STUDENTS' CONSTRUCTION OF INTENSIVE QUANTITY

by

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(Under the Direction of Leslie P. Steffe)

ABSTRACT

Intensive quantities, those quantities which characterize a multiplicative relationship between two quantities, represent a critical component of students' mathematical learning. Examples of reasoning with intensive quantities include making proportional comparisons, reasoning about linear functions that have constant rates of change, considering the densities of various materials, and analyzing the rates at which quantities covary. In this study, I investigated the mental schemes and operations that students used to construct and reason with intensive quantities and the covariational relationships those quantities described.

This dissertation reports the findings from a constructivist teaching experiment I conducted with two tenth-grade students from October 2013 to March 2014. As the students' primary teacher, I posed a variety of tasks designed to investigate how the students would construct and reason with constant multiplicative relationships between covarying quantities. After completing the teaching experiment, I conducted a retrospective analysis of the teaching session interactions in order to construct second-order models that accounted for the students' mathematical activity and changes they made to their quantitative reasoning over the course of the study.

Findings include the identification of seven constructive resources that facilitated the students' ability to construct intensive quantities and to make sense of constant covariational relationships: a) reasoning with three levels of units; b) incorporating a strategy of coordinated partitioning/iterating; c) the construction of a splitting scheme; d) the construction of iterable composite units; e) the construction of a process for quantifying a unit ratio; f) the construction of a simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units; and g) the ability to use one's operations recursively in order to flexibly change the measurement units of both quantities in a given ratio. In addition, these conceptual resources were involved in the construction of a reversible distributive partitioning scheme that enabled the construction of distributive reasoning. These results have implications for how researchers and teachers conceptualize goals for students' mathematical learning in schools.

INDEX WORDS: Algebraic reasoning, Distributive partitioning, Extensive quantities, Intensive quantities, Iterable composite units, Levels of units, Mathematical learning, Operations, Quantitative reasoning, Rates, Ratios, Radical Constructivism, Schemes, Splitting, Teaching experiment

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DEDICATION

I dedicate this work to my students—to my former students who inspired me to become a better teacher and with whom I developed a passion for learning how students think, to the students I worked with during this teaching experiment whose creative problem solving taught me more than I ever could have imagined, and to the students with whom I will work in the future, may we never stop learning from each other.

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CHAPTER 1

INTRODUCTION

In broad terms, this study is about students' development of algebraic reasoning. It tells the story of two high school students' efforts to use their conceptual resources to make sense of situations involving co-varying quantities. However, more than that, this is a story that specifies those conceptual resources and elaborates ways in which those resources relate to the understandings the students constructed for the situations.

More specifically, this study is about the role that particular numeric and fractional understandings play in the development of algebraic reasoning. As a high school mathematics teacher, I remember thinking that a solid understanding of numeric and fraction operations was an important component of achieving success in algebra. For instance, learning about rational functions and skillfully manipulating algebraic notation are just two examples of algebraic reasoning that require a solid foundation of numeric and fractional knowledge. However, while I suspected that the importance of these types of knowledge ran deeper than learning to use these understandings in a new context of unknown quantities and variable expressions, I was unable to articulate why these types of reasoning were so important for success in algebra.

This study is about elaborating and clarifying those reasons. It is about developing a clearer understanding of how the reasoning one uses to make sense of numbers and fractions relates to learning algebraic concepts like proportional reasoning and rates of change. Ultimately, this story is about the genesis of the ways of reasoning that enable one to understand a particular type of quantity that is central to the development of algebraic reasoning—intensive quantity.

Problem Statement and Rationale

School algebra represents a critical point in students' mathematical education. The role this course plays in a student's future success led the National Mathematics Advisory Panel (2008) to state, "Algebra has emerged as a central concern, for it is a demonstrable gateway to later achievement" (p. 3). From a content perspective, the study of mathematics in later high school and beyond uses and extends the ideas included in the school algebra curriculum. For instance, students construct understandings about rates of change in an algebra course when studying quantitative relationships such as linear, exponential, and quadratic functions. Later, the sophisticated analyses of the rates at which quantities change that occur when studying calculus are both made possible by, and intended to further elaborate and broaden, students' understandings of rates of change. From an equity perspective, success in school algebra ultimately represents a civil rights concern because of the significant impact it has on things such as college admissions and one's future economic opportunities (Moses & Cobb, 2001). The importance of students' success in algebra raises two important questions: which mathematical concepts represent the foundational understandings that students use in the construction of rich and powerful algebraic ways of knowing and reasoning, and how do students construct these understandings?

Constructing, interpreting, and making use of rates of change in the study of quantitative relationships is one such foundational component of school algebra. For instance, one of the high school algebra standards in the Common Core State Standards for Mathematics (CCSSM) states that students should "Create equations in two or more variables to represent relationships between quantities" (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CSSO], 2010, p. 65). The quantitative relationships studied in pursuit of this standard in school algebra frequently follow linear, quadratic, or exponential patterns. In regard to modeling with these quantitative relationships, the CCSSM state that students should "Recognize situations in which one quantity changes at a constant rate per unit interval relative to another" (NGACBP & CSSO, 2010, p. 70). Similarly, the *Principles and Standards for School Mathematics* high school algebra standards state that all students should "Analyze change in various contexts" and "Use symbolic algebra to represent and explain mathematical relationships" (National Council of Teachers of Mathematics, 2000, p. 296). Considered broadly, standards documents such as these indicate that school algebra entails studying the relationships between changing quantities, recognizing patterns within the rates of change, using symbolic notation to model these quantitative relationships, and learning techniques for transforming this notation in ways that preserve the symbolized relationships and patterns of change.

However, while reasoning with and about rates of change is a central component of the school algebra curriculum, the sequence and wording of recent standards documents suggests an assumption that students will have already constructed several foundational understandings regarding rate prior to the study of algebra and that ratio and rate reasoning are unproblematic for high school students. For instance, the CCSSM sixth grade standards outline several key understandings regarding rates and ratios. Among them, students should "Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship" and should also "Use ratio and rate reasoning to solve real-world and mathematical problems…" (NGACBP & CSSO, 2010, p. 42). Further, the understandings of ratios and rates implied by the standards become increasingly sophisticated. By the end of middle school, students are expected to "Understand the connections between proportional relationships, lines, and linear equations" (NGACBP & CSSO, 2010, p. 54).

Thus, foundational understandings regarding rates of change are taken as a given by the time students begin an algebra course in the ninth grade. This assumption is reinforced through the specific wording of the high school standards such as in the CCSSM description of the goals for high school students' modeling of quantitative relationships that states, "A model can be very simple, such as writing total cost as a product of unit price and number bought…" (NGACBP & CSSO, 2010, p. 72).

Yet, even this supposedly simple model entails several understandings that have proven difficult for students to construct. For instance, consider purchasing grapes from a store that sells 3 pounds of grapes for 4 dollars. Modeling the quantitative relationship between total cost and number of pounds of grapes bought with an algebraic equation requires several complex mathematical schemes and operations. In constructing this model, one must a) assimilate the situation as involving a multiplicative and proportional relationship between total price and the number of pounds of grapes bought, b) transform the ratio of 3 pounds of grapes for 4 dollars into the unit ratio of four-thirds dollars per pound of grapes, and c) symbolize the quantitative relationship abstractly as $d = (4/3)g$ where d represents the total cost in dollars and g represents the number of pounds of grapes purchased.

However, the operations involved in assimilating this situation as a proportional relationship and in constructing the unit ratio are highly sophisticated quantitative operations that not all high school students have constructed (Steffe, Liss II, & Lee, 2014; Steffe & Olive, 2010; P. W. Thompson & Saldanha, 2003). Even some college students struggle to appropriately symbolize this type of quantitative relationship with an algebraic equation (Clement, 1982).

Thus, while standards documents can serve to define the types of understandings teachers intend for students to develop, this example demonstrates that those same documents provide little guidance regarding the challenges students might encounter and the processes by which students might construct those understandings.

Alternatively, in-depth studies of students' cognition can provide greater insight into the cognitive processes that support particular ways of reasoning about rates and ratios as well as into potential constructive trajectories for these understandings. For example, in-depth teaching experiments conducted with middle school students resulted in several characterizations of students' conceptions of speed and rate. P. W. Thompson and Thompson (1992) found differences between how students reasoned with relationships between distance and time and used these to characterize four images of speed with which students operate. These images of speed range from an image that treats speed as a distance to a more sophisticated image of speed as a rate in which distance and time accrue simultaneously, continuously, and within a proportional relationship. Conceptualizing speed as a rate in this fashion involves constructing time as an extensive quantity so that one can reason with accruals of time and distance as well as learn to reason multiplicatively with both accruals and total accumulations of time and distance (P. W. Thompson, 1994; P. W. Thompson & Thompson, 1992).

Because images contain records of having operated and, thus, are shaped by the mental schemes and operations one has available, these characterizations of the images of speed that individuals use when reasoning about relationships among distance, time, and speed are useful for describing the results of particular mental schemes and operations. For example, an image of speed as a distance involves considering distance accrued additively where each distance unit implicitly corresponds to one time unit. In contrast, an image of speed as a rate involves considering distance and time accruing simultaneously in such a way that preserves a proportional relationship (P. W. Thompson & Thompson, 1992). Thus, more sophisticated understandings of speed and rate involve constructing multiplicative relationships between accruals of time and distance.

Efforts to use these images of speed to guide instruction provide additional clarity regarding the nature of the mental operations that produce these images. During a teaching experiment involving a sixth grade student and her classroom teacher, several miscommunications arose between the two that hindered the progress of the interaction (A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994). The researchers concluded that through his own mathematical experiences, the teacher had encapsulated highly sophisticated understandings of quantitative relationships into his symbolism and language for numeric operations. Thus, for the teacher the operation of division signified several foundational multiplicative relationships involving speed and proportionality that were not aspects of the student's schemes for dividing. While providing insight into the knowledge required for teaching about speed and rate conceptually, this line of research also indicates that understandings of proportionality and multiplicative relationships are integral components of the more sophisticated images and meanings for speed and rate (A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994; P. W. Thompson & Thompson, 1992, 1994).

However, while these studies have highlighted the role of proportionality and multiplicative reasoning in sophisticated images of speed as a rate, they have not focused on exploring the underlying mental schemes and operations that enable the multiplicative reasoning involved with constructing images based upon proportional relationships between quantities. P. W. Thompson and Saldanha (2003) suggest that how students understand numbers and numeric operations significantly impacts the ways in which they assimilate mathematical situations as well as their future constructive possibilities. In particular, these authors describe specific understandings of measurement, multiplication, and division that are involved in constructing a multiplicative understanding of fractions that supports future mathematical development. Further, their analysis suggests that multiplicative understandings of each of these topics are grounded in proportional reasoning (P. W. Thompson & Saldanha, 2003).

Other studies of students' construction of number concepts and operations have identified particular mental schemes that students use to produce and operate with quantities in ways that are compatible with the multiplicative understandings grounded in proportionality that P. W. Thompson and Saldhana (2003) describe. For example, the iterable unit of one—a hallmark of the explicitly nested number sequence—allows a student to assimilate a composite unit such as nine as nine iterations of one, or equivalently, as a number that is nine times as large as one (Steffe, 1988; Steffe & Olive, 2010). Thus, a child's whole number concepts become multiplicative with the construction of an explicitly nested number sequence.

In much the same way, an iterable unit fraction signifies the construction of the iterative fraction scheme (Olive & Steffe, 2002; Steffe & Olive, 2010). This allows a child to reorganize his/her fraction concepts into multiplicative concepts. Hence, the construction of an iterative fraction scheme allows a child to conceive of a fraction such as nine-sevenths multiplicatively as a number that is nine times as large as one-seventh.

Considered together, these areas of research suggest that one must consider the relationships among schemes and operations involved in number, fraction, and proportionality in order to investigate the underlying multiplicative schemes and operations involved in the construction of rate. While this suggests a complex web of relationships among these multiplicative concepts, the construction of these ways of operating multiplicatively is a component of the construction of a new type of quantity for students—intensive quantity.

Density, speed, and rates of change between quantities are all examples of intensive quantities, and these have measures that do not depend upon the amount of the quantities within the system that are used to form the intensive quantity. In contrast, extensive quantities consist of things such as length or mass and have measures that vary with the amount of the quantity (Jahnke, 1983; Schwartz, 1988). For example, the mass of a pile of carbon varies with changes in the quantity of carbon contained in the pile. However, the density of the carbon in the pile is constant regardless of the particular amount of carbon present. The construction of a multiplicative relationship between the mass and volume of the carbon accounts for the invariant magnitude of the intensive quantity. This dissertation study investigates which mental schemes and operations enable one to construct this type of multiplicative relationship.

Previous research also suggests that many critical multiplicative schemes and operations are involved in the construction of sophisticated understandings of fractions (Olive & Steffe, 2002; Steffe & Olive, 2010; P. W. Thompson & Saldanha, 2003). Additionally, other scholars have called for further research to investigate how students' fraction schemes and operations might support their construction of quantitative and algebraic reasoning (Norton & Hackenberg, 2010). In this dissertation, I respond to this call and investigate how the multiplicative schemes and operations students construct for operating with numbers, fractions, and extensive quantities are involved in the construction of intensive quantities.

Research Questions

Taken together, these notions of quantity and the multiplicative operations they involve suggest that a more thorough understanding of how students construct intensive quantities is needed for educators to better facilitate students' construction of foundational algebraic concepts. Toward this end, I attempt to answer the following research questions in this dissertation:

1. What conceptual constructs, including extensive quantitative schemes and operations, can explain each student's assimilation, as well as any changes in that student's assimilation, of quantitative situations involving intensive quantities?
2. What aspects of the mathematics of each participant, including extensive quantitative schemes and operations, impede or facilitate that participant's ability to work with quantitative situations involving intensive quantities?
3. What conceptual constructs, including extensive quantitative schemes and operations, are involved in the construction of intensive quantitative schemes and operations?

It is important to recognize that the phrase "quantitative situations involving intensive quantities" is written from my perspective as the researcher who designed the quantitative situations. Thus, while I understand the situations as involving intensive quantities, I make no claim that intensive quantities are inherent in the situations themselves or that others necessarily assimilate the situations as involving intensive quantity. Rather, an explicit goal of this study was to understand and explain the extent to which the participants also understood these situations as involving intensive quantities.

CHAPTER 2

FRAMEWORKS, CONCEPTUAL CONSTRUCTS, AND BACKGROUND

My development of, and attempts to answer, these research questions have been guided by a collection of complementary frameworks. These range from global philosophical positions that underpin my worldview to very specific conceptual constructs that frame my interpretation and analysis of the students' mathematical activity. This chapter provides an overview of these frameworks and proceeds from broad notions of knowledge and reality to the more specific constructs that underpin the design, conduct, analysis, and conclusions of this study.

An Unknowable Reality

> Certain truth [about God or the world] has not and cannot be attained by any man; for even if he should fully succeed in saying what is true, he himself could not know that it was so. (Xenophanes, Fragment 34; as cited in von Glasersfeld, 1995, p. 26)

Central to all aspects of my life is the belief that an ontological reality is unknowable. This is not to deny that reality exists in some form and fashion nor to suggest that there are not certain truths about the nature of the reality we live in. Rather, this is simply a claim that from the position of an individual acting within and experiencing reality first-hand, evaluating the veracity of one's knowledge of reality presents an impossible task. To do so would require direct access to reality to provide some basis for comparison between the knowledge constructed from one's experiences and the ontological truths of reality. However, as individuals, our connection to this unknowable reality exists solely in our experiences. Hence, our knowledge of reality is both developed from, and constrained by, our perceptions of our experiential reality and our creativity in organizing these experiences in meaningful ways.

In terms of my role as a teacher and mathematics education researcher, this means that I take students' mathematics (Steffe, 2007) as the starting point for the way I think about teaching, learning, and research in mathematics education. However, the implication of an unknowable reality is that not only does my experiential reality not provide access to ontological truths, but it also does not provide direct access to my students' mathematical understandings. Thus, taking students' mathematics seriously means accepting that students have mathematical realities that are independent of my own ways of knowing. Accepting students' mathematical realities, I regarded this study into students' construction of intensive quantity as an opportunity to contribute to the construction of the mathematics of students, which entails building second-order models that are derived from my experiences of students' mathematical activity (Steffe, 2007). However, to describe what these models consist of and how I understand them first requires the elaboration of several more specific conceptual constructs that fit within this global framework of an unknowable reality.

A Foundation of Radical Constructivism

Given these perspectives on the nature of reality, I find that the theory of Radical Constructivism provides a framework that adheres to these overarching perspectives yet also offers a way to consider how an individual comes to develop mathematical knowledge. In characterizing the theory of radical constructivism, von Glasersfeld (1995b) stated, "Radical constructivism is intended as a model of rational knowing" (p. 24). Two aspects of this quote are central to my approach. First, I continually find von Glasersfeld's characterization of the theory as a model of *rational* knowing as orienting for my thinking. It reminds me to consider a student's ways of reasoning as rational and to build models of his/her reasoning that can account for this internal rationality. Secondly, by focusing on modeling the processes of knowing and learning, the theory of radical constructivism highlights the perspective of an individual and, thus, is well suited for the type of questions I set out to answer.

As a consequence of my worldview, I believe that knowledge exists within the minds of individuals, not out in the world as some pre-existing knowledge we come to learn. This viewpoint is both compatible with, and informed by, the central tenets of the radical constructivist perspective. According to von Glasersfeld (1984), "Knowledge does not reflect an 'objective' ontological reality, but exclusively an ordering and organization of a world constituted by our experience" (p. 24). Given that it is the individual who carries out this ordering and organization, knowledge cannot be the result of a passive receiving but instead originates as the product of an active subject's cognitive activity. However, in light of the quote above, this activity must be interpreted not as activity with objects that possess properties and structure prior to being experienced. Rather, the activity that builds up these properties and structures is called *operating*, and "it is the operating of the cognitive entity which, as Piaget has so succinctly formulated, organizes its experiential world by organizing itself" (pp. 31–32). Thus, I view mathematical constructions, like all constructions, as created by the mind and as constructions that serve to structure and filter one's experience of mathematical situations and tasks.

The Mechanisms of Learning

However, more than providing a general theory of knowing and learning that aligns with my worldview, the theory of radical constructivism elaborates several conceptual constructs that I use in my attempts to explain students' ways of operating. In particular, von Glasersfeld developed a theory of knowledge as existing in three-part goal-oriented schemes and elaborated concepts such as assimilation, accommodation, and reflective abstraction as the mechanisms by which one implements and modifies these schemes in the face of new experiences. Each of these concepts plays a central role in my thinking. Thus, the following sections provide an overview of how I understand these cognitive mechanisms and attempt to clarify how I use particular terminology.

Schemes and operations.

According to von Glasersfeld (1995b), the knowledge that an individual constructs from organizing his/her experiences exists in three-part action schemes. In short, a scheme consists of a perceived situation, an activity, and an expected result. Further, whenever mental schemes and operations are used, it is with a specific purpose in mind. Thus, this action is considered a goal-directed response (von Glasersfeld, 1980).

More specifically, the first part of a scheme includes a perceived situation or collection of situations to which the scheme applies. Further, given the goal-directed nature of schemes, an individual uses a scheme in an attempt to accomplish something. Thus the perceived situation includes specific goals for which an individual considers a scheme useful. I refer to the second part of a scheme as *the activity of a scheme* or *the operations of a scheme*, and this includes a specific activity associated with the perceived situation that the individual anticipates will accomplish his/her goals. The activity of a scheme can include both mental and/or physical activity and actions that an individual carries out in pursuit of his/her goals. The final component of a scheme includes an expectation that the perceived situation and associated activity will lead to some expected and previously experienced desirable result that will accomplish one’s goals.

There are a few important things to note regarding the nature of schemes and what I mean by my use of this terminology. First, while I find schemes to be a useful way of characterizing knowledge, an individual is often unaware of the particular features of the schemes with which he/she operates. Thus, a scheme is not so much a construct that explains literally how people store and access knowledge, but rather it is an observer's model that attempts to account for the process of knowing. Secondly, schemes can vary greatly in their complexity. The more or less complicated the perceived situation and goal, the more or less complex the activity of the scheme can be.

I consider *mental operations* as schemes that reside on the most basic end of this spectrum of scheme complexity. By using "basic," I do not mean to convey that operations lack complexity or that they are necessarily constructed and used by all individuals in the same way. Rather, I use basic in the sense that use of a mental operation accomplishes something fundamental and can be thought to happen in an all-at-once fashion.

For example, consider the unitizing operation. von Glasersfeld (1981) defines *unitizing* as consisting of "The differential distribution of focused and unfocused attentional pulses" (p. 87). This mental operation of differentially focusing one's attention on different sensory-motor signals is posited as the operation that enables one to isolate specific aspects from the flow of one's perceptible experience. Thus, the re-presentation of past experiences in one's visualized imagination is made possible in part by use of the unitizing operation. Similarly, the unitizing operation plays a central role in models of the construction of object concepts, numerical quantities, and children's counting schemes (Steffe, 2010c; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

As schemes become increasingly complex, the activity of the schemes similarly grows in complexity. For complex schemes such as those I am investigating related to the construction of intensive quantity, it is common for the activity of these schemes to include combinations of mental operations, physical action within one's environment, and even other schemes. Thus, when I characterize one scheme as *incorporating* a second scheme, I am referring to the second scheme becoming a component of the activity of the first scheme.

Assimilation, accommodation, and abstraction.

While I view the above notions of schemes and operations as useful ways of characterizing students' knowledge, I find that the constructs of assimilation, accommodation, and abstraction are helpful for capturing the dynamic nature of learning and knowledge in development. I draw my understandings of these concepts from several of von Glasersfeld's characterizations of radical constructivism (von Glasersfeld, 1984, 1991, 1995b).

In general terms, I consider *assimilation* to be a process that explains perception. In the course of experience, one constantly receives sensory input from the environment. Assimilation is the process by which one interprets the various incoming sensory material in relation to his/her existing knowledge. Further, assimilation is an active process rather than a passive one. As von Glasersfeld (1980) explains, "The process of assimilation does not discover recurrent sensory patterns but it imposes them by disregarding differences" (p. 82). Thus, it is through assimilation that one develops a mental perception of an experience and, in doing so, activates knowledge schemes relevant to that perception.

However, more than constituting perception, assimilation can also serve as a tool of learning. For example, suppose that you had constructed a concept of apples but were just now encountering a green apple for the first time. In assimilating this experience, you might decide that even though you had never seen this particular color of apple before, the other available sensory input matches all other aspects of your apple concept. Thus, you assimilate this never before experienced object to your existing apple scheme, decide to taste the apple, and observe that the flavor and texture match your expectations developed from previous experiences with apples. In turn, the range of experiences to which your apple concept applies has expanded as a result of this generalizing assimilation.

Accommodation represents a second mechanism of learning in which one modifies his/her conceptual structures on the basis of negative feedback rather than confirmation of one's expectations. For example, suppose that in the previous example the experience of biting the apple failed to match your expectations. This might lead to a re-evaluation of the available sensory material and a recognition of new features of the fruit that were previously unattended to. The process of accommodation consists of the incorporation of these newly noticed features into one's existing knowledge. This process can lead to either a modification of existing schemes and object concepts or the creation of new ones that account for the new information. Regarding accommodation, von Glasersfeld (1995b) also states, "The same possibilities are opened, if the review reveals a difference in the performance of the activity, and this again could result in an accommodation" (p. 66).

Central to these discussions of assimilation and accommodation is the issue of one's expectations. In assimilating a situation to an existing scheme, one is essentially saying, "I think I recognize this situation, so I know how to act, and I think I know what will happen when I carry out that activity." Accommodation only happens if carrying out the planned activity fails to meet some aspect of the result one anticipates will occur. I use "*some* aspect of the result" because I believe that often a combination of these processes occurs simultaneously. Consider again the green apple. The green apple may meet your expectations for general taste and texture. However, you may also notice this green apple is tarter than any other apple you have ever tasted before. This state of partial confirmation of your expectations could lead to a generalizing assimilation that this green apple is still an apple and also an accommodation to your knowledge via the creation of a sub-scheme for green apples.

Also critical in the development of mental schemes and operations are processes of *reflection* and *abstraction*. Reflection on experiences is made possible by the ability of the mind to unitize sensory input and re-present (i.e., present again) that information to oneself mentally. Further, this re-presentation includes not only the perceived situation but also the individual's activity and its consequences. These processes allow one to compare two experiences side by side as if they co-occurred and makes abstracting concepts and schemes from these experiences possible.

However, this abstraction from experiences has multiple meanings. One might make an empirical abstraction by isolating commonalities from one's experience of things to form permanent object concepts. Alternatively, abstraction can also involve constructing relations between previously constructed object concepts and schemes and, hence, be considered a reflective abstraction. Reflective abstractions can be further categorized into one of three types—reflected, reflective, and pseudo-empirical—depending upon whether or not one is aware of the abstractions and whether the abstractions depend upon particular sensory-motor signals, respectively (von Glasersfeld, 1991). These distinctions become important to me when attempting to model subtleties in my students' mathematical constructions.

In relation to my purposes for this study, the construction of intensive quantities, which characterize relationships between two other quantities, falls into this category of arising from reflective abstractions. Further, intensive quantities represent an example of what Piaget referred to as a logical-mathematical abstraction. He described these abstractions in the following way:

> In the case of logical-mathematical abstraction…what is given is a set of the subject's own, already available actions or operations and their results. […] Properly speaking,

> then, logical-mathematical construction is neither invention nor discovery; as it comes about through reflective abstraction, it is a construction in the proper sense of that term, which is to say, it produces new combinations. (Piaget, 1967; as cited in von Glasersfeld, 1982, p. 628)

I cannot emphasize enough the critical role this characterization of logical-mathematical abstractions plays in shaping my approach to investigating how students learn intensive quantities. To me, it suggests that intensive quantities are not simply invented to describe some completely new phenomenon, nor are they discovered as some pre-existing fact of nature. Rather, intensive quantities are constructed in the mind as a new relationship between other schemes and operations that one has already constructed. Therefore, the construction of intensive quantities can be considered a reorganization of one's existing ways of operating. Thus, my approach throughout the study involved attempting to understand what existing schemes and operations the students had available and how they used these to make sense of, and operate rationally in, situations involving intensive quantity.

Dynamic equilibrium and the viability of knowledge.

von Glasersfeld (1980) uses the concept of an active or dynamic equilibrium as an analogy to the role of cognition in maintaining stable conceptual structures. Much like a bird maintaining its perch on a wire on a windy day, cognition actively modifies and maintains one's conceptual structures in the face of a constant stream of incoming sensory data from one's experiences. Further, just as a strong gust of wind requires some compensatory act for the bird to maintain its balance, incoming sensory data that do not fit neatly into one's conceptual structures must be dealt with to maintain cognitive equilibrium. Cognition, then, serves to mitigate these *perturbations* through the processes of assimilation and accommodation in order to maintain stable knowledge structures and ways of operating.

Considered in relation to the theory of an unknowable reality, the implication of considering knowledge as existing in a state of dynamic equilibrium is that viability and fit define this equilibrium rather than match. Accordingly, the viability of one's knowledge does not depend upon matching some pre-existing truths about reality but rather upon its usefulness for achieving one's goals and for maintaining a state of cognitive equilibrium (von Glasersfeld, 1980, 1984, 1995b). In turn, the effective mental schemes and operations that survive and remain viable are purposive rules that were established through one's previous attempts to organize his/her experiential reality and are maintained because of their use in neutralizing perturbations and in enabling one to make sense of new experiences.

In consideration of these processes, perturbations play an essential role in attempting to understand and model students' conceptual structures. According to von Glasersfeld (1980), "Organisms act as a result of perturbations—and perturbations are not just inputs but only such inputs as upset the organism's equilibrium" (p. 76). Further, these perturbations could arise either as a consequence of one's assimilation or of one's attempts to use his/her available schemes. For instance, one could assimilate a situation and form a particular goal but find himself/herself in a state of disequilibrium because of a lack of the schemes and operations needed to accomplish the goal. Alternatively, one might use a particular scheme to try to accomplish the goal but obtain a result that fails to match his/her expectations. Because perturbations are only those sensory inputs that upset one's dynamic equilibrium, identifying what one considers perturbing helps to delineate the limits of his/her conceptual structures. Thus, the better one becomes at identifying and predicting situations and tasks that will and will not perturb a student's ways of reasoning, the closer one is to having a viable model of that student's ways of reasoning.

Building Models of Student's Knowledge

Having elaborated the terminology I use to describe knowledge and learning, I turn my attention to characterizing the types of models I attempt to build of my students' mathematical knowledge. Steffe (2011) argues that as mathematics educators, rather than attempting to teach students adult conceptions of mathematical concepts, we would be better served by learning "how to engender children's productive mathematical thinking and how to build explanatory models of that thinking" (p. 19). This sentiment captures the essence of my goals during the teaching experiment and analysis, respectively.

One of the central concerns that arises in the course of building models of mathematical thinking is the issue of perspective. One way of conceptualizing models of thinking is to consider them as the models that an individual creates to organize and control his or her own experience. Alternatively, if that experience includes another person, the individual might also impute ways of reasoning different from his or her own to the other and, hence, build a model of the other's thinking (von Glasersfeld, 1995a).

The concepts of first- and second-order models provide a language for distinguishing between these perspectives, respectively (Steffe, 2007). Necessarily, everyone only has access to his/her own experiences and, thus, constructs his/her own first-order knowledge. Just as the truths of an ontological reality are not directly accessible, so too with the first-order models of another. Thus, the personal first-order mathematical knowledge of students is referred to as *students' mathematics*. This first-order knowledge consists of the mathematical constructions one considers viable ways of structuring and managing his/her mathematical experiences. In contrast, any models I build to characterize and explain the first-order mathematics of my students represent second-order models and are referred to as the *mathematics of students*. In terms of this study, this means that my goal is to make inferences about my students' mathematics to develop second-order models that characterize and explain their understandings of intensive quantities.

However, this begs the question of what basis these inferences will be drawn from and the models built upon. Practically speaking, students' observable activity—their verbal descriptions, utterances, inscriptions, drawings, physical movements, etc.—constitute the only available evidence of their first-order understandings. In light of this, I find a quote from Wittgenstein orienting when trying to interpret my observations of students' mathematical activity. He states, "What is the criterion for the way the formula is meant? Presumably the way we always use it" (Wittgenstein, 1983, p. 36). Replacing "formula" with any other mathematical concept or object of interest, this suggests to me that one's observable activity holds clues for understanding his/her perception of the concept.

Thus, while a student's observable activity represents the starting point for my inferences about his/her mathematical thinking, the actual inferences I draw pertain to the student's assimilation of particular situations. Because the student's assimilation involves fitting the current experience into the conceptual structures he/she has already constructed, the student's assimilation of a situation both activates, and is constrained by, his/her existing knowledge. Consequently, making inferences regarding one's assimilation of a situation is tantamount to making inferences about the conceptual schemes and operations he/she had available at the time. Essentially, I use my observations of a student's activity to make inferences about the conceptual schemes and operations he/she had available in the moment of assimilation such that those inferred conceptual constructs could account for my observations of his/her observable activity. von Glasersfeld (1995b, p. 78) refers to this cyclical process as *conceptual analysis*.

Because second-order models consist of these inferred schemes and operations, they allow for making predictions about students' activity. Then, through continued interaction one is able to compare these predictions to students' actual mathematical activity as a way of testing and refining the models. Ultimately, a second-order model consists of conceptual constructs from my own first-order knowledge and has been continually constrained and oriented by my observations of students' activity until it becomes stable and viable. As such, social interaction is critical for constructing these models of mathematical knowing, and they only remain viable so long as they are useful for understanding, explaining, and predicting students' mathematical activity.

In essence, I consider the process of constructing a second-order model of a student's mathematics as analogous to the mechanics of conversation. My view of communication of ideas between individuals finds its basis in the concept of *reciprocal assimilations* in which the two individuals attempting to communicate reach a point where each feels he/she understands the content of the conversation in mutually compatible ways (Steffe & Thompson, 2000a; P. W. Thompson, 2013). Thus, communication between individuals is not a direct exchange of ideas but rather an act of constantly adjusting one's model of the other based upon his/her actions and responses until an equilibrium is reached and the model one has constructed of the other remains viable (von Glasersfeld, 1982).

Communication characterized in this way essentially involves four steps: a) Forming a model of the other person in the conversation based upon your prior experiences with that individual and what he/she has said; b) Responding to the other person based upon this model, which could include asking probing questions to see if the other responds in the way the model predicts; c) When necessary, modifying the model to resolve any inconsistencies that arise between the model and the unfolding conversation; and d) Repeating the first three steps until both individuals feel satisfied that they have reached reciprocal assimilations of the topic of conversation (Steffe & Thompson, 2000a; P. W. Thompson, 2010, 2013). At that point, one could be said to understand the other until some future event upsets the equilibrium.

Except, rather than having a conversation between two individuals, constructing a second-order model is like having a conversation between the researcher and the model rather than the researcher and the student directly. Further, this “conversation” with the model is mediated by the observable activity of the student just as would be the case in a conversation with the student directly. However, incorporating all aspects of the student’s mathematical activity, particularly his/her nonverbal activity, allows one to make a model of the student’s mathematics even in cases where the student remains unaware of his/her own ways of reasoning. The second-order models that result from this process of interaction and communication represent the author’s description of the inferred reciprocal assimilation that results from this back-and-forth interaction.

Because it is impossible to see the world through the students’ eyes, my goal is to develop a model of the mathematical schemes and operations that could reasonably explain and account for the assumed internal rationality of a students’ mathematical activity. However, even once a possible model is developed, the model does not represent the only possible explanation, nor is it considered “truth” in the ontological sense. Rather, the model presents one possible way of knowing in which the child’s actions and activities do make sense. Thus, when imputing a scheme to a student, I am not claiming that his/her mental activity carries on exactly as described. Rather, I am making a claim that I have no indication to suggest that the student’s activity functions differently than the model suggests.

Ultimately, the goal of pursing models of students' ways of reasoning and knowing is to construct scientific explanations of students' understandings of intensive quantities. von Glasersfeld's (1982) characterization of second-order models speaks to their potential use in scientific explanations of knowledge:

> When certain cognitive structures, then, prove viable not only in the subject's organizing and ordering of its own experience, but also as the means of organizing ascribed to the models the subject constructs of "others" and *their* effort to organize and order *their* experience – then these doubly viable constructs acquire a value that can be called "objectivity." (p. 632)

Thus, second-order models become objective in the sense that they are not only viable in the model-maker's own mind, but they are also continually constituted and refined through interaction with others.

Further, I consider second-order models to be scientific explanations in the sense that they satisfy Maturana's (1987) requirements of a scientific explanation. First, they provide a mechanism by which the mathematical understandings they pertain to are constructed by describing mental schemes and operations that account for those understandings. Second, Maturana (1987) also states that the proposed mechanism must be a useful tool for prediction and that the proposed mechanism should "generate not only the phenomenon you want to explain, but other phenomena that you may observe as well." Considered with respect to models of students' mathematics, this statement has two implications. In particular, the models that I construct should allow me to make predictions regarding a) how a student might act in situations different than the one from which the models were abstracted; and b) how other students to which the model has been attributed might reply in the same situation from which the model was abstracted.

Specific Conceptual Constructs

While the central constructs of the radical constructivist theory of knowing provide tools with which to explain the mechanism of learning, they do not speak to the substance of learning. Thus, in order to use concepts such as schemes, operations, assimilation, accommodation and abstraction in my attempts to develop second-order models students' mathematics, I turn to the works of previous scholars who have elaborated models of particular types of mathematical knowing.

Lamon (2007) points out that previous studies have incorporated a variety of definitions for terms such as ratio and rate and urges researchers to define their specific use of terminology in order to clarify their intended distinctions. I find that making the distinction between students' mathematics and the mathematics of students has a significant influence on how I define several terms relevant to the current study of intensive quantity. With students' mathematics in mind, I adhere to definitions of mathematical terms and concepts that take into account the perspective of an individual knower. The following sections specify those concepts that play a central role in the current study.

Quantity

In considering reasoning with various types of quantities and the quantitative operations one employs during such reasoning, I align with P. W. Thompson's (1994) definition of a quantity as a scheme consisting of an object concept, a measureable property of that object concept, and an appropriate unit and process for assigning a numeric value to the property. This definition situates a quantity as a construction of an individual knower rather than as an entity that exists in an ontological reality independent of one's experience of it.

I find this characterization of quantity as crucial in that it allows for the possibility that different individuals can perceive quantities differently within a mathematical context even though each assimilates the “same” experiential situation. This includes the possibility that different individuals perceive of different quantities (e.g., distance, time, speed, etc.) and that they might form different conceptions of the same quantity. In terms of this study, even though I considered all the mathematical tasks I presented my students as involving intensive quantities, I make no claim that my students necessarily considered them as such. As a result, it is more precise to say that through assimilation I imbued intensive quantities to my perception of the mathematical tasks, and my goal was to find out whether or not this was the case for my students as well. Indeed, investigating and attempting to account for differences in perception of the quantities involved in various mathematical tasks was the central goal of this study.

Quantitative Operations

My primary method of describing and accounting for these differences involved developing models of my students’ available quantitative operations. My concept of quantitative operations is again informed by P. W. Thompson (1994) but also includes a subtle, yet important distinction. He defines a quantitative operation as “A *mental operation* by which one conceives a new quantity in relation to one or more already-conceived quantities” (p. 185). Mental operations such as unitizing or segmenting (Steffe, 1991b) and additive or multiplicative combinations and comparisons (P. W. Thompson, 1994) are examples of quantitative operations.

Further, P. W. Thompson (1994) describes quantitative operations as the mental operations responsible for one’s comprehension of a situation and distinguishes them from the numerical operations one uses to evaluate a quantity. This formulation leads to a distinction between the quantitative operation of *combining two quantities additively* and the numeric operation of *addition*. P. W. Thompson (1994) goes on to provide an example that helps to clarify what he might intend by this distinction, stating

> Therefore, arithmetic notation has come to serve a double function. It serves as a formulaic notation for prescribing evaluation, and it reminds the person using it of the conceptual operations that led to his or her inferences of appropriate arithmetic. (p. 188)

Following this reasoning, in the case of addition, the symbol “+” is often assumed to refer to both conceiving of a combined collection that has some unknown numerosity as well as the operations used to actually make definite the unknown numerosity of the collection.

However, in my thinking, the distinction between one’s perception of a situation and the activity one carries out based upon that perception does not always clearly differentiate between quantitative and numeric operations. Therefore, in order to clarify my way of using these terms, I offer the following example: A student is given a collection of 17 buttons, then shown a second collection of 8 additional buttons, and asked to determine how many buttons there are in total. To solve this task the student writes the following on his/her paper, “$+\begin{array}{r} 1 \\ 17 \\ 8 \\ \hline 25 \end{array}$”, and decides upon 25 buttons.

What counts as quantitative operations, and what counts as numeric operations? I make the claim that the student *could* be considered as using quantitative operations in conceiving of the sum and in quantifying the sum. To justify this claim, consider the following hypothetical explanation the student might give to explain his/her reasoning:

> I decided to add because I wanted to figure out how many buttons I would have in total if I had 8 more than the 17 I already had. Because I know that 7 and 3 are 10, I carried the 1 to keep track of that 10. Then, I only had 5 ones left, and along with the 2 tens I knew there would be 25 buttons total.

In this case, while the student’s written notation appears procedural, the reasoning that it referred to was anything but. Rather, in using strategic reasoning to carry out the addition, I would infer that the student in this case used quantitative operations both to conceive of and to quantify the sum. This example is intended to demonstrate that even the symbol "+" *could* symbolize quantitative operations for an individual as the student conceives of a new quantity (i.e., 25) in relation to one or more already conceived of quantities (i.e., 17 and 8).

As a result, I do not consider a student's observable activity and use of symbolic notation, such as "+", as evidence that differentiates between quantitative and numerical operations. Rather, for me the distinction between quantitative and numerical operations makes the most sense when considered with respect to school mathematics. As I stated earlier, the student *could* have used quantitative operations both in conceiving of, and quantifying, the sum. However, I believe that it is also possible that the student did not rely upon quantitative operations for either of these. To explain my reasoning, consider the following alternative hypothetical explanation for the student's activity:

> Initially I wasn't quite sure what to do. But in class we've been doing a lot of problems where we just add up the numbers in the word problem to find the answer, and so I decided to just try adding the 17 and 8. Then I just added 7 and 8 to get 5, carried the 1, and added 1 plus 1 to get 2. So that's how I got 25.

In this case, both the student's conception of the situation and the computational activity relied upon routine and learned procedures and were not based upon some underlying reasoning with the quantities. Thus, I would consider this student to have used numeric operations both in conceiving of and quantifying the sum. I consider these as numeric operations because the student's reasoning was about the learned computational procedure of addition per se rather than anything to do with the relationships between the quantities.

The elaboration of this example is intended to demonstrate that for me the distinction lies not with the specific observable activity one carries out to evaluate a quantity but rather with the manner in which he/she assimilates the situation to those activities. Consequently, I consider quantitative operations to include arithmetic operations so long as, for the individual using them, those operations refer to quantities and their relationships rather than simple marks on a page involved in carrying out some computational procedure that is divorced from the quantities themselves. In the case where a student's activity involves a learned procedure connected to genuine quantitative assimilative operations, one can refer to that student's scheme as a *procedural scheme* (L. P. Steffe, personal communication, May 8, 2015). If I infer that the basis for one's activity stems from a quantitative comparison of the quantities, then I refer to the activity one carries out as quantitative reasoning and the mental operations responsible for the content of that reasoning as quantitative operations.

Extensive and Intensive Quantities

This study was not about how students develop conceptions of quantities in general but rather how students use their available quantitative operations to construct operative understandings of a particular type of quantity whose common examples include rate of change, density, and concentration. Thus, I found it useful to distinguish two different categories of quantities and quantitative operations—extensive and intensive. While I have previously presented a characterization of this distinction (cf. Chapter 1), I would like to briefly elaborate upon these constructs as a way of framing the results of previous efforts to develop second-order models of students' mathematics.

One way to distinguish between extensive and intensive quantities is based upon the additive or multiplicative nature of the quantities, respectively (cf. Jahnke, 1983; Kaput, 1985; Schwartz, 1988). Thus, given a uniform pile of coffee beans, the extensive quantities *price* and *pounds* both decrease if the size of the pile shrinks while the intensive quantity *price per pound* remains constant. This characterization defines the concept of an intensive quantity on the basis of the structural relationships between quantities and their referents.

However, while I find this way of characterizing extensive and intensive quantities personally useful as a first-order concept, I also find it limiting when attempting to build second-order models of students' mathematics. For example, consider the language Schwartz (1988) uses when distinguishing these two types of quantities. He states, "Referent transforming compositions *force us* (emphasis added) to distinguish between two rather different kinds of quantity, *extensive* and *intensive* quantity" (p. 41). Thus, a student using the referent transforming numerical operations of multiplication and division would be considered to necessarily produce an intensive quantity. However, this description positions an intensive quantity as existing a priori and leaves no room for considering that students may not assimilate it as such.

When considering what a second-order model of extensive and intensive quantities might consist of, I turned to von Glasersfeld and Richards (1983). They provide an orienting interpretation of how Gauss understood extensive quantities as relations and state

> These relations he calls "arithmetical" and in arithmetic, he explains, quantities are always defined by how many times a known quantity (the unit), or an aliquot part of it, must be repeated in order to obtain a quantity equal to the one that is to be defined, and that is to say, one expresses it by means of a number. (pp. 58–59)

The central distinction between this conception and those presented above is that extensive quantities are defined through a relationship to a unit rather than as a quantity that behaves additively. Interpreting this in terms of a second order model for intensive quantities, I consider a student to have conceived of an extensive quantity if I am able to infer that this conception is based upon a relationship between a quantity's measure and a repeatable unit.

Building upon this, I make a similar distinction regarding intensive quantities. I accept the notion of an intensive quantity as a multiplicative relationship between two extensive quantities. However, I locate the creation of that multiplicative relationship within the mind of an individual rather than within a structural analysis of tasks or the use of the symbols "•" or "÷." Thus, I consider a student to have conceived of an intensive quantity if I am able to infer the following: a) That the student has imbued a multiplicative relationship to two quantities involved in a task; and b) That the quantities involved in that multiplicative relationship are extensive quantities in the sense described in the previous paragraph.

Extensive and Intensive Quantitative Unknowns and Variables

Regarding these extensive and intensive quantities, an important distinction exists between quantitative unknowns and variables on the basis of the static or dynamic nature of the relationships between quantities (Steffe, Liss II, et al., 2014). With respect to extensive quantities, an extensive quantitative unknown refers to the potential result of measuring a *fixed* but unknown extensive quantity before actually measuring it. In contrast, an extensive quantitative variable is based upon Russell's (1903) characterization of variable in which 'any number' can be thought of as representing any but no particular number. Thus, an extensive quantitative variable is the potential result of measuring a *varying* but unknown extensive quantity at *any but no particular time*.

In terms of intensive quantities, I make similar distinctions between an intensive quantitative unknown, an intensive quantitative variable, and a basic rate scheme (Steffe, Liss II, et al., 2014). Given a particular ratio of two extensive quantities, an intensive quantitative unknown refers to the potential result of enacting the operations that produce a fixed but unknown equivalent ratio. Further, there is a sense of definiteness to the unknown because the construction of an intensive quantitative unknown implies the availability of the operations needed to produce the equivalent ratio and, thus, is considered a proportionality scheme. I have also observed students who acted with an awareness of an equivalent but unknown ratio yet did not have the quantitative operations available to make the unknown ratio definite. In these cases, I consider the student to have constructed an *awareness of proportionality*. In contrast, given a particular ratio of two extensive quantities, a child who has constructed an intensive quantitative variable can enact the operations to transform this basic ratio into any but no particular equivalent ratio using any but no particular partitioning. Lastly, given two extensive quantities that covary in such a way that conserves a unit ratio, a rate is the result of enacting the operations that produce a ratio equivalent to the unit ratio at any but no particular time.

The primary distinction between the last two constructs lies in the variability. With the intensive quantitative variable, there is an explicit awareness regarding the variability of the partitioning activity, and the produced ratios at least implicitly imply change in the extensive quantities. In contrast, with the basic rate scheme, there is an explicit awareness with respect to the covariation of the extensive quantities, and changes in the partitioning activity needed to produce the equivalent ratio only become explicit when the covariation stops and the extensive quantities are measured.

To exemplify these three intensive quantitative concepts, consider a lemonade mixture scenario in which 2 tablespoons of lemonade powder are mixed into 3 cups of water. If asked for the number of tablespoons of lemonade powder that would be needed for 5 cups of water in order to produce a lemonade with the same taste, a student who has constructed an intensive quantitative unknown would have the operations available to produce the unknown, but equivalent, ratio of ten-thirds tablespoons of lemonade powder per 5 cups of water. In contrast to the intensive quantitative unknown in which a specified value of one of the extensive quantities (e.g., 5 cups) defines a specific partitioning, with the intensive quantitative variable there is an awareness that any but no particular partitioning can be chosen to produce another equivalent ratio. Lastly, suppose that the lemonade is being mixed at a factory where water and lemonade powder are pouring into a bulk lemonade mixing tank so that no matter when one stops adding lemonade powder and water the taste of the lemonade mixture will always be the same. An individual who has constructed a basic rate scheme will be aware that at any but no particular point in time the ratio of the two quantities will be equivalent to the unit ratio of two-thirds tablespoons of lemonade powder for 1 cup of water. However, the particular partitioning that produces the equivalent ratio remains implicit until a specific value of the measure of one of the quantities is given.

This distinction between quantitative unknowns and variables is similar to the distinction P. W. Thompson (1994) makes between ratios and rates. In contrast to many other scholars who have investigated these quantities (cf. Kaput & West, 1994; Lesh, Post, & Behr, 1988; Ohlsson, 1988; Schwartz, 1988; Vergnaud, 1988), P. W. Thompson distinguishes them on the basis of the particular mental operations that produce them rather than on any particular features of the task or numeric operations involved. P. W. Thompson defined a ratio as the result of a multiplicative comparison of two quantities and a rate as a reflectively abstracted constant ratio. Given this perspective, the types of quantities involved in the comparison do not determine whether a particular comparison is a ratio or a rate. Rather, the multiplicative nature of the comparison and the degree to which the result of this comparison has been abstracted from the particular situation in which it was constructed represent the relevant factors in defining the concepts.

Further, a ratio can be thought of as the result of a multiplicative comparison of two static quantities whereas a rate signifies a multiplicative comparison that characterizes a constant relationship between two dynamic quantities that covary (P. W. Thompson, 1994). Considering this in light of the previous characterizations of unknowns and variables, the result of determining an intensive quantitative unknown can be considered a ratio by P. W. Thompson's definition. Similarly, the result of implementing one's basic rate scheme can be considered a rate according to P. W. Thompson's characterization.

In each case, this static or dynamic quality of the quantities is conceptually introduced by the individual and not by the nature of the quantity, context, or task per se. For instance, a speed such as 60 miles per hour is typically assumed to represent an example of a rate. However, if a student assimilates 60 miles per hour as indicating that one has travelled exactly 60 miles in exactly 1 hour, then he/she has not formed a dynamic conception of the quantities in which the speed characterizes the relationship for any conceivable measure of the quantities. Hence, I would not consider his/her conception of speed in this case a rate.

Specific Second-Order Models of Quantitative Schemes and Operations

Previous research regarding the ontogenesis of students' mathematical knowledge has contributed a broad collection of schemes and operations that characterize students' reasoning with quantities (cf. Hackenberg, 2007, 2010; Steffe, 1988, 1992; Steffe & Olive, 2010; Steffe et al., 1983). This study draws upon that research base to help clarify the role that these schemes and operations play in the construction of intensive quantities. However, in pursuing an understanding of the ontogenesis of intensive quantities, it has been helpful to classify the schemes and operations that have been elaborated in previous research based upon the extensive or intensive nature of the quantities involved in each. What follows is not an exhaustive categorization of all previously identified quantitative schemes and operations. Rather, I have included only those schemes, operations, and conceptual constructs most relevant to the current study.

Pre-extensive quantitative schemes.

There are some quantitative schemes that children construct that can be considered as pre-extensive quantitative schemes. These are still quantitative schemes. It is just that the quantities students operate with when using these schemes are not yet extensive quantities. For instance, consider the child's initial construction of number (Steffe, 2010c). For a child who has constructed an initial number sequence, the number sequence represents a series of interiorized counting acts and, thus, a number word such as *seven* signifies the act of counting from one up to seven. This interiorization of one's counting activity is what allows a child to take an initial segment of a counting activity as given and count on from that number to determine, say, what number is 3 more than 7 by counting "Seven…eight, nine, ten".

However, as an interiorization of the activity of counting pluralities of perceptual or figurative unit items, providing meaning for a number word with an initial number sequence involves an awareness of a plurality of individual unit items that, if counted, would have numerosity equal to the given number word. The implication is that numbers produced by a child's use of his/her initial number sequence are not yet extensive quantities as they do not represent relations to a unit in the sense described above.

For similar reasons, an equi-partitioning scheme can also be considered a pre-extensive quantitative scheme. An equi-partitioning scheme is a second-order model of the reasoning one uses to partition a whole into any given number of parts, say *n* parts, and construct one of those parts as one-n^{th} of the original whole (Steffe, 2010e). Supposing *n* is seven, to accomplish this a child could use his/her number concept for seven as a partitioning template for marking off one of the seven equal parts of a continuous whole. Further, the child could conceptually disembed this marked-off part without destroying the whole from which it was produced. Lastly, the child could engage in iterating this part and use progressive integration operations to produce a 7-part segmented whole to compare with the original continuous whole to test the adequacy of his/her initial mark. Provided that the 7-part segmented whole and the original unit are of equal length, the child would then know that the marked off part was one-seventh of the original continuous unit. Even though the child engages in iterating the part to produce this awareness, the basis for calling the marked off part one-seventh comes from a part-whole comparison of the number of parts. For this reason, it does not satisfy the characterization of an extensive quantity presented above, and I consider an equi-partitioning scheme a pre-extensive quantitative scheme.

Further, I also consider a partitive fraction scheme (Steffe, 2010e), which incorporates an equi-partitioning scheme as its constitutive operation, a pre-extensive quantitative scheme. Similar to the unit fractions produced with an equi-partitioning scheme, for a child that has constructed a partitive fraction scheme, the meaning for a fraction such as three-fifths resides in a part-whole comparison rather than in relation with a unit fraction. Thus, to a child operating with a partitive fraction scheme, three-fifths refers to 3 out of the 5 partitions of a continuous unit rather than a number that is 3 times as large as one-fifth of the continuous unit.

