Optimal Pairs Trading Rules and Numerical Methods

by

Phong Thanh Luu

(Under the Direction of Jingzhi Tie and Qing Zhang)

Abstract

Pairs trading involves two correlated securities. When divergence is underway, i.e., one stock moves up while the other moves down, a pairs trade is entered consisting of a short position in the outperforming stock and long position in the underperforming one. Such a strategy bets the "spread" between the two would eventually converge. The main advantage of pairs trading is its risk neutral nature, i.e., it can be profitable regardless the general market condition. In this dissertation, a difference of the pair is studied. When the difference is governed by a mean-reversion model, the trade will be closed whenever the difference reaches a target level or a pre-determined cutloss limit. On the other hand, when it satisfies a regime-switching model, the trade will be determined by two conditional probability threshold levels. The objective is to identify the optimal threshold levels so as to maximize an overall return. We apply stochastic control theories to solve these optimal pairs trading problems. Many techniques have been implemented, including ordinary differential equation, stochastic approximation, and viscosity solution approaches. The effectiveness of these methods is examined in numerical examples.

Index words: Geometric Brownian motion, mean reversion model, regime switching model, HJB equation, stochastic approximation, viscosity solution.

Optimal Pairs Trading Rules and Numerical Methods

by

Phong Thanh Luu

B.S., Stony Brook University, 2010

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2016

Optimal Pairs Trading Rules and Numerical Methods

by

Phong Thanh Luu

Approved:

Major Professors: Jingzhi Tie
Qing Zhang

Committee: Edward Azoff
Ming-Jun Lai
Robert Varley

Electronic Version Approved:

Suzanne Barbour
Dean of the Graduate School
The University of Georgia
August 2016

Dedication

To my parents, brother's family, and sister.

To my wife and son, Vi Ai Do and Paxton GiaPhat-Do Luu.

Acknowledgments

First and foremost, I would like to express my deepest appreciation and gratitude to my dear advisors, Professor Jingzhi Tie and Professor Qing Zhang who have patiently walked me through every step of my research work. Professor Tie and Professor Zhang are great teachers and mentors who not only care about students academically but also care about them personally. I am fortunate to be one of their students.

I am also grateful to the committee members, Professor Edward Azoff, Professor Alexander Petukhov, Professor Ming-Jun Lai, and Professor Robert Varley for providing me with many helpful suggestions and comments, and serving in my committee. In addition, Professor Azoff has given me timeless and fruitful mathematical discussions and social conversations since my first day at the University of Georgia. Having him with us, the Department of Mathematics has ever been warmer. Likewise, Professor Lai and Professor Petukhov have introduced me many useful techniques of numerical analysis. I especially appreciate Professor Varley's willingness and friendliness, and it is always a pleasure to see him.

I am always grateful to my teaching mentors, Professor Edward Azoff and Professor Daniel Krashen for multiple reasons as to why I am going to be a real educator. I also owe my deepest debt of gratitude to my teachers, Phuc Gia Nguyen and Chuong Van Truong who spent days and nights by the light of an oil lamp to feed me with background knowledge in chemistry, mathematics, and physics at Tan Quoi high shool, who helped me develop my confidence and ability in mathematics, and who had a great impact on my decision to pursue the subject in the long term.

I am deeply indebted to the Department of Mathematics at the University of Georgia for the financial support during my last six years of graduate study. I would also like to thank Ms. Laura Ackerley, Mr. Jerry Daniel, Mr. Jose Malagon, Ms. Christy McDonald, Ms. Julie McEver, Ms. Connie Poore, and Ms. Gail Suggs for being always helpful.

My gratitude also goes to many great friends in Athens, Hartwell, and Lilburn, Georgia, with whom I have had wonderful time together. I would also like to acknowledge Kevin Kuo, Duy Nguyen, Eric Perkerson, and Kate Thompson, with whom I collaborated to achieve the results appearing in chapters 3 of this dissertation.

I have been receiving constant encouragement and support from my dear family and friends. I would like to give my special thank to my parents, Duc Luu and Tiem Pham; my sister, Tran Luu; my brother and his family, An, Linda, Valerie, and Vanessa Luutran; my great friends, Richard and Annette Gentile, and Joe Pastore; and my wife and son, Vi Do and Paxton Luu for their love, support, and understanding. I would not have accomplished thus far without them.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Mathematical trading rules have been studied for many years. For example, Zhang [43] considered a selling rule determined by two threshold levels: a target price and a cutloss limit. In [43], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [17] studied the optimal selling rule under a model with switching geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. Note that these papers are only concerned with geometric Brownian motion type models.

The idea of pairs trading was introduced by Bamberger and followed by Tartaglia's quantitative group at Morgan Stanley in the 1980s. The most attractive feature of pairs trading is its risk neutral nature in the sense that it can be profitable under any market conditions. For related literature and detailed discussions on the subject, we refer the reader to the paper by Gatev et al. [16], the book by Vidyamurthy [35], and references therein. In pairs trading, a pair of cointegrated stocks are selected and monitored; see Gatev et al. [16] and Liu and Timmermann [25] for related discussions. When the "spread" of the stock prices increases to a certain level, the pairs trade would be triggered: to short the stronger stock and to long the weaker one, betting the eventual convergence of the "spread". Another similar strategy bets the eventual divergence of the “spread”. When the “spread” decreases to a certain level, the pairs trade is entered by longing the stronger stock and shorting the weaker one.

Traditional pairs trading uses mean-reversion models, and closed-form solutions are often derived. In this dissertation, both mean-reversion and regime-switching models are studied, but much attention will be focused on the later one. Regime-switching models complicate the problems since the Markov chain incorporates a source of uncertainty into the models.

Mean-reversion models are one of the popular choices in financial markets to capture price movements that have the tendency to move towards an “equilibrium” level. We refer the reader to Cowles and Jones [5], Fama and French [12], and Gallagher and Taylor [15] among others for studies in connection with mean reversion stock returns. See also Hafner and Herwartz [18] for mean-reversion stochastic volatility and Blanco and Soronow [2] for asset prices in energy markets.

Other often used models are regime-switching models, which better reflect random market environment. The models incorporate parameters to describe the trends of the market which switches among a finite number of states, for instance, the uptrend (bull market) and the downtrend (bear market). Regime-switching models were first introduced by Hamilton [19] in 1989 to describe time series. The models have also been employed by Zhang [43] for optimal stock selling rules, Yin and Zhang [39] for applications in portfolio management, and Yin and Zhou [41] for dynamic Markowitz problems. Unlike these papers, this dissertation does not use geometric Brownian motions. A switching Itô diffusion of the form $dZ_t = \mu(\alpha_t)dt + \sigma dW_t$ is used instead.

In many optimal trading problems, HJB equations are derived. Techniques of stochastic control theories have been employed, such as ODE, PDE, smooth-fit technique, and viscosity solution methods to solve these equations. However, the associated HJB equations may involve complicated PDEs and classical solutions are very hard to obtain. To avoid solving these complicated HJB equations, stochastic approximation method is used. Recent references on stochastic approximation can be found in [3, 23]. These techniques have been implemented in this dissertation.

The dissertation is organized as follows. Chapter 2 is aimed at one round trip trading (buying low and selling high) a mean-reversion asset. The associated HJB equations for the value functions are derived, and the closed-form solutions and the optimal trading rules are obtained by means of ODE methods and smooth-fit techniques [17]. Chapter 3 focuses on pairs trading selling rule (when to exit a pairs position). A pair of historically correlated securities are selected and monitored. A difference of the pair is governed by a mean-reversion model. The same method as in chapter 2 is applied to obtain the closed-form solutions of the associated HJB equations. Chapter 4 is concerned with numerical methods for trading a pairs position. The log difference of the pair satisfies a regime-switching model. Unlike chapter 3, one round trip of trading is allowed. The optimal trading rules are obtained without solving the complicated HJB equations. In particular, a stochastic approximation algorithm is used to compute the probability threshold levels. Chapter 5 focuses on more general setups of pairs trading. A sequence of trades are allowed. The derived HJB equations involve a second-order, nonlinear parabolic PDE. Closed-form solutions of this PDE are very hard to obtain, and there is no guarantee of smooth solutions of these equations. A viscosity solution approach is considered. Indeed, we show that the value functions are the unique viscosity solutions of the HJB equations. This will enable an alternative numerical scheme to be constructed to approximate the probability threshold levels.

1.1 Problem 1: An ODE Approach to Mean-Reversion Trading

Chapter 2 is concerned with trading a single stock governed by a mean-reversion model. The objective is to buy low and sell high the underlying stock sequentially to maximize a discounted reward function. The ODE method is used to solve the associated HJB equations for the value functions, and the optimal stopping times can be determined by three threshold levels.

Let $S_t = \exp(X_t)$ be the asset price at time $t \geq 0$, where $X_t \in \mathbb{R}$ is a mean-reversion process governed by

$$dX_t = a(b - X_t)dt + \sigma dW_t,\ \ X_0 = x,$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion.

The generator $\mathcal{A}$ of X_t is given by

$$\mathcal{A} = a(b - x)\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}.$$

The associated HJB equations have the form

$$\begin{aligned}
\min\Big\{\rho v_0(x) - \mathcal{A}v_0(x),\ \ v_0(x) - v_1(x) + e^x + K\Big\} &= 0,\\
\min\Big\{\rho v_1(x) - \mathcal{A}v_1(x),\ \ v_1(x) - v_0(x) - e^x + K\Big\} &= 0,
\end{aligned}$$

where ρ is the given discount factor.

The closed-form solutions of the HJB equations can be obtained by means of ODE methods.

1.2 Problem 2: An ODE Approach to Pairs Trading Selling Rule

Chapter 3 is a joint work with K. Kuo, D. Nguyen, E. Perkerson, K. Thompson, and Q. Zhang [22]. This chapter is devoted to pairs trading selling rule (when to exit a pairs position). A difference of the pair is governed by a mean-reversion model. When divergence is underway, a pairs trade is entered consisting of a short position in the outperforming stock and a long position in the underperforming one. Such a strategy bets the "spread" between the two would eventually converge. The same method as in Problem 1 is applicable, and the optimal stopping times can be determined by two thresholds: a target and a pre-determined cutloss level.

Consider pairs trading that involves two stocks X_t^1 and X_t^2 at time $t \geq 0$. Trading a fraction of a share is allowed. Let $K_1 = 1/X_0^1$ shares of X^1 in the long position and $K_2 = 1/X_0^2$ shares of X^2 in the short position. The corresponding price of the position is given by

$$Z_t = K_1 X_t^1 - K_2 X_t^2.$$

$Z_t \in \mathbb{R}$ is assumed to satisfy a mean-reversion process governed by

$$dZ_t = a(b - Z_t)dt + \sigma dW_t, \; Z_0 = z,$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ the volatility, and W_t is a standard Brownian motion.

The reward function $v(z)$ satisfies the two-point-boundary-value second order ODE

$$\begin{cases} \rho v(z) = \dfrac{\sigma^2}{2}\dfrac{d^2 v(z)}{dz^2} + a(b-z)\dfrac{dv(z)}{dz}, \\ v(z_1) = z_1,\ v(z_2) = z_2. \end{cases}$$

The closed-form solution of this ODE can also be obtained.

1.3 Problem 3: A Stochastic Approximation Approach to Pairs Trading

Chapter 4 involves trading one round trip of a pairs position. When two stocks converge, a pairs trade is initiated by longing the outperforming stock and shorting the underperforming one. This strategy bets the "divergence" of the two stocks. The log difference of the pair is governed by a regime-switching model. A stochastic recursive algorithm is applied to compute the two probability threshold levels.

Let $S_t^i,\ i = 1, 2$, be the stock prices at time $t \geq 0$, with S^1 and S^2 being the stronger and weaker stocks respectively. Assume their log difference $Z_t = \log S_t^1 - \log S_t^2$ is a switching process in $\mathbb{R}$ governed by

$$dZ_t = \mu(\alpha_t)dt + \sigma dW_t,\ Z_0 = Z,$$

where $\mu(i) = \mu_i, i = 1, 2$, the expected rates of return with $\mu_1 > 0 > \mu_2$; $\sigma > 0$, the volatility; W_t, a standard Brownian motion; and α_t, the Markov chains with state space $\mathcal{M} = \{1, 2\}$.

The process α_t represents the market mode at each time t, where $\alpha_t = 1$ indicates a bull market and $\alpha_t = 2$ a bear market. The generator for α_t is denoted by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}, \text{ for some } \lambda_1 > 0,\ \lambda_2 > 0.$$

Let $p_t = P(\alpha_t = 1 | Z_s : 0 \le s \le t) \in [0, 1]$ denote the conditional probability of $\alpha_t = 1$ (bull market) given the stock price up to time t.

We need to find the optimal probability threshold levels x^1 (buy level) and x^2 (sell level) with $x^1 > x^2$ so as to maximize the reward function $g(x)$:

$$g(x) = g(x^1, x^2) = E\left[\int_{\tau_1(x)}^{\tau_2(x)} [(\mu_1 - \mu_2)p_t + \mu_2]dt\right],$$

where

$$\tau_1(x) = \inf\{t : p_t \ge x^1\},$$
$$\tau_2(x) = \inf\{t > \tau_1(x) : p_t \le x^2\}.$$

The reward function is expressed in form of expectation. Classical numerical methods such as Newton method among others do not perform well when the expectations require evaluation via simulation. Therefore, some stochastic approximation method is needed.

1.4 Problem 4: A Viscosity Solution Approach to Pairs Trading

Chapter 5 considers pairs trading in more general setups where a sequence of trades are allowed. The log difference of the pair is governed by a regime-switching model as described in problem 3. The reward function is maximized subject to

$$\begin{cases} dZ_t = [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma d\widehat{W}_t & Z_0 = Z, \\ dp_t = [-(\lambda_1 + \lambda_2)p_t + \lambda_2]dt + \dfrac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma} d\widehat{W}_t & p_0 = p, \end{cases}$$

where $(Z, p) \in \mathbb{R} \times (0, 1)$. The associated HJB equations have the form

$$\begin{aligned} \min\Big\{(\rho - \mathcal{A})v_0, v_0 - v_1 + (Z + K)\Big\} = 0, \\ \min\Big\{(\rho - \mathcal{A})v_1, v_1 - v_0 - (Z - K)\Big\} = 0, \end{aligned}$$

in $\mathbb{R} \times (0, 1)$, where

$$\begin{aligned} \mathcal{A}f(Y) = \; & \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial Z^2}(Y) + (\mu_1 - \mu_2)p(1-p)\frac{\partial^2 f}{\partial Z \partial p}(Y) + \frac{1}{2}\Big[\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma}\Big]^2\frac{\partial^2 f}{\partial p^2}(Y) \\ & + \Big[(\mu_1 - \mu_2)p + \mu_2\Big]\frac{\partial f}{\partial Z}(Y) + \Big[-(\lambda_1 + \lambda_2)p + \lambda_2\Big]\frac{\partial f}{\partial p}(Y), \end{aligned}$$

with $Y = \begin{pmatrix} Z \\ p \end{pmatrix} \in \mathbb{R} \times (0, 1)$.

The generator $\mathcal{A}$ in the HJB equations involve a second-order, nonlinear parabolic partial differential equation. Closed-form solutions of this PDE are very hard to obtain. At this point, we cannot prove there are smooth solutions to these equations. We can, however, show that the value functions are the unique viscosity solutions of these HJB equations. This enables an alternative numerical scheme to approximate the probability threshold levels.

1.5 Mathematical Preliminaries

This section summarizes certain background materials and a number of results used in the dissertation. These results and their proofs can be found in [1, 6, 10, 23, 24, 28, 36, 39].

1.5.1 Stochastic Processes

Definition 1.5.1 (Stochastic Processes). *A stochastic process is a collection of random variables $\{X(t)\}_{t\in\Lambda}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, where Λ is some indexing set.*

Typically, Λ is either the non-negative integers $\Lambda = \mathbb{Z}_+ = \{0, 1, 2, ...\}$ or the half line $\Lambda = \mathbb{R}_+ = [0, \infty)$. When $\Lambda = \mathbb{Z}_+$, we call such a process a discrete-time stochastic process. When $\Lambda = \mathbb{R}_+$, we call it a continuous-time stochastic process. Also, $X(t)(\omega)$ is sometimes written as $X_t(\omega)$ or $X(t, \omega)$ for notational convenience.

Definition 1.5.2 (Brownian Motion). *A standard one-dimensional Brownian motion is a process $\{B(t)\}_{t\in\mathbb{R}_+}$ such that*

1. *$B(0) = 0$ a.s.*

2. *$B(t)$ has independent increments, i.e., if $0 < t_1 < t_2 < ... < t_n$ then the random variables $B(t_1) - B(0), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent.*

3. *For all $s \geq 0$, $B(t + s) - B(s)$ equal in distribution to a normal r.v. with mean 0 and variance t, i.e., a r.v. with density*

$$p(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}.$$

4. *$t \to B(t)$ is continuous a.s.*

Definition 1.5.3 (Itô Diffusion). *A (time-homogeneous) Itô diffusion is a stochastic process* $X_t(\omega) = X(t, \omega) : [0, \infty] \to \mathbb{R}^n$ *satisfying a stochastic differential equation of the form*

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad t \geq s, \; X(s) = x, \tag{1.1}$$

where $B(t)$ *is* m*-dimensional Brownian motion, and* $b : \mathbb{R}^n \to \mathbb{R}^n, \; \sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ *satisfy*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|, \quad x, y \in \mathbb{R}^n,$$

i.e., $b(\cdot)$ *and* $\sigma(\cdot)$ *are Lipschitz continuous.*

For some fixed s, we will denote by $X^{s,x}(t)$, for $t \geq s$, the solution to (1.1) with initial condition $X(s) = x$ a.s. If $s = 0$, we write $X^x(t)$ for $X^{s,x}(t)$.

Let Q^x be the probability law of a given Itô diffusion $\{X(t)\}_{t \in \Lambda}$ when its initial value is $X(0) = x \in \mathbb{R}^n$. The expectation with respect to Q^x is denoted by $E^x[\cdot]$. Hence, we have

$$E^x[f_1(X(t_1)) \cdots f_k(X(t_k))] = E[f_1(X^x(t_1)) \cdots f_k(X^x(t_k))]$$

for all bounded Borel functions $f_1, \cdots, f_k$ and all times $t_1, \cdots, t_k \geq 0$, $k = 1, 2, ...$

Theorem 1.5.4 (Markov Property for Itô Diffusions). *Let* f *be a bounded Borel function from* $\mathbb{R}^n \to \mathbb{R}$. *Then for* $t, h \geq 0$,

$$E^x[f(X(t + h))|\mathcal{F}_t](\omega) = E^{X(t,\omega)}[f(X(h)].$$

Definition 1.5.5 (Filtration). *A filtration of the* σ*-algebra* $\mathcal{F}$ *is an increasing sequence of sub-*σ*-algebra* $\{\mathcal{F}_t\}_{t \in \Lambda}$*, i.e.,* $\mathcal{F}_s \subset \mathcal{F}_t$ *for all* $s \leq t$. *A stochastic process* $\{X(t)\}_{t \in \Lambda}$ *is adapted to the filtration* $\{\mathcal{F}_t\}_{t \in \Lambda}$ *if for each* $t \in \Lambda$, $X(t)$ *is* $\mathcal{F}_t$*-measurable.*

Definition 1.5.6 (Stopping Time/Markov Time)**.** *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space with filtration* $\{\mathcal{F}_t\}$*. A function (random variable)* $\tau : \Omega \to [0, \infty]$ *is called a stopping time w.r.t. (adapted to)* $\{\mathcal{F}_t\}$ *if*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$*.*

If $H \subset \mathbb{R}^n$ is any set, we define the first exit time from H, τ_H, as follows

$$\tau_H = \inf\{t > 0 : X_t \notin H\}.$$

Note that τ_H is a stopping time for any Borel set H.

Definition 1.5.7. *Suppose* τ *is a stopping time adapted to a filtration* $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$*, and let* $\mathcal{F}_\infty$ *denote the smallest* σ*-algebra containing the whole collection* $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$*. Define the* σ*-algebra* $\mathcal{F}_\tau$ *to be the* σ*-algebra generated by all sets of the form* $B \cap \{\tau \leq t\}$ *where* $B \in \mathcal{F}_\infty$ *and* $t \in \mathbb{R}_+$*.*

Theorem 1.5.8 (Strong Markov property for Itô Diffusions)**.** *Let* f *be a bounded Borel function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*, and* τ *be a stopping time w.r.t.* $\{\mathcal{F}_\tau\}$*,* $\tau < \infty$ *a.s. Then for all* $h \geq 0$*,*

$$E^x[f(X(\tau + h))|\mathcal{F}_\tau] = E^{X(\tau)}[f(X(h))].$$

Definition 1.5.9 (Generator of an Itô Diffusion)**.** *Let* $\{X(t)\}$ *be a (time-homogeneous) Itô diffusion in* $\mathbb{R}^n$*. The (infinitesimal) generator* $\mathcal{A}$ *of* $X(t)$ *is defined by*

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{E^x[f(X(t))] - f(x)}{t}, \; x \in \mathbb{R}^n.$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ *such that the limit exists at* x *is denoted by* $\mathcal{D}_\mathcal{A}(x)$*, while* $\mathcal{D}_\mathcal{A}$ *denotes the set of functions for which the limit exists for all* $x \in \mathbb{R}^n$*.*

Theorem 1.5.10 (Generator of an Itô Diffusion)**.** *Let $X(t)$ be the Itô diffusion satisfying*

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t).$$

If $f \in C_0^2(\mathbb{R}^n)$ then $f \in \mathcal{D}_A$ and

$$\mathcal{A}f(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Theorem 1.5.11 (Dynkin's formula)**.** *Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time with $E[\tau] < \infty$. Then*

$$E^x[f(X(\tau))] = f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X(s))ds\right].$$

1.5.2 Martingales

Definition 1.5.12 (Martingale and Martingale Difference)**.** *An n-dimensional stochastic process $\{X(t)\}_{t\in\mathbb{R}_+}$ is said to be a martingale on $(\Omega, \mathcal{F}, P)$ w.r.t. a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ if*

(i) $X(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$,

(ii) $E[\|X(t)\|] < \infty$ for all t, and

(iii) $E[X(t)|\mathcal{F}_s] = X(s)$ w.p. 1 for all $t \geq s$.

The sequence $\{X(t)\}_{t\in\mathbb{Z}_+}$ is called a martingale difference sequence if the condition (iii) above is replaced by $E[X(t)|\mathcal{F}_{t-1}] = 0$ w.p. 1.

If $\{X(t)\}$ is a martingale difference then for $j \leq k$,

$$E[X(j)X(k+1)] = EE[X(j)X(k+1)|\mathcal{F}_k] = E[X(j)]E[X(k+1)|\mathcal{F}_k] = 0.$$

Therefore, martingale difference sequence is uncorrelated.

1.5.3 The Martingale Problem

If $dX(t) = b(X(t)) + \sigma(X(t))dB(t)$ is an Itô diffusion in $\mathbb{R}^n$ with generator $\mathcal{A}$, and if $f \in C_0^2(\mathbb{R}^n)$ and $X(0) = x$ a.s. then

$$f(X(t)) = f(x) + \int_0^t \mathcal{A}f(X(s))ds + \int_0^t \nabla f^T(X(s))\sigma(X(s))dB(s).$$

Define $M_f(t) = f(X(t)) - \int_0^t \mathcal{A}f(X(s))ds.$

We say that $X(t)$ solves the martingale problem for generator $\mathcal{A}$ if $M_f(t)$ is a martingale for each f in $C_0^2(\mathbb{R}^n)$.

Theorem 1.5.1. *$M_f(t)$ is a $\mathcal{F}_t$-martingale, where $\mathcal{F}_t = \sigma(\{X(s), s \leq t\})$.*

Suppose that $\mathcal{A}$ is the generator for a diffusion or jump diffusion process and let $X(0) = x$ a.s. Then for any integer k time points $0 \leq t \leq t+s,\ t_i \leq t,\ i = 1, ..., k$, bounded continuous $f(\cdot)$ and smooth bounded $h(\cdot)$ with compact support, it follows that

$$Eh(X(t_i), i \leq k)[M_f(t+s) - M_f(t)] = 0. \tag{1.2}$$

On the other hand, if $X(t)$ is a process satisfying (1.2) for each such $h(\cdot), k, t, t+s$, and t_i then

$$E[M_f(t+s) - M_f(t)|X(r), r \leq t] = 0$$

for each t and $s > 0$, which proves that $M_f(t)$ is a martingale. Thus $X(t)$ solves the martingale problem for generator $\mathcal{A}$.

1.5.4 Weak Convergence

Definition 1.5.13. *Let P and P_n, $n = 1, 2, ...,$ denote the probability measures defined on a metric space $\mathbb{F}$. The sequence $\{P_n\}$ converges weakly to P and write $P_n \Rightarrow P$ if*

$$\int f dP_n \to \int f dP$$

for every bounded and continuous function f on $\mathbb{F}$. Suppose that $X_n(t)$ and $X(t)$ are random variables associated with P_n and P respectively. The sequence $X_n(t)$ converges weakly to $X(t)$ if for any bounded and continuous function f on $\mathbb{F}$,

$$Ef(X_n(t)) \to Ef(X(t))$$

as $n \to \infty$.

1.5.5 Tightness

The Space $D^d[0, \infty]$.

Skorohod theory is extended to the space $D^d_\infty = D^d[0, \infty)$ of cadlag functions on $[0, \infty)$: the space of functions from $[0, \infty)$ into R^d that are right continuous on $[0, \infty)$ and have finite left limit on $(0, \infty)$. Denote $D_\infty = D^1_\infty$. We endow D^d_∞ with the (Skorohod) J_1-topology. There is a metric m_{J_1} on D^d_∞ which induces this topology and under which the space is a complete, separable metric space (i.e., a Polish space). For our purposes, we do not need to know the precise form of this metric. References on Skorohod topology can be found in [1, 23]. For later use, for each $T \geq 0$, we define

$$\|x\|_T = \sup_{t \in [0,T]} |x(t)|, \text{ for } x \in D^d.$$

Definition 1.5.14 (Tightness). *A probability measure P defined on a metric space $\mathbb{F}$ is said to be tight if for each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{F}$ such that $P(K_\varepsilon) \geq 1 - \varepsilon$. A family of probability measures $\mathcal{M}$ defined on a metric space $\mathbb{F}$ is tight if for each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{F}$ such that*

$$\inf_{P \in \mathcal{M}} P(K_\varepsilon) \geq 1 - \varepsilon.$$

A sequence of random variables $\{X_n\}$ associated with a probability measure P defined on a metric space $\mathbb{F}$ is tight if for each $\varepsilon > 0$, there exist a compact set $K_\varepsilon \subset \mathbb{F}$ such that

$$\inf_n P\{X_n \in K_\varepsilon\} \geq 1 - \varepsilon.$$

Theorem 1.5.15 (Aldous' Tightness Criterion).

Let $\{X_n\}$ be a sequence of d-dimensional stochastic processes of D_∞. $\{X_n\}$ is tight if the following conditions hold for each $T > 0$:

(a) $\lim_{a \to \infty} \limsup_n P\{\|X_n\|_T \geq a\} = 0.$

(b) For each $\varepsilon > 0$, $\eta > 0$, there are positive constant $\delta_{\varepsilon,\eta}$ and $n_{\varepsilon,\eta}$ such that for all $0 < \delta \leq \delta_{\varepsilon,\eta}$ and $n \geq n_{\varepsilon,\eta}$,

$$\sup_{\tau \in \Lambda_{[0,T]}} P\{|X_n(\tau + \delta) - X_n(\tau)| \geq \varepsilon\} \leq \eta,$$

where $\Lambda_{[0,T]}$ denotes the set of all stopping times relative to the filtration generated by X_n that take values in a finite subset of $[0, T]$.

Theorem 1.5.16 (Prohorov)**.**

Let $\{X_n\}$ be random variables with values in B.

(a) If $\{X_n\}$ is tight then it is relatively compact, i.e, it contains a weakly convergent subsequence. If B is complete and separable, tightness is equivalent to relative compactness.

(b) If $X_n \Rightarrow X$ then $Ef(X_n) \to Ef(X)$, where $f(\cdot)$ is any real-valued bounded and measurable on B, that is continuous w.p. 1 under measure P_X.

1.5.6 Optimal Control

Definition 1.5.17. *A measurable function $f : \mathbb{R}^n \to [0, \infty]$ is called supermeanvalued (smv) (w.r.t. $X(t)$) if*

$$f(x) \geq E^x[f(X(\tau))]$$

for all stopping times τ and all $x \in \mathbb{R}^n$.

In addition, if f is also lower semincontinuous then f is called l.s.c. superharmonic (shm) or just superharmonic (w.r.t. $X(t)$).

Definition 1.5.18. *Let f be a real measurable function on $\mathbb{R}^n$. If f is a shm (smv) function and $f \geq h$, we say that f is a shm (smv) majorant of h (w.r.t. $X(t)$). The function*

$$\bar{h}(x) = \inf_f f(x), \ x \in \mathbb{R}^n,$$

where the inf *being taken over all smv majorants f of h, is called the least smv majorant of h. Similarly, suppose there exists a function $\widehat{h}$ such that*

(i) $\widehat{h}$ is a shm majorant of h, and

(ii) if f is any other shm majorant of h then $\widehat{h} \leq f$.

Then $\widehat{h}$ is called the least shm majorant of h.

Lemma 1.5.19. *(a) If f is shm (smv) and $\alpha > 0$ then αf is shm (smv).*

(b) If f_1, f_2 are shm (smv) then $f_1 + f_2$ is shm (smv).

(c) If $\{f_j\}_{j\in J}$ is a family of smv functions then $f(x) := \inf_{j\in J}\{f_j(x)\}$ is smv if it is measurable (J is any set).

(d) If $f_1, f_2, ...$ are shm (smv) functions and $f_k \uparrow f$ pointwise then f is shm (smv).

(e) If f is smv and $\sigma \leq \tau$ are stopping times then $E^x[f(X(\sigma))] \geq E^x[f(X(\tau))]$.

(f) If f is smv and H is a Borel set then $\tilde{f}(x) = E^x[f(X(\tau_H))]$ is smv.

Definition 1.5.20. *If $g : \mathcal{O} \to \mathbb{R}$, $x \in \mathcal{O}$, and*

$$g(y) \leq g(x) + < p, y - x > + \frac{1}{2} < X(y - x), y - x > + o(|y - x|^2) \text{ as } \mathcal{O} \ni y \to x$$

holds, we say $(p, X) \in J^{2,+}_{\mathcal{O}} g(x)$, the second-order superjet of g at x.

Similarly, the second-order subject of g at x is defined as

$$J^{2,-}_{\mathcal{O}} g(x) = -J^{2,+}_{\mathcal{O}}(-g(x)).$$

Definition 1.5.21. *The closure of $J^{2,+}g(x)$ is*

$$\bar{J}^{2,+}g(x) = \left\{(a, b) = \lim_{n\to\infty}(a_n, b_n),\ \text{where } (a_n, b_n) \in J^{2,+}g(x_n) \text{ and } \lim_{n\to\infty}(x_n, g(x_n)) = (x, g(x))\right\}.$$

Similarly, the closure of $J^{2,-}g(x)$ is

$$\bar{J}^{2,-}g(x) = \left\{(a, b) = \lim_{n\to\infty}(a_n, b_n),\ \text{where } (a_n, b_n) \in J^{2,-}g(x_n) \text{ and } \lim_{n\to\infty}(x_n, g(x_n)) = (x, g(x))\right\}.$$

Theorem 1.5.22. *If $x \in \mathcal{O}$ then the set of second-order superjet is*

$$J^{2,+}g(x) = \left\{ \left(\frac{\partial \varphi}{\partial x}(x,y), \frac{\partial^2 \varphi}{\partial x^2}(x,y) \right) : \varphi \in C^2(\mathbb{R} \times \mathbb{R}), \right.$$
$$\left. \text{and } g - \varphi \text{ has a global maximum at } (x,y) \right\},$$

and the set of second-order subjet is

$$J^{2,-}g(x) = \left\{ \left(\frac{\partial \varphi}{\partial x}(x,y), \frac{\partial^2 \varphi}{\partial x^2}(x,y) \right) : \varphi \in C^2(\mathbb{R} \times \mathbb{R}), \right.$$
$$\left. \text{and } g - \varphi \text{ has a global minimum at } (x,y) \right\}.$$

Theorem 1.5.23. *Let $\mathcal{O}_i,\ i = 1,2,$ be a locally compact subset of $\mathbb{R}$, and $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$.*

Let $g_i(x,y) : \mathcal{O} \to \mathbb{R},\ i = 1,2$ be upper semicontinuous functions, and $\varphi(x,y)$ be twice continuously differentiable in a neighborhood of $\mathcal{O}$.
Set

$$G(x,y) = g_1(x) + g_2(y) \text{ for } (x,y) \in \mathcal{O}.$$

Suppose $(x_0,y_0) \in \mathcal{O}$ is a local maximum of $G - \varphi$ relative to $\mathcal{O}$. Then for each $\delta > 0$ there exists $X, Y \in \mathbb{R}$ such that

$$\left(\frac{\partial \varphi}{\partial x}(x_0,y_0), X \right) \in \bar{J}^{2,+}_{\mathcal{O}_1} g_1(x_0) \text{ and } \left(\frac{\partial \varphi}{\partial y}(x_0,y_0), Y \right) \in \bar{J}^{2,+}_{\mathcal{O}_2} g_2(y_0),$$

and the block diagonal matrix with entries X, Y satisfies

$$-\left(\frac{1}{\delta} + \|A\| \right) I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \delta A^2,$$

where $A = H\varphi(x_0,y_0)$ is a 2×2 Hessian matrix. The norm of the symmetric matrix A is $\|A\| = \sup\{|\lambda| : \lambda$ is an eigenvalue of $A\} = \sup_{|\xi| \leq 1}\{| < A\xi, \xi > |\}$.

Chapter 2

An ODE Approach to Trading under a Mean-Reversion Model

2.1 Introduction

This chapter is concerned with trading an asset which has random fluctuation in its price but tends to move towards an "equilibrium" level. A mean-reversion model is often used to capture such movements. Financial market trading traditionally aims at buying low and selling high. However, identifying these low and high levels is a challenging task in practice. This chapter examines how to quantify lows and highs when the underlying asset price is governed by a mean-reversion model.

The objective is to buy and sell the underlying asset sequentially to maximize a discounted reward function. A fixed slippage (or commission) cost is associated with each transaction. Slippage affects most trading activities especially those with frequent transactions and those with larger orders. The associated HJB equations (quasi-variational inequalities) for the value functions are derived and the closed-form solutions are obtained using ODE methods. The optimal stopping times can be determined by three threshold levels x_0, x_1, and x_2. Algebraic equations for these threshold levels are obtained by means of smooth-fit techniques [17]. The optimal trading rule can be given in terms of two intervals: $I_1 = [x_0, x_1]$ and $I_2 = [-\infty, x_2]$. The idea is to initiate a trade (buy) whenever X_t enters I_1 and hold the position till X_t exits I_2. Similarly, we initiate a sell whenever X_t enters I_2 and hold the position till X_t enters I_1. We provide a Verification Theorem with a set of sufficient conditions that guarantee the optimality of the corresponding optimal stopping times. We also vary parameters to examine the affects on the threshold levels in numerical examples.

This chapter is organized as follows. In §2, the problem is formulated. In §3, some properties of the value functions are established. In §4, the associated HJB equations, and their solutions are obtained. In §5, a Verification Theorem is provided with a set of sufficient conditions that guarantee the optimality of the trading rule. In §6, numerical examples are discussed.

