K-Cohomology of

Generalized Severi-Brauer Varieties

by

Patrick Kevin McFaddin

(Under the direction of Daniel Krashen)

Abstract

In 1992, Merkurjev and Suslin provided an explicit description of the group of K_1-zero-cycles of the Severi-Brauer variety associated to a central simple algebra A. This description was given in terms of the group $K_1(A)$ and yields a cohomological description of pairs consisting of a maximal subfield of A together with an element of this subfield. In this thesis, we compute the group of K_1-zero-cycles of the second generalized Severi-Brauer variety of a central simple algebra A of index 4 in terms of elements of $K_1(A)$ and their reduced norms. Analogously, this group gives a cohomological description of the quadratic subfields of the degree 4 maximal subfields of the algebra A. To give such a description, we utilize work of Krashen to translate our problem to the computation of cycles on involution varieties. Work of Chernousov and Merkurjev then gives a means of describing such cycles in terms of Clifford and spin groups and corresponding R-equivalence classes. We complete our computation by giving an explicit description of these algebraic groups.

Index words: Algebraic K-theory, K-cohomology, algebraic cycles, central simple algebras, algebraic groups, homogeneous varieties, Severi-Brauer varieties.

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Contents

Chapter 1

Introduction

This thesis focuses on computing the group of K_1-zero-cycles on generalized Severi-Brauer varieties. For a central simple algebra of index 4 and arbitrary degree, we present an explicit description of the group of K_1-zero-cycles of the second generalized Severi-Brauer variety of the algebra. This group can be realized as the collection of pairs consisting of an element of the first K-group of our algebra together with a square root of its reduced norm. This computation utilizes results of Chernousov-Merkurjev on K-cohomology of involution varieties by means of algebraic groups and R-equivalence and work of Krashen relating homogeneous varieties arising from exceptional isomorphisms.

The theory of algebraic cycles on homogenous varieties has seen many useful applications to the study of central simple algebras, quadratic forms, and Galois cohomology. Significant results include the Merkurjev-Suslin Theorem, the Milnor and Bloch-Kato Conjectures (now theorems of Orlov-Vishik-Voevodsky and Rost-Voevodsky-Weibel) [Voe], and Suslin's Conjecture, recently proven by Merkurjev [Mer14]. These results have been put in the context of $\mathbb{A}^1$-homotopy and motivic cohomology, giving computations in K-theory and K-cohomology a healthy amount of relevance in modern mathematics.

Despite these successes, a general description of Chow groups (with coefficients) remains elusive, and computations of these groups are done in various cases. The computations given in this thesis are a continuation of the work of e.g. Chernousov, Karpenko, Krashen, Merkurjev, Suslin, Swan, and Zainoulline, who have computed K-cohomology groups of various homogeneous varieties [Bro, CGM, CM01, CM06, Kra10, Mer95, Mer, MS92, PSZ, Swa]. Many of these computations concern K_0-zero-cycles, i.e., Chow groups of zero cycles modulo rational equivalence, but in general, very few results on the group of K_1-zero-cycles are known.

The study of K-theory goes back to Grothendieck. Higher K-theory and K-cohomology were defined by Quillen in his seminal paper [Qui], where the first functorial definition of higher K-groups was given. It was in this paper that Quillen exhibited a spectral sequence which computes the K-groups of a scheme in terms of K-cohomology groups. Computations of K-groups and K-cohomology groups were then taken up by a number of researchers over the next two decades (e.g., Levine, Merkurjev, Sherman, Srinivas, Suslin, Weyman).

Along with their celebrated result on the norm-residue homomorphism [MS82], Merkurjev and Suslin computed the group of K_1-zero-cycles of a Severi-Brauer variety in terms of the first K-group of the underlying central simple algebra [MS92]. One useful interpretation of this result is that the pointed maximal subfields of a central simple algebra A, which are parametrized by a functorially defined K-cohomology group, are likewise encoded by the group $K_1(A)$, given by the abelianized arithmetic of A.

The aim of this present work is to extend this result to *generalized* Severi-Brauer varieties. Indeed, we have succeeded in giving an explicit description of the group of K_1-zero-cycles on the second generalized Severi-Brauer variety of an algebra of index 4, with a mild restriction on the characteristic of the center.

Theorem (5.4.2)**.** *Let A be a central simple algebra of index 4 over a field F of characteristic not 2, and let X be the second generalized Severi-Brauer variety of A. The group $A_0(X, K_1)$ can be identified as the group of pairs $(x, \alpha) \in K_1(A) \times F^\times$ satisfying* $\mathrm{Nrd}_A(x) = \alpha^2$.

By giving a description of the group of K_1-zero-cycles for such varieties, we aim to explore the more subtle question concerning the parametrization of pointed intermediate subfields of a central simple algebra, i.e., those fields which lie strictly between a maximal subfield and the center of our algebra. Furthermore, such a study aids in our understanding of which elements may be realized as norms from these intermediate subfields.

This thesis is organized as follows. In Chapter 2 we provide background on the theory of central simple algebras over a field and involutions (order 2 anti-automorphisms) defined on them. The study of central simple algebras with involution should be viewed as a generalization or twisted analogue of classical linear algebra, with central simple algebras playing the role of matrix algebras and involutions playing the role of bilinear forms. As such, we may associate many similar or analogous invariants to aid in their study. Two such invariants are the Clifford and discriminant algebras associated to such algebras with involution. Classically, the Clifford algebra has played a large part in the study of quadratic forms and therefore symmetric bilinear forms when the characteristic of the ground field is not 2. The discriminant algebra is a natural construction which aids in the study of induced forms on exterior powers of a given vector space.

In Chapter 3 we define some algebraic groups and homogeneous varieties, i.e., varieties which carry an action of an algebraic group. Specifically, we define unitary groups, the Clifford group, and the spin[1] group associated to certain types of algebras with involution. Again, these are analogous of their classical counterparts which are defined in terms of forms on vectors spaces. We also present exceptional isomorphisms of certain invariants of groups of type A_3 and D_3. We then define generalized Severi-Brauer and involution varieties, both of which carry the action of an algebraic group. The former is in some sense the main object of our investigation, although it is completely determined by its underlying central simple algebra (and field of definition). We finish the chapter by introducing the notion of R-equivalence, which we use in the following chapter to relate algebraic groups to K-

[1]This group is often called the *spinor* group throughout the literature.

theory and K-cohomology groups. The concept of R-equivalence is an algebraization of a homotopy-theoretic concept, where paths (or embeddings $[0,1] \hookrightarrow X$) are given by rational morphisms $\mathbb{P}^1 \dashrightarrow X$, and may be defined in terms of valuations and specializations.

In Chapter 4 we introduce the tools we shall use to study Severi-Brauer varieties and in turn their underlying central simple algebras. We begin by defining two flavors of algebraic K-theory, namely those definitions given by Milnor and Quillen. Each definition has their advantages and appropriate uses. The first is concretely defined and admits study by elementary algebraic means. The second gives a functorial definition which can be utilized to study schemes and which can be put in the context of modern algebraic topology. We give a brief account of this perspective. This viewpoint has also been used to give Quillen K-theory a realization in terms of motivic homotopy. After giving a few examples of K-group computations, we introduce the Brown-Gersten-Quillen spectral sequence which provides a means of computing the K-theory of schemes in terms of K-cohomology groups. The study of K-cohomology was axiomatized by Markus Rost and we present a brief description of this beautiful theory. The study of "Chow groups with coefficients" allows one to define cohomology groups with coefficients in any "cycle module" analogous to varying coefficient systems in singular homology.

In Chapter 5 we give a full account of known results on the group of K_1-zero-cycles for (generalized) Severi-Brauer varieties. We begin with the statement and proof of the result of Merkurjev and Suslin which yields an isomorphism between the group of K_1-zero-cycles on a Severi-Brauer variety $\mathrm{SB}(A)$ and the first K-group $K_1(A)$ of the underlying central simple algebra. We then state a few theorems which exhibit the relationship between R-equivalence and computations of K-theory and K-cohomology groups. In particular, we present two results of Chernousov and Merkurjev computing K_1-zero-cycles from algebraic groups and R-equivalence. The first result recovers the aforementioned theorem of Merkurjev and Suslin and the second provides a similar isomorphism for involution varieties of small index. We then come to the main computation of this thesis, treating the case of the second generalized Severi-Brauer variety associated to a central simple algebra of index 4. We begin with a reduction to algebras of square degree, showing that the cycle computation respects Brauer classes. We then utilize results of Krashen to transfer our computation from type A_3 homogeneous varieties (generalized Severi-Brauer varieties) to type D_3 (involution varieties). This translation in conjunction with the results of Chernousov and Mekurjev yield a description of our desired cohomology groups in terms of algebraic groups. We complete our computation by giving an explicit description of these groups.

Chapter 2

Central Simple Algebras and Involutions

Over the past century, the topic of central simple algebras has had a great impact on a vast subset of mathematics. While seemingly simple objects, their study has in part generated the development of geometric and homotopical approaches to algebraic questions and is given by a beautiful blend of algebra and geometry. As we shall see in forthcoming chapters, one method of study relies on associating geometric objects which encode the one-sided ideal structure of a given algebra, analogous to the prime spectrum of a commutative ring. With these associated geometric objects in hand, we further attach algebraic invariants, such as groups and graded rings, to distinguish which structures may arise in this fashion.

The power and insight that algebraic geometry has granted to the study of commutative rings through scheme theory must be adjusted to fit these noncommutative objects. Indeed, the prime spectrum is not suitable for noncommutative rings as it ignores one-sided ideals. Fortunately, varieties do exist which encode this ideal structure and we will make great use of them below, even though they do not serve as a true parallel or dual theory as in the commutative case. To supplement this deficiency, we make use of linear algebra by introducing involutions and corresponding algebraic invariants.

2.1 Preliminaries on Central Simple Algebras

All algebras considered will be finite-dimensional, associative, and contain a multiplicative identity. For an F-algebra A and a field extension L/F, we denote $A \otimes_F L$ by A_L. If F_{sep} denotes a separable closure of F, we write A_{sep} for $A \otimes_F F_{\mathrm{sep}}$. The *center* of an algebra A is defined to be

$$Z(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$$

and is a commutative subalgebra of A. The *opposite algebra* of A is a central simple algebra over F given by $A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\}$, with addition defined by $a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}}$, multiplication by $a^{\mathrm{op}} \cdot b^{\mathrm{op}} = (ba)^{\mathrm{op}}$, and scalar multiplication by $\alpha a^{\mathrm{op}} = (\alpha a)^{\mathrm{op}}$, for $a, b \in A$ and $\alpha \in F$. An algebra is a *division algebra* if every non-zero element is invertible.

Definition 2.1.1. A *central simple algebra* over a field F is an F-algebra with no two-sided ideals other than (0) and (1) and whose center is precisely F.

Example 2.1.2. The matrix algebra $M_n(F)$ is a central simple F-algebra.

Example 2.1.3. Any division algebra D satisfying $Z(D) = F$ is a central simple F-algebra.

Example 2.1.4. For $a, b \in F^\times$, let $(a, b)_F$ denote the *generalized quaternion algebra*. It is an F-algebra generated by $1, i, j, ij$ which satisfy the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$. It is a central simple F-algebra of F-dimension 4. Taking $F = \mathbb{R}$ and $a = b = -1$, we recover Hamilton's quaternions.

Example 2.1.5. More generally, let $a, b \in F^\times$ and suppose that F contains a copy of the group μ_n of n^{th} roots of 1. Let ζ be a primitive n^{th} root of unity. Let $A_\zeta(a, b)$ be the algebra given by generators i and j satisfying $i^n = a$, $j^n = b$, and $ij = \zeta ji$. Then $A_\zeta(a, b)$ is a central simple algebra of F-dimension n^2 and is one example of a *cyclic algebra* since it splits over the cyclic extension $F(\sqrt[n]{b})$.

Example 2.1.6. Even more generally, let L/F be a cyclic extension of degree n with Galois group $\mathrm{Gal}(L/F)$ generated by σ. Let $b \in F^\times$. Let $A(\sigma, b)$ be the algebra with basis $1, X, X^2, ..., X^{n-1}$ satisfying $X^n = b$ and $X\alpha = \sigma(\alpha)X$ for $\alpha \in L$ in addition to

$$X^i X^j = \begin{cases} X^{i+j} & \text{if } 1 \leq i, j \text{ and } i+j < n \\ bX^{i+j-n} & \text{if } 1 \leq i, j \leq n-1 \text{ and } i+j \geq n \end{cases}$$

Then $A(\sigma, b)$ is the *cyclic algebra* associated to L/F, $\sigma \in \mathrm{Gal}(L/F)$ and $b \in F^\times$.

Theorem 2.1.7 (Wedderburn). *For an algebra A over a field F, the following are equivalent:*

1. *A is central simple.*

2. *There is a finite-dimensional central division algebra over F and an integer n such that $A \cong M_n(D)$.*

3. *If K/F is an algebraically closed field, then $A_K \cong M_n(K)$ for some n.*

4. *There is a field extension L/F such that $A_L \cong M_n(L)$ for some n.*

Moreover, if any of these conditions hold, all simple left (or right) A-modules are isomorphic, and D is given uniquely up to isomorphism as $D = \mathrm{End}_A(M)$.

Proof. See [KMRT, Theorem 1.A] for references. □

One corollary of the above result is that the dimension of a central simple algebra is a square, and we define the *degree* of A to be $\deg(A) = \sqrt{\dim A}$. We define the *index* of A to be $\mathrm{ind}(A) = \deg(D)$. We say two central simple algebras over F are *Brauer-equivalent* if their underlying division algebras are isomorphic. Any field L satisfying $A_L \cong M_n(L)$ is called a *splitting field* of A. An algebra which is isomorphic to a matrix algebra over a field is called *split*.

A basis-free version of Wedderburn's Theorem states that for every central simple algebra A, there is a division algebra, unique up to isomorphism, and a D-vector space V so that $A \cong \mathrm{End}_D(V)$. In what follows, it will be useful to obtain a description of right ideals of A in terms of D-subspaces of V [KMRT, §1].