It is important to note that these three number, partitioning, and fraction schemes represent significant constructive achievements for children in their own right (Steffe, 2013). However, because the quantities involved in each do not satisfy my definition of extensive quantities, each falls into the category of pre-extensive ways of operating. Thus, the reorganization of one’s quantitative schemes and operations into extensive quantitative schemes and operations should not be taken as given and involves several metamorphic accommodations to the ways of operating described here.[1]

Extensive quantitative schemes.

In contrast, constructing one's schemes and operations as extensive quantitative schemes reorganizes them in such a way that the assimilating structures and the activity of these schemes become multiplicative in nature. As described earlier (cf. Chapter 1), constructing an explicitly nested number sequence reconstitutes one's numbers as multiplicative relations to an iterable unit of one. This allows a student to assimilate a composite unit such as nine as 9 iterations of 1, or equivalently, as a number which is 9 times as large as 1. As a result, it makes sense to think of the numbers of one's explicitly nested number sequence, which students produce through unit iteration, as extensive quantities and the associated schemes as extensive quantitative schemes (Steffe, Liss II, et al., 2014).

I also consider the splitting scheme to be an extensive quantitative scheme because its construction reconstitutes one's equi-partitioning scheme to incorporate a multiplicative relation. The primary distinction between these two schemes is that the mental operations of partitioning and iterating that are carried out sequentially in the equi-partitioning scheme become available to the student simultaneously during assimilation with the splitting scheme (Steffe, 2010d). The implication of this accommodation is that assimilating an experience as a situation of one's splitting scheme simultaneously implies a completed partition into a given number of parts, say *n* parts, and that iterating any one of those parts *n* times would reproduce the split unit. Thus, "The result of the scheme would be an inverse multiplicative relation between the part and the partitioned whole" (Steffe, Liss II, et al., 2014, p. 56). Consequently, a student who has

[1] See (Steffe & Olive, 2010) for in-depth analyses of the accommodations involved in constructing more sophisticated number and fraction schemes and operations.

constructed a splitting scheme can name one partition of this split one-n^{th} of the continuous unit on the basis of this multiplicative comparison rather than the part-whole comparison made using the equi-partitioning scheme.

Constructing an iterative fraction scheme similarly transforms one's fractional concepts into multiplicative concepts. Much like the partitive fraction scheme incorporates the equi-partitioning scheme, the iterative fraction scheme incorporates the splitting scheme as its constitutive operation (Olive & Steffe, 2010b; Steffe, 2010e). Thus, the iterative fraction scheme inherits the inverse multiplicative awareness as well. This transforms one's fractions into fractional numbers (Olive & Steffe, 2010b). For instance, a fraction such as three-fifths is understood as a number that is 3 times as large as one-fifth of a unit. This reorganization of one's fraction concepts allows an individual to conceive of fractions, which now include those beyond the whole, as numbers that exist in a multiplicative relationship to an iterable unit fraction. Hence, the iterable fraction scheme is considered an extensive quantitative scheme.

While use of an explicitly nested number sequence or an iterative fraction scheme transforms one's numbers and fractions into extensive quantities, these quantities are based upon a limited set of iterable units, namely the unit of one and the set of unit fractions. In contrast, a generalized number sequence (Steffe, 2010c) and an iterative fraction scheme for connected numbers (Olive & Steffe, 2010a) are schemes that make use of an expanded set of iterable units. As opposed to the iterable units of one involved in an explicitly nested number sequence, a child with a generalized number sequence can take iterable composite units as given in assimilation and operating. The availability of iterable composite units expands the multiplicative relationships available to the child in that he/she can consider 12 multiplicatively as a number 4 times as large as a composite unit of 3. Similarly, the construction of an iterative fraction scheme for connected numbers enables a child to utilize his/her splitting scheme for connected numbers to take non-unit fractions as iterable units in assimilation and operating. Thus, a fraction such as fifteen-sevenths can be understood as a number that is 5 times as large as three-sevenths.

Levels of units.

There is another important factor in the construction of quantitative schemes and operations that is interwoven throughout all of the pre-extensive and extensive quantitative schemes discussed above yet which has gone unstated to this point—one's ability to construct and reason with multiple *levels of units* structures (Hackenberg, 2010; Steffe & Olive, 2010). In general terms, I understand the concept of one's levels of units to refer to the number of distinct nested units that one can conceive of *simultaneously* in assimilation and operating.

For example, students who have interiorized one level of units have constructed a sequence of elemental unit items (see Figure 2.1). This is akin to an initial number sequence in which the meaning for one's number words resides in the activity of counting from 1 up to the given number (in this example 12). In contrast, students who have interiorized two levels of units, a unit of units, can take a composite unit containing a sequence of elemental unit items as a given in reasoning (see Figure 2.2). For example, a student reasoning with two levels of units could consider 12 as a single composite unit and as a sequence of individual units. Reasoning with two levels of units is consistent with how a student who has constructed an equi-partitioning scheme is able to use composite units as templates for partitioning a continuous unit into a given number of individual units. Lastly, students who have interiorized three levels of units have constructed a composite unit containing a sequence of units of units (see Figure 2.3). As a result, these students can conceive of 12 as a single composite unit, as a sequence of 4 composite units each containing 3 units, and as a sequence of 12 individual units. Recursive partitioning operations are the mental operations that allow one to assimilate situations using this type of structure where all three levels of units are conceived of simultaneously (Steffe & Olive, 2010).

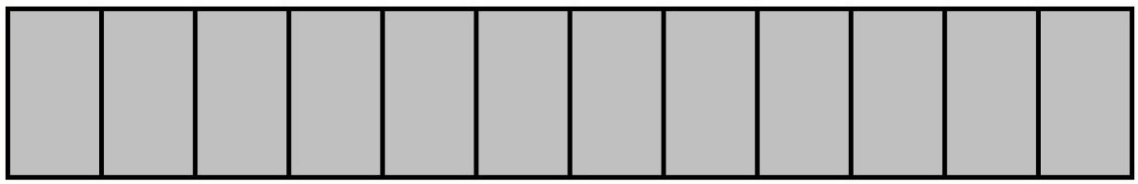

Figure 2.1. One level of units—a sequence of individual units.

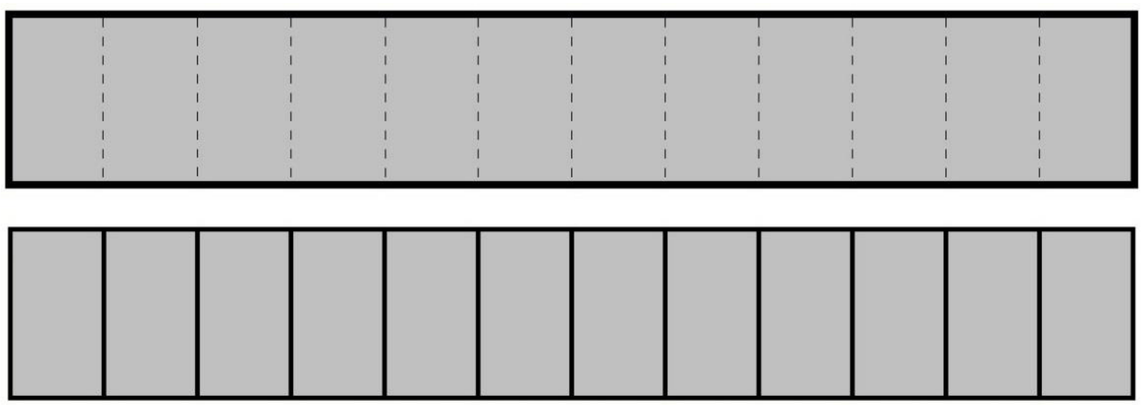

Figure 2.2. Two levels of units—a composite unit containing a sequence of individual units.

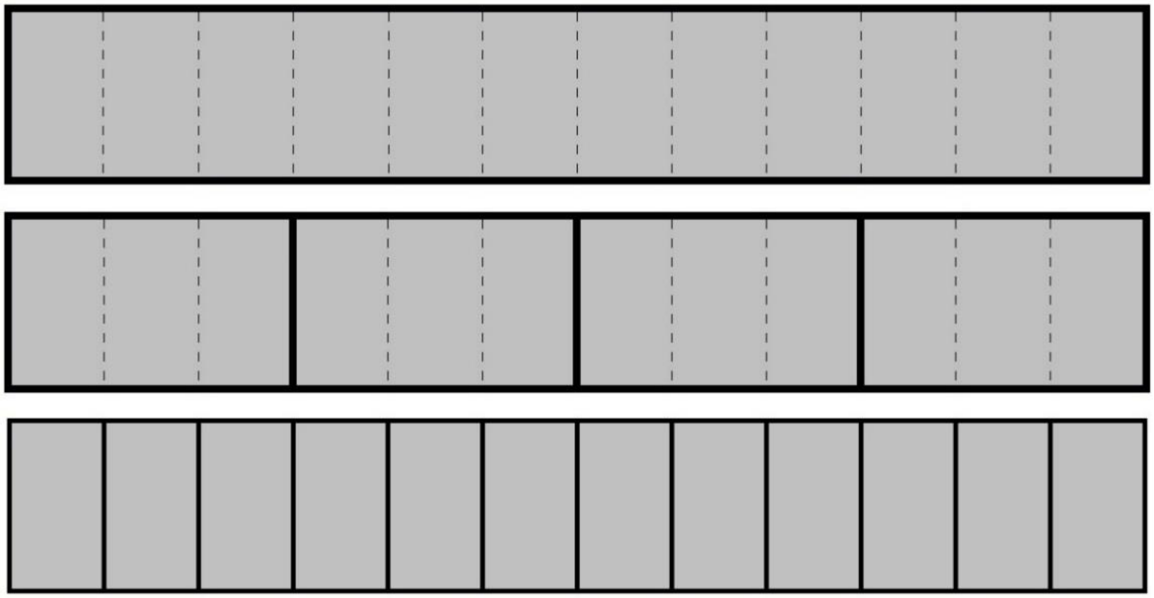

Figure 2.3. Three levels of units—a composite unit containing a sequence of composite units.

It is important to note that Figures 2.1–2.3 present models of levels of units with respect to the consideration of continuous quantities. However, the construct of reasoning with a given number of levels of units applies more generally, and reasoning with a given number of units does not necessarily require continuous units or physical or drawn models. The essential point is that in claiming a student can reason with *x* levels of units, I infer that a student can take *x* levels of units as simultaneously given in assimilation and operating within mathematical contexts.

Occasionally, a student will construct an additional level of units in the course of carrying out mathematical activity that was not initially available to him/her during assimilation of the task. For example, suppose a student with two levels of units was presented the task of determining how many groups of 3 he/she could make from a collection of 12 units. This student would be aware of a composite unit containing 3 individual unit items. Further, the student could engage in iterating this composite unit, use progressive integration operations to incorporate each iteration into a composite unit containing the previous iterations, and monitor their iterating activity to create a structure similar to that presented in Figure 2.3. Thus, the activity would create a three levels of unit structure, and the student could determine that he/she could make 4 groups of 3 from 12 units. However, the important distinction is that the third level of units was not available to the student during assimilation but rather created during the process of carrying out his/her activity. In this case, I would say that the student has two levels of units available in assimilation and operating, and he/she can construct three levels of units *in activity*.

Of the quantitative schemes previously described, having three levels of units available in assimilation is required for the construction of a splitting scheme, an iterative fraction scheme, a generalized number sequence, and an iterative fraction scheme for connected numbers. In their own way, each of these schemes involves constructing and reasoning with a composite unit containing a sequence of units of units. For example, a composite unit of 12 constructed by the operations of a generalized number sequence could consist of a unit of 3 composite units, each containing 4 individual unit items. Because a student with a generalized number sequence imposes this structure upon the situation prior to operating, abstracting this number sequence relies upon having constructed three levels of units. Similarly, each of the other extensive quantitative schemes described earlier also makes use of three levels of units as available in assimilation.

Intensive Quantitative Schemes—Hypotheses That Guided the Study

One of my central questions in this study concerns the mental schemes and operations students use to conceive of intensive quantities. In pondering this question, the research team and I conducted a conceptual analysis (P. W. Thompson, 2008) during the planning and development of this study. Building upon the work of Silvio Ceccato, von Glasersfeld (1995b) characterized a conceptual analysis as a thought experiment concerned with answering the question, "'What mental operations must be carried out to see the presented situation in the particular way one is seeing it'[?]" (p. 78). This characterization relates to constructing second-order models of students' ways of reasoning. However, P. W. Thompson (2008) also describes a second way of characterizing conceptual analysis. He states

> There is a second way to employ Glaserseld's method of conceptual analysis. It is to devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise. (p. 43)

These hypothesized ways of reasoning could be drawn from one's previous experiences working with children, from one's own first-order mathematical knowledge, as well as from previous research that elaborated second-order models of students' mathematical activity that are relevant to one's current context. Thus, not only did the practice of conceptual analysis serve as my modus operandi for building second-order models of my students' mathematics, but it also served the research team in developing hypotheses about the ways of reasoning we might encounter during the study.

In particular, we posed the question, "What ways of reasoning could a student engage in to construct an intensive quantity as a relationship between two extensive quantities?" For a specific example, we considered a context in which 4 Dutch kroner were equivalent to 3 Mexican pesos. After conducting a conceptual analysis of this task, we decided that distributive partitioning operations could feasibly play a role in establishing the unit ratio three-fourths pesos per kroner. The reasoning behind this decision went as follows. Knowing that 4 kroner were equivalent to 3 pesos implied that 1 kroner, which is one-fourth of 4 kroner, would be equivalent to one-fourth of 3 pesos. Thus, establishing the unit ratio necessitated finding one-fourth of 3. Not wanting to appeal to computational procedures for multiplication, we turned to distributive partitioning as a possible way in which a student could establish the result of finding one-fourth of 3 on the basis of his/her quantitative schemes and operations.

Previous research on children's partitioning and fractional operations offered background on students' use of distributive reasoning. For example, while exploring the role of unitizing in children's partitioning strategies, Lamon (1996) found that some students used a distribution strategy. Students using this approach marked all the pieces of a whole and distributed the parts fairly. Building upon this work, Steffe (2010a) refers to the strategy of partitioning *n* items among *m* shares by partitioning each of the *n* items into *m* parts and distributing one part from each of the *n* items to the *m* shares as distributive partitioning.

Applied to the context of pesos and kroners, using distributive partitioning would feasibly allow a student to coordinate the measures of the quantities in such a way that would preserve the given ratio and convert it to a unit ratio. At the point of trying to determine one-fourth of 3 pesos, a student could then decide to partition each of the 3 pesos into four parts. Then, because taking one-fourth of each peso is equivalent to taking one fourth of all 3 pesos, the student could determine that 1 kroner is equivalent to three-fourths of a peso, or three-fourths of a peso per kroner (see Figure 2.4).

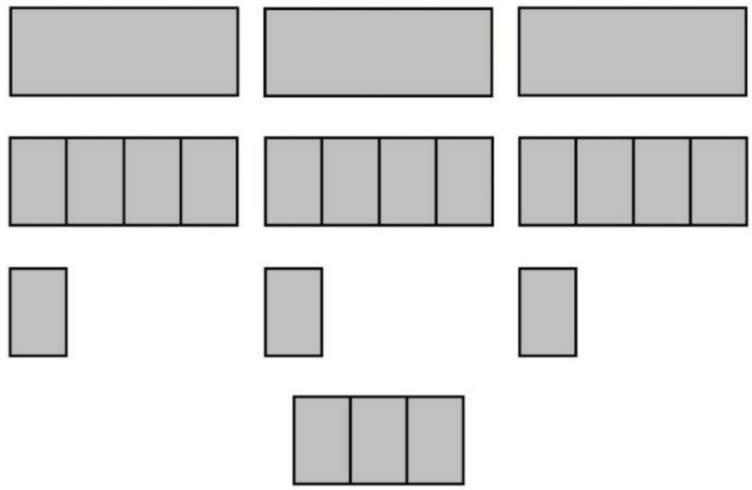

Figure 2.4. A hypothesized model for using distributive partitioning to establish one-fourth of 3 pesos as three-fourths of 1 peso.

As a result of this conceptual analysis, we hypothesized that distributive partitioning represented one particular way of reasoning that would be fortuitous for students to construct.

Further, the mathematical activity of the students who participated in a pilot study conducted prior to this teaching experiment indicated that some students had constructed distributive partitioning operations similar to those described above. In particular, we observed three variations in the types of understandings that students could construct while using their distributive partitioning operations. We observed many students who could devise a strategy for producing m equal shares of n units by distributing their activity of partitioning into m shares across each of the n units. However, the students interpreted the fractional meaning of the results of this activity in three distinct ways.

One group of students focused on the number of pieces produced by the partitioning rather than on the fractional size of the shares in relation to the units. A second group of students could accomplish this and could also establish the fractional amount of one share in relation to one unit. The third group of students could reason as the second group but could also establish the fractional amount of one share in relation to the total number of units. Considering these observed ways of reasoning with respect to the example of splitting the 3 pesos into four equal parts, the first group of students would view the results of the distributive partitioning activity as three-twelfths of the pesos, the second group could construct the result as both three-fourths of 1 peso and three-twelfths of all the pesos, and the third group could establish the result of their activity as three-fourths of a peso and as both three-twelfths and one-fourth of all the pesos.

Having observed these ways of reasoning during the pilot study, I was aware of these three categories of students' distributive partitioning operations at the beginning of the teaching experiment. Following the completion of the teaching experiment, I conducted a retrospective analysis of the students' distributive partitioning activity across both the pilot study and the teaching experiment. Based upon that analysis, I constructed second-order models of the quantitative schemes and operations that could account for each type of fractional awareness. I now refer to the first type of activity as indicative of a *distributive sharing scheme*, the second type as indicative of a *distributive partitioning scheme*, and the third type as indicative of a *reversible distributive partitioning scheme* (Liss II, 2014).

However, it is important to note that while conducting the teaching experiment for this dissertation, I was not fully aware of the different quantitative operations responsible for each type of understanding. Yet, for clarity of presentation, I will use the terminology presented in the previous paragraph throughout this dissertation even though we did not use that terminology at the time we were actually conducting the teaching experiment. In addition, I use *distributive partitioning operations* to refer to this category of conceptual operations more generally when I do not intend to single out a particular distributive partitioning scheme.

Furthermore, it is most accurate to say that during data collection for the teaching experiment, I made the distinctions based upon observable differences in the students' fractional comparisons but had not yet constructed a second-order model of each type of reasoning. Thus, one goal of the teaching experiment was to establish what other quantitative schemes and operations students engaged in when using a reversible distributive partitioning scheme to reason in the manner described above in my conceptual analysis of the pesos and kroner task.

More generally, each of the notions of quantity presented in previous sections and the schemes and operations one uses to reason with them informs the central hypothesis underlying this research. The reorganization hypothesis that guided this study into the construction of intensive quantity states that "Children's intensive quantitative schemes can emerge as accommodations in their numerical schemes if the numerical schemes[2] are constructed as extensive quantitative schemes" (Steffe, Liss II, et al., 2014, p. 3). In particular, I hypothesized that both number sequences and fractions schemes, in addition to other quantitative schemes and operations such as distributive and reversible schemes, played a role in students' construction of intensive quantity. These hypotheses represented starting points for the research as I aimed to clarify important distinctions regarding the construction of extensive quantitative schemes and operations that impact students' construction of intensive quantitative schemes and operations.

[2] The use of numerical schemes here is intended to include both children's number sequences, as well as their fraction schemes.

CHAPTER 3

METHODOLOGY

I decided to conduct a small scale constructivist teaching experiment (Steffe & Thompson, 2000b) in order to investigate my research questions related to students' construction of intensive quantity. While a variety of factors influenced this decision, principal among them was the fact that this approach involved working directly with students over an extended period of time. I viewed this as critical as it would allow for my participants' mathematical thinking to become a part of my experiential reality in the sense that the participants' ways of reasoning would play a vital role in defining the direction and nature of the interactions throughout the teaching experiment. Then, as an element of my experience, these interactions could provide an experiential basis for constructing the mathematics of each participant as a conceptual construct in my own thinking and eventually as second-order explanatory models of their quantitative reasoning. Furthermore, the ongoing nature of these interactions, which focused on bringing forth the students' creative and productive uses of their available schemes and operations, would support a process of continual refinement of my models as I continually adjusted to the mathematical activity of the students.

In the remainder of this chapter, I describe in greater detail the methodological approach I used for this study. First, I discuss the defining components of the constructivist teaching experiment methodology and how they relate to the goals of this study. Then, I outline the specific characteristics of my teaching experiment, provide some detail regarding the problem contexts used throughout the teaching experiment, and provide a timeline for the study.

The Constructivist Teaching Experiment

Basic Goals

The basic goal of conducting a constructivist teaching experiment is for a researcher to experience students' mathematical thinking and reasoning in order to construct a rational model of students' mathematics (Cobb & Steffe, 1983; Steffe & Thompson, 2000b). These models are considered rational in the sense that the things that students say and do in the context of solving tasks and interacting with the researcher, which I refer to as students' *mathematical activity*, are assumed to be rational consequences of the students' current cognitive structures regardless of whether or not they appear as rational to an observer. Developing these models involves conducting what von Glasersfeld (1995b) refers to as a *conceptual analysis* and making hypotheses about the conceptual operations a student could feasibly have used to interpret the task in the way that he/she did. Thus, the results of a teaching experiment are second-order models (cf. Chapter 2) that describe mathematics as constructed by individual knowers rather than characterizing mathematics as a collection of given truths that exist independent of human activity.

Further, these rational models of students' reasoning are intended to help develop insight regarding the *epistemic subject*, which consists of "that which is common to all subjects at the same level of development, whose cognitive structures derive from the most general mechanisms of the coordination of actions" (Piaget, 1966, p. 308). In this sense, a particular second-order model may arise as a characterization of one student's reasoning and then persist if it is found useful in characterizing and predicting the mathematical activity of other students judged to have similar conceptual structures (Steffe & Norton, 2014). Specifically with regard to algebraic reasoning, an *epistemic algebraic student* refers to "a conceptual model of what we observe as characteristic mathematical activity of students that is taken to define a level of development in the algebraic activity of the students in the context of mathematics teaching" (Steffe, Moore, & Hatfield, 2014, p. ix)

The Role of *Teaching* in a Constructivist Teaching Experiment

At times during a teaching experiment a researcher takes on a role similar to the approach an interviewer would take when conducting a clinical interview (Cobb & Steffe, 1983). The modern clinical interview method developed primarily as a response to standardized tests, which psychologists found limiting in the sense that they provided no way of revealing the thought processes behind students' answers (Ginsburg, 1997). Drawing heavily upon Piaget's methods for working with children, Ginsburg (1997) outlined three underlying goals of a clinical interview: "to depict the child's 'natural mental inclination,' to identify underlying thought processes, and to take into account the larger 'mental context'" (p. 48). In pursuit of these goals an interviewer might begin with a set of tasks or questions but also form on the spot hypotheses about the students' thinking and pose follow-up questions to test them.

While each of these descriptions also characterizes a component of the researcher's role within a teaching experiment, the goal of facilitating and analyzing change in the students' ways of reasoning over time represents the point of departure between a clinical interview and a teaching experiment. Steffe (1991a) characterizes the purpose of teaching within a teaching experiment as follows:

> From the researcher's perspective, the purpose for engaging children in goal-directed activity that includes problem solving is not simply the solution of specific problems. The primary reason is to encourage the interiorization and reorganization of the involved schemes as a result of the activity. (p. 187)

The extent to which one can infer that students actually makes advancements in their ways of reasoning then informs his/her model of the students' mathematics.

In addition, working with a small number of students over an extended period of time allows a researcher to not only investigate the students' "natural mental inclination" (Ginsburg, 1997, p. 48) but also to study the changes those natural inclinations undergo over time and what role the teacher might play in influencing those changes (Steffe, 1991a; Steffe & Thompson, 2000b). Further, the constructivist teaching experiment methodology involves specifying a model of the conceptual schemes and operations that could account for a particular natural mental inclination (Cobb & Steffe, 1983; Steffe, 1991a; Steffe & Thompson, 2000b). Doing so enables a researcher to investigate how students might use those conceptual constructs in novel ways or modify and reorganize them in service of new goals.

In essence, the role of teaching within a teaching experiment serves two interrelated functions. First, teaching is intended to provoke learning so that the learning can be studied within the context of the interactions and the students' mathematical activity as each of these develops over the course of the entire teaching experiment. Second, students often respond in surprising or unexpected ways, and the teaching interaction rarely progresses exactly as the researcher envisioned. These unexpected occurrences create points of contact with the students' mathematics and represent constraints for the researcher to negotiate. In either instances of successful interaction and learning or unexpected replies that pose constraints to the researcher's activity, the underlying goal remains promoting "the greatest progress possible in all participating students" (Steffe & Thompson, 2000b, p. 276).

The Dual Role of a Teacher-Researcher

When conducting a teaching-experiment, one takes on the dual role of teacher-researcher in order to accomplish these goals (Steffe, 1991a; Steffe & Thompson, 2000b). In the role of researcher one designs tasks that he/she thinks might reasonably support specific advancements in the students' ways of reasoning. The reasons for these decisions often arise as a combination of insights gained from previous research efforts to model students' mathematics as well as the teacher-researcher's previous experiences working with students.

However, even though one develops tasks with particular hypotheses and a potential model of the students' thinking in mind, in the moment of working with students in a teaching session one takes on the role of teacher and gives prominence to the students' activity and ways of reasoning while setting aside his/her own personal hypotheses (Steffe & Thompson, 2000b). In doing so, rather than reflecting his/her ways of thinking onto the students and the students trying to learn to think like the researcher, it is the teacher-researcher who attempts to learn to think like the students (Cobb & Steffe, 1983). In the moment of these interactions, the teacher-researcher responds intuitively to the students and alters tasks or designs new ones while attempting to explore the students' reasoning and facilitate their learning. This orientation, which constantly seeks to foreground the students' mathematics, lies at the heart of the conceptual analysis and model-building efforts that occur during a teaching experiment: "This method—setting research hypotheses aside and focusing on what actually happens—is basic in the ontogenetic justification of mathematics" (Steffe & Thompson, 2000b, p. 276).

After completing a teaching session one reflects upon the success of these interactions in order to modify existing hypotheses or form new ones regarding the conceptual schemes and operations of the students and how the teacher-researcher might support their increasing sophistication over time (Cobb & Steffe, 1983; Steffe & Thompson, 2000b). These reflections are of a slightly different nature than the intuitive responses developed while in the midst of a teaching interaction. Rather, the reflections that occur between teaching sessions involve taking a step back from the interactions themselves to compare the students' actual mathematical activity to that predicted by one's developing experiential model of the students' mathematics. This allows for both the refinement of one's models and the development of new tasks that one reasonably expects will support the students' conceptual development going forward.

The teaching experiment comes to life in the back and forth interplay between developing hypotheses regarding the students' mathematics and modifying them to account for the nature of the ongoing interactions. In this sense, the roles of teacher and researcher are mutually supportive. Each contributes to the development of more robust models of the students' mathematics that in turn enable more effective teaching interventions. This iterative process of forming and testing research hypotheses, both prior to and within a teaching session, creates the context in which one develops first experiential, and later second-order, models that account for the teacher-researcher's observations of the students' mathematical activity. Steffe and Norton (2014) describe this progression of a teacher-researcher's modeling activity as follows:

> The living, experiential models consist of the acting or interacting students and/or re-presentations of them throughout the duration of engagement with the students. The eventual and long-term goal of the teacher is to construct mathematical schemes and changes in them that explain the living, experiential models. (p. 318)

These conceptual schemes and hypotheses regarding their development constitute one's second order models and represent the source from which one constructs epistemic students.

A constructivist teaching experiment also includes a witness-researcher who helps the teacher-researcher to negotiate these roles and purposes (Steffe & Thompson, 2000b). The witness fulfills a variety of roles throughout the teaching experiment. For instance, as an external observer, the witness might notice elements of the interaction unattended to by the teacher or reflect upon those elements in a way that the teacher cannot while immersed in the constructive activity with the students. In addition, there are times at which a teacher and witness may form different opinions about an interaction. In both cases, the perceptions of the witness provide another point of view to consider when determining the direction for teaching sessions. These could arise as interjections the witness makes during the teaching session or more commonly while reflecting upon previous interactions and planning subsequent teaching sessions.

The Role of Interaction

The interactions that take place within a teaching experiment define a learning environment that is co-constructed by both the teacher-researcher and the student. This is not to say that the two experience the learning environment in the same way. In fact, I intend roughly the opposite. Both the teacher-researcher and the student construct his/her own experiential learning environments. Yet, those personally constructed learning environments are constantly constrained and thus also influenced by one’s interpretation of the actions of the other. Provided that both remain engaged in the interaction, neither the teacher-researcher nor the student defines his experience independently or freely.

To help clarify the central issue related to constructing the learning environment in a teaching session, consider an example from the field of cybernetics. Cyberneticists encountered a challenge while trying to use the theories of control systems to account for the relationship between an organism’s inner experiences and their observable behavior. Powers (1978) describes that cybernetics failed to provide new directions for psychology in part because it failed to account for how an organism’s purposes are “set by processes inside the organism and are not accessible from the outside” (p. 419). In terms of a teaching experiment, this issue applies equally to the teacher-researcher and the student—neither has direct access to the goals, intentions, and constructed realities of the other.

It is in light of this constraint that one can better understand how the learning environment emerges within a teaching experiment. The teacher-researcher designs the tasks and problem situations with specific goals in mind for how the student might assimilate those situations and what he/she might learn from that experience. Thus, the teacher-researcher defines the context in which the interactive communication will take place and hence constructs the "possible learning environment" (Steffe, 1991a, p. 189). However, lacking direct access to the teacher's intentions, the student constructs his/her own perception of the teacher-researcher designed experiences and this perception defines the "learning environment" for the student (Steffe, 1991a, p. 189).

Differences between the teacher-researcher's expectations and his/her inferences regarding the student's actual learning environment create opportunities to refine one's model of the student's reasoning to account for the processes by which the student constructs his/her learning environment and the tasks he/she sets out to solve (Cobb & Steffe, 1983; Steffe & Thompson, 2000b). For example, if a student can use a teacher-researcher's suggestion productively, that might indicate a way of reasoning that was available to the student but not previously active. Alternatively, if a student does not use particular suggestion appropriately, that indicates a dissonance between the teacher-researcher's and the student's construction of the learning environment. Negotiating these successes and struggles leads to a better understanding of the conceptual structures that can account for the student's mathematics.

Speaking from my own perspective, I consider teaching and interaction as related, yet distinct, components of a teaching experiment. Teaching is a process I undertake with specific goals in mind for how the designed tasks and questions might provoke advancements in the students' ways of reasoning. As such, the conceptual constructs of teaching and possible learning environment are inherently linked, and both fall within my personal experiences. However, within the interaction it is the students that determine the usefulness of my teaching interventions. Further, my observations of the students' mathematical activity provide my only insights to the students' constructed learning environments. Thus, similar to how I consider a second-order model the result of an internal conversation (cf. Chapter 2), I consider my experience of the learning environment as co-constructed in my own experience through teaching and interaction. The former gets defined by my conceptual constructs and the latter by the students' mathematics. It is through this internal process of reconciling the teaching and my experience of the interaction that the students' mathematics gains a reality within my personal experiences and thinking.

Analysis Techniques

In accordance with the prolonged duration of a teaching experiment, ranging anywhere from six weeks to a year or more, the teaching experiment methodology includes two primary types of analyses—ongoing and retrospective (Steffe & Thompson, 2000b). I briefly consider each with respect to the teacher-researcher's models of the students' mathematics.

Ongoing analysis and the construction of experiential models.

Ongoing analysis refers to the formation and testing of hypotheses that occur throughout the teaching and interactions described in previous sections. These analyses result in the development and continual refinement of one's experiential models of the students' mathematics (Steffe & Thompson, 2000b). These experiential models allow a teacher-researcher to predict how students might respond to a given task or situation and serve as the source of ideas for subsequent teaching sessions. In addition, one's experiential models remain viable until disconfirming evidence is found in the students' mathematical activity, at which point the models can be revised and tested again in an iterative process throughout the teaching sessions. Thus, the models are "humble theories" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 10) in the sense that they are theories about students' mathematical cognition that are subservient to and open to revision based upon the students' subsequent mathematical activity.

Retrospective analysis and the construction of second-order models.

Retrospective analysis occurs after the completion of a teaching experiment and refers to the activity of reflecting back upon one's experiential models with respect to the public records of the interactions that occurred during the teaching experiment (Steffe & Thompson, 2000b). These public records include data collected during the experiment such as students' written work, transcripts and video-recordings of the teaching sessions, and the teacher-researcher's journal. These records can serve to activate the researcher's recollections of his/her experiences with the students during the study and prompt his/her recall of the thinking that went into the development of the experiential models in use at various stages of the teaching experiment.

During retrospective analysis the teacher-researcher seeks to use these data sources as a starting point for developing "…insight into the students' actions and interactions that were not available to the teacher-researcher when the interactions took place" (Steffe & Thompson, 2000b, p. 293). In addition, during a retrospective analysis one has the benefit of being able to reflect upon the significance of a given interaction both with respect to those interactions that came before and after. Thus, interpreting the records of interaction both retrospectively and prospectively enables the researcher to "…set the child in a historical context and modify or stabilize the original interpretations, as the case may be" (Steffe & Thompson, 2000b, p. 293). Ultimately the goal of conducting a retrospective analysis is to make explicit the experiential models one constructed during the experiment and revise them to form stable second-order models of the students' mathematics that account for their mathematical activity over the course of the teaching experiment. These models comprise what Steffe and Thompson (2000b) refer to as an “explanatory framework” (p. 294) that accounts for patterns that help to understand each student’s mathematics in relation to the others.

Analysis techniques and the students’ zones of potential and actual construction.

Running parallel to these two types of analysis techniques are the researcher’s efforts to define each student’s zones of potential and actual construction. A student’s *zone of potential construction* refers to a teacher’s or researcher’s hypotheses regarding the modifications a student might be able to make to his/her current ways of operating (Steffe, 2010f, pp. 17–18; cf. zone of proximal development in Vygotsky, 1978). Thus, in a teaching experiment the zone of potential construction is a hypothetical construct of the researcher that defines his/her intentions for teaching a given student at any particular time. Through actually interacting with the student, his/her inferred zone of potential construction gets reconstituted as a *zone of actual construction* to reflect the researcher’s observations and interpretations of the mathematical activity the student actually performed. This activity may open up new constructive possibilities for the student and in turn necessitate modifications to the student’s zone of potential construction.

The zones of potential and actual construction that a researcher develops for his/her students constantly evolve through the processes of ongoing and retrospective analysis. While conducting a teaching experiment, the specific modifications the researcher hopes to engender constitute the current zones of potential construction for the students during a particular teaching interaction. Reflecting upon these interactions while conducting an ongoing analysis results in the formation of experiential models of the students’ mathematics, and these models comprise their zones of actual construction at that time. As one’s experiential models evolve throughout the experiment, so too do the students’ zones of potential and actual construction. With respect to retrospective analysis, the evolution of these constructs becomes an explicit focus of one’s analysis as he/she reconsiders the records of the teaching experiment. Lastly, the second-order models one constructs as a result of his/her retrospective analysis are intended "…to establish the zones of actual construction of the participating students and to specify the independent mathematical activity of the students in these zones" (Steffe & Thompson, 2000b, p. 290)

Specific Characteristics of My Teaching Experiment

These guiding principles of the constructivist teaching experiment served as the framework for the design and conduct of a teaching experiment that I completed in order to learn about students' construction of intensive quantity. Throughout the study, I worked with a team of researchers to plan and conduct the teaching experiment that provided the data presented in this dissertation. This research team consisted of myself, my advisor, and six other mathematics education doctoral students who contributed to various aspects of the planning and conduct of the teaching experiment. Further, both my advisor and the doctoral students took turns serving as the witness-researcher for the teaching sessions. At times the witness interjected his/her ideas during teaching sessions. However, most often he/she shared his/her observations with me while planning for subsequent teaching sessions. The sections that follow elaborate specific characteristics of the teaching experiment we conducted.

Participants

Pilot study.

In preparation for the teaching experiment, the research team and I conducted a pilot study during spring 2013. During that semester we worked with four ninth grade students (Mike, Blake, Jill, and Jack) at a rural high school located in the southeastern United States.[3] At the time of the pilot study, each of the four students was enrolled in a section of the school's most

[3] All student names used throughout this dissertation are pseudonyms.

commonly taken algebra course. During the pilot study, each student participated in 10–12 approximately 20-minute teaching sessions over a 10-week span from February 2013 through April 2013.

Teaching experiment.

The research team and I conducted a teaching experiment at the same high school with four students during fall 2013–spring 2014. We decided to continue working with the students from the pilot study who were able to continue their participation because we had already developed experiential models of those students' mathematics that could inform our work during the teaching experiment. However, two of the four students from the pilot study were unable to continue their participation. Thus, the four students who participated in the teaching experiment included two who participated in the pilot study (Blake and Jack) and two who did not (James and John). Like the students in the pilot study, both James and John were enrolled in a section of the school's most commonly taken algebra course during spring 2013.

We completed an individual initial interview with each of the four students that included a pre-determined set of tasks (see Appendix A) to allow the research team to identify the partitioning operations and levels of units that each student could use in reasoning at the beginning of the study. These initial interviews followed a semi-structured approach (Bernard, 2002) with follow-up questions and prompts based upon the specific ways in which each individual student responded to the scripted set of tasks. The goal of these initial interviews was to develop an understanding of the types of mental schemes and operations that each student had available at the beginning of the teaching experiment. We were particularly interested in the students' distributive partitioning operations and their abilities to coordinate multiple levels of units.

Based upon similarities in their reasoning, we decided to try to teach the students during the teaching experiment in the following pairs: a) Jack and John; and b) Blake and James. Jack and John both solved all of the partitioning tasks, could take three levels of units as given in assimilation, and demonstrated more sophisticated units coordinating activity than Blake and James (cf. Chapter 4 for in-depth analyses of Jack's and John's initial interviews). Blake and James could solve some, but not all, of the distributive partitioning tasks. In addition, we inferred that James could take three levels of units as given in assimilation while Blake used two levels of units in assimilation and could construct a third level of units in activity.

In this dissertation I present my models of Jack's and John's mathematics. I have focused on the mathematics of these two students for several reasons. First, because my research questions focused on identifying conceptual constructs that facilitate the construction of intensive quantities, I chose to focus on the pair of students that had the more sophisticated foundation of conceptual resources to draw upon so that I could investigate which of those would prove important to the students in constructing and operating with intensive quantities. Secondly, I served as the teacher-researcher for all but 3 of the 23 sessions involving either Jack, John, or both students.[4] Thus, my experiential models of these students' mathematics included my own memories of the goals and intentions I had while defining the possible learning environments for these students. Lastly, Jack and John were more successful at making sense of the situations involving intensive quantities throughout the experiment. Thus, I chose to focus my analysis on the development of those students' reasoning to develop a model of the quantitative schemes and

[4] The school had a small window of time available for us to work with the students and so we often worked with both pairs of students at the same time in different rooms. Thus, I was only able to serve as the teacher-researcher for eight of the 24 sessions involving either Blake, James, or both students. Two other members of the research team served as teacher-researcher for the other teaching sessions.

operations that could account for their success and hence provide insight for answering my research questions.

Data Sources

Each of the teaching sessions lasted approximately 25–35 minutes and I collected a variety of "public records…of the interactive mathematical communication" (Steffe & Thompson, 2000b, p. 292). These included video recordings and annotated transcripts of the interactions, my research journal, and the students' written work. Each of the data sources, described in more detail below, contributed to both the ongoing and retrospective analyses.

Video recordings and annotated transcripts.

Each teaching session was video-recorded from multiple perspectives. First, I used two digital video cameras to capture different camera angles in order to develop a restored view (Hall, 2000) of the interactions—a wide view to capture the entire interaction and a close-up view to capture details of the students' work as it developed. In addition, I used screen capture software to collect videos of any work the students carried out on a computer. This included times the students worked within the JavaBars (Biddlecomb & Olive, 2000) computer environment or with interactive animations developed in The Geometer's Sketchpad (Jackiw, 2012). After each teaching session, I mixed the multiple video viewpoints into a single split screen format to allow for the researchers to view all camera angles simultaneously. During the teaching experiment, I watched every teaching session at least once prior to the next teaching session as part of my ongoing analysis that informed both task design and my development of experiential models of the students' mathematics.

After completing data collection, I created annotated transcripts of each teaching session. These transcripts included a verbatim record of all spoken interactions. However, the students' activity often involved creating and describing drawn diagrams. Thus, I annotated the transcripts to include descriptions of all non-verbal actions in order to clarify how the students' verbal descriptions corresponded to the development of their written work. During retrospective analysis, I watched the mixed video of each teaching session multiple times and used the annotated transcripts to easily locate and reflect upon particular interactions that proved important to my analyses.

My research journal.

Throughout the study I developed a digital *research journal* to maintain a written record of my ideas and reflections throughout all phases of the teaching experiment. I use the term journal somewhat loosely in that I am not referring to a single, hand-written volume. Rather, I use this term to describe the collection of all written documents I produced to document my current ways of thinking about each student's mathematics as the study progressed.

This digital research journal included two types of documents—those developed while conducting the teaching experiment and those created during my retrospective analysis of the data. First, during data collection I developed a document to record my plans for each teaching session. This document contained a written description of my goals for each teaching session, which included the particular advancements I had hoped each student could make to his reasoning. In addition, the document included all planned tasks for each teaching session as well as anticipated modifications I might make depending upon how the students responded. Then, after completing each teaching session I added additional comments in order to document my reflections on how the teaching sessions actually progressed in relation to my goals and expectations.

Secondly, during retrospective analysis I created an outline document for each teaching session that included an outline of the tasks posed, the time intervals from the video-recordings that corresponded to each task, and a brief summary of the students' activity for each task. These served as a useful tool for comparing a single student's activity across multiple teaching sessions and for comparing multiple students' responses to the same task. Also, while watching the videos during retrospective analysis I added additional ideas and reflections to these outline documents in order to make note of important tasks that either provided additional evidence to support my analyses of previous teaching sessions, indicated disconfirming evidence that would require me to revise my analyses, or identified places in which the students' reasoning changed from their previous activity.

The students' written work.

During the teaching experiment, I collected all physical records of the students' activity. This included any written work or diagrams the students produced as well as any waxed string that the students cut or wrote upon while explaining their reasoning. After completing data collection, these were scanned to create digital images of the work and stored with the video and transcript data.

Method of Ongoing Task Development

Following the initial interviews, I developed the plans for the next teaching session by first forming hypotheses regarding the conceptual schemes and operations that could account for the students' mathematical activity in previous teaching sessions. From these I formed goals that characterized my intentions for teaching the students. For example, at times my teaching goals included investigating whether a student could adapt an established way of reasoning to a novel context. Other times my goals involved posing tasks and questions that I thought might prove helpful for a student to reorganize his thinking to construct a new way of operating that would alleviate a perturbation or constraint.

Throughout the teaching experiment the research team played a vital role in my ongoing analyses of the students' activity and my development of the teaching plans. Typically I would reflect upon previous teaching sessions and create a potential plan for the tasks I wanted to use to try to achieve my teaching goals. Then we would meet as a research group in between teaching sessions with the students. During these meetings we would often watch videos from previous teaching sessions to share our ongoing analyses, to compare and contrast our ideas, and to refine our upcoming teaching goals. In addition, I would share the potential plans I had created and we would discuss ways to modify them or additional tasks that could be used as alternatives. Then I would incorporate the research team's feedback and any new ideas I developed as a result of our discussions in order to create the final set of tasks that we used for each teaching session.

The Three Primary Quantitative Contexts

Over the course of the teaching experiment we created tasks within three primary quantitative contexts: filling up a swimming pool with water, allocating amounts of highway to volunteer organizations for the Adopt-A-Highway program, and comparing inch worm crawling speeds. Each context was chosen to allow for particular types of tasks and questions. In the following three sections I briefly describe each context and include samples of the types of tasks we created for each. These are not intended to be an exhaustive description of all the tasks we used. Rather, I include these descriptions here in order to provide the reader with a general awareness of the tasks used throughout the teaching experiment prior to reading descriptions of the students' mathematical activity in the analysis chapters. Throughout the analysis chapters I include additional details regarding the particular tasks we used in relation to our teaching goals at the time.

The swimming pool context.

The swimming pool context was designed around the scenario of filling an empty swimming pool with water. The tasks within this context focused on reasoning with water pumping rates and coordinating changes in the extensive quantities pool depth and pumping duration. I chose this context because I thought it afforded a sense of constancy to the covariation in that once the water pump was turned on, one could imagine that it would continue pumping water at the same rate indefinitely. In addition, I thought that the dynamic nature of the scenario made it easy to imagine the covariation of the extensive quantities—as one imagined the pool continuing to fill, the extensive quantities would accumulate in tandem. We designed tasks to investigate things such as the students' abilities to form unit ratios as measures of the pumping rate and to use them in service of finding various intensive quantitative unknowns given varying changes in water depth or pumping duration.

> *Sample task 1:* Imagine a large swimming pool. Suppose that we had to drain the swimming pool for cleaning and repairs and now it's time to fill the pool back up. The pool maintenance supervisor turns on the water pump and leaves the water running to fill up the pool. After a little while he goes to check on the progress and finds that the water depth is 3 inches after running the water for 5 minutes. How many inches deeper would the water level be if the pool maintenance supervisor let the pump run for only 1 minute?

This task provides measurements of the extensive quantities and investigates how the students might coordinate those measurements to find a unit ratio for the number of inches per minute. Variations and follow-up questions to this task involved asking about various other changes in depth or pumping duration to investigate how the students would use the unit ratios in further reasoning: How much would the water level rise in 127 minutes of pumping? If the maintenance supervisor continued to let the water run, how much deeper would the pool get from 45–49 minutes? How long would it take to raise the pool level 63 inches?

> *Sample task 2:* Unfortunately, the pump that was used to pump the water into the pool broke and so the pump operator is looking into buying a replacement pump. When the operator is out shopping, he finds two different pumps that he could use as replacements and compares them both to the original broken pump. The first pump will raise the pool level a greater amount in the same length of time as the old broken pump did. The second pump will raise the pool level by the same amount as the broken pump did, but in a shorter amount of time. Would Pump 1 be a better pump than the broken pump and why? Would Pump 2 be a better pump than the broken pump and why? Do you think either one of these two replacement pumps would be better than the other one and why?

This task provides an opportunity for the students to reason qualitatively about pumping rates in the absence of specific measurements for the extensive quantities. To follow-up these comparisons we provided specific details for each pump (e.g., Pump 1 could raise the pool level 4 inches for every 5 minutes of pumping). Then we asked questions such as the following: How many inches per minute is Pump 1 pumping? What would be some other changes in pool depth and pumping duration that would result in pumping water at the same rate as Pump 1?

The Adopt-A-Highway context.

The Adopt-A-Highway context was designed around the scenario of allocating various amounts of highway to different numbers of volunteer organizations to determine measures of the unit ratios for the number of miles per organization. I developed this context in response to the students' activity. In particular, I began to notice that the students reasoned differently with discrete countable items such as two cakes compared to continuous measurements such as 3 inches. Thus, I developed this context because it allowed for us to vary the manner in which the number of miles was presented to the students. For example, at various times we presented the miles to be allocated as individual one-mile sections (see Figure 3.1), as a continuous *n*-mile section (See Figure 3.2), as combinations of both (see Figure 3.3), and as situations presented verbally with no map given. We used these types of tasks to investigate the quantitative schemes and operations the students would use to quantify the number of miles per organization. In addition, our teaching goal was to try to help the students develop ways of reasoning that were sufficiently powerful that the students could adapt to both discrete and continuous images of the units upon which they were operating and eventually create their own figurative material for measured quantities that they could operate upon.

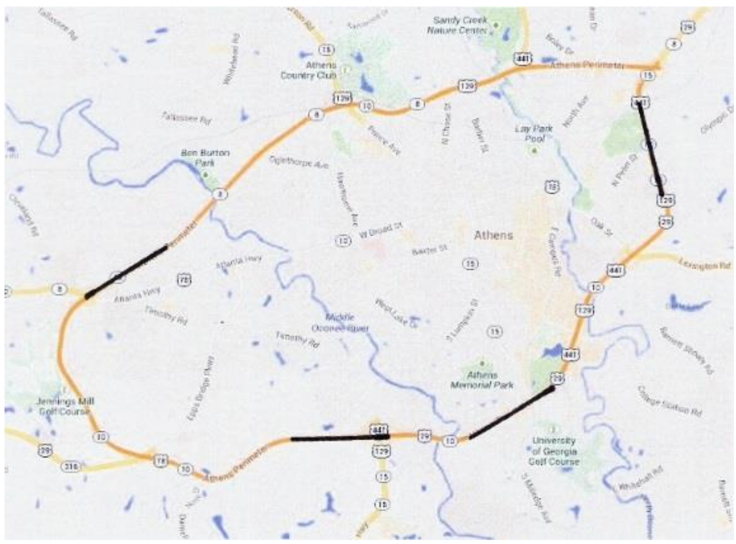

Figure 3.1. The map provided for the task of allocating four one-mile sections to nine organizations.

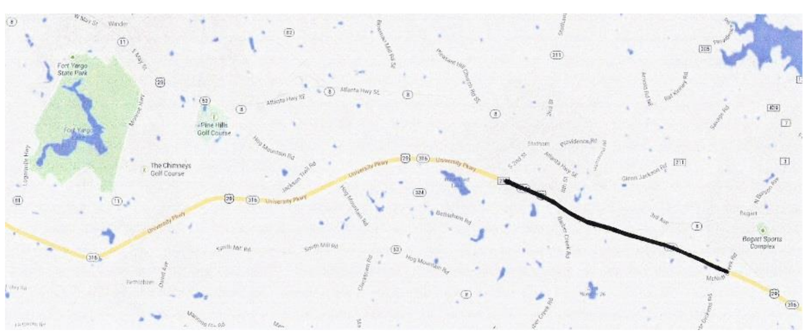

Figure 3.2. The map provided for the task of allocating 3 miles to five organizations.

Figure 3.3. The map provided for the task of allocating a four-mile, a three-mile, and a one-mile section to eight organizations.

With each task we focused on asking two types of questions related to the number of miles per organization. First, we asked the students to quantify each organization's allocation as a fractional amount of 1 mile, such as four-ninths of a mile per organization. In addition, we asked the students to quantify each organization's allocation as a fractional amount of the total number of miles, such as one-ninth of the 4 miles. Our goal was that these tasks would support the students' abstraction of ways of reasoning that would allow them to quantify, for instance, one-ninth of 4 as four-ninths of 1.

The inch worm context.

The inch worm context was developed around the scenario of analyzing the crawling speeds of various inch worms as they completed time trial races. I chose this context as a follow-up to the Adopt-A-Highway context for several reasons. First, while the Adopt-A-Highway context allowed for us to focus on the formation of unit ratios and the students' images of the measured quantities, it did not allow for a sense of covariation of the quantities. Thus, the inch worm races provided a new context that supported reasoning about the covariational relationship between the quantities. Secondly, I chose inch worm races because I felt that the students would be able to leverage their previous experiences with speeds, such as car speeds, within a novel context. Further, I hoped that the nature of inch worms' movements, crawling one body length at a time, would engender a sense of constancy for the inch worms' crawling speeds.

We used this context to explore a wider range of tasks than we pursued in the swimming pool context. For instance, we designed tasks that focused on comparing crawling speeds of different inch worms, on creating graphs or diagrams to represent a given crawling speed, and on creating algebraic equations to represent particular relationships. In addition, many of the tasks involved interactive computer animations that allowed us to ask questions about crawling speeds while leaving the choices about which measures of the extensive quantities would be useful up to the students.

> *Sample Task 1:* Using the interactive animation for Abby's time trial race, I want you to try to decide whether or not Abby is crawling at a constant, steady speed throughout the race? How did you decide?

> *Sample Task 2:* Suppose that Matt wants to know exactly how fast he was actually moving during the race so he could tell his friends how fast he can crawl. How could you figure out the crawling speed for Matt?

> *Sample Task 3:* Al is another inch worm and we found from his time trial race that he crawls at a constant speed of seven-thirds seconds per cm. I would like you to make a graph that stands for the speed of seven-thirds seconds per cm. How did you make the graph? How would someone else looking at your graph know that it represented a speed of seven-thirds seconds per cm?