2.2 Formulating the Problem

Let $X_t \in \mathbb{R}$, $t \geq 0$, denote a mean-reversion process governed by

$$dX_t = a(b - X_t)dt + \sigma dW_t, \ X_0 = x, \tag{2.1}$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion. The asset price at time t is given by $S_t = \exp(X_t)$.
Let

$$0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \cdots \tag{2.2}$$

denote a sequence of stopping times where buying is at τ_n and selling is at σ_n, $n = 1, 2, \ldots$.

We restrict to the case where the net position at any time is either flat (with no share of stock holding) or long (with one share of stock holding). If initially the net position is long ($i = 1$) then one must sell the stock before buying any shares. In this case, the sequence of stopping times is denoted by $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$. Similarly, if initially the net position is flat ($i = 0$) then one must first buy a stock before selling any shares, and the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$.

Given the initial state $X_0 = x$ and initial net position $i = 0, 1$, the reward functions of the decision sequences, Λ_0 and Λ_1, are given as follows:

$$J_i(x, \Lambda_i) = \begin{cases} E\left\{ \displaystyle\sum_{n=1}^{\infty} \left[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K) \right] \right\}, & \text{if } i = 0, \\ E\left\{ e^{-\rho\sigma_1}(S_{\sigma_1} - K) \right. & \\ \quad \left. + \displaystyle\sum_{n=2}^{\infty} \left[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K) \right] \right\}, & \text{if } i = 1, \end{cases} \tag{2.3}$$

where $K > 0$ denote the slippage cost per transaction, and $\rho > 0$ the discount factor.

For simplicity, the term $E\sum_{n=1}^{\infty} \xi_n$ for random variables ξ_n is interpreted as

$$\limsup_{N\to\infty} E \sum_{n=1}^{N} \xi_n.$$

Let $V_i(x)$ denote the value functions with the initial net positions $i = 0, 1$ and initial state $X_0 = x$. That is,

$$V_i(x) = \sup_{\Lambda_i} J_i(x, \Lambda_i). \tag{2.4}$$

Remark 2.2.1. The equalities in (2.2) allow buying and selling simultaneously. However, due to the existence of positive slippage cost K, these actions cause negative returns and therefore are automatically eliminated by optimality conditions.

Remark 2.2.2. Using the Monte-Carlo, we generate sample paths for X_t and S_t using $a = 0.8, b = 2, \sigma = 0.5$, and $X_0 = 0.5$. It can be seen that X_t, and S_t fluctuate around the equilibrium level $b = 2$, and $e^b = 7.388$ respectively.

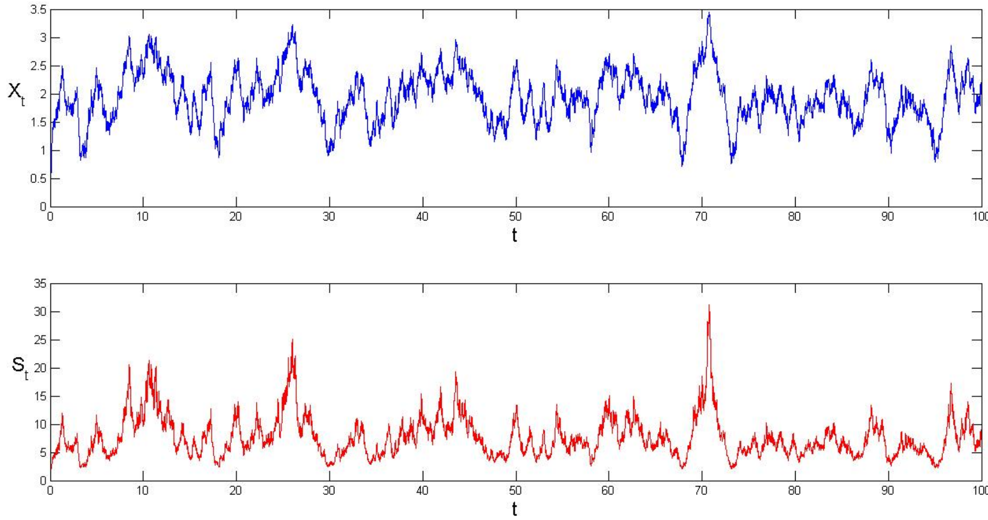

Figure 2.1: Sample paths of X_t and S_t.

2.3 Properties of the Value Functions

We now establish various properties of the value functions and solve the corresponding optimal stopping problem using the smooth-fit technique.

We observe that the sequence $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$ can be regarded as a combination of a buy at τ_1 and then followed by the sequence of stopping times $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$. Therefore,

$$\begin{aligned} V_0(x) \geq\ & J_0(x, \Lambda_0) \\ =\ & E\left\{ e^{-\rho\sigma_1}(S_{\sigma_1} - K) + \sum_{n=2}^{\infty} \left[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K) \right] \right\} \\ & -Ee^{-\rho\tau_1}(S_{\tau_1} + K) \\ =\ & J_1(X_{\tau_1}, \Lambda_1) - Ee^{-\rho\tau_1}(S_{\tau_1} + K). \end{aligned}$$

Setting $\tau_1 = 0$ (recall that $S_t = \exp(X_t)$), and taking supremum over all Λ_1, we get

$$V_0(x) \geq V_1(x) - e^x - K. \tag{2.5}$$

Similarly,

$$\begin{aligned} V_1(x) \geq\ & J_1(x, \Lambda_1) \\ =\ & J_0(X_{\sigma_1}, \Lambda_0) + Ee^{-\rho\sigma_1}(S_{\sigma_1} - K). \end{aligned}$$

By setting $\sigma_1 = 0$, and taking supremum over all Λ_0, we get

$$V_1(x) \geq V_0(x) + e^x - K. \tag{2.6}$$

The bounds for $V_i(x)$ can be obtained in the next lemma.

Lemma 2.3.1. *There exist constants K_0 and K_1 such that*

$$0 \le V_0(x) \le K_0 \text{ and}$$
$$V_1(x) \le e^x + K_1.$$

Proof. The lower bound of V_0 is clear from the definition ($V_0 = 0$ when the price stays low ($x < x_0$) all the time and there is no transaction). For the upper bound, we have

$$d(e^{-\rho t}S_t) = -\rho e^{-\rho t}S_t dt + e^{-\rho t} dS_t.$$

Using Itô's formula, we have

$$d(S_t) = d(e^{X_t}) = e^{X_t}dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 = e^{X_t}dX_t + \frac{1}{2}e^{X_t}\sigma^2 dt.$$

Hence,

$$\begin{aligned} d(e^{-\rho t}S_t) = &\ -\rho e^{-\rho t}S_t dt + e^{-\rho t}(e^{X_t}dX_t + \frac{1}{2}e^{X_t}\sigma^2 dt) \\ = &\ e^{-\rho t}S_t(-\rho dt + dX_t + \frac{\sigma^2}{2}dt) \\ = &\ e^{-\rho t}S_t(-\rho dt + a(b - X_t)dt + \sigma dW_t + \frac{\sigma^2}{2}dt) \\ = &\ e^{-\rho t}S_t\left[(\frac{\rho^2}{2} + ab - \rho - aX_t)dt + \rho dW_t\right] \\ = &\ e^{-\rho t}S_t(A - aX_t)dt + \rho e^{-\rho t}S_t dW_t, \end{aligned} \tag{2.7}$$

where $A = \frac{\sigma^2}{2} + ab - \rho$.

Integrate both sides of the equality in (2.7) from σ_n to τ_n, and then take expectation and note that $\int_{\tau_n}^{\sigma_n} e^{-\rho t}e^{X_t}dW_t$ is an Itô integral (hence martingale w.r.t. $\mathcal{F}_t = \sigma(\{W_s : s \le t\})$) to get

$$Ee^{-\rho\sigma_n}S_{\sigma_n} - Ee^{-\rho\tau_n}S_{\tau_n} = E\int_{\tau_n}^{\sigma_n} e^{-\rho t}e^{X_t}(A - aX_t)dt.$$

Note that the function $e^x(A - ax)$ is bounded above on $\mathbb{R}$. Let C be an upper bound. It follows that

$$Ee^{-\rho\sigma_n}S_{\sigma_n} - Ee^{-\rho\tau_n}S_{\tau_n} \leq CE\int_{\tau_n}^{\sigma_n} e^{-\rho t}dt. \tag{2.8}$$

Using the definition of $J_0(x, \Lambda_0)$, we have

$$\begin{aligned} J_0(x, \Lambda_0) \leq & \sum_{n=1}^{\infty}\left(Ee^{-\rho\sigma_n}S_{\sigma_n} - Ee^{-\rho\tau_n}S_{\tau_n}\right) \\ \leq & \sum_{n=1}^{\infty} CE\int_{\tau_n}^{\sigma_n} e^{-\rho t}dt \\ \leq & \; CE\int_0^{\infty} e^{-\rho t}dt = \frac{C}{\rho} := K_0. \end{aligned}$$

Taking supremum of the inequality over all Λ_0, we obtain $V_0(x) \leq K_0$.

For the bound of $V_1(x)$, using the definition of $J_1(x, \Lambda_1)$, we have

$$J_1(x, \Lambda_1) \leq J_0(x, \Lambda_0) + Ee^{-\sigma\rho_1}(S_{\sigma_1} - K) \leq K_0 + Ee^{-\sigma\rho_1}S_{\sigma_1}.$$

Similarly, integrate both sides of the equality in (2.7) from 0 to σ_1, and then take expectation to get

$$Ee^{-\rho\sigma_1}S_{\sigma_1} - e^x \leq \frac{C}{\rho}.$$

It follows that

$$J_1(x, \Lambda_1) \leq 2K_0 + e^x.$$

Taking supremum of the inequality over all Λ_1, we obtain $V_1(x) \leq e^x + K_1$, where $K_1 := 2K_0$.

Alternatively, use $V_0(x) \geq V_1(x) - e^x - K$. □

2.4 The HJB Equations

2.4.1 Establishing the HJB equations

Here we derive the HJB equations.

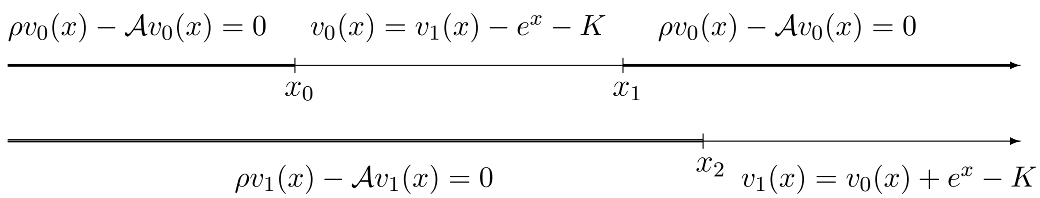

Figure 2.2: Continuation regions (darkened intervals)

The generator $\mathcal{A}$ of X_t is given by

$$\mathcal{A} = a(b-x)\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}.$$

For the rest of the chapter, we assume $v_i \in C^2(\mathbb{R})$, $i = 0, 1$.

Lemma 2.4.1. *For $i = 0, 1$, and for any $0 \le \theta_1 \le \theta_2$,*

$$E\left[\int_{\theta_1}^{\theta_2} e^{-\rho t}(\rho - \mathcal{A})v_i(X_t)dt\right] = Ee^{-\rho\theta_1}v_i(X_{\theta_1}) - Ee^{-\rho\theta_2}v_i(X_{\theta_2}).$$

Proof. For $i = 0, 1$,

$$d(e^{-\rho t}v_i(X_t)) = -\rho e^{-\rho t}v_i(X_t)dt + e^{-\rho t}d(v_i(X_t)).$$

Using Itô formula and $(dX_t)^2 = \sigma^2 dt$, we obtain

$$\begin{aligned} d(v_i(X_t)) = & \; v'(X_t)dX_t + \frac{1}{2}\sigma^2 v''(X_t)dt \\ = & \; v'(X_t)\left[a(b - X_t)dt + \sigma dW_t\right] + \frac{1}{2}\sigma^2 v''(X_t)dt \\ = & \left[a(b - X_t)v'(X_t) + \frac{1}{2}\sigma^2 v''(X_t)\right] dt + \sigma v'(X_t)dW_t \end{aligned}$$

Hence,

$$\begin{aligned} d(e^{-\rho t}v_i(X_t)) = & \; -\rho e^{-\rho t}v_i(X_t)dt + e^{-\rho t}\left\{\left[a(b - X_t)v_i'(X_t) + \frac{1}{2}\sigma^2 v_i''(X_t)\right] dt + \sigma v'(X_t)dW_t\right\} \\ = & \; e^{-\rho t}\left[-\rho v_i(X_t) + a(b - X_t)v_i'(X_t) + \frac{1}{2}\sigma^2 v_i''(X_t)\right] dt + \sigma e^{-\rho t}v_i'(X_t)dW_t \\ = & \; e^{-\rho t}(-\rho + \mathcal{A})v_i(X_t)dt + \sigma e^{-\rho t}v_i'(X_t)dW_t. \end{aligned} \tag{2.9}$$

Integrate both sides of equation (2.9) from θ_1 to θ_2, and then take expectation and note that $\int_{\theta_1}^{\theta_2} \sigma e^{-\rho t}v_i'(X_t)dW_t$ is an Itô integral (hence martingale w.r.t. $\mathcal{F}_t = \sigma(\{W_s : s \leq t\})$) to obtain

$$Ee^{-\rho\theta_2}v_i(X_{\theta_2}) - Ee^{-\rho\theta_1}v_i(X_{\theta_1}) = E\left[\int_{\theta_1}^{\theta_2} -e^{-\rho t}(\rho - \mathcal{A})v_i(X_t)dt\right].$$

□

If the net position is flat ($i = 0$) then one should only buy when the price is low but not much smaller than K (say between x_0 and x_1). Note that when starting at x in $[x_0, x_1]$, one should buy immediately ($t = 0$). In this case, $V_0(x) = V_1(x) - e^x - K$. The corresponding continuation region should include $(-\infty, x_0) \cup (x_1, \infty)$.

On the other hand, if the net position is long ($i = 1$) then one should only sell when the price is high (greater than or equal to x_2). In this case, $V_1(x) = V_0(x) + e^x - K$ and the continuation region should include $(-\infty, x_2)$. These continuation regions are darkened in Figure 2.2.

Moreover, one should not establish any new position in continuation regions. In other words, the values does not change throughout these regions. Hence, $Ee^{-\rho\theta_1}V_i(X_{\theta_1}) = Ee^{-\rho\theta_2}V_i(X_{\theta_2}), i = 0, 1$, for any X_{θ_1} and X_{θ_2}, $\theta_1 \leq \theta_2$, in those regions. In view of Lemma 2.4.1, $e^{-\rho t}(\rho - \mathcal{A})V_i(X_t) = 0$ almost surely or $(\rho - \mathcal{A})V_i(X_t) = 0$, $i = 0, 1$, almost surely.

Therefore, the associated HJB equations should have the form

$$\begin{aligned} &\min\Big\{\rho v_0(x) - \mathcal{A}v_0(x),\ v_0(x) - v_1(x) + e^x + K\Big\} = 0, \\ &\min\Big\{\rho v_1(x) - \mathcal{A}v_1(x),\ v_1(x) - v_0(x) - e^x + K\Big\} = 0, \end{aligned} \tag{2.10}$$

where $v_i(x), i = 1, 2$, have to satisfy the following conditions:

$$\begin{cases} v_0(x) \geq v_1(x) - e^x - K & \text{on } (-\infty, x_0) \cup (x_1, \infty), \\ v_1(x) \geq v_0(x) + e^x - K & \text{on } (-\infty, x_2), \\ (\rho - \mathcal{A})(v_1(x) - e^x - K) \geq 0 & \text{on } (x_0, x_1), \\ (\rho - \mathcal{A})(v_0(x) + e^x - K) \geq 0 & \text{on } (x_2, \infty), \end{cases} \tag{2.11}$$

and

$$\begin{cases} v_0(x) = v_1(x) - e^x - K & \text{on } [x_0, x_1], \\ v_1(x) = v_0(x) + e^x - K & \text{on } [x_2, \infty), \\ (\rho - \mathcal{A})v_0(x) = 0 & \text{on } (-\infty, x_0] \cup [x_1, \infty), \\ (\rho - \mathcal{A})v_1(x) = 0 & \text{on } (-\infty, x_2]. \end{cases} \tag{2.12}$$

2.4.2 Examining Conditions (2.11)

We examine the conditions in (2.11) on intervals $(-\infty, x_0)$, (x_0, x_1), (x_1, x_2), and (x_2, ∞).

First, on $(-\infty, x_0)$, it requires the first two inequalities of (2.11), i.e.,

$$|v_1(x) - v_0(x) - e^x| \leq K \text{ on } (-\infty, x_0). \tag{2.13}$$

On (x_0, x_1), it requires the second and third inequalities of (2.11). Note that on this interval, $v_0(x) = v_1(x) - e^x - K$ or $v_1(x) = v_0(x) + e^x + K \geq v_0(x) + e^x - K$. Therefore, the second inequality is automatically satisfied. Using $(\rho - \mathcal{A})v_1(x) = 0$, the third inequality can be simplified to

$$e^x(-ax + ab - \rho + \sigma^2/2) - \rho K \geq 0 \text{ on } (x_0, x_1). \tag{2.14}$$

On (x_1, x_2), it requires the first two inequalities of (2.11). That is,

$$|v_1(x) - v_0(x) - e^x| \leq K \text{ on } (x_1, x_2). \tag{2.15}$$

On (x_2, ∞), it requires the first and last inequality of (2.11). Note that on this interval, $v_1(x) = v_0(x) + e^x - K$ or $v_0(x) = v_1(x) - e^x + K \geq v_1(x) - e^x - K$. Therefore, the first inequality is automatically satisfied. Using $(\rho - \mathcal{A})v_0(x) = 0$, the last inequality can be simplified to

$$e^x(ax - ab + \rho - \sigma^2/2) - \rho K \geq 0.$$

Equivalently,

$$ax - ab + \rho - \sigma^2/2 - \rho K e^{-x} \geq 0 \text{ on } (x_2, \infty). \tag{2.16}$$

In (2.14), let $f(x) = e^x(-ax+ab-\rho+\sigma^2/2)-\rho K$. Then $f'(x) = e^x(-ax+ab-a+\sigma^2/2-\rho)$. This implies $f(x)$ has the only local and absolute maximum at $x_{max} = (ab-a+\sigma^2/2-\rho)/a$. Therefore, it is necessary $f(x_0) \geq 0$, and $f(x_1) \geq 0$.

In (2.16), let $g(x) = ax - ab + \rho - \sigma^2/2 - \rho K e^{-x}$. Then $g'(x) = a + \rho K e^{-x} \geq 0$. Therefore, it is necessary $g(x_2) \geq 0$.

2.4.3 Solving the HJB Equations

We will obtain the threshold levels (x_0, x_1, x_2) by solving the HJB equation in (3.4). We first solve the equations $\rho v_i(x) - \mathcal{A} v_i(x) = 0$ with $i = 0, 1$.

Let

$$\begin{cases} \phi_1(x) = \int_0^\infty \eta(t) e^{-\kappa(b-x)t} dt \\ \phi_2(x) = \int_0^\infty \eta(t) e^{\kappa(b-x)t} dt \end{cases}$$

where $\kappa = \sqrt{2a}/\sigma$, $\lambda = \rho/a$, and $\eta(t) = t^{\lambda-1} \exp(-t^2/2)$. Then the general solution of $\rho v_i(x) - \mathcal{A} v_i(x) = 0$ is given by a linear combination of these functions (details can be found in Eloe et al. [9]).

Note that $\phi_1(\infty) = \infty$ and $\phi_2(-\infty) = \infty$. We will derive $v_i(x), i = 1, 2$, on each continuation region.

First, consider the interval (x_1, ∞). Suppose $v_0(x) = A_1\phi_1(x) + A_2\phi_2(x)$, for some A_1 and A_2. By Lemma 2.3.1, $v_0(\infty)$ should be bounded above. This implies $v_0(x) = A_2\phi_2(x)$.

On the interval $(-\infty, x_0)$, suppose $v_0(x) = B_1\phi_1(x) + B_2\phi_2(x)$, for some B_1 and B_2. By Lemma 2.3.1, $v_0(-\infty)$ should be bounded above. This implies $v_0(x) = B_1\phi_1(x)$.

On the interval $(-\infty, x_2)$, suppose $v_1(x) = C_1\phi_1(x) + C_2\phi_2(x)$, for some C_1 and C_2. By Lemma 2.3.1, $v_0(-\infty)$ should be bounded above. This implies $v_1(x) = C_1\phi_1(x)$.

Since $v_i(x), i = 1, 2$, are twice continuously differentiable on their continuation region, we can follow the smooth-fit method. In particular, it requires $v_0(x)$ to be continuously differentiable at x_0 and x_1, and $v_1(x)$ to be continuously differentiable at x_2. Therefore,

$$\begin{cases} B_1\phi_1(x_0) = C_1\phi_1(x_0) - e^{x_0} - K, \\ B_1\phi_1'(x_0) = C_1\phi_1'(x_0) - e^{x_0}, \end{cases}$$

$$\begin{cases} A_2\phi_2(x_1) = C_1\phi_1(x_1) - e^{x_1} - K, \\ A_2\phi_2'(x_1) = C_1\phi_1'(x_1) - e^{x_1}, \end{cases}$$

and

$$\begin{cases} C_1\phi_1(x_2) = A_2\phi_2(x_2) + e^{x_2} - K, \\ C_1\phi_1'(x_2) = A_2\phi_2'(x_2) + e^{x_2}. \end{cases}$$

Let

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix}.$$

The systems above are equivalent to

$$\begin{cases} (B_1 - C_1)\phi_1(x_0) = -e^{x_0} - K \\ (B_1 - C1)\phi_1'(x_0) = -e^{x_0}, \end{cases} \tag{2.17}$$

$$\Phi(x_1) \begin{pmatrix} C_1 \\ -A_2 \end{pmatrix} = \begin{pmatrix} e^{x_1} + K \\ e^{x_1} \end{pmatrix}, \tag{2.18}$$

$$\Phi(x_2) \begin{pmatrix} C_1 \\ -A_2 \end{pmatrix} = \begin{pmatrix} e^{x_2} - K \\ e^{x_2} \end{pmatrix}. \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\begin{pmatrix} C_1 \\ -A_2 \end{pmatrix} = \Phi^{-1}(x_1) \begin{pmatrix} e^{x_1} + K \\ e^{x_1} \end{pmatrix} = \Phi^{-1}(x_2) \begin{pmatrix} e^{x_2} - K \\ e^{x_2} \end{pmatrix}. \tag{2.20}$$

In (2.17), since e^{x_0} is non-zero, so is $(B_1 - C_1)\phi_1'(x_0)$. Hence, divide the first equation by the second to get $\dfrac{\phi_1(x_0)}{\phi_1'(x_0)} = \dfrac{e^{x_0}+K}{e^{x_0}}$. This implies

$$\phi_1(x_0)e^{x_0} = \phi_1'(x_0)(e^{x_0}+K). \tag{2.21}$$

Solve (2.20) to get x_1 and x_2, and then obtain A_2 and C_1. Also, solve (2.21) to get x_0, and then obtain B_1 from (2.17). Hence, the values function are determined by

$$v_0(x) = \begin{cases} B_1\phi_1(x) & \text{on } (-\infty, x_0), \\ C_1\phi_1(x) - e^x - K & \text{on } [x_0, x_1), \\ A_2\phi_2(x) & \text{on } [x_1, \infty). \end{cases} \tag{2.22}$$

$$v_1(x) = \begin{cases} C_1\phi_1(x) & \text{on } (-\infty, x_2), \\ A_2\phi_2(x) + e^x - K & \text{on } [x_2, \infty). \end{cases} \tag{2.23}$$

Additionally, due to the existence of the slippage cost, if the stock was bought at $S_1 = e^{x_1}$ and sold at $S_2 = e^{x_2}$ then it requires that $e^{x_2} - K > e^{x_1} + K$. Equivalently,

$$e^{x_2} - e^{x_1} > 2K. \tag{2.24}$$

2.5 A Verification Theorem

We will summarize the analysis in section (2.4) in a Verification Theorem, and show that the solution $v_i(x)$, $i = 0, 1$, of equation (3.4) is equal to the value functions $V_i(x)$, $i = 0, 1$, respectively, and sequences of optimal stopping times can be constructed by using the triple (x_0, x_1, x_2).

Theorem 2.5.1. *Let* $f(x) = e^x(-ax + ab - \rho + \sigma^2/2) - \rho K$ *and* $g(x) = ax - ab + \rho - \sigma^2/2 - \rho K e^{-x}$*. Let* (x_0, x_1, x_2) *be a solution to (2.20) and* (2.21) *satisfying*

$$e^{x_2} - e^{x_1} > 2K.$$

and $f(x_0)$*,* $f(x_1)$*, and* $g(x_2)$ *are all non-negative.*

Let A_2, B_1, C_1 *be constants given by* (2.20) *and* (2.17)*.*

Let

$$\begin{cases} v_0(x) = \begin{cases} B_1\phi_1(x) & \text{on } (-\infty, x_0), \\ C_1\phi_1(x) - e^x - K & \text{on } [x_0, x_1), \\ A_2\phi_2(x) & \text{on } [x_1, \infty). \end{cases} \\ v_1(x) = \begin{cases} C_1\phi_1(x) & \text{on } (-\infty, x_2), \\ A_2\phi_2(x) + e^x - K & \text{on } [x_2, \infty). \end{cases} \end{cases}$$

Assume $v_0(x_2) \geq 0$*, and* $|v_1(x) - v_0(x) - e^x| \leq K$ *on* $(-\infty, x_0) \cup (x_1, x_2)$*. Then*

$$v_i(x) = V_i(x), \; i = 0, 1.$$

Moreover, if initially $i = 0$*, let* $\Lambda_0^* = (\tau_1^*, \sigma_1^*, \tau_2^*, \sigma_2^*, \ldots)$*, where the stopping times* $\tau_1^* = \inf\{t \geq 0 : \; X_t \in [x_0, x_1]\}$*,* $\sigma_n^* = \inf\{t \geq \tau_n^* : \; X_t = x_2\}$*, and* $\tau_{n+1}^* = \inf\{t > \sigma_n^* : \; X_t \in [x_0, x_1]\}$ *for* $n \geq 1$*. Similarly, if initially* $i = 1$*, let* $\Lambda_1^* = (\sigma_1^*, \tau_2^*, \sigma_2^*, \tau_3^*, \ldots)$*, where the stopping times* $\sigma_1^* = \inf\{t \geq 0 : \; X_t \geq x_2\}$*,* $\tau_n^* = \inf\{t > \sigma_{n-1}^* : \; X_t \in [x_0, x_1]\}$*, and* $\sigma_n^* = \inf\{t \geq \tau_n^* : \; X_t = x_2\}$ *for* $n \geq 2$*.*

Then Λ_0^* *and* Λ_1^* *are optimal.*

The following lemmas will be used in the proof of the Theorem 2.5.1.

Lemma 2.5.2. *Given z_1 and z_2, let $\theta_1 = \inf\{t : X_t \geq z_1\}$ and $\theta_2 = \inf\{t : X_t \leq z_2\}$. Then*

$$P(\theta_1 < \infty) = P(\theta_2 < \infty) = 1.$$

Lemma 2.5.3. *For any stopping times θ_1 and θ_2, if $0 \leq \theta_1 \leq \theta_2$, a.s., and $\rho v_i(X_t) - \mathcal{A}v_i(X_t) \geq 0$, for all $t \in [\theta_1, \theta_2]$, for $i = 0, 1$, then*

$$Ee^{-\rho\theta_1} v_i(X_{\theta_1}) \geq Ee^{-\rho\theta_2} v_i(X_{\theta_2}).$$

In particular, if $\theta_1 = 0$, $\theta_2 = \tau \geq 0$, and $X_0 = x$ then $v_i(x) \geq Ee^{-\rho\tau} v_i(X_\tau)$.

The equalities happen when $\rho v_i(X_t) - \mathcal{A}v_i(X_t) = 0$ for all $t \in [\theta_1, \theta_2]$.

Proof. Use Lemma 2.4.1 and the hypothesis $\rho v_i(X_t) - \mathcal{A}v_i(X_t) \geq 0$, for all $t \in [\theta_1, \theta_2]$, to obtain

$$Ee^{-\rho\theta_1} v_i(X_{\theta_1}) \geq Ee^{-\rho\theta_2} v_i(X_{\theta_2}).$$

Set $\theta_1 = 0, \theta_2 = \tau$ to obtain

$$v_i(x) \geq Ee^{-\rho\tau} v_i(X_\tau).$$

Moreover, if $\rho v_i(X_t) - \mathcal{A}v_i(X_t) = 0$ for all $t \in [\theta_1, \theta_2]$ then Lemma 2.4.1 gives the equalities.

□

Lemma 2.5.4. *If the position is $i = 0$ and $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$ then for all $N \geq 1$,*

$$Ee^{-\rho\tau_1} v_0(X_{\tau_1}) \geq Ee^{-\rho\sigma_N} v_0(X_{\sigma_N}) + E\sum_{n=1}^{N} \Big[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K)\Big].$$

Similarly, if the position is $i = 1$ and $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$ then for all $N \geq 2$,

$$Ee^{-\rho\sigma_1} v_1(X_{\sigma_1}) \geq Ee^{-\rho\sigma_N} v_0(X_{\sigma_N}) + Ee^{-\rho\sigma_1}(S_{\sigma_1} - K) + E\sum_{n=2}^{N} \Big[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K)\Big].$$

Proof. Since $v_i, i = 1, 2$ are solutions of the HJB equations in (3.4), we have for all $t \geq 0$,

$$\begin{aligned} v_0(X_t) &\geq v_1(X_t) - e^{X_t} - K, \\ v_1(X_t) &\geq v_0(X_t) + e^{X_t} - K. \end{aligned} \tag{2.25}$$

It follows, for the position $i = 0$, that

$$\begin{aligned} Ee^{-\rho\tau_1}v_0(X_{\tau_1}) \geq\ & Ee^{-\rho\tau_1}\left(v_1(X_{\tau_1}) - S_{\tau_1} - K\right) \\ =\ & Ee^{-\rho\tau_1}v_1(X_{\tau_1}) - Ee^{-\rho\tau_1}(S_{\tau_1} + K) \\ \geq\ & Ee^{-\rho\sigma_1}v_1(X_{\sigma_1}) - Ee^{-\rho\tau_1}(S_{\tau_1} + K) \\ \geq\ & Ee^{-\rho\sigma_1}(v_0(X_{\sigma_1}) + S_{\sigma_1} - K) - Ee^{-\rho\tau_1}(S_{\tau_1} + K) \\ =\ & Ee^{-\rho\sigma_1}v_0(X_{\sigma_1}) + E\Big[e^{-\rho\sigma_1}(S_{\sigma_1} - K) - e^{-\rho\tau_1}(S_{\tau_1} + K)\Big] \\ \geq\ & Ee^{-\rho\tau_2}v_0(X_{\tau_2}) + E\Big[e^{-\rho\sigma_1}(S_{\sigma_1} - K) - e^{-\rho\tau_1}(S_{\tau_1} + K)\Big]. \end{aligned}$$

In the expressions above, the third line uses Lemma 2.5.3 for $\tau_1 \leq \sigma_1$. The fourth line uses (2.25). The last line uses Lemma 2.5.3 for $\sigma_1 \leq \tau_2$.

Similarly,

$$Ee^{-\rho\tau_2}v_0(X_{\tau_2}) \geq Ee^{-\rho\sigma_2}v_0(X_{\sigma_2}) + E\Big[e^{-\rho\sigma_2}(S_{\sigma_2} - K) - e^{-\rho\tau_2}(S_{\tau_2} + K)\Big]. \tag{2.26}$$

Continue this way to obtain the first inequality of Lemma 2.5.4:

$$Ee^{-\rho\tau_1}v_0(X_{\tau_1}) \geq Ee^{-\rho\sigma_N}v_0(X_{\sigma_N}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K)\Big] \text{ for all } N \geq 1.$$

For the second inequality, we use similar computations for the position $i = 1$.

$$\begin{aligned}
Ee^{-\rho\sigma_1} v_1(X_{\sigma_1}) \geq & \; Ee^{-\rho\sigma_1}\left(v_0(X_{\sigma_1}) + S_{\sigma_1} - K\right) \\
= & \; Ee^{-\rho\sigma_1} v_0(X_{\sigma_1}) + Ee^{-\rho\sigma_1}(S_{\sigma_1} - K) \\
\geq & \; Ee^{-\rho\tau_2}(v_0(X_{\tau_2}) + Ee^{-\rho\sigma_1}(S_{\sigma_1} - K) \\
\geq & \; Ee^{-\rho\sigma_2} v_0(X_{\sigma_2}) + E\Big[e^{-\rho\sigma_2}(S_{\sigma_2} - K) - e^{-\rho\tau_2}(S_{\tau_2} + K)\Big] + Ee^{-\rho\sigma_1}(S_{\sigma_1} - K).
\end{aligned}$$

In the expressions above, the first line uses (2.25). The third line uses Lemma 2.5.3 for $\sigma_1 \leq \tau_2$. The last line uses (2.26).

Continue this way to obtain the second inequality of Lemma 2.5.4. $\square$

Lemma 2.5.5. *If the position is $i = 0$ and $\Lambda_0^* = (\tau_1^*, \sigma_1^*, \tau_2^*, \sigma_2^*, \ldots)$ is defined as in the Theorem, then for all $N \geq 1$,*

$$Ee^{-\rho\tau_1^*} v_0(X_{\tau_1^*}) = Ee^{-\rho\sigma_N^*} v_0(X_{\sigma_N^*}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n^*}(S_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(S_{\tau_n^*} + K)\Big].$$

Similarly, if the position is $i = 1$ and $\Lambda_1^* = (\sigma_1^*, \tau_2^*, \sigma_2^*, \tau_3^*, \ldots)$ is defined as in the Theorem, then for all $N \geq 2$,

$$Ee^{-\rho\sigma_1^*} v_1(X_{\sigma_1^*}) = Ee^{-\rho\sigma_N^*} v_0(X_{\sigma_N^*}) + Ee^{-\rho\sigma_1^*}(S_{\sigma_1^*} - K) + E\sum_{n=2}^{N}\Big[e^{-\rho\sigma_n^*}(S_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(S_{\tau_n^*} + K)\Big].$$

Proof. Since $v_i, i = 1, 2$ are solutions of the HJB equations in (3.4), they have to satisfy (2.12), i.e.,

$$\begin{cases}
v_0(x) = v_1(x) - e^x - K & \text{on } [x_0, x_1], \\
v_1(x) = v_0(x) + e^x - K & \text{on } [x_2, \infty), \\
(\rho - \mathcal{A})v_0(x) = 0 & \text{on } (-\infty, x_0] \cup [x_1, \infty), \\
(\rho - \mathcal{A})v_1(x) = 0 & \text{on } (-\infty, x_2].
\end{cases}$$

Note, in view of Lemma 2.5.2, that $\tau_n^* < \infty$, and $\sigma_n^* < \infty$ a.s. for $n \geq 1$.

