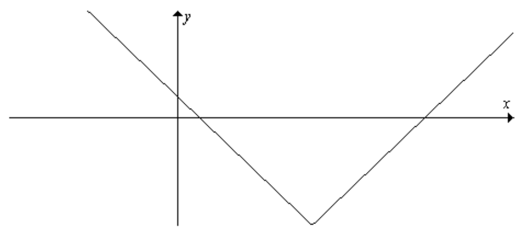
P Codes. Prototypical Uses of Function in the Task

Code	Name and description	Examples
CER	*Cause/effect relationship* Used to code content that refers to physical phenomena other than time related and in which the behavior of one variable is	Atmospheric pressure vs. boiling point Density of water/ice vs. temperature Depth under the surface of the earth vs. temperature (^oC) Electrical resistance vs. length of wire Force on spring vs. number of units compressed

Code	Name and description	Examples
	an effect of the behavior of the other (it is a directional relationship)	Force to stretch a string vs. weight (Hooke's Law) Force vs. elongation of a rubber band Height of ball dropped vs. height of bounce Length of pendulum vs. period Mass vs. volume Ohm's law Pressure (pascals) that a diver supports vs. depth (m) under the sea Resistance vs. intensity (current) Voltage vs. intensity (current) Volume of a gas vs. atmospheric pressure Weight vs. elongation of a spring
CR	*Constructed relationship* Used to code content that refers to "real life" situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables produces equally valid—for the context—relationships.	Amount of cereal vs. number of coupons Amount of gas (liters) vs. distance traveled (km) Amount of heat lost vs. width of window Amount of money vs. number of coins Amount of monthly shopping vs. month Amount of paint vs. area of walls Amount of string in a ball Area of page vs. number of pages in booklets Canadian dollars vs. American dollars Cost of buying balloons vs. whistles Cost of gas and telephone consumption Cost of installing pipelines vs. km Cost of pears vs. weight Cost of printing books vs. number of books Cost of telephone calls vs. minutes Discounts vs. price Distance vs. taxi fare Distances on a Ferris wheel Divide jackpot among buyers Goods bought vs. discount Grades in test vs. standardization Grades vs. number of pupils Growth of British railways vs. year Height (cm) vs. height (in) Height of ski jump vs. horizontal distance Height of water in bottle vs. volume Height vs. weight Imports vs. country Interest for capital vs. number of days Length of fencing vs. length of paddock (m) Liters of water vs. cost Meters of fabric vs. cost Miles vs. decimeters Miles vs. gallon of gas Net value of fishery Number of athletes vs. time to run 100m Number of batteries vs. duration

Code	Name and description	Examples
		Number of books vs. cost Number of boys vs. number of girls Number of bulbs vs. lifetime Number of cards vs. time to produced them (hours) Number of cars sold in Canadian or American dollars Number of catering hours vs. price Number of copies vs. time Number of cuts in a string vs. number of resulting pieces Number of days vs. number of workers needed to finish a job Number of dm^3 of copper vs. mass Number of frames vs. cost Number of hours vs. cost of renting a truck Number of hours vs. fitness club rates Number of lamps vs. connections to circuit breakers Number of leaves vs. length Number of liters of orange juice vs. liters of water Number of machines to do a job vs. number of days Number of nails vs. weight Number of nights vs. price Number of pencils vs. cost Number of people watching a game vs. hours of game Number of pills vs. weight (kg) Number of players vs. weight Number of pulse beats vs. time Number of representatives vs. votes obtained Number of stereos vs. sales Number of tickets vs. profits Number of tickets vs. subscription price Number of videos to rent vs. number of vouchers Number of women working vs. year Percent of meat vs. price Percentage paid vs. amount left to pay Percentages of discounts Postage vs. weight Pounds of grapes per price Price of car vs. age of car (or time owned) Price of car vs. year produced Price of different metals Price per pack vs. number of goods Price vs. weight of packages Quantities of recipe for 4, 6, 8, 12 people Rent vs. number of months Scale in a map Score vs. goal conversion Ship type vs. use Tax deduction over incidental costs Taxi fare vs. km traveled Temperature (^oC) vs. region Temperature above vs. below surface of water

Code	Name and description	Examples
		Temperature Celsius vs. Fahrenheit or vs. Réamur Temperature Celsius vs. temperature Fahrenheit Time vs. cost of parking Type of cars vs. speeds Units of gas vs. cost Units of power per user vs. cost Units of power vs. cost Volume of classroom vs. number of students Volume of water vs. height in a tube Wage vs. hours worked Weight (kg) vs. weight (lb.) Weight vs. age (months) of babies
DPPR	*Direct proportion/proportion relation.* Used to code content where there is an explicit reference to a proportion or a direct proportion without context.	Fill a table in such a way that there is a direct proportion between the entries
DRR	*Data reduction relation* Used to code statistical situations; in situations involving two variables it may be possible to have more than one outcome for a given value of a variable.	Amount of cement vs. sand Assets from a total Change of price of movie vs. year Consumer price index vs. year Cost of parking vs. hour Diameter of number of tree trunks Diameter of tree trunk vs. age Distance (m) vs. height (km) (of mountains) Fuel consumption vs. make of vehicle Grades in two subjects (math & physics) Height of number of people Height of mother and father vs. height of girl and boy Height vs. age of a tree Interest rates vs. year Number of acceleration units of a train model vs. number of coaches in the train Number of cards produced vs. hour Number of children vs. family Number of children stacking chairs vs. time in minutes Number of computers vs. price Number of employees vs. salaries Number of inhabitants per year in a town (or population vs. year) Number of papers delivered vs. time Number of planes leaving vs. day of the week Number of hours of sun vs. day Number of tractors vs. country Number of turkeys vs. country Number of turns vs. distance traveled vs. a wheel Number of units vs. price

Code	Name and description	Examples
		Parent’s vs. son’s age People in line vs. time to help each Population in five countries Population in one country vs. five years Pounds vs. dollars Precipitation (rainfall) vs. month in two places Price index vs. year Price of advertisement vs. number of items announced Price vs. height of Christmas tree Prize vs. number of winners Students vs. make of father’s car Temperature of a person at 4 times a day for two days Vehicles used vs. number of people Weight of number of people Year of car vs. price Year vs. change in salary
GDR	*Graph defined relation* Used to code content where the relation is presented in a graph whose two axes are neither labeled nor numbered.	
GR	*Geometrical relation.* Used to code content that refers to geometric figures and their characteristics.	Angle vs. hours in a clock Height of a tower of cubes vs. number of cubes, visible or invisible faces, edges, and vertices Intersection of straight lines vs. perpendicularity Length of circumference vs. radius Length of edge of a cube vs. sum of length of edges, surface area and volume of cube Length of secant vs. distance to radius Length of sides of a square vs. area Linear projection of a segment Measure of angle vs. height of ray in unit circle Measure of angles of a triangle are proportional to a sequence of numbers Order of points or lines vs. distance to a point Perimeter of rectangle with fixed area vs. length of sides Point is on line Position of squares or cubes Radius vs. area of circle; radius squared vs. area of circle; area of circle vs. volume of cylinder Side vs. area of largest kennel Similarity: Measure lengths in parallel lines cut vs. transversal lines, produce drawings at different scales, different sizes of projections (height, area) vs. distance to projector Surface area of box