Any D-subspace $U \subset V$ determines a right ideal $\mathrm{Hom}_D(V, U) \subset \mathrm{End}_D(V)$. Indeed, to any map $V \to U$, composing with $U \hookrightarrow V$ defines an endomorphism of V which is an element of $A = \mathrm{End}_D(V)$. We may thus identify

$$\mathrm{Hom}_D(V, U) = \{f \in \mathrm{End}_D(V) \mid \mathrm{Im}(f) \subset U\}.$$

For any map $f \in \mathrm{Hom}_D(V, U)$ and any map $g \in \mathrm{End}_D(V)$, the composite $f \circ g$ is an element of $\mathrm{Hom}_D(V, U)$ since we clearly have $\mathrm{Im}(f \circ g) \subset U$. Thus, $\mathrm{Hom}_D(V, U)$ is a right ideal of $\mathrm{End}_D(V)$.

Proposition 2.1.8 ([KMRT], Prop. 1.12)**.** *The map $U \mapsto \mathrm{Hom}_D(V, U)$ defines a one-to-one correspondence between subspaces of V of dimension d and right ideals of reduced dimension $d\,\mathrm{ind}(A)$ in $A = \mathrm{End}_D(V)$.*

The *reduced dimension* of an ideal $I \subset A$ is given by $\mathrm{rdim}(I) = \dim(I)/\deg(A)$ (see Definition 3.4.1).

Given a right ideal $I \subset A$, we may write $I = \mathrm{Hom}_D(V, U)$ for some D-subspace $U \subset V$. Choosing a complementary subspace U' in V, so that $V = U \oplus U'$, let $e : V \to U$ be the projection onto U parallel to U'. Notice that $e \in \mathrm{End}_D(V) = A$ is an idempotent element, i.e., $e^2 = e \circ e = e$. Using this prescription, we obtain the following result.

Proposition 2.1.9 ([KMRT], Cor. 1.13)**.** *For every right ideal $I \subset A$ there exists an idempotent $e \in A$ such that $I = eA$.*

While matrix algebras constitute a nice subcollection of central simple algebras, it is useful to view central simple algebras as generalized matrix algebras. Indeed, they enjoy many features present in matrix algebras, and much of the theory of linear algebra carries over to central simple algebras (with involution). In particular, there are analogues of the determinant (called the reduced norm) and the trace (called the reduced trace), which we now discuss, following [KMRT, 1.6].

For any central simple F-algebra A of degree n, extension of scalars gives an isomorphism $A_{\mathrm{sep}} \cong M_n(F_{\mathrm{sep}})$. The association $a \mapsto a \otimes 1 \in A_{\mathrm{sep}}$ thus defines a map $A \hookrightarrow M_n(F_{\mathrm{sep}})$. This map is injective since A is simple, so we may view every element $a \in A$ as a matrix in $M_n(F^{\mathrm{sep}})$.

Definition 2.1.10. The *reduced characteristic polynomial* of a

$$\mathrm{Prd}_{A,a}(x) = x^n - s_1(a)x^{n-1} + s_2(a)x^{n-1} - \cdots + (-1)^n s_n(a)$$

is the characteristic polynomial of the matrix representation of a described above. We define the *reduced norm* $\mathrm{Nrd}_A(a)$ and the *reduced trace* $\mathrm{Trd}_A(a)$ of a to be

$$\mathrm{Nrd}_A(a) = s_n(a) \qquad\qquad \mathrm{Trd}_A(a) = s_1(a).$$

This same definition is valid using any splitting field of A [Wei, III.1.2.4]. Indeed, as discussed in [Mer92, §2], the reduced norm may be defined by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Nrd}_A"] \arrow[d] & F \arrow[d] \\ A_L \arrow[r, "\det"] & L \end{tikzcd}$$

for any algebra A and any splitting field L/F of A.

The association $a \mapsto \mathrm{Nrd}_A(a)$ defines a homomorphism $\mathrm{Nrd}_A : A^\times \to F^\times$, which clearly factors through the abelianization $A^\times/[A^\times, A^\times] = K_1(A)$ (see Example 4.3.4 below). The resulting map $K_1(A) \to F^\times$ is also denoted by Nrd_A and its kernel by $SK_1(A)$. The reduced norm and the group $SK_1(A)$ have been successfully used to study division algebras and their K-groups. See [Mer92] for a good overview. One particularly nice property is that the reduced norm is compatible with field norms. If L is a subfield of a central simple algebra A satisfying $[L : F] = \deg(A)$, then we have the following commutative diagram, where the map $L^\times = K_1(L) \to K_1(A)$ is induced by the inclusion of L into A:

$$\begin{tikzcd} K_1(L) \arrow[rr] \arrow[dr, "N_{L/F}"'] & & K_1(A) \arrow[dl, "\mathrm{Nrd}_A"] \\ & K_1(F) & \end{tikzcd}$$

2.2 Involutions

In order to transfer a greater number of linear algebraic constructions to the study of central simple algebras, we define analogues of bilinear and quadratic forms on vector spaces.

Definition 2.2.1. An *algebra with involution* is a pair (A, σ) where A is a central simple algebra and $\sigma : A \to A$ is an anti-automorphism satisfying $\sigma^2 = \mathrm{id}_A$.

If $A = \mathrm{End}_F(V)$, then an involution σ on A is the same data as a bilinear form b_σ on V, so that central simple algebras with involution are twisted analogues of vectors spaces and bilinear forms (see [KMRT, Ch. I]). An involution *of the first kind* satisfies $\sigma|_F = \mathrm{id}_F$, while an involution *of the second kind* induces a nontrivial degree 2 automorphism of F. We refer to involutions of the second kind as *unitary involutions*. An involution of the the first kind which is a twisted form of a symmetric bilinear form is *orthogonal*. Otherwise, it is *symplectic*.

If $\mathrm{char}(F) \neq 2$, symmetric bilinear forms and quadratic forms determine one another, so that orthogonal involutions simultaneously give twisted versions of bilinear and quadratic forms. For $\mathrm{char}(F) = 2$, quadratic and bilinear forms do not coincide, and one must make use of *quadratic pairs* [KMRT, §5.B] to define twisted analogues of quadratic forms.

For an algebra (A, σ) with involution of the first kind let

$$\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}, \quad \mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}.$$

Definition 2.2.2. A *quadratic pair* on a central simple algebra A is a pair (σ, f) where σ is an involution of the first kind on A and $f : \mathrm{Sym}(A, \sigma) \to F$ is a linear map satisfying

1. $\dim_F(\mathrm{Sym}(A, \sigma)) = \frac{n(n+1)}{2}$ and $\mathrm{Trd}_A(\mathrm{Skew}(A, \sigma)) = \{0\}$

2. $f(x + \sigma(x)) = \mathrm{Trd}_A(x)$ for all $x \in A$.

Notice that if $\mathrm{char}(F) \neq 2$, for any $a \in \mathrm{Sym}(A, \sigma)$, we may write $a = \frac{1}{2}(a + \sigma(a))$. The map f is thus uniquely determined by

$$f(a) = f(\tfrac{1}{2}(a + \sigma(a)) = \tfrac{1}{2}f(a + \sigma(a)) = \tfrac{1}{2}\mathrm{Trd}_A(a).$$

Furthermore, the involution σ must be orthogonal [KMRT, p. 56]. This shows that algebras with quadratic pair are simply algebras with orthogonal involution over fields of characteristic not 2.

It will be useful to extend the notion of “unitary involution” to include semi-simple F-algebras of the form $A_1 \times A_2$, where each A_i is a central simple over F. The center L of an algebra with unitary involution (A, σ) will generally be an étale quadratic extension of F, i.e., either $L \cong F \times F$ or L/F is a separable quadratic field extension. In the first case, $A \cong A_1 \times A_2$ as above, and in the second case A is a central simple algebra over L. We will refer to such an algebra as a *central simple algebra with unitary involution*, even though the algebra is not necessarily simple and its center is not F (see introduction to [KMRT, §2.B]).

Proposition 2.2.3 ([KMRT], Prop. 2.14)**.** *Let (A, σ) be a central simple F-algebra with involution of the second kind with center $L \cong F \times F$. Then there is a central simple F-algebra E such that $(A, \sigma) \cong (E \times E^{\mathrm{op}}, \varepsilon)$, where the involution ε is defined by $\varepsilon(x, y^{\mathrm{op}}) = (y, x^{\mathrm{op}})$ and called the exchange involution.*

2.3 The Clifford Algebra

Given an algebra with quadratic pair (A, σ, f), the *Clifford algebra* $C(A, \sigma, f)$ is an F-algebra which is a quotient of the tensor algebra of A. Its multiplication is defined in terms of the quadratic pair (σ, f) and it is a twisted form of the even Clifford algebra associated to a quadratic space. Together with its canonical involution, the Clifford algebra enjoys the structure of a central simple algebra with unitary, orthogonal, or symplectic involution, depending on its degree and the characteristic of F (see Proposition 2.3.3). We give a brief definition, and refer the reader to [KMRT, §8.B] for a complete account of the Clifford algebra.

Let $\underline{A}$ denote the underlying vector space of the algebra A. The "sandwich" isomorphism

$$\mathrm{Sand} : \underline{A} \otimes \underline{A} \xrightarrow{\sim} \mathrm{End}_F(\underline{A})$$

is defined as $\mathrm{Sand}(a \otimes b)(x) = axb$ for $a, x, b \in A$. We define $\sigma_2 : \underline{A} \otimes \underline{A} \to \underline{A} \otimes \underline{A}$ by the following prescription. For any fixed $u \in \underline{A} \otimes \underline{A}$, the association

$$x \mapsto \mathrm{Sand}(u)(\sigma(x))$$

defines a linear map $\varphi_{u,\sigma} : A \to A$, so can be realized as an element of $\mathrm{End}_F(\underline{A})$. The sandwich isomorphism identifies $\mathrm{End}_F(\underline{A})$ with $\underline{A} \otimes \underline{A}$, and so there is a corresponding element of $\underline{A} \otimes \underline{A}$, which we denote $\sigma_2(u)$. That is, σ_2 is a map defined by the adjoint relation

$$\mathrm{Sand}(u)(\sigma(x)) = \mathrm{Sand}(\sigma_2(u))(x).$$

Definition 2.3.1. The *Clifford algebra* $C(A, \sigma, f)$ is the quotient algebra of the tensor algebra $T(\underline{A})$ of $\underline{A}$:

$$C(A, \sigma, f) = \frac{T(\underline{A})}{J_1(\sigma, f) + J_2(\sigma, f)}.$$

The ideal $J_1(\sigma, f)$ is generated by elements of the form $s - f(s)$ where $s \in \mathrm{Sym}(A, \sigma)$. The ideal $J_2(\sigma, f)$ is generated by elements of the form $u - \mathrm{Sand}(u)(\eta)$ such that $\sigma_2(u) = u$, and where η is an element of $\underline{A}$ satisfying $f(s) = \mathrm{Trd}_A(\eta s)$ for all elements $s \in \mathrm{Sym}(A, \sigma)$.

Remark 2.3.2. For $\mathrm{char}(F) \neq 2$, the map f is completely determined by σ, and we write $C(A, \sigma)$ for $C(A, \sigma, f)$.

The Clifford algebra comes equipped with a so-called canonical involution induced by the involution σ. Define this involution $\underline{\sigma} : T(\underline{A}) \to T(\underline{A})$ via

$$\underline{\sigma}(a_1 \otimes \cdots \otimes a_n) = \sigma(a_n) \otimes \sigma(a_{n-1}) \otimes \cdots \sigma(a_1).$$

One checks that the ideals $J_1(\sigma, f)$ and $J_2(\sigma, f)$ are mapped to themselves under $\underline{\sigma}$, so that we obtain the desired involution, which we also denoted by $\underline{\sigma}$.

The pair $(C(A, \sigma, f), \underline{\sigma})$ enjoys a variety of structures depending on the degree of A and the characteristic of F.

Proposition 2.3.3 ([KMRT], Prop. 8.12)**.** *Let (A, σ, f) be a central simple algebra with quadratic pair of degree $n = 2m$ over a field F. The canonical involution $\underline{\sigma}$ on $C(A, \sigma, f)$ is*

1. *unitary if m is odd,*
2. *orthogonal if $m \equiv 0 \pmod 4$, and $\mathrm{char}(F) \neq 2$,*
3. *symplectic if $m \equiv 2 \pmod 4$ or $\mathrm{char}(F) = 2$.*

2.4 The Discriminant Algebra

If (A, σ) is an algebra with unitary involution of degree $n = 2m$, the associated *discriminant algebra* $D(A, \sigma)$, is a central simple algebra of degree $\binom{n}{m}$ which plays the role of the exterior algebra. Indeed, extending scalars to F_{sep} yields $A_{\mathrm{sep}} \cong \mathrm{End}_{F^{\mathrm{sep}}}(V)$ and $D(A, \sigma)_{\mathrm{sep}} = \mathrm{End}_{F_{\mathrm{sep}}}(\bigwedge^m V)$. Just as in the case of the Clifford algebra, the discriminant algebra comes equipped with a so-called canonical orthogonal involution, denoted $\underline{\sigma}$. We give an abbreviated discussion of the discriminant algebra, referring the reader to [KMRT, §10] and specifically §10.E for the definition. We begin by defining λ-powers of central simple algebras and collecting some facts about their structure.

Definition 2.4.1. Let A be a central simple algebra of degree n over a field F. Let $\lambda^1 A = A$ and for every integer $k = 2, ..., n$, define the k^{th} λ-*power* of A to be

$$\lambda^k A = \operatorname{End}_{A^{\otimes k}}(A^{\otimes k} s_k),$$

where the elements $s_k = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) g_k(\pi)$ are defined as "averages" over elements of the symmetric group. Here $g_k : S_k \to (A^{\otimes k})^\times$ is a group homomorphism from the symmetric group into the group of units of $A^{\otimes k}$ [KMRT, §10.A].