First, consider the position $i = 0$. Note that $X_{\tau_1^*} \in [x_0, x_1]$. Hence,

$$\begin{aligned} Ee^{-\rho\tau_1^*} v_0(X_{\tau_1^*}) = &\ Ee^{-\rho\tau_1^*}\big(v_1(X_{\tau_1^*}) - S_{\tau_1^*} - K\big) \\ = &\ Ee^{-\rho\tau_1^*} v_1(X_{\tau_1^*}) - Ee^{-\rho\tau_1^*}(S_{\tau_1^*} + K). \end{aligned}$$

Note that $X_t \in (-\infty, x_2]$ for all $t \in [\tau_1^*, \sigma_1^*]$ and $X_{\sigma_1^*} = x_2$. This implies $(\rho - \mathcal{A})v_1(X_t) = 0$ for all $t \in [\tau_1^*, \sigma_1^*]$ and $v_1(X_{\sigma_1^*}) = v_0(X_{\sigma_1^*}) + e^{X_{\sigma_1^*}} - K$. Lemma (2.5.3) implies

$$\begin{aligned} Ee^{-\rho\tau_1^*} v_1(X_{\tau_1^*}) = &\ Ee^{-\rho\sigma_1^*} v_1(X_{\sigma_1^*}) \\ = &\ Ee^{-\rho\sigma_1^*}\big(v_0(X_{\sigma_1^*}) + S_{\sigma_1^*} - K\big) \\ = &\ Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) + Ee^{-\rho\sigma_1^*}(S_{\sigma_1^*} - K). \end{aligned}$$

Therefore,

$$Ee^{-\rho\tau_1^*} v_1(X_{\tau_1^*}) = Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) + E\Big[e^{-\rho\sigma_1^*}(S_{\sigma_1^*} - K) - e^{-\rho\tau_1^*}(S_{\tau_1^*} + K)\Big].$$

Note that $X_t \in [x_1, \infty)$ for all $t \in [\sigma_1^*, \tau_2^*]$. This implies $(\rho - \mathcal{A})v_0(X_t) = 0$ for all $t \in [\sigma_1^*, \tau_2^*]$. Lemma (2.5.3) implies $Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) = Ee^{-\rho\tau_2^*} v_0(X_{\tau_2^*})$.

Similarly,

$$Ee^{-\rho\tau_2^*} v_0(X_{\tau_2^*}) = Ee^{-\rho\sigma_2^*} v_0(X_{\sigma_2^*}) + E\Big[e^{-\rho\sigma_2^*}(S_{\sigma_2^*} - K) - e^{-\rho\tau_2^*}(S_{\tau_2^*} + K)\Big]. \qquad (2.27)$$

Continue the procedure to obtain

$$Ee^{-\rho\tau_1^*} v_1(X_{\tau_1^*}) = Ee^{-\rho\sigma_N^*} v_0(X_{\sigma_N^*}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n^*}(S_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(S_{\tau_n^*} + K)\Big] \text{ for all } N \geq 1.$$

For the second equality, we use similar computations for the position $i = 1$ as follows.

Note that $X_{\sigma_1^*} \in [x_2, \infty)$. Hence,

$$\begin{aligned} Ee^{-\rho\sigma_1^*} v_1(X_{\sigma_1^*}) = & \; Ee^{-\rho\sigma_1^*} \big(v_0(X_{\sigma_1^*}) + S_{\sigma_1^*} - K \big) \\ = & \; Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) + Ee^{-\rho\sigma_1^*} (S_{\sigma_1^*} - K). \end{aligned}$$

Note also that $X_t \in [x_1, \infty)$ for all $t \in [\sigma_1^*, \tau_2^*]$. This implies $(\rho - \mathcal{A}) v_0(X_t) = 0$ for all $t \in [\sigma_1^*, \tau_2^*]$. Lemma 2.5.3 implies $Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) = Ee^{-\rho\tau_2^*} v_0(X_{\tau_2^*})$. Use (2.27) to obtain

$$Ee^{-\rho\sigma_1^*} v_0(X_{\sigma_1^*}) = Ee^{-\rho\sigma_2^*} v_0(X_{\sigma_2^*}) + E\Big[e^{-\rho\sigma_2^*} (S_{\sigma_2^*} - K) - e^{-\rho\tau_2^*} (S_{\tau_2^*} + K) \Big].$$

Similarly,

$$Ee^{-\rho\sigma_2^*} v_0(X_{\sigma_2^*}) = Ee^{-\rho\sigma_3^*} v_0(X_{\sigma_3^*}) + E\Big[e^{-\rho\sigma_3^*} (S_{\sigma_3^*} - K) - e^{-\rho\tau_3^*} (S_{\tau_3^*} + K) \Big].$$

Continue the procedure to obtain the second equality. $\square$

Proof of Theorem 2.5.1. The proof is divided into two steps. In the first step, we show that $v_i(x) \geq J_i(x, \Lambda_i)$ for all Λ_i. Then in the second step, we show that $v_i(x) = J_i(x, \Lambda_i^*)$. Therefore, $v_i(x) = V_i(x)$, and Λ_i^* is optimal.

For the first step, first note that

$$v_0(X_{\sigma_N}) = v_0(x_2) \text{ for } \sigma_N \in \Lambda_0, N \geq 1,$$

and

$$v_0(X_{\sigma_N}) = v_0(x_2) \text{ for } \sigma_N \in \Lambda_1, N \geq 2.$$

In view of Lemma 2.5.4 and the assumption $v_0(x_2) \geq 0$, we have

$$\begin{aligned} Ee^{-\rho\tau_1}v_0(X_{\tau_1}) \geq\ & E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n}(S_{\sigma_n}-K) - e^{-\rho\tau_n}(S_{\tau_n}+K)\Big] \text{ for } N \geq 1, \text{ and} \\ Ee^{-\rho\sigma_1}v_1(X_{\sigma_1}) \geq\ & Ee^{-\rho\sigma_1}(S_{\sigma_1}-K) + E\sum_{n=2}^{N}\Big[e^{-\rho\sigma_n}(S_{\sigma_n}-K) - e^{-\rho\tau_n}(S_{\tau_n}+K)\Big] \text{ for } N \geq 2. \end{aligned}$$

Moreover, since $v_i, i = 0, 1$, satisfy the quasi-variational inequalities in (3.4), $\rho v_i(X_t) - \mathcal{A}v_i(X_t) \geq 0$ for all $t \geq 0$. Let $X_0 = x$. Use Lemma 2.5.3 to obtain

$$v_0(x) \geq Ee^{-\rho\tau_1}v_0(X_{\tau_1}) \text{ and } v_1(x) \geq Ee^{-\rho\sigma_1}v_1(X_{\sigma_1}).$$

Sending $N \to \infty$, we obtain $v_0(x) \geq J_0(x, \Lambda_0)$ for all Λ_0, and $v_1(x) \geq J_1(x, \Lambda_1)$ for all Λ_1. This implies that $v_0(x) \geq V_0(x)$ and $v_1(x) \geq V_1(x)$.

For the second step, we establish the equalities. Note that $v_0 \in C^2(\mathbb{R} \setminus \{x_0, x_1\})$, $v_1 \in C^2(\mathbb{R} \setminus \{x_2\})$, and both v_0 and v_1 are in $C^1(\mathbb{R})$.

Note that if the position is $i = 0$ then $X_t \in (-\infty, x_0] \cup [x_1, \infty)$ for all $t \in [0, \tau_1^*]$, which implies $(\rho - \mathcal{A})v_0(X_t) = 0$ for all $t \in [0, \tau_1^*]$.

Similarly, if the position is $i = 1$ then $X_t \in (-\infty, x_2]$ for all $t \in [0, \sigma_1^*]$, which implies $(\rho - \mathcal{A})v_1(X_t) = 0$ for all $t \in [0, \sigma_1^*]$.

Using Lemma 2.5.3, we get

$$v_0(x) = Ee^{-\rho\tau_1^*}v_0(X_{\tau_1^*}) \text{ and } v_1(x) = Ee^{-\rho\sigma_1^*}v_0(X_{\sigma_1^*}).$$

In view of Lemma 2.5.5, it remains to show $Ee^{-\rho\sigma_N^*}v_0(X_{\sigma_N^*}) \to 0$.

Note that $X_{\sigma_N^*} = x_2$ for all $N \geq 0$, therefore, $v_0(X_{\sigma_N^*}) = v_0(x_2)$.

Thus, it suffices to show $Ee^{-\rho\sigma_N^*} \to 0$ as $N \to \infty$. To this end, note that $v_0(x_2) \geq 0$ by assumption, so let $N \to \infty$ in the first equation of Lemma 2.5.5 to obtain

$$\begin{aligned}
v_0(x) \geq & \sum_{n=1}^{\infty} E\Big[e^{-\rho\sigma_n^*}(S_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(S_{\tau_n^*} + K)\Big] \\
= & -Ee^{-\rho\tau_1^*}(S_{\tau_1^*} + K) + \sum_{n=1}^{\infty} E\Big[e^{-\rho\sigma_n^*}(S_{\sigma_n^*} - K) - e^{-\rho\tau_{n+1}^*}(S_{\tau_{n+1}^*} + K)\Big] \\
\geq & -Ee^{-\rho\tau_1^*}(S_{\tau_1^*} + K) + \sum_{n=1}^{\infty} E\Big[S_{\sigma_n^*} - S_{\tau_{n+1}^*} - 2K\Big]e^{-\rho\sigma_n^*} \quad (\text{Use } e^{-\rho\sigma_n^*} \geq e^{-\rho\sigma_{n+1}^*}) \\
= & -Ee^{-\rho\tau_1^*}(S_{\tau_1^*} + K) + (e^{x_2} - e^{x_1} - 2K)\sum_{n=1}^{\infty} Ee^{-\rho\sigma_n^*}.
\end{aligned}$$

The last equality uses $X_{\sigma_n^*} = x_2$ and $X_{\tau_{n+1}^*} = x_1$ for all $n \geq 1$ in the position $i = 0$.

Furthermore, $e^{x_2} - e^{x_1} - 2K > 0$ by assumption and $\tau_1^* < \infty$ a.s. by Lemma 2.5.2.

Also, it can be seen from the definition of $v_0(x)$ in the Theorem that $v_0(x)$ is bounded. These imply the convergence of $\sum_{n=1}^{\infty} Ee^{-\rho\sigma_n^*}$. Therefore, $Ee^{-\rho\sigma_n^*} \to 0$ as $n \to \infty$. □

2.6 Numerical Examples

In this section, we consider a numerical example with the following specifications:

$$a = 0.8,\ b = 2,\ \sigma = 0.5,\ \rho = 0.5,\ K = 0.1.$$

Solving the quasi-algebraic equations (2.20) and (2.21) gives $(x_0, x_1, x_2) = (-4.58, 1.22, 1.7)$. Note that all the threshold levels x_0, x_1, and x_2 are below the equilibrium $b = 2$. This equilibrium serves as a pulling force that lifts the trajectory X_t from anywhere below $b = 2$. The price levels $S_0 = \exp(x_0) = 0.01$ and $S_1 = \exp(x_1) = 3.39$ are considered to be the low and the price $S_2 = \exp(x_2) = 5.47$ is the high. Two main factors affect the overall return: (i) the probability for the price to go from S_1 to S_2; (ii) the frequency for the price to travel from S_1 to S_2. It can be seen in Figure 2.1 that both the price levels $S_1 = 3.78$ and $S_2 = 5.12$ were crossed several times that offered good profit opportunities.

The corresponding value functions $V_0(x)$ and $V_1(x)$ are plotted in Figure 2.3. It is clear from this picture that $V_0(x)$ is uniformly bounded and $V_1(x)$ has an exponential growth rate (see Lemma 2.3.1).

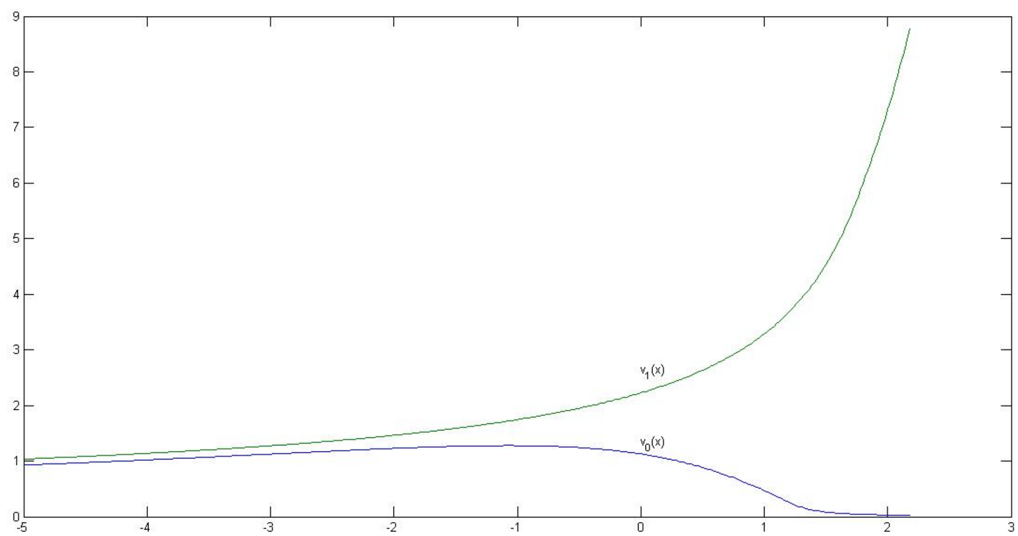

Figure 2.3: The value functions $V_0(x)$ and $V_1(x)$.

We next vary one of the parameters at a time and examine the dependence of (x_0, x_1, x_2).

In Table 2.1, we compute the threshold levels (x_0, x_1, x_2) associated with varying b. Intuitively, larger b would result larger threshold levels (x_1, x_2) and the potential of going higher from low price becomes bigger. It can be seen from Table 2.1 that the pair (x_1, x_2) is monotonically increasing, and x_0 is monotonically decreasing in b.

b	1	1.5	2	2.5	3
x_0	-4.36	-4.46	-4.58	-4.66	-4.74
x_1	0	0.64	1.22	1.78	2.34
x_2	0.74	1.22	1.7	2.18	2.66

Table 2.1: (x_0, x_1, x_2) with varying b.

In Table 2.2, we vary a. A larger a implies larger pulling rate back to the equilibrium level $b = 2$ which would encourage more transactions. It can be seen in Table 2.2 that the lower buying level x_0 decreases and the higher buying level x_1 increases in a. This leads to a larger buying interval $[x_0, x_1]$ resulting greater buying opportunities. The selling level x_2 increases but the the interval $[x_1, x_2]$ decreases which suggests one should take profit sooner as a gets bigger.

a	0.6	0.7	0.8	0.9	1
x_0	-4.2	-4.4	-4.58	-4.72	-4.86
x_1	0.98	1.14	1.22	1.3	1.36
x_2	1.58	1.62	1.7	1.74	1.78

Table 2.2: (x_0, x_1, x_2) with varying a.

In Table 2.3, we vary the volatility σ. Larger σ implies greater range for the stock price $S_t = \exp(X_t)$ which results higher profit associated with each buying and selling transaction. Table 2.3 shows that x_0 stays flat, and the intervals $[x_0, x_2]$ and $[x_1, x_2]$ are increasing, which imply higher profit as σ increases.

σ	0.3	0.4	0.5	0.6	0.7
x_0	-4.56	-4.56	-4.58	-4.58	-4.6
x_1	1.12	1.16	1.22	1.28	1.36
x_2	1.54	1.62	1.7	1.8	1.9

Table 2.3: (x_0, x_1, x_2) with varying σ.

In Table 2.4, we vary the discount rate ρ. Larger ρ means smaller in values or reward functions. Intuitively, larger ρ would result smaller threshold levels (x_1, x_2). Table 2.4 shows that (x_1, x_2) are monotonically decreasing as ρ gets larger. Smaller threshold levels (x_1, x_2) are compensated by bigger interval $[x_1, x_2]$. Moreover, the smaller buying interval $[x_0, x_1]$ and bigger interval $[x_1, x_2]$ discourage stock transactions.

ρ	0.3	0.4	0.5	0.6	0.7
x_0	-5	-4.86	-4.58	-4.32	-4.12
x_1	1.56	1.36	1.22	1.06	0.9
x_2	1.94	1.84	1.7	1.56	1.42

Table 2.4: (x_0, x_1, x_2) with varying ρ.

In Table 2.5, we vary the slippage rate K. Intuitively, larger K discourages stock transactions. Table 2.5 shows that the buying interval $[x_0, x_1]$ is decreasing and x_0 is increasing. It also suggests that x_1 is decreasing and x_2 is increasing in K. This is because the discouragement of stock transactions has to be compensated by larger interval $[x_1, x_2]$.

K	0.01	0.05	0.1	0.5	1
x_0	-5	-5	-4.58	-2.56	-1.6
x_1	1.46	1.3	1.22	0.7	-0.46
x_2	1.56	1.68	1.7	1.78	1.86

Table 2.5: (x_0, x_1, x_2) with varying K.

Chapter 3

An ODE Approach to Pairs Trading Selling Rule under a Mean-Reversion Model

3.1 Introduction

In pairs trading, a pair of historically correlated securities is selected and monitored. When the "spread" of the stock prices increases to a certain level, the pairs trade would be triggered: to short the stronger stock and to long the weaker one, betting the eventual convergence of the "spread". Like the previous chapter, the closed-form solutions to the associated HJB equations and the optimal selling rule are obtainable by means of ODE methods and smooth-fit techniques.

The attention is focused on when to exit a pairs position assuming the entry of a pairs position is triggered when the "spread" reaches two standard deviations as in [16]. The objective is to identify the optimal threshold levels determining when to sell (close) the pairs position. In particular, a selling decision is made when either the "spread" reaches a target or a pre-determined cutloss level, whichever comes first. Given the cutloss level, the goal is to decide when to lock in profits if the pairs performs as expected. In this chapter, the 'normalized' difference of the pairs is assumed to satisfy a mean reversion model, a differential equation with two point boundaries is solved, and the corresponding optimality is obtained. The results are also compared with the traditional optimal stopping approach and show that as the cutloss level vanishes, the optimal target is identical to selling threshold of the associated optimal stopping problem. In addition, the dependence of these quantities is also examined on various parameters in a numerical example. Finally, a pair of stocks is used to implement for demonstration.

This chapter is organized as follows. In §2, we formulate the pairs selling problem under consideration. In §3, we study the optimal selling rule and obtain near equivalence with the corresponding optimal stopping problem. In §4, we obtain expected holding time and profit probability. Numerical examples are presented in §5.

3.2 Formulating the Problem

Consider pairs trading that involves two stocks X^1 and X^2. The pairs position consists of a long position in X^1 and short position in X^2. Let X_t^1 and X_t^2 denote their respective prices at time $t \geq 0$. For simplicity, we allow trading a fraction of a share and consider the pairs position consisting of $K_1 = 1/X_0^1$ shares of X^1 in the long position and $K_2 = 1/X_0^2$ shares of X^2 in the short position. The corresponding price of the position is given by $Z_t = K_1 X_t^1 - K_2 X_t^2$.

We assume that Z_t is a mean-reversion (Ornstein-Uhlenbeck) process governed by

$$dZ_t = a(b - Z_t)dt + \sigma dW_t, \ Z_0 = z, \tag{3.1}$$

where $a > 0$ is the rate of reversion, b the equilibrium level, $\sigma > 0$ the volatility, and W_t a standard Brownian motion. In addition, the notation Z represents the corresponding pairs position. One share long in Z means the combination of K_1 shares of long position in X^1 and K_2 shares of short position in X^2. Similarly, for $i = 1, 2$, X_t^i represents the price of stock X^i. Lastly, Z_t is the value of the pairs position at time t. Note that Z_t can be negative.

Remark 3.2.1. Here, for convenience, the initial value $Z_0 = z$ is not limited to be zero and can take any value in $\mathbb{R}$. In addition, assuming two standard derivation entry rule, we require $Z_0 = z \leq b - 2\sigma_Z$, where the ergodic variance of Z_t is given by $\sigma_Z^2 = \sigma^2/(2a)$.

Assuming a pairs position was in place, the objective is to decide when to close the position. We consider the selling rule determined by two threshold levels: the target and a cutloss level. In particular, let z_1 denote the cutloss level and z_2 the target. The selling time is given by the exit time τ of Z_t from (z_1, z_2), i.e., $\tau = \inf\{t : \ Z_t \notin (z_1, z_2)\}$.

In Gatev et al. [16], the threshold levels $z_1 = -\infty$ and $z_2 = b$ are used to determine when to close a pairs position. Note that in practice a cutloss level is often imposed to limit possible undesirable events in the marketplace. It is a typical money management consideration. It can also be associated with a margin call due to substantial losses.

Given the initial state $Z_0 = z$, the corresponding reward function is

$$v(z) = v_{\{z_1,z_2\}}(z) = E[e^{-\rho\tau} Z_\tau | Z_0 = z]. \tag{3.2}$$

Here $\rho > 0$ is a given discount (impatience) factor.

Remark 3.2.2. In mean reversion models with $b = 0$, a useful concept in connection with the rate of reversion is *half life*, which is the time required for Z_t to change from z_0 to $z_0/2$. In practical literature the value $(\ln 2)/a$ is often used to represent such quantity. Note that the half life τ should be a stopping time. We can relate τ with $(\ln 2)/a$ as follows:

Solve the equation (3.1) in terms of W_t:

$$e^{at}Z_t = Z_0 + \int_0^t e^{as}\sigma dW_s.$$

Take $t = \tau$. Then $Z_\tau = z_0/2$. Taking expectations of both sides, we have $Ee^{a\tau} = 2$. Using Jensen's inequality, we have $e^{aE\tau} \leq Ee^{a\tau} = 2$. Therefore $E\tau \leq (\ln 2)/a$, i.e., the expected half life is typically smaller than $(\ln 2)/a$.

Example 3.2.3. In pairs trading one often choose pairs from the same industry sector because they are expected to share similar dips and highs. In this example we consider two companies from the retail industry: Wal-Mart Stores Inc. (WMT) and Target Corp. (TGT). If the price of WMT were to go up a large amount while TGT stayed the same, a pairs trader would buy TGT and sell short WMT betting on the convergence of their prices.

In this example, we use 3000 daily closing prices of X^1=WMT and X^2=TGT (from 2001/01/25 to 2012/12/31). In Figure 3.1, the process $Z_t = X_t^1/X_0^1 - X_t^2/X_0^2 = X_t^1/44.84 - X_t^2/31.86$ is plotted.

In addition, the entire data is divided into two equal halves. The first half is used to calibrate the model and the second half to backtest our results.

To estimate parameters a, b, and σ, note that, for any $0 \leq t_0 \leq t$, we have

$$Z_t = e^{-a(t-t_0)}Z_{t_0} + \int_{t_0}^t abe^{-a(t-s)}ds + \int_{t_0}^t \sigma e^{-a(t-s)}dW_s.$$

We discretize this equation with step size δ and obtain

$$Z_{n\delta} = \alpha Z_{(n-1)\delta} + \beta + \varepsilon_n,$$

where

$$\alpha = e^{-a\delta}, \beta = b(1 - e^{-a\delta}), \text{ and } \varepsilon_n = \int_{(n-1)\delta}^{n\delta} \sigma e^{-a(n\delta - s)} dW_s.$$

It is easy to see that $\varepsilon_n \sim N(0, \Sigma^2)$ with $\Sigma^2 = \sigma^2(1 - e^{-2a\delta})/(2a)$.

In view of this, $Z_{n\delta}$ is an autoregressive process of order 1. Following standard AR(1) estimation (least square) method, we obtain the estimates for α, β, and Σ, which lead to $a = 1.0987$, $b = -0.2288$, and $\sigma = 0.2669$.

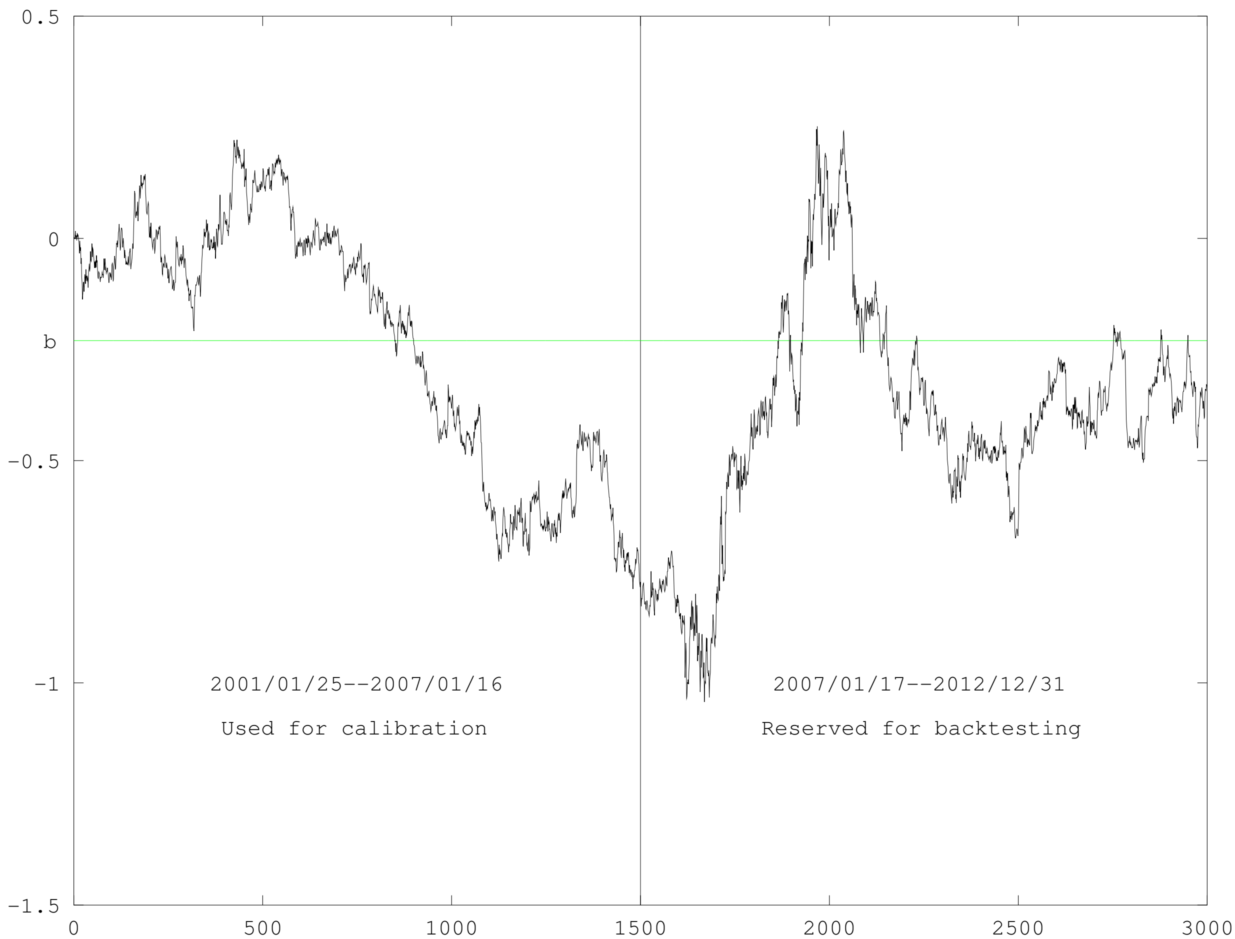

Figure 3.1: WMT and TGT (2001–2012).

3.3 An Optimal Selling Rule

Following a similar approach as in Zhang [43], we can show the reward function $v(z)$ satisfies the two-point-boundary-value differential equation

$$\begin{cases} \rho v(z) = \dfrac{\sigma^2}{2}\dfrac{d^2 v(z)}{dz^2} + a(b-z)\dfrac{dv(z)}{dz}, \\ v(z_1) = z_1,\ v(z_2) = z_2. \end{cases} \tag{3.3}$$

Let $\kappa = \sqrt{2a}/\sigma$ and $\eta(t) = t^{(\rho/a)-1}e^{-t^2/2}$. Then the general solution of (3.3) can be given in terms of a linear combination of independent solutions:

$$v(z) = C_1 \int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt + C_2 \int_0^\infty \eta(t) e^{\kappa(b-z)t} dt,$$

for some constants C_1 and C_2 Note that these constants are (z_1, z_2) dependent, i.e., $C_1 = C_1(z_1, z_2)$ and $C_2 = C_2(z_1, z_2)$.

Taking $z = z_1$ and $z = z_2$ respectively, we have

$$\begin{pmatrix} v(z_1) \\ v(z_2) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z_1)t} dt & \displaystyle\int_0^\infty \eta(t) e^{\kappa(b-z_1)t} dt \\ \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z_2)t} dt & \displaystyle\int_0^\infty \eta(t) e^{\kappa(b-z_2)t} dt \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Let $\Phi(z_1, z_2)$ denote the above 2×2 matrix. We claim that this matrix is non-singular. In fact, let

$$\phi(x) = \frac{\displaystyle\int_0^\infty \eta(t) e^{\kappa x t} dt}{\displaystyle\int_0^\infty \eta(t) e^{-\kappa x t} dt}.$$

Then it is easy to see that $\phi(x)$ is strictly increasing in x. Therefore,

$$\phi(b - z_1) > \phi(b - z_2).$$

On the other hand, we have

$$\det \Phi(z_1, z_2) = (\phi(b - z_2) - \phi(b - z_1)) \int_0^\infty \eta(t) e^{-\kappa(b-z_1)t} dt \int_0^\infty \eta(t) e^{-\kappa(b-z_2)t} dt,$$

which is strictly less than zero. Therefore $\Phi(z_1, z_2)$ is invertible and the constants C_1 and C_2 can be expressed in terms of z_1 and z_2 as follows:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi^{-1}(z_1, z_2) \begin{pmatrix} v(z_1) \\ v(z_2) \end{pmatrix} = \Phi^{-1}(z_1, z_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Given the initial value $Z_0 = z$, the corresponding reward function

$$v(z) = C_1 \int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt + C_2 \int_0^\infty \eta(t) e^{\kappa(b-z)t} dt.$$

The optimization problem is to choose $z_2 \geq z$ to maximize $v(z)$.

A 'nearly' equivalent optimal stopping problem

In this section, we consider a related optimal stopping problem. Full details of the description and treatment of this section can be obtained following the step-by-step approach in [42]. Let $\mathcal{F}_t = \sigma\{Z_r : r \leq t\}$ and τ be an $\mathcal{F}_t$ stopping time. The problem is to choose τ to maximize $J(z, \tau) = E[e^{-\rho\tau} Z_\tau | Z_0 = z]$. Let $V(z)$ denote the corresponding reward function, i.e., $V(z) = \sup_\tau J(z, \tau)$. Then, the associated Hamilton-Jacobi-Bellman equation is given by

$$\min \left\{ \rho V(z) - \frac{\sigma^2}{2} \frac{d^2 V(z)}{dz^2} - a(b - z) \frac{dV(z)}{dz}, V(z) - z \right\} = 0. \tag{3.4}$$

To solve this equation, we first consider $\rho V(z) - \mathcal{A} V(z) = 0$, where

$$\mathcal{A} V(z) = \frac{\sigma^2}{2} \frac{d^2 V(z)}{dz^2} - a(b - z) \frac{dV(z)}{dz}.$$

Its general solution is given by

$$V(z) = A_1 \int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt + A_2 \int_0^\infty \eta(t) e^{\kappa(b-z)t} dt,$$

for some constants A_1 and A_2. Suggested by the results in [42], we expect this solution to be bounded. In addition, $\rho V - \mathcal{A} V = 0$ is expected to hold on $(-\infty, z^*)$ for some z^* and $V = z$ on (z^*, ∞). In view of this, A_2 must be equal to zero. Furthermore, the solution to (3.4) should be continuously differentiable. The smooth-fit conditions demand

$$\begin{cases} A_1 \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z^*)t} dt = z^*, \\ A_1 \displaystyle\int_0^\infty (\kappa t) \eta(t) e^{-\kappa(b-z^*)t} dt = 1. \end{cases}$$

Eliminate A_1 to obtain

$$\int_0^\infty \eta(t) e^{-\kappa(b-z^*)t} dt = z^* \int_0^\infty (\kappa t) \eta(t) e^{-\kappa(b-z^*)t} dt.$$

We show next that this equation has a unique solution $z^* > 0$. In fact, let

$$\phi(z) = \int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt - z \int_0^\infty (\kappa t) \eta(t) e^{-\kappa(b-z)t} dt.$$

Then it is easy to see that $\phi(z) > 0$ for $z < 0$, and its derivative is

$$\phi'(z) = -z \int_0^\infty (\kappa t)^2 \eta(t) e^{-\kappa(b-z)t} dt.$$

It follows that $\phi(z)$ is strictly decreasing on $(0, \infty)$ and $\phi'(z) \to -\infty$, as $z \to \infty$. Therefore, $\phi(z)$ must have a unique zero $z^* > 0$. Again, following a similar argument as in [42], we can show that the optimal selling point is $\tau^* = \inf\{t : Z_t \geq z^*\}$. The corresponding reward function is

$$V(z) = \begin{cases} \dfrac{z^* \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt}{\displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z^*)t} dt} & \text{if } z \le z^*, \\ z & \text{if } z > z^*. \end{cases}$$

Returning to our (z_1, z_2)-optimization problem, intuitively as $z_1 \to -\infty$ (or when the cutloss is removed), the (z_1, z_2)-optimization problem should be equivalent to the the above optimal stopping problem. It can be shown by direct computation that as $z_1 \to -\infty$,

$$v_{\{z_1,z_2\}}(z) \to v_{\{-\infty,z_2\}}(z) = \frac{z_2 \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z)t} dt}{\displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-z_2)t} dt}.$$

This function reaches its maximum at $z_2 = z^*$, which recovers the value function for the optimal stopping problem.

Remark 3.3.1. Let $\tau = T$ be a deterministic selling time. Then $J(z, \tau) = J(z, T) = e^{-\rho T} E(Z_T) \to 0$ as $T \to \infty$. It follows that $V(z) = \sup_\tau J(z, \tau) \ge \lim_{T\to\infty} J(z, T) = 0$. In addition, let $\theta = \inf\{t \ge 0 : Z_t \ge z^*\}$. Then it can be shown as in [42] that $P(\theta < \infty) = 1$. That is, any target level z^* is reachable in finite time. Therefore, in order to have nonnegative $V(z)$, one must have $z^* \ge 0$. Of course, the holding time tends to get longer to reach z^*, especially when b is very negative. This tendency can be seen from Table 4 in §5.

In view of the above comparison, it is clear that the (z_1, z_2)-optimization problem is more intuitive and easier to work with. In addition, it is also easier to compute the expected holding time and corresponding profit probability. These are the subjects of the next section.

3.4 Expected Holding Time and Profit Probability

First we consider the expected holding time. For each $Z_0 = z$, let $\tau = \tau(z) = \inf\{t \geq 0 : Z_t \notin (z_1, z_2)\}$ and $T(z) = E[\tau | Z_0 = z]$. Then $T(z)$ satisfies the differential equation:

$$\begin{cases} \frac{\sigma^2}{2}\frac{d^2T(z)}{dz^2} + a(b-z)\frac{dT(z)}{dz} + 1 = 0, \\ T(z_1) = 0,\ T(z_2) = 0. \end{cases} \tag{3.5}$$

Let $\gamma(z) = \exp\left(\frac{a}{\sigma^2}(z-b)^2\right)$. Then its reciprocal is $\gamma^{-1}(z) = \exp\left(-\frac{a}{\sigma^2}(z-b)^2\right)$. It is easy to check that

$$\frac{d(\gamma^{-1}(z)T'(z))}{dz} = -\frac{2}{\sigma^2}\gamma^{-1}(z),$$

where T' is the derivative of T. Integrate both sides to obtain

$$T'(z) = -\frac{2}{\sigma^2}\gamma(z)\int_0^z \gamma^{-1}(u)du + \gamma(z)\gamma^{-1}(0)T_0,$$

where T_0 is a constant. Using the boundary conditions $T(z_1) = T(z_2) = 0$ and integrating both sides again, we have

$$T(z) = -\frac{2}{\sigma^2}\int_{z_1}^{t}\left(\gamma(t)\int_0^t \gamma^{-1}(u)du\right)dt + T_0\int_{z_1}^{z}\left(\gamma(t)\gamma^{-1}(0)\right)dt$$

and

$$T_0 = \frac{2\displaystyle\int_{z_1}^{z_2}\left(\gamma(t)\int_0^t \gamma^{-1}(u)du\right)dt}{\sigma^2\displaystyle\int_{z_1}^{z_2}\left(\gamma(t)\gamma^{-1}(0)\right)dt}.$$

Given the initial $Z_0 = z_0$, let $\tau_0 = \tau(z_0)$. Then the expected holding time is given by $E\tau_0 = T(z_0)$.