Code	Name and description	Examples
PR	*Pattern relation* Used to code content in which given a sequence the question is to find the general term (or an expression for the *n*th element) of the sequence.	Expression for triangular numbers Number of sides of a polygon vs. number of diagonals Number of triangles inside an n-sided polygon (diagonals from a vertex) Sum of consecutive odd numbers
RR	*Rule relation* Used to code content in which an input is transformed by certain procedure to obtain an output and in which a context is not provided.	Absolute value All polynomials Computer programming work: the student needs to write a computer/calculator program to produce particular outputs Function machines Does the x-axis represent a linear application? (The student needs to produce an expression for the application and justify that it is a linear application) Multiply by an operator $P(x)/Q(x)$, where P and Q are polynomials with coefficients in $\mathbf{R}$. Periodic functions with period 1, piece-wise on [a, a+1), a in R. Piece-wise functions Radical functions Step functions (defined parametrically, $f(x) = a$, x in $(a - 1, a]$, a in Z) Trigonometric $xy = k$
SOP	*Set-of-ordered-pairs relation* Used to code content where a list of ordered pairs is given or requested.	All possible sums of integer numbers less than 30 Any pair assignment Brotherhood Cantons vs. regions Family tree vs. color blindness Inequalities in the plane Intersection and subsets Invention vs. inventor Is a divisor of Is a fifth of Is a third of Is equivalent fraction Is factor of Is half of Is less than Is older than Is smaller than Is the same age as Is the same as Is two less than Is younger than Locate points in a Cartesian plane

Code	Name and description	Examples
		Relatives (father, wife, husband, daughters, boys, grandfather, descendant) Rivers vs. states Siblings and brothers Ways of drawing cards from a box *xRy* if the sum of the digits of *x* is equal to sum of the digits of *y*
TR	*Time relation.* Used to code content that refers to physical phenomena where time is involved and the variable is treated continuously.	Acceleration vs. number of trains Height of candle vs. time (h) it takes to burn out Reaction time to catch a bar vs. distance where the bar is caught Rpm of a pulley vs. diameter Speed (km/h) vs. distance (m): of a car before stopping once the brake is pressed, of sound of a lightning flash, of an echo Speed (m/s) vs. time: attained vs. a car after the accelerator is pushed completely, of filling a container with some liquid; of different birds, of joggers; according to the number of faucets needed to fill a tank Speed vs. rpm Speed vs. speed (of two objects/people moving) Time (h, m, s) vs. temperature (^{o}C, ^{o}F) Time in minutes sun reaches highest point in sky after 12m Time vs. distance: vs. a freely moving/falling object, vs. a ball rolling on an inclined surface, vs. a car on the road, of a hot-air balloon Time vs. height of water in a jar Time vs. volume for filling a pool

Each task will be given only one P code. If more than one code seems to apply, write them down, and then *choose the one* that in your view is more predominant. Provide an explanation for whatever code you choose.

O Codes: Operations

Operations refer to the activities that the students need to do in finding a solution to the problem. Note that a student might or might not follow a standard way to solve a problem; nevertheless, the information that is provided along with the problem is intended to give an idea of the most likely alternatives that a student may choose. He or she might choose a simpler way or repeat what was given in the text. I am looking for those strategies that would be privileged in the textbook.

This list is very long even though it is not comprehensive. If you think that there are operations that are not listed, make a note explaining why none of the O codes given is appropriate. Then choose the code or codes that are closest to what you intend to express, and tell me what is missing in them. Table C2 presents a list of the O codes, with a definition and some examples.

Table C2

O Codes. Operations That the Students Can Perform to Solve the Task

Code	Definition
CALC	*Calculate* The student needs to operate with numbers (e.g., add, subtract, multiply, divide, square a number, or find the square root, take log).
CDU	*Compute with different units* The student needs to convert units by applying a proportional relationship between measures or a formula to express a result (e.g., speed given in m/s and answer is needed in k/h)
CF	*Change between symbolic forms* The student needs to perform algebraic manipulations on a symbolic expression to obtain another one.
COE	*Carry out experiment* The student needs to perform a series of steps to collect data (either statistical or physical). Applies also to the case in which the student needs to write a computer program.
CWC	*Comparison without calculation* The student needs to produce a conjecture based on the observation of information available in task. There is no proof or calculation required (e.g., segments are proportional, lines with a slope of 0 are horizontal, one group outperforms another one).
DDR	*Determine domain or range* The student needs to find the domain, the range, or both of a relation. This can be part of a process and might not be explicitly stated in the task.
DSCP	*Describe shape in a graph* The student needs to describe the shape obtained after joining certain points in a Cartesian plane (e.g., squares, stars, and triangles). This includes examining the graph as a whole and also looking at particular intervals (e.g., observe minimum and maximum values, intercepts, sections where the relation is increasing or decreasing and so forth) with the aim of describing those features.
DTR	*Determine type of relationship* The student needs to determine whether the relation between two sets of numbers is direct, indirect, linear, or nonlinear, or whether there is no relation.
FA	*Find average* The student needs to find the average of a set of data.
FC	*Find the composite of two functions* The student needs to determine the function that is the composite of two functions, with no restriction on the representation used.

Code	Definition
FI	*Find inverse relation* The student needs to determine the inverse relation from a given one. This can be part of a process and might not be explicitly stated in the task. It does not apply to the case when only a pre-image is sought (in which case FIP is used).
FIP	*Find element of the range or of the domain of a relationship* The student needs to find in the range of the relation a value (or element) associated with a given element of the domain, or find a domain element associated with a range element, or both. This includes finding one more ordered pair of the relationship, where the student might need to choose an element of the domain and find its corresponding value in the range through the relation. It includes algebraic manipulations that involve solving for x in $f(x) = k$, where k is a given value, or finding $f(m)$ where m is an algebraic expression, finding the solution of $f(x) = f^{-1}(x)$, finding asymptotes. This code is also used when the student needs to find the function that results from the operation of two given functions; the process can be carried out by operating component by component in a table or by operating on the expressions that define the relation. This includes finding, for example, the image when $x = 0$ and the pre-image when $y = 0$, with all algebraic manipulation that may be required. There is no restriction on the representation used for the pair.
FISX	*Function is <certain characteristic>* The student is given a function, and he or she needs to establish if it satisfies a given characteristic (e.g., is bijective, is injective, is surjective, has inverse, is the identity, is constant, is increasing, is decreasing).
FNEC	*Find non-explicit characteristic* The student needs to demonstrate or prove that a particular object in the situation has a certain characteristic (e.g., a parallelogram is a square, intercept with y-axis has an abscissa of 0, there is a rectangle with maximum area, or $f^{-1}(3)$ is the solution of $f(x) = 3$).
FPN	*Find percentage or number* The student is given a situation, in which he or she needs to determine a percentage of occurrences of a certain event, or he or she is given the percentage and needs to find the number that would correspond to it in the situation.
FR2N	*Find the relation between two (sets of) numbers* The student needs to explain or produce an expression or a description of the relation between two given numbers or between two sets of numbers. The operation includes the variable identification and the process of writing down the expression (using any representation). To find the relation the student may recall previous knowledge (e.g., formulas for areas, volumes, perimeters) or base the solution on the information given (e.g., use proportionality). The relation can be given in any representation as described in the section on representations.
FS	*Find slope* The student needs to find (by a calculation, by a formula, by inspection) or locate (e.g., in a symbolic expression) the slope of a straight line.
FT	*Fill table* The student needs to either create or complete a partially filled table of values. If a relation is given or asked for (via any representation), this operation has to go together with FIP because the student will need to find images and pre-images through a relation in order to fill out the table. FT goes by itself when it is a step inside a data collection process: the relation is to be determined afterwards, using the information in the table.