Teaching Session Timeline

Lastly, I include an overview of the teaching experiment timeline to provide some context for the data and analyses I present in subsequent chapters. We tried to teach Jack and John as a pair because of the similarity of their available conceptual schemes and operations at the beginning of the experiment. We assumed that this would allow for student-student interactions in addition to the planned teacher-student interactions. However, in contrast to his consistent attendance during the pilot study, Jack had sporadic school attendance during the teaching experiment. Thus, while I planned the teaching sessions to work with the students as a pair, I would teach John individually on days that Jack was absent from school. This made it increasingly difficult to plan teaching sessions that were appropriate for both students as their level of experience within each context, as well as my zones of potential construction for each student, diverged. Thus, while I made every effort to teach the students in pairs and create opportunities for student-student interactions, the opportunities for these interactions were less frequent than initially intended.

In total, I conduced an initial interview with each student, a check-up interview with Jack, and a total of 20 subsequent teaching sessions involving Jack, John, or both students. Jack participated in nine teaching sessions and John participated in 15 teaching sessions, with four of those teaching sessions involving both students (see Figure 3.4). Each teaching session lasted approximately 25–35 minutes. We attempted to work with the students twice per week and included breaks for ongoing analyses that coincided with breaks in the school calendar and times during which the students were unavailable because of other school commitments or weather related cancellations.

		Student Attendance	
Task Context	Date	Jack	John
Initial/Check-up Interview	2/14/2013		
	10/2/2013		
	10/3/2013		
Swimming Pool	10/10/2013		
	10/11/2013		
	10/22/2013		
	10/28/2013		
	10/30/2013		
	11/6/2013		
Adopt-A-Highway	11/12/2013		
	11/14/2013		
	11/19/2013		
	11/21/2013		
Inch Worm	1/21/2014		
	1/23/2014		
	1/28/2014		
	2/6/2014		
	2/18/2014		
	2/20/2014		
	2/25/2014		
	2/27/2014		
	3/4/2014		
	3/6/2014		

Figure 3.4. A timeline of the teaching experiment, organized by task context and student attendance.

CHAPTER 4

INITIAL INTERVIEWS

A member of the research team conducted an initial interview with each student individually in order to identify several characteristics of his/her ways of reasoning at the beginning of the teaching experiment. In particular, we designed tasks that would allow us to explore each student's abilities to coordinate multiple levels of units and to identify the nature and extent of their partitioning operations (see Appendix A for the complete initial interview guide). Based upon our conceptual analysis of how a student might form a unit ratio as a measure of an intensive quantity (cf. Chapter 2) and the analysis of pilot study data, we hypothesized that both sophisticated units coordinating and partitioning operations were necessary for the construction of intensive quantity. Thus, one of my goals was to use the initial interviews to identify two students who could engage in reasoning with three levels of units. In addition, I wanted these students to have constructed distributive partitioning operations that allowed them to share n units among m people and interpret one share as n/m of one unit and as $1/m$ of all n units. Lastly, I wanted to explore each student's proportional reasoning to determine his/her facility at coordinating changes in the amounts of two co-varying quantities at the beginning of the teaching experiment.

The first two goals were approached through presenting the students a variety of partitioning tasks designed to provide evidence regarding a student's construction of equi-partitioning, recursive partitioning, and distributive partitioning schemes. In addition, the ways in which a student interpreted a given task and the type of activity he/she carried out in an attempt to solve the task provided evidence of the number of levels of units that he/she could take as given in assimilation. Lastly, to investigate the students' proportional reasoning abilities, we presented the students a lemonade mixtures scenario and asked them to vary the amounts of water and lemonade powder while still maintaining the taste of the lemonade.

Jack's Initial Interview

Jack participated in both the pilot study during spring 2013 and the dissertation study during fall 2013–spring 2014. Thus, his initial interview was conducted at the beginning of the pilot study on February 14, 2013. Given that Jack agreed to continue his participation in the study, I conducted a check-up interview with Jack on October 2, 2013, that was designed to re-evaluate Jack's fraction and partitioning operations at that point in time. This allowed me to obtain confirmation of schemes and operations I had previously attributed to Jack during the pilot study and to learn if he had made any advancements in his ways of operating prior to starting the teaching experiment in fall 2013. In the sections that follow, I include excerpts from Jack's check-up interview to provide a characterization of his levels of units coordination and distributive partitioning operations immediately prior to the start of the teaching experiment. However, because I did not pose the lemonade mixtures scenario to Jack a second time, I include excerpts from his initial interview in February 2013 to exemplify what I knew about Jack's proportional reasoning at the beginning of the teaching experiment.

Jack's Levels of Units

Throughout Jack's work in the pilot study, including his initial interview on February 14, 2013, he consistently demonstrated the ability to assimilate situations using three levels of units as given. During the check-up interview on October 2, 2013, the first confirmation of this ability occurred during a task that involved sharing a share of a strip of fruit-by-the-foot.[5]

Protocol 4.1: Jack determines the fractional amount of a share of a share.

D: Have you ever had fruit-by-the-foot?
Jack: [Nods affirmatively.]
D: Okay. So why don't you go ahead and open that up. [Jack opens the package of fruit-by-the-foot and unrolls it onto the table.] […] So, let's imagine that you want to cut this up a little bit so that you can break it up into smaller pieces. So I want you to kind of keep this in mind. I'm going to cover it up, but you can just kind of think about that as I ask you these questions.
Jack: Um hmm.
D: [Covers the strip of fruit-by-the-foot with a handkerchief.] So, imagine that you took that whole roll and you cut off a piece that was one-third of the strip. Okay, got that in mind?
Jack: Yep.
D: Alright. Now you take that piece and it's still kind of big so you want to cut it up some more. And then say that you cut off one-seventh of that new piece you had. So kind of just imagine doing that.
Jack: Can you repeat that again please?
D: Okay. So you have your original strip. You cut off one-third of the whole strip.
Jack: Um hmm.
D: Now you take that piece and cut off one-seventh of that piece.
Jack: Um hmm.
D: Alright. Do you kind of have that in mind?
Jack: Um hmm.
D: So what I'm wondering is what amount of the original strip do you have with that last piece you cut off?
Jack: [Thinks for 15 seconds.] One-twenty-ones.
D: Why do you say that?
Jack: Because there was three strips, and each of them is cut into sevenths. And you take one. And then what you have out of that is 21 because 7 times 3 is 21.

In determining that one-seventh of the one-third share was one-twenty first of the original strip of fruit-by-the-foot, Jack demonstrated his construction of a recursive partitioning scheme and the ability to take three levels of units as given in assimilation. Consider that the task directs

[5] In each of the selected protocols in this dissertation, W, Jack, and John will be used to indicate the witness-researcher, Jack, or John, respectively. Further, D will be used in cases that I acted as the teacher-researcher while T will be used when another member of the research team served as the teacher-researcher. In addition, ellipses that occur within typed dialogue indicate brief pauses in speech, the symbol, […], indicates that dialogue has been omitted, and all descriptions of actions and non-verbal details are set apart in brackets.

Jack to imagine cutting off only two pieces of the original strip of fruit-by-the-foot—a piece that is one-third of the strip and a piece that is one-seventh of the first cut-off piece. However, Jack's comment, "Because there were three strips, and each of them is cut into sevenths," indicates that while he was thinking silently he had imagined both partitioning the whole strip into thirds and also mentally partitioning the each of these partitions into seven parts in service of his fractional goal, which is precisely the activity of a recursive partitioning scheme (Steffe & Olive, 2010). Jack's use of whole number language, "One-*twenty-ones*" (emphasis added), and his response "…because 7 times 3 is 21" together suggest that Jack relied upon his whole number multiplicative reasoning to coordinate the results of mentally partitioning his initial partitions.

The critical aspect of Protocol 4.1 is that Jack demonstrates being simultaneously aware of the final piece as both one-seventh of one-third and as one-twenty first of the original strip. Jack's intuition to partition each of the original thirds into seven parts indicates that he could flexibly switch his focus between these views without destroying the part-whole relationships that each entailed. In contrast, students who take two levels of units as given in assimilation are typically unable to solve this task. Rather, these students might reply that the final piece is one-third of the whole (a result of focusing on the actual number of pieces made) or one-eighth of the whole (a result of considering seven pieces made in the cutoff one-third plus the remaining un-partitioned two-thirds of the strip as the eighth piece). In either case, only having two levels of units available during assimilation hinders their ability to coordinate the sevenths, within each of the thirds, within the original unit.

Lastly, it is important to note that Jack did not need to engage in actual partitioning but rather solved this task by operating mentally upon a hypothetical strip in his visualized imagination. Covering the tangible strip of fruit-by-the-foot provides evidence that Jack was not simply making pseudo-empirical abstractions from physical cutting/partitioning activity and thus lends further support to the hypothesis that Jack assimilated this task with three levels of units.

In addition to enabling Jack's reasoning with three levels of units, his construction of a recursive partitioning scheme is also significant because it could provide a conceptual basis for symbolic unit fraction multiplication. To evaluate if Jack had made this abstraction, immediately after he had solved the fruit-by-the-foot partitioning task I asked him how he would multiply one-seventh times one-third. My goal with this task was to investigate whether or not Jack would assimilate fraction multiplication as a situation of the same conceptual operations that he had used when solving the fruit-by-the-foot task.

Protocol 4.1: Continuation.

Jack: One-seventh times one-third?
D: Um hmm.
Jack: [Thinks for approximately 20 seconds and then leans back in his chair.]
D: What are you thinking about?
Jack: I'm trying to [remember]. Because when you multiply fractions I can't remember if it's cross multiplication or if it's just multiply. It's still one-twenty-one if it's not cross-multiply. […] But if you cross-multiply, it ends up being three-sevenths, I believe. I could be wrong. I don't know.

Jack's reply indicates that he assimilated the fraction multiplication task as a situation of his numeric operations rather than the quantitative operations he used to solve the fruit-by-the-foot task. Had he assimilated this task as a situation of his recursive partitioning scheme, then one-seventh times one-third would necessarily have been one-twenty first. However, Jack's reasoning in this excerpt lacked this sense of necessity. Rather, the fact that he considered both "just multiply[ing]" to get one-twenty first and "cross-multiply[ing]" to get three-sevenths suggests that Jack perceived my fraction multiplication question as a call to carry out a previously learned procedure. Following this exchange I asked Jack to consider the fraction multiplication question in relation to the fruit-by-the-foot task, and he was able to reconcile his numeric and quantitative operations in order to decide that the answer should be one-twenty first.

I included this interaction to highlight that while Jack had constructed sophisticated quantitative reasoning, such as a recursive partitioning scheme that supported making fractional comparisons with three levels of units, fraction language and arithmetic operations did not yet symbolize those same ways of reasoning. The research team and I remained sensitive to this distinction throughout the teaching experiment. As a result, our approach often involved first attempting to help the students develop more sophisticated ways of reasoning quantitatively and then transitioning to more formal and symbolic ways of describing the quantities as we attempted to help the students develop numeric notation and language as abstractions of their quantitative reasoning.

Throughout the remainder of the check-up interview, Jack provided further evidence of his ability to assimilate tasks using three levels of units as given when solving another fraction composition task (one-fourth of three-fifths) and the distributive partitioning tasks. This activity during the check-up interview confirmed my experiential model of Jack's levels of units that I had developed during the pilot study. Thus, I am confident claiming that throughout the teaching experiment Jack could take three levels of units as given in assimilation and reasoning.

Jack's Partitioning Operations

During the first two tasks of the check-up interview, Jack's activity provided evidence to confirm my inferences from the pilot study that he had constructed equi-partitioning (not presented here) and recursive partitioning schemes (see Protocol 4.1). In addition, during the pilot study Jack demonstrated the most advanced distributive partitioning operations of any of the four students involved in the pilot study, and he was able to construct the fractional comparisons described in the introduction to this chapter. To re-evaluate Jack's thinking at the beginning of the dissertation study, during the check-up interview I presented Jack several variations of distributive partitioning tasks. First, I presented Jack with the task of fairly sharing two unequally-sized chocolate cakes amongst three people. Similar to Protocol 4.1, after introducing the task I covered the two cakes with a handkerchief to encourage Jack to operate mentally with imagined cakes rather than literally cutting and reasoning with the cakes in front of him.

Protocol 4.2: Jack shares two unequally-sized chocolate cakes among three people.

D: [The interviewer places two differently sized chocolate Play-Doh cakes on separate plates in front of Jack.] I'm wondering, how would you cut it up so that we could all get the same fair amount?
Jack: Of both cakes?
D: Um hmm. [Places a handkerchief over the cakes.] So I'm just going to cover them up and I want you to think about it.
Jack: You have… [Thinks silently for 7 seconds.] You would just cut each of them into three, thirds.
D: Okay.
Jack: Because there's two cakes, it's not necessarily a whole cake, it's two different cakes. So, each of us would get two-sixths of both cakes.
D: Okay. So, yeah, and why? Why do you think two-sixths?
Jack: Because there's two cakes and each of them are split into three. And then we're splitting equally among three people. Then you have two—one from this cake and one from that cake, one from this cake one from that cake—so it's two-sixths because there's six pieces all together but there's two coming out of them.

Jack's responses indicate that as a result of his assimilation of my question about how to share the cakes fairly, he formed the goals of both sharing the cake fairly and also determining the fractional amount of all the cake that each share received. To accomplish the first of these goals, Jack determined that cutting each cake into thirds would enable him to produce three fair shares of all the cake. Having mentally carried out this partitioning activity, he then decided that each share was two-sixths of the cake because each share received two of the six pieces, one from each cake. This type of reasoning, which anticipates a sharing strategy without experimentation and which uses the number of pieces as the basis for fractional comparisons, is consistent with his a distributive sharing scheme.

Suspecting that Jack had constructed schemes that would allow him to reason about the fractional relationships in other ways, I decided to question him further about the amount of cake that each share represented. However, before sharing that protocol, I want to highlight one aspect of Jack's description of each share that, in retrospect, I believe indicates that he had assimilated the task with a more sophisticated distributive partitioning scheme than his initial descriptions suggested. In particular, after deciding that cutting each cake into thirds would allow him to achieve a fair sharing, Jack stated, "Because there's two cakes, *it's not necessarily a whole cake* [emphasis added], it's two different cakes." Jack's reference to "a whole cake" provides an important clue for inferring the mental operations he used to assimilate the task. To better understand what Jack meant by this phrase, consider his response to my follow-up questioning.

Protocol 4.2: Continuation.

[The teacher researcher acknowledged that Jack was correct that each share is two of the six *pieces* [emphasis added]. Then Jack described again that each of the three shares would include two pieces—one piece from each cake.]

D: So if you think, is there another way of thinking about it? What amount of all the cake does each of us have then?

Jack: Are you combining both of the cakes into one cake now or are they still two different cakes? Because if they're still two separate cakes it would still be…two-sixths.

D: Does it change the way you think about it?

Jack: [Nods in agreement.]

D: If you combine them, how do you think about it?

Jack: It would be, umm, one-third.

D: Why do you say one-third?

Jack: Because you're just splitting it among three people so if it's just one cake, it would still be one-third. It would just be a piece that would be bigger than they were if it was just two separate cakes.

From this exchange, I infer that Jack's initial assimilation of the task presented in Protocol 4.2 involved simultaneously holding two understandings of the fractional relationships.

Notice that for a second time Jack questioned whether we were considering two separate cakes or combining them into one cake. When viewed as two cakes, Jack reiterated his view of each share as two-sixths of the pieces formed from sharing each cake (see Figure 4.1a). In contrast, when he imagined "combining both of the cakes into one cake" each person would receive a piece that was one-third of all the cake because it was one larger cake being shared among three people (see Figure 4.1b). Thus, I believe that Jack's initial reference to "a whole cake" in Protocol 4.2 implied considering the two individual cakes together as a composite unit that encompassed all of the cake to be shared.

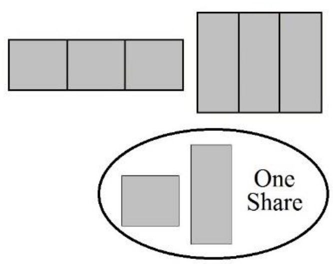

Figure 4.1a. One share viewed as a fair share from each cake.

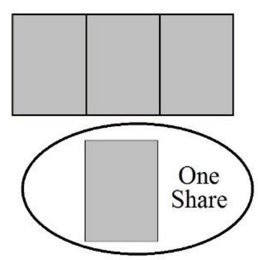

Figure 4.1b. One share viewed as a fair share from a combined larger cake.

However, it is not clear from this exchange how Jack considered these two views in relation to each other. Jack did state that the piece he imagined from the combined cake "would just be a piece that would be bigger than they were if it was just two separate cakes." From this I infer that he imagined that the piece from the larger combined cake would be larger than either of the pieces that came from sharing the individual cakes. However, even though he made a comparison of the relative size of an individual piece formed from each view of the total amount of cake, Jack never explicitly compared the fair shares generated from each view (the individual piece that was one-third of all the cake and the two pieces that were two-sixths of all the pieces of cake).

Unfortunately, I did not ask Jack to make this explicit comparison of the shares generated from each view. However, during the subsequent task Jack did provide further clarification of how he considered the results of these two views in relation to each other. Protocol 4.3 includes an excerpt from when I asked Jack about fairly sharing two equally-sized strawberry cakes among three people. Immediately prior to this excerpt, Jack described cutting each cake into thirds and taking two of the six pieces as one share. In an attempt to ensure that Jack and I were thinking about the same pieces, I asked Jack to actually cut the Play-Doh cakes and distribute the cake as he had described.

Protocol 4.3: Jack shares two equally-sized strawberry cakes among three people.

Jack: [Cuts off a piece of each cake that is roughly one-third of each cake and puts those pieces on his plate.]

D: Okay. So now, if you think of all the cake that you get to eat [points toward the pieces on Jack's plate], what fraction, or what amount, is that of one of the original cakes?

Jack: Of one of the original cakes—it's two-thirds of the original cake, of one of the original cakes.

[…]

D: What amount is that [referring to the two pieces on Jack's plate] of all this pink [strawberry] cake?

Jack: Two-sixths.

D: Is there another fraction you can think about for that?

Jack: Umm…one-third if you're combining both of them.

D: Okay. Can you say a little bit more about that? What did you mean when you say "if you're combining both of them"?

Jack: Because you combine both of them and instead of being split, splitting it into sixths, you're just splitting it into three. And you're basically just adding two of these pieces to make one piece.

D: Sure. So when you say combining them you're kind of just thinking of it all as cake [referring to imagining the two pieces on his plate as combined into one] and then you'd have that three times.

Jack: Um hmm. [Nods in agreement.]

Jack's descriptions in this protocol clarify his understanding of the fractional relationships. His comment, "Instead of […] splitting it into sixths, you're just splitting it into three. And you're basically just adding two of these pieces to make one piece," is particularly revealing. From this, I infer that he not only imagined combining the two cakes to form a composite unit representing all of the cake, but he also imagined combining the two individual thirds of each cake into a composite unit representing one share. Thus, I claim that Jack understood each share simultaneously as two-sixths and as one-third of all the cake depending upon whether he considered all of the cake as two individual cakes or as one composite cake, respectively (see Figure 4.2). Further, given that the strawberry cakes were equally-sized, Jack was also able to interpret one share as two-thirds of an individual cake. Thus, Jack demonstrated the ability use his distributive partitioning operations to understand one-third of two cakes as two-thirds of one cake, and vice versa.

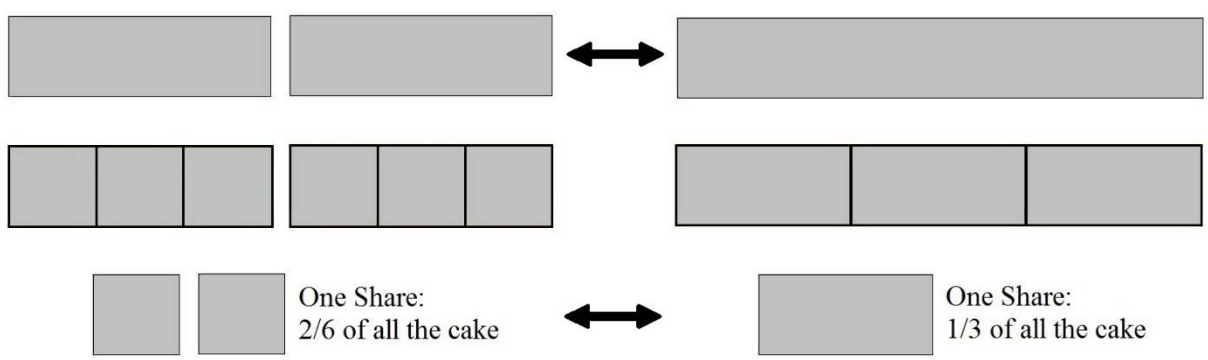

Figure 4.2. Jack's two interpretations of a fair share of the cake.

In addition to exemplifying Jack's distributive partitioning operations, these fractional comparisons provide additional confirming evidence of his ability to take three levels of units as given in assimilation. When considering two individual cakes in Protocol 4.3, Jack understood each share as two pieces, within the three pieces in one cake, within the six pieces of both cakes. Further, when considering all the cake as a composite unit, Jack mentally coordinated two-thirds of a cake, within one share, within the three shares comprising all of the cake.

Given that Jack's understandings and intentions became clear over the course of multiple tasks, it is reasonable to question whether the two views Jack elaborated in Protocol 4.3 (see Figure 4.2) were available to him upon assimilation of the task or if he only constructed the understanding of each share as one-third of the total amount of cake as a result of carrying out mental activity within the context. Steffe (2010f, pp. 20–24) explains that when one has constructed a scheme, the operations of that scheme are used in assimilation. I interpret this feature of a constructed scheme as what accounts for a person's intuition regarding the activity he/she should carry out after assimilating the situation. Thus, this question ultimately concerns whether or not Jack had constructed these ways of reasoning as schemes available to him in assimilation or simply could achieve the fractional coordination in activity.

In Jack's case, I consider his statement in Protocol 4.2, "Because there's two cakes, it's not necessarily a whole cake, it's two different cakes," as particularly significant. While the meaning of "it's not necessarily a whole cake" did not become clear until later in the interview, I take this comment to suggest that Jack was aware of this distinction immediately upon assimilating my initial question and that his choice to treat the cakes as separate cakes was simply that—a choice. This choice was likely influenced by the manner in which the task was presented (with two separate cakes), but it was a choice nonetheless. The fact that during the continuation of Protocol 4.2 Jack could explain the implications of considering the two cakes as one without hesitation supports this inference and suggests that these were not understandings he was constructing in the moment but rather understandings available to him based upon the operations he used in assimilating the task.

This analysis points to a nuance of Jack's distributive partitioning operations that I did not notice during data collection and only first observed during retrospective analysis of his activity. As I mentioned in the chapter introduction, we were trying to identify two students who could view one share as both n/m of one unit and as $1/m$ of n units. As such, Jack's work throughout the pilot study, and again during the check-up interview, demonstrated that he had schemes available to him in assimilation with which to construct those fractional comparisons. However, my retrospective analysis of Jack's reasoning in Protocols 4.2–4.3, compared to his reasoning later in the teaching experiment, revealed that Jack actually constructed two distinct ways of using his operations to construct these fractional understandings. The primary difference in these constructions lies in the manner in which Jack used composite units to construct his understandings. Because I did not observe the second way of operating until later in the teaching experiment, I will only present my analysis of how I believe Jack used his operations during the check-up interview in this chapter while presenting his second way of operating in subsequent data analysis chapters.

Jack's incorporation of composite units into his distributive partitioning operations enabled him to understand one share as one-third of the entire two cakes. In reconciling that two-thirds of one cake is equivalent to one-third of both cakes in Protocol 4.3, I infer that Jack formed two composite units—a composite unit of both of the cakes to represent all of the cake and a composite unit of one piece from each cake to represent one person's share. However, Jack's reasoning in Protocols 4.2–4.3 and during the pilot study indicated that the composite unit representing both cakes was primary in his reasoning. After mentally forming this composite unit to represent all of the cake, he then used this to reinterpret one share as one-third of all the cake. Jack later reconciled this with the results of partitioning each bar and recognized that combining the individual pieces formed from sharing each cake was equivalent to the one-third of the combined two cakes (see Figure 4.3). Figure 4.3 should be interpreted sequentially from top to bottom to mimic my analysis of the way in which Jack used his composite unit operations to construct his understanding of the fractional relationships. The dashed lines are used to indicate an awareness of the individual units contained within the composite unit, even when they are not the primary focus of the reasoning.

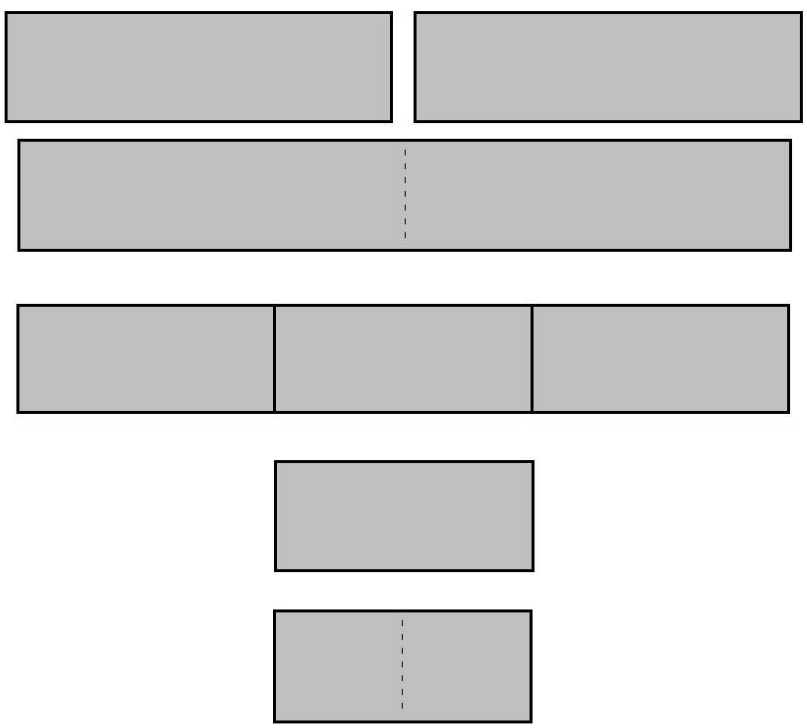

Figure 4.3. The composite two cakes is formed, partitioned into one-third of all the cake for each share, and recognized as equivalent to two-thirds of a cake per share.

Lastly, I would like to discuss one other aspect of Jack's distributive partitioning operations that become apparent during Protocols 4.2–4.3. Note that in Protocol 4.2 and its continuation, Jack repeatedly talked about each share as two-sixths of the cake. However, given unequally-sized cakes, and hence unequally-sized pieces, it is mathematically inappropriate to consider either of the pieces as one-sixth of all the cake. However, I view this as a contradiction from my point of view rather than Jack's. For instance, when justifying why each share was two-sixths in Protocol 4.2 he stated, "Because there's six pieces all together but there's two coming out of them." Thus, Jack derived meaning for his answer of two-sixths from the part-whole relationship among the *number of pieces* rather than from a comparison of the *amounts of cake* contained in each piece. For this reason, while conducting the interview I decided to validate Jack's reasoning that each share was two-sixths of the number of pieces and changed my language to asking for the amount of cake that each share represented. Further, given the sophisticated fractional understandings that Jack used during the pilot study, I remain confident that Jack would have agreed this was inappropriate had I asked him specifically if it made sense to consider either piece as one-sixth of all the cake when they were different sizes.

Jack's Proportional Reasoning

During Jack's initial interview on February 14, 2013, we presented him several tasks within the context of mixing lemonade to investigate how he could coordinate changes in the amounts of two co-varying quantities. To introduce the scenario, we had Jack mix 2 tablespoons of lemonade powder into 3 cups of water. Then, all subsequent questions within the context involved thinking about how to mix up new batches of lemonade that had different amounts of water or lemonade powder but which would always taste the same as the initial batch he had mixed.

Jack's first task was to determine how many cups of water would be needed for only 1 tablespoon of lemonade powder. To maintain the taste of the lemonade, Jack reasoned that he would need to divide 3 by 2 because he only had 1 tablespoon of lemonade powder (i.e., 2 tablespoons ÷ 2 = 1 tablespoon of lemonade powder, so he decided to divide 3 cups by 2 as well). I refer to this type of strategy as a *coordinated partitioning/iterating* strategy in that any partitioning activity Jack carried out on one of the quantities (e.g., dividing 2 tablespoons by 2) was transferred to the other quantity (e.g., dividing 3 cups by 2) in order to coordinate the changes in the amounts of the quantities.[6] This strategy was common to most of Jack's proportional reasoning within the lemonade mixtures context.

However, while referring to this strategy as a coordinated partitioning/iterating strategy adequately describes *what* Jack did with the measures of the quantities, it fails to explain *how* he made the coordination. Thus, I would like to briefly explore the underlying mental operations that enabled him to coordinate changes in the two quantities in such a way that preserved the given ratio. The following two protocols, showing Jack's most sophisticated use of this strategy and a situation in which he encountered constraints to carrying out this strategy, will provide a characterization of the range of operations he used in reasoning proportionally.

To best exemplify Jack's most sophisticated use of this strategy within the lemonade mixtures context, consider the following protocol in which Jack was asked how many tablespoons of lemonade powder would be needed for 1 cup of water.

Protocol 4.4: Jack finds the number of tablespoons needed for 1 cup of water.

W: Suppose you had one, one ah…how much powder? How many tablespoons for one cup?

---

[6] In this case, Jack only used partitioning operations in deciding to split the measure of each quantity into two parts. However, as will become more apparent in subsequent protocols, I use the strategy of *coordinated partitioning/iterating* to refer to instances in which students coordinate either partitioning or iterating activity between two quantities.

[…]

Jack: One cup? If you had three it takes two to make…wait, it takes… [Thinks silently for 25 seconds and then he continues.] If three is two…and then… you're looking…gonna have [writes 1 ½ on his paper]. So one and a half cups will make 1 cup…or is 1 tablespoon. So that equals 1 tablespoon. [Writes "1 ½ = 1" on his paper.] And, see we only needed to make 1 tablespoon…so that means that three halves make up 1 tablespoon…so you'd only need two-thirds of a tablespoon to make 1 cup of lemonade.

In determining that he needed two-thirds of a tablespoon to make 1 cup of lemonade, I infer that Jack formed a goal of finding the unknown equivalent ratio, some unknown amount of tablespoons for 1 cup, and then carried out the operations required to determine the unknown equivalent ratio. Thus, Jack's way of operating in this protocol is consistent with the construction of an intensive quantitative unknown (Steffe, Liss II, et al., 2014).

Significantly, the extensive quantitative operations he used were multiplicative in nature rather than additive operations such as subtracting one unit from the measure of each quantity. To better understand what I mean by that, consider the following example. After thinking silently about the task for approximately 30 seconds, Jack stated, "So one and a half cups will make 1 cup…or is 1 tablespoon." From this I infer that Jack knew he could split 2 tablespoons in half to obtain 1 tablespoon and thus used his splitting scheme to intuitively split the 3 cups of water into one and a half cups. I consider Jack's use of his splitting scheme in this instance to be multiplicative in nature in that splitting each quantity into the same number of parts inherently maintains the initial ratio between the quantities.

Having established 1 tablespoon of lemonade powder as requiring one and a half cups of water, Jack carried out the rest of his coordinated partitioning/iterating strategy to make definite the unknown equivalent ratio. In doing so, I infer that Jack took three-halves cups of water as input for his reversible fraction scheme to produce 1 cup of water as 2 of the 3 one-half cups. I then claim that Jack transferred this activity to the equivalent quantity 1 tablespoon to determine that 2 of the 3 parts of 1 tablespoon of lemonade powder, or two-thirds of a tablespoon, would be needed for 1 cup of water (see Figure 4.4 for a model of this activity).

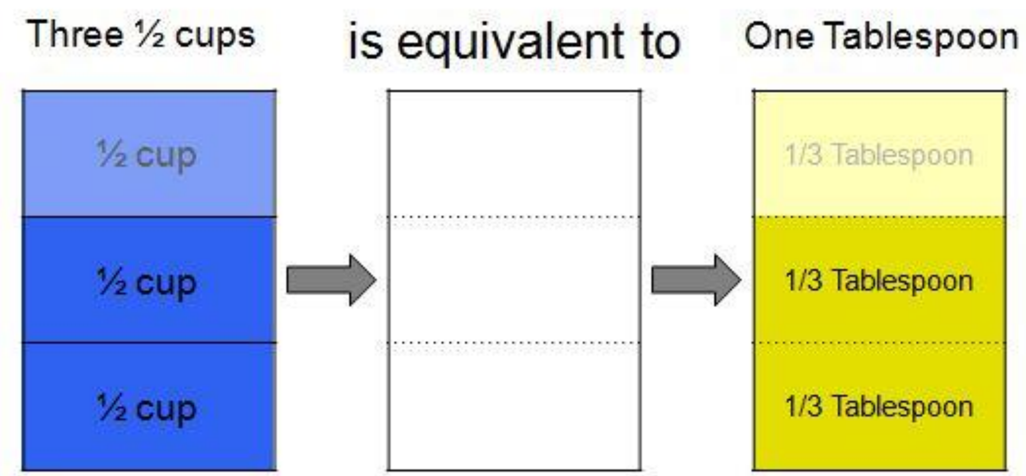

Figure 4.4. A model of Jack's coordinated partitioning/iterating activity.

Jack's coordination of the ratio of water to lemonade powder from 3:2, to 1 ½:1, to 3/2:1, to 1:2/3 indicates that he had abstracted the three for two relation between the number of cups of water and tablespoons of lemonade powder. To maintain this ratio he appears to have repeatedly decided how to operate on one quantity and then performed those same operations on the other quantity.

Additionally, Jack's reasoning in Protocol 4.4 supports my characterization of his coordinated partitioning/iterating strategy as involving transferring his partitioning operations from one quantity to the other. There are two instances in Protocol 4.4 where Jack referenced a different quantity than the numeric relationships he had used would suggest. For instance, after deciding to split the 3 cups and 2 tablespoons in half he initially said that one and a half cups will make "1 cup" rather than 1 tablespoon. Further, when Jack said, "And, see we only needed to make 1 tablespoon," I believe he meant we only needed to make 1 cup. These instances in which Jack described one quantity but stated the other suggests that the quantity upon which he decided how to operate (e.g., deciding to split 3 cups in half) remained active in his thinking as he carried out the same partitioning activity on the measure of the other quantity (e.g., splitting the 2 tablespoons in half). This is why I refer to this strategy as coordinated partitioning/iterating as it involved coordinating his activity between two quantities and using both partitioning and iterating operations to achieve the transformations.

However, while Jack typically executed his coordinated partitioning/iterating strategy successfully, he also encountered some constraints while enacting this strategy. For example, prior to asking Jack about 1 cup of water (Protocol 4.4), the teacher researcher had asked Jack how many tablespoons of lemonade powder would be needed for 5 cups of water.

Protocol 4.5: Jack attempts to find the number of tablespoons for 5 cups of water.

T: So now we want to make 5 cups. Okay. Can you tell me how many tablespoons we would need?

Jack: Okay. So we want to make 5 cups of lemonade. We'd have three. But we're only making five so it wouldn't be…if you wanted to, 4 tablespoons would make 6 cups. But we only want to make 5 cups. So you'd have the 2 tablespoons for the first 3 [cups], and then if you take the one-half tablespoon it make 5. So it would be—or no. Two cups make three and then you have one and a half would make, would go from three…three to four. No. [Shakes his head no.]

Jacks assertion that 4 tablespoons would make 6 cups again exemplifies his coordinated partitioning/iterating strategy. However, in contrast to the fraction operations he used in Protocol 4.4 to decrease the measures of each quantity, in this case he assimilated the task of enlarging the measures of each quantity to his whole number multiplicative operations to double the measures of both quantities.

However, we also see that once Jack recognized these operations would scale the quantities beyond the desired 5 cups of water, he struggled to find a way to implement his available operations to achieve his goals. Consider Jack's statement, "Two cups make three and then you have one and a half would […] go from three…three to four." From this I infer that Jack again attempted to use his coordinated partitioning/iterating strategy to split each quantity into two equal parts. However, in this instance I infer that he then conflated the quantities that each measure referred to when attempting to coordinate that result with the 3 cups for 2 tablespoons. Thus, Jack increased the number of cups of water by one and the number of tablespoons of lemonade powder by one and a half rather than the other way around. Jack seemed aware that this was not correct, uttering "No" and shaking his head. Yet, he was unable to solve this task even after thinking for about it for several more minutes. Thus, this protocol both highlights Jack's repeated attempts to use his partitioning and multiplicative operations to solve proportional comparison tasks and also signifies some of the challenges he encountered while enacting those operations.

Summary of Jack's Mathematics

Jack's initial and check-up interviews indicate that he had constructed rather sophisticated quantitative operations prior to the start of the teaching experiment. First, Jack's check-up interview provided confirmation of earlier indications that he had constructed three levels of units as given in assimilation. This feature of his reasoning was central to his partitioning schemes and his demonstrated ability to coordinate changes in quantities. In addition, Jack indicated having constructed sophisticated distributive partitioning operations that allowed him to understand one of m shares of n units as simultaneously n/m of one unit and $1/m$ of n units. Lastly, he demonstrated a coordinated partitioning/iterating proportional reasoning strategy that he could at times use to find intensive quantitative unknowns. Further, the quantitative operations he used to carry out his coordinated partitioning/iterating varied depending on the particulars of the task and included whole number multiplicative schemes, reversible fraction schemes, splitting, equi-partitioning, and iterating.

One other noteworthy aspect of Jack's mathematics is that he consistently took a very persistent approach to problem solving. Often Jack's operations where sufficiently powerful for him to assimilate our tasks and decide how to proceed rather quickly. For example, in the fruit-by-the-foot and cake sharing tasks, Jack never needed more than 15–20 seconds to assimilate the situation and carry out his operations mentally. However, at times when Jack did not immediately assimilate the situation to an operative scheme, he never gave up on tasks. Rather, he would often think intently for as long as several minutes to come up with a strategy he could use to solve the task. While this persistence did not always pay off (e.g., Jack never solved the task in protocol 4.5 despite thinking about the situation for several minutes), Jack always seemed inclined to use his available quantitative operations to engage deeply with the situations.

John's Initial Interview

Unlike Jack, John did not participate in the pilot study. Consequently, I conducted his initial interview immediately prior to the dissertation study on October 3, 2013. In order to learn about John's available ways of reasoning, I used the same initial interview protocol with John as I did with Jack at the start of the pilot study. In the following sections I will characterize what I inferred about John's units coordination, distributive partitioning, and proportional reasoning abilities, respectively.

John’s Levels of Units

Comparing John’s work across all tasks from the initial interview, I infer that John had also constructed three levels of units that he could take as given in assimilation and reasoning. The clearest indication of this ability occurred during the splitting task when John was asked to imagine a piece of string that stood in a multiplicative relationship with a given piece of waxed string.

Protocol 4.6: John mentally splits a piece of waxed string.

D: [Picks up a piece of waxed string and sets it in front of John on the table.] So this is my piece of string, and I want you to imagine that you’ve got a piece of string so that my string is 5 times longer than yours. Okay?
John: Okay. So that one’s 5 times longer than mine. So, okay.
D: Just take a moment and picture what your string would look like. So this [points to the string on the table] is my string and it’s 5 times longer than the string you’re going to imagine.
John: It would be really small.
D: How…so do you want to go ahead, umm, describe it for me? What are you thinking about?
John: It’s one-fifth the size.
[…]
D: How much would it be of this string? [Points to the string on the table.]
John: One-fifth of the size. Pretty much.
D: One-fifth of the size?
John: Because it [referring to the waxed string on the table] is 5 times bigger than mine.
D: So you could cut off a piece and then how would you check if the piece you cut off was the right size?
John: If it equals five of it. [Holds his fingers apart by a width that would be roughly one-fifth of the given string]. Like if five…like if you add it five times, like with the size, it equals the original. [Keeping his fingers held apart, he taps this on the table while explaining that five should equal the size of the original.]

Students that struggle with this task will often assimilate the initial question as a call to make a string that is 5 times longer and hence iterate the given string to accomplish this goal. In contrast, to solve this task John posited a hypothetical piece of string that would be “one-fifth the size” and so that the given string “equals five of it.” While John used the word “adding” to describe what he meant by this, I infer his reference to adding corresponded to the mental activity of iterating the piece 5 times and repeatedly adding its measured length (i.e., its "size"). This inference is also supported by the fact that earlier in the interview John used a similar strategy of iterating a piece of marked off waxed string to test if it was a fair share of a whole. In consideration of that earlier activity and John's descriptions in the above protocol, I believe that the multiplicative relationship between the given string and the hypothetical string he imagined was inherent to his assimilation of the task. In other words, I infer John assimilated the task as a situation of his splitting scheme.

John's construction of a splitting scheme provides evidence that he had constructed three levels of units that he could use in assimilation (Steffe, 2010d). Further, similar to Jack in Protocol 4.1, John never engaged in actually partitioning the string or iterating a piece to verify and construct the multiplicative comparison. Rather, he operated on the given string hypothetically and created a new string in his mind that satisfied the desired relationships. This ability to hypothetically carry out one's schemes and take the results as input for further operating provides additional evidence that John had assimilated the task using three levels of units rather than constructing them in activity. Further, John's activity in other tasks during the initial interview also suggested the availability of three levels of units in assimilation.

However, while I believe John assimilated tasks with three levels of units as a given, he also consistently seemed unaware of his quantitative operations. For example, consider John's response to the sharing a share of fruit-by-the-foot task. To begin the task, John unrolled the strip of fruit-by-the-foot and placed it on the table. After explaining that they would be imagining sharing the strip of fruit-by-the-foot, the teacher-researcher covered up the strip with a handkerchief to test if John could create figurative material upon which he could operate hypothetically and use in further reasoning.

Protocol 4.7: John reasons about a share of a share.

D: So you start with the whole thing.
John: Okay.
D: And you cut off one-fifth of it. Alright?
John: Oh, one-fifth. Okay.
D: Alright. Now you take that piece—do you have that piece in mind?
John: Um hmm.
D: And you share that piece among three people. Okay so you're going to cut that again and you've got another piece. That last piece you have—what amount is that of the whole?
John: So I'm going to be sharing it, right?
D: Um hmm.
John: [Thinks for 8 seconds. While he is thinking he moves his finger in the air above his lap as if it were a pencil. He appears to "write" 5×3 in the air before answering.] One-fifteenth.
D: Why one-fifteenth?
John: Because you cut it from one-fifth and divide, you give it to three persons.
D: Okay.
John: So I did 3 times 5, which is 15 so you get one-fifteenth of the whole entire length of the candy.

From this task I infer that John was indeed able to create and operate hypothetically on imagined figurative material. Further, it is clear that he assimilated this task as a situation of his whole number multiplicative scheme. However, John's motivation for carrying out this multiplication and the meaning it held for him within the context of sharing the fruit-by-the-foot remain less clear. Thus, to further investigate how he assimilated the task, I questioned him about the multiplication.

Protocol 4.7: First continuation.

D: So when you were saying 3 times 5, what were you [thinking]? Why were you thinking about the 3 times 5? What did that mean to you?
John: Because you divide the whole thing by 5. It's like divided by 5 because of the one-fifth. And you only have, like, 1 of 5. And I just divided by 3 again. Was it times or was it divided? [Puts his head in his hand.]
D: That's okay. Take your time.
John: [Thinks for 11 seconds.] Because I was trying to make it to a whole number. Like a fraction number.
D: Okay. Yeah, and what you said—I was just trying to understand when you were saying the 3 times 5, kind of, what that meant to you.

John: It meant like, how, like how I split pieces. Like because I'm trying to put like, if it's five—because the whole length was 5 out of 5, which is still one. I'm just trying to put it so it could both the in the…it would be under one.

D: What do you mean by that?

John: Like, because you only got one-fifth of it and you divide by 3. Then I did 3 times 5 because…because…hmm…I don't really know how to explain it. Because I learn math, the way I learn math I learn it, like, kind of different from how other people learn it.

I infer that the source of John's multiplicative reasoning came from an intuitive use of his splitting scheme and, hence, having three levels of units available in assimilation. For instance, consider his initial reply, "Because you divide the whole thing by 5. It's like divided by 5 because of the one-fifth. And you only have, like, 1 of 5." Interpreted through the lens of the splitting scheme, this statement suggests John's simultaneous awareness that the first cut-off piece is one-fifth of the whole strip and that the strip is 5 times as large as this piece. Similarly, sharing that piece among three would imply that the final share was one-third of the initial piece and the initial piece was 3 times as large as the final share. Hence, I believe that the multiplicative awareness that stems from the splitting scheme accounts for his decision to multiply 3 times 5 when considering a split of a split.

Yet, while this explanation accounts for John's intuition regarding the multiplication, it is also clear that he struggled to construct a justification for why multiplication was sensible. In fact, his attempts to explain his reasoning reveal some conflicting understandings. In particular, I infer that John holds two meanings for division that he has not consciously distinguished between—dividing as splitting a continuous unit and dividing as a numeric operation one carries out with two numbers. With one exception, I think John relied upon the former meaning throughout Protocol 4.7 and its first continuation. The exception is when John said, "And I just divided by 3 again. Was it times or was it divided?" In the first statement I infer John meant, and understood, divided by 3 as splitting the initial cut-off piece into three parts. However, in the second statement I infer John used “times” and “divided” in the sense of numerical operations rather than quantitative operations. Putting his head in his hands underscores his confusion between the two statements he just made and supports the conclusion that John had yet to distinguish between these two usages of dividing.

This conflation appears to have put John in a state of perturbation—why did he multiply 3 times 5 to determine one-fifteenth when he was dividing (from my perspective splitting) the one-fifth piece into three parts? John’s other statements in the first continuation of Protocol 4.7 demonstrate his unsuccessful attempts to reconcile this perturbation. Further, these conflicting understandings appeared to create a stressful situation for John in which he put his head in his hands and tried to explain how he learned differently in response to being unable to alleviate the perturbation. I decided to encourage John to carry out his mental operations on the actual strip of fruit-by-the-foot in hopes this activity would help him to alleviate this perturbation.

Protocol 4.7: Second continuation.

D: That’s okay. So, let’s just go back to the strip for a moment here. [Uncovers the piece of fruit-by-the-foot.] So can you just kind of put your hand on where you have made that first cut about for the one-fifth of the whole strip?

John: [Moves his hand in the air above the strip of fruit-by-the-foot and pauses 4 times while spanning the whole strip.] Like probably right about here. [Places his hand down at a place that marks off roughly one-fifth of the strip.]

D: Okay. And then I asked you to take and share that strip—that one-fifth part—among three people. Right?

John: Um hmm. It would get, like, smaller, to like right here. [Places his other hand down at a place that marks off roughly one-third of the piece his other hand was marking off.]

D: Okay. So, umm, so how many pieces like that one on the end then [points to the final piece he had just marked off with his second hand] would you be able to make in this whole bar and why?

John: There should be about 15.

D: And why do you think 15?

John: Because…hmm, I don’t know. Because…let’s see.

D: That’s okay. You’re doing good.

John: [After thinking for 23 seconds, John replied.] Because you could, you basically…it has to do something with the 3 times 15.

D: Three times 15?

John: Oh, I mean 3 times 5. Yeah. Because you do the opposite of…what was it, divide. I think. Then I times it. Because for every five there's three so I just times by 3 for every [Moves his hand along the strip of fruit-by-the-foot pausing at each of the five one-fifths of the whole strip], like to get 15 pieces.
D: There you go. Okay so that makes sense with 3 times 5, right?
John: Yeah.

Initially the activity of marking the two cuts with his hands did not help. Even though John knew "there should be about 15" of the small shares in the whole strip, after thinking for over 20 seconds he still was unsure how to explain the relevance of 3 times 5. However, eventually John found a way to justify his use of multiplication and alleviate his perturbation. His statement, "Because for every five there's three" and his subsequent activity of pausing at each of the fifths of the original strip suggest that he constructed meaning for his multiplication through recursively partitioning each of the fifths into three parts. However, while he established a partitioning meaning for the task, it is also apparent that this was not the way he was thinking about the task initially. Further, even though his use of recursive partitioning appeared to resolve his perturbation, I do not think John ever reconciled his view of dividing as splitting with his view of dividing as a numeric operation.

I included Protocol 4.7 and its continuations in this discussion of John's levels of units because I believe it reveals a critical feature of his mathematics and an important implication of this feature. First, these excerpts exemplify how John often reasoned with quantities intuitively, in ways that suggest assimilating with three levels of units, without being explicitly aware of how to explain this intuition. However, despite his lack of awareness, I consider these intuitive moments as evidence of his construction of underlying quantitative schemes and operations. As was the case with the splitting scheme in Protocol 4.7, accounting for these underlying quantitative schemes can explain John's intuitive reasoning. The fact that John remained unaware of these ways of operating speaks more to the degree to which he had abstracted these operations rather than their availability. Specifically, using the language of von Glasersfeld and Piaget, John's quantitative schemes and operations often appeared to be on the level of reflective abstractions rather than reflected abstractions of which he was aware (cf. Chapter 2).

An important implication of this lack of awareness of his quantitative schemes and operations is that John often assimilated requests to solve a task and requests to explain his solution of a task differently. As I claimed above for Protocol 4.7, John typically assimilated the former as situations of his quantitative schemes and operations. However, the latter were often assimilated as questions about numeric operations. As was the case with division, there were times in which John's quantitative reasoning conflicted with his numeric reasoning constructed from school mathematics. Having not consciously distinguished between these two for himself, the interplay of these forms of reasoning plays a central role in the story of John's mathematics throughout the teaching experiment.

Lastly, it is important to note that some of my understandings of these distinctions were first developed during retrospective analysis of John's teaching sessions. During data collection I was aware of John's tendency to explain his thinking by appealing to numeric operations such as multiplying and dividing. However, I have since developed a deeper understanding of the nuances to John's ways of assimilating and reasoning with tasks. As a result, it is most accurate to say that during the teaching experiment I operated with a general awareness of this distinction and have developed a more thorough understanding of the phenomenon through retrospective analysis.

John's Partitioning Operations

In addition to suggesting John's construction of three levels of units, Protocols 4.6 and 4.7 also provide important evidence regarding his partitioning operations. In particular, John's construction of a splitting scheme indicates that the scheme he uses to assimilate situations involving sharing/partitioning an individual unit is an extensive quantitative scheme (cf. Chapter 2). In addition, John eventually made use of a recursive partitioning scheme to justify the need to multiply 3 times 5 in Protocol 4.7.

In addition to constructing these partitioning schemes, John also solved all of the distributive partitioning tasks during the initial interviews. In particular, he demonstrated that when sharing two equally-sized cakes among three people, he could interpret one share as two-thirds of one cake and as one-third of all the cake. However, while both Jack and John constructed these mathematical relationships among the quantities, I infer that John's basis for these understandings was slightly different than Jack's reasoning as described above. To exemplify John's characteristic way of reasoning, consider his response to the task of sharing two equally-sized strawberry cakes among three people.

Protocol 4.8: John shares two equally-sized strawberry cakes among three people.

D: So you want to share all of this—three people. Say the three of us want to eat this. We want to eat it all up. What I want you to think about is how would you share it so that we all got a fair amount?

John: Cut them into three equal pieces. Like, cut this one into two-thirds and that one into two-thirds.

D: Say a little bit more. What do you mean by the two-thirds?

John: Because if you cut it into two-thirds there would be one-third left and you could, the other person could, share with the one-third on that one and the one-third on that one.

From this brief exchange, I infer that John assimilated my sharing question as a situation of his quantitative operations and formed a goal of finding the fractional amount of one cake that each person would receive. However, it is less clear what operations account for John's almost immediate awareness of each of these three shares as two-thirds of a cake.

The key to understanding the operations that John used to solve this task resides with his meaning for "them" in his reply "cut them into three equal pieces." Two plausible explanations exist. John could have used “them” to refer to each cake individually and, hence, imagined cutting each individual cake into three shares. Alternatively, “them” could also have been a reference to the total collection of cake, as in cut the total amount of cake into three equal shares. I briefly consider the implications of each possibility before discussing my hypothesis for which interpretation reflects John’s ways of operating at the time of his initial interview.

Suppose that “them” referred to each cake individually. Then, John would have imagined making a total of six pieces by cutting each cake into three parts, which would indicate that he distributed his partitioning activity across two individual cakes. This, in combination with his awareness of each share as two-thirds of a cake, would indicate John had constructed at least a distributive partitioning scheme.[7] Further, to construct each share as two-thirds of one cake, John would have reconstituted the six pieces from two units (i.e., cakes) containing three pieces to a new structure of six pieces as three units (i.e., shares) containing two pieces. Reorganizing one three levels of units structure into a different, but equivalent, three levels of units structure in this fashion is indicative of the reasoning made possible by one’s construction of a generalized number sequence (Steffe & Olive, 2010). Thus, if “them” referred to each cake individually, this protocol would provide evidence that John had constructed a generalized number sequence and at least a distributive partitioning scheme.