Proposition 2.4.2. *The λ-powers satisfy the following properties:*

1. *The algebra $\lambda^k A$ is central simple over F, Brauer-equivalent to $A^{\otimes k}$, and has degree* $\deg \lambda^k A = \binom{n}{k}$.
2. *There is a natural isomorphism $\lambda^k \operatorname{End}_F(V) = \operatorname{End}_F(\bigwedge^k V)$.*

The λ-powers of a central simple algebra come equipped with canonical involutions, which we now describe. Recall that A is an algebra of degree $n = 2m$. By the definition of $\lambda^m A$, we have a natural isomorphism

$$\lambda^m A \otimes_F \lambda^m A = \operatorname{End}_{A^{\otimes n}}(A^{\otimes n}(s_m \otimes s_m)).$$

In fact, the element $s_n \in \lambda^n A$ lies in the ideal $A^{\otimes n}(s_m \otimes s_m)$ [KMRT, Lem. 10.7], and we may consider the right ideal $I = \{f \in \operatorname{End}_{A^{\otimes n}}(A^{\otimes n}(s_m \otimes s_m)) \mid f(s_n) = \{0\}\}$. Of course, under the isomorphism stated above, this is an ideal of $(\lambda^m A)^{\otimes 2}$.

Definition 2.4.3. Let A be a central simple algebra of degree $n = 2m$. The *canonical involution* γ on $\lambda^m A$ is the involution of the first kind corresponding to the ideal I under the correspondence of F-linear anti-automorphisms of $\lambda^m A$ and ideals of $(\lambda^m A)^{\otimes 2}$ satisfying $(\lambda^m A)^{\otimes 2} = I \oplus (1 \otimes \lambda^m A)$ [KMRT, Thm. 3.8].

Remark 2.4.4. If $A = \mathrm{End}_F(V)$, the canonical involution γ on $\lambda^m A = \mathrm{End}_F(\bigwedge V)$ is the adjoint involution associated to the bilinear pairing $\bigwedge^m V \times \bigwedge^m V \to \bigwedge^n V \cong F$.

To any involution τ on A satisfying $(\tau|_F)^2 = \mathrm{id}_F$, one may associate an involution $\tau^{\wedge k}$ on $\lambda^k A$. We omit the specifics on the existence of these involutions and instead refer the reader to [KMRT, §10.D] for a full treatment.

The involution $\sigma^{\wedge m}$ induced by σ commutes with the canonical involution γ [KMRT, Lemma 10.27], so that $\theta := \sigma^{\wedge m} \circ \gamma$ defines an order 2 automorphism of $\lambda^m A$.

Definition 2.4.5. Let (A, σ) be a central simple algebra with involution of the second kind, and assume that $\deg(A) = n = 2m$. The associated *discriminant algebra* $D(A, \sigma)$ is the F-subalgebra of θ-invariant elements of $\lambda^m A$. It is a central simple algebra of degree $\binom{n}{m}$

The involutions γ and $\sigma^{\wedge m}$ are identical when restricted to the discriminant algebra, and we denote the common involution by $\underline{\sigma}$, continuing to refer to it as the canonical involution on $D(A, \sigma)$.

Remark 2.4.6. We make specific mention of the case $L = F \times F$ for use in the proof of Proposition 5.4.1. In this case, $A = E \times E^{\mathrm{op}}$ for a central simple algebra E over F of degree $2m$ and σ is given by the exchange involution ε (see Proposition 2.2.3). By the discussion preceding [KMRT, 10.31], we have an isomorphism

$$D(A, \varepsilon) = \{(x, \gamma(x)^{\mathrm{op}}) \mid x \in \lambda^m(E)\} \cong \lambda^m E.$$

We note, in particular, that by Proposition 2.4.2, the discriminant algebra $D(A, \varepsilon)$ is Brauer-equivalent to $E^{\otimes m}$. Its canonical involution will be denoted by $\underline{\varepsilon}$, as opposed to $\underline{\sigma}$.

Chapter 3

Algebraic Groups and Homogeneous Varieties

By a *scheme* over F we will mean a separated scheme of finite type over F. An *algebraic variety* is an integral scheme. We will freely use the terminology and notation found in [Har, Hart] concerning, e.g., schemes, varieties, function fields, rational points, rational morphisms, and birationality. If X and Y are F-schemes, we will denote the product $X \times_{\operatorname{Spec} F} Y$ by $X \times Y$ and occasionally by X_Y.

Let F be a field. An *algebraic group* G over F is a smooth affine scheme with a group structure that is compatible with its geometric structure. That is, the multiplication and inverse maps $m : G \times G \to G$ and $\iota : G \to G$ defined by $m(g, h) = gh$ and $\iota(x) = x^{-1}$ are morphisms of schemes. Alternatively, an algebraic group is a group object in the category of smooth affine schemes over F.

Using the associated *functor of points*, schemes and varieties may be viewed as functors with domain the category of commutative rings. We will make use of this viewpoint, more often restricting our attention to fields. In particular, we frequently consider an algebraic group G as a functor $L \mapsto G(L)$ from the category of field extensions of F to the category of groups. Following [CM01], we will utilize $\mathrm{GL}_1(A)$, $\mathrm{SL}_1(A)$, $\mathrm{Spin}(A, \sigma)$, etc., to denote the groups of F-points of the corresponding algebraic groups $\mathbf{GL}_1(A)$, $\mathbf{SL}_1(A)$, $\mathbf{Spin}(A, \sigma)$, etc. Our main focus will be on these collections of F-points as we often suppress the full algebraic group structure, although we will continue to refer to these groups as “algebraic groups.” We only make proper use of this extra structure in the discussion on the role of R-equivalence in K-cohomology computations in Section 5.2. A good reference for the following material is [KMRT].

Let A be a central simple algebra over F, and consider the group $\mathrm{GL}_1(A) := A^\times$ of invertible elements in A, called the *general linear group* of A. The kernel of the reduced norm homomorphism (see Definition 2.1.10) $\mathrm{Nrd}_A : \mathrm{GL}_1(A) \to F^\times$ is denoted $\mathrm{SL}_1(A)$, called the *special linear group* of A.

3.1 Unitary Groups

Let (A, σ) be a central simple algebra with involution.

Definition 3.1.1. A *similitude* of (A, σ) is an element $g \in A^\times$ such that $\sigma(g)g \in F^\times$. The collection of all similitudes of (A, σ) is denoted $\mathrm{Sim}(A, \sigma)$ and is a subgroup of $\mathrm{GL}_1(A)$. The scalar $\mu(g) := \sigma(g)g$ is called the *multiplier* of g, and the association $g \mapsto \mu(g)$ defines a group homomorphism $\mu : \mathrm{Sim}(A, \sigma) \to F^\times$.

If the involution σ is of unitary type, we denote $\mathrm{Sim}(A, \sigma)$ by $\mathrm{GU}(A, \sigma)$, and call it the *general unitary group* of (A, σ). For such a pair (A, σ) and with $\deg(A) = 2m$, define the *special general unitary* and *special unitary* groups

$$\mathrm{SGU}(A, \sigma) = \{g \in \mathrm{GU}(A, \sigma) \mid \mathrm{Nrd}_A(g) = \mu(g)^m\}$$

$$\mathrm{SU}(A, \sigma) = \{u \in \mathrm{GU}(A, \sigma) \mid \mathrm{Nrd}_A(u) = 1\}.$$

Remark 3.1.2. In the case where A has center $L \cong F \times F$, there is a central simple algebra E over F such that $(A, \sigma) \cong (E \times E^{\mathrm{op}}, \varepsilon)$, where ε is the exchange involution (Proposition 2.2.3).

The special general, and special unitary groups of (A, σ) are then given by [KMRT, §14.2]

$$\mathrm{SGU}(E \times E^{\mathrm{op}}, \varepsilon) = \{(x, \alpha) \in E^\times \times F^\times \mid \mathrm{Nrd}_E(x) = \alpha^m\}$$

$$\mathrm{SU}(E \times E^{\mathrm{op}}, \varepsilon) = \{x \in E^\times \mid \mathrm{Nrd}_E(x) = 1\} = \mathrm{SL}_1(E).$$

3.2 The Clifford and Spin Groups

The Clifford Group

The multiplicative group of the Clifford algebra $C(A, \sigma)$ contains a group $\Gamma(A, \sigma)$, called the *Clifford group*, whose action on $C(A, \sigma)$ fixes $A \subset C(A, \sigma)$. The definition of this action takes some care to define, and we give a brief discussion following [KMRT, §13].

Let (A, σ, f) be a central simple algebra with quadratic pair. There is a linear map $\gamma = \oplus \gamma_n : T(\underline{A}) \to T(\underline{A})$ induced from a representation $S_{2n} \to \mathrm{GL}(A^{\otimes n})$ [KMRT, Prop. 9.4]. In the split case, where $(A, \sigma, f) = (\mathrm{End}_F(V), \sigma_q, f_q)$, the identification of A with $V \otimes_F V$ gives a defition of γ:

$$\gamma(v_1 \otimes \cdots \otimes v_{2n}) = \gamma_n(v_1 \otimes \cdots \otimes v_{2n}) = v_2 \otimes \cdots \otimes v_{2n} \otimes v_1.$$

Consider the subset $T_+(\underline{A})$ of the tensor algebra, given by $T_+(\underline{A}) = \bigoplus_{n \geq 1} A^{\otimes n}$. The vector space $T_+(\underline{A})$ is naturally a left and right $T(\underline{A})$-module, using the multiplication of the tensor algebra. Let us consider a new left module structure of $T_+(\underline{A})$. For $\alpha \in T(\underline{A})$ and $v \in T_+(\underline{A})$, define $\alpha * v = \gamma^{-1}(u \otimes \gamma(v))$.

Again, we may describe this map explicitly in the split case. Identifying A and $V \otimes_F V$, we have $(\alpha_1 \otimes \cdots \otimes \alpha_{2i}) * (v_1 \otimes \cdots \otimes v_{2j}) = v_1 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{2i} \otimes v_2 \otimes \cdots \otimes v_{2j}$. Notice that this is just the usual product in $T(\underline{A})$ except v_1 is transplanted to the font of the product.

Definition 3.2.1. The *Clifford bimodule* of (A, σ, f) is

$$B(A, \sigma, f) = \frac{T_+(\underline{A})}{[J_1(\sigma, f) * T_+(\underline{A})] + [T_+(\underline{A}) \cdot J_1(\sigma, f)]}.$$

Here, the ideal $J_1(\sigma, f)$ is the two-sided ideal of $T(\underline{A})$ given in Definition 2.3.1.

For any $a \in A$, we can realize a as a degree 1 element of $T_+(\underline{A})$. This defines a map $b : A \to B(A, \sigma, f)$. The following result shows that this map recovers some of the essential structure of A and gives some nice properties of $B(A, \sigma, f)$.

Theorem 3.2.2 ([KMRT], Theorem 9.7)**.** *Let (A, σ, f) be a central simple F-algebra with quadratic pair.*

1. *The F-vector space $B(A, \sigma, f)$ carries a natural $C(A, \sigma, f)$-bimodule structure, where the left action is given by $*$, as well as a natural left A-module structure.*

2. *The canonical map $b : A \to B(A, \sigma, f)$ is an injective homomorphism of left A-modules.*

Definition 3.2.3 ([KMRT], Def. 13.11)**.** The *Clifford group* $\Gamma(A, \sigma, f)$ is defined by

$$\Gamma(A, \sigma, f) = \{c \in C(A, \sigma, f)^\times \mid c^{-1} * A^b \cdot c \subset A^b\}.$$

where A^b denotes the image of A under the map $b : A \to B(A, \sigma, f)$.

Thus, the Clifford group is given by those invertible elements of the Clifford algebra which conjugate A (as a module) into itself. Let us compare this to the classical case. For a nonsingular quadratic space (V, q), the *special Clifford group* $\Gamma^+(V, q)$ is given by

$$\Gamma^+(V, q) = \{c \in C_0(V, q)^\times \mid c \cdot V \cdot c^{-1} \subset V\}.$$

Of course, the Clifford algebra (as defined in Section 2.3) is a twisted form of the even Clifford algebra $C_0(V, q)$. This invariance under the action of the Clifford algebra translates to the underlying vector space in this case.

The Spin Group

We refer the reader to [KMRT, § 8, 13] for all pertinent definitions. As discussed in Section 2.3, the Clifford algebra $C(A, \sigma, f)$ comes equipped with its canonical involution $\underline{\sigma}$. There is a corresponding multiplier map $\underline{\mu} : C(A, \sigma, f) \to F^\times$ defined by $\underline{\mu}(g) = \underline{\sigma}(g)g$ (Definition 3.1.1). Denote the restriction of this multiplier map to the Clifford group $\Gamma(A, \sigma, f) \subset C(A, \sigma, f)$ also by $\underline{\mu}$.

Definition 3.2.4. The *spin group* $\mathrm{Spin}(A, \sigma, f)$ is defined to be the kernel of the multiplier map $\underline{\mu}$, i.e.,

$$\mathrm{Spin}(A, \sigma, f) = \{\gamma \in \Gamma(A, \sigma, f) \mid \underline{\mu}(\gamma) = 1\} \subset \Gamma(A, \sigma, f)$$

3.3 Exceptional Identifications

We now wish to make precise the relationship between the Clifford, spin, and unitary groups arising from the exceptional identifications of algebras of types A_3 and D_3. We summarize the discussion on equivalences of groupoids found in [KMRT, §15.D]. We will restrict to the case where $\mathrm{char}(F) \neq 2$ for the sake of clarity, as we only make use of the results of this section under this assumption.

Let A_3 denote the category of central simple algebras with unitary involution of degree 4 with morphisms given by maps of F-vector spaces. Let D_3 be the category of central simple algebras with orthogonal involution of degree 6 and maps of F-vector spaces. These categories are in fact *groupoids*, i.e., all morphisms are invertible, which follows from the fact that vector spaces of equal dimension are isomorphic. Let

$$\mathbf{C} : \mathsf{D}_3 \longrightarrow \mathsf{A}_3$$

be the functor which maps an algebra with orthogonal involution (A, σ) to its Clifford algebra with canonical involution $(C(A, \sigma), \underline{\sigma})$, as defined in Section 2.3. Let

$$\mathbf{D} : \mathsf{A}_3 \longrightarrow \mathsf{D}_3$$

be the functor which maps an algebra with unitary involution (B, τ) to its discriminant algebra with canonical orthogonal involution $(D(B, \tau), \underline{\tau})$, as defined in Section 2.4.

Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is an *equivalence* if F induces a bijection on morphism sets $\mathrm{Mor}_{\mathcal{A}}(X, X') \cong \mathrm{Mor}_{\mathcal{B}}(F(X), F(X'))$ (fully-faithful) and every object Y of $\mathcal{B}$ is isomorphic to an object of the form $F(X)$ for some object X of $\mathcal{A}$ (essentially surjective).

Theorem 3.3.1 ([KMRT], Theorem 15.24)**.** *The functors* $\mathbf{C}$ *and* $\mathbf{D}$ *define an equivalence of groupoids* $\mathsf{A}_3 \simeq \mathsf{D}_3$*.*

Under this equivalence, one may recover invariants of algebras with orthogonal involution by means of invariants of algebras with unitary involution.

Proposition 3.3.2 ([KMRT], Prop. 15.27)**.** *Let* $(A, \sigma) \in \mathsf{A}_3$ *and* $(B, \tau) \in \mathsf{D}_3$ *correspond to one another under the groupoid equivalence of* A_3 *and* D_3*. Then we have identifications* $\Gamma(A, \sigma) = \mathrm{SGU}(B, \tau)$ *and* $\mathrm{Spin}(A, \sigma) = \mathrm{SU}(B, \tau)$.

3.4 Severi-Brauer Varieties

The notion of generalized Severi-Brauer variety was first given by Blanchet in [Bla], and the following information may also be found in [KMRT, Kra10]. Let A be a central simple algebra of degree n. Consider the Grassmannian $\mathrm{Gr}(nk, A) = \mathrm{Gr}(nk, n^2)$ of nk-dimensional subspaces of A, where we consider A as an F-vector space of dimension n^2. Among the elements of $\mathrm{Gr}(nk, A)$ are the right ideals of A of dimension nk. The property of being a right ideal is given by polynomial equations [KMRT, 1.C], so that the collection $\mathrm{SB}_k(A)$ of all right ideals of A of dimension nk is closed subvariety of $\mathrm{Gr}(nk, A)$.

Definition 3.4.1. For any integer $1 \leq k \leq n = \deg(A)$, the k^{th} *generalized Severi-Brauer variety* $\mathrm{SB}_k(A)$ of A is the variety of right ideals of dimension nk in A. For an ideal $I \in \mathrm{SB}_k(A)$, the integer k is called the *reduced dimension* of I. Thus, $\mathrm{SB}_k(A)$ is the variety of right ideals of A of reduced dimension k. It is a homogeneous variety under the action of the algebraic group $\mathbf{SL}_1(A)$.

One may also associate Severi-Brauer varieties to ideals of a central simple algebra, as discussed in [Kra10, Def. 4.7]. For J a right ideal of A, define $\mathrm{SB}_k(J)$ to be the collection of right ideals of A of reduced dimension k which are contained in J. The variety $\mathrm{SB}_k(J)$ may be realized as the collection of ideals of an algebra closely related to A, as the following theorem and proceeding remark dictate.

Theorem 3.4.2 ([Kra10], Theorem 4.8)**.** *Let A be a central simple algebra and let $J \lhd A$ be a right ideal of reduced dimension k. Then there exists a degree k algebra D, Brauer-equivalent to A such that* $\mathrm{SB}_k(J) = \mathrm{SB}_k(D)$.

Remark 3.4.3. A porism of the above result is that the algebra D is given by eAe, where e is the idempotent element of A which corresponds to J (see Proposition 2.1.9).

Proposition 3.4.4 ([KMRT], Prop. 1.17)**.** *The Severi-Brauer variety* $\mathrm{SB}_k(A)$ *has a rational point over an extension L of F if and only if* $\mathrm{ind}(A_L) \mid k$*. In particular,* $\mathrm{SB}(A) = \mathrm{SB}_1(A)$ *has a rational point over L if and only if L splits A.*

In general, the algebra A does not necessarily split over L if $\mathrm{SB}_k(A)$ has an L-rational point. However, the existence of rational points guarantees that the geometry of $\mathrm{SB}_k(A)$ is still quite favorable.

Proposition 3.4.5 ([Bla], Prop. 3)**.** *The variety* $\mathrm{SB}_k(A)$ *has an L-rational point if and only if the free composite $L \cdot F(\mathrm{SB}_k(A))/L$ is rational (i.e., purely transcendental).*

Remark 3.4.6. Notice that the field $L \cdot F(\mathrm{SB}_k(A))$ is the function field of $\mathrm{SB}_k(A)_L$, considered as an L-variety. Thus, $\mathrm{SB}_k(A)$ has an L-rational point if and only if $\mathrm{SB}_k(A)_L$ is a rational L-variety, i.e., birational to $\mathbb{P}_L^{k(n-k)}$.

The generalized Severi-Brauer variety $\mathrm{SB}_k(A)$ is a *twisted form* of the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional subspaces of F^n, which will be made precise in the following theorem [Bla, Cor. 2(i)]. For $k = 1$, $\mathrm{SB}_1(A) = \mathrm{SB}(A)$ is the usual Severi-Brauer variety of A, which is a twisted form of projective space $\mathbb{P}^{n-1} \cong \mathrm{Gr}(1,n)$. It was first defined by Châtelet [Cha].

Theorem 3.4.7 ([KMRT], Theorem 1.18)**.** *For $A = \mathrm{End}_F(V)$, there is a natural isomorphism $\mathrm{SB}_k(A) \cong \mathrm{Gr}(k,V)$. In particular, for $k = 1$, $\mathrm{SB}(A) \cong \mathbb{P}(V)$.*

In view of this result, we can view Severi-Brauer varieties from a cohomological perspective. Since generalized Severi-Brauer varieties are twisted forms of Grassmannians, they are parametrized by the Galois cohomology set $H^1(F, \mathrm{Aut}(\mathrm{Gr}(k,n))$. A theorem of Chow gives a description for the automorphism group of the Grassmannian over F:

$$\mathrm{Aut}(\mathrm{Gr}(k,n)) = \begin{cases} \mathrm{PGL}_n(F) & \text{if } 2k \neq n \text{ or } k = 1 \\ \mathrm{PGL}_n(F) \times \mathbb{Z}/2 & \text{if } 2k = n \text{ and } k > 1 \end{cases}$$

In particular, we see that Severi-Brauer varieties are parametrized by $H^1(F, \mathrm{PGL}_n)$.

Example 3.4.8. As we have seen above, for $A = M_n(F)$, we have $\mathrm{SB}_k(A) = \mathrm{Gr}(k,n)$. In particular, $\mathrm{SB}(A) = \mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$.

Example 3.4.9. Let $A = (a,b)_F$ be a quaternion algebra (Example 2.1.4). Then $\mathrm{SB}(A)$ is given by $\{(x : y : z) \mid ax^2 + bx^2 - z^2 = 0\} \subset \mathbb{P}^2$.

Our main focus will be the case $k = 2$, the second generalized Severi-Brauer variety $\mathrm{SB}_2(A)$ associated to a central simple algebra, which is a form of the Grassmannian $\mathrm{Gr}(2,n)$.

3.5 Involution Varieties

Involution varieties were first defined in [Tao], and the following may also be found in [Kra10]. Let (A, σ) be an algebra of degree n with orthogonal involution and let I be a right ideal of

A. The *orthogonal ideal* of I with respect to σ is given by

$$I^{\perp} = \{x \in A \mid \sigma(x)y = 0 \text{ for } y \in I\}.$$

We say that an ideal I is *isotropic* if $I \subset I^{\perp}$. Alternatively, an ideal is isotropic if $\sigma(I) \cdot I = (0)$. Let $\mathrm{IV}(A, \sigma)$ denote the collection of isotropic right ideals of A of dimension n. We clearly have inclusions

$$\mathrm{IV}(A, \sigma) \subset \mathrm{SB}(A) \subset \mathrm{Gr}(n, A) = \mathrm{Gr}(n, n^2).$$

In fact, $\mathrm{IV}(A, \sigma)$ is a closed subvariety of $\mathrm{SB}(A)$, and the above inclusion morphism is defined by forgetting the isotropy condition. Just as $\mathrm{SB}(A)$ is a twisted form of projective space, $\mathrm{IV}(A, \sigma)$ is a twisted form of a projective quadric. Indeed, if (A, σ) is split (so that $A \cong \mathrm{End}_F(V)$), then σ is given by the adjoint involution corresponding to a bilinear form b_σ on V. Such a form defines a quadratic form q_σ, and the involution variety is given by $\mathrm{IV}(A, \sigma) = \{q_\sigma = 0\}$.

Involution varieties may also be viewed from a cohomological perspective. Since the automorphism group of a projective quadric is given by the projective orthogonal group PO_n, the cohomology set $H^1(F, \mathrm{PO}_n)$ parametrizes involution varieties $\mathrm{IV}(A, \sigma)$, with $\deg(A) = n$.

One may formulate a definition in characteristic 2 [CM01] utilizing quadratic pairs (see Definition 2.2.2). Let (A, σ, f) be a central simple algebra of degree n with quadratic pair. A right ideal $I \subset A$ is called *isotropic* if $\sigma(I) \cdot I = 0$ and if $f(a) = 0$ for all $I \cap \mathrm{Sym}(A, \sigma)$. We may then define the *involution variety* $\mathrm{IV}(A, \sigma, f)$ as the collection of all isotropic ideals of dimension n. If $\mathrm{char}(F) \neq 2$, then $\mathrm{IV}(A, \sigma, f) = \mathrm{IV}(A, \sigma)$. The involution variety is a homogeneous variety for the algebraic group $\mathbf{O}^+(A, \sigma, f)$ [KMRT, p. 351, §26].

In very special (indeed, exceptional) cases, the varieties defined above coincide. The following result gives one manifestation of the exceptional identifications of Section 3.3. Here we assume that the characteristic of F is not 2.

Lemma 3.5.1 ([Kra10], Lem. 6.5). *Let A be a central simple algebra of degree 4 over F. Then $\mathrm{SB}_2(A)$ is isomorphic to an involution variety $\mathrm{IV}(B, \sigma)$ corresponding to a degree 6 algebra with orthogonal involution.*

Remark 3.5.2. One porism of Lemma 3.5.1 is that the index of B is at most 2. This fact will allow us to utilize results of Chernousov-Merkurjev concerning K-cohomology of involution varieties (see Section 5.2).

3.6 *R*-Equivalence

We follow the discussion given in [CM01, §1.1]. For an irreducible F-variety Y and point $y \in Y$, consider the function field $F(Y)$ and local ring $\mathcal{O}_y$ at y. Given any other F-variety X, we have an inclusion $X(\mathcal{O}_y) \subset X(F(Y))$. We will say that an element $f \in X(F(Y))$ is *defined at y* if it lies in the subset $X(\mathcal{O}_y)$. The quotient $\mathcal{O}_y \to F(y)$ induces a map $X(\mathcal{O}_y) \to X(F(y))$. For an element f defined at y, we denote the image of f under this map by $f(y)$, and call it the *value of f at y*.

Taking $Y = \mathbb{P}^1_F$, with function field $F(t)$, an element $f \in X(F(t))$ is defined at a rational point $\alpha \in \mathbb{P}^1_F(F)$ if it is defined at the corresponding point of $\mathbb{P}^1_F$.

Definition 3.6.1. Let G be an algebraic group over F. An element $x \in G(F)$ is *R-trivial* if there is an element $f \in G(F(t))$ defined at $t = 0$ and $t = 1$ so that the value of f is the identity at 0 and x at 1, i.e., $f(0) = 1$ and $f(1) = x$.

More geometrically, a point $x \in G(F)$ is called R-trivial if there is a rational morphism $f : \mathbb{P}^1 \dashrightarrow G$, defined at 0 and 1, and with $f(0) = 1$ and $f(1) = x$. We can thus view R-trivial elements $x \in G(F)$ as those which can be connected via a rational curve to the identity element of $G(F)$.

The collection of all R-trivial elements of $G(F)$ is denoted $RG(F)$ and is a normal subgroup of $G(F)$. The quotient $G(F)/R = G(F)/RG(F)$ is called the group of R-equivalence classes (for G). Given any extension L/F, the group $G(L)/R$ is defined to be the group of

R-equivalence classes of the L-variety G_L. A group G is *R-trivial* if $G(L)/R$ is trivial for all extensions L/F.

Lemma 3.6.2 ([CM98], Lem. 1.2)**.** *Let H be a closed normal subgroup of an algebraic group G. Then the collection $RH(F)$ of R-trivial elements of H is a normal subgroup of $G(F)$.*

This lemma will help justify the results of Section 5.2. In particular, we will consider the cases where $G = \mathbf{GL}_1(A)$ and $H = \mathbf{SL}_1(A)$ (Theorem 5.2.6), in addition to $G = \mathbf{\Gamma}(A, \sigma, f)$ and $H = \mathbf{Spin}(A, \sigma, f)$ (Theorem 5.2.7).

Chapter 4

Algebraic K-Theory and K-Cohomology

Here we present the tools which will be used to study the aforementioned homogeneous varieties, obtained by associating algebraic invariants to these geometric objects. Information obtained in this way may in turn be used to study our original algebraic structures of interest, namely central simple algebras (with involution).

Two flavors of algebraic K-theory are defined below, using constructions given by Milnor and Quillen. Milnor K-theory of fields has a very simple definition and is given by explicit presentations in low degrees. Although its definition is claimed to be ad hoc, it arises naturally in the study of Galois and motivic cohomology, and we present one such manifestation. Quillen K-theory is defined functorially and gives a means of viewing algebraic K-theory through the lens of modern algebraic topology (as well as motivic homotopy theory), and a brief summary of this viewpoint is included. We then define K-cohomology, first in a geometric and sheaf-theoretic attitude, followed by a more axiomatic approach. Throughout this chapter, all rings and schemes are assumed to be noetherian.