Code	Definition
GC	*Give cardinal* The student needs to count the number of elements of a set.
GD	*Give definition* The student needs to produce a definition based on the reading of the text (e.g., define *argument*, *ordered pair*, *abscissa*).
GECE	*Give examples and counterexamples* The student needs to find examples of cases that satisfy a given situation, cases where a proposed situation does not hold, or both. Used also when the student needs to make up a story about a particular situation (e.g., a bath in a bathtub, or changing speeds of racers)
GP	*Give period* The student needs to find or give the period of a function that is known to be periodic.
GU	*Give unit* The student needs to give the units in which a certain element is measured.
LER	*List the elements of a relation* The student needs to produce a listing of all the elements of the relation. Note that this applies to relations where the domain is a finite set (e.g., a family tree with grandparents, parents, and children).
LPCP	*Locate points in graph* The student needs to locate points in a graph; a graph can be any of the types defined in the section on representations. Whenever a Cartesian plane is involved, the code must be applied if both elements of the ordered pair are known and need to be located. If that is not the case (e.g., the time at which the temperature is 50°C), then use the operation FIP. LPCP always requires NPOX when a Cartesian plane is involved.
M	*Measure* The student needs to apply a measurement procedure (e.g., in the Cartesian plane, variables in an experiment).
NPOX	*Name point on axis* The student needs to determine a number on an axis, either by reading it from the scale given or by doing an interpolation. It may or may not require a calculation (by means of adding a certain number a needed number of times).
NV	*Name variables* The student needs to identify the given variables in a situation (or representation) or to establish them. It is not necessary to use this code if FR2N is used.
OISX	*Operation is* The student needs to verify that an operation between relations satisfies a certain property (e.g., the operation is commutative, associative, etc.).
P	*Prove* The student needs to produce a proof of a statement either given in the text or produced by the student.
RIF	*Relation is function?* The student needs to determine if a relation is a function or not.
RPCP	*Read points from graph* The student needs to read the coordinates of a point or a set of points from a graph. A graph can be of any of the types defined in the section on representations. Whenever a Cartesian plane is involved, the code must be applied if both elements of the ordered pair have to be determined (e.g., the coordinates of the maximum value of a relation). If that is not the case, then use the operation FIP (e.g., the time at which the temperature is 5°C). RPCP always requires NPOX when a Cartesian plane is involved.

Code	Definition
TRIG	*Apply trigonometric identities/formulas* The student needs to use basic trigonometric relations between angles and sides of triangles to find unknown values of sides or angles.
TRIL	*Trace identity line* The student needs to draw in a Cartesian plane the line $y = x$ for a particular purpose (e.g., compare with lines of different slope, find an inverse, find a composite function).
TRL	*Trace regression line* The student needs to trace a regression line through a cloud of points.
UPWE	*Use proportionality within entries* The student needs to use the fact that a given relation is proportional.

R Codes: Representations

These codes refer to the representations that are present or required in a task. I have identified 9 different possible representations, which are described in Table C3. Each task can receive more than one R code because through the solution the student may need to use several of them.

Table C3

R Codes. Representations Enacted in the Task

Code	Description
AD	*Arrow diagrams* The task uses arrow diagrams, in either one or two bubbles.
G	*Graph in two axes* The task uses a Cartesian plane, frequency diagram, histogram, broken line (time series), or scatter-plot.
N	*Numerical* The task does not require any symbols; instead, it requires numbers.
NL	*Number line* The task requires a number line.
P	*Pictorial* The task uses drawings of machines, maps, geometrical shapes and figures, photos, or pictograms (frequency diagram where the y-axis is not present), pies (only one variable is sketched) or any other kind of drawing.
SS	*Semi symbolic* The task uses expressions that contain both words and symbols. (e.g., cost of x pounds of apples = $0.3 \times x$).
S	*Symbolic* The task uses expressions with only symbols. This includes arithmetical notation, sets (e.g., $\{x \mid x > 0, x \in \mathbf{N}\}$), ordered pairs, equations (e.g., $f(x) = x + 1$; $y = x + 1$, $f(2) = x + 1$), mappings ($f: x \rightarrow x + 1$), or intervals.
T	*Tabular* The task uses a table. The table can be given, asked for, or a requisite for the process of keeping track of the entries.

Code	Description
V	*Verbal* The task uses a description of a situation using natural language (e.g., a pound of apples costs 30 cents) or requires the student to interpret a situation with natural language.

C Codes: Controls

Controls codes are used to characterize the process of solving a task considering the ways in which the student can control that the solution produced is correct or not. The codes attempt to answer the question, How does the student know that his or her answers are correct? Even though these codes strongly depend on what is given in the text, and I acknowledge that coding a task without its context makes this assignment difficult, I think that there are activities that belong to the task that can help to establish the correctness of a result. Consider the case of locating points of a linear pattern in a Cartesian plane. If at the end the student finds out that not all the points lie on a straight line, then he or she can tell that there is a mistake in the operation of locating points in the graph. That is the nature of the control function. The codes are listed in Table C4.

Table C4

C Codes. Controls Available to Determine the Correctness of a Solution

Code	Description
CLC	*Use calculator or computer to check the answer* The student is told to create a program so that he or she can check his or her answers, or to observe a graph in the calculator.
CPE	*Compare previous examples or problems* The student has performed similar operations and can use them in this particular situation. (Note: This code can be assigned only when more than one problem is available to code from the same section in a textbook.)
CON	*Continuity is assumed* The student uses the fact that a function is continuous to determine the likeliness or unlikeness of a result. Also the situation is such that if continuity is not assumed, the problem cannot be solved (e.g., the student has a table of values—of a polynomial function—and he or she needs to find the image of a value that is not in the table)
DC	*Double check* The student either repeats the process used to obtain the answer (e.g., relocates points in the Cartesian plane) or reverses the process to get something that is given in the statement of the task (undoes the sequence of operations).

Code	Description
LFLUR	*Look for likely or unlikely results* The student can use indicators in the statement of the task (e.g., the student obtains a number too big or too small for a given scale in a Cartesian plane, or he or she is getting decimals or negative numbers when whole or positive numbers are expected, or a set of points in a Cartesian plane are not aligned on a line) or can use previous knowledge (e.g., the sides of a square have the same length).
MT1P	*More than one point (vertical line test)* The student has to determine if an element of the domain of a relation has one and only one element assigned from the co-domain of the relation.
UAR	*Use alternative (given or not given) representations* The student can use other representations (e.g., results in a table vs. results with a formula or a graph, a set of ordered pairs as an arrow diagram). These can be explicitly given in the statement of the task or can be result of something the student was asked to do.
UCP	*Use checkpoints* The statement of the task offers answers to previous questions, warns the student about what is not a correct answer, suggests looking at the following tasks, or suggests checking with a partner who is doing the same task.
UGI	*Use given information.* The statement of the task gives additional information that can be used to test a result and that might not be relevant to the solution of the problem (e.g., if there is only $100 to spend—then the domain of the relation has to be restricted)

APPENDIX D

MESSAGES FOR CODERS

The first test of the coding procedure was carried out by electronic mail. The messages inviting people to participate, giving the details about the coding process and summarizing the experience, are provided here.

Electronic message inviting people to collaborate in the coding process:

Fri, 12 Nov 1999
Hello There,
I would like to know who of you would be interested in helping me with a test of a coding system that I developed for my study. I will send you 'clear' specifications as what to do to code four mathematics problems related to functions taken from textbooks around the world (languages include English, French, German, Portuguese, and Spanish, although the samples for this test may contain up to 3 different languages--of course I will be sensitive to your preferences). The procedure includes to read the problem, to solve it as an 7th to 8th grader would solve it, assign 4 types of codes, and then give me some feedback on how you did the assignment. I anticipate a two-hour work total. The problems are not difficult at all.
I would appreciate any help that you can give me, I will provide more details for those who might be interested.
THANKS A LOT!
Vilma Mesa.

Electronic message explaining the process of coding:

Mon, 15 Nov 1999
Dear All,
Thanks for your willingness to help me with this test of the coding process.
With this message I am sending an acrobat file that contains the codes that I have developed together with instructions for applying them.
You will receive three or four tasks to code in one acrobat file attached to a second e-mail. 11 tasks were randomly selected from a pool of 541 problems. I made groups taking into consideration your language proficiency. (I will elaborate on this point in each particular case).
For each of you the procedure is the same:
0. Read through the coding procedure. If you have questions, e-mail me, I will answer them right away.
1. Be sure that you can read the tasks. For some of you it will be difficult because the picture included it is not very clear. If that is the case for you, please write down what you can 'read' from the text of the task--it does not have to be an exact or literal translation, but what are you 'reading' in it. This will allow me to contrast to the notes of other coders with my notes about those cases.
2. Solve the task. Keep notes as long as you proceed; that will help you to justify your code assignment.
3. Proceed with the code assignment as indicated in the coding procedure.