Alternatively, suppose that “them” referred to the total collection of cake. Then, John’s reasoning would indicate that he used unitizing operations to unite the two individual cakes into a single composite whole. This supported his assimilation of the task as a situation of his scheme for finding thirds of a single unit and is consistent with forming a goal of splitting the entire

[7] To make the judgment that he had constructed a *reversible* distributive partitioning scheme, John would have to also provide evidence that he understood each share as one-third of both cakes. Because that question was not part of this protocol, such a judgment is not possible from this excerpt.

amount of cake into only three pieces for the three fair shares. Having two individual cakes from which to construct those three shares, John could then imagine cutting one fair share off each cake and then combining the leftover pieces to form the third fair share. In this case, that meant cutting two-thirds of a cake from the first cake, two-thirds of a cake from the second cake, and then combining the leftover one-thirds of each cake to form the third share of two-thirds of a cake (see Figure 4.5; the dashed line indicates his awareness that the total amount of cake consisted of two cakes). Thus, rather than indicating a distributive partitioning scheme and a generalized number sequence, Protocol 4.8 would provide evidence of John's ability to use uniting operations to form composite units that he could use in further operating.

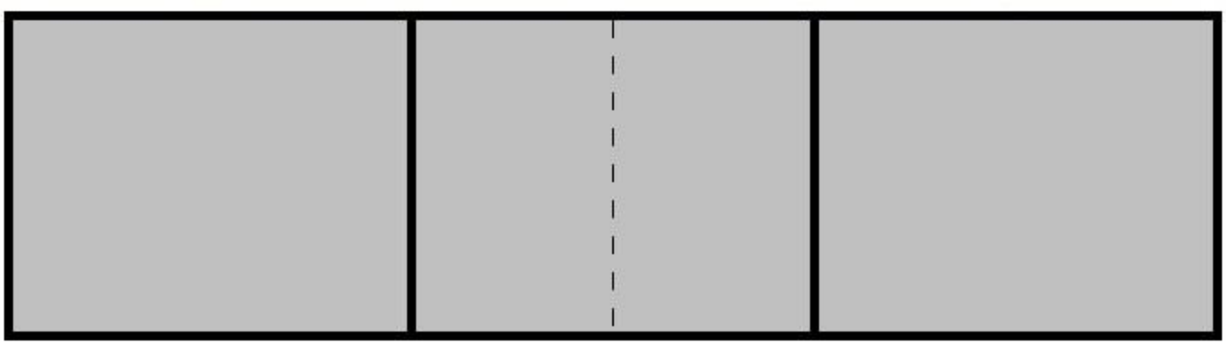

Figure 4.5. A model of John's reasoning: Three shares of two-thirds comprise all the cake.

Reflecting upon these two possibilities, I claim that the latter more accurately reflects the reasoning John used during Protocol 4.8. First, consider that John said "Cut this one into *two-thirds* [emphasis added]" rather than describing cutting the cake into *thirds*. This is consistent with only imagining producing three shares, rather than with producing six pieces that could later be reconstituted as three shares. Further, referring to each piece as two-thirds could suggest that he conceived of two-thirds of a cake as a single composite unit, rather than as two individual one-third pieces.[8] Thus, cutting two-thirds off the first cake would not produce three pieces and John would not have engaged in distributive partitioning. Instead, by treating two-thirds of a cake as a single unit John would have imagined creating two pieces in each cake—a two-thirds piece and a leftover one-third piece. This is consistent with his explanation that the third person could "share with the one-third on that one and the one-third on that one."

However, considered in isolation, the interaction in Protocol 4.8 does not provide enough evidence to claim either of these two alternative analyses is correct with a high degree of certainty; yet, as Steffe and Thompson (2000b) describe, one important aspect of a retrospective analysis is one's ability to consider an interaction prospectively with respect to the student's activity that came afterwards. Thus, I considered this interaction with respect to John's reasoning throughout the remainder of the teaching experiment and found that his activity here was very similar to the interaction is Protocol 6.10 (see Chapter 6). In both cases, my inference is that he did not engage in distributive partitioning but rather treated the fractional unit two-thirds as a single unit. As a result, my retrospective analysis of John's activity in protocol 4.8 is that he relied upon his unitizing operation to construct and reason with composite units to solve the task.

Supposing this second hypothesis, one question remains—whether or not John anticipated the result prior to acting. For example, I claim that John formed a goal of splitting a composite two cakes into three parts. This would entail finding the fractional amount of each cake such that three iterations of this amount would comprise the complete two cakes. John's replies in Protocol 4.8 clearly indicate that he established two-thirds of a cake as the amount that

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[8] By virtue of referring to each share as "two-thirds" of a cake, I believe that John was at least tacitly aware of each cake as containing three parts. However, the claim is that he treated two-thirds of a cake as a single unit and thus did not explicitly focus on partitioning each cake into three parts in service of partitioning all the cake into three parts.

would accomplish this goal. In doing so, John established two cakes as a composite unit containing three composite units, each of size two-thirds. Thus, he established a three levels of units structure on the basis of an iterable/iterating fractional unit, which is a significant constructive achievement.

The question that remains is whether John anticipated this structure prior to acting or whether he constructed this awareness in activity. His reply of two-thirds appeared very intuitive to me, both in the moment of the interaction as well as when watching the videotape during retrospective analysis. I believe John's intuition in this case was supported by the fact that the task included only two cakes and three people. In other situations during the teaching experiment when the numbers of units and shares were greater, his activity did not initially indicate this same type of intuition. For this reason, my inference is that he most likely considered two-thirds as a possibility and quickly recognized that two-thirds of a cake would achieve his fractional and sharing goals.

Returning again to John's initial interview, recognizing that he had established each of the three shares as two-thirds of a cake, I decided to question him further to see if he had established a multiplicative relationship between this share and the entire amount of cake. His replies to these questions also help to clarify the operations that John may have used to decide upon each share as two-thirds of a cake in the first place.

Protocol 4.8: Continuation.

D: What amount of all of the strawberry cake would you get to eat?
John: Like, amount. Umm, like what kind of amount? Of the whole two?
D: So, yeah, of the whole two. So you said you would have two-thirds of one cake. Right?
John: Yeah.
D: What amount of all the cake would you get to eat?
John: Umm…pretty much…33%.
D: Why do you say 33%?

John: Because there's…because one person, like, 33 point 33 [i.e., "33.33"]. One person get 33.33 and the other person get 33.33 and it's close to 100 when you add those up.
D: Okay. So you're thinking about it with the numbers and splitting up 100 percent.
John: Um hmm.
D: Umm…what about a fraction? What fraction would you get of all the cake do you think?
John: One-third of the cake—of two whole cakes.
D: So why does that make sense to you?
John: Because, I'm more like a visual learner. I learn from how things work by seeing what people do and [inaudible].
D: Okay. So when you say you're thinking about it visually, what kind of visual do you have in mind when you're thinking about that one-third?
John: Hmmm…well just cutting it into two-thirds so a person would get…each—because two-thirds plus two-thirds and plus two-thirds equals two. So I think of just cutting it into equals will make three persons happy.

Here we see that John interpreted one share as 33.33% of all the cake and as one-third of all the cake. While it is possible he established the share as one-third of all the cake because he knew that 33.33% was an approximation of the fraction one-third, I think that both replies actually stemmed from the same interpretation of considering all the cake as a composite whole. For instance, John explained how three shares of 33.33% and three shares of two-thirds of a cake add up to the total amount of cake (100% and two cakes, respectively). In addition, he characterized himself as a visual person and imagined "just cutting it into equals." Each of these components of his reasoning is consistent with the hypothesis that John's understandings in both Protocol 4.8 and its continuation stemmed from same conception of the total amount of cake as a composite whole rather than operating upon each cake individually.

My model of John's reasoning during the initial interview is that he assimilated the task by forming a composite unit to represent all the cakes, split that composite into three parts, and then tried to decide upon the size of each part so that three iterations would comprise all the cake. Further, I infer that he established two-thirds of a cake as the amount of cake that would quantify the size of each of these three shares of the composite whole because as he says, "two-thirds plus two-thirds and plus two-thirds equals two." Thus, he formed a multiplicative relationship between each share and the total amount of the cake on the basis of treating the composite total amount of cake as a single composite unit and applying his splitting scheme. John's initial response to the task of sharing two unequally-sized cakes among three people lends further support to this hypothesis.

Protocol 4.9: John shares two unequally-sized chocolate cakes among three people.

D: So that was with strawberry cake. Remember they were both the same? Kind of a similar scenario. But what if…what if we have two cakes that are different sizes? [Places two unequally-sized chocolate cakes in front of John.]

John: Well you can't really divide it equally…if you don't know their—hmm, what is it? How much…like their size. We don't know if the sizes are equal.

D: Well let's think about it for a little bit. So, umm, you had some really good ideas with the strawberry cakes. So this time we have two chocolate cakes—different sizes from each other—but the three of us still want to eat all of this cake up. […] So I want you to first think about—is there a way you could share it so that we used all of the cake but we all got a fair amount?

John: [Thinks silently for 8 seconds.] You could probably find a value or just, yeah the value.

D: Well, what do you mean by that? What are you thinking about with the value?

John: Like how much…like…never mind. Not like, not the value. Well I can't really think of a way. Pretty much.

I believe John's concern of not knowing the cakes "size" or "if their sizes were equal" indicates that he wanted to form a composite unit of all the cakes but was not sure how given that the cakes were different sizes and shapes. Further, this is consistent with John's initial reference to, and eventual decision to give up on, finding a "value." I infer that the value he had in mind was a value he could assign to each person so that three of those shares would comprise all of the cake. However, because the cakes were different sizes there was no clear value to assign to the total amount of cake compared to the previous protocol where John knew the three shares together had to form two cakes.

After deciding this strategy of finding the value would not work in this case, John attempted some trial and error cutting of the cakes and eventually experienced a moment of intuition.

Protocol 4.9: Continuation.

> [The interviewer gave John a knife and encouraged him to experiment with how he could cut the cakes. After 105 seconds of experimentation, John continued with the following strategy.]

John: Well, I would cut…like probably cut this [points to the smaller cake] into three and this [points to the larger cake] into threes.
D: Okay, why do you say that? Or what would you do once you did that?
John: Because they would get a fair amount of each one.
D: Okay. Good. So if you did that, what would—what pieces would you end up putting on this plate then?
John: A big one and a small one.
D: Okay. So what amount would that person get of all the chocolate cake?
John: One-third.
D: Why one-third?
John: Because the two [points alternately at both of the cakes]…well because one-thirds umm, because it's three person and you're spreading it between three person[s]. So you divide it by 3.

Thus, we see that after moving on from his initial assimilation of the task, John eventually recognized that fairly sharing each cake would allow him to produce a fair share of all the cake. In contrast to the previous protocols, in this case John does engage in a distributive partitioning strategy in the sense that he consciously decided to partition each cake to achieve his sharing goal. However, while this strategy allowed him to envision the fair shares, his fractional comparisons still relied upon the notion of a composite unit comprising all of the cake "because it's three person and you're spreading it between three person[s]."

Considered together, Protocols 4.8–4.9 provide evidence that John understood the fractional comparisons I was looking for during the initial interviews. Thus, at the time I considered his activity during the initial interview to indicate that John, like Jack, had constructed a reversible distributive partitioning scheme that he could use in assimilation.

However, in retrospect it is not entirely accurate to call his way of reasoning reversible distributive partitioning. Prior to this continuation of Protocol 4.9, John never explicitly engaged in distributing the partitioning activity across the multiple cakes. Rather, I infer that John formed a composite unit to represent all the cake and also conceived of each share as a single composite unit. This supported his ability to conceptually split the cake for three people and to conceive of each share as one-third of all the cake. For this reason, I say that John *achieved the results* of a reversible distributive partitioning scheme but that he did not engage in the activity of distributing the partitioning across multiple units to construct his fractional understandings. In contrast, during the continuation of Protocol 4.9 I infer that John *did* carry out a distributive partitioning of the cakes to accomplish his sharing goal. Thus, I infer John had distributive partitioning operations available, but at the time of conducting the teaching experiment I had overestimated the extent to which he had abstracted these operations.

John's Proportional Reasoning

We also engaged John in the lemonade mixtures context to explore how he could use his available quantitative operations to coordinate changes in two quantities as they varied in a fixed relationship. Considered as a whole, I would characterize John's strategy as coordinated partitioning/iterating in that like Jack, I interpret John's strategy as one of deciding how to operate on one of the quantities before transferring this operating to the other quantity. However, the specific quantitative operations that supported this strategy varied from the operations Jack relied upon.

In particular, in retrospect I found that John's reasoning within the lemonade task was consistent with the ways of operating exemplified in Protocols 4.6–4.9 above. For example, John's first task was to determine how many cups of water would be needed for only 1 tablespoon of lemonade powder. John immediately responded 1.5 cups and explained that he divided by 2 because it was 3 cups for every 2 tablespoons. I infer that John assimilated this task as a situation of his splitting scheme and was able to rely upon the multiplicative awareness it entails to intuitively know that he needed to split the quantity of water in half as well.

At times John also appeared to use the same quantitative operations he demonstrated in the distributive partitioning tasks and relied upon reasoning with a composite whole. For example, consider his replies when reasoning about the number of tablespoons needed for 1 cup of water.

Protocol 4.10: John finds the number of tablespoons needed for 1 cup of water.

John: [Thinks silently for 5 seconds.] Probably, you need one-third of it.
D: What do you mean? Can you say a little bit more about that? What do you mean by "One-third of it"? Or why do you think one-third of it?
John: Wait, I meant two-thirds. Wait, let me see. One…
D: Just talk your way through it.
John: [Thinks silently for 20 seconds.] So for one, right?
D: Yeah.
John: For just one and make it taste the same.
D: One cup of water. How much lemon powder to make it taste the same as what we started with?
John: Probably two-thirds now.
D: So why two-thirds?
John: Because for every one-thirds it equals…like, let me see. I think every one-thirds equals one [points to the one-cup line on the pitcher that currently has three cups of lemonade in it] and every another one-third equals one [Points to the two-cup line on the pitcher] and one-third for one [points to the three-cup line] for the water.

Unfortunately the bell rang signaling the end of class before I could follow-up on John's solutions of one-third and two-thirds. However, my inference is that John assimilated this task to the same quantitative operations as those he used in Protocol 4.8 above. I don't believe that this was simply a recognition that the same coordination was involved (i.e., 2 tablespoons split into 3 cups compared to two cakes split among three people). Rather, I believe that John assimilated the change in the quantity of water in this case as a situation of his splitting scheme. Hence, he could use his awareness that the new amount of water was one-third of the original amount of water to decide that he would need one-third of the total amount of lemonade powder as well. And while he never explained why he also viewed this as two-thirds of a tablespoon, my inference is that he identified two-thirds as the number of tablespoons that he could mentally insert into each one-third of the lemonade powder to produce the original 2 tablespoons.

In addition to forming composite wholes to represent the total amount of a given quantity, John's activity in the lemonade mixtures context also revealed another aspect of his reasoning with composite units. For example, consider his reply to the following task.

Protocol 4.11: John's attempt to find the number of tablespoons for 15 cups of water.

D: So the scenario is we have 3 cups of water for every 2 tablespoons. And now I'm thinking about if we wanted to make up a batch with 15 cups of water, how many tablespoons of powder would you need?
John: [Thinks silently for 10 seconds.] Fifteen…five!
D: Why five?
John: Because…hmmm, let's see. Because—how do I say it…I divided by 3 to get 5 because for every 2—oh no—for every 3 you add 2 so you could add, just add 2. Just keep adding 2 for every 3 cups pretty much. Like multiplication.

John's activity here provides evidence that he had constructed iterable composite units that he could use to structure his assimilation of situations. In particular, I infer that his intuitive response of "five!" indicated to John the number of groups of 3 cups that would be needed in order to produce 15 cups of water. Thus, his composite unit three was constructed as a countable item and, hence, an iterable composite unit (Steffe, 2010c). His awareness of the relationship between dividing and multiplying in this case lends additional support to this hypothesis. Splitting 15 into 5 groups of 3 implied he could "just keep adding […] like multiplication" to reconstitute the 15 cups of water.

John's activity in Protocol 4.11 also provides additional evidence regarding his use of the coordinated partitioning/iterating strategy. I infer that while reasoning with the 15 cups of water, John temporarily suspended his focus on the relationship between the quantities and first considered the 15 cups in relation to the 3 cups. Then, after constructing the change in the amount of water as a multiplicative relationship he returned his attention to the 3:2 relationship. To explain what his result of "five" meant, John stated, "For every 3 you add 2 so you could add, just add 2. Just keep adding 2 for every 3 cups." From this I infer that he transferred the multiplicative relationship he had constructed for the amount of water to the lemonade powder as well. Thus, he used his operations with iterable composite units to assimilate the change from 3 to 15 cups of water, and then he used these same operations to decide how to transform the 2 tablespoons of lemonade powder.

However, John's attempts to carry out this strategy of adding 2 for every 3 following Protocol 4.11 indicated that he had some difficulty maintaining the 3 for 2 relationship while enacting these operations on both quantities simultaneously.

Protocol 4.11: Continuation.

D: Okay. So why don't you do that out loud? Just kind of keep track of it as you go.
John: Three…three then two. [On his left hand he puts up one finger and on his right hand he puts up two fingers.] Then six four. [His left hand was not visible, but he held up four fingers on his right hand.] Nine…eight. Wait, no I messed up again I think.
D: That's okay. Take your time. You're doing good.
John: [Thinks silently for 6 seconds.] I think I might need to write it.
D: [Slides paper and a marker over to John.]
John: So it's 15. Three and you have two for it [Writes "3 2" on his paper]. So it would be six and four [Writes "6 4" on his paper. Then writes "9 6"]. Six. Twelve and eight. Fifteen and 10. [Finishes writing "12 8" and "15 10" while talking.]
D: Um hmm. So then how many tablespoons would you need for the 15 cups of water?
John: 10.

We see that John initially attempted to mentally keep track of iterating both three and two, but partway through he lost track of this counting activity. I infer that his difficulty resided in maintaining the three for two relation while carrying out the five iterations rather than any limitation to his strategy per se. Iterating a single composite unit, such as three, involves progressively integrating additional composite threes while simultaneously monitoring the number of iterations. However, in the context of a three for two relationship one has to iterate two composite quantities while also monitoring the number of iterations. This units coordination proved difficult for John to accomplish mentally. However, he carried out his strategy meaningfully and easily with paper and pencil. This task was likely novel to John, and the simultaneous iteration of two composite units a type of coordination he had not attempted before. I believe that given more opportunities to solve tasks such as this, John could have constructed ways of carrying out his coordinated iterating strategy.

Summary of John's Mathematics

John's initial interview suggests that, like Jack, he had constructed sophisticated quantitative operations prior to the start of the teaching experiment. In fact, despite some challenges explaining his thinking, John successfully solved every task we presented him during the initial interviews. Based upon his reasoning throughout the initial interview, I infer that John had constructed three levels of units that he could take as given in assimilation and operating. This was most evident in John's use of his splitting scheme throughout the interview and his clear understanding of the multiplicative relationship it produces between a split part and the original whole. In addition to his splitting scheme, John consistently used his unitizing operation to form and reason with composite units in a way that allowed him to reason about an entire collection of units as a single entity.

I believe that these two features of John's mathematics account for his ways of operating in both the cake sharing and the lemonade mixtures tasks. Using these ways of reasoning John was able to construct one of m shares of n units as simultaneously n/m of one unit and $1/m$ of n

units. And while his use of composite units alleviated the need to distribute the partitioning activity across multiple units in the task of sharing two equally-sized cakes, he did carry out a distributive partitioning in the case of two unequally-sized cakes. My conclusion from these observations is that John's use of splitting and composite units was sufficiently powerful to alleviate the need to engage in distributive partitioning in most cases. The implications of this observation are something I investigated during my retrospective analysis of the rest of John's work throughout the teaching experiment.

In addition, throughout the interview John's reasoning appeared very intuitive in that he often quickly came up with a solution that seemed reasonable to an observer. However, John frequently found it difficult to explain the reasoning that led to these intuitive results and to verbalize the necessity of the relationship his intuitive result leveraged. Thus, while John reasoned quite powerfully, my inference is that he had not abstracted all of his quantitative operations to the level of being explicitly aware of the activity of those operations. As was the case in Protocols 4.7, the power of John's quantitative operations at times put him in a state of perturbation as a result of a disconnection between the quantitative reasoning that he used in assimilation and the numerical operating he used when attempting to explain his intuition.

CHAPTER 5

THE MATHEMATICS OF JACK

My goal in this chapter is to characterize the mathematics of Jack. I initially developed my model of Jack's reasoning while interacting with him during data collection and later refined it while retrospectively analyzing every teaching session and interaction. I have divided my presentation of the results of these analyses into sections based upon the three primary contexts in which I worked with the students (cf. Chapter 3). Within each of these sections, I've selected specific excerpts from the teaching sessions that capture important aspects of Jack's ways of reasoning, and I use these excerpts as a way of situating my analyses of Jack's mathematics within the context of his mathematical activity. In addition, I've chosen excerpts that allow me to analyze Jack's characteristic ways of reasoning, the successes and struggles he encountered, and the changes I observed in his ways of reasoning over the course of the teaching experiment. In particular, my analyses characterize Jack's reasoning by drawing inferences regarding the goals he developed, by explaining the mathematical activity he engaged in while attempting to achieve those goals, and by developing a model of Jack's mathematics that accounts for each of these.

The Swimming Pool Context

Following the completion of the initial interviews, I engaged the students in a series of tasks within the context of filling up a swimming pool with water. The tasks involved a focus on pumping rates and coordinating changes in the quantities pool depth (measured in inches) and pumping duration (measured in minutes). My overarching goal at this point in the teaching experiment was to put Jack in problematic situations to investigate how he would use his available quantitative operations creatively to solve tasks.

The first of Jack's three teaching sessions working within this context occurred on October 10, 2013. For this teaching session, I designed the initial questions to investigate Jack's ability to operate with a given measurement of each quantity to form a unit ratio in this context as a measure of the quantity inches per minute. Previously, in the lemonade mixtures tasks and other similar recipe tasks during the pilot study, Jack had demonstrated some success abstracting a given ratio relation and using his quantitative operations to reason proportionally and find a unit ratio such as two-thirds of a tablespoon per cup of water (cf. Protocol 4.4). Thus, I anticipated that he would be able to use his partitioning and fraction operations to form a unit ratio in this context as well, and I planned the other questions for this session to investigate how Jack would use the resulting unit ratio in further reasoning about other intensive quantitative unknowns.

Jack Incorporates His Coordinated Partitioning/Iterating Strategy

To introduce the context, I showed Jack an animation of the water level of the pool rising as time passed and stopped the animation at a time of 5 minutes (see Figure 5.1). After telling him that the water measured 3 inches deep after 5 minutes, I asked him how deep the water would be if we imagined continuing to fill to pool and checking the depth after 10 minutes, and again after 25 minutes, had elapsed. Jack immediately replied 6 inches deep and 15 inches deep, respectively, and provided justifications that relied upon his whole number multiplicative reasoning. For example, he explained that in the case of 25 minutes he multiplied 3 times 5 because "You're doing it by 5 minutes each," there were 5 fives in 25, and the water was rising 3 inches every 5 minutes.

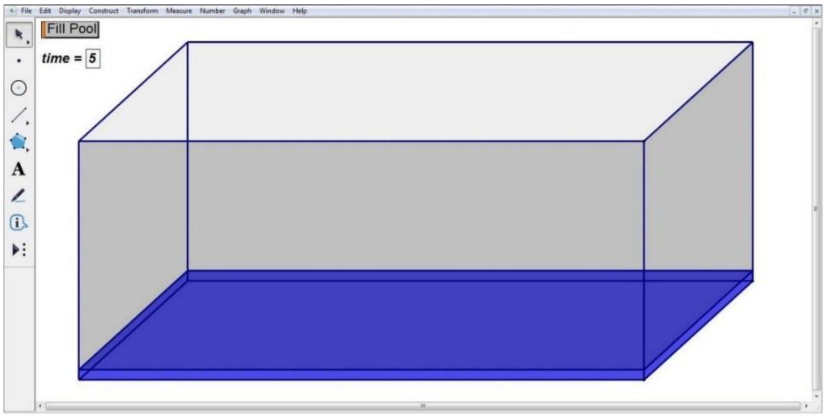

Figure 5.1. A screen shot of the animation stopped at a time of 5 minutes.

I infer that Jack's production of these equivalent ratios (i.e., 6 inches in 10 minutes and 15 inches in 25 minutes) was based upon his construction of composite units as iterable units and on a substitution of 3 inches for 5 minutes in reasoning with these composite units. Thus, Jack's solution to these tasks represents another use of his coordinated partitioning/iterating strategy. He assimilated the transformed time measurements as some number of iterations of 5 minutes and then used this assimilation to guide his activity with the concomitant quantity 3 inches.

Jack's reasoning with these initial questions in this context, while brief, demonstrates two important features of his conceptual constructs that were typical of his efforts to reason with situations involving covarying quantities throughout most of the experiment. First, it confirms his ability to abstract and operate with the numeric relation between the quantities. As the research team and I learned during the pilot study, this ability is not a trivial construction and is supported by the operations responsible for producing three levels of units (Steffe, Liss II, et al., 2014). Further, as he had done previously with the lemonade tasks, Jack assimilated the given information using conceptual operations that preserve the multiplicative relationship between the quantities. In this case, the iterability of his composite units accounts for his multiplicative reasoning and the preservation of the given ratio.

Secondly, and more importantly for understanding Jack's reasoning throughout the rest of the teaching experiment, I believe that Jack's brief explanation suggests a critical aspect of his reasoning with variation and changes in covarying quantities. In particular, one hypothesis that I advance throughout this chapter is that Jack's image of variation is one of completed uniform motion. For example, recall that when reasoning about the water depth after 25 minutes of pumping water Jack stated, "You're doing it *by 5 minutes each* [emphasis added]." This statement suggests that the completed change in the quantities that the initial animation indicated, 3 inches in 5 minutes, formed the basis for his reasoning about the changes in the covarying quantities. As I will highlight throughout my analyses of Jack's mathematics, this remained a central aspect of his reasoning throughout a majority of the teaching experiment.

However, it is vital that I follow up this assertion with a few clarifying remarks. First, Jack's use of this reasoning here was expected. Indeed, I had intentionally chosen to ask questions about multiples of the given 5 minute measurement and anticipated that Jack would abstract the numeric relation between the quantities and successfully use his operations with whole number composite units to transform the given ratio. My goal in doing so was to provide Jack with an opportunity to form a mental re-presentation of his experiences with the animation as a means for abstracting the numeric relation between the measurements within the context of mental operations that I had confidence he had already constructed (i.e., iterable composite units).

In addition to being expected, Jack's conceptualization of the measurements as indicating completed uniform motion was also productive. Jack's successful use of his whole number operations suggested that he had indeed abstracted the numeric relation as a mental object he could use in further reasoning. This observation provided me with confidence going forward with subsequent tasks of the teaching session because they all relied upon Jack's construction of this relation as a mental object. Thus, abstracting this relation as an indication of completed changes in the quantities was not a hindrance to Jack's reasoning in this case. Quite the opposite, it underpinned his reasoning about accumulations of the respective quantities given an imagined continuation of the variation.

Lastly, this characterization of Jack's image of variation was never a component of my experiential model of Jack during the teaching experiment. Rather, I first identified this feature of Jack's reasoning during retrospective analysis of a task from Jack's second to last teaching session on February 20, 2014. While my analysis of that task will follow in due course, after forming this hypothesis about Jack's reasoning I found that it provided a way of consistently accounting for Jack's reasoning, particularly his activity for which I previously had no unifying explanatory model. Further, as with this task, I retrospectively found that even in cases when Jack reasoned successfully and I could explain the operations he used to construct his understandings, this characterization remained consistent with Jack's reasoning. Thus, while the implications of this aspect of Jack's mathematics do not become apparent until later in the teaching experiment, I found tendrils of this way of reasoning throughout the data.

Jack Demonstrates Differing Levels of Success Determining Unit Ratios

Jack struggles to use his splitting scheme to establish a unit ratio.

Having confirmed Jack's ability to abstract the numeric relationship between measures of the quantities pool depth and pumping duration, I turned my attention to investigating Jack's ability to establish a unit ratio and use that result in further reasoning. While I anticipated Jack could use his partitioning and fraction operations to establish the unit ratio, his reasoning in the following protocol indicates a constraint to his reasoning that I had not yet accounted for in my model of his mathematics. This interaction also occurred during the October 10th teaching session and occurred immediately after the discussion of 10 and 25 minutes.

Protocol 5.1: Jack's attempt to find a unit ratio for the number of inches per minute.

D: How much deeper would the water get if we let the water run for just 1 minute?
Jack: [Thinks silently for 47 seconds.] Umm, I don't know. I can't think of it.
D: Well, what were you thinking about? It seemed like you were thinking about some things there. Do you want to describe how you were thinking about it?
Jack: Just how much it would fill up in 1 minute. I was thinking like 75% of 1 inch in each minute. But it would be 3 in 4 minutes if it was like that.
D: Tell me a little bit more. How'd you know that would be 3 [inches] in 4 minutes?
Jack: Because 75 times 4 is 300.
D: So, then 300—what does that mean? Like 300 what?
Jack: It would be 300, so that's 3 inches.
D: Okay, so that would be 3 inches in 4 minutes. That's close to what we have. We've got 3 inches in 5 minutes. Umm, how did you get the 75?
Jack: I was trying to split the 3. I was trying to split it equally into each minute to see how much you would get in 1 minute.

As Jack states, he was trying to find "how much it would fill up in 1 minute." Based upon this and Jack's final comment, I infer that he had assimilated the task using his splitting scheme and formed a goal of splitting the 3 inches into five equal parts such that any part could be iterated five times (one for each minute) to produce the original 3 inches. However, Jack's inability to carry out this goal after 47 seconds of thinking, combined with his trial and error reasoning about 75 percent of an inch per minute, suggests that this goal produced a perturbation for Jack.

I claim that this perturbation stemmed from Jack's attempt to use his splitting scheme in a novel context. For instance, using his splitting scheme, Jack had previously demonstrated the ability to mentally split a continuous unit into a given number of parts and know that any one of those parts could be iterated the given number of times to produce a connected whole equivalent to the original continuous unit. However, in this case the continuous unit Jack attempted to split was a composite unit of 3 inches rather than a continuous single whole.

Recall that during the initial interview, Jack intuitively split 3 cups into two parts, each containing one and a half cups of water (that he later recognized as 3/2 cups of water and used in further reasoning, cf. Protocol 4.4). In contrast, we see here that Jack could not carry out his goal of splitting a continuous unit of three into five parts. I claim that in the lemonade mixtures context Jack's intuitive split was supported by his use of a dyadic attentional pattern. Steffe (2010a, 2010b) explains that such a pattern represents a foundational component of children's construction of number and accounts for children's earliest forms of fragmenting continuous quantities into two equal parts. However, the dyadic pattern on its own would be insufficient for Jack to identify one and a half cups as the measure of each of these parts. To accomplish this quantification, I infer that Jack incorporated his number sense (i.e., $1 \frac{1}{2} \cdot 2 = 3$) into his intuitive split.

Thus, while I am confident Jack possessed an attentional pattern that he could use to split a continuous unit of three into five parts, his number sense did not support his quantification of this split into five parts as it had previously with only two parts. At the time, I hypothesized that Jack might be able to alleviate this perturbation if he were to assimilate this task as a situation of his distributive partitioning operations. Thus, I rephrased the discussion in terms of fractions rather than percentages and decimals in hopes that Jack would incorporate his other available quantitative operations as a means of accomplishing his goal.

Protocol 5.1: Continuation.

D: Would it help if you thought of it as a fraction instead of a percent? Is there a way you can think of it that way?
Jack: It would be one-fifth.
D: What would be one-fifth?

Jack: It would fill up one-fifth of, umm, in 1 minute it fills up one-fifth of the way. No, that would be the time. Time is one-fifth but the water is only three…so there's five of them. I have no idea.

[…]

D: Okay. I have one other question I want to go back to a little bit. So before I asked you about 1 minute, right, and you said something about one-fifth. So what were you thinking about with the one-fifth? That was one-fifth of what?

Jack: Of, one-fifth of the minutes.

D: Okay. So, yeah, if we had 1 minute that would be one-fifth of the 5 minutes. What fraction of the water do you think we would have then at 1 minute?

Jack: Two-thirds.

Unfortunately, rephrasing the question in terms of fractions did not help Jack eliminate his perturbation and quantify the result as three-fifths of an inch per minute. Apparently, knowing that "in 1 minute it fills up one-fifth of the way" did not activate Jack's distributive partitioning operations.

However, this continuation of Protocol 5.1 does help to clarify several important aspects of Jack's reasoning. First, this excerpt provides additional evidence that Jack assimilated the task as a situation of his splitting scheme. Jack's comment that "time is one-fifth" indicates his awareness of 1 minute as one-fifth of the time compared to the initial measurement of 5 minutes. This awareness is consistent with having assimilated the time as a continuous unit split into five parts, one for each minute, each of which is one-fifth of the whole duration of time. Further, Jack talked about trying to split the 3 inches equally and replied, "…but the water is only three…so there's five of them." Thus, I infer that he attempted to transfer this activity of splitting into five parts to the concomitant quantity 3 inches in an attempt to carry out his coordinated splitting strategy. However, as in Protocol 5.1, his activity indicates that he could not use his quantitative operations to quantify the results of this split. Ultimately, I believe that Jack's reply of two-thirds, much like the 75% of one inch, represented a reasonable estimate rather than a result of his splitting and fraction operations.

In addition, my retrospective analysis of the continuation of Protocol 5.1 revealed a difference in the underlying nature of the fraction words used throughout this protocol. For instance, with respect to the relationship between the desired and initial time measurement, I've already claimed that Jack provided meaning for the fraction one-fifth through use of his splitting scheme. Hence, one-fifth referred to the multiplicative relationship between 1 and 5 minutes. However, when Jack said "in 1 minute it fills up one-fifth of the way," he used one-fifth as an operator (Kieren, 1976, 1980, 1993). Thus, rather than referring to the result of a multiplicative comparison, one-fifth in this quotation referred to a goal of acting upon another quantity to find one-fifth of that quantity. The former conception speaks to Jack's assimilation of the task while the latter speaks to his goal of finding one-fifth of 3 inches. While splitting operations would enable Jack to find one-fifth of a single continuous unit, we see here that these same operations proved insufficient for quantifying one-fifth of a composite unit. As a result, I do not believe that Jack had fully constructed the notion of a fraction as an operator.

This issue manifested itself in some slight confusion and miscommunication between Jack and I. Regarding the confusion, consider Jack's full statement describing the meaning of one-fifth: "It would fill up one-fifth of, umm, in one minute it fills up one-fifth of the way. No, that would be the time. Time is one-fifth but the water is only three…so there's five of them. I have no idea." The fact that Jack initially transferred his meaning from having one-fifth of the time to needing to find one-fifth of the water suggests that constructing a unit-fraction as an operator lay within his zone of potential construction. However, he also seemed unsure of this statement and vacillated between one-fifth referring to the number of inches and to the multiplicative comparison of the two time states. Lacking the operations to implement both meanings for one-fifth left Jack in a state of uncertainty.

In retrospect, I do not believe that Jack had explicitly made a distinction between these two meanings. This accounts for the apparent miscommunication at the end of the continuation of Protocol 5.1. I asked my final question, "What fraction of the water do you think we would have then at 1 minute?", with the intention of having Jack make the goal of finding one-fifth of the water depth explicit. However, his reply of two-thirds suggests that he interpreted my question as asking for the fractional amount of 1 inch rather than the fractional amount of the entire quantity. The former characterizes the result of a completed fractional comparison, whereas the latter would refer to a goal of operating on a quantity in a particular way.

Protocol 5.1 and its continuation occurred over the course of approximately 7 minutes of real time during the teaching session. Because Jack was in a state of perturbation for nearly that entire time, I decided to not question him further about this task despite the fact that his reply of two-thirds was not an accurate quantification of the number of inches per minute for the water pump. Presumably, had I asked, Jack would have been able to determine that two-thirds of an inch per minute would not be equivalent to 3 inches in 5 minutes much like he had determined that 75% of 1 inch in a minute was not correct. However, Jack never carried out any activity to test the appropriateness of two-thirds of an inch per minute, and his activity in subsequent tasks suggests he accepted this result as equivalent to the initial measurement.

Jack establishes a unit ratio using his reversible fractional reasoning.

Following the interaction in Protocol 5.1, I decided to ask Jack about the number of minutes the pump needed to run per inch of water depth. My goal in doing so was to give Jack another opportunity to creatively use his quantitative operations to quantify a unit ratio but in the context of a new task for which he was not already in a state of perturbation. I found Jack's immediate solution to this task somewhat surprising in the moment and also revealing regarding the nature of his fractional operations.

Protocol 5.2: Jack determines how long it would take to raise the pool level 1 inch.

D: What if the pool maintenance guy wanted to know how much time it would take to go up just 1 inch? So here [referring to the previous task] he was measuring time and thinking about depth. What if he just wanted to know how long it would take to go up 1 inch?
Jack: [Thinks silently for 4 seconds.] One minute and 30 seconds.
D: How'd you think about that?
Jack: Because if 1 minute total was two-thirds, that means it's…30 seconds is one-third and you're just adding another third to make it 1 whole inch.

In contrast to the previous protocol, Jack quantified this unit ratio almost immediately. However, rather than using the initial measurement of 3 inches in 5 minutes as the basis for coordinating the two quantities, Jack used his previous result of two-thirds of an inch per minute as the starting point for his reasoning. I infer that Jack formed the same overarching goals in both Protocols 5.1 and 5.2—to transform a given ratio to an equivalent ratio having a unit value for one of the quantities. Further, in each case he incorporated his coordinated partitioning/iterating strategy by attempting to apply the same quantitative transformation to both quantities.

The difference in starting ratios between these two unit ratio tasks (3 inches in 5 minutes versus two-thirds of an inch per minute) had a significant impact upon the operations Jack used to assimilate the task. Considering the two tasks specifically, from Protocol 5.1 I inferred that Jack assimilated transforming 5 minutes into 1 minute as a situation of his splitting scheme and attempted to transfer this split to 3 inches. However, his partitioning and numeric operations did not support quantifying the result of this split. In contrast, Jack's activity in Protocol 5.2 indicates that he relied upon his reversible fraction schemes to accomplish his goal of transforming two-thirds of an inch into 1 inch. Using essentially the same reasoning as he had during the lemonade mixtures task in the initial interview (cf. Protocol 4.4), Jack conceptually split the two-thirds into two parts to identify one-third of an inch. Then, he united together three one-thirds of an inch to obtain the desired quantity. Coordinating this partitioning activity with the concomitant quantity produced the ratio of 1 minute 30 seconds per inch.

Accounting for the observed differences in Jack's ability to establish unit ratios.

One possible explanation for the extreme difference in Jack's ability to produce these two unit ratios is that his reversible fraction scheme was more sophisticated than his splitting scheme. The implication of this would be important for deciding how to help Jack make progress: If Jack could develop a way to transform any given ratio into a fractional comparison, he would be able to leverage his reversible fraction operations to solve a broader range of tasks than he could on the basis of his whole number operations.

While plausible given the stark contrast between Jack's prolonged challenge to quantify the results of his splitting operation and the ease with which he quantified the results of his reversible fraction scheme, I believe these differences are symptoms of a more subtle difference that more completely accounts for Jack's activity. I hypothesize that the nature of the units upon which Jack attempted to operate in relation to the desired fraction operations accounts for the observed differences in his ability to quantify the results of his mental operations. Recall that based upon Protocol 5.1, I claimed that Jack had not fully constructed fractions as operators because he could not use his concept of one-fifth to operate on the composite unit 3 inches. Neither whole number operations nor an intuitive split are sufficient for this operation. Rather, to quantify one-fifth of a composite three requires partitioning the composite three into units different than the three units already defined by the measurement itself.

In comparison, Jack's activity in Protocol 5.2 shows what he can do when there is no need engage in re-partitioning a composite unit. The concomitant quantity in that case was a single continuous unit of 1 minute. Because Jack provided his answer in terms of minutes and seconds rather than purely minutes, it is not clear whether he converted minutes into seconds before, or after, transforming the 1 minute. Regardless, transferring the operations he used to transform two-thirds of an inch into 1 inch onto the quantity 1 minute would not require re-partitioning a composite unit.

Consider both cases. If Jack reasoned in terms of minutes, the 1 minute could be split to one-half minute for one-third of an inch. He could then unite this result with the original 1 minute for two-thirds of an inch to generate one and a half minutes, or 1 minute and 30 seconds, for 1 inch. Alternatively, if Jack reasoned in terms of seconds, then the composite 60 seconds could be split into 30 seconds for one-third of an inch and then united with the original 1 minute for two-thirds of an inch to generate 1 minute and 30 seconds for 1 inch. While this method does involve splitting a composite unit of 60 seconds, this splitting respects the partition that is defined by the measurement and does not require re-partitioning the 60 seconds into units other than individual seconds. Thus, whole number multiplicative operations are sufficient to quantify the result of splitting a composite 60 seconds into two parts.

A few examples might help to clarify my meaning. Suppose that one's goal is to use one-fourth as an operator to transform another quantity. If that quantity is a single continuous unit, such as 1 second, then no partitioning has been defined on the unit. As a result, one is free to split it into four parts to produce one-fourth of the 1 second. In cases such as this, Jack had previously demonstrated the ability to use both unit and non-unit fractions to operate upon individual continuous units.

The issue lies with using a fraction as an operator upon a composite unit. Suppose that one wanted to find one-fourth of 20 seconds. Having constructed at least two levels of units, 20 seconds can be assimilated as a composite unit containing 20 individual seconds. Thus, the measurement defines a partition upon the continuous quantity—in this example, a partition into 20 parts. Using one's conception of one-fourth to operate upon this 20 seconds can be accomplished using whole number multiplicative reasoning without changing the 20-part partition that was defined by virtue of the measurement. Knowing that $4 \cdot 5 = 20$, one can determine that one-fourth of 20 seconds is 5 seconds.

Lastly, suppose that one wanted to find one-fourth of 7 seconds. The measurement in this case defines a partition of seven parts upon the quantity. However, whole number multiplication proves insufficient, and one cannot find one-fourth of 7 while operating solely within the constraints of the given partitioning unit (i.e., seconds). Rather, finding one-fourth of 7 units requires one to re-partition the composite number of seconds into some partition other than the given unit of individual seconds so that the appropriate fraction of 1 second can be identified.

Thus, an important issue with constructing fractions as operators that one can use to act upon composite units lies with whether or not one can operate within the structure defined by the measurement unit. In the case of a composite 20 seconds, using one-fourth as an operator could be accomplished by operating within the constraints of the partitioning defined by the measurement unit (i.e., whole seconds). However, in the case of a composite 7 seconds, using one-fourth as an operator requires one to create a new partition different than the one defined by the measurement unit (i.e., partial seconds).

While these examples gloss over the issue of whether or not one has constructed fraction notation and language as symbolic of mental operations such as splitting, that is not my point in this example. Rather my point is to clarify why I believe Jack struggled to form a unit ratio in Protocol 5.1 yet almost immediately produced a unit ratio in Protocol 5.2. I claimed that Jack assimilated these tasks with different quantitative operations (a splitting scheme versus a reversible fraction scheme, respectively). However, my hypothesis is that the issue was not these differences in assimilation but rather with differences in the nature of the quantities upon which Jack attempted to use those quantitative operations to act. Jack's operations were sufficiently sophisticated to use his reversible fraction scheme to operate upon the quantity 1 minute but insufficient for splitting the composite quantity 3 inches into five parts. While Jack's reasoning in these protocols did not provide any evidence as to what quantitative operations could enable him to alleviate these constraints, these excerpts do help to clarify the limits of his ways of reasoning and to account for his prolonged perturbation in Protocol 5.1.

Jack Quantifies Some Intensive Quantitative Unknowns

In Protocol 5.2, Jack had successfully produced a unit ratio for the quantity *minutes per inch* that maintained the multiplicative relationship between the quantities using the information he took as given for the starting point of his reasoning (i.e., the two-thirds of an inch per minute). Following this task, I asked Jack several other questions about the pumping duration required to raise the pool level various amounts of water depth so that I could investigate his ability to use this newly established unit ratio in further reasoning. Jack successfully (with the exception of one computational error) found the time it took to raise the pool level 2 inches, 4 inches, 2 feet, and 111 inches. In each case, Jack used a building up approach and combined the necessary iterations of the ratios one and a half minutes per inch and 5 minutes per 3 inches[9] as well as iterations of other ratios he had constructed as solutions to previous questions.

[9] While these are not equivalent ratios, Jack accepted them as such. Because my goal at this point was to investigate how he might use his constructed unit ratio in further reasoning, I intentionally did not attempt to perturb Jack's assumption of this equivalence. His activity throughout the rest of the teaching session suggests he considered them to be equivalent characterizations of the same pumping rate.

For example, when considering 111 inches Jack combined four iterations of the time it took for 2 feet, one iteration of the time it took for 1 foot, and one iteration of the time it took for 3 inches. Thus, he used his whole number operations with iterable composite units to determine the desired result. However, I found it somewhat surprising that he did not scale directly to 111 inches from the 1 minute 30 seconds per inch as a duration 111 times as long as the duration of time needed for 1 inch. However, I do not view this as a necessary constraint to his ways of operating but rather a result of Jack carrying out his calculations mentally. As a matter of practicality, it is more efficient to build up to 111 inches using iterations of 40 minutes per 2 feet[10] rather than only iterations of the unit ratio. Further, Jack did reason multiplicatively with this intermediate ratio knowing that three iterations of 2 feet would take 3 times as much time, or 120 minutes.

Using his constructed ratios to solve these tasks indicates Jack's ability to quantify some intensive quantitative unknowns at this point in the teaching experiment. The construction of iterable composite units accounts for his success in coordinating accruals of the quantities as he imagined continuing the variation and filling the pool to various other depths. Further, his activity remained consistent with considering the ratios he had available to him as indications of completed change. However, the limitation of being unable to use his quantitative operations to quantify the results of using his fractions as operators on composite units remained a constraint in Jack's ways of reasoning throughout the remainder of this teaching session as well as the next.

Jack Provides Evidence of Reasoning With the Intensive Quantity *Inches per Minute*

Jack's next teaching session occurred on October 28, 2013, (he was absent for the teaching session on October 22, 2013) and the tasks focused on comparing the pumping rates of

[10] This ratio was generated from iterations of the original measurement 5 minutes per 3 inches.

two different replacement water pumps. One of my goals with this session was to provide a slightly different context for the students to creatively use their quantitative operations. In addition, I designed the teaching session around comparing different water pumps because I wanted to investigate the extent to which the students were reasoning with the intensive quantity pumping rate as opposed to comparing the extensive quantities pool depth and pumping duration and simply coordinating changes in their values.

Jack's reasoning throughout this teaching session remained consistent with the characterizations provided above in relation to Protocols 5.1 and 5.2. However, two interesting aspects of his reasoning in this teaching session contributed to my model of his thinking. The first occurred while trying to decide which replacement pump would be the better water pump.

Protocol 5.3: Jack compares the pumping rates of two replacement water pumps.

D: [Jack picks up a marker and paper to record the information.] So Pump 1 can raise the level of the pool 4 inches in 5 minutes, and Pump 2 can raise the level of the pool 3 inches in 4 minutes. […] I want you to try to decide which of the two pumps would be better and why do you think it would be a better pump. Alright. So, just go ahead and think about that for a little while first and then we'll share in a little bit. [After thinking for 35 seconds, Jack looks up.]
[…]

Jack: The second one is better. Because even though it has one less—it has 3 inches in 4 minutes, but the 4 inches in 5 minutes. Because it's not every minute…every minute it doesn't go up 1 inch. So even though this is 5 [minutes] and it goes 4 [inches]. But this is four five. If this goes to 6 minutes [referring to the first pump] it's only going to be like…it's going to be like…I guess you could say it's even. I guess. Because, like, even if this goes up to 6 minutes then this would be 4 minutes. Four, four point something minutes. Or five point something minutes. Somewhere around there. But this one [Pump 2]—if he goes up to 5 minutes he goes up to four point something. Because each minute doesn't add exactly 1 [inch], it adds more, or like…each minute doesn't add exactly 1 inch. Because if it did, then it would be five five, four four [Referring to 5 inches in 5 minutes and 4 inches in 4 minutes for Pumps 1 and 2, respectively.]

D: Okay. So, you're saying if this one [Pump 1] went up to 6 minutes would it be more or less than 5 inches?

Jack: It would be…it would be less because it would be four point something.

D: And what about this one [Pump 2], if it went up to like 5 minutes, would it be more or less [than 4 inches]?

Jack: It would be three point something.

[…]

W: I'm sorry, I don't know what Jack said. I didn't hear what pump he thinks is better. I forgot.

Jack: The first one.

W: Why is that?

Jack: Because even though…because that one [Pump 2] goes 3 inches in 4 minutes and that one [Pump 1] goes 4 inches in 5 minutes. But, if you added a minute to four and three, three wouldn't go up 1 whole inch. It would just go up a fraction of an inch. And five to six would be the same thing, but it would be greater than the four and three.

Jack's explanation provides evidence that he did, in fact, reason about the intensive quantity pumping rate. Had he based his decision on a comparison of the changes in water depth at a common multiple of the pumping durations given, such as 20 minutes, I would have considered his reasoning as an example of an extensive quantitative comparison. However, even though he did not quantify a specific pumping rate for each replacement pump, the pumping rate of 1 inch per minute formed the basis of Jack's comparisons rather than specific measures of each extensive quantity.

While initially incorrect about which pump had a more favorable pumping rate, he appeared to change his mind during the course of explaining his thinking. At first Jack seemed to think the replacement pumps had pumping rates greater than 1 inch per minute stating, "But this one [Pump 2]—if he goes up to 5 minutes he goes up to four point something. Because each minute doesn't add exactly 1 [inch], it adds more." However, then Jack changed his mind and decided that both pumps had pumping rates less than 1 inch per minute because continuing to pump at that rate would result in accumulations of "five-five" and "four-four." As a result, he knew that continuing to run each pump for an additional minute would yield less than a full inch of change in water depth for each pump. Accordingly, he replied to the witness researcher that he then thought that Pump 1 represented the better replacement option. I infer that Jack based this decision on knowing that in 5 minutes Pump 2 would raise the pool depth three and "a fraction of an inch," whereas Pump 1 was defined as pumping 4 inches in 5 minutes.

Jack Uses Division to Refer to a Splitting Goal

In addition to revealing that he reasoned with the intensive quantity, pumping rate, this task revealed an additional limitation to Jack's efforts use his mental operations to quantify the values of intensive quantities. Because Jack had decided that each replacement pump changed the pool depth less than 1 inch per minute, I elected to again see if Jack could creatively determine a strategy for quantifying the numerical value of the specific pumping rate.

Protocol 5.4: Jack attempts to quantify Pump 1's specific pumping rate.

D: Okay. So, I guess what I'm wondering—is there a way to figure out how much that would be if they each went up 1 minute? Why don't you think about that.

Jack: [Thinks for 100 seconds before writing "4 ÷ 5" on his paper. He then thinks for another 55 seconds without writing anything else down.]

D: What are you thinking about Jack? You wrote something down here [Points to his paper were he had written down "4 ÷ 5".] What were you thinking?

Jack: I was trying to see, because every minute…you go for 5 minutes there's 4 inches. I was trying to see what fraction of an inch it is for a minute.

Jack's activity in this excerpt suggests that division and splitting refer to the same conceptual goal for Jack. Much like in Protocol 5.1 when he wanted to split the 3 inches equally into five parts, Jack formed a goal in this case of "trying to see what fraction of an inch it is for a minute." However, unlike Protocol 5.1 in which he attempted to reason mentally with fractional quantities, here Jack assimilated this task as a situation of division. Thus, I infer that his inscription of "4 ÷ 5" indicates that for Jack the division symbol "÷" and the mental operation of splitting referred to the same goal of partitioning a composite unit into a given number of equal sub-units, any one of which could be repeated five times to produce 4 inches of water depth.

As with Protocol 5.1, Jack did not engage in any activity that would have enabled him to quantify the size of these unknown sub-units. Significantly, "4 ÷ 5" did not refer to four-fifths of an inch. I intentionally did not encourage him to carry out long division or use a calculator at this time because I wanted to encourage him to use his quantitative operations rather than computational procedures. The rest of the teaching session focused on coordinating changes in the quantities for various numbers of inches and minutes. These tasks verified Jack's ability to successfully coordinate changes in these quantities while imagining continuations of the variation but otherwise did not reveal anything new about his ways of reasoning.