4.1 Milnor $\boldsymbol{K}$-Theory

We now give some background on Milnor K-theory, which will play a critical role in the definitions of K-cohomology and the Rost complex. In a sense, Milnor K-theory is the most fundamental and simplest part of algebraic K-theory, and its utility and ubiquity continue to make it a useful object of study. Good references include [EKM, GS06].

Let F be a field. For each $n \in \mathbb{N}$, we may form the n-fold tensor product of $F^\times$, with the convention that $T^0(F) = \mathbb{Z}$:

$$T^n(F) = F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times.$$

Definition 4.1.1. The n^{th} Milnor K-group $K_n^M(F)$ is the quotient of $T^n(F)$ by the subgroup generated by elements of the form $a \otimes (1-a)$.

The expression $a \otimes (1-a)$ is not an element of $T^0(F)$ or $T^1(F)$, so no relations are imposed on these groups. It follows that $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$.

The Milnor K-theory ring $K_*^M(F)$ is the direct sum of the groups $K_n^M(F)$ over all $n \in \mathbb{N}$. It may also be described as the quotient

$$K_*^M(F) = T(F)/\langle a \otimes (1-a) \mid a \in F^\times \rangle$$

of the full tensor algebra $T(F) = \bigoplus T^n(F)$ by the ideal generated by elements of the form $a \otimes (1-a)$. We will denote the image of $a_1 \otimes \cdots \otimes a_n$ in this quotient by $\{a_1, ..., a_n\}$ and refer to these elements as *symbols*.

Example 4.1.2. As discussed above, $K_0^M(F) \cong \mathbb{Z}$ and $K_1^M(F) \cong F^\times$ for any field F.

Example 4.1.3 (Milnor K-Theory of Finite Fields)**.** For $n \geq 2$, $K_n^M(\mathbb{F}_q) = 0$, where $\mathbb{F}_q$ denotes the finite field with q elements. For a proof, see [GS06].

The the fact that $a \otimes (1-a) = 0$ in $K_*^M(F)$ is called the *Steinberg relation*, and naturally arises in the study of central simple algebras. Indeed, the generalized quaternion algebra (see

Example 2.1.4) $(a, 1-a)_F$ is split, i.e., isomorphic to a matrix algebra over F. To see this, we note that its Severi-Brauer variety is given by the solutions of $ax^2 + (1-a)y^2 - z^2 = 0$. This projective conic has rational point $(1:1:1)$, and is thus split, i.e., isomorphic to $\mathbb{P}^1$.

For what follows in Section 4.6 on K-cohomology, it will be useful to introduce the *residue homomorphisms* in Milnor K-theory.

Let L be a field with discrete valuation v, residue field κ, discrete valuation ring R and local parameter π. The discrete valuation $v : L^\times \to \mathbb{Z}$ can be realized as a map $K_1(L) \to K_0(\kappa)$. Indeed, this may be generalized to higher degree K-groups.

Proposition 4.1.4 ([GS06], §7; [EKM], §100.B)**.** *For each $n \geq 1$ there exists a unique homomorphism*

$$\partial_v : K_n^M(L) \longrightarrow K_{n-1}^M(\kappa)$$

satisfying $\partial_v(\{\pi, u_2, ..., u_n\}) = \{\overline{u}_2, ..., \overline{u}_n\}$ for all local parameters π and all $(n-1)$-tuples $(u_2, ..., u_n)$ of units of R, where $\overline{u}_i$ denotes the image of u_i in $\kappa = R/(\pi)$.

This map is called the *residue homomorphism.* In the case $n = 1$, the homomorphism ∂_v is given by the valuation v, as stated above. For $n = 2$, the map ∂_v is determined by the *tame symbol*[1] $T_v : K_2(L) \to K_1(\kappa)$, as described in [Sri, Ex. 1.15]

$$T_v(\{a, b\}) = (-1)^{v(a)v(b)}\overline{\left(\frac{a^{v(b)}}{b^{v(a)}}\right)}.$$

Broader Context

Milnor's K-groups naturally arise in the study of field arithmetic. Let ℓ be an integer with $\frac{1}{\ell} \in F$ and assume that F contains ℓ^{th} roots of unity. Let ζ be a primitive ℓ^{th} root of unity. The *Galois symbol* is a map

$$F^\times \times F^\times \to \mathrm{Br}(F)[\ell]$$

[1]Some refer to the residue homomorphism as the tame symbol, for any degree n [GS06]. However, we will reserve this moniker for the $n = 2$ case.

defined by $(a,b) \mapsto [A_\zeta(a,b)]$. Here $\mathrm{Br}(F)[\ell]$ is the group of ℓ-torsion elements in $\mathrm{Br}(F)$ and $A_\zeta(a,b)$ is the *cyclic algebra* of degree ℓ (see Example 2.1.5). One may show that the Galois symbol factors through $k_2(F) := K_2^M(F)/\ell K_2^M(F)$, and the resulting map is usually called the norm-residue homomorphism [Sri, I]. We continue to use the notation $k_n(F) = K_n^M(F)/\ell K_n^M(F)$.

Let us describe this map from another perspective. Let $\ell \in \mathbb{Z}_+$ be an integer relatively prime to $\mathrm{char}(F)$, and let μ_ℓ denote the group of ℓ^{th} roots of unity in $(F_{\mathrm{sep}})^\times$. The Kummer sequence of $G = \mathrm{Gal}(F_{\mathrm{sep}}/F)$-modules

$$0 \to \mu_\ell \to (F_{\mathrm{sep}})^\times \xrightarrow{\cdot \ell} (F_{\mathrm{sep}})^\times \to 0$$

yields an exact sequence of Galois cohomology groups

$$(F_{\mathrm{sep}})^\times \to (F_{\mathrm{sep}})^\times \to H^1(F, \mu_\ell) \to H^1(F, (F_{\mathrm{sep}})^\times).$$

Recall that $H^1(F, (F_{\mathrm{sep}})^\times) = 0$ by Hilbert's Theorem 90. Therefore, we have an isomorphism $k_1(F) \xrightarrow{\cong} H^1(F, \mu_\ell)$. Using the cup product in Galois cohomology, we have a map

$$k_1(F) \otimes_{\mathbb{Z}} k_1(F) \xrightarrow{\sim} H^1(F, \mu_\ell) \otimes_{\mathbb{Z}} H^1(F, \mu_\ell) \xrightarrow{\cup} H^2(F, \mu_\ell^{\otimes 2}),$$

which once again factors through $k_2(F)$. This is precisely the map defined above using cyclic algebras. In general, there is a map

$$k_n(F) = K_n^M(F)/\ell K_n^M(F) \longrightarrow H^n(F, \mu_\ell^{\otimes n}),$$

also called the norm-residue homomorphism [Sri, vdK]. This homomorphism has been the focus of a substantial amount of work done by a great number of people. It was shown to be an isomorphism when $n = 2$ by Merkurjev and Suslin, who utilized computations of

K-chomology groups of Severi-Brauer varieties [MS82]. Milnor conjectured that this map is an isomorphism if $\ell = 2$. It has been shown to be an isomorphism in general, a fact that was originally conjectured Bloch and Kato, and proven by Rost, Voevodsky, and Weibel [Voe].

4.2 Quillen K-Theory

We define Quillen's higher algebraic K-groups using his Q-construction. While Milnor K-groups lend to more concrete computations, Quillen K-theory provides the functoriality and naturality necessary to establish fundamental results, including dévissage, resolution, and localization sequences. Moreover, this functoriality allows one to define cohomology groups for the K-theory sheaf on the category of schemes.

Quillen's brilliant insight was in realizing the collection of K-groups as homotopy groups of a single topological space. This was a first step in viewing algebraic K-theory from an algebro-topological perspective, where it has since played a significant role. Good references for this material include [Qui, Sri, Wei], and we include terminology from [Sch]. For a realization in terms of algebraic topology, see [EKMM]. For a definition in terms of motivic homotopy, see [DLRV]. For a definition in terms of additive invariants of derived and dg-categories, see [Tab].

We begin by defining Quillen's Q-construction. By an *exact category* we will mean an additive category E embedded as a full subcategory of an abelian category A which is closed under extensions, i.e., if $0 \to X \to Y \to Z \to 0$ is an exact sequence in A with X and Z objects of E, then Y is isomorphic to an object of E. An exact sequence in E is then an exact sequence in A whose objects lie in E, referred to as a *conflation* (or admissible sequence) of E.

If a morphism $i : A \to B$ arises in a conflation $0 \to A \xrightarrow{i} B \to C \to 0$ of E, we say that i is an *inflation* (or admissible monomorphism), and denote it diagrammatically by $\rightarrowtail$. If a morphism $q : Y \to Z$ arises in a conflation $0 \to X \to Y \xrightarrow{q} Z \to 0$ of E, we say that q is a *deflation* (or admissible epimorphism), and denote it diagrammatically by $\twoheadrightarrow$.

Define $Q\mathsf{E}$ as the category having the same objects as E, but with a morphism $A \to B$ given by an equivalence class of diagrams $A \twoheadleftarrow X \rightarrowtail B$. Another such diagram $A \twoheadleftarrow X' \rightarrowtail B$ is equivalent (i.e., defines the same morphism of $Q\mathsf{E}$) if there is an isomorphism $X \to X'$ making the following digram commute:

$$\begin{tikzcd}
 & X \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, "\cong"] & \\
A & & B \\
 & X' \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}$$

Composition of morphisms $A \twoheadleftarrow X \rightarrowtail B$ and $B \twoheadleftarrow Y \rightarrowtail C$ is given by the pullback diagram

$$\begin{tikzcd}
 & X \times_B X' \arrow[r, tail] \arrow[d, two heads] & X' \arrow[r, tail] \arrow[d, two heads] & C \\
A & X \arrow[l, two heads] \arrow[r, tail] & B &
\end{tikzcd}$$

Quillen's K-groups of an exact category E are then defined as the sequence of functors

$$K_n : \mathsf{Ex} \xrightarrow{Q} \mathsf{Cat} \xrightarrow{N} \mathsf{sSet} \xrightarrow{|\cdot|} \mathsf{Top} \xrightarrow{\pi_{n+1}} \mathsf{Ab}.$$

Here, Ex denotes the category of (small) exact categories and exact functors, Cat the category of (small) categories and functors, sSet the category of simplicial sets and simplicial maps, Top the category of topological spaces and continuous maps, and Ab the category of abelian groups and group homomorphisms. The functor N denotes the nerve construction, $|\cdot|$ the geometric realization, and π_{n+1} denotes the $(n+1)^{\text{th}}$ homotopy group, taking the base point to be the image of the zero object of the exact category E.

To recover the K-theory of schemes, we can augment the above sequence of functors with the association $X \mapsto \mathsf{P}(X)$, where $\mathsf{P}(X)$ denotes the exact category of locally free sheaves of finite rank on a scheme X. We then define $K_n(X) = K_n(\mathsf{P}(X))$. Similarly, we define $G_n(X) = K_n(\mathsf{M}(X))$, where $\mathsf{M}(X)$ denotes the abelian category of coherent sheaves on X.

The K-theory (and G-theory) of a noetherian ring R may be recovered by the same prescription, taking $\mathsf{P}(R)$ to be the exact category of finitely generated projective modules and $\mathsf{M}(R)$ to be the category of finitely generated left R-modules.

Quillen's K-theory of schemes enjoys a robust structure and we highlight some aspects here [Sri, V]. For a scheme X, the fully faithful embedding $\mathsf{P}(X) \subset \mathsf{M}(X)$ induces maps $K_n(X) \to G_n(X)$ for all n. This map is an isomorphism if X is regular [Wei, Theorem V.3.4]. Waldhausen showed that there is a natural product on $\bigoplus_{n \geq 0} K_n(X)$, which endows this set with the structure of a graded-commutative ring, similar to Milnor K-theory. The set $\bigoplus G_n(X)$ is then a graded $\bigoplus K_n(X)$-module.

If $f : X \to Y$ is a morphism of schemes, the pullback functor $f^* : \mathsf{P}(Y) \to \mathsf{P}(X)$ induces a homomorphism $f^* : K_i(Y) \to K_i(X)$. Thus, K_i may be viewed as a contravariant functor from schemes to abelian groups. Similarly, if $f : X \to Y$ is a flat morphism of noetherian schemes, the exact functor $f^* : \mathsf{M}(Y) \to \mathsf{M}(X)$ induces a map $f^* : G_i(Y) \to G_i(X)$. Thus, G_i is a contravariant functor from the category of noetherian schemes and flat morphisms to the category of abelian groups.

For a field F, elements of $K_n(F)$ satisfy the Steinberg relation $a \cdot (1-a) = 0$, encountered in the definition of Milnor K-groups $K_n^M(F)$ (Definition 4.1.1). It follows that for any field F there exists a homomorphism $K_n^M(F) \to K_n(F)$ which is an isomorphism for $n = 0, 1, 2$. We will denote elements of $K_2(F)$ by $\{a, b\}$, just as with Minor K-groups.

4.3 Examples

Example 4.3.1 (Low Degree K-Theory of Fields)**.** As seen in Example 4.1.2, for a field F, we have $K_0^M(F) \cong \mathbb{Z}$ and $K_1^M(F) \cong F^\times$. Since Milnor and Quillen K-theory agree for fields (and in low degree), this recovers the Quillen K-groups. A presentation for K_2 of fields is given by the following result.

Theorem 4.3.2 (Matsumoto, see [Sri, Wei])**.** *If F is a field, then $K_2^M(F)$ is the abelian group generated by the collection of symbols $\{a, b\}$ with $a, b \in F^\times$ subject to the relations*

1. *Bilinearity: $\{aa', b\} = \{a, b\} + \{a', b\}$*
2. *Steinberg Relation: $\{a, 1-a\} = 0$ for all $a \notin \{0, 1\}$*

Example 4.3.3 (K-Theory of Finite Fields)**.** By Example 4.1.3, for a finite field $\mathbb{F}_q$, we have $K_2(\mathbb{F}_q) = 0$, and in fact $K_n^M(\mathbb{F}_q) = 0$ for all $n \geq 2$. However, Quillen K-groups of finite fields are in general nonzero in degrees higher than 2.