4. E-mail me your coding. You don't have to include the text of the task; please follow the format that I included in the coding procedure file. You can write directly in the body of an e-mail message or send your formats as an attachment, preferably in an RTF file.
If you have any difficulties, please let me know.
Thanks a lot,
Vilma Mesa

Electronic message reporting the results of the process:

Mon, 29 Nov 1999
Dear All,
Thanks a lot for your kind collaboration in the test of my coding system. The whole process was very valuable for me to unveil both problems and good points about it.
As I expected we have a good agreement on the first category, Prototypical uses of functions: 94% and not so good agreement on the others (33%, 53% and 50%, comparing your overall coding to mine), even though I was expecting higher figures these codes. I attribute this outcome to two important issues:
1. In the case of operations, the meaning of some of the codes overlap the uses of others; some of you helped me describing what you were looking for and what you used instead, which allowed me to make explicit what I was thinking on in the first place. As a result some codes have disappeared, some have been expanded to represent broader uses, and I have paired some of them to describe particular situations.
2. I did not provide you with the material that comes before the task that you solved. That puts you in a different position from mine. Because I have access to that previous work I can decide which solution--from all the possible available--is more likely to be followed by a student, which causes me to introduce codes (for example, fill a table) that you could probably find unnecessary.
On the good side, I found that the system does help to discriminate tasks (an “interesting” task is getting more codes than a not-so interesting one--I am working on this idea; right now it does not mean much).
Given this, I will carry out another experience on a 1-1 basis with people around here (an interview with audio recording). I will provide two (out of four) tasks plus the section in which that task is presented. I will ask the person to solve the task and afterwards I will provide a list for the person to chose the codes for the categories, asking for the reasons for doing that. By using this procedure I will be also checking the validity of the coding system (an issue raised by some of you).
I really appreciate the time that you gave me for doing this. Thank you very much.
Vilma Mesa.

APPENDIX E

ITEMS FROM THE TIMSS ACHIEVEMENT TEST

Ten items were selected from the released set of items for Population 2 in the TIMSS study. I provide below their identification code, content category, and the performance expectation according to the TIMSS framework (International Association for the Evaluation of Educational Achievement, 1997). I also provide the coding for each item, following the coding procedure I developed. When more than one solution was possible, the final coding aggregated all the possible codes for the operations, representations, and controls for the several solutions proposed.

Item I-04, Algebra, Solving Problems

> The numbers in the sequence 2, 7, 12, 17, 22, … increase by fives. The numbers in the sequence 3, 10, 17, 24, 31, … increase by sevens. The number 17 occurs in both sequences. If the two sequences are continued, what is the next number that will be seen in both sequences?

Coding: The relation deals with the identification of two different patterns, both numerical, whose construction procedure is known (increase by fives, increase by sevens). The prototypical use of function is *pattern relation*, PR. There are two solutions that students could have used. To find the common element in both sequences, the student might apply the construction procedures for both sequences (finding the images through the relations, adding fives or sevens, FIP). In that case, a list can be produced to find the common element (list the elements of the relation, LER). The calculations needed are coded as CALC. The student might also attempt to write an expression for the two sequences (FR2N), and using the fact that 5 and 7 are relative primes, find the number associated with the next seventh position for the first sequence and with the fifth position in the second, FIP). The item requires a numerical representation for the first solution, and symbolic and numerical representations for the second. The student can doublecheck the computations (DC), but he or she knows that the answer is found once the two sequences give the same number. The student is looking for likely results (LFLUR). The item was coded as: PR, FIP-FR2N-LER-CALC, S-N, DC-LFLUR.

Item I-08, Geometry, Solving Problems

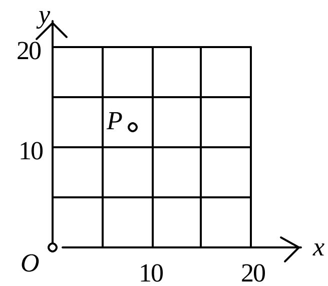
> A straight line on a graph passes through the points (3, 2) and (4, 4). Which of these points also lies on the line?
>
> A. (1,1)
> B. (2,4)
> C. (5,6)
> D. (6,3)
> E. (6,5)

Coding: In this case the student might locate the two given points on the Cartesian plane (LPCP), and from there, by trial and error, check each of the alternatives offered to see which point is more likely to be on the line that joins the two points (LPCP, G, LFLUR-UCP-CON). As an alternative, the student might follow the pattern for the abscissas 3, 4, 5, and the coordinates, 2, 4, 6. This approach uses the fact that consecutive first differences of coordinates are constant in linear functions (the slope is constant). In this case, the student lists the elements of the relation and once the pair (5, 6) is found the student knows that an answer has been found (LER-CALC, N, DC-UCP-CON). As another approach, the student might establish the linear relation and by trial and error check the alternatives offered (FR2N-FIP-CALC, S, DC-UCP-CON). The item was coded as: PR, LPCP-LER-FR2N-FIP-CALC, N-G, UCP-DC-UAR-LFLUR-CON.

Item J-16, Geometry, Performing Routine Procedures

> Which of the following are most likely to be the coordinates of point P?

A. (8,12)
B. (8,8)
C. (12,8)
D. (12,12)



Coding: There is no explicit relation given here; the item deals with managing the Cartesian plane. The content of reference is coded as relation defined by a graph (GDR). The student needs to read the coordinates of the point by naming two values on the axes; estimation is needed to identify those points, but the size of the estimation is guided by the alternatives suggested in the answers (RPCP-NPOX-CALC, N-G, UCP-DC). The item was coded: GDR, RPCP-NPOX-CALC, N-G, UCP-DC

<u>Item J-18, Algebra, Performing Routine Procedures</u>

The table represents a relation between *x* and *y*.
What is the missing value in the table?

A. 2
B. 3
C. 4
D. 5
E. 6

x	*y*
1	1
2	?
4	7
7	13

Coding: By checking that the first differences of both abscissas and coordinates are constant, (1 – 4 = 4 – 7= –3 and 1 – 7 = 7 – 13 =– 6), the student can either write an expression for the linear function, finding the slope (6/3 = 2) and the intercept (–1) and substituting 2, to get 3 (FR2N-FS-FIP), or locate the points in a Cartesian plane and assuming continuity find an approximate ordinate value for an abscissa of 2 (as 3). The item was coded as RR, FR2N-FS-FIP-CALC-LPCP, T-N-G, CON-DC-UCP-UAR.

Item L-11, Algebra, Solving Problems

> A rubber ball rebounds to half the height it drops. If the ball is dropped from a rooftop 18m above the ground, what is the total distance traveled by the time it hits the ground the third time?
>
> A. 31.5m
> B. 40.5m
> C. 45m
> D. 63m

Coding: The context of reference is a cause-and-effect relation. The student needs to apply the transformation of the heights twice in order to get the heights needed for the addition; a drawing might help to clarify the situation. The heights to be added are 18m, 9m twice, and 4.5m twice, to get 45m. The item was coded as: CER, FIP-CALC, N-P, DC-UCP.

Item L-14, Proportionality, Performing Routine Procedures

> The table shows the values of x and y, where x is proportional to y.