Jack Makes Progress Toward Splitting Composite Units to Establish Unit Ratios

Because Jack had been unable to resolve the constraint of using his quantitative operations to transform a composite unit, I planned some interventions to see which alterations to the task would enable Jack to assimilate splitting a composite unit as a situation of his distributive partitioning operations. To start the teaching session on November 6, 2013, I had Jack explain his thinking about the two replacement pumps. He again stated that he knew it would be less than an inch per minute, but he could not think of a way to determine what fraction of an inch per minute each pump would be pumping.

Jack constructs a unit ratio in a recipe context.

I decided to investigate whether Jack could adapt his distributive partitioning operations to find unit ratios for pumping rates. Previously, he had experienced some success splitting composite units by splitting each individual unit within the context of scaling recipes. Thus, to see if Jack could reason analogously between these two contexts, I asked him to think about a cookie recipe that called for 2 cups of flour for every 3 dozen cookies. However, he experienced similar constraints as those in the previous protocols and remained unsure how to split the 2 cups of flour equally among the 3 dozen cookies. Next I altered my language to rephrase the tasks in terms of sharing, suggested to Jack that he try using a diagram, and drew two horizontal segments in a row and labeled them as “2 cups” (see Figure 5.2). Protocol 5.5 demonstrates how Jack made use of this diagram.

Figure 5.2. The diagram of 2 cups that I suggested to Jack.

Protocol 5.5: Jack uses a diagram to determine the number of cups of flour per dozen.

D: How could you think about sharing those [referring to the linear segments of the diagram] and splitting up those 2 cups so that you could, umm, end up making the 3 dozen cookies? Would there be a way to, kind of, share those 2 cups amongst those 3 dozen cookies?

Jack: [Thinks for 10 seconds.]

D: Or maybe…do you have, do you have an idea? Because otherwise I’m maybe thinking of something [we could try].

Jack: I mean, you can—if it’s 2 cups you can split each of the 2 cups into thirds.

D: Um hmm.

Jack: So, two-thirds of both cups would make 1 dozen.

D: Okay. Umm, you want to show me what you were thinking on the picture?

Jack: [Picks up the marker.] Like, take it and then three pans for cookies. [Draws three circles to represent three pans for the 3 dozen cookies.] And so you’d have to split, like, each cup into thirds. [Makes vertical marks below the horizontal lines representing the 2 cups, roughly splitting each segment into three equal parts.] And then it would be one-third, one-third, one-third. Same for over here. [Labels each of the one-third cup sections in the diagram of the 2 cups of flour.] And then there’s three of these [referring to the 3 dozen cookies]. So these two go to one, these two could go to one, these two could go to one. [Draws arcs connecting two one-third cup segments to each of the circles representing the 3 dozen cookies. See Figure 5.3 for his completed diagram.]

D: Um hmm. So it would be two-thirds of a cup of flour?

Jack: Yeah, two-thirds of a cup of flour for each.

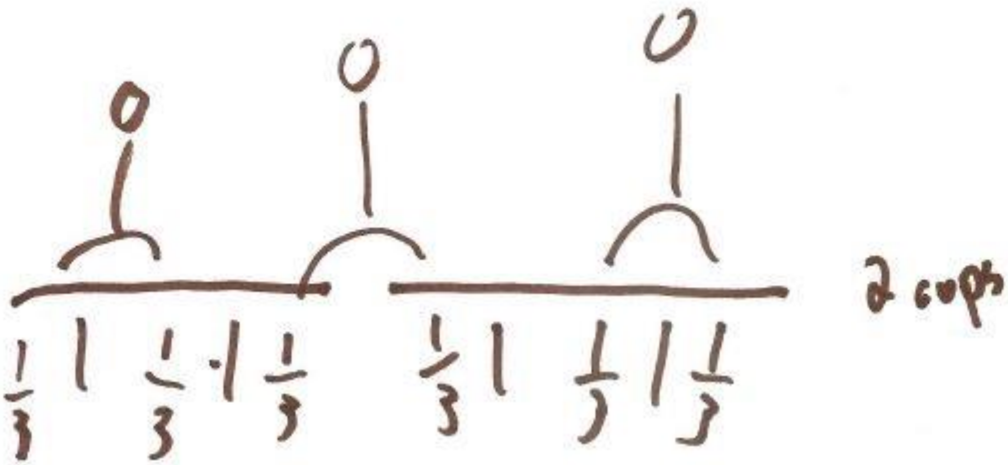

Figure 5.3. Jack's completed diagram for how he determined two-thirds cups per dozen.

In contrast to his earlier struggles, Jack successfully split the 2 cups into three equal parts after thinking for only 10 seconds. Thus, my alterations to the task had the desired effect, and Jack assimilated the task as a situation of his distributive partitioning operations. It is not clear whether the sharing language, the diagram, or a combination of both contributed to this change in Jack's assimilation of the task. However, my hypothesis is that representing the 2 cups as two separated line segments played the greatest role in this change. In particular, I infer that physically separating the two segments contributed to Jack's realization that he could split each of the individual cups into thirds in order to achieve his goal of splitting the entire quantity of flour into three equal parts. While Jack initially stated it would be "two-thirds of both cups" for 1 dozen, his explanation and accompanying diagram make it clear he was thinking of two-thirds of 1 cup of flour for each dozen cookies.

Jack attempts to adapt these ways of reasoning to the water pump scenario.

Given this advance in Jack's ways of reasoning, I returned his attention to the water pump context to see if he could reason analogously to determine the pumping rate of Pump 1. A

12 minute and 35 second interaction ensued in which Jack encountered, and eventually alleviated, several significant perturbations to determine that Pump 1 was pumping at a rate of four-fifths of an inch per minute. Jack's conceptual path from two-thirds of a cup per dozen to four-fifths of an inch per minute was rarely direct, and the protocol that follows is rather long. However, I find the entire interaction critical when accounting for the mental operations that ultimately enabled Jack to overcome his earlier constraints to successfully quantify the result of splitting a composite unit. I have broken the protocol into several parts so that I can intersperse my discussion of critical moments throughout the excerpt. In addition, I have included line numbers for this protocol to more easily identify particular phrases during my analysis of the interaction.

Protocol 5.6: Jack's determines a pumping rate by reasoning analogously to splitting 2 cups of flour among 3 dozen cookies.

D: What if we came back then to this pumping scenario. We had Pump 1, which was pumping 4 inches for each 5 minutes.

Jack: Um hmm.

D: So we were trying to think about how many inches per each minute it would be. Is there a way for you to use some similar ideas or strategies to think about that question with the inches per each minute?

Jack: [Picks up the marker.] You can have 4 inches the same way you did that. [Draws four roughly inch-long horizontal segments in a row on his paper and labels them as "4 inch".] Like that. And then have 5 minutes [Draws five circles below the segments and labels them as "five min".] Then you'd like… [Thinks for 6 seconds. Then he moves his marker in the air over all of the one-inch segments and pauses five times. Next he moves his marker back to the beginning of the row of segments and moves the marker in the air over the segments, pausing twice over each one-inch segment.] So like, you can take it and, umm, hold on. [Partitions the first one-inch segment into four parts. Then, he pauses and again moves his marker over the segments in the air, pausing several times over each segment. However, it is unclear exactly how many times he pauses over each segment this time. He then goes back to the marks he made under the first one-inch segment and crosses them out. See Figure 5.4 for his diagram after completing this activity.]

Figure 5.4. Jack’s diagram after his first attempt at splitting 4 inches into five parts.

21	D:	What are you thinking about?
22	Jack:	I was wondering if I split each one of them into four and go three, three, three.
23		But you can’t because you have one left over.
24	D:	Um hmm. Yeah, because if you did them each into four then you’d have 16
25		parts—16 fractions there of your inches.
26	Jack:	[Marks each of the four one-inch segments into three parts.]

Jack’s diagram and description in lines 7–10 show that he made a correlation between the activity he carried out with my suggested diagram during Protocol 5.5 and the current task. In both cases, the composite quantity he wanted to split was represented by individual line segments, and the concomitant quantity that defined the splitting goal was represented by circles.

However, in contrast to Protocol 5.5 where Jack appropriately decided to partition each segment into three parts, we see here that Jack’s approach to partitioning each one-inch segment was much less certain. When he paused five times while spanning his finger over the entire collection of 4 inches (lines 11–12), I infer that he imagined cutting the entire 4 inches into five equal parts, one for each minute. Afterward, he appeared to experiment with different numbers of partitions in each one-inch segment (lines 12–20). His explanation in lines 22–23 confirms that he tested splitting each one-inch segment into four parts but determined this would not support creating five equal partitions of the entire four inches. Lastly, in line 26 we see that Jack tried a new partition of three parts in each one-inch segment. I believe that Jack's trial-and-error approach to partitioning each individual segment indicates he had not yet become explicitly aware of the ways of operating that guided his activity when splitting 2 cups of flour among 3 dozen cookies.

Jack reflects upon his reasoning with the recipe scenario.

As a result, I decided to question Jack about his decision to split each cup into three parts in hopes of supporting a reflected abstraction of his previous activity. My goal was to help Jack make his partitioning activity more explicit and to connect this activity with fraction language so that he might come to understand his activity as finding one-third of the composite unit by finding one-third of each unit.

Protocol 5.6: First continuation.

D: So when you were thinking about this with the flour and the 3 dozen cookies, how did you know to—why did you decide to do each into three?
Jack: Because it was three separate things.
D: Okay. So like what amount of all the flour went into each dozen then?
Jack: Two-thirds, two-thirds.
D: Two-thirds of a cup. But like, what amount of all the flour—of all 3 cups—was that?
Jack: You mean both cups?
D: Excuse me. Both cups, yeah. The 2 cups.
Jack: What amount of all of it?
D: Yeah.
Jack: Umm… [Thinks for 10 seconds.]
D: Does my question make sense?
Jack: Yeah, it makes sense. I'm just thinking about it.
D: Okay.
Jack: [Thinks for 9 more seconds.] It would be two-sixths.
D: Why do you say that?
Jack: Because there's three in each one of them, but together it's six. And then it's just two, two-sixths.
D: Okay. Is there another name for that fraction or another way you could think about what fraction that would be?
Jack: [Thinks for 8 seconds.] So two-sixths would be one-third of 2 cups of flour.
D: Hmm. Does that make sense that it's one-third of the flour?
Jack: Yeah.
D: Why does that make sense to you?

52 Jack: Because, umm, like instead of me splitting each one into thirds, I could have
53 just split them, like, not exactly in half, but a little more than half. [Yawns.]
54 D: Um hmm. So if you thought of [picks up the marker to draw something on
55 Jack's diagram, but then hesitates]…umm, I don't know if I want to draw. But
56 I think that makes sense. When you're saying a little more than half, you're
57 picturing that it…umm—what amount is that in relation to all of the flour?
58 Jack: If like, you have, instead of having 2 cups like you do, it's like one big cup.
59 [Draws a horizontal segment that he refers to as "one big cup".] And you go
60 just a little bit more than half. [Marks this new segment into three equal parts,
61 calling each "a little bit more than half".] You'd end up [writes "1/3"
62 underneath each of the three parts he had drawn on the connected two-cup
63 segment. See Figure 5.5.]

Figure 5.5. Jack's completed "one big cup" diagram.

64 D: Each being a third. Okay. And then this line now represents what?
65 Jack: Two cups. But I added it in to make just one big cup.
66 D: Sure. So you think of…if you think of all the flour as all 2 cups you get a third
67 in each part, and then you can figure out that that was two-thirds of 1 cup in
68 each of those, right?
69 Jack: [Nods affirmatively.]

Jack's reasoning in this protocol mimics the distributive partitioning operations he used during the check-up interview (cf. Protocol 4.3). Consider the following progression of his reasoning in this excerpt. First, Jack made explicit his strategy of partitioning each cup into three parts because there were "three separate things" (i.e., 3 dozen, line 29). As a result of partitioning each cup into three parts, Jack knew each dozen contained two-thirds of a cup of flour (line 31).

Considering this in relation to the entire 2 cups, Jack initially viewed this result as two-sixths of all the flour (lines 42–45). Next he recognized this as one-third of all the flour (line 48). Lastly, Jack coordinated these results to understand that he could think of the amount of flour per dozen as one-third of all of the flour and as two-thirds of 1 cup of flour. The model of Jack's distributive partitioning operations I presented in Chapter 4 fully accounts for this reasoning and the fractional relationships he constructed.

However, there are two differences between this protocol and Jack's earlier activity with sharing the cakes that I believe are significant. First, it is likely more intuitive and natural to imagine partitioning cakes than it is to imagine partitioning cups of flour. So using his distributive partitioning operations in this context represents an advancement to his ways of reasoning. As mentioned earlier, the suggestion of the linear diagram likely supported this possible accommodation to the situations that activate his distributive partitioning operations.

Secondly, Jack's creation of the "one big cup" diagram (see Figure 5.5) made his conceptual uniting of the 2 individual cups into a composite unit representing both cups explicit. Further, this explicit representation of the 2 cups as a composite unit provided a diagram upon which Jack could operate further and coordinate the two types of fractions he was using—the fractional amount of all the flour (i.e., one-third) and the fractional amount of 1 cup of flour (i.e., two-thirds). For example, when Jack split the "one big cup" segment into three parts, he decided where to make the partitions by going "just a little bit more than half" (lines 60–61). I infer that this referred to a little bit more than one-half of an individual cup, specifically two-thirds of an individual cup, not half of the entire 2 cups. Thus, in carrying out this activity upon the diagram, Jack appeared to be simultaneously aware of the 2 cups as two individual one-cup units and as one composite two-cup unit. Further, he coordinated the fractional results produced from each view within the same "one big cup" diagram.

Jack establishes the unit ratio as four-fifths inches per minute.

Having made these ways of reasoning with the 2 cups of flour more explicit than he had done previously, I returned once again to the 4 inches in 5 minutes pumping context. I've included the remainder of the interaction in its entirety to provide the reader with a better sense of the direction of his reasoning throughout the course of the interaction. I will save the remainder of my analysis of this interaction until after the completion of the rest of the protocol.

Protocol 5.6: Second continuation.

D: Okay. So, so then what about back to here? [Points to the four segments and five circles Jack had previously drawn on his paper to represent the 4 inches and 5 minutes, respectively.] We had the 4 inches in the 5, for each 5 minutes, right. And we were trying to figure out how to decide how many inches per minute that would be. Umm…What do you think?

Jack: Like… [Draws a horizontal segment and labels it as "4 in".] You take [and] add all 4 inches into one and then you split it, [partitions the segment into five parts], into the same size. One, two, three, four, five [stated while counting the partitions]. That's five. And then, so that's about, that's one-fifth of each one of these, of the whole, of all 4 inches put together. [Labels each partition with the fraction "1/5". See Figure 5.6.]

Figure 5.6. Jack's representation of splitting the composite 4 inches into five parts.

D: Okay. So it would be one-fifth of all 4 of the inches.

Jack: Um hmm.

D: So then what about that other question? So we knew this [points to the previous work with the cups of flour and dozens of cookies] was a third of all the flour together, but we also could figure out how much it was of 1 cup. So, it would be a fifth of all 4 of those inches. But what amount of 1 inch would that be?

Jack: It would be, umm, like, how much, how—eeah [an utterance that I took to mean Jack was having trouble stating what he meant and he was clearing his mind to start again.] We're looking for how much, how many inches it would go up in 1 minute.

D: Um hmm.

Jack: It goes up 1 inch, eeah, 4 inches in 5 minutes which is basically…five…ah. So 5 minutes would be…4 inches in 5 minutes would be 5 fives or one. So, in 1 minute it would…be like one… [Thinks for 15 seconds.] It would be one-fifth. Unless you, because you get, if you add two of them that's too many. There's 4 minutes. There's one, two, three, four, five splits [counting the partitions made by his four marks]. And there's two, two, two—you can't do that. So it would have to be one and then a little bit over on this one, one and a little bit over on that one, one and then a little bit over, one and little bit over. [While saying this, he slid his finger along the four-inch segment and paused at spots slightly longer than the one-fifth marks on the segment.] To make four. You wouldn't be—you wouldn't be able to add—just keep one [one-fifth as 1 inch] because that would be too many inches. And you can't just make it two [two-fifths as 1 inch] because that would be too little inches.

D: When you say two, I don't know if I quite know what you mean there. Two what?

Jack: Like, two-fifths. Two-fifths is too big. It's more than one, it's more than 1 inch. And one-fifth is less than 1 inch. So it would have to be somewhere in between one-fifth. It would have to be, like, a slightly bigger fraction but not as big.

D: So, okay. So this is when you're thinking of all 4 inches together, right?

Jack: Um hmm.

D: So are you saying that it would be, umm, one of these would represent 1 minute? [Points to one of the "1/5" parts on Jack's four-inch segment.]

Jack: Uh uh [No]. One of these, because this, if you would had just one of these [points to one of the five "1/5" parts on his four-inch segment], it would make 5 minutes total. If you, like, counted it like that. But it's a little bit more than this is to get it to where it's not.

D: We had 5 minutes for pump one, right?

Jack: Um hmm.

D: Um hmm.

Jack: But I mean, like 4 inches because this is four [points to the four separate inch long segments he had drawn earlier]. So it wouldn't be five. Each one of these is a minute [points to each segment that he labeled as one-fifth of the four-inch segment]. So there's 5 minutes and 4 inches. But, if you just took this one [referring to one of the "1/5" parts on his four-inch segment as an inch], that'd be an inch, that'd be an inch, that'd be an inch, that'd be an inch, that'd be an inch [points at each "1/5" part in turn]. That's 5 inches not 4.

D: Yeah, so it can't. So each of these can't be a whole inch.

Jack: And then two-fifths couldn't be a whole inch. Or it is more than an inch.

D: Oh, okay. So you're saying so this part [points to one-fifth of the four-inch segment] can't be a whole inch. But if you looked at two of the minutes [traces the marker along two of the one-fifths of the four-inch segment], that would be more than an inch.

Jack: Um hmm.

D: So let me ask you a different, a related, question. So this is when we're thinking about all of the 4 inches as a whole group, right. [Points to the four-inch segment split into five parts, each one-fifth of the whole. See Figure 5.6]. Well, what if we kind of think of them separate a little bit. What if we have our 4 inches, right? [Draws four separate one-inch long segments.]

Jack: Um hmm.

D: So rather than kind of thinking about it all as one group, we've got our 4 inches. We know that we fill up that amount per each 5 minutes.

Jack: Um hmm. You, I kind of, bleah [another tongue-tied type utterance].

D: Now how might you split those up so you can figure out how much is in each minute?

Jack: [Makes two marks in the first one-inch segment. Then Jack pauses and looks back at the previous work with the four-inch segment, see Figure 5.6, and appears to count something. He then returns to the first one-inch segment and makes two additional marks in the segment producing five partitions in the first one-inch segment. He continues on to partition each of the remaining three one-inch segments into five parts. He then writes the fraction "1/5" underneath one of those partitions in the last one-inch segment and then draw lines connecting the written fraction "1/5" to one partition from each of the four one-inch segments. See Figure 5.7 for his diagram after completing this activity.] So each one of those is five-fifths—is one-fifth—because each one is split into one, two, three, four, five. [Counts the five partitions in the first inch.]

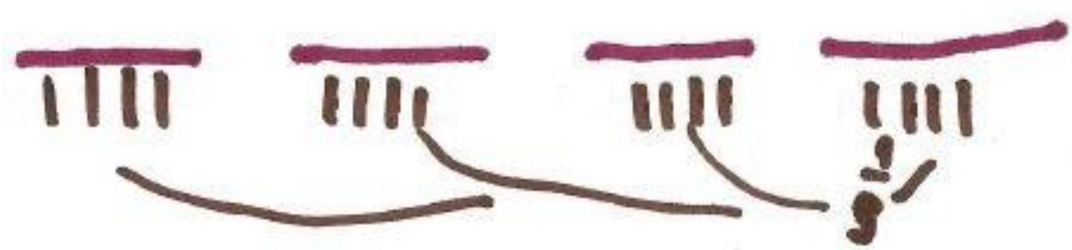

Figure 5.7. Jack's representation of splitting the four individual one-inch units into five parts each.

D: Okay.

Jack: And, it would be four-fifths of each one. Because four; and one, two, three, four [said while counting the small partitions of the individual one-inch segments]; one, two, three, four; one, two, three, four; one, two, three, four. That’s 5, that’s 5 minutes.

D: Um hmm. So when you were doing this and counting these off—one, two, three, four—what were you checking?

Jack: Making sure it was four-fifths of each one.

D: So if you did four-fifths, and then…when you were, when you were counting, what were you trying to get?

Jack: Like, I was trying to see how many would be left over. So one, two, three, four—that’s four [places a dot in each of the first four partitions of the first inch segment].

D: Um hmm.

Jack: And then one, two, three, four—that’s four [places a dot in each of the next four partitions of the inch segments]. And then…that’s four [places a dot in the next four partitions]. [Places a dot in each of the next four partitions.] That’s four. [Places a dot in each of the remaining four partitions.] That’s four. [See Figure 5.8. I have changed the color of the dots from his original diagram to make it more clear what Jack was counting with each group of four.]

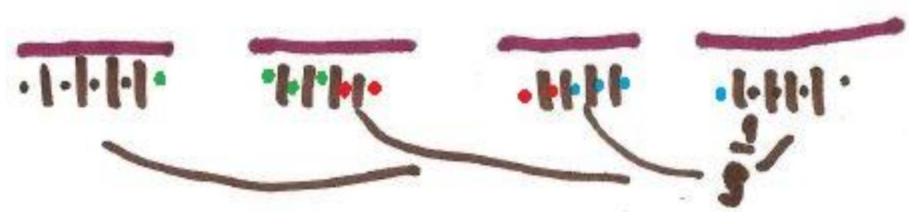

Figure 5.8. Jack’s representation of splitting the four individual one-inch units.

D: So each group, each group represents what then? Each group of four.

Jack: Equals 1 minute.

D: So how many inches per minute?

Jack: It would be four-fifths of an inch per minute.

D: [Gives Jack a high-five.] Nice job! That was good! Good work on that! So four-fifths. So now you know it’s less than an inch a minute. And you can tell me that it’s four-fifths of an inch-per-minute. Nice!

Importantly, this protocol represents the first time that Jack constructed a unit ratio in the context of pumping rates. Jack initially identified this unit ratio as one-fifth of each split inch and then transitioned to understanding this as four-fifths of 1 inch, suggesting he viewed each split inch, and its partitions, as identical (lines 152–162). At the end of this protocol, I infer that Jack understood that pumping 4 inches of water per every 5 minutes was equivalent to pumping less than an inch per minute, that the depth in 1 minute would be one-fifth of the depth in 5 minutes, and that this pumping rate would be four-fifths of an inch per minute. Yet, developing these understandings clearly was not a straightforward and simple achievement for Jack.

Accounting for the progression of Jack's reasoning.

While there are many interesting aspects of this protocol that I could analyze in greater detail, I want to focus specifically on one aspect of Jack's reasoning that I believe accounts for both his struggles and his achievements—the nature of the units with which Jack was operating throughout Protocol 5.6 and its two continuations. Jack's alternating treatment of a measured quantity as a single composite unit and as a sequence of individual unit items characterizes his reasoning throughout this entire protocol.

Consider the following summary of the progression of his reasoning. Prior to this teaching session, the evidence suggests that Jack assimilated measured quantities as a single composite unit (cf. Protocol 5.1). True to form, Jack initially could not quantify the result of splitting a composite 2 cups of flour into three equal shares (before Protocol 5.5). Next, incorporating my suggested diagram of two physically separated linear units, Jack reasoned with each cup individually to achieve the quantification of this split as two-thirds of a cup per dozen (Protocol 5.5). Returning to the pumping rates, Jack similarly assimilated the measured 4 inches as four individual one-inch segments but used a trial-and-error approach in an attempt to achieve splitting the quantity into five parts, one for each minute (Protocol 5.6, lines 1–26). Then, when I intervened to try to help him become more explicitly aware of his reasoning about the two-thirds of a cup per dozen, Jack first focused on two individual units (Protocol 5.6, lines 27–31) and later reasoned with them as a composite quantity that he called “one big cup” (Protocol 5.6, lines 32–69). Returning for a second time to the pumping rates, Jack maintained this focus on the composite quantity and reasoned with a single linear unit that represented all 4 inches (Protocol 5.6, lines 70–135). However, he struggled to locate 1 inch within the context of the fifths of 4 inches he had constructed. Hence, I suggested Jack return to considering individual units and he successfully identified the unit ratio as four-fifths of an inch per minute (Protocol 5.6, lines 136–184).

Thus, throughout the entire Protocol Jack’s focus *alternated* between reasoning with a measured quantity as a single composite unit and as sequence of individual unit items. Further, each change in his assimilation of the measured quantities can be attributed to a teacher-researcher intervention, either a question or a suggested diagram. This suggests that Jack’s assimilation of the measured quantities was constrained to whatever conception he was operating within at the time.

However, even though Jack did not independently switch his focus between these two views of the measured quantities, his successful use of my suggestions indicates that each way of reasoning was available to him as a viable way of assimilating a measured quantity. Moreover, each perspective appears to have been important in Jack’s eventual construction of the two unit ratios. When viewing the measured quantity as a single composite unit, I infer that Jack assimilated the situation with his splitting scheme. Much like in Protocol 5.1, this enabled Jack to form a goal of finding a number of inches that was one-fifth of 4 inches. Alternatively, when viewing the measured quantity as a sequence of individual unit items, I infer that Jack assimilated the situation with his distributive partitioning operations. Then, as in the context of sharing two cakes, Jack reasoned by splitting each individual unit. Yet, Jack's struggles throughout Protocol 5.6 highlight that either view on its own proved insufficient.

In consideration of this progression of Jack's reasoning, I claim that the only two times in which Jack became *simultaneously* aware of these two conceptions of the measured quantity occurred at the ends of the first and second continuations of Protocol 5.6. At the end of the first continuation, Jack coordinated his result of two-thirds of an inch per minute within the context of his "one big cup" diagram that represented the composite unit of both cups (lines 48–69). Then at the end of the second continuation, I infer that Jack's previous splitting of the composite four-inch segment into five one-fifths informed his decision to split each individual one-inch segment into five parts (lines 147–155). Thus, constructing the unit ratio four-fifths of an inch per minute apparently involved coordinating these two alternative views to become simultaneously aware of each characterization of the measured quantity and its implications.

This simultaneous awareness accounts for Jack's ability to overcome his earlier constraints and finally quantify the results of splitting a composite unit. Further, it provides a model of the quantitative operations needed to construct unit fractions as operators one can use to conceptually transform both individual and composite units.

This protocol suggests that in all of Jack's previous uses of his distributive partitioning operations, I had taken for granted his simultaneous awareness of a composite unit as a collection of individual unit items and as a single composite unit. Rather than being a given way of reasoning available in assimilation, I infer that Jack was constructing this simultaneous awareness anew within each context. Jack's struggles to construct this awareness in the context of splitting 4 inches into five parts suggest that this is not a trivial construction. Further, Jack's understanding at this point of the experiment seems to be defined by an alternating awareness of measured quantities as composite wholes and as sequences of individual unit items. The particular awareness to which he assimilates a given task seems dependent upon contextual factors of the task or the teacher-researcher's questions.

There is another important simultaneous awareness that relates to Jack's coordinated partitioning/iterating strategy—the simultaneous awareness that the one-fifth in Jack's composite four-inch diagram (see Figure 5.6) characterized the desired relationship for both quantities. I infer that Jack's initial decision to split the four-inch segment into five parts stemmed from the fact that he wanted to know the change in water depth *per minute*. Thus, on the basis of his splitting operation, finding one-fifth of the four-inch segment corresponded to his knowledge that 1 minute constituted one-fifth of 5 minutes. Further, he knew that each group of four-fifths of an inch "equals 1 minute" (line 179).

However, in the process of carrying out these complex operations, Jack at times conflated the quantities. For example, in lines 115–118 Jack used "minutes" even though his description suggests he was thinking about the number of inches. While he was able to quickly resolve this conflation in the course of our interaction, this example indicates that Jack's explicit awareness of the simultaneous nature of one-fifth was at times suppressed as a result of the significant cognitive demand placed upon his conceptual resources.

Investigating Jack's abstraction of these ways of reasoning.

I hypothesized that abstracting the operations Jack used during Protocol 5.6 to the level that they became available during assimilation would reduce the cognitive demand and enable Jack to more easily coordinate his operations on both quantities. To that end, following Protocol

5.6 I asked Jack if he could determine the pumping rate for Pump 2, which was pumping 3 inches of water in every 4 minutes.

Protocol 5.7: Jack determines a unit pumping ratio equivalent to 3 inches per every 4 minutes.

Jack: [Draws a diagram of three disconnected horizontal segments and partitions each into four parts.]
D: So what do you think?
Jack: It would be, umm… [Jack then puts a dot in each of the 12 partitions he had made, pausing after placing dots in each group of three partitions. See Figure 5.9 for his completed diagram. As with Figure 5.8, I have again changed the color of his dots to reflect the groups in which he produced them.] Three-fourths.
D: Um hmm. Three-fourths what then? What does that mean?
Jack: Three-fourths, um, of an inch in 1 minute.

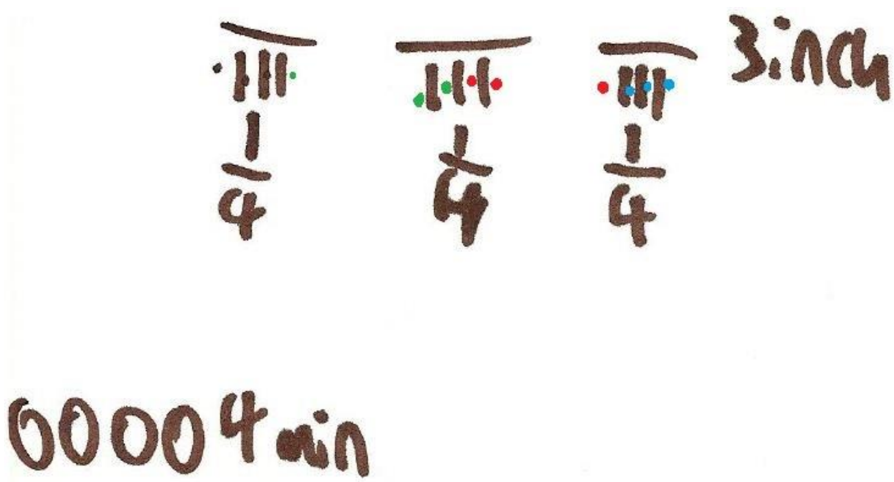

Figure 5.9. The diagram Jack constructed while determining the unit ratio three-fourths of an inch per minute.

I find this protocol significant as a follow-up to Protocol 5.6 given the stark contrast between the relative ease with which he constructed the unit ratio for Pump 2 and the struggles he encountered while determining the unit ratio for Pump 1. In fact, this entire protocol lasted only 65 seconds, a majority of which were spent drawing the diagram. Thus, we see that Jack successfully assimilated the pumping information for the Pump 2 as a situation of the same schemes he used to find the unit ratio in the previous protocol. Further, he appropriately accommodated his strategy to account for having 4 inches in 3 minutes without resorting to any trial-and-error partitioning.

Due to its proximity to the previous task, I do not consider this as an indication that Jack had completed the abstraction of these ways of reasoning. Rather, given the nature of these interactions, I conclude that we operated within Jack's zone of potential construction and that the nature of the interaction supported at least a *pseudo-reflective abstraction* that he relied upon in this protocol.[11] I call this a pseudo-reflective abstraction rather than a reflective abstraction because Jack achieved this functional accommodation in activity, and it remained unclear if this way of reasoning would persist as an assimilating structure more generally. The judgment that Jack had not yet reflectively abstracted these ways of reasoning was made in consideration of my prospective knowledge that during the next teaching session Jack needed to re-construct this sequence of operations when solving a novel, but closely related task.

However, I consider Jack's abstraction as something different than a *pseudo-empirical abstraction*. According to von Glasersfeld (1995c), a pseudo-empirical abstraction refers to "a coordination or pattern of the subject's own activities and operations […] [that] can take place *only if suitable sensorimotor material is available* [emphasis added]" (p. 105). Further, pseudo-empirical "has been used when there are actions involved without specifically specifying if the actions are operations" (L. P. Steffe, personal communication, May 30, 2015). The critical difference in this case is that Jack produced the sensorimotor material upon which he operated

[11] I am indebted to Leslie P. Steffe for suggesting this terminology to refer to an abstraction that falls somewhere between a pseudo-empirical and a reflective abstraction.

rather than operating in response to particular given sensorimotor material. Additionally, the activity he carried out with the diagram served as a stand in for his conceptual distributive partitioning operations.

However, while certainly indicating an advancement in Jack's ways of reasoning, his activity in this protocol leaves several unanswered questions. First, it is unclear if Jack understood three-fourths of an inch per minute as one-fourth of 3 inches per 4 minutes. Further, his diagram suggests the same conception of the measured quantity that he used at the end of the previous task (i.e., a sequence of individual units). Thus, while he produced his own perceptual material upon which to operate, this task does not reveal whether or not he also constructed a simultaneous awareness of both conceptions of the measured quantity in this case.

The Adopt-A-Highway Context

To investigate these unanswered questions further, I designed a series of tasks related to allocating various amounts of roadway to volunteer organizations in the Adopt-A-Highway context. My primary goal in this context was to provide the students opportunities to abstract their ways of operating upon composite quantities in order to make these ways of reasoning more explicit and flexible. In particular, I wanted students to develop and solidify the mental operations necessary to form a goal of finding a fractional amount of a composite quantity and to quantify that amount as a fraction of one unit.

Jack Reconstructs His Ways of Operating

I would characterize Jack's reasoning in the various Adopt-A-Highway context tasks as one step back, followed by three steps forward. Further, the types of reasoning Jack used throughout this context remained consistent with the reasoning he exemplified during previously presented protocols. However, his facility with these operations, as well as the range of situations to which they applied, improved quickly.

One step back.

In the first task of the November 14, 2013, teaching session, Jack initially struggled to coordinate the quantities, and his reasoning was reminiscent of the struggles he experienced at the beginning of Protocol 5.6. The task involved determining the amount of miles each group would be responsible for when allocating 4 miles of highway to nine volunteer organizations. In addition, I provided Jack with a map that had identified the 4 miles as four one-mile sections (see Figure 5.10).

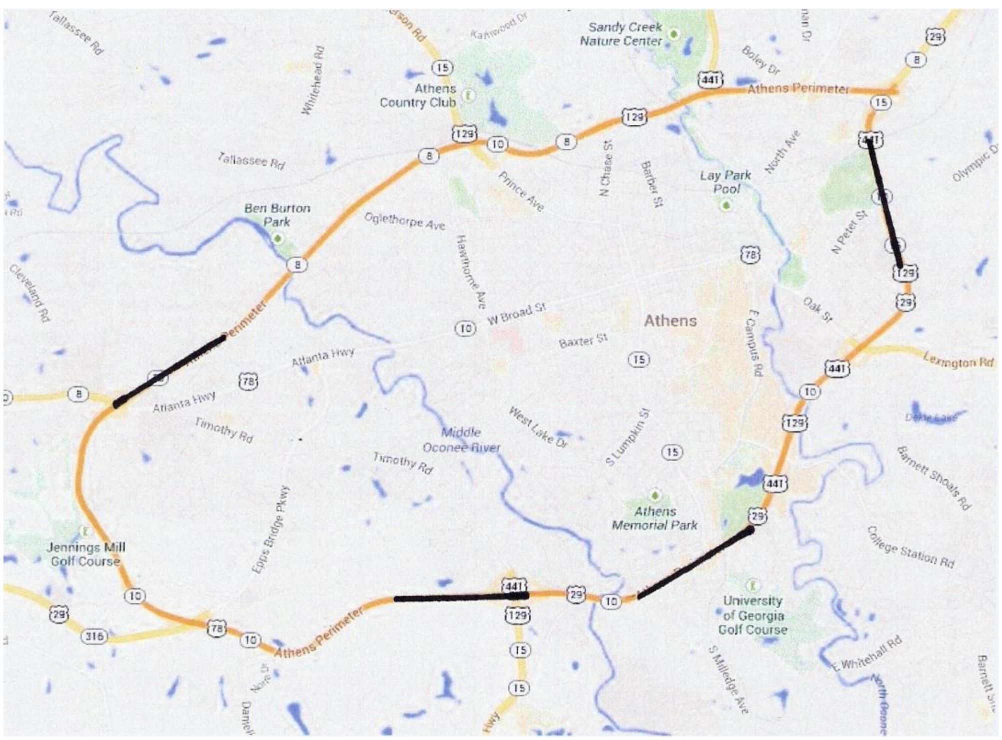

Figure 5.10. The map provided for the task of allocating four one-mile sections to nine organizations.

However, before describing how Jack solved this task it is important to note that Jack never wrote anything down nor made any marks on the map. Rather, he constructed figurative material upon which he carried out his operations and reasoning. I find this significant because it indicates that Jack's reasoning was not constrained to perceptual material or particular diagrams and provides further evidence that he had made more than a pseudo-empirical abstraction of his ways of operating. Thus, even though Jack frequently used diagrams to document and describe his reasoning on subsequent tasks in the Adopt-A-Highway context, I infer that those diagrams provided a context for Jack to physically enact the operations he used to mentally operate upon figurative material.

In solving the given task, it appeared that Jack's goal of finding nine equal shares guided his initial reasoning. To accomplish this goal, he mentally split three of the one-mile sections into three parts each and split the final one-mile section into nine parts. As a result, he decided that each group would get one-ninth of the combined 3 miles and one-ninth of the final mile.

While allocating the highway in this fashion accomplished his goal of fairly allocating all 4 miles, Jack struggled to quantify the number of miles this represented for each group. On the basis of his whole number multiplicative operations, he stated that one-ninth of 3 miles would be one-third of a mile because each of the 3 miles would be split into three parts to create the nine parts of the 3 miles total. The perturbation arose for Jack when he attempted to combine the one-third and one-ninth of a mile into a total amount for each group. He thought about this unsuccessfully for approximately 90 seconds.

Next, I asked Jack if he could figure out how many ninths would be in each one-third of a mile. Jack apparently assimilated this question as a situation of his recursive partitioning scheme and quickly determined that one-third was equivalent to three-ninths of a mile. This realization then enabled him to unite the two sections of highway together in order to produce the result four-ninths of a mile for each organization.

I interpret this activity as confirmation of my previous assertion that Jack had not yet completed the reflective abstraction of the ways of reasoning he used when finding unit ratios in Protocols 5.6 and 5.7. However, his reasoning here was different than previous tasks in that rather than carrying out the same operations upon each one-mile section, Jack created a composite three-mile section to go along with the remaining one-mile section. Yet, he experienced constraints similar to previous tasks when trying to reconcile how to combine one-ninth of 3 miles with one-ninth of 1 mile. Splitting each one-mile section into nine-parts allowed Jack to alleviate his perturbation and identify each group's share as four-ninths of a mile. It took Jack several minutes to resolve this perturbation, indicating that his reasoning was not as anticipatory as it was at the end of the previous teaching session during Protocol 5.7. For this reason, I characterized his reasoning as taking a slight step back.

One step forward.

However, recognizing anew that splitting each individual unit enabled him to quantify his goal of splitting a composite unit apparently played a critical role in Jack's reasoning. In subsequent tasks within the Adopt-A-Highway context, Jack's reasoning became both more anticipatory and applicable to a wider range of situations. His activity in the next several tasks suggests that he made progress in constructing a simultaneous awareness of measured quantities as both single composite wholes and as sequences of individual units. Further, the advances in Jack's ability to solve tasks within this context provide insight regarding the conceptual operations that could feasibly account for his construction of this simultaneous awareness.

Consider Jack's response to the next task of trying to allocate 5 miles of highway evenly to eight different volunteer organizations for the Adopt-A-Highway program. Prior to the interaction in Protocol 5.8, I gave Jack a map with five separate one-mile sections of highway shaded, and he promptly proceeded to partition each of the five one-mile sections into eight parts. After completing the partitioning he thought for 10 seconds and then wrote "5/8 for each group." Protocol 5.8 begins with Jack's explanation for how he thought about the task.

Protocol 5.8: Jack's solution to allocating 5 miles to eight volunteer organizations.

Jack: There was eight groups and there's 5 miles. So you split each mile into eighths. So, it's…for 1 mile, it's five-eighths. And, for—you get five-eighths per group for 1 mile. But for all the miles it would be… [Writes "8 · 5 = 40" and then the fraction "5/40" on his paper.] So it will be five over forty for every group.
D: So they'd get five out of forty of
Jack: Five miles.
D: Of these parts?
Jack: Yeah.
D: Okay. One follow-up question on that. These forty, what's one of those forty? [Points to the denominator of his fraction "5/40".] Can you point to it?
Jack: One of those forty?
D: Yeah. They're fortieths. Umm, that's…
Jack: Like, just one piece of a mile. [Points to one of the small partitions he had made in the fifth one-mile section.]
D: Okay. And how big is this one piece? What part of a mile?
Jack: It's one-eighth, or one-eighth of a mile.
D: Okay. So there's five of those pieces that are an eighth of a mile for each group.
Jack: Um hmm.
D: Umm…yeah. So you think it would be…so five-eighths of a mile per each group. And one last question on yours. Which…when, ah, so what would be the—can you show me, like, what you would give to the first group? Say we had like group A, B, C, D, E… What would the first group get?
Jack: [Counts five of the small partitions in the last one-mile section and circles them.]

There are a few aspects of Jack's reasoning that stand out in this task. First, Jack's partitioning had again become intentional, and he assimilated finding one of the eight organization's share of the highway as a situation of his distributive partitioning operations. Secondly, Jack understood the results of his partitioning both in terms of each individual mile (five-eighths) and in terms of all the miles (five-fortieths). Further, he had constructed a relationship between these two views and could explain that the fortieths in his written fraction "5/40" were each "one-eighth of a mile."

Both Jack and John were present for this teaching session. Thus, following this excerpt John explained that he also got five-eighths of a mile. Further, he stated that each group would be responsible for one-eighth of a mile because "There's eight groups and there's five. You'll…you'll get the fraction five over forty. Which I did, but I just simplified it to one-eighth".[12] Because John had identified each organization's share as one-eighth of all 5 miles, I returned to question Jack further about the fractional amount of the entire 5 miles that Jack had allocated to each organization.

Protocol 5.8: Continuation.

D: Could we use your diagram to think about, um, why this would also be an eighth? Do you agree that this is—that we can think of it as an eighth too?

Jack: Yeah, cause all he did is simplify the fraction.

D: So if we think about the fraction we can reduce it, but what about the picture? Is there a way to figure out on your picture that it would be one-eighth?

Jack: If you just split the entire thing into eighths.

D: Can you say a little bit more about that? What do you mean by split the entire thing into eighths?

Jack: Instead of having each individual one split into eighths, you have the *whole* [Jack's emphasis] 5 miles split all in—split it—you have like one big five miles and they were all together, and you could split it into eighths.

D: Um hmm, sure. So could you show me on your picture where the, um, the eighths would be? Even though you've already split it up into smaller parts?

Jack: It would be like that. [Points to the five-eighths of a mile share that he had previously circled for one of the organizations.]

D: You want to just take the marker and show me?

Jack: [Makes a mark at the left endpoint of the five-eighths mile segment he had previously circled.] Right here. And then one, two, three, four, five. [Proceeding from the mark he just made, counts five-eighths of a mile for the second group and makes a second mark.] One, two, three, four, five. [Makes another mark.]…[Jack continues counting groups of five-eighths miles and making marks on his diagram until he has exhausted the entire 5 miles (see Figure 5.11). He then counts the groups of five-eighths miles.] One, two, three, four, five, six, seven, eight.

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[12] Because Jack's mathematics is the focus of this Chapter, I have included John's response only for context regarding the interaction in which the excerpts for these protocols took place.

Figure 5.11. Jack's diagram after marking off each organization's five-eighths mile section.

I infer that Jack had both conceptions of the measured quantity available—5 miles as a single composite unit and 5 miles as a sequence of individual units. For example, when describing why each share would be one-eighth of all 5 miles Jack stated, "Instead of having each individual one split into eighths, you have the *whole* [Jack's emphasis] 5 miles split […], you have like one big five miles and they were all together and you could split it into eighths." Thus, assimilating the quantity as a composite whole, "one big five miles," each share was one-eighth of the whole on the basis of Jack's splitting operations. Alternatively, assimilating the quantity as a sequence of individual units, each share was five-eighths on the basis of Jack's distributive partitioning operations.

Accounting for the operations that support Jack's simultaneous awareness.

Significantly, Jack's activity in this excerpt also provides an important clue regarding the operations that enable him to construct this simultaneous awareness of the measured quantity. I infer that Jack united the results of his distributive partitioning activity to construct a composite unit of five-eighths of a mile per group. The significance of the formation of this composite fraction is that Jack then used this newly formed composite unit to restructure the 40 one-eighth mile sections of highway into eight sections of five-eighths of a mile.

The implication is that this method of restructuring the fractional result effectively changed the measurement unit. Rather than measuring the total number of miles in terms of miles (i.e., a quantity 5 times as large as 1 mile), Jack's activity of iterating the composite fraction reconstituted the quantity in terms of five-eighths of a mile (i.e., a quantity that is 8 times as large as five-eighths of a mile). This use of his unitizing and iterating operations enabled Jack to provide meaning for one share as one-eighth of the entire 5 miles on the basis of his quantitative operations rather than having to appeal to a procedure for simplifying fractions.

Jack's activity in this protocol indicates his construction of the reversible distributive partitioning scheme (Liss II, 2014). Iterating the composite fractional unit provided a way to reconcile the fractional results of his distributive partitioning operations with those of his splitting operation. In particular, I infer that Jack assimilated the task as a situation of his splitting scheme, formed the goal of splitting 5 miles into eight parts, and enacted his distributive partitioning operations to accomplish that goal. Then, iterating the composite fraction five-eighths enabled Jack to take the result of his distributive partitioning scheme (i.e., five-eighths of a mile per organization) and use it to reconstruct the original situation (i.e., 5 miles split into eight parts).

As a result, I claim that the construction of composite fractions (in this case five-eighths of a mile) as iterating units accounts for Jack's construction of a simultaneous understanding of a measured quantity both as a composite whole and as a sequence of individual unit items. I refer to Jack's five-eighths as an *iterating unit* (cf. Steffe, 2010c, pp. 41–42) because I infer that he constructed the composite fractional unit five-eighths of a mile per group in activity. Yet, this accomplishment in activity suggests that constructing an iterable composite fractional unit that he could use in assimilation lay within Jack's zone of potential construction at this point of the teaching experiment.

A second step forward.

Jack's response to his final two tasks in this teaching session suggest that he indeed was on his way to constructing these as assimilating operations. Consider Jack's solution to the task of allocating a three-mile section to five volunteer organizations.

Protocol 5.9: Jack assimilates with a simultaneous awareness of a measured quantity both as a single composite unit and as a sequence of individual units.

D: [Places a new map in front of each student.] We've got a three-mile section here that they want to allocate among five groups. Okay. So this shaded part—that's a three-mile long section. And we want to figure out, umm, I want you to think about how much road, how much of 1 mile, would each group be responsible for if we have these 3 miles to allocate to five groups. Okay. So, feel free to think about it, write some things down. Kind of just…
Jack: Three-fifths.
D: …give me a heads up when you're ready. [Chuckles.]
Jack: Honestly, these are…
D: Why don't you think about it for yourself for a moment? I want to give John a second to think as well. Okay?
Jack: [Nods affirmatively.]
D: And if you could write something down that would help you explain, go for it.
Jack: [Starting at the left of the given segment, Jack partitions the given segment and writes his results underneath the diagram. See Figure 5.12 for his completed diagram.]
[…]
D: So why don't you tell me how you thought about the first one? Because you did that really fast.
Jack: Like, the first one it's just—it's all together so you split it up into fifths. And it's…one fifth of each mile. But in 1 mile it would be three-fifths of 1 mile because if you take the miles separate and you split them into fifths—you're splitting it for five people—it's three-fifths of 1 mile.
D: Um hmm. Sure. And then, so when you were thinking about this one, […] how are you thinking about that? [Points to the "1/5 = 3 mil" on his paper.]
Jack: Because you're just splitting, doing the same thing, except they're all together in one so you just split it in five so it would be one-fifth of 3 miles.
D: So a fifth of 3 miles is how much of 1 mile?
Jack: Is three-fifths of 1 mile.

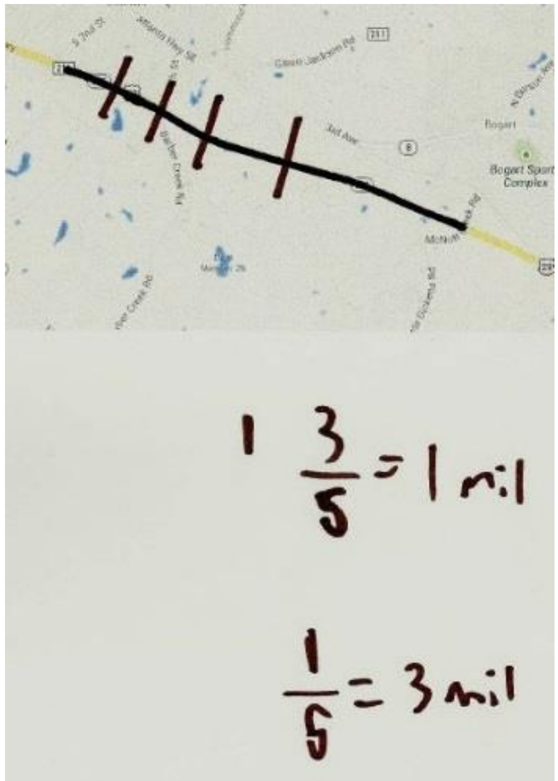

Figure 5.12. Jack’s completed diagram for allocating a three-mile section of highway to five volunteer organizations.

In contrast to previous tasks in which Jack at times required several minutes to construct an understanding of the situation, here Jack solved the task mentally before I could even finish explaining the task. The fact that Jack solved this task so quickly suggests that the ways of operating he constructed in the course of Protocol 5.8 remained active in his thinking and were available as assimilating operations for this task. This enabled Jack to so quickly recognize one-fifth of 3 miles as three-fifths of 1 mile. Further, while previous protocols left some doubt as to the extent to which Jack’s reasoning required the appropriate question or diagram to initiate his activity, here Jack solved the task mentally and only carried out operations on the diagram to satisfy the interviewer’s curiosity.

A third step forward.

To further test this theory that Jack's ways of reasoning were becoming abstracted from any particular context, I phrased Jack's final task of the teaching session in terms of fractions rather than allocating/sharing language.

Protocol 5.10: Jack finds one-fifth of 7 miles.

D: So Jack, I want you to think about one more question today. So, how would you find, umm, what would be, um, a fifth of 7 miles?

Jack: For five…are you still using it like that? [Points to the previous map with the connected 3 mile section of highway shaded.]

D: Umm, if that helps you think about it you can. But I'm just kind of thinking, kind of in general—how would you find a fifth of 7 miles? And however you want to think about it that helps you, go for it.

Jack: [Jack picks up his marker and immediately draws seven separated horizontal segments. He then writes "5" on his paper and proceeds to partition each of the seven segments into five parts. Then after about 5 seconds he writes "1/5 = 7 mil". After about 25 seconds he writes "1" and then "7/5". Then he crosses that out and writes "1 2/5".]
[The end-of-period bell rings.]

Jack: I'm done. […] One-fifth of 7 miles is one mile and two-fifths of a second one.

D: So how did you determine it? Like, what was your strategy?

Jack: You just split each mile into five and then take it and, um…put seven over five because there's 7 miles.

Jack's solution suggests that the operations described above provide a viable way of constructing one's fractions as operators. Previously, Jack had struggled to assimilate fraction language questions as situations of his quantitative operations (c.f. the continuation of Protocol 4.1). In addition, his initial attempts to construct unit ratios in the swimming pool context demonstrated that even when he did form a goal of using his fractional concepts to operate upon measured quantities (e.g., finding one-fifth of 3 inches in Protocol 5.1) his ways of reasoning were insufficient for quantifying the results of those operations. However, in this case I infer that Jack assimilated my question as a call to use his quantitative operations to act upon the quantity 7 miles and activated the conceptual resources necessary to quantify one-fifth of 7 miles as seven-fifths of 1 mile. Further, Jack's creation of his own perceptual material to operate upon suggests that his assimilation of the task called forth these ways of operating rather than the perceptual material being a given part of the task presentation.