Example 4.3.4 (Local Rings [Sri])**.** Let R be a (not necessarily commutative) local ring, i.e., R has a unique maximal ideal. Then $K_0(R) = \mathbb{Z}$, generated by the free module of rank 1. We have $K_1(R) \cong R^\times/[R^\times, R^\times]$, where $[R^\times, R^\times]$ denotes the commutator subgroup of $R^\times$.

Example 4.3.5 (Non-Separated Scheme [Wei], II.8.2.4)**.** Let X be affine n-space with a double origin, obtained by identifying two copies of $\mathbb{A}^n$ along $\mathbb{A}^n \setminus \{0\}$. Then $G_0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, while $K_0(X) \cong \mathbb{Z}$.

Example 4.3.6 (Projective Space [Wei], V.1.5.1)**.** Let X be a quasi-projective scheme and $\mathbb{P}_X^n = \mathbb{P}_F^n \times_{\operatorname{Spec} \mathbb{Z}} X$ projective space over X. There is an isomorphism of rings

$$K_*(\mathbb{P}_X^n) \cong K_*(X) \otimes_{\mathbb{Z}} K_0(\mathbb{P}_{\mathbb{Z}}^n) \cong K_*(X)[z]/(z^r + 1).$$

Example 4.3.7 (Severi-Brauer Varieties [Wei], V.1.6.6)**.** If X is the Severi-Brauer variety $\mathrm{SB}(A)$ corresponding to a central simple algebra A of degree n, there is an isomorphism

$$K_*(X) \cong \bigoplus_{i=0}^{n-1} K_*(A^{\otimes i}).$$

4.4 An Algebro-Topological Perspective

As discussed above, Quillen's higher K-groups are given as the composition of functors

$$\mathsf{Ex} \xrightarrow{Q} \mathsf{Cat} \xrightarrow{B} \mathsf{Top} \xrightarrow{\pi_{n+1}} \mathsf{Ab},$$

where $B = |\cdot| \circ N$ is the classifying space functor. The topological space obtained prior to applying π_{n+1} will be referred to as the *K-theory space*, denoted $K(\mathsf{E})$.

To enter the realm of modern algebraic topology, we may instead consider the *K-theory spectrum*, which we will also denote by $K(\mathsf{E})$, along with its corresponding stable homotopy groups. This definition was first given by F. Waldhausen in [Wal], and applies to *Waldhausen categories* or *categories with cofibrations and weak equivalences*, a generalization of the notion of exact category. We refer the reader to [Wei, IV.8]. In summary, one uses Waldhausen's iterated $wS.$-construction to produce a delooping of the K-theory space obtained in Quillen's work. This delooping realizes the K-theory spectrum as an Ω-spectrum or an *infinite loop space* [Wei, IV.6].

The main advantage of using Waldhausen's definition of higher K-theory is that one may functorially associate a symmetric spectrum $K(\mathsf{E})$ to any exact category E so that tensor products determine smash products of spectra (see introduction of [Jar]).

With this framework in place, for any scheme X defined over a field F, one may view the association $X \mapsto K(\mathsf{VB}(X))$ as a presheaf on the category Sch_F of F-schemes with values in the category Spt of spectra [Jar, §3], where $\mathsf{VB}(X)$ denotes the exact category of vector bundles on X for the big Zariski site of F. We denote this presheaf of spectra by $\mathbf{K}$. The category of presheaves of spectra has a nice model structure [Hov], and one may topologize this presheaf with respect to any choice of topology on Sch_F by taking an appropriate stably fibrant model (see the discussion in [Jar, §4]). In particular, Quillen's algebraic K-groups of an F-scheme X are obtained as the homotopy groups of the spectrum $R\mathbf{K}(X) = RK(\mathsf{VB}(X))$, where $R\mathbf{K}(X)$ a stably fibrant model of $\mathbf{K}(X)$ relative to the model structure on the big Zariski site $(\mathsf{Sch}_F)_{\mathrm{Zar}}$.

4.5 The Brown-Gersten-Quillen Spectral Sequence

We now discuss a spectral sequence which computes K-groups of a regular scheme from the K-theory of fields which occur on its first page. This naturally leads to definition of K-cohomology groups and their importance in K-theory computations. A good reference for the following information is [Sri, Ch. 5].

Given a scheme X, let K_n^X (resp. G_n^X) denote the Zariski sheaf on X associated to the presheaf $U \mapsto K_n(U)$ (resp. $U \mapsto G_n(U)$), for U an open subset of X. Using the usual general framework [Hart, III], we may consider the cohomology groups of X with coefficients in K_n^X (resp. G_n^X), which we denote by $H^p(X, K_n)$ (resp. $H^p(X, G_n)$) and refer to as K-*cohomology* groups.

Theorem 4.5.1 (Brown-Gersten-Quillen)**.** *Let $X^{(p)} \subset X$ be the set of points of codimension p in X. There is a fourth-quadrant spectral sequence of cohomological type*

$$E_1^{p,q}(X) = \bigoplus_{x \in X^{(p)}} K_{-p-q}(F(x)) \Rightarrow G_{-p-q}(X),$$

which is convergent if X has finite (Krull) dimension.

The utility of this result is in the fact that the K-theory of schemes may be recovered by K-theory of fields. Note that for X regular, G-theory and K-theory coincide, so the above sequence abuts to K-groups. These groups may be computed by making use of K-cohomology, as the following result asserts.

Proposition 4.5.2. *Let X be a regular scheme, so that $\mathcal{O}_{X,x}$ is a regular local ring for each $x \in X$. Then there are canonical isomorphisms*

$$E_2^{p,q}(X) = H^p(X, K_{-q}),$$

where $E_2^{p,q}(X)$ is the second page of the Brown-Gersten-Quillen spectral sequence.

We can thus realize the K-cohomology groups $H^p(X, K_q)$ of a d-dimensional scheme X as cohomology of the *Gersten complex*

$$\bigoplus_{x \in X^{(0)}} K_{-q}(F(x)) \to \bigoplus_{x \in X^{(1)}} K_{-(q+1)}(F(x)) \to \cdots \to \bigoplus_{x \in X^{(d)}} K_{-(q+d)}(F(x)),$$

obtained from the first page of the BGQ spectral sequence by fixing the index q.

Recall that for a regular scheme X of finite type over F, the *Chow group* $\mathrm{CH}^p(X)$ of cycles of codimension p on X modulo rational equivalence is the cokernel the divisor map [Ful]

$$\bigoplus_{x \in X^{(p-1)}} F(x)^\times \longrightarrow \bigoplus_{x \in X^{(p)}} \mathbb{Z}.$$

Of course, $F(x)^\times = K_1(F(x))$ and $K_0(F(x)) = \mathbb{Z}$, so that $\mathrm{CH}^p(X)$ is recovered by K-cohomology groups. In fact, we have Bloch's formula:

Theorem 4.5.3 (Bloch's Formula [Qui])**.** *Let X be a regular scheme of finite type over a field F. For each $p \geq 0$, there are natural isomorphisms*

$$H^p(X, K_p) \cong \mathrm{CH}^p(X).$$

Remark 4.5.4. If one instead considers dimension p cycles as opposed to codimension p cycles, Bloch's formula takes the form $H_p(X, K_{-p}) \cong \mathrm{CH}_p(X)$.

4.6 Chow Groups with Coefficients

We now present a discussion of cycle modules as developed by Markus Rost. The Gersten complex encountered above may be generalized via an axiomatic approach to produce a complex for any assignment of "graded abelian group" to "extension of the ground field" and which satisfies certain data and compatibility rules. These cycle modules are the correct notion of coefficient system for the algebro-geometric analogue of singular homology, i.e., Chow groups of cycles of a given dimension on a scheme or variety. This explains the label "Chow groups with coefficients."

Cycle Modules

Cycle modules were first introduced by Rost in [Ros] and good references are [GMS] for use in producing cohomological invariants and [EKM] for the case of Milnor K-theory.

Definition 4.6.1. A *cycle module* M over F is a function assigning to every field extension L/F a graded abelian group $M(L) = M_*(L)$, which is a graded module over the Milnor K-theory ring $K_*^M(F)$ satisfying some data and compatibility axioms. This data includes

1. For each field homomorphism $L \to E$ over F, there is a degree 0 homomorphism $r_{E/L} : M(L) \to M(E)$ called *restriction*.

2. For each field homomorphism $L \to E$ over F, there is a degree 0 homomorphism $c_{E/L} : M(E) \to M(L)$ called *corestriction* (or norm).

3. For each extension L/F and each (rank 1) discrete valuation v on L, there is a degree -1 homomorphism $\partial_v : M(L) \to M(\kappa(v))$ called the *residue homomorphism*, where $\kappa(v)$ is the residue field of v.

These homomorphisms are compatible with the corresponding maps in Milnor K-theory. See D1-D4, R1a-R3e, FD, and C of [Ros, Def. 1.1, Def. 2.1].

Remark 4.6.2. The term "residue homomorphism" is derived from the case where the cycle module M is given by Milnor K-theory (Proposition 4.1.4).

Example 4.6.3. Utilizing Milnor and Quillen K-theory, the assignments $L \mapsto K_n^M(L)$ and $L \mapsto K_n(L)$, for any extension L/F, define cycle modules over F, which are denoted K_*^M and K_*, respectively.

Let X be an F-variety and M a cycle module over F, and set

$$C_p(X, M_q) = \bigoplus_{x \in X_{(p)}} M_{p+q}(F(x)),$$

where $X_{(p)}$ denotes the collection of points of X of dimension p. Using the residue homomorphim ∂_v, we define a map

$$\partial_X : C_p(X, M_q) \to C_{p-1}(X, M_q)$$

as follows. For any pair of points $y \in X_{(p)}$ and $x \in X_{(p-1)}$, let $Y = \overline{\{y\}}$, a closed subvariety of X of dimension p. If $x \in Y$ (so that x is a specialization of y), then the local ring $\mathcal{O}_{Y,x}$ of Y at x is a discrete valuation ring, with valuation denoted v_x. Set the (x, y)-component $\partial_X^{(x,y)}$ of

$$\partial_X : \bigoplus_{y \in X_{(p)}} M_{p+q}(F(y)) \to \bigoplus_{x \in X_{(p-1)}} M_{p+q-1}(F(x))$$

to be the residue map

$$\partial_{v_x} : M_{p+q}(F(y)) \to M_{p+q-1}(F(x)).$$

For any pair $(x', y') \in X_{(p-1)} \times X_{(p)}$ which do not satisfy the above conditions, set $\partial_X^{(x',y')} = 0$.

Homology and Cohomology

We now begin our discussion of invariants produced by the study of cycle modules. The following result was first shown by Kato for $M_* = K_*^M$ [Kat].

Proposition 4.6.4 ([Ros], Lem. 3.3)**.** *For every F-scheme X and cycle module M over F, the map ∂_X is a differential of $C_*(X,M)$, i.e., $(\partial_X)^2 = 0$.*

We thus obtain a complex

$$\cdots \to C_{p+1}(X, M_q) \xrightarrow{\partial_X} C_p(X, M_q) \xrightarrow{\partial_X} C_{p-1}(X, M_q) \to \cdots$$

with differentials ∂_X induced by the residue homomorphisms ∂_{v_x}. This is often referred to as the (homological) *Rost complex*. We denote the homology group at the middle term by $A_p(X, M_q)$.

By this prescription we define K-*homology* groups $A_p(X, K_q^M)$ of an irreducible F-variety by taking $M_* = K_*^M$. We define K-cohomology groups $A^p(X, K_q^M)$ of a variety of dimension d by the formula [EKM, §56]

$$A^p(X, K_q^M) := A_{d-p}(X, K_{d-p}^M).$$

Alternatively, to any F-variety we may associate a cohomological Rost complex similar to the homological case, and the groups $A^p(X, K_q^M)$ are given by the cohomology of this complex [Ros, §5]. Taking the cycle module M_* to be defined by Quillen K-theory K_*, we obtain the Gersten complex (see Section 4.5) as a special case of the (cohomological) Rost complex.

The assignment $X \mapsto A_*(X, K_*)$ defines a (covariant) functor from the category of schemes and proper morphisms to the category of bigraded abelian groups and bigraded homomorphims [EKM, §52.A]. That is, for a proper morphism $f : X \to Y$ of schemes there is a *push-forward homomorphism*

$$f_* : A_p(X, K_q) \to A_p(Y, K_q).$$

The assignment $X \mapsto A_*(X, K_*)$ also defines a contravariant functor from the category of schemes and flat morphisms to the category of abelian groups [EKM, §52.B]. If $g : Y \to X$

is a flat morphisms of relative dimension d, there is a *pull-back homomorphism*

$$g^* : A_p(X, K_q) \to A_{p+d}(Y, K_{q-d}).$$

Furthermore, if $g : Y \to X$ is a morphism of smooth schemes, then the pull-back homomorphism for K-homology induces a pull-back $g^* : A^p(X, K_q) \to A^p(Y, K_q)$ [EKM, §56]. Thus, the assignment $X \mapsto A^*(X, K_*)$ defines a contravariant functor from smooth schemes to abelian groups.

One extremely useful tool is the Projective Bundle Theorem for K-cohomology. If $E \to X$ is a vector bundle, we let $\mathbb{P}(E)$ denote the associated projective bundle.

Theorem 4.6.5 ([EKM], Theorem 53.10)**.** *For a vector bundle $E \to X$ of rank r there is an isomorphism*

$$\bigoplus_{i=1}^{r} A_{*-i+1}(X, K^M_{*+i-1}) \to A_*(\mathbb{P}(E), K^M_*).$$

Remark 4.6.6. Notice that in the case of zero-cycles, the Projective Bundle Theorem yields an isomorphism $A_0(X, K^M_*) \cong A_0(\mathbb{P}(E), K^M_*)$.