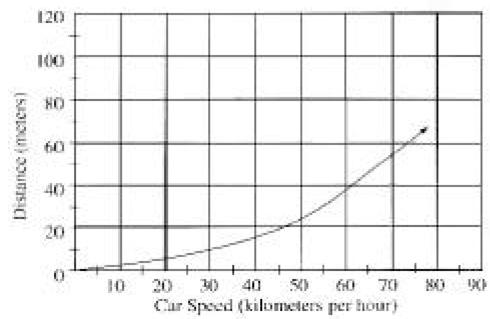
x	3	6	P
y	7	Q	35

> What are the values of P and Q?
>
> A. $P = 14$ and $Q = 31$
> B. $P = 10$ and $Q = 14$
> C. $P = 10$ and $Q = 31$
> D. $P = 14$ and $Q = 15$
> E. $P = 15$ and $Q = 14$

Coding: The prototypical use of function is coded as direct proportion relation, because there is no context (DPPR). The student needs to find the constant of proportionality (UPWE) and then use that number to find the values of P and Q (FIP). The treatment is basically numerical, and the student can doublecheck his or her answer if what is obtained is not in the list of possible answers. The final coding was: DPPR, FIP-UPWE, T-N, DC-UCP.

<u>Item O-01, Data Representation, Analysis and Probability, Solving Problems</u>

The graph shows the distance traveled before coming to a stop after the brakes are applied for a typical car traveling at different speeds.



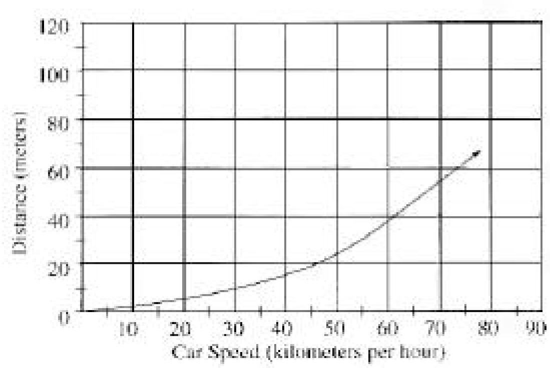
A car traveling on a highway stopped 30m after the brakes were applied. About how fast was the car traveling?

A. 48 km/h
B. 55 km/h
C. 70 km/h
D. 160 km/h

Coding: This is a relation that involves time as a continuous variable (TR). The student needs to find the pre-image of 30 under the relation given by the graph. The procedure is based on the graph and the student can doublecheck his or her answer if what is obtained is not in the list of possible answers. The final coding of the item was TR, FIP-NPOX-RPCP, G, DC-UCP.

<u>Item R-08, Data Representation, Analysis and Probability, Solving Problems</u>

The graph shows the distance traveled before coming to a stop after the brakes are applied for a typical car traveling at different speeds.



A car is traveling 80 km per hour. About how far will the car travel after the breaks are applied?

A. 60 m
B. 70 m
C. 85 m
D. 100 m

Coding: This item received the same codes as the previous one, except for the control activities. The student needs to assume that the graph can be extended continuously to be able to find the image of 80 (the graph ends near the point (75, 60)). The coding was TR, FIP-NPOX-RPCP, G, DC-UCP-CONT.

Item S-01, Algebra, Solving Problems

Here is a sequence of three similar triangles. All of the small triangles are congruent.

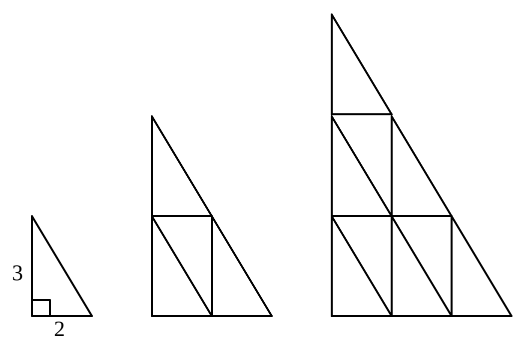


Figure 1 Figure 2 Figure 3

a. Complete the chart by finding how many small triangles make up each figure:

Figure	Number of small triangles
1	1
2	
3	

b. The sequence of similar triangles is extended to the 8th Figure. How many small triangles would be needed for Figure 8?

Coding: The prototypical use of function is pattern; the student can follow the pattern some steps more (COE) to fill in the appropriate cells of the table. The pattern at figure *n* corresponds to the sum of the first *n* odd numbers. This expression can be used to find the number of triangles for the eighth figure. The item was coded as PR, FIP-COE-FT, T-S-N-P, DC-UCP.

<u>Item V-02, Data Representation, Analysis, and Probability, Solving Problems</u>

> The following two advertisements appeared in a newspaper in a country where the units of currency are *zeds*.

Building A	Building B
Office space available	Office space available
85-95 square meters 465 *zeds* per month	35-260 square meters 90 *zeds* per square meter per year
100-120 square meters 800 *zeds* per month	

> If a company is interested in renting an office of 100 square meters in that country for a year, at which office building, A or B, should they rent the office in order to get the lower price? Show your work.

Coding: The prototypical use of function in this item is as constructed relation between the variables area for rent (in square meters) and cost of renting the space (in zeds). The student needs to find the value of one variable (cost) given the value of the other (100 square meters). He or she needs to establish that 100 belongs to the domain of the relation and in the case of Building A, that only one of the alternatives can be used. All the information is provided without any recourse to symbols. Two calculations are needed in order to make a decision. The final coding was CR, FIP-DDR-CALC, V-N, DC.

APPENDIX F

TYPES AND ANTITYPES

The results of the CFA program are presented below in an edited form of the actual listing produced by the program. A four-digit number represents the configurations, the second column of the table. The first digit refers to the prototypical use of function (1=set of ordered pairs; 2=physical; 3=rule; 4=figural; 5=social). The second digit refers to operations (1=manipulate only; 2=appreciate only; 3=manipulate and appreciate only; 4=calculate with manipulate or appreciate or alone; 5=other operations not in manipulate, appreciate, or calculate). The third digit refers to representations (1=symbolic only; 2=symbolic combined with another representation; 3=any combination of representations not including the symbolic). The fourth digit refers to the control activities (1=based on the process only; 2=based on the process and the contract; 3=based on the content only or in combination with process, or contract-based control activities). For example the configuration 1121 indicates an use of function as set of ordered pairs, requiring only manipulation operations, using symbolic representation in combination with other representations, and basing the control of the correctness of the answers on the process of solution only.

The third column gives the observed frequency of the configurations. The fourth configuration gives the expected frequency under the assumption of independence of assignments. Statistic is the value of the *z*-test statistic that was chosen for this particular analysis. The test-wise α is protected using a Bonferroni adjustment, calculated as α/t, where t is the number of tests. In this case, with a $5 \times 5 \times 3 \times 3$ table, $\alpha^* = 0.00022$. The next column gives the one-tailesd probabilities of the tests statistic. Type indicator gives the designation of a given configuration as a type or antitype. Log P is the Poisson probability that the observed cell frequency is smaller than the expected cell frequency.

The configurations are ranked according to the size of this probability, the smaller the probability, the higher the rank.

Number	Config-uration	Obs.erved frequency	Expected frequency	Statistic	Probability	Type indicator	Log P	Rank
1	1121	54	31.013	4.128	.00001832	Type	3.978	18
2	1122	35	15.592	4.915	.00000044	Type	4.806	12
3	1211	24	6.136	7.211	.00000000	Type	7.429	6
4	1223	13	4.797	3.746	.00009001	Type	2.854	24
5	1233	24	3.369	11.240	.00000000	Type	12.533	4
6	1411	19	6.162	5.172	.00000012	Type	4.603	14
7	1432	17	5.354	5.033	.00000024	Type	4.341	16
8	2133	14	2.969	6.402	.00000000	Type	5.520	10
9	2333	10	1.931	5.806	.00000000	Type	4.458	15
10	2431	16	4.569	5.348	.00000004	Type	4.616	13
11	2433	8	1.451	5.436	.00000003	Type	3.867	19
12	2531	3	.340	4.563	.00000252	Type	2.294	29
13	3111	127	22.066	22.338	.00000000	Type	52.327	1
14	3131	8	38.138	-4.880	.00000053	Antitype	6.629	8
15	3321	65	35.320	4.994	.00000030	Type	5.341	11
16	3322	59	17.758	9.787	.00000000	Type	13.992	3
17	3331	1	24.809	-4.780	.00000088	Antitype	6.503	9
18	3332	0	12.473	-3.532	.00020649	Antitype	2.799	25
19	4323	12	1.658	8.032	.00000000	Type	6.707	7
20	4332	18	1.843	11.902	.00000000	Type	11.784	5
21	4432	6	1.385	3.921	.00004403	Type	2.518	28
22	4531	2	.205	3.966	.00003661	Type	1.737	41
23	4533	1	.065	3.664	.00012399	Type	1.201	51
24	5121	4	22.306	-3.876	.00005313	Antitype	3.799	20
25	5232	43	3.835	19.999	.00000000	Type	29.305	2
26	5233	10	2.423	4.867	.00000057	Type	3.663	21
27	5322	20	7.295	4.704	.00000128	Type	4.118	17
28	5333	11	3.238	4.314	.00000803	Type	3.264	22