Jack Adapts His Ways of Reasoning to a Novel Context

In his next teaching session on November 21, 2013, the last teaching session before winter break, I posed Jack one more task within the Adopt-A-Highway context in order to evaluate the permanence and flexibility of the operations Jack had used so powerfully during the November 14, 2013 teaching session. For this task, I presented Jack with a map containing a continuous two-mile section and a separate continuous three-mile section, and I asked him how much of 1 mile each of seven organizations would be responsible for (see Figure 5.13). Jack initially partitioned each section into seven parts and stated that each organization would be responsible for one-seventh of 2 miles and one-seventh of 3 miles.

Figure 5.13. The provided map for the task of finding one of seven organization's allotment of a three- and a two-mile section of highway.

When asked how many miles each group would receive in total, Jack initially hesitated and asked, “Is it going to be a fraction or does it have to be the exact amount?” In retrospect, Jack’s question suggests that even though he had constructed sophisticated fraction operations, he did not view the results of these operations as exact representations of quantities. This represents one of several instances during the teaching experiment in which Jack expressed concern over whether to use fractions or decimals to express his results. I include this quotation here to highlight that Jack had conceptually separated decimal and fractional quantities and relied upon different quantitative operations to reason with each.[13] At times he expressed a preference for reasoning with fractional quantities while at other times he preferred to use decimals.

In this case, I encouraged Jack to use fractions rather than trying to find a decimal value for the number of miles of highway for each group. Jack then stated it would be like 5 miles split among seven groups, created a diagram of five individual segments and partitioned each into seven parts, and determined each organization’s allotment. From his activity, I infer that he accomplished his goal of finding one-seventh of all 5 miles by finding one-seventh of each mile. Thus, while his production of these results was not as automatic as it was at the end of the previous teaching session, Jack’s ability to adapt his ways of reasoning to solve a situation involving finding one-seventh of two distinct composite quantities indicates that Jack was able to constructively bring forth these operations and use them in ways that were unavailable to him prior to his work in the Adopt-A-Highway context.

[13] In Chapter 7, I present an argument for how the quantitative operations described in this dissertation provide a pathway to reconciling one’s fraction and decimal quantities.

The Inch Worm Context

The Adopt-A-Highway context proved beneficial for helping Jack abstract ways of reasoning that allowed him to produce unit ratios and to operate more flexibly upon composite quantities. However, the context did not allow for considering a covariation of the quantities. Thus, following winter break, the remainder of the teaching sessions involved tasks designed within the context of reasoning about inch worm crawling speeds.

The tasks I developed within this context took on a slightly different focus from those in the previous task contexts. For example, the tasks in the swimming pool and the Adopt-A-Highway contexts focused primarily on the conceptual operations involved with transforming specific measurements into unit ratios that characterized the relationship between the quantities. In contrast, tasks within the inch worm context focused less explicitly on the construction of unit ratios. Rather, I designed the remaining teaching sessions to explore the meanings the students held for unit ratios and the extent of the students' understanding of the covariational relationships they characterized. In addition, many of the tasks incorporated dynamic computer animations that simulated various inch worm races and speed time trials. Against this backdrop, questions ranged from thinking about what measurements one would need to measure an inch worm's crawling speed, to making diagrams and graphs that represented a particular crawling speed, and to writing equations that characterized the relationships amongst the quantities.

Jack's Initial Assimilation of Crawling Speeds in the Inch Worm Context

Unfortunately, Jack's school attendance, while sporadic throughout the teaching experiment, diminished significantly during spring 2014; at one point he missed four of five scheduled teaching sessions. The one he attended during that stretch was a joint teaching session conducted with both Jack and John together on January 23, 2014. Because this was Jack's first day reasoning within this context, I had Jack start to think about speeds in the absence of any measurements. My first task for him involved watching an animation of a race between two inch worms, Speedy and Flash, and thinking about who was faster. The animation started with Speedy slightly ahead of Flash, the race progressed horizontally from left to right, and I stopped the race with both inch worms at the same horizontal location on the screen (see Figure 5.14).

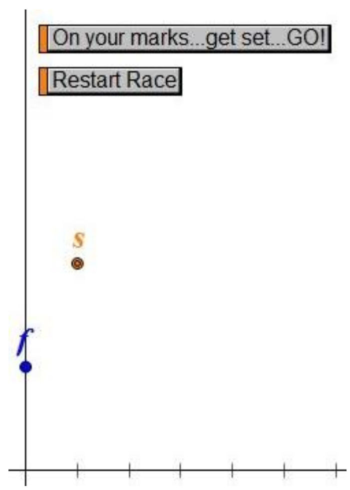

Figure 5.14a. Screen shot of the start of the race animation.

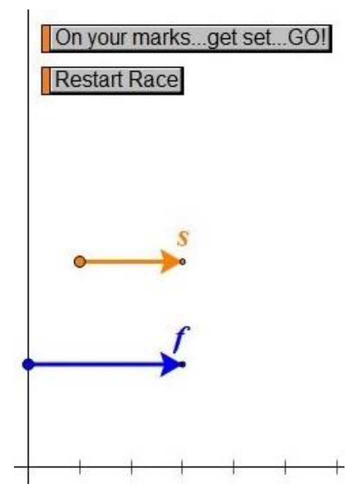

Figure 5.14b. Screen shot of the end of the race animation.

Protocol 5.11: Jack qualitatively compares two inch worm's crawling speeds.

D: What do you think? Who do you think is faster?
Jack: Flash.
D: Why Flash?
Jack: Because even though they started at the same time and Speedy's a little bit ahead of him, he still caught up to Speedy after. And it's got a longer distance in the same amount of time so it's [inaudible].
D: Okay. So even though they're at the same place on the sidewalk now they're not tied? Like, Speedy?
Jack: Like they're tied but, when they, when they started Speedy was ahead of him and now they're tied. So that means Flash is faster than him.

Jack's explanation for why Flash was faster indicates that he assimilated crawling speeds in terms of both changes of distance and time. For example, Jack's assertion that Flash was faster despite having the same finishing distance as Speedy indicates that he focused on each worm's change in position, or distance traveled, rather than simply a gross comparison of their endpoints. But his decision was based upon more than the distance each travelled, as he also qualified that Flash was faster because he had, "A longer distance in the same amount of time."

To move the teaching session toward quantifying crawling speeds, I next asked Jack about finding Flash's crawling speed. His reply was both surprising and significant.

Protocol 5.12. Jack considers which quantities are needed to quantify crawling speed.

D: Good. Okay. So Flash is the faster one. So suppose Flash says, "Well I'm faster." But he wants to know exactly how fast he is so he can tell his friends, "Hey, this is my crawling speed." What kind of information would you need to know to figure out Flash's actual speed?
Jack: [Shakes his head no.] I have no idea. I know we did that last year but I can't remember it.
D: So, like, would there be some things that we might be able to measure that would help you think about his speed?
Jack: [Sighs.] It's like velocity times speed or something like that. Or no, velocity times time.
D: Okay. Would knowing the time, umm, help?
Jack: Yeah. Yeah. I would need the time and the distance.
D: Okay. Would you—would just one of those be enough or do you need to know both time and distance?
Jack: Both.
D: Why do you say both?
Jack: Because if you just do that you know the time, it doesn't mean anything. It doesn't help anything. And if you just know the distance, it doesn't help either because you don't know how long it took him.

I found Jack's reply surprising in that even though he based his previous judgment that Flash was faster upon comparing changes in distance and duration of travel, Jack did not immediately recognize these quantities as useful for determining Flash's actual crawling speed. Instead, he assimilated my question as related to his school mathematics experiences. For Jack, the task of determining speed brought forth images of formulas involving velocity, speed, and time. However, I infer that he did not remember the specific formula and that Jack's identification of both distance and time as important stemmed from his quantitative comparison in the previous protocol.

I find this significant for two reasons. First, it again suggests a divide between Jack's quantitative operations and his school mathematical knowledge. Secondly, while there is no way to know exactly how much Jack's school experiences influenced his reasoning during the teaching experiment, I infer that a majority of his activity with us stemmed from creative uses of his quantitative operations rather than rehearsed ways of operating he learned in school. In this former category I include responses such as Jack's quantification of a fraction composition on the basis of his recursive partitioning operations (cf. Chapter 4) or his initial comparison of the crawling speeds in Protocol 5.11. In the latter category I include responses such as Jack's attempts to recall procedures for multiplying fractions or formulas he used in school related to speed.

Jack Uses Time and Distance Measurements to Quantify Crawling Speeds

Because Jack had decided that he needed both a time and distance measurement to find a crawling speed (John had determined this as well in a previous teaching session), I transitioned the teaching session to using measurements of these quantities to compare speeds. The setting for subsequent tasks involved imagining that the inch worms had set up a time trial track and were taking turns going down the track to check their crawling speeds. The computer screen had a centimeter grid overlay allowing the students to track distances from the starting line to the finish line at 25 centimeters, and I presented the animations as recordings of these time trials that the students could start, stop, rewind, etc. I also gave the students stopwatches so that they could have the experience of monitoring elapsed time while the inch worms crawled down the track.

Prior to this following protocol, I asked each student to collect a specific measurement from their given inch worm time trial. John measured that his inch worm, Abby, took 10 seconds to crawl 4 centimeters and Jack found that Matt needed 21 seconds to crawl 7 centimeters. After writing down each other's measurements, I asked them to try to decide which inch worm crawled at a faster speed. The students initially worked independently before I later asked them to share their thinking with each other. While I will focus my analysis of this protocol on Jack's reasoning, I have included John's replies for comparison.

Protocol 5.13: Jack and John use time and distance measurements to compare speeds.

John: Just based on this measurement?
D: Um hmm. Yeah, so you've got one measurement for each inch worm. Is that—can you tell from that which of those two, Matt or Abby, is faster?
John: Well, you could tell, but at the same time you can't because they're not moving at a constant speed. It depends if they're moving constant speed or not.
D: [Nods.] Okay. Umm…so if it was a constant speed, would you be able to tell?
John: Um hmm.
D: Okay. Why don't you write down how you would tell if it was a constant speed and then you can tell me about the not constant speed next. So, kind of, how would you decide? [Turns towards Jack.]
Jack: [On his paper Jack wrote "Matt - 21 - seven = 3" and "aby - 10 - four = ".] Do you have a calculator?
D: Umm….what are you trying to calculate? I might be able to just help you.
Jack: This. 21 divided by 7. I think it's…
D: What do you think that is?
Jack: 21. Or 21 divided by 7 is like 3.
D: Um hmm. Yeah, you're right.
Jack: And then 10 divided by 4 is…like 2…
D: If you need to draw, draw a picture or something to kind of figure out how many times, what 10 divided by 4 is, you can do that.
Jack: I know how much it is. I just…kind of…I'm just going to do fractions. That's easier. [Finished the statement for Abby which now reads: "aby - 10 - four = 2 ½".]
D: Yep. That's perfectly fine.
Jack: Okay
D: How did you figure out that it was a half?
Jack: Because 4…2 is half of 4. And after 8 it only goes into 10 two more times.
John: [On his paper John writes "If it was constant speed I can tell because their moving at the same speed each sec. Ex like 3 cm per 10 sec.".] Well, that's only part of it. Part [of what] I got.
D: Okay. So, this is part of what you got if they're going at a constant speed?

John: Yeah.
D: So let's, let's maybe kind of share now. I think you guys both maybe got an idea now. So John why don't you tell us how you decided. Or first of all who did you think was—can you tell who's faster?
John: Well…not really. Not right now.
D: Okay. So what are you thinking about?
John: Because, if they get tired they could slow down. Or if that guy is just waiting for the red one to go faster.
D: So maybe if
John: Kind of like, a real life situation.
D: Okay.
John: And never moving at a constant speed.
D: If their speed is changing you maybe wouldn't be able to tell. Okay. [To Jack:] Do you agree with that?
Jack: Somewhat.
D: Somewhat?
Jack: But like for the problem, because 21, 21 seconds is how long it takes Matt to go 7 centimeters and Abby takes 10 seconds to go 4. But if you make it, if you make her go 8 centimeters, which it would only be 20 [seconds] instead of Matt which would be like…3. So he goes 3 centimeters, or like, it would be 1 centimeter for every 3 goes into 21. Or every 7 that goes into 21. It would be, umm…he'd end up going 24 [seconds] to get to 8 [centimeters] and Abby would only be 20 [seconds] to get to 8 [centimeters]. So, Abby is going faster.

Accounting for Jack's quantification of crawling speeds.

Considering first Jack's reasoning, I conclude that his activity here is consistent with the models of his reasoning that I developed previously to characterize his activity in the filling the swimming pool and Adopt-A-Highway contexts. For example, we see that he formed a goal of finding unit ratios for each inch worm's crawling speed, used division to quantify these ratios, and successfully incorporated them into further reasoning. I infer that Jack's use of division to produce the unit ratios "3" and "2 ½" indicates his goal of splitting the quantities 21 and 10 seconds into seven and four composite parts, respectively, of unknown numerosity. I view this as analogous to his assimilation of the task in Protocol 5.4 in which Jack wrote "4 ÷ 5" to describe his goal of quantifying the number of inches per minute of Pump 1. Thus, considering only Matt's measurements, my hypothesis is that Jack had formed a goal of splitting 21 seconds into seven composite parts, each containing some unknown number of seconds. Carrying out the division enabled him to identify the time required for each inch worm to crawl 1 centimeter.

I claim that the constant in Jack's activity across the previous contexts and the present protocol is his coordinated partitioning/iterating scheme. In this case he formed a goal of restructuring a given measured quantity (the time measurement) in terms some new composite unit of unknown size. Further, I infer that this goal arose from assimilating the change in the concomitant quantity (a change from 7 to 1 centimeters) using his splitting scheme. I consider these *composite* units of unknown size because unlike changing the concomitant quantity to a single unit such as 1 centimeter, the unit used to restructure the given measured quantity is itself a unit of units. (e.g., the five-eighths of a mile in Protocol 5.8). In essence, in this case Jack's splitting goal involves reconstituting the 21 seconds in terms of some composite unit of unknown numerosity such that seven iterations of this composite unit would result in 21 seconds (and similarly with 10 seconds and four composite parts).

The difference across this and previous tasks is the particular quantitative operations Jack used to carry out his coordinated partitioning/iterating strategy. In Protocol 5.13, to accomplish the desired restructuring of the time measurements I infer that Jack relied upon his whole number operations to quantify the results of his division statements. In contrast, when Jack's whole number operations were insufficient for accomplishing this task, such as with 4 inches in 5 minutes, Jack demonstrated that he had constructed distributive partitioning operations that he could implement to achieve his splitting goals.

Further evidence of Jack's conception of covariation as based on completed change.

However, while this accounts for his division and production of unit ratios, I also find the above excerpt revealing about Jack's conception of quantitative covariation. Notice that in

Protocol 5.13, John immediately expressed concerned about whether or not each inch worm traveled at a constant speed. When asked if he agreed with John's concern, Jack stated, "Somewhat." However, the activity he carried out after this comment to find the time it would take each inch worm to travel 8 centimeters indicates that Jack's assimilation of the situation assumed a constant speed.

This assimilation is consistent with my earlier assertion that Jack's conception of covariation was one of completed change. Jack took the completed change as given and used it in further reasoning. However, I consider him not sharing John's concern as indicating he was not reasoning about the inch worm trips in progress but rather focused on the completed trip. Focusing on the completed change over the entire duration is similar to reasoning with the average rate of change over the interval, and I infer that this accurately characterizes Jack's predominant way of reasoning throughout the teaching experiment.

My purpose in pointing this out here is not to disparage Jack's reasoning in this task. On the contrary, Jack leveraged this conception quite successfully in this case as he demonstrated the ability to use the results of his ways of operating to quantify the amount of time needed for imagined continuations of the covariation. In the excerpt above, Jack found the time each inch worm needed to travel 8 centimeters. In addition, following this protocol Jack found the time it would take each inch worm to travel 28 centimeters by multiplying 28 by both 3 and 2 ½.

Rather, my goal in identifying this feature of Jack's reasoning here was twofold. First, it provides additional evidence for the claim that an image of completed change guided a majority of Jack's reasoning within covariational situations. Secondly, I find this to be a useful lens through which to interpret the successes and struggles that Jack experienced during the remainder of the teaching sessions.

A Characterization of My Goals for Working With Jack for the Remainder of the Study

Jack was absent for the next two teaching sessions and so I worked with John individually on those days. As a result, I found that it became increasingly difficult to plan teaching sessions that would be appropriate for both Jack and John and made the decision to teach the students individually for the rest of the teaching experiment.

For the remainder of the teaching sessions we took a slightly different approach. Rather than starting with measurements and asking Jack to find unit ratios as we had been doing, we presented him with different types of starting information and investigated how he used the given information. For example, at times we provided Jack the inch worm's speed and designed tasks to explore his meaning for the given speed. Later in the teaching experiment we showed Jack graphical representations of several inch worm time trials and asked questions about what he thought the graphs indicated about the inch worm's speeds. The goal of these types of tasks was to develop an understanding of Jack's meanings for crawling speed, the types of tasks Jack could solve using them, and the ways of assimilating the tasks that supported these meanings and activity.

Jack's Conception of Constant Quantitative Covariation

Jack constructs linear graphs to represent constant speeds.

I first developed the characterization of Jack's assimilation of quantitative covariation as indicating completed uniform motion within the context of these new types of tasks that we used toward the end of the teaching experiment. Jack could use this way of reasoning quite powerfully. For example, he could both construct and interpret graphical representations of constant speeds. During the February 18, 2014, teaching session, we told Jack that an inch worm named Al had completed his time trial and we found that he had a crawling speed of seven-thirds seconds per centimeter. The teacher-researcher had Jack close his eyes to imagine what Al's trip would be like and also had him start the animation. After watching part of the animation, he stopped the race and Al's distance and time measurements recorded on the screen read 6.10 centimeters and 14.23 seconds. Then, the teacher-researcher asked Jack to make a graph that would represent Al's crawling speed. Jack explained that he had thought of two graphs, one based upon the speed and one based upon the animation, and created both graphs (see Figure 5.15).

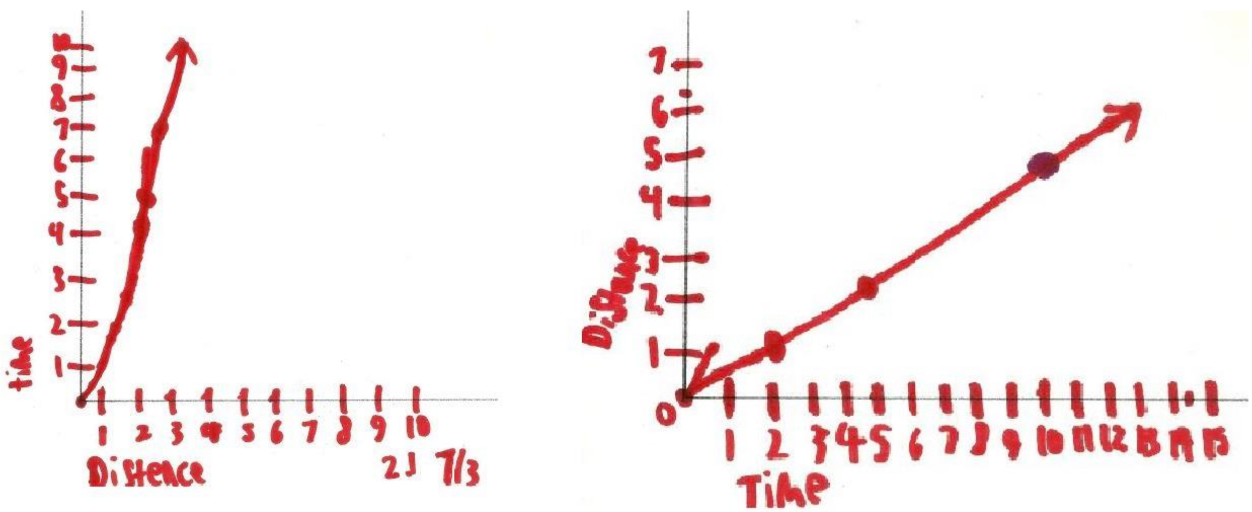

Figure 5.15. Jack's graphs of Al's speed based upon seven-thirds seconds per cm and the completed animation, respectively.

Jack's descriptions and method for producing each graph.

Before sharing some of Jack's descriptions of his graphs, it is important to note that Jack thought that seven-thirds was equivalent to the decimal "2.1". The teacher-researcher did not correct or inquire about this incorrect decimal conversion at the time so as not to interrupt Jack's thinking. Further, while constructing these graphs, a majority of Jack's activity remained non-verbal. Thus, to provide the reader with a sense of how Jack thought about each graph, I have included brief descriptions of how Jack produced and discussed each graph.

To produce the first graph (see Figure 5.15, left), Jack started at the origin and drew a line segment, pausing roughly at 2.1 seconds. I infer from his actions and utterances that he intended this time value to occur at 1 centimeter. He then extended this segment stopping at a little over 4 seconds and roughly at 2 centimeters, and he continued to extend the line in this fashion. Jack described this graph as follows:

> It's up two…two point one. It's seven-thirds seconds per centimeter. So, it's like here for the first one and then here for the second one and here for the third one. [While stating this Jack tapped his marker at three points on the line that I infer had distance values of 1, 2, and 3 centimeters.]

With regard to the second graph (see Figure 5.15, right), Jack explained that he chose his scales of 7 and 15 to accommodate the distance (6.10 cm) and time (14.23 seconds) measurements that were shown on the paused animation. After labeling the axes Jack explained:

> He's still going two point one. So you just take this and be like…because this is your distance so it would be going like this. [Jack then drew a line from the origin to roughly the point (2.1 seconds, 1 cm).] And then it would just keep going up. [While saying this he extended the line segment.]

Lastly, when comparing the second graph to the first Jack stated, "So it would still be going the same rate. It's just time is on the bottom and distance is on that instead of the way I did it before. […] They're representing the same speed but different distances."

Accounting for Jack's production of the linear graphs.

All of Jack's activity and descriptions with these two graphs remained consistent with my characterization that he assimilated speeds as indications of completed uniform motion in that he operated in terms of the measurement unit defined by the given speed (i.e., one centimeter intervals). For instance, when producing both graphs Jack drew a line segment from the origin to the point representing a distance of one centimeter and 2.1 seconds. Further, his activity and descriptions suggest that he imagined continuing the graphs by accumulating the quantities in a ratio of 2.1 seconds per each additional completed centimeter. I hypothesize that for Jack the given speed explicitly entailed traveling 1 centimeter, and his iterable units account for his continued accrual of the quantities (e.g., locating points on the graph at 1, 2, and 3 centimeters). Thus, using this conception Jack knew that the constant speed would produce a linear graph by virtue of his iterating the time and distance measurements defined by the unit rate 2.1 seconds per centimeter.

At this point in the teaching experiment, it remained unclear if Jack could coordinate the accruals of the quantities by iterations of other measurement units (i.e., partial centimeters rather than 1 centimeter). Doing so would indicate that he had constructed an intensive quantitative variable. However, his activity provides no such evidence. Further, after considering this interaction prospectively, I infer that Jack did not consider the process of change from zero to 1, 2, and 3 centimeters. Instead, I hypothesize that he assimilated the situation as inherently involving uniform motion, much like he did when he disregarded John’s concern that the crawling speeds might not have remained constant (cf. Protocol 5.13).

The hypothesis that Jack assimilated speeds as indicating completed uniform motion also accounts for why Jack produced two graphs. Using this conception, Al’s speed would refer to completing 2.1 seconds per each completed centimeter. Accordingly, Jack started the first graph by labeling time on the vertical axis and then distance on the horizontal axis. Next, Jack’s activity made it clear that he based his production of the second graph on the measurements listed in the paused animation. These were located on the screen so that the distance was shown above the time measurement, and when Jack referenced the measurements he talked first about the distance and second about the time. Accordingly, Jack made his second graph by labeling the vertical axis as distance and the horizontal axis as time. Thus, my hypothesis is that Jack saw the need for two different graphs because the information he had available, the given speed and the measurements from the animation, indicated different completed uniform motions. Hence, for Jack they required different graphs. I infer that this is why Jack viewed the two graphs as representing the same speed, "…but different distances."

A constraint emerges in Jack's reasoning.

While I claim that Jack leveraged this conception of speed as completed uniform motion to appropriately produce and interpret the linear graphs, this way of reasoning also had some limitations. These started to become more apparent in the next teaching session on February 20, 2014, when we asked Jack to compare two speeds.

Protocol 5.14: Jack compares two different speeds.

T: So Al was crawling at the speed of seven-thirds seconds per centimeter. Do you want to write it down? Feel free to write it down if you want to. And today we have Ryan. And Ryan is crawling at the speed of seven-fifths seconds per centimeter.
Jack: [Picks up the marker and records the information about Ryan and writes "7/5 sec".]
T: That's per centimeter. And…
Jack: [Adds the "per cm" to his label.]
T: Um hmm. Centimeter. And Al was seven-thirds seconds per centimeter. Do you remember that?
Jack: Um hmm. [Writes the fraction "7/3" and then labels the both fractions with the appropriate inch worm's name.]
T: Al and Ryan. And then just why don't you take a moment and think about, like, how they [are] going to move? Okay, so…we got the image in your head?
Jack: Yeah.
T: Okay. So, umm, who do you think is faster?
Jack: [Thinks for 12 seconds.] Ryan.
T: Ryan. Can you tell me why you think so?

Jack: Because it takes Ryan one point… [Thinks for approximately 10 seconds]…1.2 seconds to get 1 centimeter and it takes Al 2.1 seconds to get 1 centimeter.[14]
T: Umm.
W: You want to work those out, on paper?
Jack: [Writes "1.2" and "2.1" on his paper after Ryan and Al's respective speeds.]
W: So you think Ryan is faster?
Jack: Yeah.
T: Okay, so…how do you know?
Jack: Because it only takes—if they're both going one centimeter, it only takes him [Ryan] 1.2 seconds and it takes him [Al] 2.1 seconds to get 1 centimeter. It takes Al 2.1 seconds to get 1 centimeter. It takes Ryan 1.2 centimeters, ah seconds, to get 1 centimeter.
T: Well that's cool. Umm, so can you tell me if Ryan is faster, can you tell me how much faster he is than Al?
Jack: [Thinks for approximately 8 seconds.] 0.9 seconds faster.

Given these responses, it was unclear whether Jack assimilated the stated speeds as describing a single time and distance pair or a relationship that characterized an infinite collection of time and distance pairs. For instance, notice that when the teacher-researcher asked him to write down Al's and Ryan's speed, Jack recorded them as a number of seconds rather than seconds per centimeter. I do not believe that Jack was unaware that the time measurements corresponded to 1 centimeter of travel. Rather, it is more that his reasoning was explicitly about the time measurements while the distance of travel often remained implicit in this case. Further, even though he qualified his comparison of the time measurements with "if they're both going 1 centimeter," he also stated that Ryan was 0.9 seconds faster rather than 0.9 seconds per centimeter faster.

To investigate this ambiguity in Jack's replies further, the teacher-researcher asked him to draw a diagram that stood for Ryan's speed. Initially Jack drew a diagram that indicated a 1 centimeter segment and labeled it as "7/5 or 1.2 sec per cm" (see Figure 5.16, left). While this is

[14] Jack converted the fractional speeds to decimals by using whole number and remainder reasoning. Because five goes into seven once with a remainder of two, Jack viewed the fraction 7/5 as equivalent to 1.2. Similarly, he converted the fraction 7/3 to the decimal 2.1.

not incorrect, as a single instantiation of time and distance it does not convey whether or not Jack was thinking of a broader relationship than this one time/distance pair. Thus, the teacher-researcher asked Jack if he could draw another diagram that still stood for the same speed. After thinking for 10 seconds, he drew a second diagram that indicated a 2 centimeter segment and labeled it as “2.4 sec. per 2 cm” (see Figure 5.16, right). Further, when asked if he could draw other diagrams that stood for the same speed, Jack replied that he could make as many as he wanted.

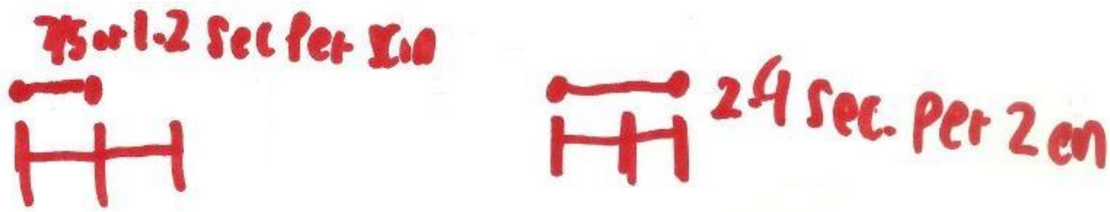

Figure 5.16. Jack’s two diagrams to represent the speed 1.2 seconds per centimeter.

These responses indicate that even though Jack’s diagrams gave prominence to a single time/distance pair, he was aware of the broader relationship and could use his iterable units to imagine the quantities continuing to accrue in multiples of the given ratio. Yet, given Jack’s way of labeling his two diagrams, it remained unclear to what extent he considered his two diagrams as indicating the same speed. I infer that he doubled the time and distance measurements to produce his second diagram. For instance, Jack labeled this second diagram as “2.4 sec. per 2 cm.” While this is an appropriate label, it also seems to suggest that because the particular values of the extensive quantities changed, the value of the intensive quantity had to change as well.

However, it would have been just as appropriate to label this second diagram as 1.2 seconds per centimeter as well. Unsure whether Jack recognized this distinction, the teacher-researcher questioned Jack further.

Protocol 5.15: Jack considers the possibility of traveling at a speed of seven-fifths seconds per centimeter for only one-half centimeter.

T: So does he have to crawl 1 centimeter to be going at a speed of seven-fifths seconds per centimeter?
Jack: Yeah.
T: Umm.
Jack: Because he's at a constant speed.
T: He's at the constant speed.
Jack: So and he, to go one point—it would only take him 1.2 seconds to go 1 centimeter. So unless he's going slower than his normal time, then it would have—he would have to be going 1 centimeter to get that much [time].
W: Would you have to change the numbers? Suppose he goes a half a centimeter.
Jack: A half a centimeter.
W: Would he be going seven-fifths seconds per centimeter?
Jack: It would mean he'd be going faster than what it says he's going.

This task suggests that Jack may not have made this distinction between measures of the intensive and extensive quantities. Further, I believe this protocol exemplifies a limitation of his assimilation of speeds as indicating completed uniform motion. While Jack previously demonstrated that he could conceptually continue the completed change to form a new completed change, to Jack this also seems to have meant that it was no longer appropriate to label these with the same measure of speed. For example, at one point Jack said, "So unless he's going slower than his normal time […] he would have to be going 1 centimeter to get that much [time]." From this I infer that he was thinking that the only way to accumulate 1.2 seconds while traveling a greater distance would be to take less time to travel 1 centimeter. Hence, he would no longer be going the same speed, but would rather be "…going faster than what it says he's going."

It is important to point out that while perplexing to the research team, this discrepancy did not appear to represent a perturbation for Jack. Rather, this issue with a measured speed only pertaining to a single time and distance appeared as a non-contradictory (from Jack's perspective) consequence of his ways of reasoning. Further, this was not the only time Jack reasoned this way during the teaching experiment; he experienced a similar constraint when reasoning about the possibility of going 60 miles per hour for 1 second. This is why I have been so consistent in characterizing Jack's assimilation of speed as indicating completed uniform motion. Jack's reasoning within Protocol 5.15 follows logically if one accepts this model of Jack's assimilation of speed. Even though there are likely language and interactional issues at play in these instances (e.g., Jack may have thought the researcher's intended that the same time would elapsed while traveling only one-half centimeter), that is partially my point. I infer that the issues in communication arose because of the way that Jack assimilated the speeds rather than the other way around.

Jack resolves the constraint.

The teacher-researcher decided to have Jack watch the first few centimeters of the animation to see if that would help Jack to alleviate this lacuna in his reasoning.

Protocol 5.15: Continuation.

T: [Turns on the computer animation of Ryan's time trial.] So this is Ryan's time trial. So I think that this is a good way to check if he, umm, if he crawls faster at, you know crawling for a half centimeter or one or if they are a different speed or not. So why don't you click this button—start and stop race.
Jack: [Starts the animation and stops it at 4.33 seconds and 3.09 cm.]
T: So did you see that the, ah, Ryan is passing
Jack: The halves.
T: A half, and then one.
Jack: Yeah.
T: Do you think that he's crawling at different speeds or the same speed.
Jack: He's crawling at the same speed.
T: Why do you say that?

Jack: Because even if he just goes half, he's still going the same speed because once he gets to that 1 [centimeter] it's the same time every time.

T: Can you tell me more about what do you mean by once he gets to that 1?

Jack: Like, if he goes…like you have 2 centimeters. You have your halves in each one of them. [Draws another 2 centimeter diagram on his paper and puts a dot in the center of both the first and second centimeters on his diagram.] And he goes from here to here to here. [Above the 2 centimeter diagram, Jack draws a dot in line with the 0, 1/2, and 1 cm marks.] It only takes him 0.6 seconds to get to there [labels the 1/2 centimeter point with "0.6"] and then 1.2 seconds to get to there [labels the 1 centimeter point with "1.2", see Figure 5.17]. So it takes him [point] six seconds to get here [pointing to the 1/2 cm point] and then [point] six seconds to get the rest of the way [to 1 cm].

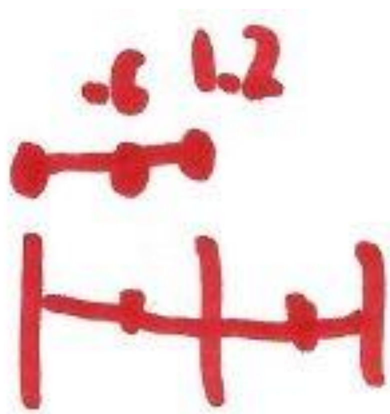

Figure 5.17. Jack's modified diagram of the speed 1.2 seconds per centimeter.

Jack's changes to his diagram and the ways in which he talked about speed in this excerpt indicate his assimilation of the situation changed to incorporate an image of the covariation as it progressed. I infer that as Jack watched the animation progress, he actively monitored the accumulation of time and distance. These experiences enabled Jack to reconstitute his earlier diagram of 1.2 seconds per centimeter in terms of *the process* of advancing from 0 to 1 centimeters. Thus, he identified one-half centimeter within the first centimeter and coordinated this with finding one-half of the accumulated time, or 0.6 seconds for one-half centimeter. What's more, carrying out this activity supported his recognition that this was the same speed as 1.2 seconds per centimeter because in Jack's words, "…once he gets to that 1 [centimeter] it's the same time every time."

My hypothesis is that Jack's construction of *both* quantities as divisible quantities in this case accounts for this advance in his ability to conceptualize speed in a more sophisticated way. In particular, by partitioning the measurements he previously took as indicating completed change, Jack was able to step inside that completed change to imagine its construction. In doing so, he conceptualized a new accumulation of the quantities that if continued, would produce the same accumulations as the given speed. I am not claiming that Jack *could not* have reasoned in the way previously. In fact, I infer that he quantified the time for one-half centimeter using quantitative operations he already had available (in this case, splitting 1 centimeter and 1.2 seconds in half). Rather, I'm claiming that he *did not* reason in this way previously. Doing so in this case immediately supported Jack's resolution of the communication issue that previously existed.

Further, I claim that constructing both quantities as divisible in this instance was important because it enabled Jack to change the measurement unit and step outside the world of centimeters. Previously, I inferred that Jack assimilated the speed as an indication of completed motion. In this case, that specifically meant an amount of time for travelling 1 centimeter. Then, on the basis of unit iteration it was not a problem for Jack to imagine continuing the variation and accruing the quantities in multiples of the given ratio. However, while this changed the values of each extensive quantity, it never changed the measurement unit; Jack still had centimeters, he just had more than 1 centimeter. In contrast, splitting given measures of the extensive quantities in half opens the door to reconstituting the covariation in terms of a new unit—half centimeters. I infer that it was this change of unit, supported by constructing both quantities as divisible in this case, that enabled Jack to conceptualize the covariation in a more sophisticated way.

Admittedly, splitting the measurement unit into two parts to form half-centimeter units is likely insufficient to support a more general abstraction of one's image of covariation. As Jack demonstrated previously, at times splitting into two parts can enable one to operate intuitively without necessarily producing a more generalizable way operating.[15] However, in this case I believe that this activity opened up new constructive possibilities for Jack that he used to explain the coordination of the extensive quantities in new ways that were previously unavailable to him. Constructing a concept of rate more generally would likely require using one's splitting operations recursively to enable one to reconstitute the accruals of the quantities in terms of *any but no particular* measurement unit.

Unfortunately, this advancement appeared to be temporary. During Jack's last teaching session on March 4, 2014, Jack again incorporated reasoning that indicated a conception of covariation as completed uniform motion. Despite this, I find Protocol 5.15 critical for understanding the conceptual resources that Jack had available throughout the teaching experiment because of the contrast it provides between his typical ways of operating and the reasoning he used in that interaction.

[15] For example, compare Jack's success in Protocol 4.4 (supported by his use of a dyadic attentional pattern) to his struggle to adapt those ways of reasoning to accomplish his goal of splitting 3 inches into 5 parts in Protocol 5.1.

Jack's Attempts to Write Symbolic Equations for Constant Covariational Relationships[16]

Finally I wish to discuss one final aspect of Jack's mathematics—his efforts to construct equations to characterize constant covariational relationships. We first explored this with Jack during the teaching session on February 18, 2014.

Jack reasons with unknown quantities.

The following task occurred at the end the same teaching session (February 18, 2014) as when Jack produced the graphs for Al's speed (see Figure 5.15). As before, Jack considered Al's crawling speed to be 2.1 seconds per centimeter.

Protocol 5.16: Jack reasons with unknowns.

T: Can you figure it out that how far, umm, Al's going to crawl at any given time?
Jack: Like, what do you mean? Like if you just name a time?
T: Yeah. Then can you figure out the distance that he crawls?
Jack: Yeah.
T: How do you do that?
Jack: Like, because you have 21 and it takes him… Like if you give me a time.
T: Um hmm.
Jack: Like, because it still takes him 21, or 2.1. You just divide 2.1 in the time you gave me and then you get, and then you get that—the, umm, the time. Then however many times that number goes in to that distance you gave me, it will tell me how far he went—like how many centimeters he went.

Jack's replies in this protocol suggest that he could reason in terms of an unknown quantity. Specifically, given some unknown time, Jack described dividing this time by 2.1. I infer that Jack's goal in dividing these quantities was quotitive in that he wanted to find how many times 2.1 went into the given time. Essentially, this is tantamount to measuring the given time in terms of units of size 2.1 seconds. And because Jack knew that Al crawled 2.1 seconds per centimeter, the result of this division would be the number of centimeters Al crawled.

[16] I use *constant covariational relationships*, both here and throughout the remainder of the dissertation, to refer to covariational relationships in which the quantities vary in such a way that ratios of the accruals of each quantity and ratios of the total accumulations of each quantity remain constant. A mathematically knowledgeable observer would likely call these linear relationships whose covariational relationship is defined by a constant rate of change.

Following this excerpt, Jack accurately described how he would carry out this strategy if given a time of 16 seconds.

Jack struggles to symbolize his reasoning with an algebraic equation.

Because Jack had demonstrated the ability to reason in terms of an unknown value of time, the teacher-researcher continued on to see if Jack could write a symbolic equation that characterized this relationship.

Protocol 5.16: Continuation.

T: Okay. Well, that's great. Let's say that the, umm…I'll give you a variable instead of the specific number. So…what if I give you a time *T*?
W: [Speaking to the teacher-researcher:] *D*.
T: *D*?
W: A distance.
T: Distance. Alright. So instead of giving you the time, what if I give you a distance, *D*? Then can you find a time that the Al's going to take to crawl the distance *D*?
Jack: I'd have to know the time. Because even if you give me just a variable I can't figure anything out. It's like taking *D* [writes "*D* =" on his paper] and be like, "What does this equal?" Bleah. A problem. Because you said if you give me *D*, like you just say distance. Like, you find the distance from distance. That's like, okay so you have…umm…well, it's not necessarily enough information. Because 28—because I can say 1 second, I can say 2 seconds because I don't have a, like, a time that he went. To find the distance I have to know the time. To find the time I have to know the distance.
T: Yeah, what if I give you the time, *T*? Can you find the distance that the Al's crawled? The time *T*.
Jack: No. [Laughs.]
T: No?
Jack: No, because still you'd have to know the distance to find the time if you gave me *T*.

In contrast to Jack's ability to reason with an unknown time in the previous excerpt, Jack experienced constraints when asked to reason in terms of a variable symbol such as *D* or *T*. I claim that this constraint stemmed from the manner in which Jack assimilated variable symbols (e.g., *D* and *T*). Further, his previous school mathematics experiences likely influenced this interaction as well.

In particular, I infer that the researchers and Jack assimilated these variable symbols very differently. For example, when the researchers said they were giving Jack *D*, to them this meant that the distance was any but no particular distance. Thus, their question intended to investigate how Jack might use his knowledge of Al's crawling speed to operate upon this quantity to find *T* and write an equation to characterize the relationship between the speed and the two covarying quantities. However, I infer that when the researchers told Jack they were giving him *D*, he interpreted this as a call to *find D* rather than to *operate upon D*. This accounts for Jack's apparent confusion with the researcher's questions—how could he find the distance if he was never told the time, and vice versa?

This miscommunication highlights a contradiction between Jack's quantitative operations and the operations he used to operate upon formal algebraic notation. We saw that the former proved sufficient for Jack to devise a strategy to use his conception of speed to transform an unknown, and potentially variable, number of seconds into the appropriate unknown number of centimeters. Seemingly, this activity should support the eventual abstraction of this reasoning to the level of an algebraic equation that symbolized those ways of operating. However, while Protocol 5.16 showed that Jack could reason with unknown variable quantities, the continuation of this protocol exemplifies that Jack did not assimilate algebraic notation as symbolic of variables, but rather as unknowns to find.

Jack constructs an algebraic equation as symbolic of his quantitative reasoning.

Jack did successfully use algebraic notation to write an equation characterizing the relationship between distance, time, and speed during his final teaching session on March 4, 2014. However, to explain the conceptual resources that I think supported Jack's accomplishment of this task, I must include several tasks within the same protocol.

Prior to reading the protocol, take note that Jack used division language during this teaching session differently than I normally would expect.[17] However, by having him carry out some of his division statements on a calculator I deduced that when Jack said things such as, "I'd divide 10 into 6" he really mean $6 \div 10$. Thus, whenever Jack described dividing a quantity *into* another, such as *A* into *B*, I infer that he meant what I would characterize as B divided by A. To help clarify this issue, throughout the protocol I include my comments regarding what I infer Jack meant in brackets and use the notation, $B \div A$, when I intend B divided by A. Also, I have eliminated some of the redundancies in our conversation so as to only include the relevant interactions in an attempt to make the interaction clear despite the differences in language Jack and I initially used. Places were transcript has been removed are indicated with the symbol, […].

Immediately prior to the following protocol Jack was reasoning about an inch worm named Sam's crawling speed based upon a graph of her completed time trial (see Figure 5.18). After describing that he would need to know Sam's distance and time measurements to figure out the speed that the graph indicated, I introduced a graph that had a variable point that could be moved anywhere along the line. Further, the sketch included measurements for the time and distance values of this variable point. At the start of this protocol, the variable point measurements indicated a time of 6 seconds and a distance of 10 centimeters. I have included the entire protocol uninterrupted by my analysis in order to provide a sense of the progression of the questions and Jack's reasoning about the relationship among measurements of distance, time, and speed.

[17] Jack used his language for division differently in this teaching session compared to all of his other teaching sessions. Based upon considering his statements with respect to his activity and the tasks, I infer that this was the only day for which Jack and I had difficulty communicating about Jack's intentions regarding division.

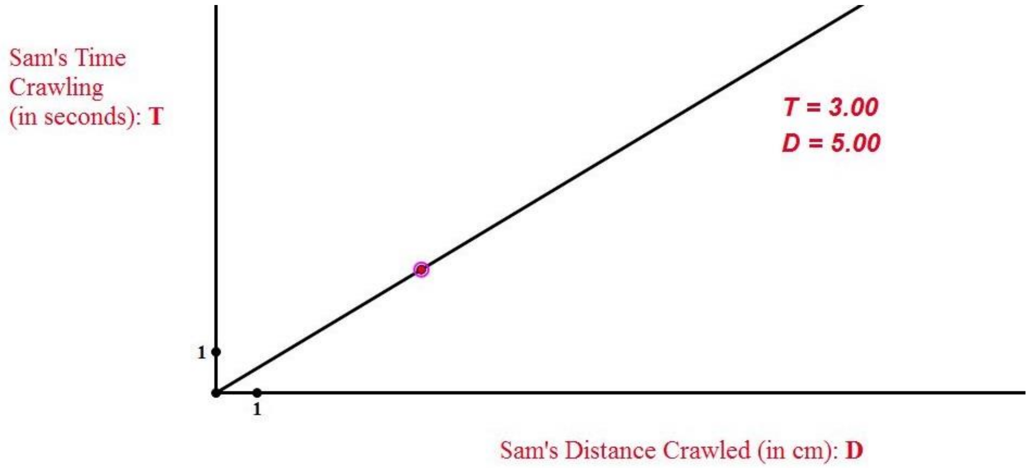

Figure 5.18. A screen capture of the interactive graph of Sam's time trial data.

Protocol 5.17: Jack constructs a symbolic equation for the relationship between speed, time, and distance.

D: What I'm wondering is could you—how could you use that information, the measurement, umm, to figure out a measure for Sam's speed?
Jack: I'd divide 10 into 6. [I later clarified that he meant "6 ÷ 10".] […] I'm putting 10 into 6 [again referring to "6 ÷ 10"] because if you [do] 6 into 10 it doesn't give you the right things. You need to divide distance into time. So, that's what I did and it gave me how…how much time it took her to get 1, umm, centimeter.
[…]
D: Well we could take that measurement though from any point in the race. Right?
Jack: Yeah.
D: We could go anywhere here. [Moves the variable point around along the line, changing the measurements for the time and distance.] Like say we stopped at 3. Umm, it looks like 3 [seconds] 5 [centimeters]. Umm…
Jack: I could still figure it out. All's you do is 3 into 5. Or 5 divided by 3. No. Three divided by 5. I mixed that up. [Jack meant "3 ÷ 5" in all three cases.]
D: Talk me through that again. How come 3 divided by 5?
Jack: Because it's still gives you the same. If you divide 3 into 5 it still gives you how long it took him to get 1 second. It's just smaller numbers.
D: How long it… So it gives you how long it takes him to do what?
Jack: To go 1 centimeter.
D: Okay. So if we did that anywhere?
Jack: Anywhere…you could still figure the same out. You just need to divide the distance by the time. [I infer that Jack meant "$T \div D$".]
D: Okay. Well let's, let's go ahead and do that once and umm… [Passes Jack the mouse for the computer.] Do you want to go up to measure—or excuse me, umm,

number. And let's just calculate that. [Jack opens the "Calculate" menu on The Geometer's Sketchpad (Jackiw, 2012).] […] You can click on these [the on-screen measurements] if you want to do it time. [Jack clicks on the time measurement]. And then, like, you've got all of your operations here [in the on-screen calculator]. Divided by or, you know, whatever. [Jack clicks the division symbol and then the distance measurement on the calculator, and then "OK". The result "$\frac{T}{D} = 0.60$" appears on the screen.]

D: So what does that [mean]?

Jack: Point 60.

D: So what does that mean?

Jack: It takes 0.60 seconds to get 1 centimeter.

D: Um hmm. And, ah—so go ahead and move that point around. If you move that point around, what do you think is going to happen?

Jack: It's going to change the [Starts to reach for the mouse to click and drag the point].

D: Hold on one second! [Tries to stop Jack from moving the point before he can finish describing what he thinks is going to happen, but he already clicked the point and moved it slightly.] Well, go ahead.

Jack: It's going to change the time, like the distance. It's the same. Because all of it just divides. All of it equals 0.60 because it's the same—it's going the constant speed, not changing. So it will always be…the, ah…0.60 seconds per centimeter.

D: Okay. Perfect. [Deletes just the "$\frac{T}{D} = 0.60$" calculation from the screen.] So one question then. So you know the relationship between the time and the distance and her speed. And she's going at that constant speed. I'm wondering, umm, could you write an equation that would describe this relationship?

Jack: What do you mean?

D: So say we—because if we go at different points here in the race [clicks the variable point and drags it along the line so the time and distance measurements change], we keep getting different time and distance measurements. Like if we actually went to her race…

Jack: Any time and distance would still make it the same.

D: Yeah. But the same what?

Jack: The same, umm, time it took him to get. Because no matter what distance it is, it's the same time. Because it's just every se, se, bleah, centimeter, it's 0.60 seconds. So it never changes. It doesn't matter how many, how much your numbers change. It's always going to be the same as long as the person is going at a constant speed.

D: The same speed. Sure. So…what I'm wondering is, maybe, umm, like what if we used variables. So because the time and distance at every different point in the race are always different measurements, but yet there's always this constant speed like you're telling me.

Jack: Um hmm.

D: What if we used *T* for the time and *D* for the distance? Could you write an equation that would describe these relationships?

Jack: Like, that would show, like, his constant speed?

D: Umm, sure, that would kind of capture what's going on here. The relationship between time and distance for Sam.

Jack: Like…I don't know. The only thing I could think of would be… [Writes 9.60 ÷ 16.01, which is the time measurement divided by the distance measurement that was shown on the screen for the variable point at its current location.] That's just—and then you just, like *T* divided by *D*. That's it. And that gives you the, ahh, which equals speed. [Jack writes the long division statement "$T \div D = S$", see Figure 5.19, and I infer that he truly meant it.] […] And then you, when you get your speed, that's it. You just need the speed.

D: And in this case we know that, right?

[…]

Jack: It would be distance divided by time would equal point 60 [Writes the long division statement "$T \div D = 0.60$". See Figure 5.19.]

[…]

D: Okay. And if, if you had a calculator, umm…calculate. [Opens the GSP calculator.] Do you want to type this one in and just show me how you would type that one in? You can click on the numbers.

Jack: It would be nine point six zero divided by 16 point zero one. [Uses the mouse to click those numbers as he speaks].

D: Alright, perfect.

Jack: It's still the same. It's still 0.60.

D: Yeah, so that would be equal to your speed.

Jack Um hmm.

D: Awesome! That makes a lot of sense.

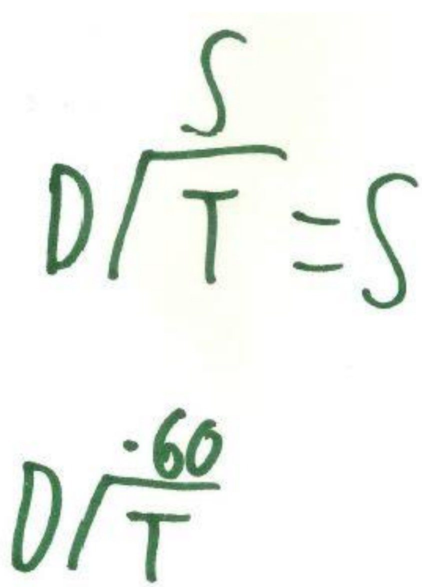

Figure 5.19. The equations Jack wrote to characterize the relationship among measures of time, distance, and speed in the given context.

In contrast to Protocol 5.16, we see here that Jack successfully determined an equation to describe the covariational relationship among the quantities. I find that the progression of Jack's reasoning over the course of this protocol helps to account for this success. First, I infer that the initial part of the protocol was important in that Jack computed the speeds for multiple time and distance measurements. In addition, the incorporation of the interactive graph enabled me to ask Jack questions about what would happen if the values changed and actually instantiate those changes several times. My hypothesis is that these two activities provided a context that supported Jack's abstraction of the constancy of the speed, in this case 0.60 seconds per centimeter. After computing this speed for 6 seconds and 10 centimeters, and again for 3 seconds and 5 centimeters, Jack began to reason that no matter which measurements we used from the time trial graph, the result would be the same.

Second, I consider Jack's construction of the equation "$T \div D = S$" to be a product of his quantitative operating rather than an application of previously learned ideas. Recall that at various points throughout the teaching experiment Jack had previously attempted, unsuccessfully, to recall formulas for speed that he had learned in school. However, I do not think this equation here relied upon that reasoning at all. First, his production and explanation of the formula followed on very clearly from his thinking in the early parts of Protocol 5.17. Further, I remain fairly confident that Jack would not have learned a formula for speed as time divided by distance in school. By focusing on the more unconventional way of characterizing the relationship between distance and time, seconds per centimeter, I feel confident that Jack constructed the formula in the course of his activity and on the basis of his quantitative operations.