One can recover Chow groups of a scheme X using K-homology. Indeed, by definition the group $\mathrm{CH}_p(X)$ is the free abelian group generated by subvarieties of X of dimension p up to rational equivalence (include fulton definition) Thus,

$$\mathrm{CH}_p(X) = \operatorname{coker}\left(\bigoplus_{x \in X_{(p+1)}} K_1(F(x)) \xrightarrow{\partial_X} \bigoplus_{x \in X_{(p)}} K_0(F(x))\right),$$

since $K_1(F(x)) = F(x)^\times$ and $K_0(F(x)) = \mathbb{Z}$. This Chow group is thus the K-homology group $A_p(X, K_{-p})$ [EKM, §57.A] (compare to Remark 4.5.4).

The focus of our main result (Theorem 5.4.2) will be the group of K_1-zero-cycles. This group is given by the following characterization, following our above definition:

$$A_0(X, K_1) = \operatorname{coker}\left(\bigoplus_{x \in X_{(1)}} K_2(F(x)) \xrightarrow{\partial_X} \bigoplus_{x \in X_{(0)}} K_1(F(x))\right).$$

Remark 4.6.7. Since Quillen and Milnor K-groups of fields agree in degrees $0, 1$, and 2, the group $A_0(X, K_1)$ coincides with $H_0(X, K_1)$, defined in Section 4.5. Furthermore, these groups may be identified with the group $A_0(X, K_1^M)$ for (see the end of Section 4.2).

In concrete terms, the group of K_1-zero-cycles on a scheme X is given by the collection of equivalences classes of formal sums $\sum(\alpha_x, x)$, where x is a closed point on X and $\alpha_x \in K_1(F(x)) = F(x)^\times$. Equivalence of cycles is then induced by tame symbols associated to discrete valuation rings coming from 1-dimensional subvarieties of X and their specializations. In general, a K_n^M-zero-cycle is a sum $\sum(\alpha_x, x)$ where $x \in X$ is closed point and $\alpha_x = \{a_1, ..., a_n\} \in K_n^M(F(x))$, with equivalence similarly induced by residue homomorphisms.

In view of our focus being on groups of zero-cycles, the utility of the following result is clear and will be used in the proof of Lemma 5.3.1.

Proposition 4.6.8 ([KM], Cor. RC.13)**.** *For a cycle module M, the groups $A_0(X, M)$ and $A^0(X, M)$ are birational invariants of the smooth complete variety X.*

We complete this subsection with an example of a cycle module defined using K-homology.

Example 4.6.9. Let X be an F-variety. The assignment $L \mapsto A_0(X_L, K_*^M)$ defines a cycle module over F, with the graded structure induced by the graded structure of K_*^M. We will denote this cycle module by $A_0[X, K_*^M]$. It was first defined in [Ros, §7] and further studied in [CM01, Mer14].

Norms

Analogous to the reduced norm homomorphism $\mathrm{Nrd}_A : K_1(A) \to K_1(F)$ and its kernel $SK_1(A)$, there are norm homomorphisms defined on groups of K_n^M-zero-cycles and the study of their kernels has been a useful endeavor. This information may be found in [CM01, §1.5] and will be used in Section 5.2.

Let X be a complete variety over F. For any $n \geq 0$ there is a norm homomorphism $N_n : A_0(X, K_n^M) \to K_n^M(F)$ defined by

$$N_n \left(\sum (\alpha_x, x) \right) = \sum N_{F(x)/F}(\alpha_x),$$

where $N_{F(x)/F}$ denotes the field norm corresponding to the the extension $F(x)/F$, and $N_{F(x)/F}(\alpha_x) = \{N_{F(x)/F}(a_1), ..., N_{F(x)/F}(a_n)\} \in K_n^M(F)$ for $\alpha_x = \{a_1, ..., a_n\}$. We denote the kernel of N_n by $\overline{A}_0(X, K_n^M)$.

We make note that in the case $n = 0$, we obtain the degree homomorphism

$$N_0 : A_0(X, K_0) = \mathrm{CH}_0(X) \to K_0(F) = \mathbb{Z},$$

whose image coincides with $m(X)\mathbb{Z}$ where $m(X) = \gcd([F(x) : F])$ taken over all closed points $x \in X$ [CM06].

For $n = 1$ and X the Severi-Brauer variety of a central simple algebra A, there is an isomorphism $A_0(X, K_1) \cong K_1(A)$, and the group $\overline{A}_0(X, K_1)$ coincides with the group $SK_1(A)$. This will be discussed in detail in Section 5.1.

Chapter 5

Chow Groups with Coefficients and Generalized Severi-Brauer Varieties

With all objects and tools for investigation in order, we come to the final chapter and main results of this thesis. We begin by recalling the theorem and proof of Mekurjev and Suslin showing that the group of K_1-zero-cycles of $\mathrm{SB}(A)$ and the group $K_1(A)$ are isomorphic. In some sense, the collection of maximal subfields $L \subset A$ and their arithmetic, encoded in the distinguished element $\alpha \in L^\times = K_1(A)$, can be recovered by $K_1(A)$. Indeed, elements of $A^\times$ generically generate maximal subfields and the subfield arithmetic is recorded by the abelianized arithmetic of A.

To exhibit how R-equivalence naturally arises in the study of K-theory and K-cohomology groups, we present a discussion of Chernousov and Merkurjev relating R-equivalence classes of algebraic groups to the group of K_1-zero-cycles for some homogeneous varieties. We then present our calculation of the group of K_1-zero-cycles for the second generalized Severi-Brauer variety of an algebra of index 4, beginning with a reduction to algebras of square degree.

5.1 The Group of K_1-Zero-Cycles on Severi-Brauer Varieties

The inspiration for the main result of this thesis is attributed to the following theorem. As such, we include a proof. Recall that the group $H_0(X, K_1)$ coincides with the groups $A_0(X, K_1)$ and $A_0(X, K_1^M)$ (see Remark 4.6.7).

Theorem 5.1.1 ([MS92]). *Let A be a central simple algebra over a field F, let $X = \mathrm{SB}(A)$ be the corresponding Severi-Brauer variety. Then there exists a commutative digram*

$$\begin{tikzcd} H_0(X, K_1) \arrow[rr, "p", "\sim"'] \arrow[dr, "N_1"'] & & K_1(A) \arrow[dl, "\mathrm{Nrd}"] \\ & K_1(F) & \end{tikzcd}$$

with a natural isomorphism p. In particular, $\ker(H_0(X, K_1) \xrightarrow{N_1} K_1(F)) \cong SK_1(A)$.

Proof. We present the proof found in [Mer92]. We begin by treating the case of a division algebra D of degree n. For any closed point $x \in X$, the residue field $F(x)$ splits D; hence, $[F(x) : F] = kn$ for some k. Therefore, $F(x)$ can be imbedded in the matrix algebra $M_k(D)$ and there is an induced map on K-groups

$$K_1(F(x)) \to K_1(M_k(D)) \cong K_1(D).$$

One checks that we have a commutative diagram

$$\begin{tikzcd} \bigoplus_{x \in X_{(0)}} K_1(F(x)) \arrow[rr] \arrow[dr, two heads] & & K_1(D) \\ & H_0(X, K_1) \arrow[ur, "p"'] & \end{tikzcd}$$

with surjective homomorphism p.

For the proof of injectivity of p, one constructs a map in the opposite direction. But we first wish to understand the structure of the set of closed points of X. Using the associated functor of points of X, for any field extension L/F, the set $X(L) = \mathrm{Mor}(\operatorname{Spec} L, X)$ of L-points of X is in natural bijection with the set of right ideals of D_L of dimension n.

Let $L \subset D$ be a maximal subfield. The L-vector space $\mathrm{Hom}_L(D, L)$ is a right ideal in $\mathrm{End}_L(D) \cong D_L$ of dimension n and hence defines a morphism $\operatorname{Spec} L \to X$. Let x_L be a point of its image. Then clearly $F(x_L) \hookrightarrow L$, and since $F(x_L)$ splits D, we have $[F(x_L) : F] \geq n$. Hence, $F(x_L) = L$ and $\deg(x) = n$. The association $L \mapsto x_L$ thus defines a map from the set of maximal subfields of D to the set of closed points in X of smallest degree n. This map is in fact a bijection.

Consider the Zariski-dense subset $S = \{\alpha \in D \mid F(\alpha) \text{ is a maximal subfield in } D\}$ of $D^\times$. For $\alpha \in S$, $F(\alpha)$ is a maximal subfield and hence defines a closed point $x_{F(\alpha)} \in X$ with $F(x_{F(\alpha)}) = F(\alpha)$. The inverse of p is first defined on the set $S \subset D^\times$ as

$$q : S \to H_0(X, K_1), \qquad \alpha \mapsto (\alpha, x_{F(\alpha)}),$$

and then extended to $D^\times$. It remains to show that q is a homomorphism, i.e., that

$$(\alpha, x_{F(\alpha)}) + (\beta, x_{F(\beta)}) = (\alpha\beta, x_{F(\alpha\beta)}) \in H_0(X, K_1).$$

This can be shown by a specialization argument. Consider the field $F' = F(t)$, the algebra $D' = D(t)$, and elements $\alpha \in S$, $f(t) = \beta t + 1 - t \in D(t)^\times$, where $\beta \in S$. Let

$$w = \big(\alpha f(t), x_{F'(\alpha f(t))}\big) - \big(\alpha, x_{F'(\alpha)}\big) - \big(f(t), x_{F'(f(t))}\big) \in H_0(X_{F(t)}, K_1).$$

We have $w(0) = 0$ and $w(1) = (\alpha\beta, x_{F(\alpha\beta)}) - (\alpha, x_{F(\alpha)}) - (\beta, x_{F(\beta)})$ and we wish to show that $w(1) = 0$. Consider the commutative diagram

$$\begin{tikzcd} H_0(X, K_1) \arrow[r, "i"] & H_0(X_{F(t)}, K_1) \arrow[r] \arrow[d, "p"] & \bigoplus_{y \in \mathbb{A}^1} H_0(X_{F(y)}, K_0) \arrow[d, "p_0"] \\ & K_1(D(t)) \arrow[r] & \bigoplus_{y \in \mathbb{A}^1} K_0(D_{F(y)}) \end{tikzcd}$$

where the upper row is the localization exact sequence. The group $\mathrm{CH}_0(X) = H_0(X, K_0)$ is known to be infinite cyclic and therefore has no torsion. Hence, p_0 is injective. A diagram chase shows that $w \in \mathrm{Im}(i)$, and hence, the specialization of w at all points coincide; therefore, $w(1) = w(0) = 0$. $\square$

An alternative proof of this theorem was given by Chernousov and Merkurjev [CM01], utilizing R-equivalence and cohomological invariants, the statement of which is included in the next section. For any reductive algebraic group G and character $\rho : G \to \mathbb{G}_m$, there is a map $\widetilde{\alpha}_F : G(F)/RH(F) \to A_0(X, K_1)$, where $H = \ker(\rho)$ and X is a variety encoding algebraic information of G. In certain cases, this map is an isomorphism, e.g., when X is the involution variety of a central simple algebra of small degree with quadratic pair, or when X is the Severi-Brauer variety of a central simple algebra.

5.2 R-Equivalence and K-Theory

We now give a description of the relationship between R-equivalence and K-theory, beginning with an example showing that R-equivalence naturally arises in the computation of algebraic K-groups. We then give a more general description of work of Chernousov-Merkurjev relating R-equivalence and K-cohomology using the theory of cohomological invariants (see [GMS, Mer99]).

Example 5.2.1 ([CM01, Vos]). Let A be a central simple algebra. The abelianization map $\mathrm{GL}_1(A) = A^\times \to A^\times_{ab} = K_1(A)$ induces an isomorphism

$$\mathrm{GL}_1(A)/R\mathrm{SL}_1(A) \cong K_1(A).$$

Here we define a homomorphism from the group of R-equivalence classes of an algebraic group to the group of K_1-zero-cycles of a corresponding homogeneous variety, following [CM01]. This map yields an isomorphism which allows us to compute K_1-zero-cycles on an involution variety using algebraic groups.

Let M be a cycle module over F, X an irreducible F-variety of dimension d with generic point x, and $y \in X$ a point of codimension 1. An element $u \in M(F(X))$ is *unramified at* y if u is in the kernel of the (x, y)-component

$$(\partial_X)_d^{(x,y)} : M_n(F(X)) = M_n(F(x)) \to M_{n-1}(F(y))$$

of the differential $(\partial_X)_d : C_d(X, M_{n-d}) \to C_{d-1}(X, M_{n-d})$ of the Rost complex (Section 4.6).

Remark 5.2.2. The collection of all elements in $M_n(F(X))$ which are unramified at all codimension 1 points of X is subgroup of $M_n(F(X))$ which coincides with the group $A^0(X, M_n)$ as defined in Section 4.6. This collection may thus be thought of as the group of "unramified M-valued functions" on X [Ros, p. 338].

Let G be a connected algebraic group over a field F and let M be a cycle module over F, regarded as a functor from the category of field extensions of F to the category of abelian groups.

Definition 5.2.3. An *invariant* of G in M of dimension d is a natural transformation $G \to M_d$. That is, an invariant is a collection of compatible maps $G(L) \to M_d(L)$ for any extension L/F.

We have projections and multiplication maps $p_1, p_2, m : G \times G \to G$. We say an element $\alpha \in A^0(G, M_d)$ is *multiplicative* if $p_1^*(\alpha) + p_2^*(\alpha) = m^*(\alpha)$ as elements in $A^0(G \times G, M_d)$, where the maps p_i^* and m^* are the corresponding pull-back maps discussed in Section 4.6. Any multiplicative element $\alpha \in A^0(G, M_d)$ defines an invariant $\widetilde{\alpha}$ of G in M of dimension d, as we now describe.