Next we consider the profit probability. Given $Z_0 = z$, the profit probability $P(z)$ is defined as the conditional probability of Z_t reaching z_2 before hitting z_1. Therefore, $P(z)$ satisfies the differential equation:

$$\begin{cases} \dfrac{\sigma^2}{2}\dfrac{d^2P(z)}{dz^2} + a(b-z)\dfrac{dP(z)}{dz} = 0, \\ P(z_1) = 0, \ P(z_2) = 1. \end{cases} \tag{3.6}$$

Solving this equation we have

$$P(z) = \frac{\displaystyle\int_{z_1}^{z} \exp\left(\frac{a}{\sigma^2}(u-b)^2\right) du}{\displaystyle\int_{z_1}^{z_2} \exp\left(\frac{a}{\sigma^2}(u-b)^2\right) du}.$$

With $Z_0 = z_0$, the corresponding profit probability

$$P_0 = P(z_0) = \frac{\displaystyle\int_{z_1}^{z_0} \exp\left(\frac{a}{\sigma^2}(u-b)^2\right) du}{\displaystyle\int_{z_1}^{z_2} \exp\left(\frac{a}{\sigma^2}(u-b)^2\right) du}.$$

3.5 Numerical Examples

In this section, we use the parameters of the WMT-TGT example, i.e.,

$$a = 1.0987, \ b = -0.2288, \ \sigma = 0.2669, \ \rho = 0.20.$$

The entry of the pairs position is triggered when the "spread" is two standard deviations, i.e., $z_0 = b - 2\sigma_Z = -0.59$.

Remark 3.5.1. Recall that the cutloss level in this paper is a pre-determined risk level and our objective is to choose z_2 to maximize the corresponding reward function. Alternatively, one may consider the optimization over both variables z_1 and z_2 simultaneously. For example, let us consider the optimization over $(z_1, z_2) \in [-1.60, -0.90] \times [0, 0.50]$. It can be seen following simple numerical computation that the optimal $(z_1^*, z_2^*) = (-1.60, 0.14)$, which shows that the optimal z_1^* takes the lowest value of its given interval. This is true in general due to our mean reversion formulation. In view of this, it is natural to focus on z_2 with z_1 given.

In this section, we take $z_1 = b - 6\sigma_Z = -1.31$. The maximum of $v(-0.59) = v_{\{-1.31,z_2\}}(-0.59)$ is reached at $z_2 = 0.14$. The corresponding expected holding time is $E\tau_0 = 11.74$ and profit probability is $P_0 = 0.999981$. Note that the profit probability is overwhelming in the mean reversion case. It is probably meaningful only when comparing with related cases because the model is only an approximation to the market and may never be as precise.

Next, we vary one of the parameters at a time and examine the dependence on these parameters.

Dependence of $(z_2, E\tau_0, P_0)$ on parameters

First we consider the values of $(z_2, E\tau_0, P_0)$ associated with varying a. A larger a leads to a greater pulling force back to the equilibrium b. It can be seen in Table 3.1 that the target level z_2 decreases as a increases. Also, the expected holding time $E\tau_0$ shrinks while the profit probability P_0 increases in a. The selling level z_2 decreases which suggests one should take profit sooner as a gets bigger because the potential of going higher becomes smaller.

a	0.4987	0.7987	1.0987	1.3987	1.6987
z_2	0.22	0.17	0.14	0.12	0.10
$E\tau_0$	14.89	12.48	11.74	11.67	11.10
P_0	0.999984	0.999982	0.999981	0.999979	0.999978

Table 3.1: $z_2, E\tau_0, P_0$ with varying a.

In Table 3.2, we vary the volatility σ. By and large, the volatility is the source forcing the price to go away from its equilibrium. The large the σ, the farther the price fluctuates, so is the target level z_2. The expected holding time decreases because the target is more reachable. This leads to increases in profit probability.

σ	0.1669	0.2169	0.2669	0.3169	0.3669
z_2	0.06	0.10	0.14	0.18	0.24
$E\tau_0$	27.48	15.87	11.73	9.70	9.02
P_0	0.999975	0.999978	0.999981	0.999981	0.999982

Table 3.2: $z_2, E\tau_0, P_0$ with varying σ.

Next, we vary the discount rate ρ. Larger ρ means quicker profits. This is confirmed in Table 3.3. It shows that larger ρ leads to a smaller target z_2, shorter holding time, and larger profit probability. This means more buying opportunities and quicker profit taking.

ρ	0.05	0.10	0.20	0.40	0.80
z_2	0.19	0.16	0.14	0.12	0.10
$E\tau_0$	18.79	14.05	11.73	9.90	8.44
P_0	0.999977	0.999979	0.999981	0.999982	0.999983

Table 3.3: $z_2, E\tau_0, P_0$ with varying ρ.

In Table 3.4, we vary the mean b. It can be seen in this case, a smaller b lower the equilibrium, which pushes down the target z_2, which remains above zero. In the meantime, smaller b leads to large gap between b and its target, which leads in turn increasing expected holding time $E\tau_0$ and smaller profit probability P_0.

b	-0.0288	-0.1288	-0.2288	-0.4288	-0.62880
z_2	0.24	0.19	0.14	0.08	0.05
$E\tau_0$	5.49	7.82	11.73	53.24	825.97
P_0	0.999986	0.999983	0.999981	0.999970	0.999260

Table 3.4: $z_2, E\tau_0, P_0$ with varying b.

Finally, instead of having $z_0 = b - 2\sigma_Z = -0.59$, we vary its value around $z_0 = -0.59$. As can be seen in Table 3.5, the target z_2 is not z_0 dependent. In addition, smaller z_0 leads to increasing $E\tau_0$ and smaller profit probability P_0.

z_0	-0.39	-0.49	-0.59	-0.69	-0.79
z_2	0.14	0.14	0.14	0.14	0.14
$E\tau_0$	11.09	11.39	11.73	11.82	11.98
P_0	0.999983	0.999982	0.999981	0.999979	0.999976

Table 3.5: $z_2, E\tau_0, P_0$ with varying z_0.

Remark 3.5.2. Recall that in Gatev et al. [16], the corresponding $z_1 = -\infty$ and $z_2 = b$. Clearly, the choice of $z_2 = b$ is far from being optimal. On the other hand, notice the symmetry of $(Z_t - b)$. When following the same two standard deviation entry rule, the opposite trade should be opened (i.e., short X^1 and long X^2) when $Z_t > b + 2\sigma_Z$, which forces the previous pairs position to be closed. In view of this, z_2 should satisfy the constraint $z_2 \leq b + 2\sigma_Z$. Clearly, the above constraint on z_2 does not hold automatically. In our base case, $z_2^* = 0.14$ and $b + 2\sigma_Z = 0.13$. To follow the same entry rule, we truncate the target level and use $\min\{z_2, b + 2\sigma_Z\}$ instead in our backtesting example.

Backtesting (WMT-TGT)

In this section, we backtest the pairs selling rule using the stock prices of WMT and TGT from 2001 to 2012. Let X_t^1 and X_t^2 be the daily closing prices of WMT and TGT stocks, respectively. The initial prices on 2001/01/25 are $X_0^1 = 44.84$ and $X_0^2 = 31.86$. The corresponding $Z_t = X_t^1/44.84 - X_t^2/31.86$.

Using the parameters obtained in Example 3.2.3 based on the historical prices from 2001 to 2007, we obtained the target levels in Table 3. Note that $b + 2\sigma_Z = 0.13$. We use $\min\{z_2, b+2\sigma_Z\}$ as the target in lieu of z_2. In addition, we follow the two standard deviation entry rule, i.e., open the position whenever $Z_0 = z_0 \leq b - 2\sigma_Z = -0.59$. This occurred on 2007/1/17 and the daily closing prices were 42.34 for WMT and 55.82 for TGT.

Suppose we trade with an account with the capital \$10K and allocate equal amounts to long and short positions. At the closing of 1/17, we buy 118 shares of WMT at \$42.25 and short 91 shares of TGT at \$54.65. Next, we exit the pairs position when either Z_t reaches the target z_2 or the cutloss level $z_1 = -1.31$, whichever comes first.

In Table 3.6, the exit prices and gain/loss of each trade with different choices for ρ are given.

ρ	0.05	0.10	0.20	0.50	1.00
z_2	0.19	0.16	0.14	0.12	0.10
$\min\{z_2, b + 2\sigma_Z\}$	0.13	0.13	0.13	0.12	0.10
$X^1_{\tau_0}$	47.71	47.71	47.71	46.88	47.70
$X^2_{\tau_0}$	27.83	27.83	27.83	29.18	30.43
Closed on	08/11/18	08/11/18	08/11/18	08/11/17	08/11/14
Gain/Loss	\$3084.90	\$3084.90	\$3084.90	\$2864.11	\$2847.12
(with base \$9958.65)	30.97%	30.97%	30.97%	28.76%	28.58%

Table 3.6: Testing results with different choices for ρ.

In Figure 3.2, the range from the target levels are marked together with entry level z_0 and cutloss level z_1.

Figure 3.2: WMT and TGT (2001–2012)

Chapter 4

A Stochastic Approximation Approach to Pairs Trading under a Regime-Switching Model

4.1 Introduction

This chapter is concerned with numerical methods for trading a pairs position. In particular, we focus on stochastic approximation algorithms. In addition, we limit ourselves to trading one round trip of a pairs position. The advantage of stochastic approximation is to obtain optimal trading rules without solving the complicated HJB equations.

In this chapter, a pair of correlated stocks are selected and monitored. When the "spread" of the stock prices decreases to a certain level, the pairs trade is initiated by longing the stronger stock and shorting the weaker one, betting the eventual divergence of the "spread". This strategy is a reverse of the one in the previous chapter. The objective is to buy and sell the log difference of the pair to maximize a reward function.

Traditional pairs trading often uses mean-reversion models, and closed-form solutions are derived. In this chapter, the log difference of the pair is governed by a regime-switching model instead. Regime-switching models complicate the problem since the Markov chain introduces a source of uncertainty. The models incorporate parameters to describe the trends of the market which switches among a finite number of states, for instance, the uptrend (bull market) and the downtrend (bear market). Regime-switching models were first introduced by Hamilton [19] in 1989 to describe time series. The models have also been employed by Zhang [43] for optimal stock trading rules, Yin and Zhang [39] for applications in portfolio management, and Yin and Zhou [41] for dynamic Markowitz problems. Unlike these papers, this chapter does not use geometric Brownian motions. A switching Itô diffusion of the form $dZ_t = \mu(\alpha_t)dt + \sigma dW_t$ is used instead.

In many optimization problems, the reward functions to be maximized are expressed as some expectation. Numerical methods such as Newton method among others are applicable when the function and its gradient can be observed without errors at desired values. However, these methods do not perform well when the functions involve expectations that require evaluation via simulation. To overcome the stated limitation of the classical numerical methods, Robbins and Monro introduced the following stochastic recursive algorithm, also known as stochastic approximation in 1951 [26].

Let Y_n denote the observation taken at time n at parameter value x_n, the recursive algorithm

$$x_{n+1} = x_n + a_n Y_n,\ a_n > 0$$

is used to approximate the root x^0, where a_n is the step size. This is called Robbins-Monro algorithm. Stochastic optimization algorithms that employ finite-difference methods are called Kiefer-Wolfowitz algorithms [21].

Using stochastic approximation approach to find the optimal buy and sell threshold levels for a bull-bear switching market is motivated by the studies in [8] and [33]. Unlike these two studies, this chapter focuses on pairs trading. In the following sections, the problem under consideration is formulated using a regime-switching model where the drift switches between two states of a partially observable Markov chain corresponding to either an uptrend (bull market) or a downtrend (bear market). The market is allowed to switch between bull and bear modes any number of times. Nothing except stock prices up to the current time is assumed to be observable. A stochastic recursive algorithm is then developed to compute the probability thresholds. Also, the convergence of the algorithm is verified under suitable conditions. Finally, the algorithm is tested with pairs of stocks, and the result has been shown to outperform that of the buy and hold strategy.

4.2 Formulating the Problem

Let $S_t^i,\ i = 1, 2$, be the stock prices at time $t \geq 0$, with S^1 and S^2 being the stronger and weaker stocks respectively. Assume their log difference $Z_t = \log S_t^1 - \log S_t^2$ is a switching process in $\mathbb{R}$ governed by

$$dZ_t = \mu(\alpha_t)dt + \sigma dW_t,\ Z_0 = Z, \tag{4.1}$$

where $\mu(i) = \mu_i, i = 1, 2$, the expected rates of return with $\mu_1 > 0 > \mu_2$; $\sigma > 0$, the volatility; W_t, a standard Brownian motion; and α_t, the Markov chains with state space $\mathcal{M} = \{1, 2\}$.

The process α_t represents the market mode at each time t, where $\alpha_t = 1$ indicates a bull market and $\alpha_t = 2$ a bear market. The generator for α_t is denoted by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},\ \text{for some } \lambda_1 > 0,\ \lambda_2 > 0.$$

Let $0 \le \tau_1 \le \tau_2 < \infty$ be stopping times where one buys the pair at τ_1 and sells it at τ_2. We need to determine τ_1 and τ_2 to maximize the reward function:

$$J(\tau_1, \tau_2) = E[Z_{\tau_2} - Z_{\tau_1}].$$

Let $p_t = P(\alpha_t = 1 | Z_s : 0 \le s \le t) \in [0, 1]$ denote the conditional probability of $\alpha_t = 1$ (bull market) given the stock price up to time t.

Then the Wonham filter [37] in terms of p_t satisfies the SDE equation:

$$dp_t = [-(\lambda_1 + \lambda_2)p_t + \lambda_2]dt + \frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma} d\widehat{W}_t, \tag{4.2}$$

where $\widehat{W}_t$ is an innovation process given by

$$d\widehat{W}_t = \frac{dZ_t - [(\mu_1 - \mu_2)p_t + \mu_2]dt}{\sigma}.$$

Equivalently,

$$dZ_t = [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma d\widehat{W}_t. \tag{4.3}$$

Integrating both sides of the equation from $t = \tau_1$ to $t = \tau_2$ gives

$$Z_{\tau_2} - Z_{\tau_1} = \int_{\tau_1}^{\tau_2} [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma(\widehat{W}_{\tau_2} - \widehat{W}_{\tau_1}).$$

It follows that the reward function has the form

$$J(\tau_1, \tau_2) = E[Z_{\tau_2} - Z_{\tau_1}] = E\left[\int_{\tau_1}^{\tau_2} [(\mu_1 - \mu_2)p_t + \mu_2]dt\right].$$

The trading rule is determined by two probability threshold levels: x^1 (buy level) and x^2 (sell level) with $x^1 > x^2$ so as the reward function is maximized.

Let $x = (x^1, x^2)$ and

$$g(x) = g(x^1, x^2) = E[Z_{\tau_2} - Z_{\tau_1}] = E\left[\int_{\tau_1(x)}^{\tau_2(x)} [(\mu_1 - \mu_2)p_t + \mu_2]dt\right],$$

where

$$\tau_1(x) = \inf\{t : p_t \geq x^1\},$$
$$\tau_2(x) = \inf\{t > \tau_1(x) : p_t \leq x^2\}.$$

4.3 Stochastic Recursive Algorithm and Convergence Analysis

Note that $g(x)$ is not completely observable since it involves expectation. To solve the problem, a stochastic approximation algorithm is used.

Suppose $g(x)$ can be observed with some noise, and assume $g(x) = E[G(x, \chi(x))]$, where $\chi(x)$ is the random noise at x. Then for each x, $G(x, \chi(x))$ is an estimator of $g(x)$. For notational simplicity, we write $\chi = \chi(\cdot)$ without a variable. In our simulation study, at each value of x, N sample paths of (Z_t, p_t) are generated using (4.2) and (4.3), say $\{(Z_{t,k}, p_{t,k})\}_{k=1}^N$. Then we can take

$$G(x, \chi) = \frac{\widetilde{G}(x, \chi_1) + \cdots + \widetilde{G}(x, \chi_N)}{N}, \tag{4.4}$$

where $\widetilde{G}(x, \chi_k) = \displaystyle\int_{\tau_1(x)}^{\tau_2(x)} [(\mu_1 - \mu_2)p_{t,k} + \mu_2]dt.$

Suppose the random samples $\{\widetilde{G}(x, \chi_k)\}$ are i.i.d. Then by the law of large numbers, $G(x, \chi)$ converges to $g(x)$ w.p. 1 as $N \to \infty$. In what follows, we also assume $G(x, \chi)$ depends smoothly on x.

The recursive algorithm has the form

$$x_{n+1} = x_n + a_n Y_n, \tag{4.5}$$

where

$$\begin{aligned} Y_n &= (Y_{n,1}, Y_{n,2}), \\ Y_{n,i} &= \frac{G(x_n + b_n e_i, \chi^+_{n,i}) - G(x_n - b_n e_i, \chi^-_{n,i})}{2b_n} \end{aligned} \tag{4.6}$$

for $i = 1, 2$.

The step size sequence $a_n \geq 0$ satisfies

$$\sum_{n=0}^{\infty} a_n = \infty, \;\; a_0 = 0, \;\text{ and } a_n \to 0.$$

We choose $a_n = O(n^{-1})$ and $b_n = O(n^{-1/6})$.

The interpolated time scale is defined in terms of the step size sequence. Define

$$t_n = \sum_{k=0}^{n} a_k.$$

We can assume $0 \leq a_n < 1$, and define

$$\begin{aligned} &\delta(t) = \text{ the unique value of } n \text{ such that } t_n \leq t < t_{n+1}, \\ &\delta^n(t) = \delta(t_n + t), \;\text{ for } t \geq 0. \end{aligned}$$

Note that $\delta(t)$ is constant on each interval $[t_n, t_{n+1})$, and that $\delta(t)$ may stay unchanged through consecutive intervals $[t_n, t_{n+1}) \cup [t_{n+1}, t_{n+2})$. Therefore, $\delta^n(0) \leq n$.

Define the continuous time interpolation $x^0(t)$ on $[0, \infty)$ by

$$x^0(t) = x_n, \;\text{ for } t_n \leq t < t_{n+1}.$$

Define the sequence of shifted process $x^n(t)$ by

$$x^n(t) = x^0(t_n + t), \text{ for } t \geq 0.$$

Note that $x^n(0) = x_n$.

The following figures illustrate the functions $\delta(t)$, $\delta^n(t)$, $x^0(t)$, and $x^n(t)$.

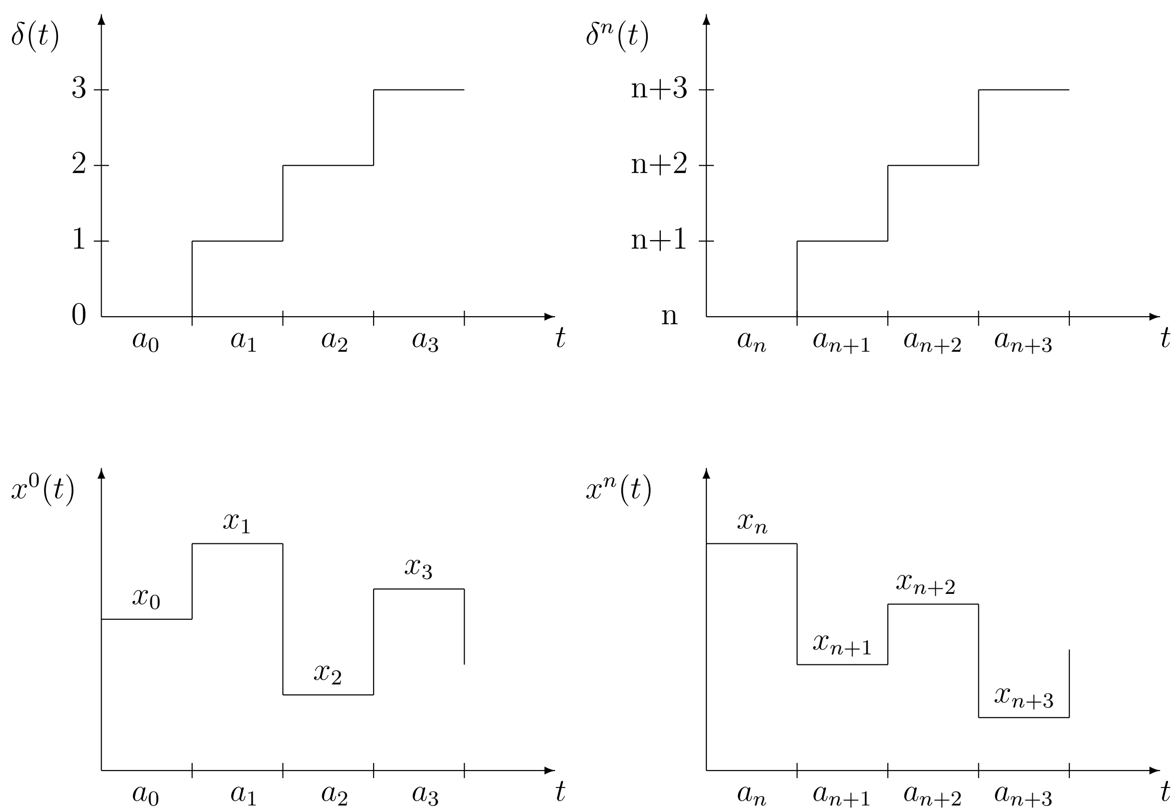

Figure 4.1: The functions $\delta(t)$, $\delta^n(t)$, $x^0(t)$, and $x^n(t)$.

Observe that $x^n(t) \in D^2_\infty$, the space of $\mathbb{R}^2$-valued functions that are right continuous and have finite left limits, endowed with the Skorohod topology (1.5.5).

For $\chi_{n,i} = (\chi^+_{n,i}, \chi^-_{n,i})$, define

$$\begin{aligned}
&\delta G_{n,i}(x_n, \chi_{n,i}) = G(x_n + b_n e_i, \chi^+_{n,i}) - G(x_n - b_n e_i, \chi^-_{n,i}),\\
&\delta M_{n,i}(x_n, \chi_{n,i}) = \delta G_{n,i}(x_n, \chi_{n,i}) - E_n[\delta G_{n,i}(x_n, \chi_{n,i})],\\
&\delta g_{n,i}(x_n) = g(x_n + b_n e_i) - g(x_n - b_n e_i),\\
&A_{n,i}(x_n, \chi_{n,i}) = E_n[\delta G_{n,i}(x_n, \chi_{n,i})] - \delta g_{n,i}(x_n),\\
&B_{n,i}(x_n) = \frac{\delta g_{n,i}(x_n)}{2b_n} - \partial_{x^i} g(x_n),
\end{aligned}$$

where $E_n = E[\cdot|\mathcal{F}_n]$, and $\mathcal{F}_n$ is the σ-algebra generated by $\{x_j, \chi^{\pm}_j : j < n\}$.

The algorithm (4.5) can be rewritten as

$$x_{n+1} = x_n + a_n \left[\frac{A_n(x_n, \chi_n)}{2b_n} + \frac{\delta M_n(x_n, \chi_n)}{2b_n} + B_n(x_n) + \nabla g(x_n) \right],$$

where

$$\begin{aligned}
&A_n = (A_{n,1}, A_{n,2}),\ B_n = (B_{n,1}, B_{n,2})\ , \delta M_n = (\delta M_{n,1}, \delta M_{n,2}),\\
&\nabla g = (\partial_{x^1} g, \partial_{x^2} g),\ x_n = (x^1_n, x^2_n),\ \chi_n = (\chi_{n,1}, \chi_{n,2}).
\end{aligned}$$

By definitions,

$$\begin{aligned}
x^n(t) = &\ \ x_n + \sum_{i=n}^{\delta^n(t)-1} a_i \left[\frac{A_i(x_i, \chi_i)}{2b_i} + \frac{\delta M_i(x_i, \chi_i)}{2b_i} + B_i(x_i) + \nabla g(x_i) \right]\\
= &\ \ x_n + A^n(t) + \delta M^n(t) + B^n(t) + \nabla g^n(t).
\end{aligned}$$

This implies

$$x^n(t+s) - x^n(t) = \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \left[\frac{A_i(x_i, \chi_i)}{2b_i} + \frac{\delta M_i(x_i, \chi_i)}{2b_i} + B_i(x_i) + \nabla g(x_i) \right]. \tag{4.7}$$

For the rest of the chapter, we assume the following conditions hold:

(A1) The second derivative of $g(x)$ is continuous.

(A2) For each compact set Ω, $\sup_n E|G(x_n, \chi_n) I_{\{x_n \in \Omega\}}|^2 < \infty$.

(A3) For each x in a bounded set and for each $0 < T < \infty$,

$$\sup_n \sum_{j=n}^{\delta^n(T)-1} \sqrt{E|E_j A_j(x, \chi_j)|} < \infty,$$

$$\lim_n \sup_{0 \le i \le \delta^n(T)} E|\Gamma_i^n| = 0,$$

where

$$\Gamma_i^n = \frac{1}{a_{n+i}} \sum_{j=n+i}^{\delta^n(T)+i-1} \frac{a_j}{2b_j} E_{n+i}[A_j(x_{n+i+1}, \chi_j) - A_j(x_{n+i}, \chi_j)] \text{ for } i \le \delta^n(T).$$

We introduce the truncated processes as follows.

For each integer M, let Π_M be a smooth real-valued function on $\mathbb{R}^2$ satisfying

$$\Pi_M(x) = \begin{cases} 1 & \|x\| < M, \\ 0 & \|x\| \ge M+1. \end{cases}$$

Define x_n^M by $x_1^M = x_1$, and for $n \ge 1$ set

$$x_{n+1}^M = x_n^M + \left[a_n \frac{A_n(x_n, \chi_n)}{2b_n} + a_n \frac{\delta M_n(x_n, \chi_n)}{2b_n} + a_n B_n(x_n) + a_n \nabla g(x_n) \right] \Pi_M(x_n^M).$$

Note that $x_n^M = x_n$ until the first time that x_n exceeds M.

For $t \ge 0$, define the continuous time interpolation $x_M^0(t)$ by

$$x_M^0(t) = x_n^M, \text{ for } t_n \le t < t_{n+1}.$$

For $t \geq 0$, define the sequence of shifted process $x_M^n(t)$ by

$$x_M^n(t) = x_M^0(t_n + t), \text{ for } t_n \leq t < t_{n+1}.$$

It can be shown in [24, p.321] that for each positive T,

$$\lim_{K\to\infty} \sup_M P\{\|x_M^n(t)\|_T \geq K\} = 0.$$

Lemma 4.3.1. *Let $\varepsilon > 0$ and $0 \leq s < \varepsilon$. The following holds:*

(a) δM_n is a martingale difference with respect to $\mathcal{F}_n$.

(b) $E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{\delta M_k}{2b_k} \Pi_M(x_k^M)\right|^2 \to 0$ *uniformly in t as $n \to \infty$.*

(c) $\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{B_k}{2b_k} \Pi_M(x_k^M) \to 0$ *uniformly in t as $n \to \infty$.*

(d) $\lim_{\varepsilon\to 0} E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \nabla g \Pi_M(x_k^M)\right|^2 \to 0$ *uniformly in t as $n \to \infty$.*

(e) $\lim_{\varepsilon\to 0} E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{A_k}{2b_k} \Pi_M(x_k^M)\right|^2 \to 0$ *uniformly in t as $n \to \infty$.*

(f) The sequence $x_M^n(t)$ is tight in D_∞^2.

Proof. Part (a) follows directly from the definition of martingale difference (1.5.12).

For part (b), since δM_n is a martingale difference, it is orthogonal. So,

$$E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{\delta M_k}{2b_k} \Pi_M(x_k^M)\right|^2 = \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(k^{-10/6}) E|\delta M_k \Pi_M(x_k^M)|^2.$$

Note that (b) follows from $\sup_k E|\delta M_k \Pi_M(x_k^M)|^2 < \infty$.

For part (c), the Taylor series approximation and the continuity of $\partial^2_{x^i} g(\cdot)$ imply

$$\begin{aligned} B_{k,i}(x_k^M)\Pi_M(x_k^M) &= \left[\frac{g(x_k^M + b_k e_i) - g(x_k^M - b_k e_i)}{2b_k} - \partial_{x^i} g(x_k^M)\right] \Pi_M(x_k^M) \\ &= O\left(\frac{|\partial^2_{x^i} g(\xi_k)| b_k^2}{2b_k}\right) = O(b_k), \end{aligned}$$

where ξ_k is between $x_k^M - b_k e_i$ and $x_k^M + b_k e_i$.

Therefore,

$$\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{B_k}{2b_k} \Pi_M(x_k^M) = \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(a_k b_k) = \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(n^{-7/6}) \to 0$$

uniformly as $n \to \infty$.

For part (d), apply Cauchy-Schwartz inequality,

$$\begin{aligned} E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \nabla g \Pi_M(x_k^M)\right|^2 &\le \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(k^{-2}) \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} E\left|\nabla g \Pi_M(x_k^M)\right|^2 \\ &\le \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(k^{-2}) \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} C \\ &\le C(t+s-t) \sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} O(k^{-2}) \to 0 \end{aligned}$$

uniformly in t as $n \to \infty$ and $\varepsilon \to 0$.

In the above, we have used $\sup_k E\left|\nabla g \Pi_M(x_k^M)\right|^2 < \infty$ due to the continuity of ∇g.

For part (e), we can show, using the same technique as in [38, Lemma 3.1], that

$$E\left|\sum_{k=\delta^n(t)}^{\delta^n(t+s)-1} a_k \frac{A_k}{2b_k} \Pi_M(x_k^M)\right|^2 \to 0 \text{ uniformly in } t \text{ as } n \to \infty.$$

Note that the Markov inequality implies convergence in probability from convergence in mean or in mean square.

For part (f), by the Aldous' tightness criterion in (1.5.15), we need to show, for any $\varepsilon > 0$, let $t \geq 0$, and $0 \leq s \leq \varepsilon$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} E|x_M^n(t+s) - x_M^n(t)|^2 = 0.$$

Using the representation in (4.7), we can write

$$x_M^n(t+s) - x_M^n(t) = \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \left[\frac{A_i(x_i, \chi_i)}{2b_i} + \frac{\delta M_i(x_i, \chi_i)}{2b_i} + B_i(x_i) + \nabla g(x_i) \right] \Pi_M(x_i^M).$$

Hence,

$$\begin{aligned} |x_M^n(t+s) - x_M^n(t)|^2 \leq \quad & 4 \left| \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \frac{A_i}{2b_i} \Pi_M(x_i^M) \right|^2 + 4 \left| \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \frac{\delta M_i}{2b_i} \Pi_M(x_i^M) \right|^2 + \\ & 4 \left| \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i B_i \Pi_M(x_i^M) \right|^2 + 4 \left| \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \nabla g \Pi_M(x_i^M) \right|^2 . \end{aligned}$$

It follows from Lemma 4.3.1 that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} E|x_M^n(t+s) - x_M^n(t)|^2 = 0.$$

□

Theorem 4.3.2. *Under conditions* (A1)-(A3)*, $x^n(t)$ converges weakly to $x(t)$, which is a solution of*

$$\dot{x} = \nabla g(x),$$

provided that the ordinary differential equation above has a unique solution for each initial condition.

Proof. Since $x_M^n(\cdot)$ is tight, by Prohorov's theorem, it is relatively compact, i.e., it contains a weakly convergent subsequence. We still write this subsequence as x_M^n for notational simplicity, and denote its limit as x_M.

Let k be an integer time points $0 \le t \le t+s,\ t_i \le t,\ i = 1, ..., k,\ f(\cdot)$ be continuously differentiable and bounded, and $h(\cdot)$ be smooth and bounded with compact support.

The weak convergence of $x_M^n(.)$ and Prohorov's theorem imply

$$\begin{aligned} &Eh(x_M^n(t_i), i \le k)[f(x_M^n(t+s)) - f(x_M^n(t))] \\ &\to Eh(x_M(t_i), i \le k)[f(x_M(t+s)) - f(x_M(t))] \end{aligned} \tag{4.8}$$

as $n \to \infty$.

On the other hand, recall that

$$x_M^n(t+s) - x_M^n(t) = \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \left[\frac{A_i(x_i, \chi_i)}{2b_i} + \frac{\delta M_i(x_i, \chi_i)}{2b_i} + B_i(x_i) + \nabla g(x_i) \right] \Pi_M(x_i^M).$$

By Lemma 4.3.1, we can write

$$x_M^n(t+s) - x_M^n(t) = \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \nabla g(\cdot) \Pi_M(x_i^M) + o^*(1),$$

where $o^*(1) \to 0$ in probability uniformly in t.

Note that, we can choose an increasing sequence of positive integers $\{m_k\}$ and a deceasing sequence of positive real numbers $\{\alpha_k\}$ with $\delta^n(t) \le m_k \le m_{k+1} - 1 \le \delta^n(t+s) - 1$ which satisfy

$$\frac{1}{\alpha_k} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{i} \to 1 \text{ as } n \to \infty.$$

Moreover, since $\nabla g(\cdot)\Pi_M(\cdot)$ is continuous and bounded, by the weak convergence of x_n^M, we have, for m_k sufficiently large,

$$E\nabla g(x_i^M)\Pi_M(x_i^M) = E\nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M) + \varepsilon_i, \text{ where } \varepsilon_i \le \frac{1}{i}, \text{ for } i \in [m_k, m_{k+1} - 1].$$

Hence,

$$\begin{aligned}
x_M^n(t+s) - x_M^n(t) = & \sum_{i=\delta^n(t)}^{\delta^n(t+s)-1} a_i \nabla g(x_i^M)\Pi_M(x_i^M) + o^*(1) \\
= & \sum_{\{k:\delta^n(t) \le m_k \le m_{k+1}-1 \le \delta^n(t+s)-1\}} \sum_{i=m_k}^{m_{k+1}-1} a_i \nabla g(x_i^M)\Pi_M(x_i^M) + o^*(1) \\
= & \sum_{\{k:\delta^n(t) \le m_k \le m_{k+1}-1 \le \delta^n(t+s)-1\}} \sum_{i=m_k}^{m_{k+1}-1} a_i \nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M) + o^*(1) \\
= & \sum_{\{k:\delta^n(t) \le m_k \le m_{k+1}-1 \le \delta^n(t+s)-1\}} \alpha_k \nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M) \frac{1}{\alpha_k} \sum_{i=m_k}^{m_{k+1}-1} \frac{1}{i} + o^*(1) \\
= & \sum_{\{k:\delta^n(t) \le m_k \le m_{k+1}-1 \le \delta^n(t+s)-1\}} \alpha_k \nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M) + o^*(1).
\end{aligned}$$

Furthermore, for some ξ in $(t, t+s)$,

$$\begin{aligned}
f(x_M^n(t+s)) - f(x_M^n(t)) = & \nabla f(x_M^n(\xi))[x_M^n(t+s) - x_M^n(t)] \\
= & \nabla f(x_M^n(\xi)) \sum_{\{k:\delta^n(t) \le m_k \le m_{k+1}-1 \le \delta^n(t+s)-1\}} \alpha_k \nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M).
\end{aligned}$$

Since ∇f is continuous and bounded, we have, by the weak convergence of x_M^n and Prohorov's theorem, and for s sufficiently small,

$$f(x_M^n(t+s)) - f(x_M^n(t)) = \sum_{\{k:\delta^n(t)\le m_k\le m_{k+1}-1\le\delta^n(t+s)-1\}} \alpha_k \nabla f(x_{m_k}^M)\nabla g(x_{m_k}^M)\Pi_M(x_{m_k}^M) + o^*(1).$$

For the same reason as in the convergence of (4.8), we have

$$\begin{aligned}&Eh(x_M^n(t_i), i\le k)[f(x_M^n(t+s)) - f(x_M^n(t))]\\ &\to Eh(x_M(t_i), i\le k)\int_t^{t+s}\nabla f(x_M(r))\nabla g(x_M(r))\Pi_M(x_M(r))dr\end{aligned}$$

as $n\to\infty$.