Lastly, I think designing this task to allow Jack the room to decide how to use the algebraic notation and what equation to write played an important role in his success. For instance, rather than giving Jack only *D* and asking him to find *T*, or vice versa, I decided to try giving Jack both variables and asking him to write an equation that described the relationship between time and distance for Sam's time trial. Thus, even though I introduced the idea of using variable notation, *T* and *D* for the measures of time and distance, it was Jack who decided how to use the notation.

Comparing the tasks in Protocols 5.16 and 5.17, I think that the former highlighted a conception of a variable as a fixed unknown by asking for the equation as an analog for how to operate upon a given quantity to find the concomitant value of the other quantity. Alternatively, I think that asking for an equation that described the relationships highlighted the variable nature of the quantities.

Ultimately, I believe that the task itself does not involve unknown quantities or variable quantities, but rather it is one's assimilation of the task that makes this determination. However, I do find it a reasonable claim to suggest that particular features of the task design or the sequence of questioning provide different sensory input and, hence, opportunities to assimilate seemingly similar questions in different ways. Jack's activity in Protocol 5.16 showed that he could think in terms of variable quantities. However, this was not brought forth in the context of writing an equation being given *T*, a variable time. In the case of Protocol 5.17, I infer that Jack developed a sense of the constancy of the unit ratio and leveraged this to construct an equation that incorporated algebraic notation as representing variables. And while I cannot know for certain which aspects of this task most contributed to Jack's assimilation of symbolic notation as representing variable quantities, I am confident that it was this assimilation that supported his successful construction of an equation to characterize the relationship among measures of speed, distance, and time.

CHAPTER 6

THE MATHEMATICS OF JOHN

I turn now to characterizing the mathematics of John and take a similar approach to that used in the previous chapter. Thus, I have split my presentation of the results into three primary sections based upon the same three primary task contexts and have developed my analyses of key tasks and interactions essentially chronologically. As with the previous chapter, my primary goal is to develop a model of the conceptual schemes and operations that can account for my observations of John's mathematical activity. To that end, I have chosen a sequence of excerpts from the teaching sessions that capture important aspects of John's mathematics. While many similarities exist between the excerpts chosen for John and those previously presented for Jack, the selected protocols in this chapter do not always include the same tasks as the protocols in Chapter 5. Rather, the excerpts chosen for this chapter are those that best represented the range of John's ways of reasoning and that enabled me to account for his successes, his challenges, and changes in his ways of reasoning over time.

Over the course of the entire teaching experiment, John experienced relatively fewer constraints than Jack. Further, the types of tasks for which John did experience constraints were often of a different nature than the situations that perturbed Jack. Thus, accounting for the conceptual operations that can explain both John's successes and his struggles provides an important point of comparison for considering what the mathematics of Jack and John suggest about the construction of intensive quantity.

The Swimming Pool Context

I began the teaching sessions with John having essentially the same goals as those I had for working with Jack—to engage John in problematic situations to investigate how he would use his available quantitative operations creatively to solve tasks and coordinate changes in co-varying quantities. During the initial interview, despite some differences in how they solved the tasks and their facility at explaining their thinking, the fact that both Jack and John successfully solved nearly every task I presented them indicated they had each constructed sophisticated schemes and operations for operating upon and coordinating quantities. Thus, while teaching John I initially focused on exploring how he would use his available operations in novel situations and investigating any constraints he might experience while reasoning about pumping rates and coordinating changes in water depth and pumping duration.

John Adapts and Abstracts His Ways of Reasoning During His First Teaching Session

John's first teaching session in the swimming pool context, which occurred on October 11, 2013, played an important role in my retrospective analysis of his mathematics for several reasons. First, his solution to the unit ratio task demonstrates one of John's characteristic ways of reasoning, which he used frequently throughout the duration of the teaching experiment. In addition, his responses to the tasks provided confirming evidence for many of the inferences I had drawn from his activity in the initial interview. His reasoning on subsequent tasks also indicated an additional quantitative scheme he had constructed that was not apparent from his initial interview but which helped me to better understand the conceptual operations that could account for John's mathematical activity. Finally, John's solutions to these tasks provide examples of him using his quantitative operations in novel contexts and actively abstracting his ways of reasoning in the course of his activity.

John's characteristic strategy for establishing unit ratios.

To exemplify these assertions, consider first John's strategy for establishing a unit ratio. After introducing the swimming pool context and watching the animation of the pool filling, I told John that the pool maintenance supervisor had measured the pool depth and found that the pool level had risen 3 inches in 5 minutes. The strategy he used to find how much deeper the water would get if the pump ran for 1 minute demonstrates John's most common way of reasoning when coordinating changes in covarying quantities to find unit ratios.

Protocol 6.1: John determines the number of inches per minute given a completed change of 3 inches in 5 minutes.

John: You take 3 divided by 5. [Computes this with long division on his paper and records his result as ".06" on his paper.] And point zero six. Point six inches.
D: Okay. Umm…so point?
John: Point six inches for 1 minute. Pretty much.
D: Okay. Point six or [point zero six]?
John: Yeah, zero point six I meant.[18]
D: Okay. So a couple of questions then. Umm, tell me what you were thinking about. Like why did you decide to do this?
John: Well, because it was 5 minutes and it rised 3 inches. But I think I did something wrong. That's the problem—it felt like I'm doing it wrong but…
D: Why do you feel like that?
John: Wait, never mind. I think I'm doing it right. Yeah.
D: Is there a way you could check?
John: Umm…you could add it up. If it equals 3 inches.
D: Go ahead and do that, go ahead and check. [John starts to write on his paper.] Can you do that in your head?
John: Umm, yeah.
D: Or just kind of do it out loud. You don't have to necessarily write that part down.
John: Well…0.6 plus 0.6 is 1.2. Plus 0.6 is 1.8.
D: Okay.
John: And…that's not 3 inches.
D: So what is the meaning of the 1.8? It's 1.8 what?
John: 1.8 inches of water.
D: Okay. And how long?
John: In 5 minutes I think. That was for 5 minutes. I probably used the wrong formula.

[18] For clarity of writing, henceforth I use typed decimal values rather than a verbatim transcript of the spoken numbers. For example, when "0.6" is used in a quotation, the speaker stated, "Point six" or "Zero point six." If the speaker stated the decimal differently than one of these two options, I then include the speaker's actual language in the transcript.

D: Go back through that one more time. Umm…how were you checking? You were doing what?

John: I was doing like…if you add it up it's supposed to equal 0.5. Like if you add it up five times.

D: Um hmm. So why don't you try that one more time and just keep track as you're going how many times you've added it.

John: Okay. 0.6, 1.2, 1.8, 2.4, 3. [While saying these values he kept track of each additional 0.6 on his fingers until he had five fingers up and then stopped.] It would be 3.

D: Okay. So 0.6?

John: It would be every 1 minute.

D: Okay. 0.6 what?

John: Zero point zero zero. [Stops and then starts over.] Point zero six inches per minute.

D: Okay. So a similar question. Umm, so this…umm…so you divided by 5 here. Right? So why were you thinking about dividing by 5? Why did that make sense to you?

John: Because…5 was the time, was how much you leave it [the water pump] on, so I was trying, like, to make 5 into 1 and like… […] Yeah, pretty much this division. [Points to the written work on his paper.] You get out a smaller one though. Like…it's kind of hard to explain.

D: That's okay. You're doing good. Take your time.

John: Let's see. [Thinks for 8 seconds.] I don't know why I divided by 5…

We see at the beginning of this protocol that John used long division to quantify the number of inches the pool level would rise in 1 minute. This was John's most commonly used strategy for finding unit ratios given non-unit measurements for the values of quantities. I claim that John's use of long division exemplifies one of his constructed quantitative schemes that he used both in assimilation and operating here, as well as throughout the remainder of the teaching experiment.

To substantiate the claim that this represents a scheme for John, I in turn consider the three components of this scheme—the conceptual operations responsible for John's assimilation of the situation, the goal-directed activity he carried out, and the results he expected this activity to accomplish. Ultimately, I believe that John's splitting scheme accounts for his assimilation of the task, the formation of his goal, and his expectation for what the results of his division activity would mean. For instance, John explained that he decided to divide because he was trying "…to make 5 into 1." The fact that John later engaged in iterating to test the sensibility of his result suggests he assimilated this transformation to 1 minute as the result of splitting 5 minutes into five one-minute parts. Then, I infer that John coordinated this awareness with the concomitant quantity and formed a goal of splitting the composite 3 inches into five parts on the basis of this assimilation. Returning to the beginning of the protocol, we see that John's computational procedure for long division acted as the activity of the scheme that accomplished this goal. Lastly, regarding John's expected results, we see that he eventually became satisfied with his result after monitoring his iterating of 0.6 inches to stop at five iterations and 3 inches.

Thus, I infer that John assimilated the transformation from 5 to 1 minutes using his splitting operation, used that relationship to form a goal of splitting the 3 inches into five parts, carried out a long division to accomplish that goal, and relied upon the understandings afforded him from his splitting scheme to make sense of the results. As a result, this way of operating accomplished John's coordinated partitioning/iterating strategy as he transferred his quantitative operations from the change in time to the change in depth. I refer to this scheme as *John's unit ratio division scheme*.

It is important to note that I consider John's unit ratio division scheme a quantitative scheme rather than a numeric scheme (cf. Chapter 2). I did not come to this characterization hastily. In fact, I vacillated throughout the experiment and my retrospective analysis between whether or not to consider instances in which John used strategies such as that in Protocol 6.1 as numeric operating because of his reliance on computational procedures learned in school to solve tasks and his struggles to explain why those procedures were sensible strategies. However, considering John's activity as numeric operating cannot account for the formation of his goals, nor his ability to meaningfully interpret the results of his computations. Thus, I consider John's unit ratio division scheme a procedural scheme (cf. Chapter 2) and the reasoning it entails quantitative reasoning because of the role that his splitting operations play in guiding his mathematical activity.

John initially lacked awareness of his ways of operating.

In addition to exemplifying one of John's characteristic ways of operating, Protocol 6.1 also provides additional evidence for the claim that he was not always explicitly aware of his ways of operating. For example, John intuitively and immediately decided to divide 3 by 5 to answer the question. Yet, when asked why he decided to carry out that activity, he initially responded that he felt like he did something wrong. Further, John struggled to explain his decision to divide by 5 even after iterating the unit ratio to verify the consistency of his result with the initial measurement. He knew his decision to divide related to transforming 5 minutes to 1 minute but seemed unsure of how to explain his activity and never verbalized how this related to transforming the pool depth measurements. The protocol ends with John stating, "I don't know why I divided by 5…" Thus, as in the initial interview, I infer that John's struggle to explain the intuition that led to the mathematical activity he carried out indicates he was not explicitly aware of the goals and operations involved in his unit ratio division scheme.

Furthermore, this protocol provides additional evidence that John at times struggled to coordinate changes in two quantities simultaneously. For example, for his initial attempt to check the validity of his result John iterated 0.6 inches three times. However, he experienced a perturbation when the resulting 1.8 inches did not match the initial measurement of 3 inches.

However, I hypothesize that the issue did not lie with John's operations with iterable units per se but rather in coordinating these operations between two different quantities (cf. to the difficulty he experienced when carrying out his coordinated iterating strategy in Protocol 4.11).

For example, I infer that in transferring his splitting from the change in duration to the change in depth, John focused on the 3 inches and only iterated the 0.6 inches three times rather the five that would stem from his original assimilation of the change in duration. Further, when I encouraged him to think through this unit iteration again he stated, “If you add it up it’s supposed to equal 0.5. Like if you add it up five times.” While John said “zero point five,” I infer that he had maintained in his mind both the 0.6 inches per minute and the 5 total minutes given in the original measurement but struggled to explicitly distinguish all of the relationships at once. The fact that he resolved his perturbation by subsequently monitoring five iterations of 0.6 suggests he was at least tacitly aware of the relationships even though he could not clearly explain them verbally.

A departure from the initial interview: John’s reasoning with decimal quantities.

John’s use of division to quantify the result of splitting a composite 3 inches into five parts represented a departure from his previous attempts to split composite units during the initial interview. For example, in previous situations in which I had inferred John formed a similar goal, he relied upon fraction operations to produce results such as two-thirds of a cake per person. Further, in contrast to his use of long division in Protocol 6.1, the operations John used in the initial interview tasks did not require him to appeal to any computational procedures. Thus, I decided to question him further to see if he could quantify the unit ratio as a fraction on the basis of his quantitative operations rather than long division.

Protocol 6.1: Continuation.

D: Yeah, so what about for 1 minute. What fractional part of an inch would that be?
John: Umm, three-fifths.
D: Why do you say three-fifths?
John: Because… Wait, for 1 minute?
D: Um hmm.
John: Because, umm…if you put three-fifths as a decimal, it is the fraction for 0.6.

D: Okay. So the three-fifths, umm, does that make sense with the situation to you? Like if you think about the decimal you can make a comparison with three-fifths and 0.6.

John: Yes.

D: But what about, umm, like if you…like the 5 minutes and the 3 inches. Does three-fifths make sense with that?

John: Well, no. Not really.

D: No, why not?

John: Because… Hmm, let me see. If you're supposed to do 5 for 3…I can't really tell if I don't, like, do it mathematical. I can't, like, [know] how much it rise in, like, 1 minute.

D: Okay. So if you were thinking of 5 for 3, without doing the division with decimals, do you have a way that you could think about how much that would be?

John: No. I don't think I would have [a way]. Probably it will take a long time. But I don't think I'm going to find a way that's real easy.

Thus, we see that John did recognize that 0.6 inches per minute was equivalent to three-fifths of an inch per minute. It is clear from his comments that John recognized that the fraction three-fifths could be converted to the decimal 0.6. I infer that for John this equivalence stemmed from an association between three-fifths and the decimal 0.6 that likely developed from previous experiences converting fractions to decimals. Having already carried out the long division to identify the pumping rate as 0.6 inches per minute, John leveraged this association to identify the pumping rate as a fractional amount of an inch per minute.

Two aspects of the latter half of this excerpt stand out to me. First, when explaining why three-fifths did not make sense to him he stated, "I can't really tell if I don't, like, do it mathematical. I can't, like, [know] how much it rise in, like, 1 minute." I infer that doing it "mathematical" referred to carrying out the long division that enabled John to quantify how much the pool level rose in 1 minute as a decimal amount. It was somewhat surprising that he could not justify the three-fifths of an inch per minute in the same way that he previously justified 0.6 inches per minute. Second, John's last comment indicates he did not assimilate the situation as a situation of the quantitative operations he had previously used during the initial interview to solve tasks such as this. It is possible that already having carried out the long division to find a suitable solution and having those operations active in his thinking limited his ability to assimilate the situation again with different operations to devise an alternative strategy.

Within the context of this teaching session, these aspects of John's mathematics did not appear to play a role in his ability to assimilate and solve subsequent tasks. Further, I do not claim that he could not think differently about the tasks to resolve these issues but rather that he did not. However, I point them out here because they speak to John's assimilation of situations and hint at a subtle yet important distinction in his mathematics that became more apparent at the end of the teaching experiment.

Investigating John's ability to use his unit ratio, 0.6, in further reasoning.

Because John had successfully operated upon the given measurement to determine a unit ratio, I decided to investigate his ability to use this ratio in service of other tasks. Essentially, I wanted to investigate whether the result of John's long division represented a ratio that pertained specifically to 1 minute or a rate that applied more generally to the covariation of time and pool depth. Immediately prior to the task in the following protocol, I asked John how many inches per minute the pool level was rising. He replied "point zero six," which, given his activity in Protocol 6.1, I inferred meant 0.6 inches per minute. Thus, with the result of his earlier reasoning activated in his mind I continued on to investigate how he would use it in further reasoning about the scenario. I told John to imagine some more time passing and progressed the pool filling animation to 17 minutes. Then I asked him how much higher the pool level would be at 18 minutes compared to the depth at 17 minutes.

Protocol 6.2: John compares the water depth after 17 and 18 minutes.

D: How much deeper would the water be?
John: Well, a little, a little bigger. One-seventeenth of the whole.
D: What's that?
John: It would be one-seventeenth bigger plus that.
D: Say that again, I'm not quite following what you mean there.

John: Like, because if the whole thing is 17 minutes it will be 1 of 17 bigger. Like if you add. It's going to be like 18 out of 17 bigger.
D: Oh, okay. So…so if you, let's say you have the depth. Like you knew the depth at 17. What would you do with that to figure out the depth at 18? I think this is kind of what you were explaining.
John: I would divide it by 17 and then add that to the total and I'll get 18 over 17.
D: So it would be eighteen-seventeenths as deep as it was before. Is that what you mean?
John: Um hmm.
D: Okay. Umm…is there a way to know how much—how many actual inches that went based upon what we've already talked about, the things we're already done?
John: Hmm…no. If I don't know the deepness or, like, I don't know the height and how much it gains, I can't tell.

John and I operated with different perceptions of the task in mind throughout this protocol. I originally intended to use this question to investigate if John had constructed 0.6 inches per minute as a result that described the change for any 1 minute interval rather than only as the change in depth that occurred during 0–1 minutes of pumping. However, John's replies suggest he focused on determining the depth at 18 minutes based upon the unknown depth at 17 minutes.[19] Despite the fact that John and I had slightly different questions in mind, his reasoning in this task indicates several important features of his mathematics.

John's reasoning demonstrates the availability of sophisticated fractional reasoning.

First, I claim that John's assertion that the depth would be "18 out of 17 bigger" indicates his construction of an iterative fraction scheme. Because of John's strong and consistent use of his splitting operation, I infer that he assimilated the depth at 17 minutes as seventeen-seventeenths, or 17 out of 17. Taking that as a given, John then integrated the additional one-seventeenth to produce his result. Thus, John's ability to assimilate quantities with three levels of units, along with his progressive integration operations, account for his construction of 18 out of

[19] The animation only quantified the duration of pumping as the height of the water level physically rose within the swimming pool. Thus, even though John had seen the animation continue to 17 minutes, the specific value for the height of the water at 17 minutes remained unknown.

17 as the result of uniting the original depth with an additional one-seventeenth of that depth. Consequently, this excerpt demonstrates that John's available fraction schemes supported reasoning beyond the whole, something not previously observed during his initial interview.

Second, this excerpt suggests that John had constructed the fraction eighteen-seventeenths as an operator. His description of how to find the depth at 18 minutes indicates that the activity of his unit ratio division scheme supported this construction. Further, this provides evidence that John could determine an intensive quantitative unknown in that he considered the depth at 17 minutes as some specific but unknown depth and described how to use it to quantify the unknown depth at 18 minutes.

John's reasoning indicates his result of 0.6 represented a ratio, but not yet a rate.

Lastly, even though John's activity in Protocol 6.2 demonstrates his ability to assimilate the task meaningfully with his fraction operations, he never made a connection between these understandings and his previous result of 0.6 inches per minute. Specifically, he stated, "If I don't know the deepness or, like, I don't know the height and how much it gains, I can't tell." Thus, while he started the excerpt by stating the water would rise one-seventeenth of the depth at 17 minutes, he did not connect this with his previous division of the depth at 5 minutes into five parts.

Following this task I asked John about the depth at 3 minutes. He replied that the water level would be lower than it was for the initial 5 minute measurement, and that he would want to measure the water level to find the depth. From these responses I inferred that the result John established during Protocol 6.1 did not represent an intensive quantity that characterized the constant covariational relationship between the quantities.

John reconstructs the meaning for his unit ratio and establishes 0.6 as a unit rate.

Because it appeared that the result of John's division, 0.6, did not yet characterize a general relationship between depth and duration, I decided to question him further about 1 minute in order to develop a better understanding of what the result of 0.6 meant to John.

Protocol 6.3: John reconstructs the meaning of 0.6.

D: Well what did you say it would be at 1 minute?
John: Point zero six. [I inferred that he meant "0.6" based upon his previous responses.]
D: Or what fraction is that?
John: Three-fifths.
D: Three-fifths. And what does that mean again to you?
John: Six…like point six of the whole number.
D: Okay. Umm…and in terms of the situation—what does that point six, or the three-fifths—what does that mean to you?
John: Umm…it means… Well pretty much it means trying to make, trying to make it even. Like divide it into even sections.
D: Okay. Okay, I think that make sense. I think I'm still thinking of something a little bit different though. But like in terms of the time and the height of the water. Like it's three-fifths what? Is that three-fifths of a minute is that three-fifths of… [John interrupted the teacher-researcher before he finished this statement.]
John: Three-fifths of a minute, like, for like for every 1 minute.
D: Say that one more time.
John: Three-fifths of every 1 minute. No. Like 1 minute equals three-fifths.
D: Three-fifths what?
John: Three-fifths inches.

This excerpt shows John making explicit three particular meanings for his previous result of 0.6. Similar to the initial interview (cf. Protocol 4.7), I infer that John initially assimilated my request to explain what the three-fifths meant to him as a question about the meaning of three-fifths as a number in relation to one unit rather than as a quantity in relation to the pool filling situation. However, as I questioned him further he began to also make explicit the meanings that underpinned his quantitative reasoning in Protocol 6.1. For example, previously John could not verbalize why dividing 3 by 5 made sense to him. However, in this excerpt he clearly explained that the result of 0.6 referred to trying to "…divide it into even sections." This represented an advancement in John's ability to explicitly describe his goals for the long division activity that produced the 0.6. Yet, as this did not make the relationship with the actual quantities of duration and water depth clear, I inquired again about the meaning of three-fifths. His replies suggest that John actively reconstructed the meaning for three-fifths within the context of the covariational situation.

I continued on to ask about several other specific instantiations of the covariational situation in order to investigate whether or not John's more explicit awareness of these relationships would support operating more meaningfully with his previous result of 0.6. When asked again about the depth at 3 minutes, John immediately replied "One point zero eight," which I infer meant 1.8. Further, he explained that he added the 0.6 three times. Similarly, when asked how the maintenance supervisor could predict the water depth after 127 minutes of pumping he stated, "I would just times it by 127, the decimal" and explained that the result of this computation would refer to the depth of the pool.

Given that he confidently and immediately produced these solutions, I infer that John maintained the numerical relationship 0.6 inches per minute and assimilated these situations using iterable units. Recall that when checking the suitability of his result during Protocol 6.1, John iterated the 0.6 inches five times to reproduce a depth of 3 inches for 5 minutes. However, John's earlier struggles to quantify the change in pool depths from 17–18 minutes and for 3 minutes indicate that the relevance of this strategy was not immediately apparent. Then, after making the meanings involved with 0.6 inches per minute more explicit during Protocol 6.3, the iterating that he previously carried out in activity became the assimilating operation for these solutions.

I hypothesize that becoming more explicitly aware of his ways of reasoning during Protocol 6.3 supported John's ability to abstract 0.6 inches per minute as a quantity that characterized the constant covariational relationship. Specifically, during that protocol John abstracted his operations to the point that he could verbalize the meaning of 0.6 in three ways: as a number in relation to one, as a consequence of completing the goal of his division activity, and as a quantity in relation to the covariational situation. Whether or not John's awareness of all three of these understandings was needed for his subsequent coordination of the quantities remains unclear.

However, I believe John's rapid success with those tasks indicates a larger point in understanding John's mathematical development. Specifically, while John operated with each of these understandings intuitively on the basis of the quantitative operations he used in assimilating the tasks in Protocol 6.1, reflecting upon his previous activity and reconstructing the meanings again during Protocol 6.3 appears to have helped him become more aware of his ways of operating. In turn, making his understandings about the meaning of 0.6 more explicit supported John's construction of 0.6 as a rate that he could use when assimilating other instances of covariational change within this context. I hypothesize that for John, the 0.6 inches per minute now characterized the relationship between time and distance at any point in the covariational process.

Because John demonstrated (both during the initial interview and Protocol 6.1) that he often operated intuitively with quantities and became more aware of his thinking upon further reflection, asking John to explain his reasoning became one of my common approaches to working with him for the remainder of the teaching experiment. Furthermore, this approach served a dual purpose in helping me to accomplish my goals as a teacher-researcher. As a teacher I inferred that these opportunities to reflect upon his mathematical activity were helpful for John to become more explicitly aware of his ways of reasoning and his quantitative goals. As a researcher, John's explanations provided me with insight into the quantitative operations he used in constructing his results.

John uses his newly established rate to structure his assimilation of subsequent tasks.

John quantifies the change in water depth for a given pumping interval.

Following John's explanations in Protocol 6.3, I decided to investigate the generality of his 0.6 inches per minute. Because the previous two questions (i.e., the depth at 3 minutes and 127 minutes) involved a total accumulation from the start of filling up the swimming pool, I created a question about an interval of change similar to the task in Protocol 6.2.

Protocol 6.4: John quantifies the change in water depth from 45 to 49 minutes.

D: So from, say, 45 minutes to 49 minutes, how much deeper would the water level get?
John: It would get…hmmm. For 45 to 49. It would get…2.4 deeper. [John thought for a total of 17 seconds before producing the answer of 2.4.]
D: Why do you say that?
John: Because…if I was supposed to add up… Like subtracting 49 from 45, then you get the total, like, how much it rise, then the difference.
D: Okay.
John: Then the difference is how much it rises.
D: So did you figure out how high it was at 49 and how high it was at 45 and subtract those? Is that what you did or did you do something different?
John: Well I did it a little different. Like, I just did it, like, kind of like the opposite where I just added 4 times.
D: And why did that make sense to you?
John: Because…just by adding. Because…it's 4 times, it's 4 times a minute longer. So, I just added that 4 times, and subtracted from—or added it 4 times and you get the difference.

John's activity here supports the hypothesis that he had constructed 0.6 inches per minute as an intensive quantity that characterized all changes within the constant covariational situation. In addition to quantifying total accumulations such as the depth at 127 minutes, we see that John could also successfully make sense of the situation to identify the change in depth over a specified duration. This represents an advancement over his previous reasoning in Protocol 6.2. In that case John's replies suggest he understood that from 17–18 minutes the water level would raise by one-seventeenth of the water level at 17 minutes, but he did not recognize this as 0.6 inches for the 1 minute change.

I claim that two aspects of John’s mathematical activity in Protocol 6.4 account for this advancement in his reasoning—the construction of an interval and using the iterable unit 0.6 inches per minute in assimilating the change on that interval. For instance, he stated, “Like subtracting 49 from 45, then you get the total, […]. Then the difference is how much it rises.” From this I infer that John had constructed an interval of change that included both the changes in duration and water level. John used *subtracting* to refer to the change in duration and *difference* to describe the change in depth. Further, John characterized the change in duration as 4 times longer than a minute and used this to justify iterating 0.6 inches 4 times, which is a use of his coordinated iterating strategy. This supports the claim that John used 0.6 inches per minute as an iterable unit in assimilation to produce his result of 2.4 inches deeper.

In addition, I believe that this was a novel task for John and his solution represented a creative use of his available quantitative operations. For example, notice that John had several stops and starts while explaining his thinking in Protocol 6.4. From this I infer that John’s initial production of 2.4 resulted from an intuitive use of his quantitative operations. Then, responding to my prompts to explain his thinking served as an opportunity for John to re-present that activity to himself and reconstruct the relationships that underpinned his activity in a more explicit way. In doing so, we see John becoming able to distinguish between the quantity *difference in depths* and the numeric operation of subtraction. The fact that this distinction was not always immediate as he sought to explain his use of addition to quantify a difference supports the conclusion that John was in the process of actively reflecting upon his activity.

John establishes the reciprocal unit ratio.

Following this task, I decided to ask John about the reciprocal quantity, minutes per inch. My goal in doing so was to provide him another opportunity to construct a unit ratio and use it in further reasoning about the covariation of duration and water depth. In addition, I asked John to use fractions rather than decimals because I wanted to encourage him to continue using his quantitative operations creatively rather than resorting to his computational procedure for division as the activity that accomplished his quantitative goals.

Protocol 6.5: John determines how long it takes for the pool level to rise 1 inch.

John: [Thinks for 24 seconds.] Umm…it's kind of hard to do it with fractions. Pretty much.
D: Why don't you tell me what you were thinking? Like, did you have any ideas? Do you want to tell me what you were thinking about?
John: Like increasing, like, you said three-fifths. Like try to increase it to six-ten, or whatever. And then try to figure it out how long it is.
D: Um hmm.
John: [Thinks for 20 seconds.] Hmm, put it in minutes. How can I put it in minutes?
D: You're doing good. Take your time. There's no rush.
John: [Thinks for 20 more seconds.] Like, probably 1 minute and 40 seconds.

At this point in the protocol we see that John successfully coordinated the quantities to solve the task. I infer that John used his quantitative operations in order to produce the mathematically appropriate result of 1 minute and 40 seconds to raise the pool level 1 inch. However, because John carried out most of his reasoning mentally, it remained unclear how he produced this result. Thus, the protocol continues with John's attempts to explain his reasoning.

Protocol 6.5: Continuation.

D: Okay. Do you want to tell me how you were thinking about that?
John: Because for every 30 seconds it rised…ahh…let's see, the decimal for three-fifths—that would be it rised three-tenths for every 30 seconds. So I was trying to split it, like, split it into seconds. Like the seconds for the total of three-fifths—wait, not three-fifths. Because adding three-fifths plus three-fifths, that would be…that'd be like in the one…how do I say it—that would be six-fifths.
D: Okay. So, I mean, you have…
John: So it should be under 2 minutes. It should be under 2 minutes.

D: Okay. So the three-fifths. If you did three-fifths of an inch and three-fifths of an inch that would be six-fifths of an inch?

John: Yeah.

D: In how long?

John: That would be in 2 minutes.

D: Two minutes. Okay. So you're thinking it should be less than 2 minutes?

John: It should be less than 2 minutes.

D: So can you go back through it one more time? You have a good idea. I don't know if I was able to follow it though. So could you explain your thinking one more time?

John: It was like, for every 30 seconds it should rise three-tenths.

D: And how do you know that?

John: Because for every 1 minute it rise three-fifths. So I just increased it. I just increased the three-fifths to six-fifths [I infer he meant six-tenths] and divide that by 2 and you get three-tenths. And that's half of a minute.

D: Okay. So three-tenths of an inch for 30 seconds?

John: Um hmm.

D: Okay. And then where'd you go from there?

John: Hmm…where'd I go from there? Let's see.

D: Yeah, because we were trying to figure out how long it would take to go up 1 inch of depth. And so far, so now you've gotten and told me that it would take

John: It rises, like, if it rised…if it rised three-fifths, I mean three-tenths for every—what is it—30 seconds. I divided by 10 to make it to, umm, like, every 10 seconds. Because…

D: Okay. So in 10 seconds what would that be? What part of an inch?

John: Let me see…10…10 divided by…like, let me see. [Thinks for a total of 16 seconds.] Not divided by 10, I meant divided by 3.

D: Divided by 3. Okay. So 30 seconds to 10 seconds.

John: Yeah.

D: So at 10 seconds…so what part of an inch would that be in 10 seconds?

John: It would be point one, point zero one for 10 seconds.

D: Point one or point zero one? Or what fraction if that's easier?

John: Umm…that would be one-tenth. Easier.

D: So then…so how do you use that then? How does that help you?

John: Well, by using fractions, well, pretty much…it help me to divide it into equal pieces. Then I just add it up to, umm, to get 1. Wait, what is it? One inches of depth.

D: Um hmm.

John: I just added it up.

D: Okay, can you explain what you added up and then how you did that?

John: Well…

D: What was it that you were adding up?

John: I was adding up…point…what was that number? I forgot the number now. I lost it in my head. Umm…let me see. I was adding up point one.

D: Okay.

John: Because that's 10 seconds. So add it up six times and you get point [points at the division of 3 by 5 on his paper], you get that decimal, which is three-sixths. I mean three-fifths of the whole thing. Then I just keep adding it up until I get 1 inches.
D: Okay.
John: The whole number. Like the whole number 1. It makes it easier.
D: Okay. So how many—yeah, so you get to 1 inch. And then what was the—how long would that take then?
John: One minute and 40 seconds. […] 100 seconds total.

Based upon these explanations, I claim that John's production of the new unit ratio, 1 minute and 40 seconds per one inch, relied primarily upon his splitting scheme, his reversible fraction reasoning, and his ability to use decimal values and their fractional equivalents interchangeably. For instance, John started his explanation by stating, "Because for every 30 seconds it rised…ahh…let's see the decimal for three-fifths—that would be it rised three-tenths for every 30 seconds." From this I infer that John mentally exchanged the fraction three-fifths for its decimal equivalent 0.6, split both the changes in water level and duration in two parts, and recognized this result as three-tenths inches per 30 seconds. Because John used decimals and fractions such as 0.6 and three-fifths interchangeably throughout this teaching session, it is unclear if his statement of "three-tenths" referred to three-tenths the fraction or 0.3 the decimal. My inference is like earlier with 0.6 and three-fifths, John knew these were equivalent numbers and he likely had both in mind. Regardless of this distinction, splitting each quantity in two parts produced the intermediate ratio three-tenths inches per 30 seconds. Based upon the rest of John's explanation, I infer he split each of these quantities into three parts to produce a second ratio of one-tenth inches per 10 seconds. Lastly, John reasoned reversibly with the fraction one-tenth inches to construct the duration for three-fifths (0.6) and 1 inch as six and ten iterations, respectively, of one-tenth inches per 10 seconds.

More than showing how John could also use reversible reasoning to coordinate changes in the quantities, this excerpt provides additional evidence that John reasoned with his quantitative operations intuitively and later actively constructed his awareness of this reasoning in activity to explain his thinking to the teacher-researcher. I included the break between Protocol 6.5 and its continuation to emphasize the contrast between these two. In Protocol 6.5, John thought for a little over a minute to produce his result of 1 minute and 40 seconds as the time required to raise the pool level 1 inch. In contrast, John's efforts to explain his strategy during the continuation of the protocol lasted roughly 5 minutes.

John determines an intensive quantitative unknown.

Because John could explain the reasoning that led to his result of 1 minute and 40 seconds for 1 inch of pool depth, in the moment I hypothesized that he would be able to use this result in assimilation as he had previously with the 0.6 inches per minute. To evaluate this hypothesis I asked John how long it would take to raise the pool level 63 inches.

Protocol 6.6: John reasons about the duration of time needed to raise the pool 63 inches.

John: [Thinks for 3 seconds.] Divide it by three-fifths. Like…well, let me see. 63 inches. 63. [Thinks for a total of 15 seconds.] Yeah, I would divide it by three-fifths.
D: Okay. Can you explain why you would divide it by three-fifths? Like, what would that mean to you?
John: It's like for every three-fifths it's a minute, so it's like a minute to me. Three-fifths is like a minute in my mind. I could try to get the number for it.
D: For how many minutes it would take?
John: For how many minutes it would take.

Here we see that John assimilated this new task in relation to the three-fifths of an inch per minute he had used so effectively in earlier tasks. Having constructed three-fifths of an inch per minute as an iterable unit, I believe that John envisioned some unknown number of iterations of three-fifths producing 63 inches. Then, I infer that he intended to use the operation of division to quantify this number of iterations, with each representing 1 minute. Thus, in contrast to the partitive division that comprised the activity of John's unit ratio division scheme in Protocol 6.1, here John used division in a quotitive sense.

However, because John did not assimilate this task in relation to his result from Protocol 6.5, 1 minute and 40 seconds per 1 inch, I validated his first strategy and asked him if he could think of any other ways to solve this task. After thinking about this for a little bit, he mentioned that he could multiply. Further, when I asked him what multiplication he had in mind he stated, "I would multiply this to the minutes [points at his earlier result of 0.6]. So like the minutes to get that inches depth [points to the "63" on his paper]." Thus, John recognized that multiplying the unknown number of minutes by 0.6 inches per minute would result in 63 inches of change in the pool depth.

This is consistent with the reasoning that underpinned his quotitive division strategy. Yet, John never used his result of 1 minute and 40 seconds per 1 inch as part of his efforts to quantify the unknown duration. Consequently, I infer that John had constructed 1 minute and 40 seconds per 1 inch as a unit ratio but not yet as a unit rate.

Accounting for John's construction of 0.6 inches per minute as a unit rate and 1 minute 40 seconds per inch as a unit ratio.

I found it both surprising in the moment and important for my retrospective analysis of John's mathematics that John did not use 1 minute and 40 seconds per 1 inch as a rate. My surprise that he did not assimilate the unknown duration in terms of 1 minute and 40 seconds per inch indicates that I had overestimated the significance of his ability to produce and explain the reasoning that led to this unit ratio in Protocol 6.5. John's activity with unit ratios in this teaching session indicates that in each case (Protocols 6.1 and 6.5), the results of his reasoning at least initially remained phenomenologically bound to the specific situations in which they were produced—1 minute and 1 inch, respectively. Thus, I infer that the process of quantifying the change in depth for 1 minute or the duration required to raise the pool level 1 inch was insufficient for John to construct each result as an intensive quantity that characterized the entire constant covariational relationship.

In addition, the fact that John did begin to use three-fifths (or 0.6) inches per minute quite productively in assimilating subsequent tasks suggests that something was different in John's reasoning with this unit ratio compared to his reasoning with 1 minute and 40 seconds per 1 inch. Comparing his reasoning with both unit ratios across Protocols 6.1–6.6, two aspects of those interactions stand out. First, I noticed a difference in the activity John carried out with each result. Specifically, John's success in using 0.6 inches per minute relied upon using that quantity as an iterable unit. In Protocol 6.1 John carried out iterating in activity to evaluate the suitability of the result while in Protocol 6.4 I inferred that he took the iterable unit as given while assimilating the tasks. In contrast, John did not use 1 minute and 40 seconds per inch as an iterable unit—neither in activity nor in assimilation of subsequent tasks.

Secondly, the nature of the explanations John produced for each unit ratio differed. Recall that during Protocol 6.3, John verbalized the meaning of 0.6 as a number in relation to one, as the number that achieved his goal of splitting the change in depth evenly, and as the number of inches for every 1 minute. Further, after this exchange John began to use 0.6 inches per minute in assimilating subsequent tasks. In contrast, his explanations in Protocol 6.5 focused on describing the specific operations and transformations he carried out to determine the result of 1 minute and 40 seconds per 1 inch. Thus, even though he explained how he carried out the transformation for the second unit ratio more clearly than he did with the first unit ratio during Protocol 6.1, these explanations never explicitly focused on describing the meaning of the result in terms of the covariational situation.

It is important that I clarify that I do not see these differences as necessary constraints to John's reasoning but rather as features of the way the interaction between John and the teacher-researcher unfolded during the teaching session. I hypothesize that had I asked John to reconsider the meaning of the 1 minute and 40 seconds he found for raising the water level 1 inch, he could have abstracted that ratio as an intensive quantity that characterized the entire constant covariational situation much as he had done previously with the three-fifths (or 0.6) inches per minute. For this reason I have been careful to say that John *did not* construct 1 minute and 40 seconds as a unit ratio, not that he *could not* do so. I decided to include Protocols 6.5, 6.6, and John's reasoning about the number of minutes per 1 inch of water because they help to clarify aspects of the interaction that supported John's construction of 0.6 inches per minute as an intensive quantity that broadly characterized the constant covariational relationship between water depth and pumping duration.

Reflecting back upon John's first teaching session.

Considered as a whole, John's mathematical activity in the October 11, 2013, teaching session revealed several critical features of his mathematics. First, his overall success in determining suitable solutions to the tasks indicates the power of his available quantitative operations. The fact that John often intuitively solved the tasks, at times with little hesitation, indicates he had already constructed the quantitative operations that supported this intuition prior to this teaching session. In particular, I inferred that John's ability to construct and assimilate with iterable composite units supported his intuitive reasoning. For example, in Protocol 6.6 John used three-fifths of an inch per minute to structure his perception of the task and guide his strategy.

However, John's explicit awareness of his ways of operating often lagged behind his ability to solve particular tasks. Further, as evidenced throughout the above protocols, his verbalization of his thinking was not nearly as intuitive as the operating he used to solve the tasks. I believe that this suggests these tasks were novel yet fell within his zone of potential construction. Further, opportunities to re-present and reflect upon his quantitative activity to make the relationships between the quantities more explicit, such as during Protocol 6.3, supported John's abstraction of his ways of reasoning and the meaning of his results.

I made the decision to include so many tasks from John's first teaching session in the swimming pool context for several reasons. First, these protocols capture most of John's characteristic ways of reasoning within this context. Secondly, they provided opportunities to examine the quantitative operations that supported John's reasoning and the ways in which he used those operations to produce and operate with the intensive quantity 0.6 inches per minute. Lastly, this sequence of protocols revealed several differences in John's assimilation of the tasks throughout the interview that helped me to better understand his mathematics.

John Abstracts His Ways of Reasoning During the Remaining Swimming Pool Tasks

As characterized above, John had demonstrated he could use his quantitative operations to coordinate changes in the values of the extensive quantities pool depth and pumping duration, but these results were not necessarily applicable beyond the immediate situations in which they were constructed. Thus, going forward, I formed the goal of working with John to investigate his ability to construct and operate with unit ratios as characterizations of the constant covariational relationship between the quantities. In addition, I attempted to design novel contexts in which John would continue to reason creatively and to provide opportunities for him to reflect upon his reasoning to become more explicitly aware of his goals and strategies.

John's reasoning became much more sophisticated during the final two teaching sessions developed around the swimming pool context (October 28, 2013, and October 30, 2013). The tasks for these sessions focused on comparing the pumping rates of different replacement water pumps and reasoning about specific changes in water depth and pumping duration that would produce the same pumping rates. John's reasoning throughout these teaching sessions remained consistent in that I infer he relied upon the same quantitative operations that characterized his mathematical activity in Protocols 6.1–6.6. However, over the course of these two teaching sessions I observed a significant change in John's understanding of the results of his activity.

John constructs unit ratios to compare two pumping speeds on October 28, 2013.

First consider John's responses to the task of comparing two different water pumps to decide which would be a better replacement for the original water pump that had broken down. Replacement Pump 1 raised the level of the pool 4 inches in 5 minutes while Pump 2 raised the level of the pool 3 inches in 4 minutes. Both Jack and John were present for the teaching session on October 28, 2013. However, because I have already presented Jack's replies to this task in Protocol 5.3, I only include John's responses here and use the symbol, […], to indicate places in which interactions between the teacher-researcher and Jack have been removed.

Protocol 6.7: John compares two possible replacement pumps.

D: I want you to try to decide which of the two pumps would be better and why do you think it would be a better pump. […] [Thinks for 40 seconds without writing anything down, then indicates he has decided.] What were you thinking John?

John: Well, I…this first one is better because if I put it in a fraction this one [Pump 1] is bigger by one, one-twentieth. Because…I did, like, 4 over 5 and 3 over 4. And then I times this by 5 over 5 and times this by 4 over 4 to get 20 on the bottom. And then I get 16 and 15. And this one is bigger. [Points to the first fraction "4/5" he had written for Pump 1. See Figure 6.1 for the written work John produced while speaking.]

$$\frac{4}{5}\ \frac{x}{x}\ \frac{4}{4}\quad \frac{16}{20}$$

$$\frac{3}{4}\ \frac{x}{x}\ \frac{5}{5}\quad \frac{15}{20}$$

Figure 6.1. John’s written work for comparing Pump 1 and Pump 2.

D: Okay. Can you tell me a little bit about what these, like what does this mean to you when you had 4 over 5? What is that—four-fifths what? What does that mean?
John: Four-fifths. Let me see. It means…to me…it’s like a bigger, well it means like… [Thinks for 8 seconds.] It’s kind of hard to explain. Well, it’s like to me it’s a unique way of how I solve. And I don’t really know how to explain my unique way.
D: Okay.
John: Because I make a—I solve problems, like, how teacher normally do not solve it.
D: Okay. So how did you know that you wanted to [do this]? [Points towards the work on his paper.]
John: Because I wanted to make them equal. Like, put it into, like…I wanted to make them like into a… [Trails off his speech.] Let me see, how can I say it? Like, compare them. Like compare them using… [Points at his written work. Thus, I infer he meant “using fractions”.]
W: Can I ask a question?
D: Um hmm.
W: Could you draw a picture of that 4 divided by 5 for me? What did you mean by that?
John: Well, 4 divided by 5.
W: Can you just draw me a picture. You’ve got 4 inches in 5 minutes. And you get four-fifths, right?
John: Um hmm.
W: Can you draw me a picture of what that means?
John: Four-fifths. A picture…
W: If you’ve got 4 inches in 5 minutes.
John: It means, well. To me, going up 4 inches.
W: What’s that 4 divided by 5 mean? Can you draw me a picture of that?
John: Of 4 divided by 5. Try to draw a picture of it.
D: Um hmm.

[…]
John: [After thinking for 70 seconds, John responds.] Well, it's, it's—I can't really draw a picture.
D: Well maybe we'll come back to that question.
W: Okay.
D: So, we'll kind of keep that one in the back of our minds. […]
John: Well…
D: Did you have an idea?
John: It pretty much means, like, 1 minute—how much it fills. Like 4 over 5. That's, like, 1 minute. How much it fills…per minute.

John's reasoning in this excerpt remained very similar to that which he carried out during the previous teaching session in the sense that he solved the task rather quickly and intuitively. Further, he struggled to explain the meaning of his activity in relation to the covariational situation and focused his explanations more on what he did rather than why he did it or how it related to the quantities. Yet, at the end of this excerpt John makes a critical realization about the meaning of his activity.

To develop a better sense of John's mathematics, I consider each of these aspects of his reasoning in this excerpt more closely. To solve this task, John appeared to make the fractional comparison mentally and then reproduced that work on paper while explaining his decision that "the first one is better." His explanation indicates that he based his decision on comparing the fractions four-fifths and three-fourths and that he chose this strategy "Because I wanted to make them equal. […] Like compare them using [fractions]." I infer that making them equal referred to converting the fractions to a common denominator to more easily compare the fractions.

Once he formed the goal of comparing the two fractions, his activity suggests that he assimilated that goal as a situation of his procedural scheme for changing denominators and comparing fractions. Thus, I consider John's execution of this strategy an example of numeric reasoning rather than quantitative reasoning because this comparative process appeared to be a routine procedure unrelated to the quantities themselves and the task context more generally.

Once he completed this activity, Pump 1 represented the better choice because it had a bigger fraction. Further, I hypothesize that John used his quantitative operations to assimilate the task initially. In particular, even though I consider the process John used to make the fractional comparison an example of numeric reasoning, numeric reasoning alone cannot account for his production of the fractions four-fifths and three-fourths in the first place.

Unfortunately, John's explanations provide little evidence as to the specific quantitative operations he used in assimilation. Further, it is not clear if John initially considered his written "4/5" as referring to the division *4 divided by 5* or the fraction *four-fifths*. He treated the notation "4/5" as a fraction to make the actual comparison, and he at times adopted the teacher- and witness-researchers' language of four-fifths and 4 divided by 5. However, considering only the language that John independently introduced, he initially stated, "I did, like, 4 over 5 and 3 over 4." In addition, at the end of the excerpt he explained that 4 over 5 meant, "How much it fills…per minute." One possible explanation is that he assimilated the pumping details for each pump as a situation of his unit ratio division scheme, which he had previously used to quantify the depth in 1 minute during Protocol 6.1. However, John's explanations provide insufficient evidence to make a strong claim that John did in fact reason this way.

While I cannot fully account for the quantitative reasoning that John used in assimilating this task, my primary reason for including this protocol is because of his struggle to explicitly describe the meaning of his activity in relation to the covariational situation. When I first asked John what the 4 over 5 he had written meant, he stated, "I don't really know how to explain my unique way." From this I infer he was more focused on trying to explain how he thought about the initial task rather than the meaning of the 4 over 5 in relation to the scenario. Thus, the teacher- and witness-researcher asked several follow-up questions to try to encourage John to reflect upon why he carried out the activity that he did and to form an image of the quantities he operated upon. John never did produce a picture or diagram. However, right before moving on to a new task John stated, “It pretty much means, like, 1 minute—how much it fills. Like 4 over 5. That’s, like, 1 minute. How much it fills…per minute.” This final comment suggests he constructed some quantitative meaning for his reasoning while thinking about the diagram.

Unfortunately, rather than follow-up on John’s realization, I engaged the students in finding combinations of changes in water depth and pumping duration that would result in pumping water at the same rate as Pump 1. Thus, I do not know if John could have produced a diagram at that point, and it remains unclear what John thought about to decide that 4 over 5 meant how much the pool was filling per minute.

John reconstitutes his unit ratios as unit rates on October 30, 2013.

We returned to water pump comparison task to start the next teaching session with John to try to gain some insight into his thinking. To begin, I asked John to describe what he remembered about comparing Pump 1 and Pump 2. First, he recalled that Pumps 1 and 2 raised the water level 4 inches in 5 minutes and 3 inches in every 4 minutes, respectively. Then he recalled that Pump 1 was the better pump “because I put it into a fraction and it’s a bigger fraction—0.8 is bigger than 0.75.” In response to this we inquired further about what these fractions and decimals meant to John. His replies suggest that the process of constructing meaning for 4 over 5 during Protocol 6.7 was both lasting and impactful.

Protocol 6.8: John explains his meaning for 0.8 and 0.75.

D: What does the 0.8 mean to you? Like when you think about the 0.8, it’s 0.8 what?
John: Like, how much it fills per minute. Pretty much.
D: Okay.
W: Have him draw a picture of that once.
D: Umm, yeah, so why don’t we—do you want to pick your color [marker]? So can you draw, kind of, a picture of that? So we had 4 inches per every 5 minutes was

the setting for that pump. Could you draw a picture of what that would look like to you?

John: Four inch.

W: Yeah, you said it was what?

John: Like…

W: Point what per inch?

John: 0.8 inches. Like, let me see. I'll just, like… [Draws a vertical segment.] Let's see. So I'm supposed to draw the four, like, put it into…let me see… [Adds onto his drawing to form a rectangle.] Like, fill. Okay this is 0.5. [Partitions the vertical segment into two parts and labels the tick mark with ".5".] And 0.8 is up here. [Puts another tick mark on the vertical segment where roughly 0.8 would be and labels it as ".8".] And it's like filled up this much. [Draws a horizontal line at the 0.8 level and shades everything below that line.] [Begins to draw another diagram for Pump 2.] And it's like this one is like…to me it's like, 0.5 is here and 0.75 is a little smaller. [Draws a second rectangle and again partitions the vertical segment, labels ".5" and ".75", draws a horizontal line at the 0.75 level, and shades below the line. See Figure 6.2 for the completed diagrams.] Pretty much just like…pretty much it's just like this. Like, and these are like a minute. [Places his finger on the top and bottom of the marked portion of the second rectangle indicating that the shaded vertical height represented 1 minute.] And I just…if you stack up it's 0.5 bigger is all [I infer he was comparing the 0.8 to the 0.75 and thus meant 0.05 bigger]. It's a little bigger every minute.

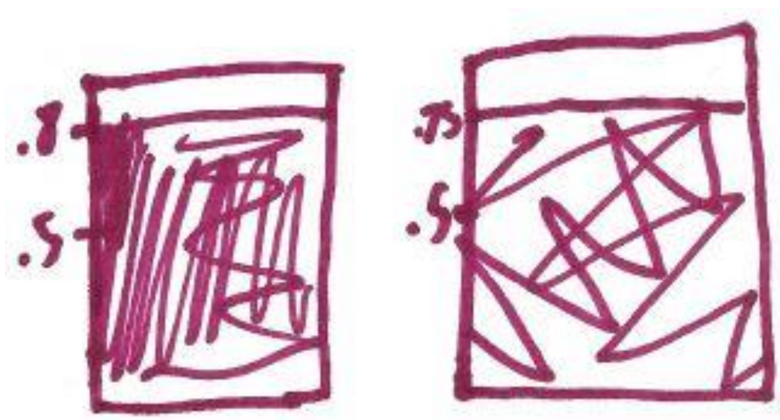

Figure 6.2. John's diagram for 0.8 and 0.75 inches per minute, respectively.

This excerpt demonstrates that the meaning John constructed for his activity with the fractions 4/5 and 3/4 had, in fact, persisted beyond the previous teaching session. In contrast to his difficulty explaining what the 4 over 5 meant during the previous protocol, here John immediately responded that the 0.8 meant "how much it fills per minute." In addition, the fact that John could draw diagrams to explain what the decimal values meant supports the inference that his revelation at the end of Protocol 6.7 indicated that he had constructed meaning for his results in terms of the covariational water pumping situation.

Furthermore, unlike earlier protocols in which John's results at times remained phenomenologically bound to the given situation, I claim that he had constructed the quantity 0.8 inches per minute as an intensive quantity that characterized the entire constant covariational relationship between the quantities. To support this claim, first consider John's consistent use of *per minute* language throughout Protocol 6.8. He used this language when initially describing the meaning of 0.8 and again when describing how his diagrams indicated both inches and minutes. Also, rather than simply making the extensive quantitative comparison that 0.8 indicated more change in depth than 0.75, he described Pump 1 as "a little bigger every minute." Second, the teacher-researcher had actually asked John to draw a diagram for the pumping rate 4 inches per 5 minutes. Yet, John's diagram explicitly entailed 1 minute. Thus, there was reason to believe that John recognized those as equivalent characterizations of the same pumping rate. To investigate this further, the teacher-researcher questioned John about the diagrams he would draw for other pumping durations.