For a point $x : \operatorname{Spec} L \to G$ in $G(L)$, set the image of α under the pullback homomorphism

$$x^* : A^0(G, M_d) \to A^0(\operatorname{Spec} L, M_d) = M_d(L)$$

to be $\widetilde{\alpha}_L(x)$. In this way, we obtain a map $\widetilde{\alpha}_L : G(L) \to M_d(L)$ for any extension L/F. The collection of all such maps defines a natural transformation $\widetilde{\alpha} : G \to M_d$. Conversely, any invariant of G in M of dimension d is obtained from a multiplicative element $\alpha \in A^0(G, M_d)$ by this prescription.

The main goal of this section to define an invariant $G \to A_0[X, K_*]$ of dimension 1, where $A_0[X, K_*]$ is the cycle module defined in Example 4.6.9. That is, we wish to define compatible maps $G(L) \to A_0(X_L, K_1)$ for any field extension L/F, for suitable choice of variety X. In fact, X is a variety which encodes the structure of purely algebraic objects, as we will see below in a few examples. Using our discussion above, this goal is achieved by exhibiting a multiplicative element of

$$A^0(G, A_0[X, K_1]) \subset A_0[X, K_1](F(G)) = A_0(X_{F(G)}, K_1),$$

where the first inclusion follows from Remark 5.2.2. In other words, we need to exhibit an unramified multiplicative element α of $A_0(X_{F(G)}, K_1)$.

The choice of element of $A_0(X_{F(G)}, K_1)$ depends on a character $\rho : G \to \mathbb{G}_m$ which satisfies certain hypotheses (see [CM01, Prop. 1.3]). One consequence of these assumptions is that the image of the generic point ξ of G, considered as an element of $G(F(G))$, under the map

$$\rho(F(G)) : G(F(G)) \to \mathbb{G}_m(F(G)) = F(G)^\times$$

is precisely the element ρ considered as a regular function on G, i.e., an element of $F(G)^\times$. The following result allows us to relate this fact to K-cohomology groups. Recall the definition of the norm homomorphisms from Section 4.6.

Proposition 5.2.4 ([CM01], Prop. 2.1)**.** *For any field extension L/F, the image of the homomorphism $\rho(L) : G(L) \to L^\times$ coincides with the image of the norm homomorphism*

$$(N_1)_L : A_0(X_L, K_1) \to K_1(L) = L^\times.$$

Since ρ is in the image of $\rho(F(G))$, there is an element $\alpha \in A_0(X_{F(G)}, K_1)$ which satisfies $(N_1)_{F(G)}(\alpha) = \rho$. By [CM01, §3.1], the element α is unrammified and multiplicative, and thus defines an invariant $\widetilde{\alpha} : G \to A_0[X, K_1]$. Evaluating $\widetilde{\alpha}$ at the base field F yields a homomorphism $\widetilde{\alpha}_F : G(F) \to A_0(X, K_1)$. We take H to be the subgroup $\ker(\rho)$ of G.

Proposition 5.2.5 ([CM01], Prop. 3.6)**.** *The homomorphism $\widetilde{\alpha}_F$ factors through a map $G(F)/RH(F) \to A_0(X, K_1)$. Furthermore, $\widetilde{\alpha}_F$ induces a map $H(F)/R \to \overline{A}_0(X, K_1)$.*

These induced homomorphisms are isomorphisms for certain characters ρ and varieties X which satisfy specified criteria [CM01, §2]. We present two cases where these hypotheses are satisfied.

Let A be a central simple algebra of degree n over F and let $G = \mathbf{GL}_1(A)$. Let the character $\rho : G \to \mathbb{G}_m$ be given by the reduced norm, i.e., for every commutative F-algebra R, the map $\rho(R) : \mathrm{GL}_1(A_R) \to R^\times$ is the reduced norm homomorphism. The kernel of ρ is $\mathbf{SL}_1(A)$. Let X be the Severi-Brauer variety of A. The following result recovers Theorem 5.1.1.

Theorem 5.2.6 ([CM01], Theorem 6.1)**.** *Let A be a central simple algebra over F, X the Severi-Brauer variety of A. Then there are canonical isomorphisms*

$$K_1(A) = \mathrm{GL}_1(A)/R\,\mathrm{SL}_1(A) \cong A_0(X, K_1),$$

$$SK_1(A) = \mathrm{SL}_1(A)/R \cong \overline{A}_0(X, K_1).$$

Now let A be a central simple algebra of degree $n = 2m$, $n \geq 4$ with quadratic pair (σ, f). Let $G = \mathbf{\Gamma}(A, \sigma, f)$ and let $\rho : G \to \mathbb{G}_m$ be given by the spinor norm [KMRT, Def. 13.30]. The kernel of ρ is $\mathbf{Spin}(A, \sigma, f)$. Let X be the involution variety $\mathrm{IV}(A, \sigma, f)$ (Section 3.5). In order for X to satisfy the required criteria, one must take A to have small index.

Theorem 5.2.7 ([CM01], Theorem 6.5)**.** *Let A be a central simple algebra over F of even dimension and index at most 2 with quadratic pair (σ, f) and let X be the corresponding involution variety. Then there are canonical isomorphisms*

$$\Gamma(A, \sigma, f)/R\,\mathrm{Spin}(A, \sigma, f) \cong A_0(X, K_1),$$

$$\mathrm{Spin}(A, \sigma, f)/R \cong \overline{A}_0(X, K_1).$$

5.3 Reduction to Algebras of Square Degree

For an algebra of index p^2, we reduce the computation of K_1-zero-cycles of $\mathrm{SB}_p(A)$ to that of $\mathrm{SB}_p(D)$, where D is the underlying division algebra of A. In case $p = 2$, the reduction to algebras of degree 4 will allow the use of involution varieties in the proof of the main theorem.

Recall that for J a right ideal of A, $\mathrm{SB}_k(J)$ is the collection of right ideals of A of reduced dimension k which are contained in J (see discussion following Definition 3.4.1).

Lemma 5.3.1. *Let p be a prime and let $A = M_n(D)$ be a central simple algebra of index p^2. Let $X = \mathrm{SB}_p(A)$, and $Y = \mathrm{SB}_p(D)$. There is an isomorphism $A_0(X, K_1) \cong A_0(Y, K_1)$.*

Proof. Since $p^2 = \mathrm{ind}(A) = \mathrm{ind}(A_F) \mid p^2$, Proposition 3.4.4 implies that $\mathrm{SB}_{p^2}(A)$ has an F-point, i.e., an ideal J of reduced dimension p^2. Let e_J be the corresponding idempotent element of A (Proposition 2.1.9). Define a rational map

$$\varphi_J : \mathrm{SB}_p(A) \dashrightarrow \mathrm{SB}_p(J)$$

by the association $I \mapsto e_J I \subset J$, for any ideal I of reduced dimension p. The map φ_J is defined on the open locus consisting of ideals I satisfying $\mathrm{rdim}(e_J I) = p$.

By Theorem 3.4.2 and Remark 3.4.3, the algebra $D := e_J A e_J$ is degree p^2, Brauer-equivalent to A and satisfies $\mathrm{SB}_p(J) = \mathrm{SB}_p(D)$. Since $\mathrm{ind}(A) = p^2$, the algebra D is division and $A = M_n(D)$. We denote the resulting map $\mathrm{SB}_p(A) \dashrightarrow \mathrm{SB}_p(D)$ also by φ_J.

Let η be the generic point of $\mathrm{SB}_p(D)$ and take $L = F(\eta)$. Let $\mathfrak{f}$ be the generic fiber of φ_J, i.e., $\mathfrak{f} = \mathrm{SB}_p(A)_L = \mathrm{SB}_p(A_L)$ is the scheme-theoretic fiber over η. We first show that $\mathfrak{f}$ is a rational L-variety. The field L satisfies $\mathrm{ind}(D_L) \mid p$. Since D is Brauer-equivalent to A, we have $\mathrm{ind}(A_L) \mid p$, so that $\mathrm{SB}_p(A)$ has an L-rational point, again by Proposition 3.4.4. By Proposition 3.4.5, the function field of $\mathrm{SB}_p(A)_L$ is purely transcendental over L, so that $\mathfrak{f} = \mathrm{SB}_p(A)_L$ is rational. Thus, $\mathrm{SB}_p(D)_{\mathfrak{f}}$ is birational to $\mathrm{SB}_p(D) \times \mathbb{P}_L^{\dim \mathfrak{f}}$.

The group of zero-cycles defines a birational invariant of smooth projective varieties (Proposition 4.6.8) yielding an isomorphism

$$A_0(\mathrm{SB}_p(D)_{\mathfrak{f}}, K_1) \cong A_0(\mathrm{SB}_p(D) \times \mathbb{P}_L^{\dim \mathfrak{f}}, K_1).$$

By the Projective Bundle Theorem (4.6.5), and specifically Remark 4.6.6, we have an isomorphism $A_0(\mathrm{SB}_p(D) \times \mathbb{P}_L^{\dim \mathfrak{f}}, K_1) \cong A_0(\mathrm{SB}_p(D), K_1)$.

The variety $\mathrm{SB}_p(A)$ is birational to $\mathrm{SB}_p(D)_{\mathfrak{f}}$, isomorphic along the open locus of definition of φ_J. Again using the fact that $A_0(-, K_1)$ is a birational invariant, and combining this fact with the above isomorphism, we have $A_0(\mathrm{SB}_p(A), K_1) \cong A_0(\mathrm{SB}_p(D), K_1)$.

□

5.4 Main Result

Having reduced to the case of (division) algebras of square degree, we utilize a result of Krashen (see Lemma 3.5.1) to transfer the computation of zero-cycles of the second generalized Severi-Brauer variety to that of an involution variety. This result also guarantees that the involution variety of interest comes from an algebra of index no greater than 2. We then make use of Theorem 5.2.7 to translate this computation into an analysis of those algebraic groups defined in Section 3. We now take F to be a field of characteristic not 2.

Proposition 5.4.1. *Let A be a central simple algebra of degree 4 over a field F and let X be the second generalized Severi-Brauer variety of A. The group $A_0(X, K_1)$ can be identified as the group of pairs $(x, \alpha) \in K_1(A) \times F^\times$ satisfying* $\mathrm{Nrd}_A(x) = \alpha^2$.

Proof. By Lemma 3.5.1, $\mathrm{SB}_2(A)$ is isomorphic to the involution variety $\mathrm{IV}(B, \sigma)$ of a degree 6 algebra B with orthogonal involution σ. In particular,

$$A_0(\mathrm{SB}_2(A), K_1) = A_0(\mathrm{IV}(B, \sigma), K_1).$$

Moreover, B is Brauer-equivalent to $A^{\otimes 2}$, so that $\mathrm{ind}(B) \leq 2$. The involution σ is obtained from the bilinear form $\wedge^2 V \times \wedge^2 V \to \wedge^4 V \cong F$ by descent.

Consider the algebra $(A \times A^{\mathrm{op}}, \varepsilon)$ over $F \times F$ with unitary involution defined by exchanging factors. By Remark 2.4.6, the associated discriminant algebra $D(A \times A^{\mathrm{op}}, \varepsilon)$ is given by $\lambda^2 A$, has degree $\binom{4}{2} = 6$, and is Brauer-equivalent to $A^{\otimes 2}$. Furthermore, the canonical involution $\underline{\varepsilon} := \varepsilon^{\wedge 2}$ on $\lambda^2 A$ is also induced by the bilinear form $\wedge^2 V \times \wedge^2 V \to \wedge^4 V \cong F$ (Remark 2.4.4), yielding an isomorphism $(B,\sigma) \cong (D(A \times A^{\mathrm{op}}, \varepsilon), \underline{\varepsilon})$ of algebras with orthogonal involution. Thus, the algebras (B,σ) and $(A \times A^{\mathrm{op}}, \varepsilon)$ correspond to one another under the groupoid equivalence of A_3 and D_3 of Theorem 3.3.1.

Since $\mathrm{ind}(B) \leq 2$, Theorem 5.2.7 gives a canonical isomorphism

$$A_0(\mathrm{IV}(B,\sigma), K_1) \cong \Gamma(B,\sigma)/R\,\mathrm{Spin}(B,\sigma). \tag{5.4.1}$$

As $(B,\sigma) \in \mathsf{D}_3$ corresponds to $(A \times A^{\mathrm{op}}, \varepsilon) \in \mathsf{A}_3$, Proposition 3.3.2 yields exceptional identifications $\Gamma(B,\sigma) = \mathrm{SGU}(A \times A^{\mathrm{op}}, \varepsilon)$ and $\mathrm{Spin}(B,\sigma) = \mathrm{SU}(A \times A^{\mathrm{op}}, \varepsilon)$. Furthermore, Remark 3.1.2 gives an explicit description

$$\mathrm{SGU}(A \times A^{\mathrm{op}}, \varepsilon) = \{(x,\alpha) \in A^\times \times F^\times \mid \mathrm{Nrd}_A(x) = \alpha^2\},$$

$$\mathrm{SU}(A \times A^{\mathrm{op}}, \varepsilon) = \mathrm{SL}_1(A),$$

with the inclusion of the latter given by inclusion into the first factor $x \mapsto (x,1)$. The quotient in Equation 5.4.1 can then be identified as

$$\{(x,\alpha) \in A^\times \times F^\times \mid \mathrm{Nrd}_A(x) = \alpha^2\}/R\mathrm{SL}_1(A),$$

and therefore consists of elements of $x \in A^\times/R\mathrm{SL}_1(A) = \mathrm{GL}_1(A)/R\mathrm{SL}_1(A) = K_1(A)$ together with a square-root α of $\mathrm{Nrd}_A(x)$ in F. $\square$

Theorem 5.4.2. *Let A be a central simple algebra of index 4 and arbitrary degree over a field F, and let X be the second generalized Severi-Brauer variety of A. The group $A_0(X, K_1)$ can be identified as the group of pairs $(x,\alpha) \in K_1(A) \times F^\times$ satisfying $\mathrm{Nrd}_A(x) = \alpha^2$.*

Proof. The reduced norm respects the canonical isomorphism $K_1(M_n(D)) = K_1(D)$. We combine the isomorphisms of Theorem 5.3.1 and Proposition 5.4.1, yielding the desired result. $\square$

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