The previous expression and (4.8) give

$$\begin{aligned}&Eh(x_M(t_i), i\le k)[f(x_M(t+s)) - f(x_M(t))]\\ &= Eh(x_M(t_i), i\le k)\int_t^{t+s}\nabla f(x_M(r))\nabla g(x_M(r))\Pi_M(x_M(r))dr.\end{aligned}$$

Let

$$M_f(t) = f(x_M(t)) - \int_0^t \mathcal{A}_M f(x_M(r))dr,$$

where $\mathcal{A}_M f(x_M(r)) = \nabla f(x_M(r))\nabla g(x_M(r))\Pi_M(x_M(r))$.

Using the previous equality gives

$$\begin{aligned}&Eh(x_M(t_i), i\le k)[M_f(t+s) - M_f(t)]\\ &= Eh(x_M(t_i), i\le k)[f(x_M(t+s)) - f(x_M(t)) - \int_t^{t+s}\mathcal{A}_M f(x_M(r))dr] = 0.\end{aligned}$$

So, by (1.5.3) and (1.2), the process x_M solves the martingale problem for the generator $\mathcal{A}_M$.

Also, we can deduce from the previous equality that

$$f(x_M(t+s)) - f(x_M(t)) - \int_t^{t+s}\mathcal{A}_M f(x_M(r))dr = 0 \ a.s.$$

Dividing both side of the equation by s and letting $s \downarrow 0$, we obtain

$$\nabla f(x_M(t))\dot{x}_M(t) = \mathcal{A}_M f(x_M(t)) = \nabla f(x_M(t))\nabla g(x_M(t))\Pi_M(x_M(t)).$$

This implies $\dot{x}_M(t) = \nabla g(x_M(t))\Pi_M(x_M(t))$.

Note that we actually take the right derivative. The left derivative can be obtained similarly.

Finally, by sending $M \to \infty$, we obtain the desired result similar to being shown in [24, p.284]. $\square$

4.4 Numerical Examples

The following steps are used to implement the stochastic approximation algorithm (4.5). Suppose the training data has T time points.

Input: $\mu_1, \mu_2, \lambda_1, \lambda_2, \sigma$ (calculated from the training data).

(1) Initiate $x_0 = (x_0^1, x_0^2)$.

(2) Use (4.2) and (4.3) to simulate $\{(Z_t, p_t)\}_{t=1}^T$ in $\mathbb{R} \times [0,1]$.

(3) For $n > 0$, compute x_n recursively as follow: let $e_1 = (1,0), e_2 = (0,1), a_n = 0(n^{-1}), b_n = 0(n^{-1/6})$.

 (3.1) Find $\tau_1(x_n + b_n e_1) < \tau_2(x_n)$, then use (4.4) to compute $G(x_n + b_n e_1, \chi_{n,1}^+)$.

 (3.2) Find $\tau_1(x_n - b_n e_1) < \tau_2(x_n)$, then use (4.4) to compute $G(x_n - b_n e_1, \chi_{n,1}^-)$.

 (3.3) Find $\tau_2(x_n + b_n e_1) > \tau_1(x_n)$, then use (4.4) to compute $G(x_n + b_n e_1, \chi_{n,2}^+)$.

 (3.4) Find $\tau_2(x_n - b_n e_1) > \tau_1(x_n)$, then use (4.4) to compute $G(x_n - b_n e_1, \chi_{n,2}^-)$.

 (3.5) Use (4.6) to compute $Y_n(x_n, \chi_n)$.

 (3.6) Use (4.5) to update $x_{n+1} = x_n + a_n Y_n$.

(4) Repeat step (3) with $n \leftarrow n+1$ until $\|x_n - x_{n+1}\|_2 < \varepsilon$ for some ε, or n is sufficiently large.

The algorithm is now demonstrated using daily closing prices of the pairs: GOOGL-YHOO (Google Alphabet Inc. and Yahoo! Inc.) and SNE-LPL (Sony Corporation and LG Display Co., Ltd.). The data is collected from Yahoo Finance from June 20$^{\text{th}}$ 2008 to May 31$^{\text{st}}$ 2016 (2000 data points). The first 1500 data points (June 20$^{\text{th}}$ 2008- June 5$^{\text{th}}$ 2014) are used as training data, and the remaining 500 data points (June 6$^{\text{th}}$ 2014 - May 31$^{\text{st}}$ 2016) are set aside for testing.

We first consider the pair GOOGL-YHOO. The following figures are the training data pair, and their interactive graph.

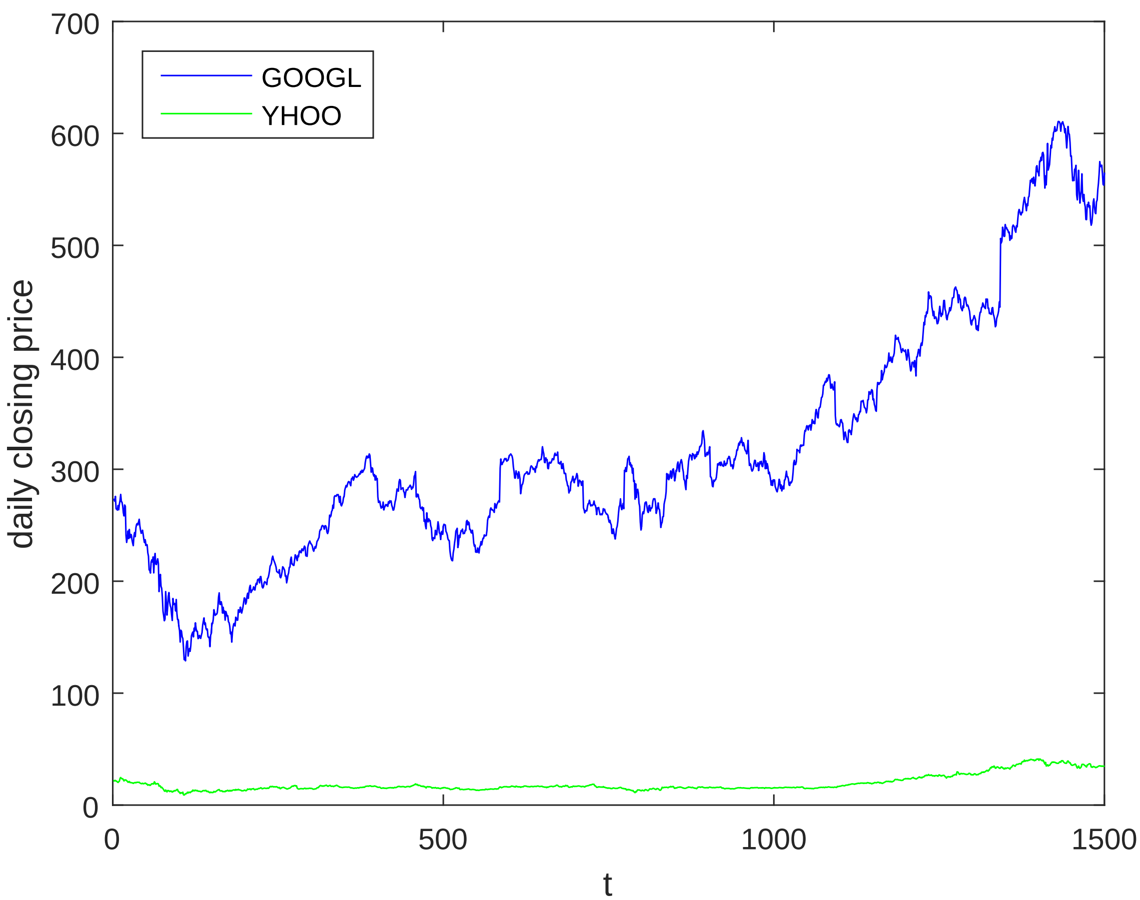

Figure 4.2: Training data pair (June 20$^{\text{th}}$ 2008–June 5$^{\text{th}}$ 2014) (GOOGL-YHOO).

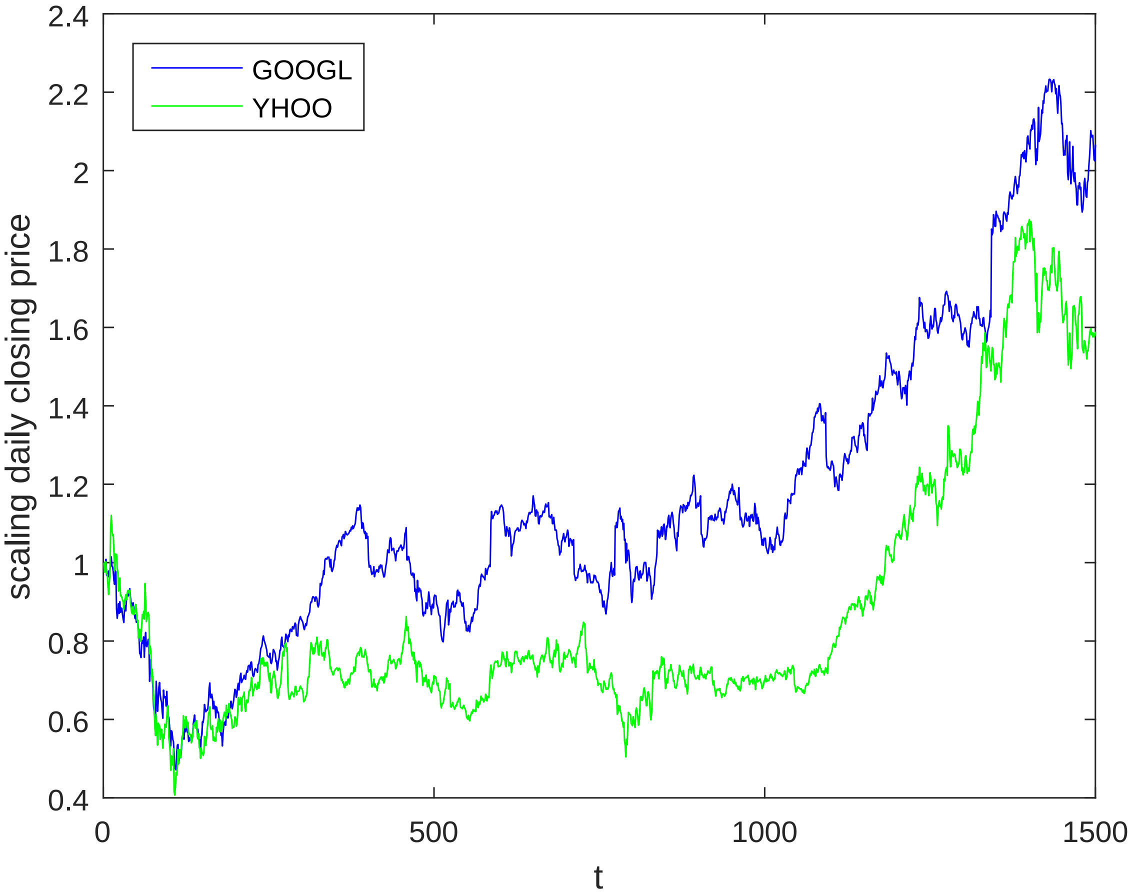

Figure 4.3: The interactive graph of the training data pair (GOOGL-YAHOO).

Figure 4.3 shows that GOOGL outperforms YHOO, so we will consider

$$Z_t = \log(\text{GOOGL}_t) - \log(\text{YHOO}_t).$$

The graph of Z_t is below.

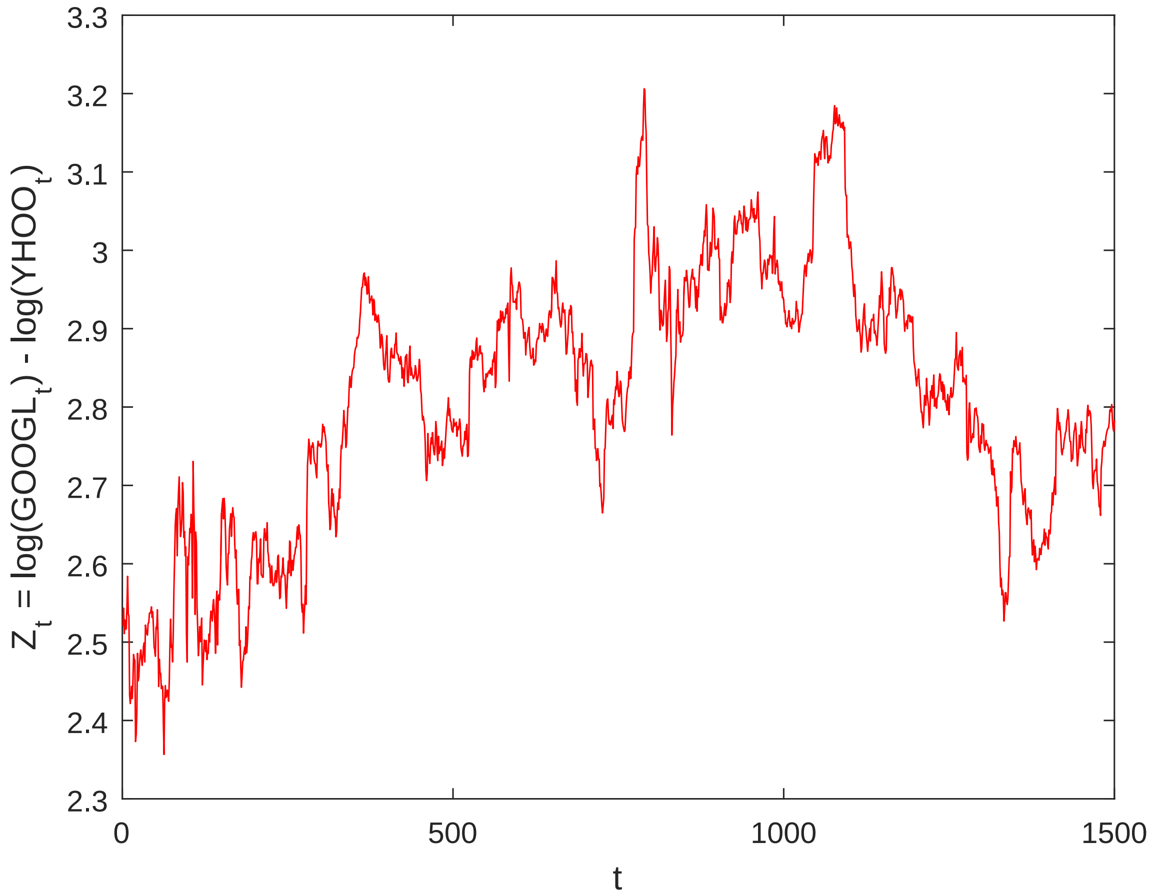

Figure 4.4: The log difference of the training data pair (GOOGL-YHOO).

Using the training data Z_t, the parameters μ_1, μ_2, λ_1, λ_2, σ (overall historical volatility), σ_1 (bull market volatility), and σ_2 (bear market volatility) are estimated corresponding to the bull-bear market below.

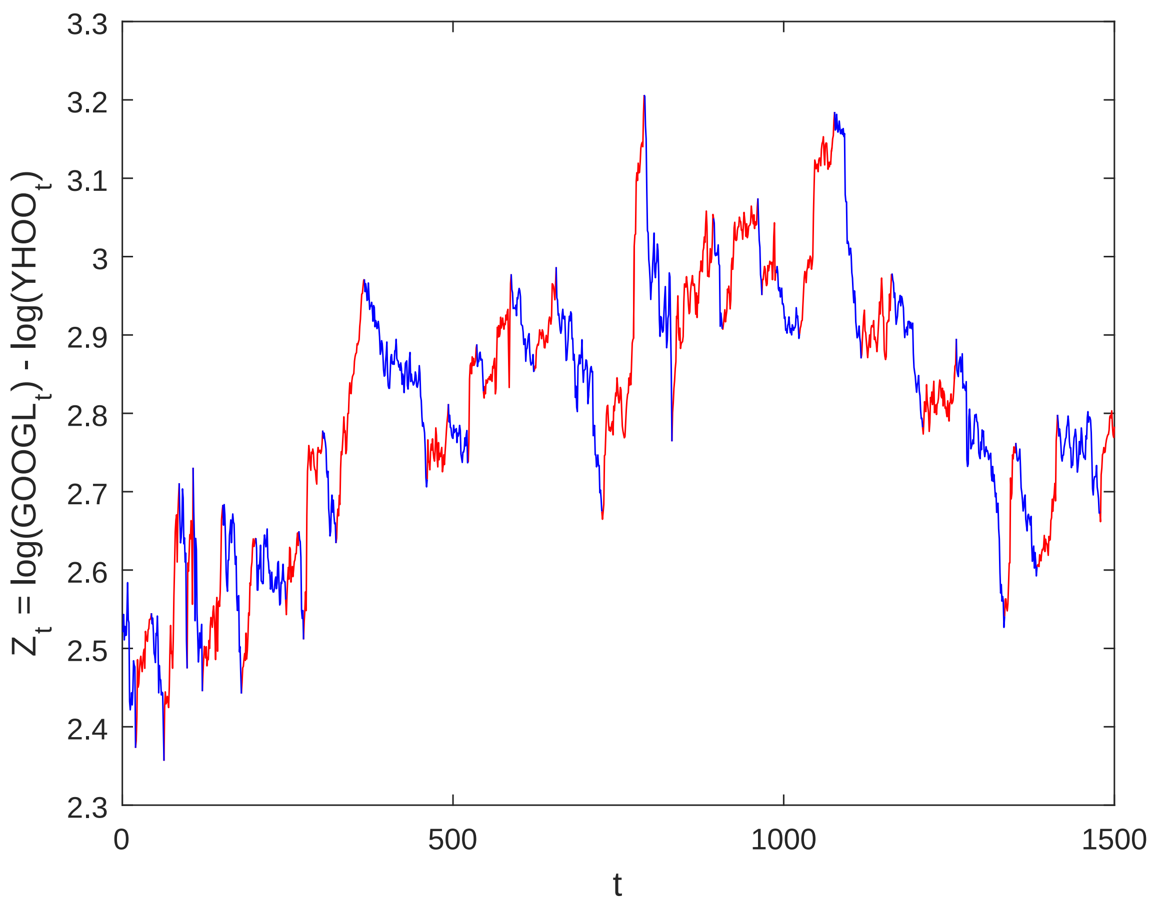

Figure 4.5: The log difference of the training data pair, and the bull (red) and bear (blue) markets (GOOGL-YHOO).

μ_1	μ_2	λ_1	λ_2	σ	σ_1	σ_2
1.846917	-2.069453	7.392473	7.284768	0.341831	0.366002	0.366265

Table 4.1: Parameter Estimation of Z_t (GOOGL-YHOO).

The stochastic approximation algorithm is implemented using the estimated $\mu_1, \mu_2, \lambda_1, \lambda_2$, and σ as input. Also, 2000 simulations of (Z_t, p_t) are used for each initial guess of the buy and sell threshold levels $x_0 = (x_0^1, x_0^2)$. With initial guess $(x_0^1, x_0^2) = (0.9, 0.5)$ and $(0.6, 0.1)$, the output shows (x_n^1, x_n^2) converges to $(0.863427, 0.159374)$.

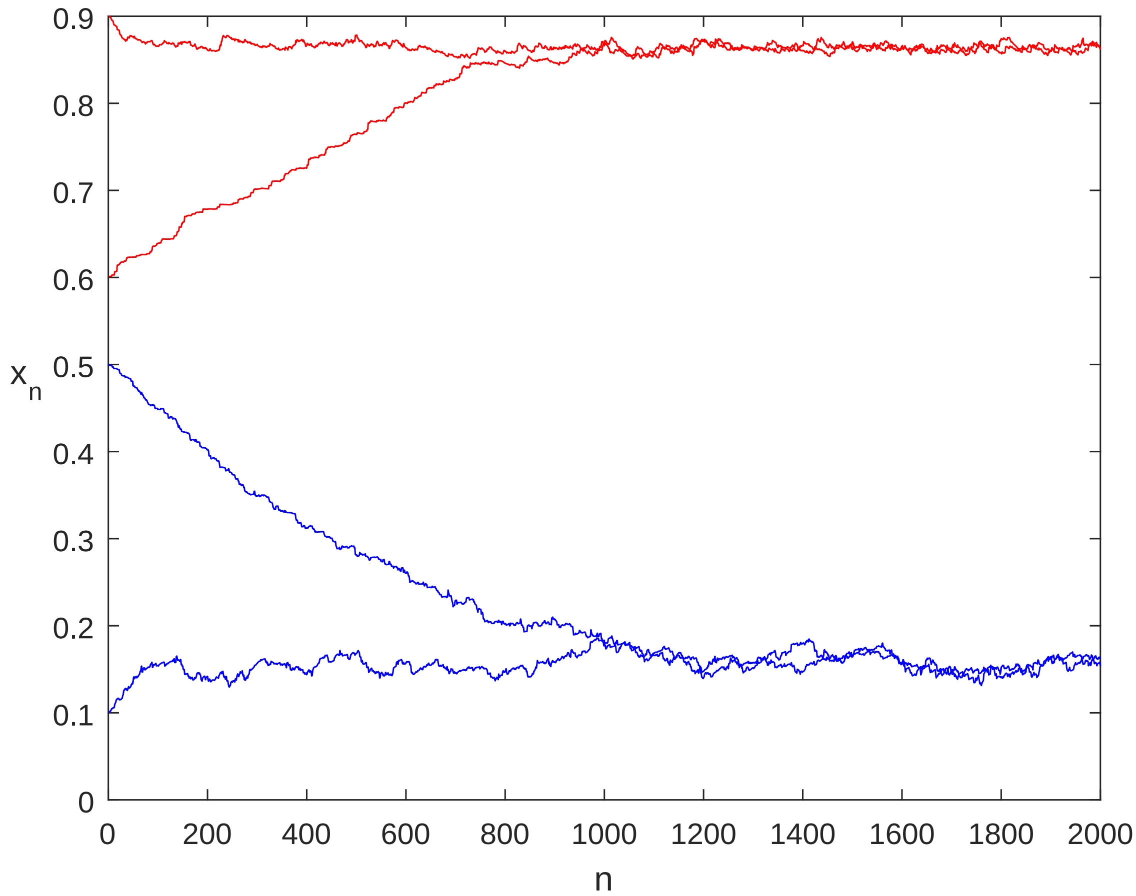

Figure 4.6: $(x_n^1, x_n^2) \to (0.863427, 0.159374)$ with initial guess $(x_0^1, x_0^2) = (0.9, 0.5)$ and $(0.6, 0.1)$ (GOOGL-YHOO).

Now, the threshold $(0.863427, 0.159374)$ is used to trade the testing data. Figure 4.7 below shows the conditional probability p_t at each time point of the testing data, and the buy and sell thresholds $x_0^1 = 0.863427$ and $x_0^2 = 0.159374$ respectively.

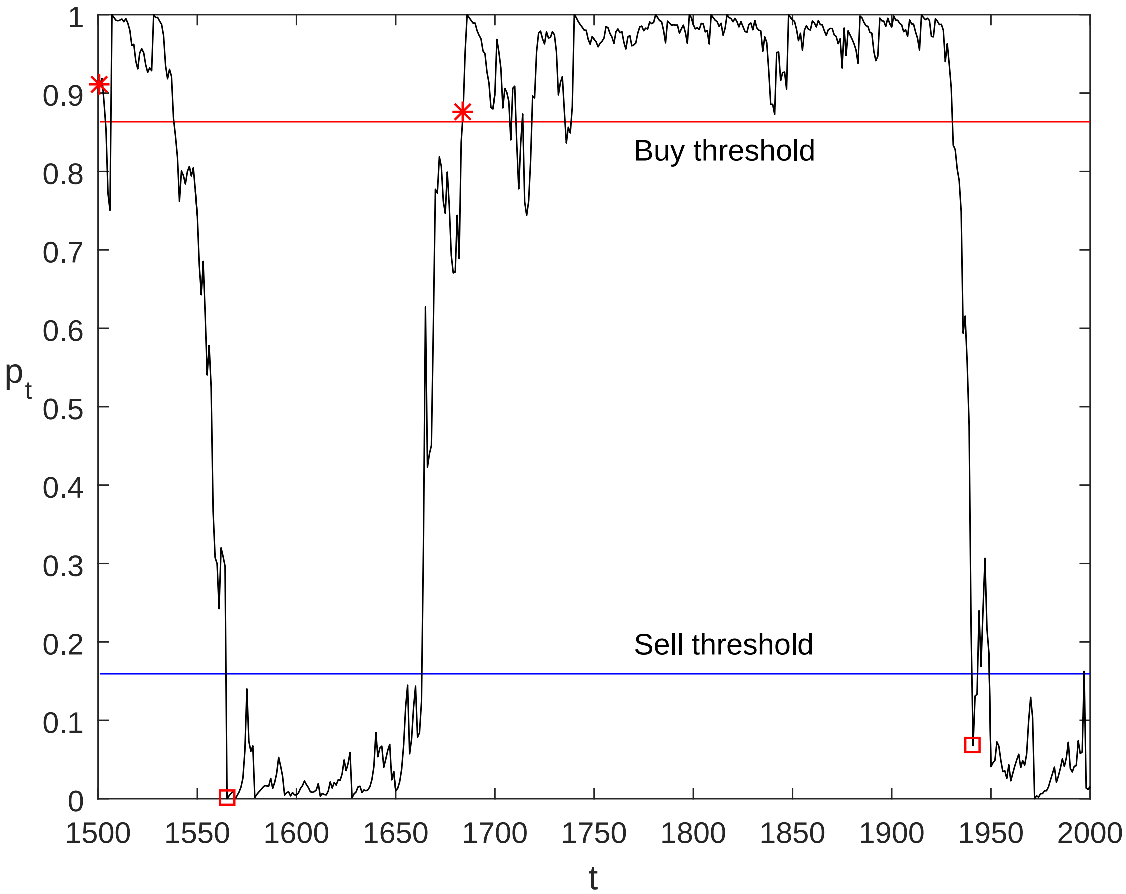

Figure 4.7: Threshold levels and the probability curve (GOOGL-YAHOO).

We use $(Z_{1500}, p_{1500}) = (2.783, 0.951218)$ to initiate the p_t for the testing data Z_t. In Figure 4.7 above, the star marks indicate the places to buy Z_t, and the square marks indicate the places to sell Z_t. There are two buy and sell round trips. These transactions are marked as stars (buy) and squares (sell) in the interactive graph of the testing pair below.

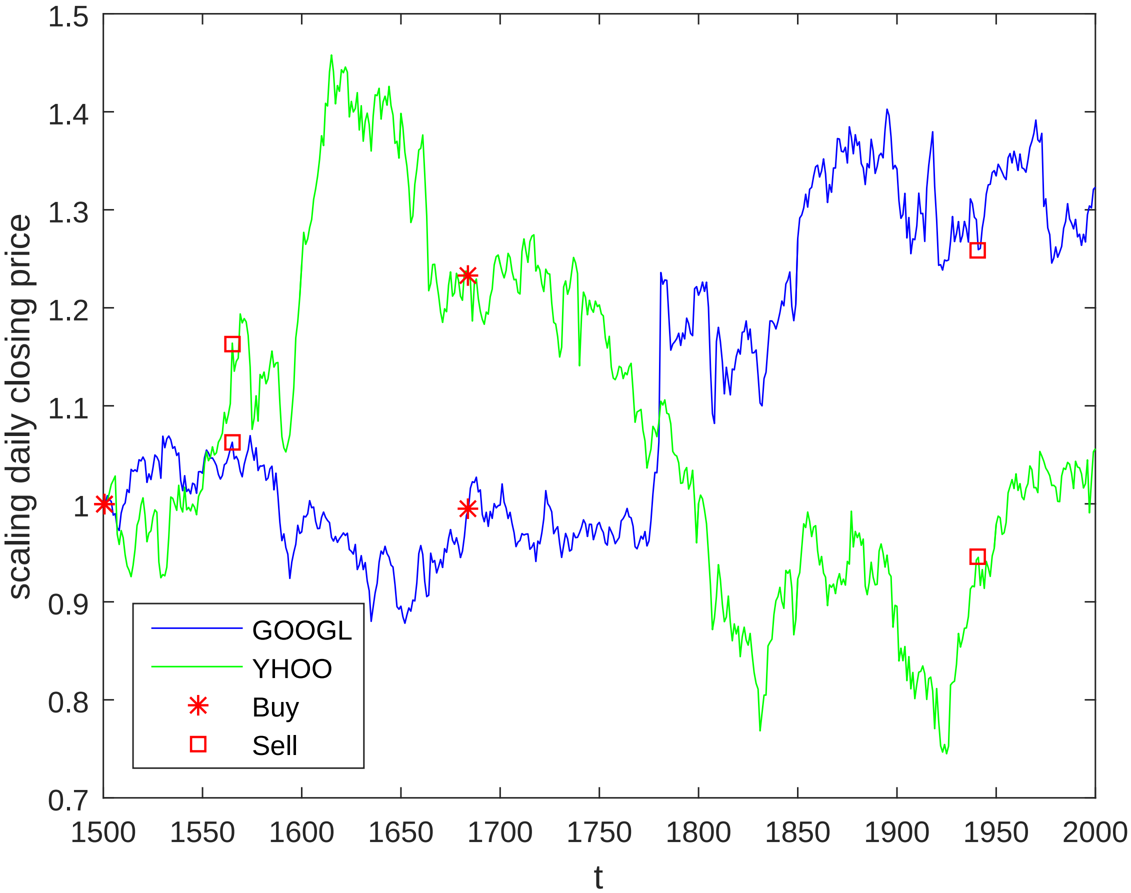

Figure 4.8: Interactive graph of the testing pair, and the buy and sell indicators (GOOGL-YAHOO).

Remark 4.4.1. As shown in Figure 4.8, the behavior of the pair in the first round trip trade follows the trend of the training data for a very short period, and turns sour: YHOO outperforms GOOGL. The algorithm limits the loss by early exit. In the second round trip trade, the algorithm enters the position when the pair converges. In this case, the pair follows the trend of the training data: GOOGL outperforms YHOO. This results in a gain.

Next, we test the performance of the stochastic approximation algorithm. For this, we start with an account of capital \$10000, and use equal amounts to long and short positions. The commission fee is \$5 per trade for longing and \$5 per trade for shorting, and an additional \$1 per share for shorting. The first round trip is initiated on 06/06/2014: buy 8 shares of GOOGL at \$566.03 per share, and sell short 135 shares of YHOO at \$35.92 per share. The trade is closed on 09/08/2014: sell 8 shares of GOOGL at \$601.63, and buy 135 shares of YHOO at \$41.81. The second round trip is initiated on 02/27/2015: buy 8 shares of GOOGL at \$562.63 per share, and sell short 104 shares of YHOO at \$44.28 per share. The trade is closed on 03/07/2016: sell 8 shares of GOOGL at \$712.80, and buy 104 shares of YHOO at \$33.96. The two round trip trades are summarized in the following table.

Buy date	$\tau_{1,i}$	Sell date	$\tau_{2,i}$	Gain	Adjusted Capital
06/06/2014	1501	09/08/2014	1565	-520.35	9479.65
02/27/2015	1684	03/07/2016	1941	2264.64	11744.29

Table 4.2: Summary of stochastic approximation strategy (GOOGL-YHOO).

The total gain is \$1744.29, which corresponds to 17.44%. For the buy and hold strategy, the total gain is \$1428.76 or 14.29%.

Note that Z_t is symmetric, and one can trade $(-Z_t)$ the same way as Z_t. However, the future market is expected to follows the trend of the training data. Therefore, the algorithm is expected to perform better when Z_t is taken by subtracting the underperforming stock from the outperforming one. To demonstrate this, we take

$$Y_t = \log(\mathrm{YHOO}_t) - \log(\mathrm{GOOGL}_t).$$

Using the reverse bull-bear market of Z_t, we obtain the following parameters for Y_t:

μ_1	μ_2	λ_1	λ_2	σ	σ_1	σ_2
2.069453	-1.846917	7.284768	7.392473	0.341831	0.366265	0.366002

Table 4.3: Parameters of Y_t (YHOO-GOOGL).

The threshold $(0.863427, 0.159374)$ is again used to trade the testing data. The graph below shows the conditional probability p_t of the testing data Y_t, and the buy and sell thresholds.

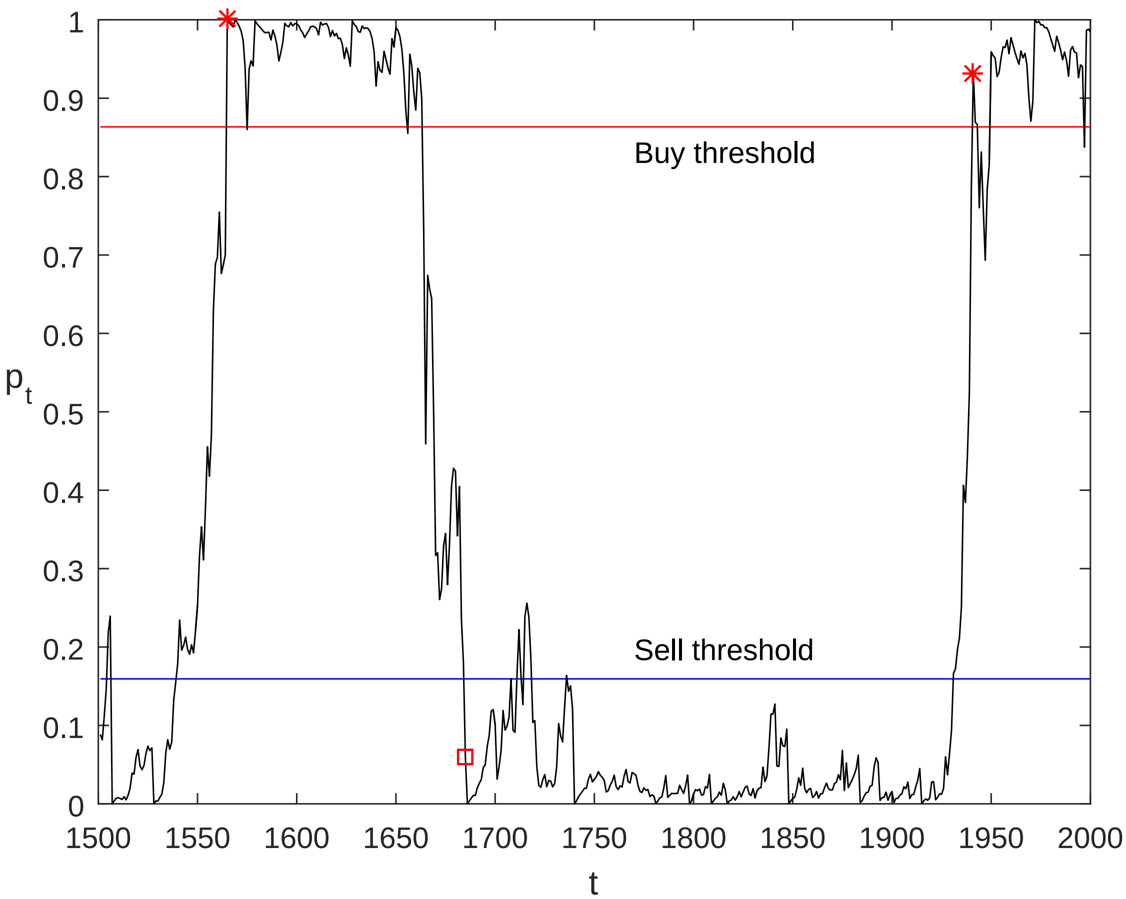

Figure 4.9: Threshold levels and the probability curve of Y_t (YHOO-GOOGL).

There is only one buy and sell round trip corresponding to the testing data. The algorithm still waits to exit (sell Z_t) from the second buy. We also start with an account of capital \$10000 to test the performance of the algorithm. The total gain in the round trip is \$476.58, which corresponds to 4.77%. This shows Z_t performs better than Y_t as expected.