APPENDIX G

CONFIDENCE INTERVALS

This appendix contains graphs of the 95% confidence intervals for the percentage of correct responses for seven of the ten TIMSS items selected, organized by groups of participating countries. Two figures, one for grade seven and the other for grade eight, followed by a table of values, are given for each item. The items are organized by their TIMSS identification, and the countries are ordered by percentage of correct responses.

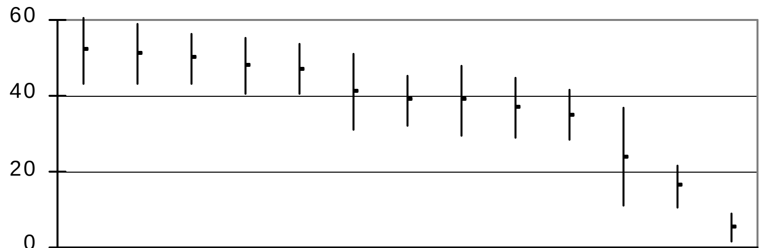
The names of the countries are abbreviated as follows:

Australia	ASL
Austria	OST
Canada	CAN
Colombia	COL
England	ENG
Hong Kong	HNK
Ireland	IRL
Portugal	POR
Singapore	SIN
South Africa	SAF
Spain	SPA
Switzerland	SWI
United States	USA

Item I-04

Item I-04, Seventh Grade

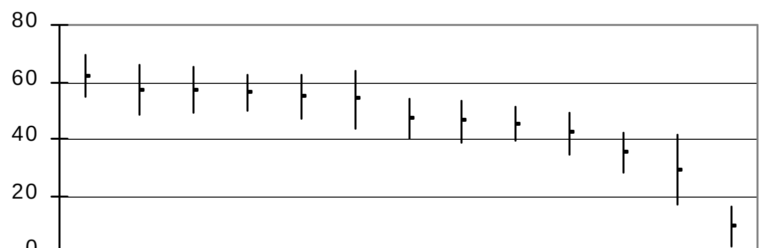
Figural



	IRL	SIN	SWI	OST	CAN	ENG	ASL	USA	HNK	SPA	COL	POR	SAF
Lower	43.5	43.1	43.5	40.9	40.5	31.1	32.5	29.9	29.1	28.5	10.7	10.3	1.3
Upper	60.5	58.9	56.5	55.1	53.5	50.9	45.5	48.1	44.9	41.5	37.3	21.7	8.7
▪ % Correct	52	51	50	48	47	41	39	39	37	35	24	16	5

Item I-04, Eighth Grade

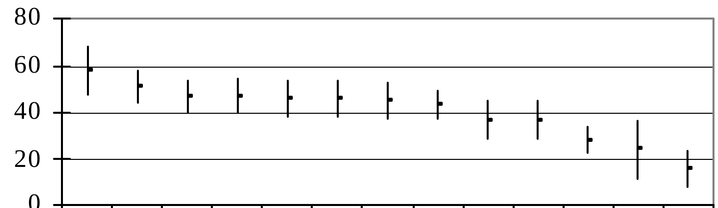
Figural



	SIN	IRL	OST	CAN	SWI	ENG	HNK	SPA	ASL	USA	POR	COL	SAF
Lower	54.6	48.2	48.8	49.5	47.4	43.8	39.9	38.6	39.1	34.6	27.9	16.8	2.2
Upper	69.4	65.8	65.2	62.5	62.6	64.2	54.1	53.4	50.9	49.4	42.1	41.2	15.8
▪ % Correct	62	57	57	56	55	54	47	46	45	42	35	29	9

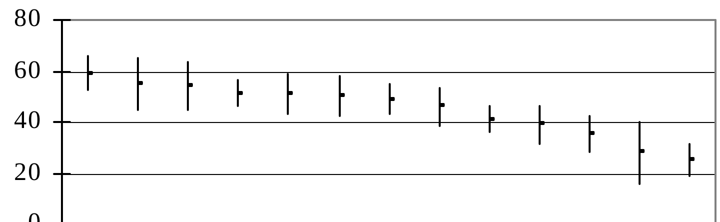
Item I-08

Item I-08, Seventh Grade
Figural



	ENG	HNK	ASL	SIN	OST	SWI	IRL	CAN	SPA	USA	SAF	COL	POR
Lower	47.8	43.9	40.2	39.6	38.1	38.1	37.4	37.1	28.8	29.1	21.8	11.0	7.4
Upper	68.2	58.1	53.8	54.4	53.9	53.9	52.6	48.9	45.2	44.9	34.2	37.0	22.6
% Correct	58	51	47	47	46	46	45	43	37	37	28	24	15

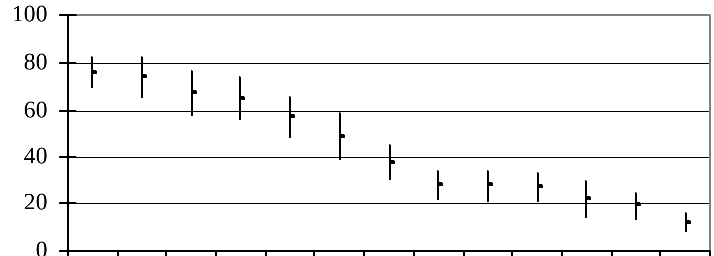
Item I-08, Eighth Grade
Figural



	SIN	ENG	OST	ASL	SWI	HNK	CAN	IRL	USA	SPA	POR	COL	SAF
Lower	52.5	44.5	44.7	45.9	43.4	42.1	43.3	38.6	35.9	31.6	27.9	15.8	18.8
Upper	65.5	65.5	63.3	56.1	58.6	57.9	54.7	53.4	46.1	46.4	42.1	40.2	31.2
% Correct	59	55	54	51	51	50	49	46	41	39	35	28	25

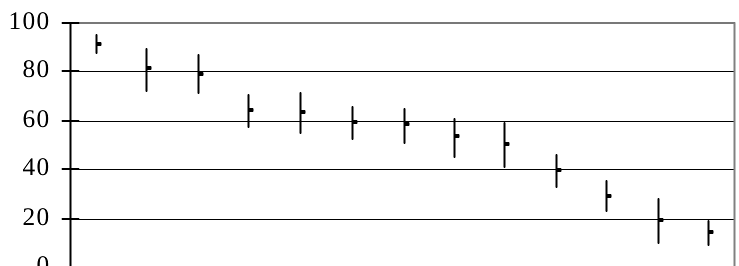
Item J-16

Item J-16, Seventh Grade
Figural



	SIN	HNK	ENG	OST	ASL	USA	CAN	SPA	SWI	IRL	COL	POR	SAF
Lower	70.1	65.5	57.7	55.9	48.5	39.4	30.4	21.5	21.2	20.8	14.4	13.6	8.3
Upper	81.9	82.5	76.3	74.1	65.5	58.6	45.6	34.5	34.8	33.2	29.6	24.4	15.7
% Correct	76	74	67	65	57	49	38	28	28	27	22	19	12

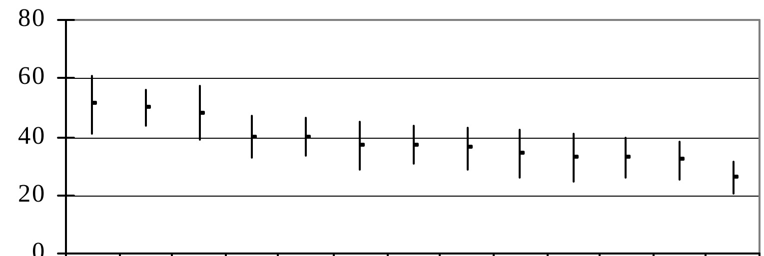
Item J-16, Eighth Grade
Figural



	SIN	HNK	ENG	ASL	OST	SWI	USA	CAN	IRL	SPA	POR	COL	SAF
Lower	87.3	72.5	71.4	57.8	55.1	52.2	50.9	45.1	40.9	32.5	22.8	10.2	8.9
Upper	94.7	89.5	86.6	70.2	70.9	65.8	65.1	60.9	59.1	45.5	35.2	27.8	19.1
% Correct	91	81	79	64	63	59	58	53	50	39	29	19	14

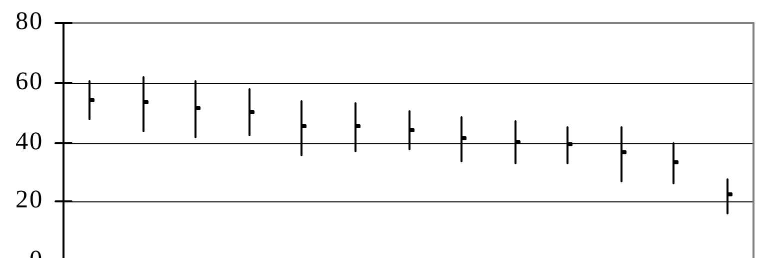
Item J-18

Item J-18, Seventh Grade
Rule



	ENG	ASL	HNK	CAN	SIN	IRL	SWI	USA	OST	COL	SPA	POR	SAF
Lower	41.1	43.8	38.7	32.9	33.8	28.8	30.5	28.9	25.8	24.8	26.2	25.5	20.3
Upper	60.9	56.2	57.3	47.1	46.2	45.2	43.5	43.1	42.2	41.2	39.8	38.5	31.7
% Correct	51	50	48	40	40	37	37	36	34	33	33	32	26

Item J-18, Eighth Grade
Rule

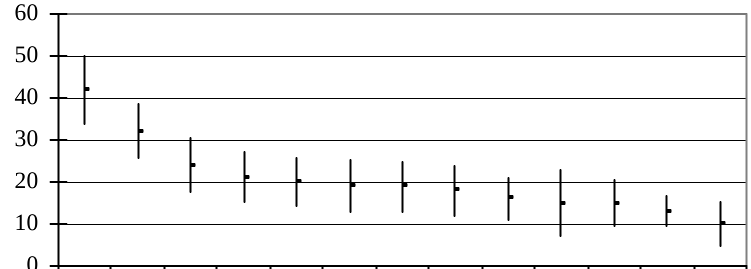


	ASL	HNK	ENG	SIN	OST	SWI	CAN	IRL	SPA	USA	COL	POR	SAF
Lower	47.8	43.9	41.4	42.1	35.9	36.8	37.5	33.6	32.6	33.1	26.7	26.2	16.3
Upper	60.2	62.1	60.6	57.9	54.1	53.2	50.5	48.4	47.4	44.9	45.3	39.8	27.7
% Correct	54	53	51	50	45	45	44	41	40	39	36	33	22

Item L-14

Item L-14, Seventh Grade

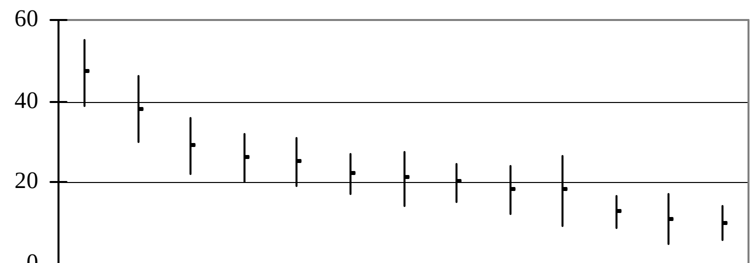
Rule



	SIN	HKN	CAN	IRL	SWI	USA	POR	ASL	SPA	OST	ENG	SAF	COL
Lower	33.8	25.5	17.5	15.1	14.1	12.8	13.1	12.1	11.2	7.1	9.6	9.3	4.6
Upper	50.2	38.5	30.5	26.9	25.9	25.2	24.9	23.9	20.8	22.9	20.4	16.7	15.4
% Correct	42	32	24	21	20	19	19	18	16	15	15	13	10

Item L-14, Eighth Grade

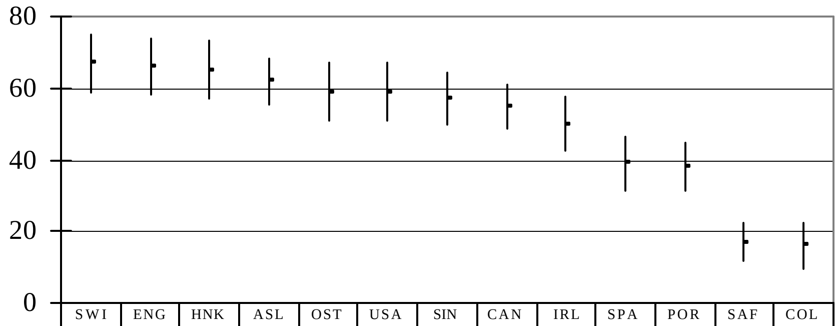
Rule



	SIN	HKN	SWI	CAN	IRL	ASL	POR	USA	ENG	OST	SAF	COL	SPA
Lower	39.1	29.8	22.2	20.1	19.1	17.2	14.5	15.5	12.1	9.5	9.0	4.8	5.8
Upper	54.9	46.2	35.8	31.9	30.9	26.8	27.5	24.5	23.9	26.5	17.0	17.2	14.2
% Correct	47	38	29	26	25	22	21	20	18	18	13	11	10

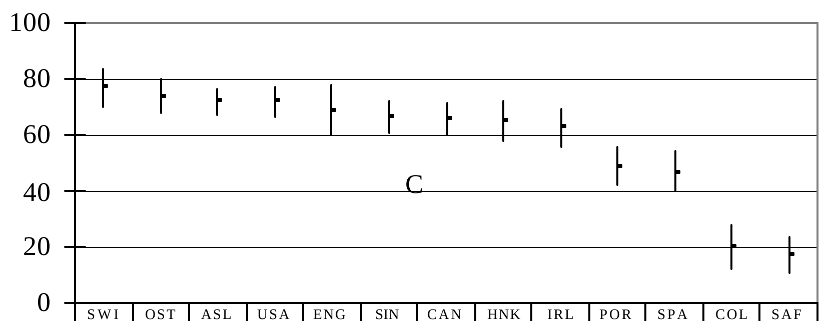
Item O-01

Item O-01, Seventh Grade
Physical



	SWI	ENG	HNK	ASL	OST	USA	SIN	CAN	IRL	SPA	POR	SAF	COL
Lower	58.8	58.1	56.8	55.5	50.8	50.8	49.9	48.8	42.6	31.4	31.2	11.6	9.8
Upper	75.2	73.9	73.2	68.5	67.2	67.2	64.1	61.2	57.4	46.6	44.8	22.4	22.2
▪ %Correct	67	66	65	62	59	59	57	55	50	39	38	17	16