Next, we consider the pair SNE-LPL with

$$Z_t = \log(\mathrm{SNE}_t) - \log(\mathrm{LPL}_t).$$

Following is the summary of the results.

μ_1	μ_2	λ_1	λ_2	σ	σ_1	σ_2
2.808430	−4.094507	5.936675	6.072874	0.471204	0.518395	0.536774

Table 4.4: Parameter Estimation of Z_t (SNE-LPL).

With initial guess $(x_0^1, x_0^2) = (0.9, 0.5)$ and $(0.6, 0.1)$, (x_n^1, x_n^2) converges to $(0.877036, 0.151839)$ as shown in Figure 4.10 below.

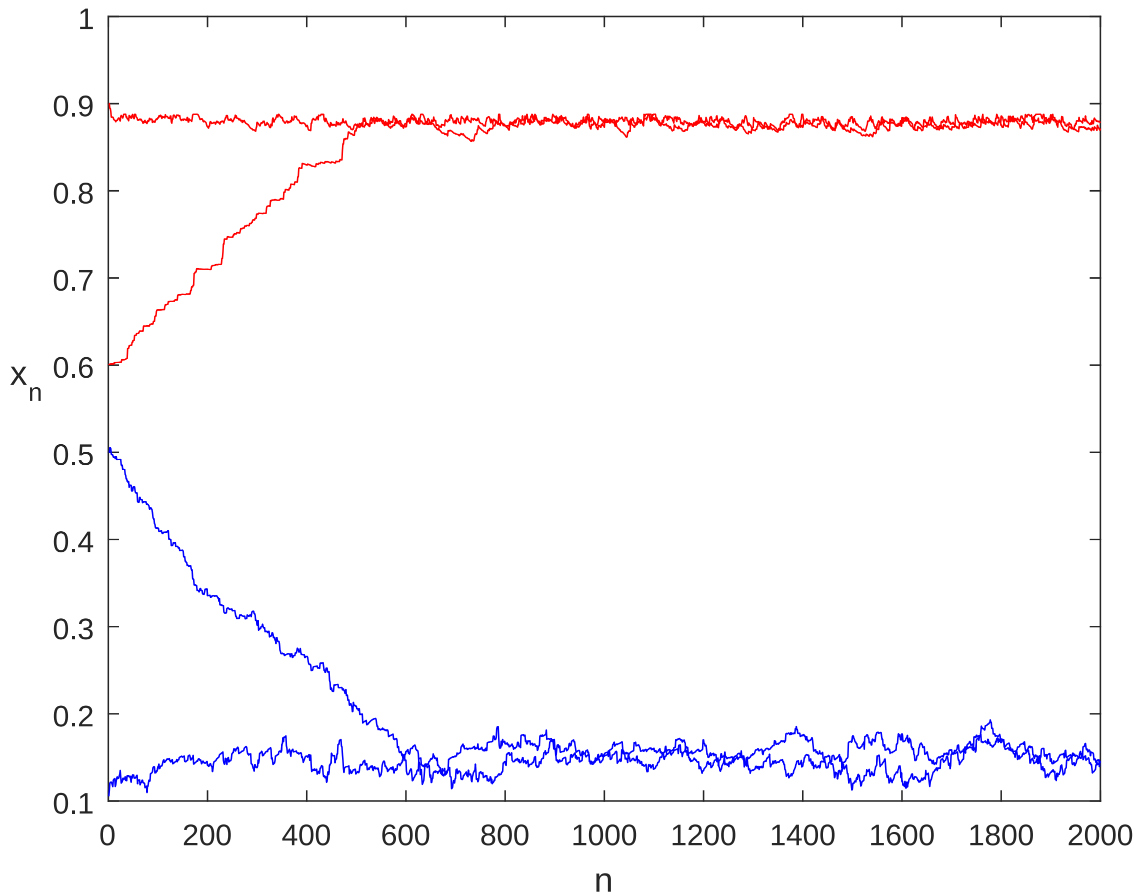

Figure 4.10: $(x_n^1, x_n^2) \to (0.877036, 0.151839)$ with initial guess $(x_0^1, x_0^2) = (0.9, 0.5)$ and $(0.6, 0.1)$ (SNE-LPL).

The threshold $(0.877036, 0.151839)$ is used to trade the testing data. The trading is summarized in the following figures and table.

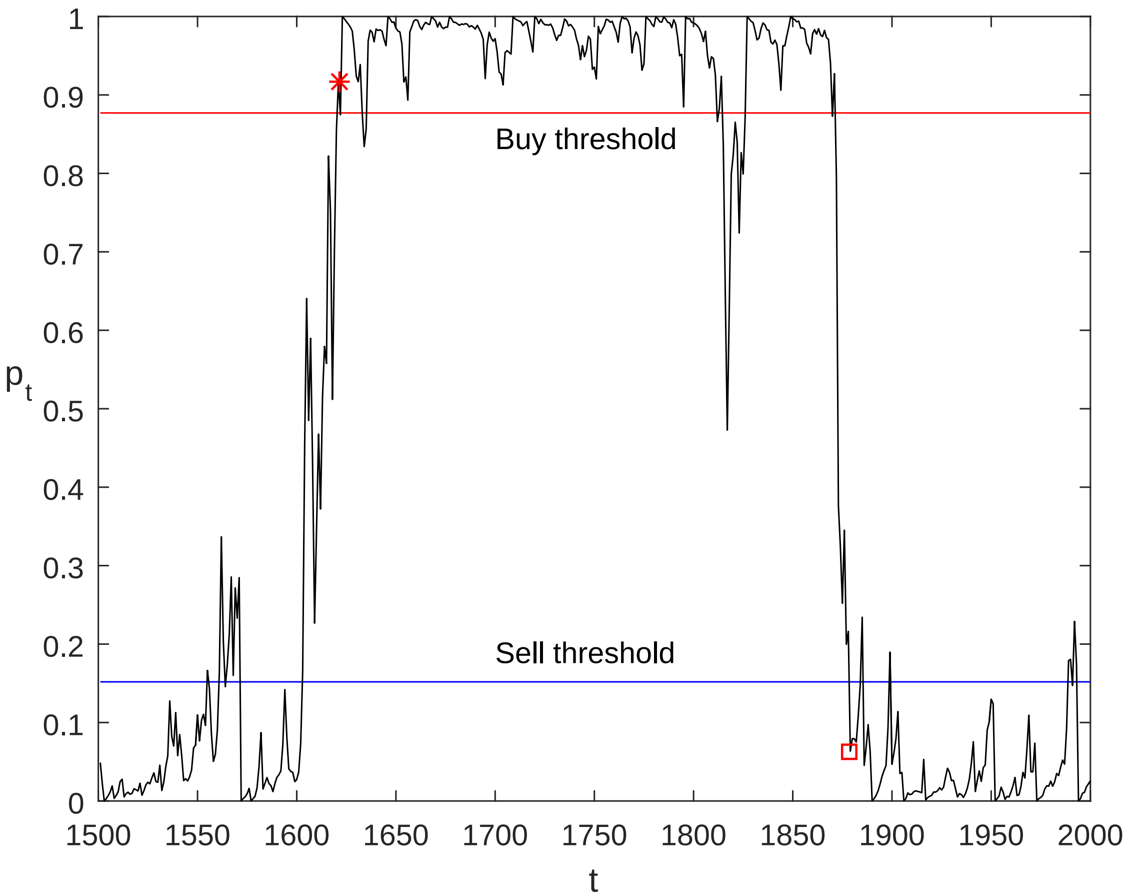

Figure 4.11: Threshold levels and the probability curve of Z_t (SNE-LPL).

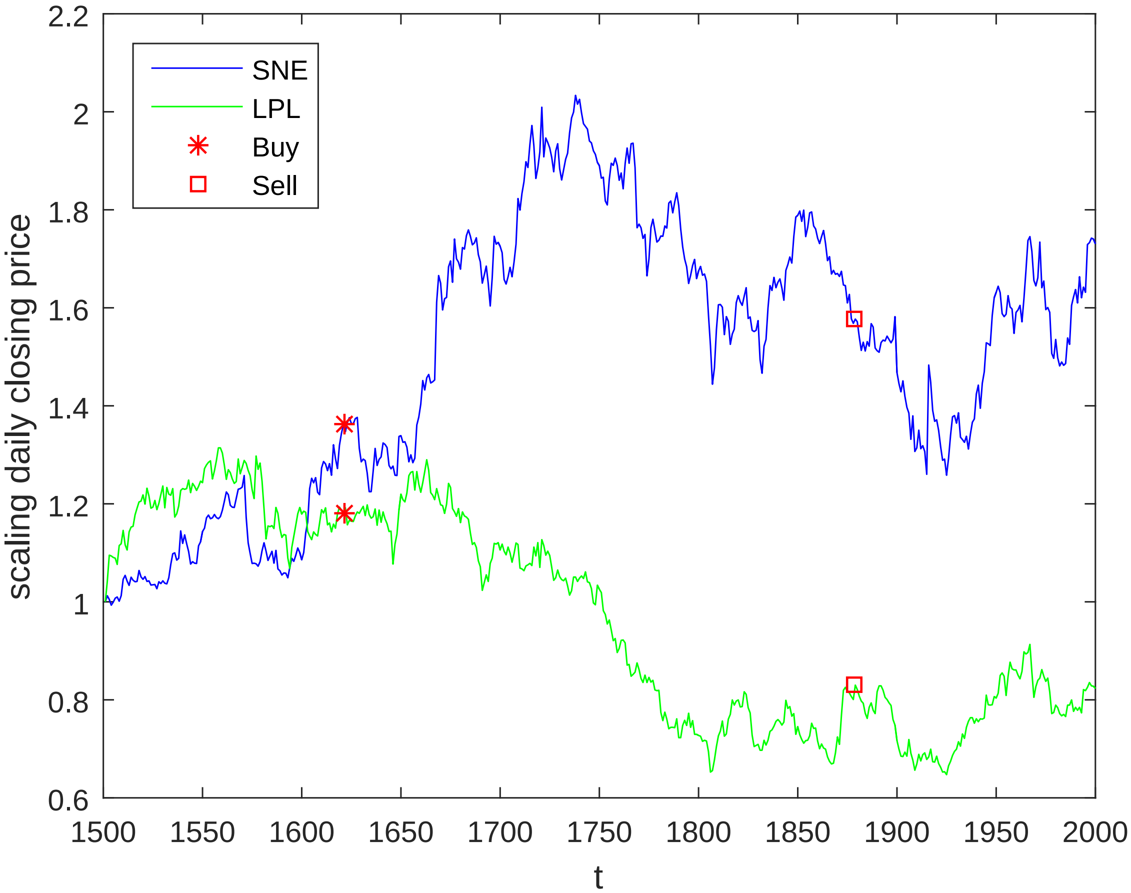

Figure 4.12: Interactive graph of the testing pair, and the buy and sell indicators (SNE-LPL).

There is only one buy and sell round trip. We also start with an account of capital \$10000 to test the performance of the algorithm.

Buy date	$\tau_{1,i}$	Sell date	$\tau_{2,i}$	Gain	Adjusted Capital
11/25/2014	1621	12/04/2015	1879	2173.42	12173.42

Table 4.5: Summary of stochastic approximation strategy (SNE-LPL).

The total gain is \$2173.42, which corresponds to 21.73%.

Remark 4.4.2. It may take very long to finish one round trip. However, this strategy of pairs trading always leaves the account in cash during the length of any round trip. This cash is equal to the amount in the account prior to entering the round trip, subtracting the commission fees. The cash can be used for other shorter term investment, at least drawing interest over the round trip length.

Chapter 5

A Viscosity Solution Approach to Pairs Trading under a Regime-Switching Model

5.1 Introduction

As a continuation study of the problem of pairs trading, this chapter focuses on more general setups. A sequence of trades are allowed. The value functions can be characterized in terms of HJB equations which involve a second-order, nonlinear parabolic partial differential equation. In this case, a closed-form solution to the associated PDE is very difficult to obtain. At this point, we cannot prove there are smooth solutions to these equations. Therefore, we resort to viscosity solutions of the HJB equations. The concept of viscosity solutions was introduced in the 1983 paper by Crandall and Lions [7] to capture positive non-smooth solutions. In the following sections, we show that the value functions are indeed the unique viscosity solutions of the HJB equations. This will enable an alternative numerical scheme to be constructed to approximate the probability threshold levels.

5.2 Formulating the Problem

Let $S_t^i, i = 1, 2$, be the stock prices at time $t \geq 0$, with S^1 and S^2 being the stronger and weaker stocks respectively. Assume their log difference $Z_t = \log S_t^1 - \log S_t^2$ is a switching process in $\mathbb{R}$ governed by

$$dZ_t = \mu(\alpha_t)dt + \sigma dW_t,\ Z_0 = Z,$$

where $\mu(i) = \mu_i, i = 1, 2$, the expected rates of return with $\mu_1 > 0 > \mu_2$; $\sigma > 0$, the volatility; W_t, a standard Brownian motion; and α_t, the Markov chains with state space $\mathcal{M} = \{1, 2\}$.

The process α_t represents the market mode at each time t, where $\alpha_t = 1$ indicates a bull market and $\alpha_t = 2$ a bear market. The generator for α_t is denoted by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},\ \text{for some } \lambda_1 > 0,\ \lambda_2 > 0.$$

Let

$$0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \cdots \tag{5.1}$$

denote a sequence of stopping times where buying is at τ_n and selling is at σ_n, $n = 1, 2, \ldots$.

We restrict to the case where the net position at any time is either flat (no stock position of either S_t^1 or S_t^2) or long with one share of Z_t (long in the outperforming stock S_t^1 and short in the underperforming one S_t^2). If initially the net position is long ($i = 1$) then one must sell Z_t before buying any shares. In this case, the sequence of stopping times is denoted by $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$. Similarly, if initially the net position is flat ($i = 0$) then one must first buy Z_t before selling any shares, and the corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$.

Given the initial state $Z_0 = Z$, $\alpha_0 = \alpha \in \{0, 1\}$, and initial net position $i = 0, 1$, the reward functions of the decision sequences, Λ_0 and Λ_1, are given as follows:

$$J_i(Z,\alpha,\Lambda_i) = \begin{cases} E\left\{\displaystyle\sum_{n=1}^{\infty}\left[e^{-\rho\sigma_n}(Z_{\sigma_n}-K) - e^{-\rho\tau_n}(Z_{\tau_n}+K)\right]\right\} & \text{if } i=0, \\ E\left\{e^{-\rho\sigma_1}(S_{\sigma_1}-K)\right. & \\ \quad \left. +\displaystyle\sum_{n=2}^{\infty}\left[e^{-\rho\sigma_n}(Z_{\sigma_n}-K) - e^{-\rho\tau_n}(Z_{\tau_n}+K)\right]\right\} & \text{if } i=1, \end{cases} \tag{5.2}$$

where $K > 0$ denote the slippage cost per transaction, and $\rho > 0$ the discount factor. Motivated by Etheridge [11], we may assume $\mu_2 < \rho < \mu_1$.

For simplicity, the term $E\sum_{n=1}^{\infty}\xi_n$ for random variables ξ_n is interpreted as

$$\limsup_{N\to\infty} E\sum_{n=1}^{N}\xi_n.$$

Let $V_i(Z,\alpha)$ denote the value functions with the initial net positions $i=0,1$, and initial state $Z_0 = Z$, $\alpha_0 = \alpha \in \{0,1\}$. Then

$$V_i(Z,\alpha) = \sup_{\Lambda_i} J_i(Z,\alpha,\Lambda_i). \tag{5.3}$$

Let $p_t = P(\alpha_t = 1 | Z_s : 0 \le s \le t) \in (0,1)$ denote the conditional probability of $\alpha_t = 1$ (bull market) given the difference of stock prices up to time t. Then the Wonham filter [37] in terms of p_t satisfies the SDE equation:

$$dp_t = [-(\lambda_1+\lambda_2)p_t + \lambda_2]dt + \frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}d\widehat{W}_t,$$

where $\widehat{W}_t$ is an innovation process given by

$$d\widehat{W}_t = \frac{dZ_t - [(\mu_1-\mu_2)p_t + \mu_2]dt}{\sigma}.$$

Equivalently,

$$dZ_t = [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma d\widehat{W}_t.$$

The problem of maximization of the discounted return becomes

$$V_i(Z, p) = \sup_{\Lambda_i} J_i(Z, p, \Lambda_i), \tag{5.4}$$

where $J_i(Z, p, \Lambda_i) = J_i(Z, \alpha, \Lambda_i)$ subject to

$$\begin{cases} dZ_t = [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma d\widehat{W}_t & Z_0 = Z, \\ dp_t = [-(\lambda_1 + \lambda_2)p_t + \lambda_2]dt + \dfrac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma} d\widehat{W}_t & p_0 = p, \end{cases} \tag{5.5}$$

where $(Z, p) \in \mathbb{R} \times (0, 1)$.

5.3 Properties of the Value Functions

We now establish various properties of the value functions.

Lemma 5.3.1.

1. $V_0(Z, p) - V_1(Z, p) + (Z + K) \geq 0.$

2. $V_1(Z, p) - V_0(Z, p) - (Z - K) \geq 0.$

Proof. We observe that the sequence $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$ can be regarded as a combination of a buy at τ_1 and then followed by the sequence of stopping times $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$.

Therefore,

$$
\begin{aligned}
V_0(Z,p) \geq\ & J_0(Z, p\Lambda_0) \\
=\ & E\left\{ e^{-\rho\sigma_1}(Z_{\sigma_1} - K) + \sum_{n=2}^{\infty}\left[e^{-\rho\sigma_n}(S_{\sigma_n} - K) - e^{-\rho\tau_n}(S_{\tau_n} + K)\right]\right\} \\
& -Ee^{-\rho\tau_1}(S_{\tau_1} + K) \\
=\ & J_1(Z_{\tau_1}, p_{\tau_1}, \Lambda_1) - Ee^{-\rho\tau_1}(Z_{\tau_1} + K).
\end{aligned}
$$

Setting $\tau_1 = 0$, and taking supremum over all Λ_1, we get

$$V_0(Z,p) \geq V_1(Z,p) - (Z + K).$$

Similarly,

$$
\begin{aligned}
V_1(Z,p) \geq\ & J_1(Z,p,\Lambda_1) \\
=\ & J_0(Z_{\sigma_1}, p_{\sigma_1}, \Lambda_0) + Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K).
\end{aligned}
$$

By setting $\sigma_1 = 0$, and taking supremum over all Λ_0, we get

$$V_1(Z,p) \geq V_0(Z,p) + (Z - K).$$

□

Lemma 5.3.2 (Bounds of the value functions)**.**

1. $0 \leq V_0(Z,p) \leq |Z| + \dfrac{\mu_1 - \mu_2}{\rho}$.

2. $Z - K \leq V_1(Z,p) \leq |Z| + \dfrac{\mu_1 - \mu_2}{\rho} + Z + K$.

Proof. The non-negativity of V_0 follows from its definition. We first establish an upper bound for part (1) of the lemma.

$$\begin{aligned} J_0(Z,p,\Lambda_0) = &\ E\left\{\sum_{n=1}^{\infty}\left[e^{-\rho\sigma_n}(Z_{\sigma_n}-K)-e^{-\rho\tau_n}(Z_{\tau_n}+K)\right]\right\} \\ \leq &\ E\left\{\sum_{n=1}^{\infty}\left[e^{-\rho\sigma_n}Z_{\sigma_n}-e^{-\rho\tau_n}Z_{\tau_n}\right]\right\} \\ = &\ E\left\{\sum_{n=1}^{\infty}\left[e^{-\rho\sigma_n}Z_{\sigma_n}-e^{-\rho\tau_n}Z_{\tau_n}\right]\right\}. \end{aligned} \tag{5.6}$$

Now,

$$\begin{aligned} d(e^{-\rho t}Z_t) = &\ -\rho e^{-\rho t}Z_t dt + e^{-\rho t}\left\{[(\mu_1-\mu_2)p_t+\mu_2]dt+\sigma d\widehat{W}_t\right\} \\ = &\ e^{-\rho t}[-\rho Z_t+(\mu_1-\mu_2)p_t+\mu_2]dt+e^{-\rho t}\sigma d\widehat{W}_t \\ \leq &\ e^{-\rho t}(-\rho Z_t+\mu_1)dt+e^{-\rho t}\sigma d\widehat{W}_t. \end{aligned}$$

Then

$$Ee^{-\rho\sigma_n}Z_{\sigma_n}-Ee^{-\rho\tau_n}Z_{\tau_n} \leq E\int_{\tau_n}^{\sigma_n} e^{-\rho t}(-\rho Z_t+\mu_1)dt. \tag{5.7}$$

Note that

$$\begin{aligned} EZ_t = &\ Z+E\int_0^t[(\mu_1-\mu_2)p_s+\mu_2]ds \\ \geq &\ Z+E\int_0^t \mu_2 ds = Z+\mu_2 t. \end{aligned} \tag{5.8}$$

(5.7) becomes

$$\begin{aligned} Ee^{-\rho\sigma_n}Z_{\sigma_n}-Ee^{-\rho\tau_n}Z_{\tau_n} \leq &\ E\int_{\tau_n}^{\sigma_n} e^{-\rho t}(-\rho Z-\rho\mu_2 t+\rho\mu_2 t+\mu_1)dt \\ \leq &\ E\int_{\tau_n}^{\sigma_n} e^{-\rho t}(-\rho Z-\rho\mu_2 t+\mu_1)dt. \end{aligned}$$

(5.6) becomes

$$\begin{aligned}
J_0(Z,p,\Lambda_0) \leq \quad & E\left\{\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} e^{-\rho t}(\rho|Z| - \rho\mu_2 t + \mu_1)dt\right\} \\
\leq \quad & E\int_0^{\infty} e^{-\rho t}(\rho|Z| - \rho\mu_2 t + \mu_1)dt \\
= \quad & |Z| + \frac{\mu_1 - \mu_2}{\rho}.
\end{aligned}$$

This implies

$$0 \leq V_0(Z,p) \leq |Z| + \frac{\mu_1 - \mu_2}{\rho}.$$

Part (2) follows from part (1) and Lemma 5.3.1. $\square$

Let $Y_t = \begin{pmatrix} Z_t \\ p_t \end{pmatrix}$ be an Itô diffusion satisfying

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\widehat{W}_t,$$

where

$$b(Y_t) = \begin{pmatrix} (\mu_1 - \mu_2)p_t + \mu_2 \\ -(\lambda_1 + \lambda_2)p_t + \lambda_2 \end{pmatrix} dt \text{ and } \sigma(Y_t) = \begin{pmatrix} \sigma \\ \dfrac{(\mu_1 - \mu_2)p_t(1-p_t) + \mu_2}{\sigma} \end{pmatrix} d\widehat{W}_t.$$

The generator of Y_t is given by

$$\mathcal{A}f(Y_t) = \sum_i b_i(Y_t)\frac{\partial f}{\partial Y_i}(Y_t) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(Y_t)\frac{\partial^2 f}{\partial Y_i \partial Y_j}(Y_t),$$

where

$$\sigma\sigma^T(Y_t) = \begin{pmatrix} \sigma^2 & (\mu_1 - \mu_2)p_t(1-p_t) \\ (\mu_1 - \mu_2)p_t(1-p_t) & \left[\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\right]^2 \end{pmatrix}.$$

That is,

$$
\begin{aligned}
\mathcal{A}f(Y_t) = \;& \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial Z^2}(Y_t) + (\mu_1-\mu_2)p_t(1-p_t)\frac{\partial^2 f}{\partial Z\partial p}(Y_t) + \frac{1}{2}\Big[\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\Big]^2\frac{\partial^2 f}{\partial p^2}(Y_t) \\
& + \Big[(\mu_1-\mu_2)p_t+\mu_2\Big]\frac{\partial f}{\partial Z}(Y_t) + \Big[-(\lambda_1+\lambda_2)p_t+\lambda_2\Big]\frac{\partial f}{\partial p}(Y_t).
\end{aligned}
\tag{5.9}
$$

For the rest of the chapter, we assume $v_i \in C^2(\mathbb{R}\times(0,1))$, $i=0,1$.

Lemma 5.3.3. *For $i=0,1$, and for any $0\le\theta_1\le\theta_2$,*

$$
E\left[\int_{\theta_1}^{\theta_2} e^{-\rho t}(\mathcal{A}-\rho)v_i(Z_t,p_t)dt\right] = Ee^{-\rho\theta_2}v_i(Z_{\theta_2},p_{\theta_2}) - Ee^{-\rho\theta_1}v_i(Z_{\theta_1},p_{\theta_1}).
$$

Proof. For $i=0,1$,

$$
d(e^{-\rho t}v_i(Z_t,p_t)) = -\rho e^{-\rho t}v_i(Z_t,p_t)dt + e^{-\rho t}dv_i(Z_t,p_t).
$$

Using Itô formula with $(dZ_t)^2=\sigma^2 dt$, $(dp_t)^2=\left[\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\right]^2 dt$, and $(dZ_t)(dp_t)=(\mu_1-\mu_2)p_t(1-p_t)$, we obtain

$$
\begin{aligned}
d(v_i(Z_t,p_t)) = \;& \frac{\partial v_i(\cdot)}{\partial Z}dZ_t + \frac{\partial v_i(\cdot)}{\partial p}dp_t + \frac{1}{2}\frac{\partial^2 v_i(\cdot)}{\partial Z^2}(dZ_t)^2 + \frac{1}{2}\frac{\partial^2 v_i(\cdot)}{\partial p^2}(dp_t)^2 + \frac{\partial^2 v_i(\cdot)}{\partial Z\partial p}(dZ_t)(dp_t) \\
= \;& \Bigg\{[(\mu_1-\mu_2)p_t+\mu_2]\frac{\partial v_i(\cdot)}{\partial Z} + [-(\lambda_1+\lambda_2)p_t+\lambda_2]\frac{\partial v_i(\cdot)}{\partial p} + \frac{\sigma^2}{2}\frac{\partial^2 v_i(\cdot)}{\partial Z^2} + \\
& \frac{1}{2}\Big[\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\Big]^2\frac{\partial^2 v_i(\cdot)}{\partial p^2} + (\mu_1-\mu_2)p_t(1-p_t)\frac{\partial^2 v_i(\cdot)}{\partial Z\partial p}\Bigg\}dt + f(Z_t,p_t),
\end{aligned}
$$

where

$$
f(Z_t,p_t) = \left[\sigma\frac{\partial v_i(Z_t,p_t)}{\partial Z} + \frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\frac{\partial v_i(Z_t,p_t)}{\partial p}\right]d\widehat{W}_t.
$$

Hence,

$$
\begin{aligned}
d(e^{-\rho t}v_i(Z_t,p_t)) = \quad & e^{-\rho t}\left\{-\rho v_i(Z_t,p_t) + [(\mu_1-\mu_2)p_t+\mu_2]\frac{\partial v_i(Z_t,p_t)}{\partial Z} + \right.\\
& [-(\lambda_1+\lambda_2)p_t+\lambda_2]\frac{\partial v_i(Z_t,p_t)}{\partial p} + \frac{\sigma^2}{2}\frac{\partial^2 v_i(Z_t,p_t)}{\partial Z^2} + \\
& \frac{1}{2}\left[\frac{(\mu_1-\mu_2)p_t(1-p_t)}{\sigma}\right]^2\frac{\partial^2 v_i(Z_t,p_t)}{\partial p^2} + \\
& \left.(\mu_1-\mu_2)p_t(1-p_t)\frac{\partial^2 v_i(Z_t,p_t)}{\partial Z \partial p}\right\} dt + e^{-\rho t}f(Z_t,p_t) \\
= \quad & e^{-\rho t}\left[-\rho v_i(Z_t,p_t) + \mathcal{A}v_i(Z_t,p_t)\right] dt + e^{-\rho t}f(Z_t,p_t).
\end{aligned}
\tag{5.10}
$$

Integrate both sides of equation (5.10) from θ_1 to θ_2, and then take expectation to obtain the required result. $\square$

5.4 The HJB Equations and A Verification Theorem

If the net position is flat ($i = 0$) then one should buy immediately ($t = 0$) when starting in the buying region. In this case, $V_0(Z,p) = V_1(Z,p) - (Z+K)$.

On the other hand, if the net position is long ($i = 1$) then one should sell immediately ($t = 0$) when starting in the selling region. In this case, $V_1(Z,p) = V_0(Z,p) + (Z-K)$.

However, one should not establish any new position in the continuation region (holding region). In other words, the values do not change throughout this region. Hence, $Ee^{-\rho\theta_1}V_i(Z_{\theta_1},p_{\theta_1}) = Ee^{-\rho\theta_2}V_i(Z_{\theta_2},p_{\theta_2})$, $i = 0,1$, for any $(Z_{\theta_1},p_{\theta_1})$ and $(Z_{\theta_2},p_{\theta_2})$, $\theta_1 \le \theta_2$, in this region. In view of Lemma 5.3.3, $e^{-\rho t}(\mathcal{A}-\rho)V_i(Z_t,p_t) = 0$ almost surely in the holding region or $(\mathcal{A}-\rho)V_i(Z_t,p_t) = 0$, $i = 0,1$, almost surely.

Therefore, the associated HJB equations have the form

$$\begin{aligned} &\min\Big\{(\rho-\mathcal{A})v_0, v_0-v_1+(Z+K)\Big\}=0,\\ &\min\Big\{(\rho-\mathcal{A})v_1, v_1-v_0-(Z-K)\Big\}=0 \end{aligned} \tag{5.11}$$

in $\mathbb{R}\times(0,1)$, and $v_i(\cdot), i=1,2$, have to satisfy the following conditions:

$$\begin{cases} v_0(Z,p)\geq v_1(Z,p)-(Z+K),\\ v_1(Z,p)\geq v_0(Z,p)+(Z-K),\\ (\rho-\mathcal{A})v_0(Z,p)\geq 0,\\ (\rho-\mathcal{A})v_1(Z,p)\geq 0, \end{cases} \tag{5.12}$$

and

$$\begin{cases} v_0(Z,p)=v_1(Z,p)-(Z+K) & \text{on Buying Region},\\ v_1(Z,p)=v_0(Z,p)+(Z-K) & \text{on Selling Region},\\ (\rho-\mathcal{A})v_0(Z,p)=0 & \text{on Holding Region},\\ (\rho-\mathcal{A})v_1(Z,p)=0 & \text{on Hoding Region}. \end{cases} \tag{5.13}$$

Lemma 5.4.1. *The solution (v_0,v_1) of (5.11) satisfies*

$$\begin{cases} (\rho-\mathcal{A})v_0=(\rho W-\mathcal{A}W)^-\\ (\rho-\mathcal{A})v_1=(\rho W-\mathcal{A}W)^+, \end{cases} \tag{5.14}$$

where $W(Z,p)=v_1(Z,p)-v_0(Z,p)$ satisfies

$$\begin{cases} (\rho-\mathcal{A})W=0 & \textit{if } Z-K<W<Z+K,\\ (\rho-\mathcal{A})W\geq 0 & \textit{if } W=Z-K,\\ (\rho-\mathcal{A})W\leq 0 & \textit{if } W=Z+K, \end{cases} \tag{5.15}$$

in $\mathbb{R}\times(0,1)$.

Proof. We first show the first equation of (5.14). We consider various cases of $v_0(Z,p) - v_1(Z,p)$. If $v_0(Z,p) - v_1(Z,p) > -(Z+K)$ then

$$(\rho - \mathcal{A})v_0 = 0 \text{ and } (\rho - \mathcal{A})W = (\rho - \mathcal{A})(v_1 - v_0) \geq 0.$$

On the other hand, if $v_0(Z,p) - v_1(Z,p) = -(Z+K)$ then $v_1(Z,p) - v_0(Z,p) = Z+K \geq Z-K$. This implies

$$(\rho - \mathcal{A})v_1 = 0 \text{ and } (\rho - \mathcal{A})W = (\rho - \mathcal{A})(v_1 - v_0) = -(\rho - \mathcal{A})v_0 \leq 0.$$

We have proved the first equation of (5.14). The second equation can be obtained similarly. We will next prove (5.15). If $Z - K < W(Z,p) < Z + K$ then (5.11) implies

$$(\rho - \mathcal{A})v_i\Big|_{(Z,p)} = 0, \text{ i=1,2.}$$

This gives the first equality of (5.15).

Similarly, if $W = Z - K$, i.e., $v_1 - v_0 = Z - K$, then $v_0 - v_1 + Z + K = 2K > 0$. Then the first equation of (5.11) implies $(\rho - \mathcal{A})v_0(Z,p) = 0$. In addition, we always have $(\rho - \mathcal{A})v_1(Z,p) \geq 0$. Therefore, we obtain the first inequality of (5.15).

Likewise, if $W = Z + K$, i.e., $v_1 - v_0 = Z + K$, then $v_1 - v_0 - Z + K = 2K > 0$. Then the second equation of (5.11) implies $(\rho - \mathcal{A})v_1(Z,p) = 0$. In addition, we always have $(\rho - \mathcal{A})v_0(Z,p) \geq 0$. Therefore, we obtain the last inequality of (5.15). $\square$

We now define different regions: Holding Region (HR), Buying Region (BR), and Selling Region (SR).

$$\begin{aligned}
HR = &\ \{(Z,p) \in \mathbb{R} \times (0,1) : Z - K < W(Z,p) < Z + K\}, \\
BR = &\ \{(Z,p) \in \mathbb{R} \times (0,1) : W(Z,p) = Z + K\}, \\
SR = &\ \{(Z,p) \in \mathbb{R} \times (0,1) : W(Z,p) = Z - K\}.
\end{aligned}$$

Lemma 5.4.2.

$$\begin{aligned}
BR \subset &\ \left\{(Z,p) \in \mathbb{R} \times (0,1) : p \geq \frac{\rho(Z+K) - \mu_2}{\mu_1 - \mu_2}\right\}, \\
SR \subset &\ \left\{(Z,p) \in \mathbb{R} \times (0,1) : p \leq \frac{\rho(Z+K) - \mu_2}{\mu_1 - \mu_2}\right\}.
\end{aligned}$$

Proof. If $(Z,p) \in BR$ then it follows from (5.15) that

$$0 \geq (\rho - \mathcal{A})(Z+K) = \rho(Z+K) - (\mu_1 - \mu_2)p - \mu_2,$$

or equivalently,

$$p \geq \frac{\rho(Z+K) - \mu_2}{\mu_1 - \mu_2}.$$

The second part of the lemma can be obtained similarly. □

We now define two free boundaries:

$$\begin{aligned}
SR^* &= \{(Z,p) \in \mathbb{R} \times (0,1) : (Z,p) \in SR \cap HR\}, \\
BR^* &= \{(Z,p) \in \mathbb{R} \times (0,1) : (Z,p) \in BR \cap HR\}.
\end{aligned}$$

Lemma 5.4.3. *For any stopping times θ_1 and θ_2, if $0 \leq \theta_1 \leq \theta_2$, a.s., and $(\rho - \mathcal{A})v_i(Z_t, p_t) \geq 0$, for all $t \in [\theta_1, \theta_2]$, for $i = 0, 1$, then*

$$Ee^{-\rho(\theta_1 - t)}v_i(Z_{\theta_1}, p_{\theta_1}, \theta_1) \geq Ee^{-\rho(\theta_2 - t)}v_i(Z_{\theta_2}, p_{\theta_2}, \theta_2).$$

In particular, if $\theta_1 = 0$, $\theta_2 = \tau \geq t$, and $Z_0 = Z$, $p_0 = p$, we have

$$v_i(Z, p) \geq Ee^{-\rho\tau}v_i(Z_\tau, p_\tau).$$

The equalities happen when $(\rho - \mathcal{A})v_i(Z_t, p_t) = 0$ for all $t \in [\theta_1, \theta_2]$.

Proof. Use Lemma 5.3.3 and the hypothesis $(\rho - \mathcal{A})v_i(Z_t, p_t) \geq 0$ for all $t \in [\theta_1, \theta_2]$ to obtain

$$Ee^{-\rho\theta_1}v_i(Z_{\theta_1}, p_{\theta_1}) \geq Ee^{-\rho\theta_2}v_i(Z_{\theta_2}, p_{\theta_2}).$$

Set $\theta_1 = 0,\ \theta_2 = \tau$ to obtain

$$v_i(Z, p) \geq Ee^{-\rho\tau}v_i(Z_\tau, p_\tau).$$

Also, if $(\rho - \mathcal{A})v_i(Z_t, p_t) = 0$ for all $t \in [\theta_1, \theta_2]$ then Lemma 5.3.3 gives the equalities. □

Lemma 5.4.4. *If the position is $i = 0$ and $\Lambda_0 = (\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots)$ then for all $N \geq 1$,*

$$Ee^{-\rho\tau_1}v_0(Z_{\tau_1}, p_{\tau_1}) \geq Ee^{-\rho\sigma_N}v_0(Z_{\sigma_N}, p_{\sigma_N}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big].$$

Similarly, if the position is $i = 1$ and $\Lambda_1 = (\sigma_1, \tau_2, \sigma_2, \tau_3, \ldots)$ then for all $N \geq 2$,

$$\begin{aligned} Ee^{-\rho\sigma_1}v_1(Z_{\sigma_1}, p_{\sigma_1}) \geq \quad & Ee^{-\rho\sigma_N}v_0(Z_{\sigma_N}, p_{\sigma_N}) + Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K) + \\ & E\sum_{n=2}^{N}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big]. \end{aligned}$$

Proof. Since $v_i, i = 1, 2$ are solutions of the HJB equations (5.11), we have for all $t \geq 0$,

$$\begin{aligned} v_0(Z_t, p_t) &\geq v_1(Z_t, p_t) - (Z_t + K), \text{ and} \\ v_1(Z_t, p_t) &\geq v_0(Z_t, p_t) + Z_t - K. \end{aligned} \tag{5.16}$$

It follows, for the position $i = 0$, that

$$\begin{aligned} Ee^{-\rho\tau_1} v_0(Z_{\tau_1}, p_{\tau_1}) \geq\ & Ee^{-\rho(\tau_1 - t)} \left(v_1(X_{\tau_1}) - Z_{\tau_1} - K\right) \\ =\ & Ee^{-\rho\tau_1} v_1(Z_{\tau_1}, p_{\tau_1}) - Ee^{-\rho\tau_1}(Z_{\tau_1} + K) \\ \geq\ & Ee^{-\rho\sigma_1} v_1(Z_{\sigma_1}, p_{\sigma_1}) - Ee^{-\rho\tau_1}(Z_{\tau_1} + K) \\ \geq\ & Ee^{-\rho\sigma_1}(v_0(Z_{\sigma_1}, p_{\sigma_1}) + Z_{\sigma_1} - K) - Ee^{-\rho\tau_1}(Z_{\tau_1} + K) \\ =\ & Ee^{-\rho\sigma_1} v_0(Z_{\sigma_1}, p_{\sigma_1}) + E\Big[e^{-\rho\sigma_1}(Z_{\sigma_1} - K) - e^{-\rho\tau_1}(Z_{\tau_1} + K)\Big] \\ \geq\ & Ee^{-\rho\tau_2} v_0(Z_{\tau_2}, p_{\tau_2}) + E\Big[e^{-\rho\sigma_1}(Z_{\sigma_1} - K) - e^{-\rho\tau_1}(Z_{\tau_1} + K)\Big]. \end{aligned}$$

In the above, the third line uses Lemma 5.4.3 for $\tau_1 \leq \sigma_1$. The fourth line uses (5.16). The last inequality uses Lemma 5.4.3 for $\sigma_1 \leq \tau_2$.

Similarly,

$$Ee^{-\rho\tau_2} v_0(Z_{\tau_2}, p_{\tau_2}) \geq Ee^{-\rho\sigma_2} v_0(Z_{\sigma_2}, p_{\sigma_2}) + E\Big[e^{-\rho\sigma_2}(Z_{\sigma_2} - K) - e^{-\rho\tau_2}(Z_{\tau_2} + K)\Big]. \tag{5.17}$$

Continue this way to obtain the first inequality of Lemma 5.4.4:

$$Ee^{-\rho\tau_1} v_0(Z_{\tau_1}, p_{\tau_1}) \geq Ee^{-\rho\sigma_N} v_0(Z_{\sigma_N}, p_{\sigma_N}) + E\sum_{n=1}^{N} \Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big] \text{ for all } N \geq 1.$$

For the second inequality, we use similar computations for the position $i = 1$.

$$\begin{aligned} Ee^{-\rho\sigma_1} v_1(Z_{\sigma_1}, p_{\sigma_1}) \geq & \; Ee^{-\rho\sigma_1}\left(v_0(Z_{\sigma_1}, p_{\sigma_1}) + Z_{\sigma_1} - K\right) \\ = & \; Ee^{-\rho\sigma_1} v_0(Z_{\sigma_1}, p_{\sigma_1}) + Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K) \\ \geq & \; Ee^{-\rho\tau_2}(v_0(Z_{\tau_2}, p_{\tau_2}) + Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K) \\ \geq & \; Ee^{-\rho\sigma_2} v_0(Z_{\sigma_2}, p_{\sigma_2}) + E\Big[e^{-\rho\sigma_2}(Z_{\sigma_2} - K) - e^{-\rho\tau_2}(Z_{\tau_2} + K)\Big] + \\ & \; Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K). \end{aligned}$$

In the above, the first line uses (5.16). The third line uses Lemma 5.4.3 for $\sigma_1 \leq \tau_2$. The last inequality uses (5.17).

Continue this way to obtain the second inequality of Lemma 5.4.4. □

Lemma 5.4.5. *Let $\Lambda_0^* = (\tau_1^*, \sigma_1^*, \tau_2^*, \sigma_2^*)$, where the stopping times $\tau_1^* = \inf\{t \geq 0 : (Z_t, p_t) \in BR\}$, and for $n \geq 1$,*

$$\begin{aligned} \sigma_n^* = & \; \inf\{t \geq \tau_n^* : (Z_t, p_t) \in SR\} \\ \tau_{n+1}^* = & \; \inf\{t \geq \sigma_n^* : (Z_t, p_t) \in BR\}. \end{aligned}$$

Let $\Lambda_1^* = (\sigma_1^*, \tau_2^*, \sigma_2^*, \tau_3^*, ...)$, where the stopping times $\sigma_1^* = \inf\{t \geq 0 : (Z_t, p_t) \in SR\}$, and for $n \geq 2$,

$$\begin{aligned} \tau_n^* = & \; \inf\{t \geq \sigma_{n-1}^* : p_t(Z_t, p_t) \in BR\} \\ \sigma_n^* = & \; \inf\{t \geq \tau_n^* : (Z_t, p_t) \in SR\}. \end{aligned}$$

Then for all $N \geq 1$,

$$Ee^{-\rho\tau_1^*} v_0(Z_{\tau_1^*}, p_{\tau_1^*}) = Ee^{-\rho\sigma_N^*} v_0(Z_{\sigma_N^*}, p_{\sigma_N^*}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n^*}(Z_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(Z_{\tau_n^*} + K)\Big],$$

and for all $N \geq 2$,

$$\begin{aligned} Ee^{-\rho\sigma_1^*} v_1(Z_{\sigma_1^*}, p_{\sigma_1^*}) = & \; Ee^{-\rho\sigma_N^*} v_0(X_{\sigma_N^*}) + Ee^{-\rho\sigma_1^*}(Z_{\sigma_1^*} - K) + \\ & \; E\textstyle\sum_{n=2}^{N}\Big[e^{-\rho\sigma_n^*}(Z_{\sigma_n^*} - K) - e^{-\rho\tau_n^*}(Z_{\tau_n^*} + K)\Big]. \end{aligned}$$

Proof. Since $v_i, i = 1, 2$, are solutions of the HJB equations (5.11), they have to satisfy (5.13), i.e.,

$$\begin{cases} v_0(Z,p) = v_1(Z,p) - (Z+K) & \text{on BR,} \\ v_1(Z,p) = v_0(Z,p) + (Z-K) & \text{on SR,} \\ (\rho - \mathcal{A})v_i(Z,p) = 0, \ i = 0,1 & \text{on HR.} \end{cases}$$

First, consider the position $i = 0$. Note that $(Z_{\tau_1^*}, p_{\tau_1^*}) \in BR$. Hence,

$$\begin{aligned} Ee^{-\rho\tau_1^*} v_0(Z_{\tau_1^*}, p_{\tau_1^*}) = &\ \ Ee^{-\rho\tau_1^*}\big(v_1(Z_{\tau_1^*}, p_{\tau_1^*}) - Z_{\tau_1^*} - K\big) \\ = &\ \ Ee^{-\rho\tau_1^*} v_1(Z_{\tau_1^*}, p_{\tau_1^*}) - Ee^{-\rho\tau_1^*}(Z_{\tau_1^*} + K). \end{aligned}$$

Note that for all $t \in [\tau_1^*, \sigma_1^*]$, $(Z_t, p_t) \in HR$. Hence, $(\rho - \mathcal{A})v_1(Z_t, p_t) = 0$. Moreover, since $(Z_{\sigma_1^*}, p_{\sigma_1^*}) \in SR$, $v_1(X_{\sigma_1^*}) = v_0(X_{\sigma_1^*}) + Z_{\sigma_1^*} - K$. Lemma 5.4.3 implies

$$\begin{aligned} Ee^{-\rho\tau_1^*} v_1(Z_{\tau_1^*}, p_{\tau_1^*}) = &\ \ Ee^{-\rho\sigma_1^*} v_1(Z_{\sigma_1^*}, p_{\sigma_1^*}) \\ = &\ \ Ee^{-\rho\sigma_1^*}\big(v_0(Z_{\sigma_1^*}, p_{\sigma_1^*}) + Z_{\sigma_1^*} - K\big) \\ = &\ \ Ee^{-\rho\sigma_1^*} v_0(Z_{\sigma_1^*}, p_{\sigma_1^*}) + Ee^{-\rho\sigma_1^*}(Z_{\sigma_1^*} - K). \end{aligned}$$

Therefore,

$$Ee^{-\rho\tau_1^*} v_1(Z_{\tau_1^*}, p_{\tau_1^*}) = Ee^{-\rho\sigma_1^*} v_0(Z_{\sigma_1^*}, p_{\sigma_1^*}) + E\Big[e^{-\rho\sigma_1^*}(Z_{\sigma_1^*} - K) - e^{-\rho\tau_1^*}(Z_{\tau_1^*} + K)\Big].$$

Note that for all $t \in [\sigma_1^*, \tau_2^*]$, $(Z_t, p_t) \in HR$. Hence, $(\rho - \mathcal{A})v_0(Z_t, p_t) = 0$. Lemma 5.4.3 implies $Ee^{-\rho\sigma_1^*} v_0(Z_{\sigma_1^*}, p_{\sigma_1^*}) = Ee^{-\rho\tau_2^*} v_0(Z_{\tau_2^*}, p_{\tau_2^*})$.

Similarly,

$$Ee^{-\rho\tau_2^*} v_0(Z_{\tau_2^*}, p_{\tau_2^*}) = Ee^{-\rho\sigma_2^*} v_0(Z_{\sigma_2^*}, p_{\sigma_2^*}) + E\Big[e^{-\rho\sigma_2^*}(Z_{\sigma_2^*} - K) - e^{-\rho\tau_2^*}(Z_{\tau_2^*} + K)\Big]. \quad (5.18)$$

Continue the procedure to obtain

$$Ee^{-\rho\tau_1^*}v_1(Z_{\tau_1^*},p_{\tau_1^*}) = Ee^{-\rho\sigma_N^*}v_0(Z_{\sigma_N^*},p_{\sigma_N^*})+E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n^*}(Z_{\sigma_n^*}-K)-e^{-\rho\tau_n^*}(Z_{\tau_n^*}+K)\Big] \text{ for all } N \geq 1.$$

For the second equality of the lemma, we use similar computations for the position $i = 1$ as follows.

Note that $(Z_{\sigma_1^*}, p_{\sigma_1^*}) \in SR$. Hence,

$$\begin{aligned} Ee^{-\rho\sigma_1^*}v_1(Z_{\sigma_1^*},p_{\sigma_1^*}) = {} & Ee^{-\rho\sigma_1^*}\big(v_0(Z_{\sigma_1^*},p_{\sigma_1^*}) + Z_{\sigma_1^*} - K\big) \\ = {} & Ee^{-\rho\sigma_1^*}v_0(Z_{\sigma_1^*},p_{\sigma_1^*}) + Ee^{-\rho\sigma_1^*}(Z_{\sigma_1^*} - K). \end{aligned}$$

Note also that for all $t \in [\sigma_1^*, \tau_2^*]$, $(Z_t, p_t) \in HR$. This implies $(\rho - \mathcal{A})v_0(Z_t, p_t) = 0$. Lemma 5.4.3 implies

$$Ee^{-\rho\sigma_1^*}v_0(Z_{\sigma_1^*},p_{\sigma_1^*}) = Ee^{-\rho\tau_2^*}v_0(Z_{\tau_2^*},p_{\tau_2^*}).$$

Use (5.18) to obtain

$$Ee^{-\rho\sigma_1^*}v_0(Z_{\sigma_1^*},p_{\sigma_1^*}) = Ee^{-\rho\sigma_2^*}v_0(Z_{\sigma_2^*},p_{\sigma_2^*}) + E\Big[e^{-\rho\sigma_2^*}(Z_{\sigma_2^*} - K) - e^{-\rho\tau_2^*}(Z_{\tau_2^*} + K)\Big].$$

Similarly,

$$Ee^{-\rho\sigma_2^*}v_0(Z_{\sigma_2^*},p_{\sigma_2^*}) = Ee^{-\rho\sigma_3^*}v_0(Z_{\sigma_3^*},p_{\sigma_3^*}) + E\Big[e^{-\rho\sigma_3^*}(Z_{\sigma_3^*} - K) - e^{-\rho\tau_3^*}(Z_{\tau_3^*} + K)\Big].$$

Continue the procedure to obtain the second equality of the lemma. $\square$

Theorem 5.4.6 (Verification Theorem)**.** *Let (v_0, v_1) be solutions to the HJB equations (5.11), and BR^* and SR^* be the associated boundaries. Then $v_0(Z,p)$ and $v_1(Z,p)$ are equal to the value functions $V_0(Z,p)$ and $V_1(Z,p)$, respectively.*
Moreover, let $(\Lambda_0^*, \tau_n^*, \sigma_n^*)$ and $(\Lambda_1^*, \sigma_n^*, \tau_n^*)$ be defined as in Lemma 5.4.5.
If

$$\lim_{N\to\infty} Ee^{-\rho\sigma_N^*} v_0(Z_{\sigma_N^*}, p_{\sigma_N^*}) = 0.$$

Then Λ_0^* and Λ_1^* are optimal.

Proof. The proof is divided into two steps. In the first step, we show that $v_i(Z,p) \geq J_i(Z,p,\Lambda_i)$ for all Λ_i. Then in the second step, we show that $v_i(Z,p) = J_i(Z,p,\Lambda_i^*)$. Therefore, $v_i(Z,p) = V_i(Z,p)$ and Λ_i^* is optimal.

For the first step, in view of Lemma 5.4.4, we have for all $N \geq 1$,

$$Ee^{-\rho\tau_1} v_0(Z_{\tau_1}, p_{\tau_1}) \geq Ee^{-\rho\sigma_N} v_0(Z_{\sigma_N}, p_{\sigma_N}) + E\sum_{n=1}^{N}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho(\tau_n - t)}(Z_{\tau_n} + K)\Big],$$

and for all $N \geq 2$,

$$\begin{aligned} Ee^{-\rho\sigma_1} v_1(Z_{\sigma_1}, p_{\sigma_1}) \geq \ & Ee^{-\rho\sigma_N} v_0(Z_{\sigma_N}, p_{\sigma_N}) + Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K) + \\ & E\sum_{n=2}^{N}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big]. \end{aligned}$$

Sending $N \to \infty$, we obtain

$$Ee^{-\rho\tau_1} v_0(Z_{\tau_1}, p_{\tau_1}) \geq E\sum_{n=1}^{\infty}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big] = J_0(Z, \Lambda_0),$$

and

$$Ee^{-\rho\sigma_1} v_1(Z_{\sigma_1}, p_{\sigma_1}) \geq Ee^{-\rho\sigma_1}(Z_{\sigma_1} - K) + E\sum_{n=2}^{\infty}\Big[e^{-\rho\sigma_n}(Z_{\sigma_n} - K) - e^{-\rho\tau_n}(Z_{\tau_n} + K)\Big] = J_1(Z, \Lambda_1).$$

Moreover, since $v_i, i = 0, 1$, satisfy the quasi-variational inequalities in (5.11), we have $(\rho - \mathcal{A})v_i(Z_t, p_t) \geq 0$ for all $t \geq 0$. Then use Lemma 5.4.3 to obtain

$$v_0(Z,p) \geq Ee^{-\rho\tau_1}v_0(Z_{\tau_1}, p_{\tau_1}) \geq J_0(Z, \Lambda_0) \text{ and } v_1(Z,p) \geq Ee^{-\rho\sigma_1}v_1(Z_{\sigma_1}, p_{\sigma_1}) \geq J_1(Z, \Lambda_1).$$

Taking supremum over all Λ_0 and Λ_1, we obtain $v_0(Z,p) \geq V_0(Z,p)$ and $v_1(Z,p) \geq V_1(Z,p)$ respectively.

For the second step, we establish the equalities as follows.

Note that if the position is $i = 0$ then for all $t \in [0, \tau_1^*]$, $(Z_t, p_t) \in HR$, which implies $(\rho - \mathcal{A})v_0(Z_t, p_t) = 0$.

Similarly, if the position is $i = 1$ then for all $t \in [0, \sigma_1^*]$, $(Z_t, p_t) \in HR$, which implies $(\rho - \mathcal{A})v_1(Z_t, p_t) = 0$. Using Lemma 5.4.3, we get

$$v_0(Z,p) = Ee^{-\rho\tau_1^*}v_0(Z_{\tau_1^*}, p_{\tau_1^*}) \text{ and } v_1(Z,p) = Ee^{-\rho\sigma_1^*}v_0(Z_{\sigma_1^*}, p_{\sigma_1^*}).$$

Sending $N \to \infty$ in Lemma 5.4.5 to obtain $v_i(Z,p) = J_i(Z, p, \Lambda_i^*)$. Therefore, $v_i(Z,p) = V_i(Z,p)$, and Λ_i^* is optimal. $\square$

5.5 Viscosity Solutions of the HJB Equations

Theorem 5.5.1. *Let $g(x,p) \geq 0$ be a bounded, continuous reward function defined on $\mathbb{R} \times (0,1)$ and $\widehat{g}$ be the least superharmonic majorant of g. Then*

$$v(x,p) = \widehat{g}(x,p),$$

where $v(x,p)$ is the associated value function.

The following lemma will be used in the proof of Theorem 5.5.1.

Lemma 5.5.2. *For $\varepsilon > 0$, let*

$$D_\varepsilon = \{(z,p) : g(z,p) < \widehat{g}(z,p) - \varepsilon\},$$
$$\tau_\varepsilon = \inf\{t > 0 : (Z_t, p_t) \notin D_\varepsilon\}.$$

Define $\widetilde{g}_\varepsilon(z,p) = E^{z,p}\widehat{g}(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon})$. Then

(a) $\widetilde{g}_\varepsilon$ is supermeanvalued.

(b) $\widetilde{g}_\varepsilon(z,p) + \varepsilon \geq g(z,p)$ for all (z,p).

Proof. For any stopping time θ,

$$\begin{aligned} E^{z,p}\widetilde{g}_\varepsilon(Z_\theta, p_\theta) = &\quad E^{z,p}E^{Z_\theta,p_\theta}\widehat{g}(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon}) \\ = &\quad E^{z,p}E^{z,p}[\widehat{g}(Z_{\tau_\varepsilon+\theta}, p_{\tau_\varepsilon+\theta})|\mathcal{F}_\theta] \\ = &\quad E^{z,p}\widehat{g}(Z_{\tau_\varepsilon+\theta}, p_{\tau_\varepsilon+\theta}) \\ \leq &\quad E^{z,p}\widehat{g}(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon}) \\ = &\quad \widetilde{g}_\varepsilon(z,p). \end{aligned}$$

In the above, the second line uses the Strong Markov property. The fourth line uses Lemma 1.5.19e. We have just proved (a).

We will prove (b) by contradiction.

Suppose for contradiction that $\delta = \sup\limits_{(z,p)\in\mathbb{R}\times(0,1)} [g(z,p) - \widetilde{g}_\varepsilon(z,p)] > \varepsilon$.

Then for any $0 < \gamma < \varepsilon$, there exists (z_0, p_0) such that

$$g(z_0,p_0) - \widetilde{g}_\varepsilon(z_0,p_0) \geq \delta - \gamma > 0. \tag{5.19}$$

Note that $\widetilde{g}_\varepsilon + \delta \geq g$, so $\widetilde{g}_\varepsilon + \delta$ is a supermeanvalued majorant of g. It follows that

$$\widetilde{g}_\varepsilon(z_0, p_0) + \delta \geq \widehat{g}(z_0, p_0).$$

Add the previous two inequalities to obtain

$$g(z_0, p_0) + \gamma \geq \widehat{g}(z_0, p_0). \tag{5.20}$$

If $\tau_\varepsilon = 0$ a.s. then by the definition of $\widetilde{g}_\varepsilon$,

$$\widetilde{g}_\varepsilon(z_0, p_0) = E^{z_0, p_0} \widehat{g}(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon}) = \widehat{g}(z_0, p_0) \geq g(z_0, p_0).$$

The last inequality contradicts (5.19).

If $\tau_\varepsilon > 0$ a.s. then for any $0 \leq t < \tau_\varepsilon$, $(Z_t, p_t) \in D_\varepsilon$. We have

$$\begin{aligned}
\widehat{g}(z_0, p_0) \geq & \; E^{z_0, p_0} \widehat{g}(Z_t, p_t) \\
\geq & \; E^{z_0, p_0}[g(Z_t, p_t) + \varepsilon] \\
\geq & \; \liminf_{t \to 0} E^{z_0, p_0}[g(Z_t, p_t) + \varepsilon] \\
\geq & \; E^{z_0, p_0}[\liminf_{t \to 0} g(Z_t, p_t) + \varepsilon] \\
= & \; g(z_0, p_0) + \varepsilon.
\end{aligned}$$

In the above, the first inequality follows from the superharmonicity of $\widehat{g}$. The second inequality uses $(Z_t, p_t) \in D_\varepsilon$. The last inequality uses Fatou's lemma.

The last inequality and (5.20) give $\gamma \geq \varepsilon$, which is a contradiction. We have proved (b). $\square$

Proof of Theorem 5.5.1. Lemma (5.5.2) implies $\widetilde{g}_\varepsilon + \varepsilon$ is a supermeanvalued majorant of g. Hence,

$$\begin{aligned}
\widehat{g}(z,p) \leq & \quad \widetilde{g}_\varepsilon(z,p) + \varepsilon \\
= & \quad E^{z,p}[\widehat{g}(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon})] + \varepsilon \\
\leq & \quad E^{z,p}[g(Z_{\tau_\varepsilon}, p_{\tau_\varepsilon}) + \varepsilon] + \varepsilon \\
\leq & \quad v(z,p) + 2\varepsilon.
\end{aligned}$$

In the above, the third line uses the definition of D_ε.

Since ε is arbitrary, $\widehat{g}(z,p) \leq v(z,p)$.

On the other hand, for any stopping time θ,

$$\begin{aligned}
\widehat{g}(z,p) \geq & \quad E^{z,p}[\widehat{g}(Z_\theta, p_\theta)] \\
\geq & \quad E^{z,p}[g(Z_\theta, p_\theta)].
\end{aligned}$$

Taking supremum over all stopping times θ to obtain

$$\widehat{g}(z,p) \geq \sup_\theta E^{z,p}[g(Z_\theta, p_\theta)] = v(z,p).$$

Therefore, $v(z,p) = \widehat{g}(z,p)$. □

Theorem 5.5.3. *Theorem 5.5.1 still holds when g is unbounded.*

Proof. Define $g_N = g \wedge N,\ N = 1, 2, ...$

Let $\widehat{g}_N$ be the least superharmonic majorant of g_N.

Let v_N be the associated value function with g_N.

Note that each g_N is bounded, so Theorem 5.5.1 gives $v \geq v_N = \widehat{g}_N$.

Note also that $\widehat{g}_{N+1} \geq g_{N+1} \geq g_N$, i.e., $\widehat{g}_{N+1}$ is a superharmonic majorant of g_N. So $\widehat{g}_{N+1} \geq \widehat{g}_N$, i.e., the sequence $\widehat{g}_N$ is increasing, and let $h = \lim_{N\to\infty} \widehat{g}_N > \lim_{N\to\infty} g_N = g$. By

Lemma 1.5.19d, h is a superharmonic majorant of g. Hence,

$$h \geq \widehat{g}.$$

Furthermore, $v \geq \lim_{N\to\infty} \widehat{g}_N = h \geq \widehat{g}$.

Using the same argument as in the proof of Theorem 5.5.1, one also has $v \leq \widehat{g}$.

Therefore, $v = \widehat{g}$. □

Definition 5.5.4. *$(v_0(z,p), v_1(z,p))$ are viscosity solutions of the HJB equations (5.11) if the following holds:*

(a) $v_i(z,p)$, $i = 0, 1$, is continuous in $\mathbb{R} \times (0,1)$, and $|v_i(z,p)| \leq C(1 + |z| + |p|)$.

(b)

$$\begin{aligned} &\min\Big\{\rho v_0(z_0,p_0) - \mathcal{A}\varphi_0(z_0,p_0), v_0(z_0,p_0) - v_1(z_0,p_0) + (z_0 + K)\Big\} \geq 0, \\ &\min\Big\{\rho v_1(z_0,p_0) - \mathcal{A}\varphi_1(z_0,p_0), v_1(z_0,p_0) - v_0(z_0,p_0) - (z_0 - K)\Big\} \geq 0, \end{aligned} \tag{5.21}$$

whenever $\varphi_i \in C^2(\mathbb{R} \times (0,1))$, $i = 0, 1$, is such that $v_i(z,p) - \varphi_i(z,p)$ has a local minimum at (z_0, p_0).

(c)

$$\begin{aligned} &\min\Big\{\rho v_0(z_0,p_0) - \mathcal{A}\varphi_0(z_0,p_0), v_0(z_0,p_0) - v_1(z_0,p_0) + (z_0 + K)\Big\} \leq 0, \\ &\min\Big\{\rho v_1(z_0,p_0) - \mathcal{A}\varphi_1(z_0,p_0), v_1(z_0,p_0) - v_0(z_0,p_0) - (z_0 - K)\Big\} \leq 0, \end{aligned} \tag{5.22}$$

whenever $\varphi_i \in C^2(\mathbb{R} \times (0,1))$, $i = 0, 1$, is such that $v_i(z,p) - \varphi_i(z,p)$ has a local maximum at (z_0, p_0).

If (a) and (b) hold, we say (v_0, v_1) are viscosity supersolutions. If (a) and (c) hold, we say (v_0, v_1) are viscosity subsolutions.

Theorem 5.5.5. *The value functions (V_0, V_1) are the viscosity solutions of the HJB equations (5.11).*

Proof. Note that $V_i,\ i = 0, 1$, is continuous by definition, so (a) follows from Theorem 5.3.2.

We will first show (V_0, V_1) are viscosity supersolutions of (5.11).

Note that $V_0(z,p) - V_1(z,p) + (z+K) \geq 0$ and $V_1(z,p) - V_0(z,p) - (z-K) \geq 0$ always hold. We only need to show $\rho V_i(z_0,p_0) - \mathcal{A}\varphi_i(z_0,p_0) \geq 0$, where $\varphi_i \in C^2(\mathbb{R} \times (0,1)),\ i = 0,1$, is such that $V_i(z,p) - \varphi_i(z,p)$ has a local minimum at (z_0,p_0) in a ball $B_r(z_0,p_0)$.

Let θ be the first exit time of (Z_t, p_t) from $B_r(z_0,p_0)$.

Define $\Phi_i(z,p) = \varphi_i(z,p) + V_i(z_0,p_0) - \varphi_i(z_0,p_0)$.

Apply Dynkin's formula to $f(Z_t,p_t) = e^{-\rho t}\Phi_i(Z_t,p_t)$, and for $\tau \leq \theta$,

$$E^{z_0,p_0} e^{-\rho\tau}\Phi_i(Z_\tau,p_\tau) = \Phi_i(z_0,p_0) + E^{z_0,p_0}\int_0^\tau e^{-\rho t}\Big[-\rho\Phi_i(Z_t,p_t) + \mathcal{A}\Phi_i(Z_t,p_t)\Big]dt.$$

Replace the definition of Φ_i into the previous equation and note that $\Phi_i(z_0,p_0) = V_i(z_0,p_o)$ and $\mathcal{A}\Phi_i(z,p) = \mathcal{A}\varphi_i(z,p)$ to obtain

$$\begin{aligned} E^{z_0,p_0}e^{-\rho\tau} & \Big[\varphi_i(Z_\tau,p_\tau) + V_i(z_0,p_0) - \varphi_i(z_0,p_0)\Big] - V_i(z_0,p_0) \\ = & \ E^{z_0,p_0}\int_0^\tau e^{-\rho t}\Big[-\rho\Big(\varphi_i(Z_t,p_t) + V_i(z_0,p_0) - \varphi_i(z_0,p_0)\Big) + \mathcal{A}\varphi_i(Z_t,p_t)\Big]dt. \end{aligned}$$

For $0 \leq t \leq \tau$,

$$V_i(Z_t,p_t) - \varphi_i(Z_t,p_t) \geq V_i(z_0,p_0) - \varphi(z_0,p_0).$$

Equivalently,

$$V_i(Z_t,p_t) \geq \varphi_i(Z_t,p_t) + V_i(z_0,p_0) - \varphi(z_0,p_0).$$

Replace this inequality into the previous equality to obtain

$$E^{z_0,p_0}e^{-\rho\tau}V_i(Z_\tau,p_\tau) - V_i(z_0,p_0) \geq E^{z_0,p_0}\int_0^\tau e^{-\rho t}\Big[-\rho V_i(Z_t,p_t) + \mathcal{A}\varphi_i(Z_t,p_t)\Big]dt.$$

By Lemma 5.5.3, V_i is superharmonic. Then

$$V_i(z_0,p_0) \geq E^{z_0,p_0}V_i(Z_\tau,p_\tau) \geq E^{z_0,p_0}e^{-\rho\theta}V_i(Z_\tau,p_\tau).$$

Hence,

$$E^{z_0,p_0}\int_0^\tau e^{-\rho t}\Big[-\rho V_i(Z_t,p_t) + \mathcal{A}\varphi_i(Z_t,p_t)\Big]dt \leq 0.$$

Multiply both sides by $\dfrac{1}{\tau}$, and let $\tau \to 0$, we obtain $-\rho V_i(z_0,p_0) + \mathcal{A}\varphi_i(z_0,p_0) \leq 0$.

We have shown (V_0,V_1) is a viscosity supersolution.

The proof of (V_0,V_1) being a viscosity subsolutions is quite similar. $\square$

Theorem 5.5.6. *Suppose* $U_i(x,y)$, $W_i(x,y)$, $i=0,1$, *are continuous on* $\mathbb{R}\times(0,1)$*, and* (U_0,U_1) *and* (W_0,W_1) *are respectively viscosity subsolutions and viscosity supersolutions of the HJB equations (5.11). If* U_i, W_i *satisfy*

$$\begin{cases} |U_1(x,y) - U_0(x,y) - x| \leq K, \\ |W_1(x,y) - W_0(x,y) - x| \leq K. \end{cases} \tag{5.23}$$

Then

$$U_i(x,y) \leq W_i(x,y)$$

for all $(x,y)\in\mathbb{R}\times(0,1)$.

The following lemma will be used in the proof of Theorem 5.5.6.

Lemma 5.5.7. *For $p \in (0,1)$, define on $\mathbb{R} \times \mathbb{R}$ the following functions:*

$$\begin{aligned}
\varphi(x,y) = &\ \ n(x-y)^2 + \tfrac{1}{m}(x^2+y^2),\ n,m = 1,2,...\\
G^p(x,y) = &\ \ U_i(x,p) - W_i(y,p),\ i = 0 \ \textit{or}\ 1,\\
G^p_\varphi(x,y) = &\ \ G^p(x,y) - \varphi(x,y).
\end{aligned}$$

Then $G^p_\varphi(x,y)$ has a global maximum at some point (x_n,y_n) and $|x_n| + |y_n| \le C_m$ for some constant C_m does not depend on n.
Consequently, there is a convergent sequence (x_n,y_n) of global maximum of $G^p_\varphi(x,y)$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^* \ \textit{and}\ \lim_{n\to\infty} n(x_n-y_n)^2 = 0.$$

Note that $G^p_\varphi(x,y)$ depends on n and m, but it is written without n and m for notational simplicity.

Proof. Note that $G^p(x,y)$ has linear bound by the definition of viscosity solutions, so

$$\lim_{|x|+|y|\to\infty} G^p_\varphi(x,y) = -\infty.$$

Moreover, since $G^p_\varphi(x,y)$ is continuous, it attains global maximum at some point (x_n,y_n). Consequently,

$$\begin{cases}
G^p_\varphi(0,0) \le G^p_\varphi(x_n,y_n),\\
G^p_\varphi(x_n,x_n) \le G^p_\varphi(x_n,y_n),\\
G^p_\varphi(y_n,y_n) \le G^p_\varphi(x_n,y_n).
\end{cases} \tag{5.24}$$

Equivalently,

$$\begin{cases}
\dfrac{1}{m}(x_n^2+y_n^2) \le G^p(x_n,y_n) - G^p(0,0) - n(x_n-y_n)^2,\\
n(x_n-y_n)^2 \le G^p(x_n,y_n) - G^p(x_n,x_n) + \frac{2}{m}x_n^2 - \frac{1}{m}(x_n^2+y_n^2),\\
n(x_n-y_n)^2 \le G^p(x_n,y_n) - G^p(y_n,y_n) + \frac{2}{m}y_n^2 - \frac{1}{m}(x_n^2+y_n^2).
\end{cases}$$

This implies

$$
\begin{cases} \dfrac{1}{m}(x_n^2 + y_n^2) \le G^p(x_n, y_n) - G^p(0,0) - n(x_n - y_n)^2, \\ 2n(x_n - y_n)^2 \le 2G^p(x_n, y_n) - G^p(x_n, x_n) - G^p(y_n, y_n). \end{cases} \tag{5.25}
$$

Then

$$
\frac{1}{m}(x_n^2 + y_n^2) \le 3G^p(x_n, y_n) - G^p(0,0) - G^p(x_n, x_n) - G^p(y_n, y_n).
$$

The linear bound of $G^p(x, y)$ gives

$$
\frac{1}{m}(x_n^2 + y_n^2) \le C(1 + |x_n| + |y_n|)
$$

for some constant C. Equivalently,

$$
\frac{x_n^2 + y_n^2}{1 + |x_n| + |y_n|} \le Cm.
$$

It follows that $|x_n| + |y_n| \le C_m$ for some constant C_m. We can extract a convergent subsequence: first extract (x_{n_k}, y_{n_k}) such that x_{n_k} converges, then extract from (x_{n_k}, y_{n_k}) a convergent subsequence. Will still call this sequence (x_n, y_n) for notational simplicity.