Item O-01, Eighth Grade
Physical

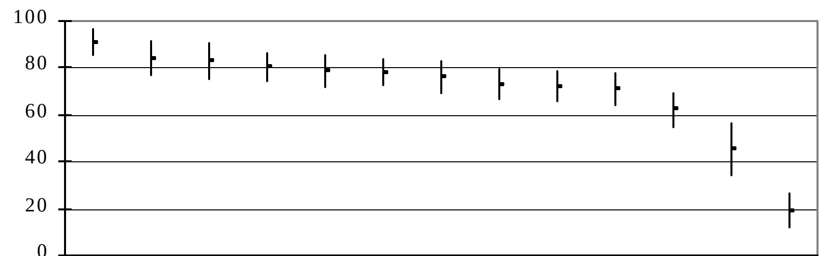


	SWI	OST	ASL	USA	ENG	SIN	CAN	HNK	IRL	POR	SPA	COL	SAF
lower	70.5	67.8	67.2	66.6	60.2	61.3	60.6	57.9	56.2	41.6	39.6	12.4	10.5
upper	83.5	80.2	76.8	77.4	77.8	72.7	71.4	72.1	69.8	56.4	54.4	27.6	23.5
▪ %Correct	77	74	72	72	69	67	66	65	63	49	47	20	17

Item S-01, Part a

Item S-01a, Seventh Grade

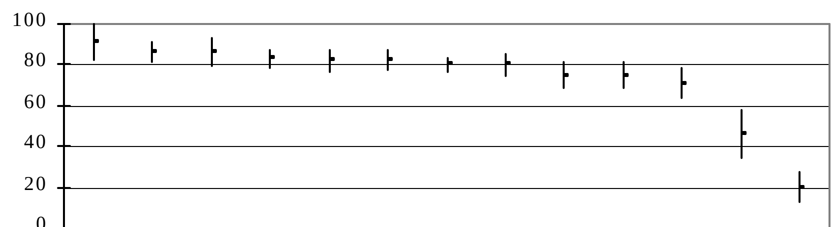
Figural



	OST	HNK	SIN	ASL	ENG	USA	SPA	IRL	CAN	SWI	POR	COL	SAF
Lower	85.6	76.6	75.4	74.1	72.2	72.3	68.9	66.8	65.8	64.2	54.6	34.0	11.9
Upper	96.4	91.4	90.6	85.9	85.8	83.7	83.1	79.2	78.2	77.8	69.4	56.0	26.1
▪ % Correct	91	84	83	80	79	78	76	73	72	71	62	45	19

Item S-01a, Eighth Grade

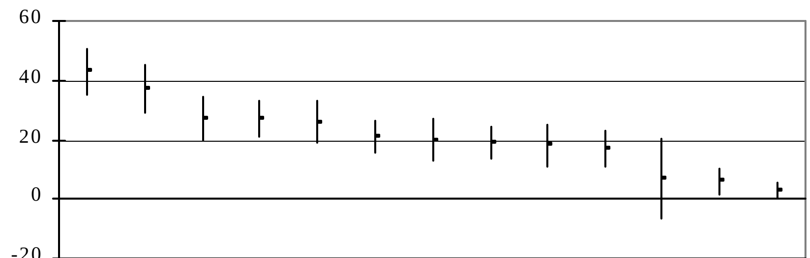
Figural



	OST	ASL	HNK	ENG	SIN	USA	SPA	SWI	CAN	IRL	POR	COL	SAF
Lower	81.9	81.2	79.2	78.8	76.6	77.2	76.3	74.3	68.5	68.8	63.6	34.1	12.9
Upper	100.1	90.8	92.8	87.2	87.4	86.8	83.7	85.7	81.5	81.2	78.4	57.9	27.1
▪ % Correct	91	86	86	83	82	82	80	80	75	75	71	46	20

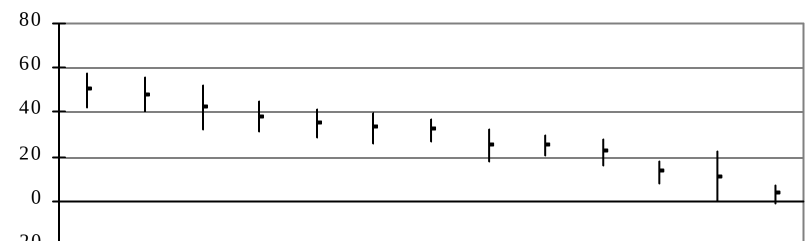
Item S-01, Part b

Item S-01b, Seventh Grade
Figural



	SIN	ENG	ASL	OST	SPA	USA	HNK	CAN	IRL	SWI	COL	POR	SAF
Lower	35.1	28.8	19.6	20.8	18.9	15.9	12.6	13.3	11.2	10.8	-6.6	1.8	0.5
Upper	50.9	45.2	34.4	33.2	33.1	26.1	27.4	24.7	24.8	23.2	20.6	10.2	5.5
▪ % Correct	43	37	27	27	26	21	20	19	18	17	7	6	3

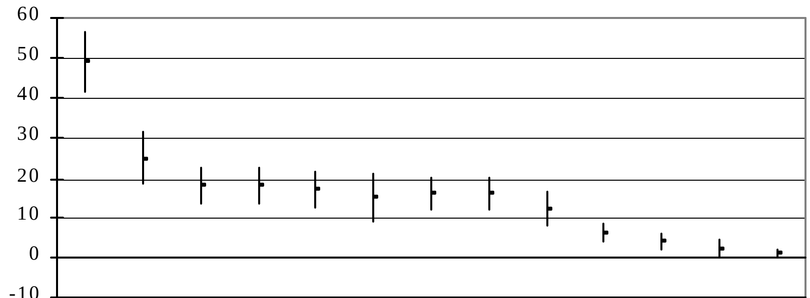
Item S-01b, Eighth Grade
Figural



	ENG	SIN	HNK	ASL	OST	USA	SPA	CAN	IRL	SWI	POR	COL	SAF
Lower	42.1	40.4	32.4	30.9	29.1	26.2	26.9	17.6	20.5	16.3	7.9	-0.6	-0.7
Upper	57.9	55.6	51.6	45.1	40.9	39.8	37.1	32.4	29.5	27.7	18.1	22.6	6.7
▪ % Correct	50	48	42	38	35	33	32	25	25	22	13	11	3

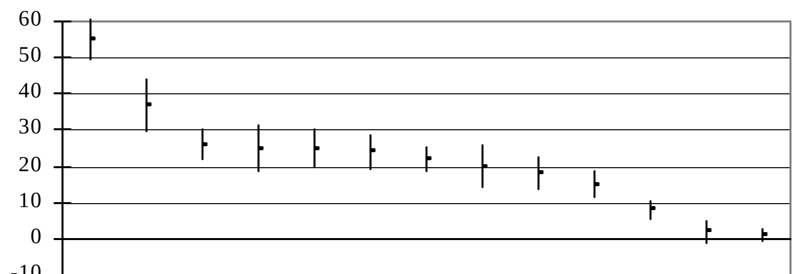
Item V-02

Item V-02, Seventh Grade
Social



	SIN	HNK	ASL	IRL	OST	USA	CAN	SWI	ENG	SPA	POR	SAF	COL
Lower	41.6	18.5	13.5	13.5	12.5	8.8	11.8	11.8	7.8	3.7	2.0	-0.3	-0.1
Upper	56.4	31.5	22.5	22.5	21.5	21.2	20.2	20.2	16.2	8.3	6.0	4.3	2.1
% Correct	49	25	18	18	17	15	16	16	12	6	4	2	1

Item V-02, Eighth Grade
Social



	SIN	HNK	SWI	IRL	OST	CAN	ASL	ENG	USA	SPA	POR	SAF	COL
Lower	49.3	29.9	21.8	18.5	19.9	19.2	18.6	14.3	13.5	11.3	5.5	-1.1	-0.4
Upper	60.7	44.1	30.2	31.5	30.1	28.8	25.4	25.7	22.5	18.7	10.5	5.1	2.4
% Correct	55	37	26	25	25	24	22	20	18	15	8	2	1