Next rewrite the second inequality of (5.25) as

$$
(x_n - y_n)^2 \le \frac{1}{2n}\Big(2G^p(x_n, y_n) - G^p(x_n, x_n) - G^p(y_n, y_n)\Big).
$$

Sending $n \to \infty$, and the linear bound of $G^p(x, y)$ give

$$
\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*.
$$

Finally, let $n \to \infty$ in the second inequality of (5.25) and use the previous result to obtain

$$\lim_{n\to\infty} n(x_n - y_n)^2 = 0.$$

$\square$

Proof of Theorem 5.5.6. Let (x_n, y_n) be a global maximum of $G^p_\varphi(x, y)$ as in Lemma 5.5.7. By Theorem 1.5.23, for each $\delta \geq 0$, there exists $X_n, Y_n \in \mathbb{R}$ such that

$$\left(\frac{\partial\varphi}{\partial x}(x_n, y_n), X_n\right) \in \bar{J}^{2,+}U_i(x_n, p), \tag{5.26}$$

and

$$\left(\frac{\partial\varphi}{\partial y}(x_n, y_n), -Y_n\right) \in \bar{J}^{2,+}\Big(-W_i(y_n, p)\Big) = -\bar{J}^{2,-}W_i(y_n, p),$$

or equivalently,

$$\left(-\frac{\partial\varphi}{\partial y}(x_n, y_n), Y_n\right) \in \bar{J}^{2,-}W_i(y_n, p), \tag{5.27}$$

where

$$\begin{aligned}\frac{\partial\varphi}{\partial x}(x_n, y_n) &= 2n(x_n - y_n) + \frac{2}{m}x_n,\\ \frac{\partial\varphi}{\partial y}(x_n, y_n) &= -2n(x_n - y_n) + \frac{2}{m}y_n.\end{aligned}$$

By Theorem 1.5.22, (5.26), and the definition of viscosity subsolution, we have

$$\min\left\{\begin{array}{l}\rho U_0(x_n, p) - \dfrac{\sigma^2}{2}X_n - [(\mu_1 - \mu_2)p + \mu_2][2n(x_n - y_n) + \dfrac{2}{m}x_n],\\ U_0(x_n, p) - U_1(x_n, p) + (x_n + K)\end{array}\right\} \leq 0,$$

and

$$\min\left\{\begin{array}{l}\rho U_1(x_n, p) - \dfrac{\sigma^2}{2}X_n - [(\mu_1 - \mu_2)p + \mu_2][2n(x_n - y_n) + \dfrac{2}{m}x_n],\\ U_1(x_n, p) - U_0(x_n, p) - (x_n - K)\end{array}\right\} \leq 0.$$

Conditions (5.23) imply

$$\rho U_i(x_n,p) - \frac{\sigma^2}{2}X_n - [(\mu_1-\mu_2)p+\mu_2][2n(x_n-y_n)+\frac{2}{m}x_n] \le 0. \tag{5.28}$$

Theorem 1.5.22, (5.27), and the definition of viscosity supersolution imply

$$\min\left\{\begin{array}{l} \rho W_0(y_n,p) - \frac{\sigma^2}{2}Y_n - [(\mu_1-\mu_2)p+\mu_2][2n(x_n-y_n)-\frac{2}{m}y_n], \\ W_0(y_n,p)-W_1(y_n,p)+(y_n+K) \end{array}\right\} \ge 0,$$

and

$$\min\left\{\begin{array}{l} \rho W_1(y_n,p) - \frac{\sigma^2}{2}Y_n - [(\mu_1-\mu_2)p+\mu_2][2n(x_n-y_n)-\frac{2}{m}y_n], \\ W_1(y_n,p)-W_0(y_n,p)-(y_n-K) \end{array}\right\} \ge 0.$$

Similarly,

$$\rho W_i(x_n,p) - \frac{\sigma^2}{2}Y_n - [(\mu_1-\mu_2)p+\mu_2][2n(x_n-y_n)-\frac{2}{m}y_n] \ge 0.$$

Equivalently,

$$-\rho W_i(x_n,p) + \frac{\sigma^2}{2}Y_n + [(\mu_1-\mu_2)p+\mu_2][2n(x_n-y_n)-\frac{2}{m}y_n] \le 0.$$

Add the previous inequality to (5.28) to obtain

$$\rho[U_i(x_n,p)-W_i(y_n,p)] - \frac{\sigma^2}{2}(X_n-Y_n) - [(\mu_1-\mu_2)p+\mu_2]\frac{2}{m}(x_n+y_n) \le 0.$$

Let $m\to\infty$ to get

$$\rho[U_i(x_n,p)-W_i(y_n,p)] \le \frac{\sigma^2}{2}(X_n-Y_n). \tag{5.29}$$

Moreover, the last part of Theorem 1.5.23 gives

$$-\left(\frac{1}{\delta}+\|A\|\right) I \leq \begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq A+\delta A^2,$$

where

$$A = \partial^2\varphi(x_n, y_n) = \begin{pmatrix} 2n+2/m & -2n \\ -2n & 2n+2/m \end{pmatrix},$$

and

$$A^2 = \begin{pmatrix} 2n+2/m & -2n \\ -2n & 2n+2/m \end{pmatrix}.$$

Let $m \to \infty$ and take $\delta = \dfrac{1}{8n}$, we get

$$\begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq \begin{pmatrix} 2n & -2n \\ -2n & 2n \end{pmatrix} + \begin{pmatrix} n & -n \\ -n & n \end{pmatrix} = \begin{pmatrix} 3n & -3n \\ -3n & 3n \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3n & -3n \\ -3n & 3n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This implies

$$X_n - Y_n \leq 0.$$

Send $n \to \infty$ in (5.29) to obtain

$$U_i(x^*, p) - W_i(x^*, p) \leq 0.$$

Now for any $x \in \mathbb{R}$,

$$G_\varphi^p(x, x) \leq G_\varphi^p(x_n, y_n).$$

Equivalently,

$$U_i(x, p) - W_i(x, p) - \frac{2}{m}x^2 \leq U_i(x_n, p) - W_i(y_n, p) - n(x_n - y_n)^2 - \frac{1}{m}(x_n^2 + y_n^2).$$

Send $m, n \to \infty$ and use Lemma 5.5.7 to obtain

$$U_i(x, p) - W_i(x, p) \leq U_i(x^*, p) - W_i(x^*, p) \leq 0.$$

$\square$

Remark 5.5.8. *The value functions (V_0, V_1) are both viscosity subsolutions and supersolutions of the HJB equations (5.11), and also satisfy (5.23). Therefore, they are the unique viscosity solutions of the HJB equations (5.11).*

5.6 Future Research Directions

As can be seen, the HJB equations (5.11) involve a second-order, nonlinear parabolic partial differential equation. At this point, we cannot show there are classical solutions to these equations. However, we can use the technique of change variables to convert this PDE into a non-degenerate parabolic PDE (see 6.2). In the next research project, we will study classical solutions of the PDE with new variables.

In addition, often noticed is the very restrictive assumption of the Black-Scholes model: the continuously compounded stock returns are normally distributed with constant volatility. This assumption does not generally hold in equities markets. Volatility is not constant in time, but tends to be inversely related to price, with high stock prices usually showing lower volatility than low stock prices. The direction of my research in the next few years will be to overcome this limitation, by allowing time-varying volatility.

One popular approach for incorporating time-varying volatility is to allow volatility to be governed by its own stochastic process. Such a model is known as stochastic volatility model, including the Heston model (1993). Prior to the Heston model, popular stochastic volatility models include Hull and White (1987), Scott (1987), Wiggins (1987), Chensey and Scott (1989), and Stein and Stein (1991) among others. Although the Heston model was not the first stochastic volatility model in pricing options, it has become the most important, and now is a standard stochastic volatility model.

Chapter 6

Appendix

6.1 Proof of Lemma 2.5.2

This proof can be found in [42].

We only show that $P(\theta_1 < \infty) = 1$ since the proof for $P(\theta_2 < \infty) = 1$ is similar.

Given $z_0 < z_1$ and define

$$\theta_0 = \inf\{t : X_t \notin (z_0, z_1)\}.$$

Let $p_{z_0}(x) = P(X_{\theta_0} = z_1 | X_0 = x)$. Then it satisfies the equation (see [43])

$$\frac{\sigma^2}{2}\frac{d^2 p_{z_0}(x)}{dx^2} + a(b - x)\frac{dp_{z_0}(x)}{dx} = 0,$$

with $p_{z_0}(z_0) = 0$ and $p_{z_0}(z_1) = 1$. The solution to this equation can be written as

$$p_{z_0}(x) = \frac{\int_{z_0}^{x} \exp\left[-\frac{2a}{\sigma^2}(bu - \frac{u^2}{2})\right] du}{\int_{z_0}^{z_1} \exp\left[-\frac{2a}{\sigma^2}(bu - \frac{u^2}{2})\right] du} = \frac{\int_{z_0}^{x} \exp\left[\frac{a}{\sigma^2}(u-b)^2\right] du}{\int_{z_0}^{z_1} \exp\left[\frac{a}{\sigma^2}(u-b)^2\right] du}.$$

Sending $z_0 \to -\infty$, we have $p_{z_0}(x) \to 1$. This implies $P(\theta_1 < \infty) \geq p_{z_0}(x) \to 1$. $\square$

6.2 Change Variables for the Generator (5.9)

Let $(x,y) \in \mathbb{R} \times (0,1)$. Suppose the generator of (x,y) is given by

$$\begin{aligned}\mathcal{A} = \ & \tfrac{\sigma^2}{2}\tfrac{\partial^2}{\partial x^2} + (\mu_1-\mu_2)y(1-y)\tfrac{\partial^2}{\partial x \partial y} + \tfrac{1}{2}\Big[\tfrac{(\mu_1-\mu_2)y(1-y)}{\sigma}\Big]^2 \tfrac{\partial^2}{\partial y^2} \\ & + \Big[(\mu_1-\mu_2)y+\mu_2\Big]\tfrac{\partial}{\partial x} + \Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]\tfrac{\partial}{\partial y}.\end{aligned}$$

For convenience, we consider

$$\begin{aligned}\mathcal{A} = \ & \sigma^2\tfrac{\partial^2}{\partial x^2} + 2(\mu_1-\mu_2)y(1-y)\tfrac{\partial^2}{\partial x \partial y} + \Big[\tfrac{(\mu_1-\mu_2)y(1-y)}{\sigma}\Big]^2 \tfrac{\partial^2}{\partial y^2} \\ & +2\Big[(\mu_1-\mu_2)y+\mu_2\Big]\tfrac{\partial}{\partial x} + 2\Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]\tfrac{\partial}{\partial y}.\end{aligned}$$

Let $a = \sigma^2,\ b = (\mu_1-\mu_2)y(1-y),\ c = \Big[\dfrac{(\mu_1-\mu_2)y(1-y)}{\sigma}\Big]^2.$

Then $\mathcal{A}f(x,y) = af_{xx} + 2bf_{xy} + cf_{yy} + \cdots$

Note $b^2 - ac = 0$.

Let $\xi = \xi(x,y),\ \eta = \eta(x,y)$. Then

$$\begin{aligned}
f_x = \ & f_\xi \xi_x + f_\eta \eta_x, \\
f_y = \ & f_\xi \xi_y + f_\eta \eta_y, \\
f_{xx} = \ & (f_\xi \xi_x)_x + (f_\eta \eta_x)_x = (f_\xi)_x \xi_x + f_\xi \xi_{xx} + (f_\eta)_x \eta_x + f_\eta \eta_{xx} \\
= \ & f_{\xi\xi}\xi_x^2 + 2f_{\xi\eta}\xi_x\eta_x + f_{\eta\eta}\eta_x^2 + f_\xi\xi_{xx} + f_\eta\eta_{xx}, \\
f_{yy} = \ & (f_\xi \xi_y)_y + (f_\eta \eta_y)_y = (f_\xi)_y \xi_y + f_\xi \xi_{yy} + (f_\eta)_y \eta_y + f_\eta \eta_{yy} \\
= \ & f_{\xi\xi}\xi_y^2 + 2f_{\xi\eta}\xi_y\eta_y + f_{\eta\eta}\eta_y^2 + f_\xi\xi_{yy} + f_\eta\eta_{yy}, \\
f_{xy} = \ & (f_\xi \xi_x)_y + (f_\eta \eta_x)_y = (f_\xi)_y \xi_x + f_\xi \xi_{xy} + (f_\eta)_y \eta_x + f_\eta \eta_{xy} \\
= \ & f_{\xi\xi}\xi_x\xi_y + f_{\xi\eta}\xi_x\eta_y + f_{\xi\eta}\xi_y\eta_x + f_{\eta\eta}\eta_x\eta_y + f_\xi\xi_{xy} + f_\eta\eta_{xy}.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}f = \ & a(f_{\xi\xi}\xi_x^2 + 2f_{\xi\eta}\xi_x\eta_x + f_{\eta\eta}\eta_x^2 + f_\xi\xi_{xx} + f_\eta\eta_{xx}) + \\
& 2b(f_{\xi\xi}\xi_x\xi_y + f_{\xi\eta}\xi_x\eta_y + f_{\xi\eta}\xi_y\eta_x + f_{\eta\eta}\eta_x\eta_y + f_\xi\xi_{xy} + f_\eta\eta_{xy}) + \\
& c(f_{\xi\xi}\xi_y^2 + 2f_{\xi\eta}\xi_y\eta_y + f_{\eta\eta}\eta_y^2 + f_\xi\xi_{yy} + f_\eta\eta_{yy}) + \cdots \\
= \ & (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)f_{\xi\xi} + 2(a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y)f_{\xi\eta} + \\
& (a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)f_{\eta\eta} + \cdots
\end{aligned}
$$

We need $af_{xx} + 2bf_{xy} + cf_{yy} + \cdots = Af_{\xi\xi} + 2Bf_{\xi\eta} + Cf_{\eta\eta} + \cdots$, where

$$
\begin{cases}
A = & a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\
B = & a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y, \\
C = & a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.
\end{cases}
\tag{6.1}
$$

$$B^2 - AC = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2 = 0.$$

We want to choose $\eta(x, y)$ so that $C = 0$, i.e.,

$$a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0.$$

Equivalently,

$$a\Big(\frac{\eta_x}{\eta_y}\Big)^2 + 2b\frac{\eta_x}{\eta_y} + c = 0.$$

Solve the last equation to get $\dfrac{\eta_x}{\eta_y} = -\dfrac{a}{b}$.

Consider the level curve $\eta(x, y) = k$, for some constant k. Differentiate with respect to x, we get

$$\eta_x + \eta_y\frac{dy}{dx} = 0.$$

This implies

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{b}{a} = \frac{(\mu_1 - \mu_2)y(1 - y)}{\sigma^2}.$$

Equivalently,

$$\begin{aligned}
\frac{dy}{y(1-y)} &= \frac{(\mu_1-\mu_2)}{\sigma^2}dx \\
\Big(\frac{1}{y}+\frac{1}{1-y}\Big)dy &= \frac{(\mu_1-\mu_2)}{\sigma^2}dx \\
\ln(y)-\ln(1-y) &= \frac{(\mu_1-\mu_2)}{\sigma^2}x+k_1 \\
\frac{y}{1-y} &= k_2 e^{\frac{(\mu_1-\mu_2)}{\sigma^2}x}.
\end{aligned}$$

We can choose $k_2 = k$ to obtain $\dfrac{y}{1-y}e^{-\frac{(\mu_1-\mu_2)}{\sigma^2}x} = k = \eta(x,y)$. This will make $C = 0$. Since $B^2 - AC = 0$ no matter what the transformation $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ will be, $B = 0$. Hence, we can choose any $\xi(x,y)$, in particular, $\xi(x,y) = y$. We obtain the system:

$$\begin{cases} \xi(x,y) = y, \\ \eta(x,y) = \dfrac{y}{1-y}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}. \end{cases}$$

Then the first equation of (6.1) gives $A = c$. It follows that

$$\mathcal{A}f = \left[\frac{(\mu_1-\mu_2)y(1-y)}{\sigma}\right]^2 f_{\xi\xi} + \text{ first order terms.}$$

Moreover,

$$\begin{aligned}
\eta_x &= -\frac{\mu_1-\mu_2}{\sigma^2}\eta, \\
\eta_{xx} &= \Big(\frac{\mu_1-\mu_2}{\sigma^2}\Big)^2\eta, \\
\eta_y &= \frac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}, \\
\eta_{yy} &= \frac{2}{(1-y)^3}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}, \\
\eta_{xy} = \eta_{yx} &= -\frac{\mu_1-\mu_2}{\sigma^2}\frac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
f_x &= -\tfrac{\mu_1-\mu_2}{\sigma^2}\eta f_\eta, \\
f_y &= f_\xi + \tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}f_\eta = f_\xi + \tfrac{1}{y(1-y)}\eta f_\eta, \\
f_{xx} &= f_{\eta\eta}\Big(\tfrac{\mu_1-\mu_2}{\sigma^2}\Big)^2\eta^2 + f_\eta\Big(\tfrac{\mu_1-\mu_2}{\sigma^2}\Big)^2\eta = \Big(\tfrac{\mu_1-\mu_2}{\sigma^2}\Big)^2(\eta^2 f_{\eta\eta}+\eta f_\eta), \\
f_{yy} &= f_{\xi\xi} + 2f_{\xi\eta}\tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x} + f_{\eta\eta}\left[\tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}\right]^2 + f_\eta\tfrac{2}{(1-y)^3}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}, \\
&= f_{\xi\xi} + \tfrac{2}{y(1-y)}\eta f_{\xi\eta} + \tfrac{1}{y^2(1-y)^2}\eta^2 f_{\eta\eta} + \tfrac{2}{y(1-y)^2}\eta f_\eta, \\
f_{xy} &= -\tfrac{\mu_1-\mu_2}{\sigma^2}\eta f_{\xi\eta} - \tfrac{\mu_1-\mu_2}{\sigma^2}\tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}\eta f_{\eta\eta} - \tfrac{\mu_1-\mu_2}{\sigma^2}\tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}f_\eta, \\
&= -\tfrac{\mu_1-\mu_2}{\sigma^2}\Big(f_{\xi\eta}\eta + \tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}f_{\eta\eta}\eta + \tfrac{1}{(1-y)^2}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}f_\eta\Big), \\
&= -\tfrac{\mu_1-\mu_2}{\sigma^2}\Big(\eta f_{\xi\eta} + \tfrac{1}{y(1-y)}\eta^2 f_{\eta\eta} + \tfrac{1}{y(1-y)}\eta f_\eta\Big).
\end{aligned}
$$

So, the first order terms

$$
\begin{aligned}
&= 2(\mu_1-\mu_2)\eta f_\eta + 2\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2\eta f_\eta - 2\frac{\mu_1-\mu_2}{\sigma^2}\Big[(\mu_1-\mu_2)y+\mu_2\Big]\eta f_\eta \\
&\quad +2\Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]\tfrac{1}{y(1-y)}\eta f_\eta + 2\Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]f_\xi \\
&= \left\{2(\mu_1-\mu_2) + 2\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 - 2\frac{\mu_1-\mu_2}{\sigma^2}\Big[(\mu_1-\mu_2)y+\mu_2\Big] + 2\Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]\frac{1}{y(1-y)}\right\}\eta f_\eta \\
&\quad +2\Big[-(\lambda_1+\lambda_2)y+\lambda_2\Big]f_\xi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{A}f(\eta,\xi) = \;& \left[\frac{(\mu_1-\mu_2)\xi(1-\xi)}{\sigma}\right]^2 f_{\xi\xi} + \\
& \Bigg\{2(\mu_1-\mu_2) + 2\left(\frac{\mu_1-\mu_2}{\sigma}\right)^2 - 2\frac{\mu_1-\mu_2}{\sigma^2}\Big[(\mu_1-\mu_2)\xi+\mu_2\Big] + \\
& 2\Big[-(\lambda_1+\lambda_2)\xi+\lambda_2\Big]\frac{1}{\xi(1-\xi)}\Bigg\}\eta f_\eta + 2\Big[-(\lambda_1+\lambda_2)\xi+\lambda_2\Big]f_\xi,
\end{aligned}
$$

with $\eta(x,y) = \dfrac{y}{1-y}e^{-\frac{\mu_1-\mu_2}{\sigma^2}x}$ and $\xi(x,y) = y$.

6.3 Parameter Estimation and Simulation for the Model (4.1)

6.3.1 Parameter Estimation

Let N be the number of time points, and T be the corresponding time length.

The normalized trading days $NoTD = N/T$. We use $NoTD = 250$ in our simulation.

1. Compute the rate of return for the bull market μ_1:

   Let $[m_i, n_i], i = 1...K$ be the time intervals corresponding to the bull markets.

   Let $d_i = (n_i - m_i + 1)/NoTD$ be the normalized number of trading days in the ith interval.

   Let $r_i = (Z_{n_i} - Z_{m_i})/d_i$ be the return rate on the i interval. Then

$$\mu_1 = \frac{r_1 + r_2 + ... + r_K}{K},$$

   where K is the number of intervals of the bull market.

   μ_2 is computed similarly.

2. Compute the transition rate λ_1 from bull to bear:

$$\frac{1}{\lambda_1} = \frac{d_1 + d_2 + ... + d_K}{K}.$$

   λ_2 is computed similarly.

3. Compute the volatility σ:

   Let $R_i = Z_{i+1} - Z_i, i = 1...(N-1)$ be the return rates. Then

$$\sigma = \sqrt{NoTD \times VARIANCE(\{R_i\})}.$$

6.3.2 Simulation of (Z_t, p_t)

Combine (4.2) and (4.3), we have

$$dp_t = f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} dZ_t, \tag{6.2}$$

where

$$f(p) = -(\lambda_1 + \lambda_2)p + \lambda_2 - \frac{(\mu_1 - \mu_2)p(1-p)((\mu_1 - \mu_2)p + \mu_2)}{\sigma^2}.$$

Discretize (6.2) to obtain

$$p_{t+1} = p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}(Z_{t+1} - Z_t), \tag{6.3}$$

where dt is the step size.

Since $p_t \in [0, 1]$, we use the truncation process instead

$$p_{t+1} = \min\left(\max\left(p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2}(Z_{t+1} - Z_t), 0\right), 1\right).$$

Discretize (4.3) to obtain

$$Z_{t+1} = Z_t + [(\mu_1 - \mu_2)p_t + \mu_2]dt + \sigma(\widehat{W}_{t+1} - \widehat{W}_t). \tag{6.4}$$

We use (6.3) and (6.4) to generate (Z_t, p_t).

Following is a simulation of $Z_t = \log(GOOGL_t) - \log(YHOO_t)$ and p_t with parameters:

μ_1	μ_2	λ_1	λ_2	σ
1.846917	-2.069453	7.392473	7.284768	0.341831

Table 6.1: Parameters of Z_t used in the simulation (GOOGL-YHOO).

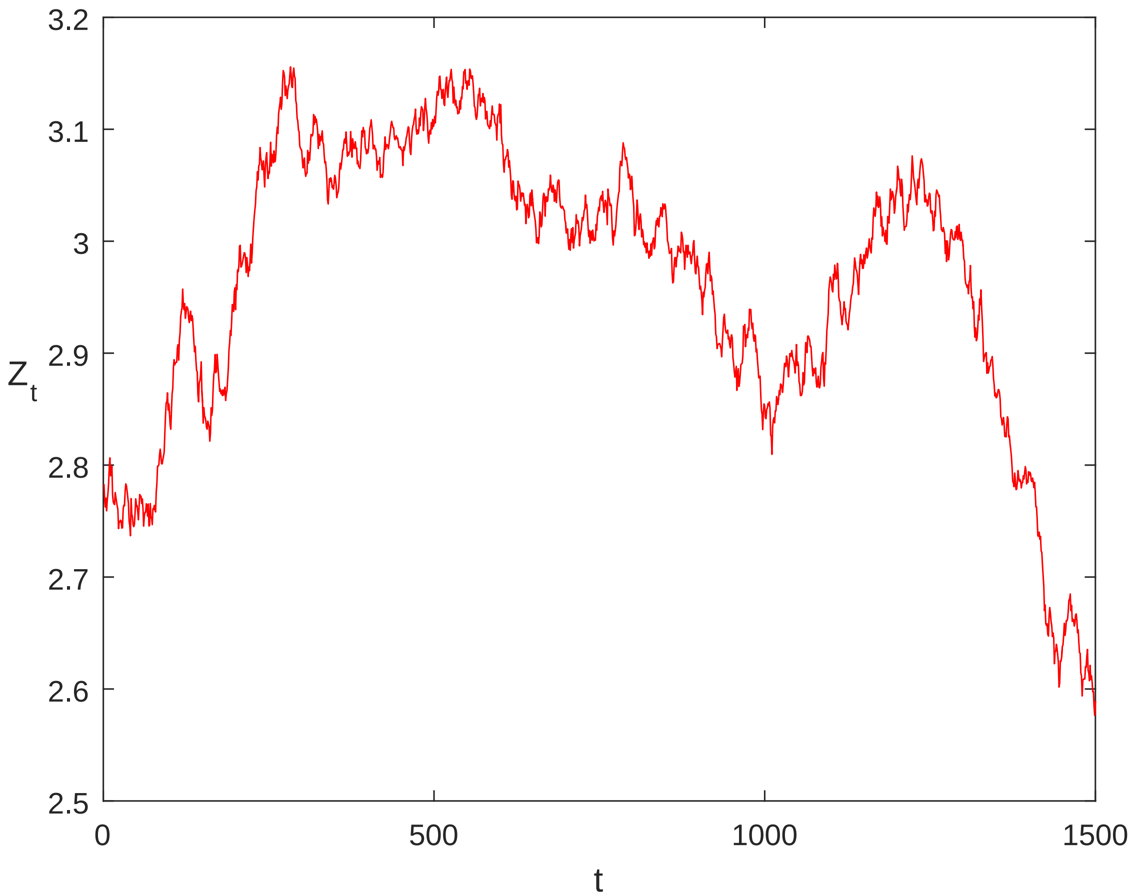

Figure 6.1: A sample path of Z_t (GOOGL-YHOO).

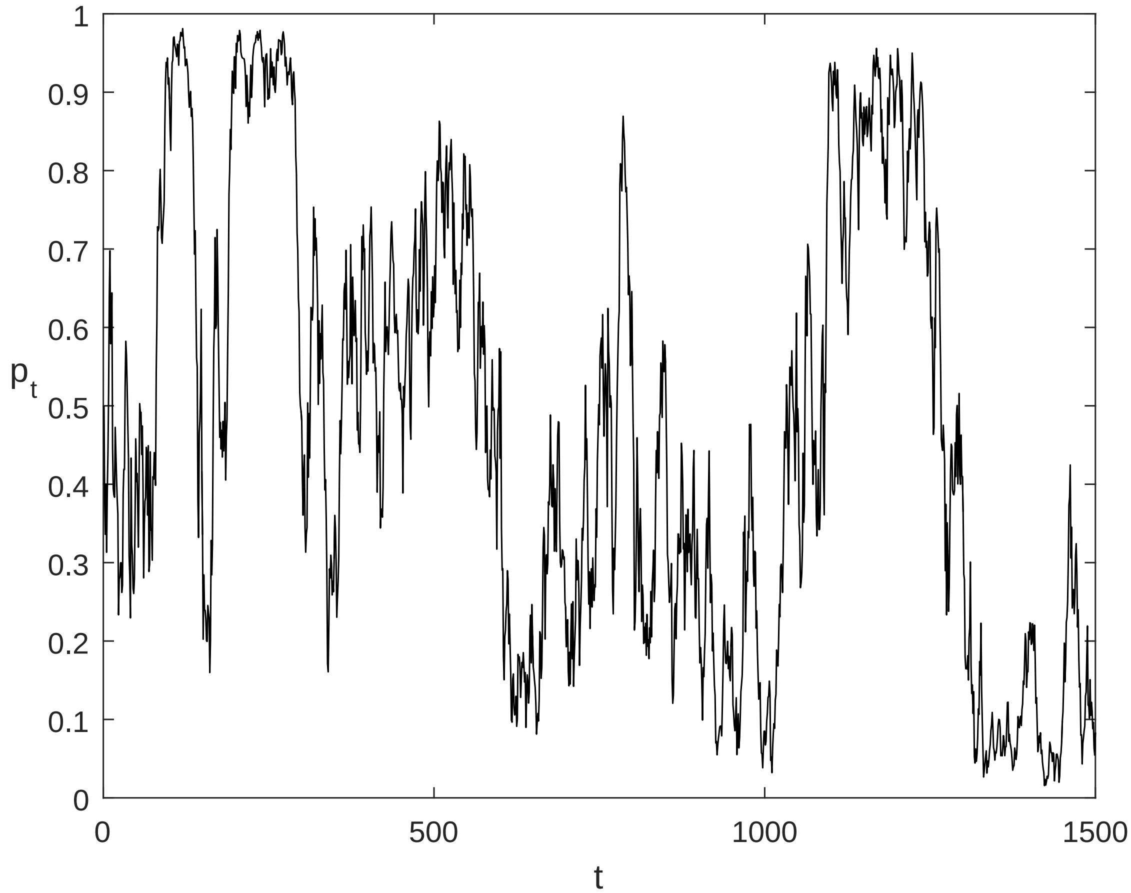

Figure 6.2: The corresponding conditional probability p_t (GOOGL-YHOO).

Bibliography

[1] P. Billingsley, *Convergence of probability measures*, 2nd Ed., Wiley, New York, 1999.

[2] C. Blanco and D. Soronow, Mean reverting processes – Energy price processes used for derivatives pricing and risk management, *Commodities Now*, 68–72, (2001).

[3] H. F. Chen, *Stochastic approximation and its applications*, Kluwer Academic, Dordrecht, Netherlands, 2002.

[4] Y. Cheng, F. Liu, R. H. Liu, G. Yin, and Q. Zhang, Stock liquidation via stochastic approximation using nasdaq daily and intra-day data, *Math. Finance*, Vol. 16, No. 1, (January, 2006).

[5] A. Cowles and H. Jones, Some posteriori probabilities in stock market action, *Econometrica*, 5, 280–294, (1937).

[6] M. G. Crandall, H. Ishii, and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *American Math. Soc.*, Vol. 27, No. 1, pp. 1–67, (July, 1992).

[7] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *American Math. Soc.*, Vol. 277, No. 1, (May, 1983).

[8] M. Dai, Q. Zhang, and Q. Zhu, Trend following trading under a regime switching model, *SIAM Journal on Financial Mathematics*, Vol. 1, pp. 780-810, (2010).

[9] P. Eloe, R. H. Liu, M. Yatsuki, G. Yin, and Q. Zhang, Optimal selling rules in a regime-switching exponential Gaussian diffusion model, *SIAM J. Appl. Math.*, Vol. 69, No. 3, pp. 810–829, (2008).

[10] S. N. Ether and T. G. Kurtz, *Markov processes: characterization and convergence*, Wiley, New York, 1986.

[11] A. Etheridge, *A course in financial calculus*, Cambridge University Press, New York, 2002.

[12] E. Fama and K. R. French, Permanent and temporary components of stock prices, *Journal of Political Economy*, 96, 246–273, (1988).

[13] W. H. Fleming and H.M. Soner, *Controlled markov processes and viscosity solutions*, 2nd Ed., Springer, New York, 2006.

[14] W. H. Fleming and R.W. Rishel, *Deterministic and stochastic optimal control*, Springer-Verlag, New York, 1975.

[15] L. A. Gallagher and M. P. Taylor, Permanent and temporary components of stock prices: Evidence from assessing macroeconomic shocks, *Southern Economic Journal*, 69, 345–362, (2002).

[16] E. Gatev, W. N. Goetzmann and K. G. Rouwenhorst, Pairs trading: Performance of a relative-value arbitrage rule, *Review of Financial Studies*, Oxford University Press for Society for Financial Studies, 19, 797–827, (2006).

[17] X. Guo and Q. Zhang, Optimal selling rules in a regime switching model, *IEEE Transactions on Automatic Control*, 50, 1450–1455, (2005).

[18] C. M. Hafner and H. Herwartz, Option pricing under linear autoregressive dynamics, heteroskedasticity, and conditional leptokurtosis, *Journal of Empirical Finance*, 8, 1–34, (2001).

[19] J. D. Hamilton, A new approach to the economic analysis of non-stationary time series, *Econometrica*, Vol. 57, pp. 357–384, (1989).

[20] J. C. Hull, *Options, futures, and other derivatives*, 5th Ed., Prentice Hall, New Jersey, 2003.

[21] J. Kiefer and J. Wolfowitz, Stochastic estimation of the maximum of a regression function, *Ann. Math. Statist.*, Vol. 23, No. 3, 462–466, (1952).

[22] K. Kuo, P. Luu, D. Nguyen, E. Perkerson, K. Thompson, and Q. Zhang, Pairs trading: an optimal selling rule, *Math. Control and Related Fields*, Vol. 5, No. 3, (Sep., 2015).

[23] H. J. Kushner, *Approximation and weak convergence methods for random processes with applications to stochastic systems theory*, MIT Press, Cambridge, MA, 1984.

[24] H. J. Kushner and G. Yin, *Stochastic Approximation and recursive Algorithms and Applications*, 2nd Ed, Springer, New York, 2003.

[25] J. Liu and A. Timmermann, Optimal convergence trade strategies, *Review of Financial Studies*, 26, 1048–1086, (2013).

[26] S. Monro and H. Robbins, A stochastic approximation method, *Ann. Math. Statist.*, Vol. 22, No. 3, 400–407, (Sep., 1951).

[27] D. Nguyen, G. Yin, and Q. Zhang, A stochastic approximation approach for trend-following trading, *International Series in Operations Research and Management Science*, Vol. 209, pp 167–184, (April, 2014).

[28] B. Øksendal, *Stochastic differential equations*, 6th Ed., Springer-Verlag, New York, 2003.

[29] M. Pemy, Q. Zhang, Optimal stock liquidation in a regime switching model with finite time horizon, *J. Math. Anal. Appl.*, Vol. 321, pp. 537–552, (2006).

[30] F. D. Rouah, *The Heston model and its extensions in Matlab and C#*, Wiley, New Jersey, 2013.

[31] S. P. Sethi, M.I. Taksar, and Q. Zhang, Capacity and production decisions in stochastic manufacturing systems: an asymptotic optimal hierarchical approach, *Production and Operations Management Soc.*, Vol. 1, No. 4, (Fall, 1992).

[32] S. P. Sethi and Q. Zhang, *Hierarchical decision making in stochastic manufacturing systems*, Birkhauser, Boston, 1994.

[33] Q. S. Song, G. Yin, and Q. Zhang, Stochastic optimization methods for buying-low-and-selling-high strategies, *Stochastic Anal. Appl.*, Vol. 27, pp. 52-542, (2009).

[34] Q. Song and Q. Zhang, An optimal pairs-trading. *Automatica*, 49, 3007–3014, (2013).

[35] G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*, Wiley, Hoboken, NJ, 2004.

[36] R. J. Williams, *Notes from Stochastic Models of Complex Networks course*, UC San Diego, California, Fall 2008.

[37] W. M. Wonham, Some applications of stochastic differential equations to optimal non-linear filtering, *SIAM J. Control*, Vol. 2, pp. 347–369, (1965).

[38] G. Yin, R. H. Liu, and Q. Zhang, Recursive algorithms for stock liquidation: a stochastic optimization approach, *SIAM J. Optim.*, Vol. 13, 240–263, (2002).

[39] G. Yin and Q. Zhang, Nearly optimal asset allocation in hybrid stock-investment models, *Journal of Optimization Theory and Applications*, Vol. 121, pp. 419–444, (May, 2004).

[40] G. Yin and Q. Zhang, *Continuous-time markov chains and applications: a two-time-scale approach*, 2nd Ed., Springer, New York, 2013.

[41] G. Yin and X. Y. Zhou, Markowitz's mean-variance portfolio selection with regime-switching: from discrete-time models to their continuous-time limits, *IEEE Transactions on Automatic Control*, Vol. 49, pp. 349–360, (March, 2004).

[42] H. Zhang and Q. Zhang, Trading a mean-reverting asset: buy low and sell high. *Automatica*, 44, 1511–1518, (2008).

[43] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Contr. Optim.*, 40, 64–87, (2001).