Protocol 6.8. Continuation.

D: Okay. What about the 4 inches per every 5 minutes. What might that look like if you drew a diagram of that?
John: Well, that was this one. [Points to the first diagram he drew.]
D: Okay. Umm…so how could you use your diagram here to think about, like, 10 minutes?
John: Well, 10 minutes.
D: Like what setting you would use for 10 minutes.
John: 10 minutes, let me see. I would just…add this [taps his marker on the ".8"]. Well, I would just add this…let me see. Oh, I would just times it by 10.

D: Why do you say that?
John: Because this is 1 minute and I would just times it by 10 and I would get 8. And it would be 8 inches, because this is inches. [Moves his marker up and down the first rectangular diagram to indicate it represents an inch.
D: Okay.
W: Would you have to draw, would you have to draw 10 boxes?
John: Umm…no.
W: Why is that?
John: Because if you draw 10 boxes you're probably just wasting time. But some people, they need to draw the boxes. But you could just add it by 0.8 ten times and still just times it by 10 to replace the drawings.

The continuation of Protocol 6.8 provides additional warrants for the claim that John had constructed 0.8 inches per minute as an intensive quantity that characterized the entire constant covariational situation. Notice that John stated his diagram did represent 4 inches per 5 minutes even though he had previously described how his diagram indicated 1 minute. I hypothesize that John did not see a need to draw different diagrams for 0.8 inches per minute and 4 inches per every 5 minutes because they represented the same pumping rate to him. Further, consider John's comments, "Oh, I would just times it by 10" and "Because if you draw 10 boxes you're probably just wasting time." These suggest that John had constructed the unit ratio, 0.8 inches per minute, as an iterable ratio. Then, the iterability of this unit ratio alleviated the need to draw additional diagrams and accounts for John's recognition that multiplying 0.8 by 10 would encapsulate the iterating activity of adding and progressively integrating 0.8 inches 10 times. Both of these aspects of John's reasoning in this excerpt support the claim that John had reconstituted the 0.8 inches per minute as a unit rate.

Furthermore, John's comparison of other pumping rates after this protocol indicate he could use these ways of reasoning to make sense of other pumping rates as well. For example, we also asked John if pumping 26 inches per every 32 minutes would be equal to the pumping rate of replacement Pump 1. He quickly said, "I'm trying to put it into, like, per minute. And like inches per minute, not 26 per 32 minutes." He then wrote "13/16," which indicates he achieved a proportional comparison, decided this was not equivalent to Pump 1 because he could not simplify the fraction any further, and described the thirteen-sixteenths as “just like the inches per minute.” In fact, Protocol 6.7 marked the last time during the teaching experiment that John’s ability to interpret his results in terms of the covariational context lagged behind his ability to produce a unit ratio for a given situation.

John demonstrates the sophistication and flexibility of his ways of reasoning.

John’s final task within the filling up the swimming pool context demonstrates the type of reasoning that characterized his approach to coordinating covarying quantities throughout the remainder of the teaching experiment. As a further test that John had abstracted the quantity 0.8 inches per minute to the point that it was freed from any specific values of the extensive quantities, we asked John if one could pump water at the rate of 0.8 inches per minute for 1 second.

Protocol 6.9: John considers the possibility of pumping water at a rate of 0.8 inches per minute for less than 1 minute.

D: Could we have it run at that rate for 1 second?
John: Hmm, yes.
D: So why do you say yes?
John: Because, you just have less time and it’s not going to fill up 0.8. You’re just going to fill up part of it. Like part of the whole 0.8 inches of the pool.
D: Okay. So when it’s 0.8 inches per minute, do we have to let it run for a minute?
John: Yes, to get to 0.8.
D: To get to 0.8. But could we run it for less than a minute and still have it be the same rate?
John: Yes.
D: So why—how do you know that? Like why do you say that?
John: Because if you have it run for less than a minute it’s just going to fill up… Let’s see, how do I say it? Umm…it’s like umm, [suppose] you let it run for like 2 seconds. That would just be a part of a minute so that would just be, umm, one-third, no one-thirtieth of the whole 1 minute. So it’s just going to fill up part of it, like one-thirtieth of 0.8.

John’s reasoning in this excerpt suggests that he had constructed an understanding of 0.8 inches per minute as an intensive quantity that implied any proportional change in the extensive quantities would result in the same pumping rate. For example, John initially replied, "Because, you just have less time and it's not going to fill up 0.8." This indicates that he could envision pumping water at that rate for less than a minute, which intuitively meant the pool would not fill all the way up to 0.8 inches. Given his responses to previous tasks, it was not surprising that he could envision coordinating changes in the extensive quantities. However, more than that, I infer from John's other comments that he also considered it appropriate to still call this smaller change in pool depth over the smaller amount of time 0.8 inches per minute.

To understand the quantitative operations that may have made this awareness possible, consider the example John created of pumping at the rate of 0.8 inches per minute for only 2 seconds. He concluded that this would result in raising the pool level by one-thirtieth of 0.8 inches in one-thirtieth of 1 minute. John's identification of 2 seconds as one-thirtieth of 1 minute suggests he was aware that splitting 1 minute into 30 parts would produce a two-second part such that 30 iterations of the part would reconstitute 1 minute. Similarly, transferring this split to the quantity 0.8 inches would produce a part, one-thirtieth of 0.8, for which 30 iterations would reproduce the given 0.8 inches.

Thus, I infer that as a result of the splitting operations John used in constructing 2 seconds as one-thirtieth of 1 minute, the resulting ratio was constituted as an iterable unit. In turn, constructing one-thirtieth of 0.8 inches per one-thirtieth of 1 minute as an iterable unit would afford an awareness that continuing to accumulate water and time in this ratio would eventually result in an accumulation of 0.8 inches in 1 minute. Thus, my hypothesis is that John's splitting scheme and his ability to construct the results of his splitting scheme as iterable composite units account for his awareness that his two-second ratio still represented the same pumping rate. In terms of John's mathematics, because of the availability of these ways of reasoning, to John the quantity 0.8 inches per minute did not represent a particular pumping duration of change in water depth, but rather all possible changes in pumping duration and water depth.

The Adopt-A-Highway Context

I transitioned to tasks within the Adopt-A-Highway context with a focus on investigating the students' quantification of unit ratios. At this point in the teaching experiment, the biggest question I had about John's mathematics pertained to the quantitative operations he used to construct unit ratios such as four-fifths or thirteen-sixteenths of an inch per minute. The above excerpts indicate that once constructed, John could use these quantities as assimilating concepts to structure his activity with subsequent tasks. Further, his activity and explanations in the swimming pool context provide evidence of the quantitative operations that support John's ability to use these unit ratios as intensive quantities in further reasoning. However, these same protocols offer few indications of the quantitative operations that led to John's formation of the ratios in the first place. Rather, his activity frequently began with an intuitive decision to use a fraction such as four-fifths to characterize the change in depth for 1 minute. John's struggles to verbalize why he decided to carry out particular operations indicate he lacked awareness of the source of this intuition as well.

As a result, within the Adopt-A-Highway context I designed tasks that focused on the construction of unit ratios and provided opportunities for John to reflect upon his ways of reasoning with the quantities. My primary goal in using this approach was to investigate the quantitative operations John used to establish unit ratios and to facilitate his abstraction of these operations so his reasoning could become more explicit and flexible. As with Jack, I wanted John to develop the operations needed to form a goal of intentionally finding a fractional amount of a composite quantity and to quantify that amount as a fraction of one. Doing so could support, for example, forming a goal of finding one-fifth of 4 and enacting a strategy for quantifying this as four-fifths of 1.

Recall that tasks within this context involved fairly allocating various sections of highway to different numbers of volunteer organizations and focused on identifying the appropriate number of miles per organization. John's first of four teaching sessions in this context occurred on November 12, 2013. His reasoning during these four teaching sessions helped to clarify my interpretation of the quantitative operations John used to construct his understandings.

John Develops Strategies for Establishing Unit Ratios in the Adopt-A-Highway Context

John uses his unit ratio division scheme.

The first task that I posed for John involved fairly allocating 4 miles of highway among seven different volunteer organizations. After explaining the scenario, I asked him how much of a mile each organization would be responsible for and provided a map that identified the 4 miles as four individual one-mile sections (see Figure 6.3). John's initial response was to ask, "So are we just talking about just they're all being equal?" After I confirmed that we wanted to allocate the highway evenly to each organization, John picked up his marker and wrote the long division of 4 divided by 7 on his paper. At that point I interrupted John before he could carry out his long division procedure to ask him what that long division meant to him. He explained that he wanted to divide to get the equal amount and that the result of the division would represent the amount of highway each organization would receive.

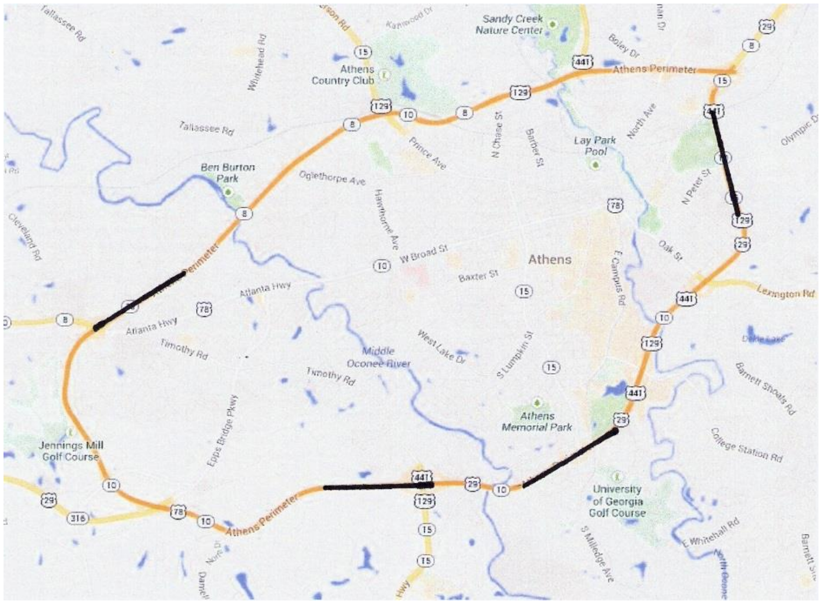

Figure 6.3. The map identifying four one-mile sections of highway to be allocated among seven volunteer organizations.

Based upon these explanations, I infer that John assimilated this task as a situation of his unit ratio division scheme. First, he had explicitly formed the goal of splitting the 4 miles into seven equal groups. Then, carrying out the division would have quantified an amount of highway that if repeated 7 times, would exhaust the 4 miles.

John experiments to construct an alternative strategy.

Because John had previously demonstrated his unit ratio division scheme, I decided to stop him before he computed a decimal value for the amount of highway per organization. Because this way of reasoning relied upon numeric reasoning as the activity of the scheme (i.e., a procedure for long division), I interrupted his strategy to investigate if John could construct a different way of determining the unit ratio that relied upon his quantitative operations rather than the numeric computation.

At the time I hypothesized that John might recognize this as similar to the cake sharing tasks during the initial interview and use his quantitative operations to construct the result as four-sevenths of a mile per organization. Instead, asking John to try to determine a different strategy appeared to place him in a state of perturbation. Within the first 30 seconds of his reasoning, he said "I can't" four times, and he appeared uncertain about what he might try. However, not wanting John to remain discouraged or to give up on the task, I encouraged him to experiment and shifted my language to asking John if he could find a way to split the miles up to find each organization's share. I hoped that this might facilitate his use of splitting operations to think about partitioning the miles and experimenting with different partitioning strategies so that he might construct a successful approach in activity that he could later abstract as a way of assimilating subsequent tasks.

Responding to my repeated encouragement to experiment and see what might happen, John began to consider some possible fractional amounts for each organization. His strategy for testing these fractional amounts proved quite informative with respect to my model of his mathematics.

Protocol 6.10: John experiments to find a strategy for allocating 4 miles of highway among seven organizations.

John: I know that it's bigger than a half a mile, but it's smaller than a mile. And it's between, like a side. [Motions towards the map and moves his marker in the air over one of the one-mile sections as if cutting it at some point in the middle of the mile.] But that's all I know. I can't.

D: Okay. Can you tell me a little bit about how you decided that? So you said you knew it was bigger than a half. How did you know that?

John: Because if you split it in half that would be equal to all—that would be for eight. Because it would be split in half equally and that would be eight.

D: That would be eight organizations. Sure.

John: Um hmm.

D: So then you know it needs to be a little bit bigger than that. So would it work to split each of them into a third of a mile sections, into three parts?

John: Into, like, one…like two-thirds of each section? That would be two… [Using his marker, he motioned in the air above the one-mile sections as if imagining cutting them. He paused once above each one-mile section on the map. Based upon where he moved his marker in the air, I infer that he was imagining cutting a two-thirds of a mile section in each of the four miles. He then thinks for approximately 10 seconds.] No. It's not going to work if you split into two-thirds.

D: How did you know? How did you decide that that wouldn't work?

John: Because it would be, umm, well, let's see. Well basically I just added it up. Two-thirds and two-thirds and then another two-thirds with these two. [While saying this, he points at one of the one-mile sections, then a second, and then at both of them while saying "…and another two-thirds with these two."] Then these two. And that's only…six. [He appears to count again the two-thirds of a mile sections that he imagined making in the map.] Yeah, six. It's only six. That's too big, I need to get to seven. So it's smaller then.

I characterize John's experimental strategy as one of choosing a possible fractional amount and testing the suitability of that fraction by determining how many groups of that size could be formed. For example, John reasoned that splitting each mile in half equally would produce eight equal shares. Considered with respect to his splitting operation, this meant each share would need to be a larger amount of 1 mile in order to produce fewer total equal shares. Further, I infer that John could test one-half mile very intuitively because imagining marking a one-half mile part in each of the 4 miles would create an equivalent one-half mile as the size of the remaining part within each mile.

Following his explanation for rejecting one-half of a mile as too small, I asked John if splitting each mile into thirds and making three parts in each section might work. My purpose in doing so was twofold. First, I wanted to investigate how John would use my suggestion of making three parts in each mile. Second, and more importantly, I did not want John's initial realization that he could not come up with the result to preclude him from attempting to use his quantitative operations creatively. This was the first time during the teaching experiment that John could not intuitively find a way to solve the given task. Thus, I asked this question in part to try to convey to him that an experimental approach and trying something new to see what would happen was perfectly acceptable and valued just as much as an using an intuitive or anticipatory approach.

I find John's strategy for evaluating the feasibility of two-thirds of a mile per organization more revealing of his underlying approach then his previous reasoning with one-half. First, notice that John assimilated my suggestion of splitting each mile into thirds, or three parts, as indicating he should consider two-thirds. This is not surprising in that he already knew that he needed more than one-half mile per organization. However, identifying each possible share as two-thirds suggests that he focused on identifying composite shares of size two-thirds within the group of 4 miles rather than considering the implications of splitting each one-mile section into three parts.

I infer that he operated with two-thirds as a composite unit and explicitly imagined marking of a single segment of size two-thirds in each mile. Implicitly, identifying shares of size two-thirds implied splitting each mile into three parts. However, I do not believe John explicitly thought about splitting each one-mile section into three parts but instead relied upon his number sense to imagine splitting 1 mile into only two parts—a composite two-thirds of a mile and the remaining part as one-third of a mile (see Figure 6.4). In this figure, the red portions of each mile indicate the composite two-thirds of a mile while the black portions represent the parts that remained after identifying these composite sections within each mile.

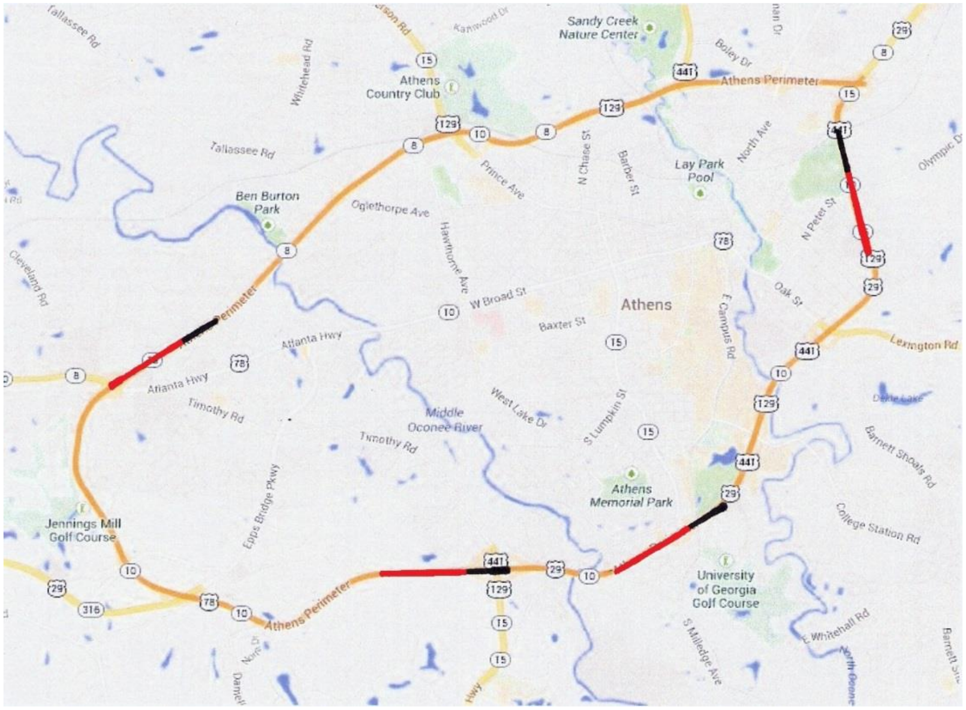

Figure 6.4. An image for my model of how John tested two-thirds of a mile per organization.

John's activity supports this inference. For example, while moving his marker over the four segments on the map he paused only once over each one-mile section rather than the two times that one would expect if he had imagined partitioning each mile into three parts. Further, John said, "Two-thirds and two-thirds and then another two-thirds with these two" while pointing at one of the one-mile sections, then a second, and then at both of them. This is consistent with imagining marking off a composite two-thirds of each of the first 2 miles and then recognizing that the remaining parts of each mile could together form another composite two-thirds.

Furthermore, retrospectively comparing this protocol to the initial interview, I found this model of John's approach consistent with his activity in the cake sharing task (cf. Protocol 4.8). In each case, John described uniting the remaining parts from two of the units to form an additional composite share. During the initial interview, this supported his conclusion that two-thirds of a cake per person would produce three fair shares of all the cake. Here, John determined that two-thirds of a mile per organization would allocate the 4 miles among six organizations rather than the desired seven.

Following this protocol, John experimented with other fractional amounts and continued to use an approach consistent with my model of his activity in Protocol 6.10. For instance, he decided to try three-fifths of a mile per organization next. To test this possibility, John wrote the fraction "3/5" on his page four times, stated, "That's four", and then wrote the fraction "2/5" on his paper four times. When considering what these results indicated he said, "That would be eight. […] I'll get three-fifths and another three-fifths and then I'll have two-fifths left." He also described that, "These two-fifths were the left over from these" and tapped his marker on each of the four "3/5" he had written on his paper. Thus, I infer that John first identified a composite share of three-fifths within each mile, combined all the remaining parts, and attempted to regroup them into additional composite shares of three-fifths. Since he could not form a total of seven composite shares of three-fifths, he rejected three-fifths miles per organization as an option. Afterward, he decided not to test four-sixths because it was equivalent to two-thirds and then established that five-sevenths of a mile per organization also would not work on the basis of the same type of reasoning.

As John explained his thinking about these fractions, two things became clear to me about his activity—I had succeeded in encouraging him to experiment, and his approach did not leverage his unit coordinating operations. Thus, as the interaction progressed I asked John more and more questions about trying to predict whether or not a particular fractional amount per organization would work prior to carrying out any activity.

In response to these questions, John developed a strategy of predicting the number of leftover pieces and deciding if he could regroup them to form the appropriate number of shares. For example, for five-sevenths per organization there would be eight-sevenths left over after identifying a composite five-sevenths in each of the miles. Because this could only create one more composite share, John rejected five-sevenths as a possibility.

While this did represent progress over his previous need to carry out the activity with paper and pencil, it did not afford him any anticipation as to whether or not a particular fraction would achieve his goal of finding seven equal shares. Thus, I decided to intervene and had John mark one-seventh of one of the miles on the map. During the interaction that followed, John experienced a moment of insight.

Protocol 6.10: Continuation.

D: How many one-sevenths of a mile like this could you make if you split up all of the highway that you have to work with? [Points to the one-seventh mile partition John had made on the map.]
John: Let's see…seven… [Taps on each of the one-mile sections with his marker.] 28. Probably 28 of these.
D: Okay. So if we split up all of the highway like we started to do there, all four of these sections we could get 28
John: One-sevenths.
D: One-sevenths of a mile. Okay. Or twenty-eight-sevenths of a mile.
John: Oh! You could split it between four now. Like four-sevenths now.
D: Say that again. What are you thinking?
John: You could split everything into four-sevenths and they will all get four-sevenths.
D: Say more about that. How did you decide on four?
John: Because if there's seven group and there's 4 miles. I just, I just pretty much…and I times it so that would be 28. So I was like, 4 goes into twenty…oh, how do I say it? It's kind of hard for me to explain… Well, 4 times 7 is 28.

I infer that John's insight, "Oh! You could split it between four now," indicates that he had made a coordination between his partitioning activity and his multiplicative reasoning. His goal throughout the all of the interactions surrounding this task was to reconstitute the 4 miles into seven equal shares. Thus, establishing that splitting each mile into sevenths would produce the fraction 28/7 created a situation of his whole number multiplicative reasoning, which he used to accomplish his splitting goal. Further, this coordination relied upon the commutative nature of John's multiplicative operations. For instance, splitting each of the four one-mile sections into sevenths produced the fraction 28/7 of a mile (i.e., $4 \cdot 7 = 28$) that he reorganized into seven groups of four-sevenths of a mile (i.e., $7 \cdot 4 = 28$). Consequently, in activity John had constructed a strategy for using his available operations to quantify the result of splitting 4 miles into seven parts.

Exploring the Implications of John's Insight

John's insightful use of his multiplicative reasoning to solve this task raised several new questions that I investigated during the remainder of John's Adopt-A-Highway focused teaching sessions. First, while his strategy enabled him to quantify one share as four-sevenths *of 1 mile*, how did he understand this result in relation to the *total 4 miles* given in the original task? Second, would this insight persist as something John could use in assimilating and structuring his activity with different tasks? Further, did he recognize why creating sevenths allowed him to find a solution while two-thirds or three-fifths did not? Lastly, if John did abstract these ways of reasoning as assimilating operations he could use to interpret subsequent tasks, what limitations, if any, might arise as John adapted these ways of reasoning to other tasks?

John constructs each organization's share as a fraction of the total number of miles.

I decided to explore the first of these questions following the interaction described in the continuation of Protocol 6.10 (still during the November 12th teaching session). John initially struggled to interpret four-sevenths as a fraction of the total number of miles. However, the manner in which he overcame his initial uncertainty proved helpful for clarifying my model of

John's mathematics. I have included the entire interaction uninterrupted in order to more clearly portray the progression of his reasoning.

Protocol 6.11: John reinterprets four-sevenths of 1 mile as a fraction of 4 miles.

D: What amount of all the road, of all four of these miles, is one group responsible for?
John: What amount…let me see? This is the amount for seven, but… Four-sevenths… There's seven… [Says these utterances under his breath while thinking for 32 seconds.] Ahh, I don't know. It's kind of hard.
D: What if I…did you have any ideas that you were thinking about?
John: I was thinking about it in a percent, but I was like…never mind.
D: Yeah, you don't need to think about percents. Fractions—let's stick with fractions today.
John: Because fractions and percents are pretty much the same things.
D: Yeah you can turn a fraction into thinking about it as a percent. So we'll just think about fractions today. That's fine. Umm, so, so let's go back to the beginning for a second. So what were we trying to do here with this scenario?
John: Trying to split it up equally among seven groups.
D: Alright. So if we know we're splitting all of this up equally among the seven groups, what amount of the road is each group going to end up responsible for?
John: Well, four-sevenths of a mile.
D: Hmm. So that's four-sevenths of 1 mile. Right? Something about like that. [Uses his fingers to span roughly four-sevenths of one of the one-mile sections on the map.]
John: Um hmm.
D: But what amount of all the 4 miles is that?
John: That's only…let's see. Four miles. That's only…let's see. [Thinks for 14 seconds.] I'm going to make a line. [Draws a horizontal line segment across most of the width of his paper. He then marks one partition. See Figure 6.5.] Four-sevenths. So that's four-sevenths of a mile.

Figure 6.5. John's diagram in progress.

D: Um hmm. What does the line represent to you? What were you thinking about?

John: Like, one group and this represents 4 miles. [Traces his marker along the length of the whole segment.]

D: Okay, okay. So you can keep going. Just kind of tell me what you're thinking as you're going.

John: Well, I'm just trying to change the picture just connecting all of the lines, pretty much. [Points at each of the individual one-mile sections on the original map.]

D: Okay.

John: [Continues making marks on the horizontal line so that he makes six marks/seven partitions.] One, two, three, four, five, six, seven. Okay. That's seven. One, two, three, four, five, six, seven. So… [See Figure 6.6.]

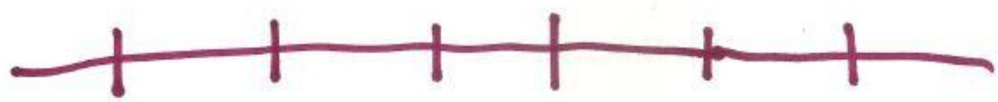

Figure 6.6. John's completed diagram.

D: So what do these different lines represent to you right now?

John: They're just for the group. How much they, umm, let's see. The lines represent the groups. Each is a group.

D: So this is [what]? [Traces his marker along the length of the second partition.]

John: Yeah.

D: And it's what about the group?

John: That's how much they take care of.

D: Okay. And then this whole line represents the 4 miles?

John: Um hmm.

D: Okay. So thinking about that. So what fraction, or what amount, of all 4 miles is each group going to end up taking care of then?

John: Umm. [Traces his finger along the horizontal segment, pausing at each mark.] One-seventh of the 4 miles.

D: Sure! How come?

John: Because they're only taking care of part of it.

D: Um hmm. So how did you know it was one-seventh though?

John: Because it was 4 miles. [Traces his marker along the horizontal segment.] This is the whole 4 miles and this [points to the first partition] is only one. And that's, putting it in a decimal, that's one-seventh—1 over 7. So that was just one-seventh.

D: Sure. Great! So we know that each group is taking care of one-seventh of the 4 miles. And how much of 1 mile is that?

John: Umm…four-sevenths.
D: Four-sevenths. Right. So one-seventh of all 4 of those miles was the four-sevenths of 1 mile.
John: Um hmm.
D: Great!

Accounting for John's solution.

I claim that John's creation of a composite unit to represent the total number of miles represents the critical feature of John's reasoning in this protocol. For example, John first explained, "I was thinking about it in a percent." Given his goal with this scenario of "trying to split it up equally among seven groups," I infer that the percentage John had in mind was a number that would equal 100% if repeated seven times. This is precisely the reasoning John used in the initial interview when sharing two cakes among three people (cf. Protocol 4.8: Continuation). In that case, John only needed three shares, and he could rely upon his number sense to identify the unknown percentage as 33.33% (which he later identified as equivalent to the fraction one-third). However, needing seven shares to equal 100%, John could not intuitively determine the unknown percentage for this task. In addition, the fact that he did not immediately know that the fraction would be one-seventh indicates he was not simply trying to convert a known fraction into the unknown percentage, but rather his goal and reasoning at that point pertained to identifying the unknown percentage. After agreeing to use fractions, we see that John constructed a linear diagram that represented the entire 4 miles, and this enabled him to resolve his constraint.

I infer that John's creation of a single continuous segment to represent the entire 4 miles indicates that he had formed a composite unit for the quantity *total number of miles*. Then, using unmarked partitions to represent the four-sevenths of a mile per organization suggests he formed a secondary composite unit to encapsulate the *number of miles per organization*. I refer to these partitions as "unmarked" in that each partition represented four-sevenths of a mile to John. Yet, neither the sevenths nor the individual miles were marked on his diagram (see Figure 6.6).

Constructing these composite units allowed John to focus explicitly on the relationship of interest while remaining tacitly aware of the structure he had already established. In particular, John's reasoning (and also his diagram in Figure 6.6) foregrounded the composite unit, total number of miles, as a two levels of units structure: a composite 4 miles consisting of seven equivalent shares. At the same time, creating the composite unit for the number of miles per organization enabled John to background each organization's share of four-sevenths of a mile, which was itself a three levels of units structure. This supported his ability to construct each share as one-seventh of the total number of miles on the basis of a part-whole comparison of these composite units while also remaining aware of each share as four-sevenths of 1 mile.

In addition, I believe that thinking about percentages and constructing the single continuous segment served the same purpose for John—to form a united composite whole that could represent the total number of miles without having to explicitly focus on the fact that there were 4 miles. In both cases, John had formed a goal of splitting the composite whole among seven groups. While this proved challenging with percentages, John's splitting operations supported his activity with the continuous segment. Thus, John's ability to reconstruct the sequence of four individual one-mile segments given in the original task as a single composite whole represents the critical aspect of John's reasoning in this protocol.

Contrasting John's solution with the reversible distributive partitioning scheme.

I consider John's use of composite units in this protocol as commensurate with Jack's reasoning during the initial interview (cf. my analysis of Protocols 4.2–4.3). In both cases, the students leveraged composite unit reasoning to establish the desired fractional relationships. In particular, foregrounding the composite unit representing the total number of units supported their activity.

However, I consider this use of composite units to be slightly different than the reasoning entailed in the reversible distributive partitioning scheme. Taking the composite unit for the total number of miles as primary, as John did here, backgrounds the three levels of units structure to create a two levels of unit structure than can be used to identify the desired result. In contrast, the activity of the reversible distributive partitioning scheme takes the composite unit for the number of miles per organization as primary and uses it to reconstitute one three level of units structure as a different, but equivalent, three levels of units structure (cf. Chapter 5, Accounting for the operations that support Jack's simultaneous awareness). Thus, I do not consider John's solution to the task as evidence that he had constructed a reversible distributive partitioning scheme.

John reconstructs these ways of reasoning in a related task.

With the few minutes remaining in the November 12th teaching session, I presented John a slightly different scenario to investigate whether the operations he used in the previous two tasks had persisted as operations he could use to structure his activity. I asked John to imagine that instead of 4 miles we had had 5 miles to allocate amongst seven volunteer organizations. Before I could even finish asking the question, John replied, "Probably, they're taking care of five-sevenths then each." He explained that because there were now 5 miles, he did 5 times seven to get 35 and then knew all seven organizations would get five-sevenths of a mile. In addition, he multiplied five-sevenths by 7 and recognized the resulting thirty-five-sevenths as five (see Figure 6.7). I infer this multiplication referred to repeating five-sevenths of a mile per organization, for seven organizations, to reconstitute the entire 5 miles.

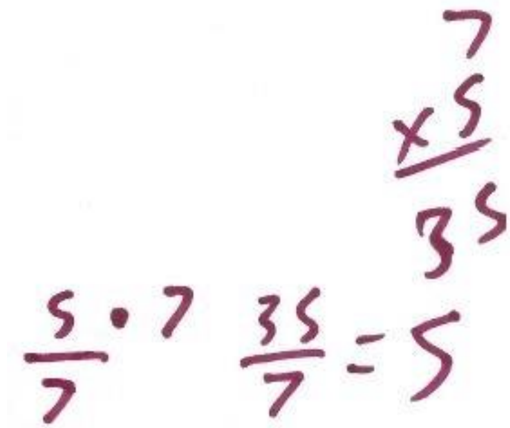

Figure 6.7. John’s written explanation for five-sevenths of a mile per organization.

I also asked John how much of the entire 5 miles each organization would receive. He thought about that for a moment and then replied, “Umm…let’s see. Now, I’m stuck again.” Because the bell had just rung I decided to stop at that point and told John we could come back to the task during the next teaching session. Yet, as I finished talking John explained, “I think it should still be one-seventh. Because it’s…it would still be one-seventh, but it’s just a bigger amount.” While I did not have time to ask John to explain this further, I infer that he used similar reasoning as before; because he was still allocating the highway to seven organizations, he still considered each share as one-seventh of all the miles.

Thus, John had clearly made progress in abstracting the ways of reasoning he used during Protocols 6.10 and 6.11. Whereas the interactions in those protocols lasted a combined 26 minutes, John solved these last two tasks almost immediately with the entire interaction, including his explanations, lasting roughly 3 minutes. However, given the similarity of these tasks to the previous two, as well as their temporal proximity, I consider John’s solution to these tasks with 5 miles for seven organizations as evidence that he had made at minimum a pseudo-reflective abstraction of his ways of operating.

John's solutions to related situations during the next three teaching sessions provided evidence to suggest that he had in fact reflectively abstracted a majority of these ways of operating to the point that they had become available during assimilation of subsequent tasks. In fact, John identified each organization's share as a fraction of 1 mile and as a fraction of all the miles for two related, yet distinct, tasks. Further, while the teacher-researcher and John spent several minutes discussing his strategy and the meaning of the fractional comparisons, in each case he determined the unknown values of the quantities in under 30 seconds. Thus, his activity had become much more anticipatory in nature.

John Successfully Reasons With Composite Units in Subsequent Teaching Sessions

John's ability to form and reason with composite units became a characteristic feature of his mathematics and accounts for his activity in each of the remaining Adopt-A-Highway context tasks. As the teaching experiment progressed, John's reliance upon reasoning with composite units supported his attempts to make sense of new situations. The first of these tasks occurred on November 14, 2013, and involved allocating 5 one-mile sections to eight volunteer organizations (see Figure 6.8). John identified each organization's share as both five-eighths of 1 mile and one-eighth of all the miles. The second task on November 14th involved allocating a continuous three-mile section to five volunteer organizations (see Figure 6.9). In this case, John identified each organization's share as three-fifths of 1 mile as one-fifth of 3 miles. In both tasks, I infer that John constructed two composite units that featured prominently in his reasoning and explanations—a single composite whole representing the total number of miles and a second composite unit representing each organization's share of 1 mile.

Figure 6.8. The map provided for the task of allocating 5 miles to eight organizations.

Figure 6.9. The map provided for the task of allocating 3 miles to five organizations.

A change emerges in John's method for quantifying unit ratios in this context.

I say that John abstracted *a majority* of his ways of operating, rather than *all*, because one difference emerged in the way John solved these types of tasks. Specifically, while John used a consistent strategy to explain and justify his results, the manner in which he quantified each organization's fractional amount of 1 mile changed. Previously, John relied upon whole number multiplicative reasoning with the total number of parts produced by partitioning each unit in order to determine four-sevenths and five-sevenths miles per organization. However, John never used that way of reasoning again while explaining his solutions to related tasks. Rather, John's quantification of the number of miles per organization became much more intuitive. Often, he could not explain the source of his intuition yet reasoned with the results quite productively.

Two excerpts in particular provide insight regarding the reasoning John used to construct his awareness of the fractional relationships. The first is from November 14, 2013, when I asked John how he decided upon five-eighths of a mile per organization.

Protocol 6.12: John explains how he determined five-eighths of mile per organization.

D: How did you decide that it would be five-eighths?
John: Well, there's eight groups and there's 5 miles. I just put five on the top and eight on the bottom. And…
D: So how'd you know to do that—to think of it as five-eighths? Like what, what were you thinking about with the miles?
John: Well, I was trying to make it even. I was thinking of making it even using a fraction. So I tried five-eighths because…it was eight groups and…let me see… They're like, they're like the *x* and the *y*—they're variable.

I interpret John's explanations as suggesting that he had formed a pattern, *number of miles* / *number of organizations*, that he used to quantify the unknown number of miles as a fraction of 1 mile for each organization.[20] I consider his reply, "They're like the *x* and the *y*—they're variable," to mean that as the values of each quantity changed, John adjusted the fraction accordingly. My inference is that he abstracted this pattern from considering his previous solutions (four-sevenths and five-sevenths) in relation to their respective tasks (4 and 5 miles split among seven organizations). Then, to accomplish "making it even using a fraction," John decided upon five-eighths of a mile per organization. Had John used the same strategy as before, I would have expected him to mention something similar to $5 \cdot 8 = 40$ and how he could allocate

[20] I use the "/" in the identified pattern to refer to a fraction, not division.

all forty-eighths by giving five-eighths to each of the eight organizations. Instead, John explained that he decided to "just put five on the top and eight on the bottom."

As a result, rather than producing five-eighths by using his quantitative operations to transform the 5 miles to five-eighths of a mile, I infer that John determined that the solution had to be five-eighths because that fit the pattern of the results from previous tasks. I was confident that this pattern-based reasoning replaced the quantitative operating he previously carried out to establish the values of unit ratios. However, at this point in the experiment it remained unclear if this pattern symbolized those operations and John simply no longer needed to enact them but could if asked.

Investigating John's pattern-based reasoning.

John uses his pattern in assimilating related, yet novel, tasks.

The second excerpt that helps to clarify the reasoning John used to construct his awareness of the fractional relationships occurred on November 19, 2013. For this task, I gave John a map with a four-mile section, a three-mile section, and a one-mile section to allocate among 11 volunteer organizations (see Figure 6.10). John thought for 5 seconds before saying, "That's 8 miles, 11 groups. Let's see. They're responsible for eight-elevenths of a mile." The following protocol includes the discussion that ensued after John produced this result.

Figure 6.10. The map provided for the task of allocating a four-mile, a three-mile, and a one-mile section to eight organizations.

Protocol 6.13: John explains why the result had to be eight-elevenths of a mile.

D: So can you tell me how you decided that? How did you figure that out?
John: Umm, I made a diagram and I put—well, in my head.
D: Is it something you can show?
John: Yeah, but it's just the same diagram. [John draws a line across his paper and partitions it into eight parts. Then he partitions the first part into subparts.] Each group is going to be responsible for this much. [John makes a larger partition to separate a group consisting of the first eight sub-parts. See Figure 6.11.]

Figure 6.11. The diagram John created to explain eight-elevenths of a mile per organization.

D: Okay. How many parts did you make there when you were doing these small ones?
John: 11. [John actually made 12 sub-parts, but here, and again later, he describes intending to make 11 sub-parts.]

D: So what were you thinking about with that?
John: Well, it says there's 11 groups. I was trying, let's see. I just…I did 8 over 11 and that's where I got the 11 from this part. That's pretty much it.
D: Okay. How did you know that this would be eight of them then? [Points to the group of 8/11 he had indicated for one group.] Like eight of these sections. Like, what were you thinking about with that?
John: I was thinking that it should be less than 1 though. It should be less than 1 mile for each.
D: Why does that make sense?
John: Because there's only 8 miles and there's 11 groups.
D: Sure. So why not something like nine though? How did you know it was eight-elevenths and not, like, nine-elevenths?
John: Well. That would be a little too big. Because, umm, how do I explain this? […] [Writes "9/11 + 9/11".] If you plus it eleven times is going to be 99 over 11, and that equals 9 miles.

This excerpt provides additional evidence to support my characterization of John's construction of the unit ratio in Protocol 6.12 as pattern-based. For example, consider John's explanation for why he thought about making elevenths: "Well, it says there's 11 groups. […] I did 8 over 11 and that's where I got the 11 from this part." This is consistent with my inference that John had formed a pattern that he used to quantify the number of miles per organization. In this case, 8 miles for 11 organizations implied 8 over 11, which implied partitioning the first mile into 11 parts.

Accounting for John's conviction in the results of his pattern-based reasoning.

While I infer that John used pattern-based reasoning to decide upon five-eighths of a mile per organization, I also believe that as soon as he decided upon this fraction he was convinced it must be correct; John never expressed doubt that it might be something other than five-eighths of a mile per organization. For example, consider John's response to my suggestion of nine-elevenths. John explained this could not be the case because nine iterations of nine-elevenths would produce 9 miles in total. Thus, nine-elevenths "would be a little too big."

I hypothesize that John's sense of conviction stems from the iterability of his composite fractions. This enabled John to anticipate that 11 iterations of the composite five-eighths of a mile would comprise 8 miles. The fact that John did not actually have to carry out the iterating in order to be convinced of his results suggests that he had these operations available during his assimilation of the task. In this sense, I consider the fractional amounts he produced to quantify the number of miles per organization as necessary results for John, where the necessity stemmed from assimilating the fractional amount per organization as an iterable composite fraction.

John leverages his iterable composite units to reconstitute the total number of miles.

In addition to supporting John's conviction in his unit ratios, constructing the number of miles per organization as an iterable composite fraction enabled John to reconstitute the relationships and identify each organization's share as a fractional amount of the total number of miles. For example, consider John's reasoning with the task of allocating a three-mile section to five volunteer organizations. Immediately prior to this task, I introduced John to the JavaBars (Biddlecomb & Olive, 2000) computer program so that he had a context in which he could carry out mental operations such as partitioning or iterating. John first partitioned 1 mile into five parts, pulled out three of those parts, and stated that each organization would receive three-fifths of 1 mile. Next, when I questioned John about this fraction in relation to the total 3 miles, he stated that each share of three-fifths of a mile could be thought of as either three-fifteenths of the whole or as one-fifth of the whole. The following protocol includes his explanation for how he knew this.

Protocol 6.14: John justifies three-fifths of 1 mile as one-fifth of 3 miles.

D: Why do you say one-fifth?
John: Well, it's…umm…you could put three of these, I mean you could put five of these [moves his mouse over the pulled out three-fifths] to make the whole thing. So it's one-fifth.
D: You want to show me what you're thinking about when you said that? I'm wondering if you could show me what you meant there.
John: [John moves the pulled-out three-fifths above the original bar representing the 3 miles. He then makes copies of the fifths and places another three-fifths above the

original three-mile bar and changes the color of this new three-fifths. See Figure 6.12.]

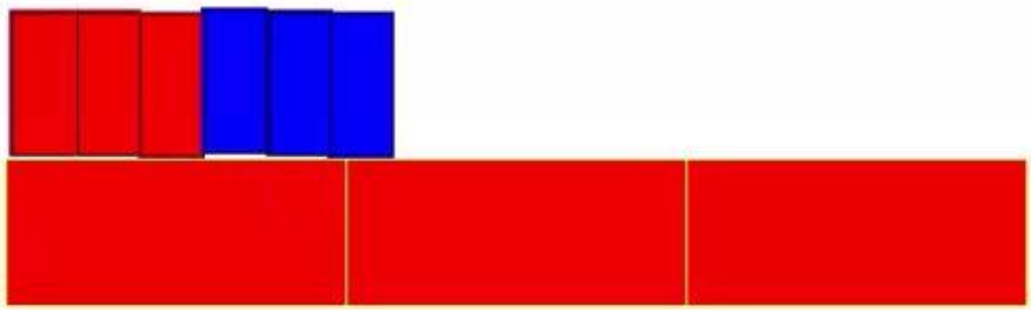

Figure 6.12. A screen capture of what John created in JavaBars (Biddlecomb & Olive, 2000).

D: So why did you make those a different color?
John: Because this is one for each group and this is a different group. And if you add on more it's going to be equal. It's going to be the same amount.
D: So how many groups like that could you make?
John: Five.
D: Five. So then each one of those groups would be how much of the whole?
John: One-fifth of the whole.
D: One-fifth of the whole. Good. And then, so we've got one-fifth of the whole and then how much of a mile is one group going to get then?
John: How much of a mile?
D: Um hmm. How much of 1 mile?
John: Oh. Three-fifths of 1 mile.

John's reasoning in this excerpt exemplifies how he could leverage the iterability of his composite fractions to reinterpret the number of miles per organization as a fractional amount of the total number of miles. John's immediate answer to my question was that three-fifths of a mile was equivalent to one-fifth of 3 miles because "you could put five of these to make the whole thing." Later, producing another copy of three-fifths and changing its color provided a clear visual representation of the mental coordination that I infer John had already made prior to carrying out these actions in JavaBars (Biddlecomb & Olive, 2000). Thus, John's operations with iterable composite fractions supported his ability to understand each organization's share as both three-fifths of 1 mile and one-fifth of 3 miles.

A model of John's characteristic ways reasoning in the Adopt-A-Highway context.

Considering John's activity more generally, my model of the reasoning he used to construct these fractional relationships involves three components. First, pattern-based reasoning accounts for John's production of unit ratios for the number of miles per organization. Second, he consistently assimilated the total number of miles as a single composite whole. Lastly, John's construction of each share as an iterable composite fraction supported his ability to understand one share simultaneously as a fraction of 1 mile and as a fraction of the total number of miles. I found that this model accurately characterized John's reasoning throughout a range of tasks and teaching sessions in the Adopt-A-Highway context. Thus, while I have chosen three particular excerpts for Protocols 6.12–6.14 that I felt best demonstrated these components of my model, the model provides a characterization of John's typical ways of operating with tasks in this context.

Next, I provide some brief elaborations of each of these three components to clarify several details of how I intend this model. Regarding the first component, I believe that an association between the specifics of the task and the fractional result defined the pattern that John used to quantify the unit ratios. Further, I found no evidence that the pattern symbolized the ways of reasoning John constructed to solve the task in Protocol 6.10. As a result, I infer that this association did not arise as an abstraction from using quantitative operations to transform the total number of miles to an amount for each organization, and I do not consider the pattern to represent quantitative reasoning. However, I also believe that John's use of quantitative schemes and operations to make sense of the unit ratios he produced as results of this pattern accounts for his conviction that the pattern worked.

Concerning the second component, John consistently assimilated the total number of miles as a single composite whole. The tasks within this context presented the units as separated one-mile sections, as continuous multiple-mile sections, or as some combination thereof. Yet, regardless of how we presented the units on the maps, John's success with each task can be attributed in part to the fact that he conceptually united all of the miles into a single composite whole that he could use in further reasoning.

Lastly, John's construction of the number of miles per organization as an iterable composite fraction served a dual purpose in his reasoning. First, knowing that he could iterate the fraction once for each organization provided a sense of necessity to the unit ratios John produced using his pattern. Second, his anticipation that this potential iteration would reconstitute the total number of miles allowed him to interpret each share as a fractional amount of this total length. Furthermore, this second use of his composite fractions is compatible with the reasoning involved in reversible distributive partitioning scheme.

Lastly, I wish to conclude this section by highlighting the distinction I observed between John's quantification of the unit ratios and his use of those ratios in further reasoning. I characterized the former as pattern-based reasoning and the latter as quantitative reasoning. Admittedly, this distinction is somewhat blurry as I have previously discussed how John's operations with iterable units supported the meaning he attributed to the unit ratios he generated. Yet, while the distinction is subtle, I believe it is also quite important. Later in the teaching experiment, John experienced a constraint that I believe highlights a limitation to the use of pattern-based reasoning. However, within the Adopt-A-Highway context, where each of the tasks had similar structures and foci, the fact that John based his formation of the unit ratios on a pattern posed no issues. On the contrary, as the above protocols indicate, he leveraged his abstracted pattern in combination with his quantitative operations quite powerfully to solve a range of tasks. Characterizing his reasoning in this fashion enabled me to create a consistent and explanatory account of John's mathematics.

The Inch Worm Context

Within the inch worm context, I learned relatively less about the particular operations that John could take as givens in reasoning than I did in the swimming pool and Adopt-A-Highway contexts. However, I learned more about the ways in which John could use those operations to understand and reason successfully with situations involving constant covariational relationships. As with Jack, the primary emphasis of the tasks within this context transitioned away from focusing on quantifying unit ratios to allow more time for tasks that provided the students opportunities to operate creatively within a wide range of novel contexts. This approach enabled me to investigate the interplay between how John leveraged his available quantitative operations and the meanings he constructed for the quantities and the relationships among them.

As a result, the focus for my presentation of the results shifts slightly in this context. I include some excerpts that provide additional evidence to support previous claims regarding my model of John's mathematics. However, the data included in this section focus primarily upon excerpts from the teaching sessions that reveal new aspects of John's ability to use his quantitative operations in novel situations, and I emphasize the affordances and constraints of his ways of reasoning.

John's initial exposure to the inch worm context involved a variety of questions that focused on the concept of crawling speed. In previous contexts, the tasks typically started with providing measurements of the extensive quantities and investigating how John could operate upon them to construct and make sense of an intensive quantity. However, while this proved useful for learning about John's quantification of unit ratios and intensive quantities, all decisions about which quantities and measurements would be useful were made for John by virtue of the task setup. Thus, to develop a better sense of how John independently thought about the quantity crawling speed, I designed a series of tasks using dynamic computer animations of inch worm races. These animations allowed me to ask John questions about speed while leaving the decisions to him regarding which quantities and measurements would be useful.

John's Conceptualization of Quantitative Covariation

John assimilates crawling speed as a dynamic relationship.

John's first teaching session in this context, which occurred on January 21, 2014, provided some evidence regarding how he conceptualized quantitative covariation. For example, the first task involved identifying the faster inch worm, Flash or Speedy, based upon an animation of their race (see Figure 6.13 for screen shots from the start and end of the animation). John quickly decided that Flash was faster and explained, "Because Speedy started first, but Flash is catching up. So Flash has got to be going faster because if you keep letting them go Flash will eventually pass Speedy." From this I infer that John created a mental re-presentation of the situation and reasoned about the race as it progressed. Describing Flash as *catching up* and stating he *will eventually pass* Speedy suggests that, at minimum, John actively monitored the distance between the two and could anticipate how this quantity would change if the race were to continue.

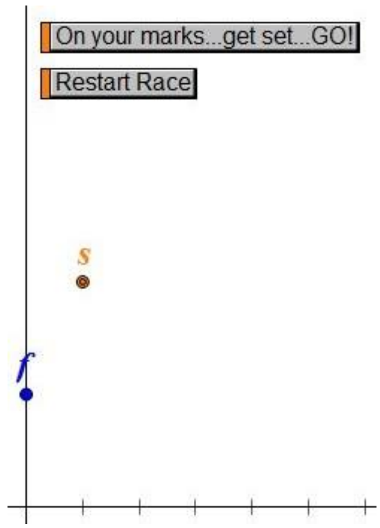

Figure 6.13a. Screen shot of the start of the race animation.

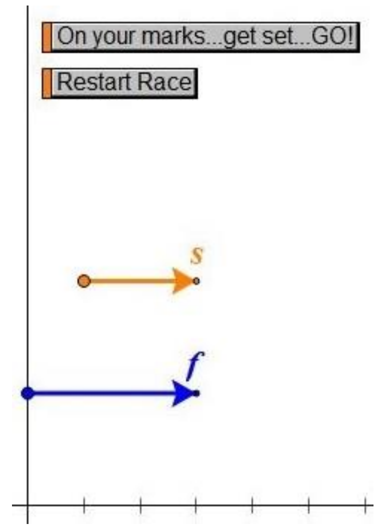

Figure 6.13b. Screen shot of the end of the race animation.

John's explanation during the subsequent task clarifies how he conceptualized the quantities while actively monitoring the race in progress. For this next task, I asked what kind of things he would need to know to figure out Flash's speed. John provided two responses: a) "Let's see. You would need the time in seconds per centimeter"; and b) "The distance traveled per second, or like per minute since they're traveling kind of slow." Thus, more than monitoring the distance between the two inch worms as the race progressed, I infer that John focused on both changes in time (e.g., "if you keep letting them go") and the distance between the inch worms (e.g., "Flash will eventually pass Speedy"). John's replies to these tasks indicate that his assimilation of the tasks and his conception of crawling speed involved a dynamic relationship between elapsed time and distance crawled.

Given John's conception of crawling speeds as relationships, I transitioned to tasks that involved reasoning about crawling speeds using measurements of the accumulations of distance and time. For the next teaching session on January 23, 2014, the tasks focused on comparing the crawling speeds of two different inch worms, Abby and Matt. Both John and Jack were present for this teaching session. Thus, I provided each student with a computer showing an animation for one of the inch worms and a stopwatch for collecting measurements from the race animation. After John measured that Abby took 10 seconds to travel 4 centimeters and Jack measured that Matt took 21 seconds to travel 7 centimeters, I asked the students to compare the inch worms' crawling speeds based upon the one measurement they had collected for each inch worm.

John's initial reaction to this task provides additional evidence regarding how he conceptualized the covariation of distance and time. As soon as I asked the question, John voiced concern that he only had one measurement from each inch worm's race. For instance, he stated, "It depends if they're moving [at a] constant speed or not" and compared the situation to real life where the inch worms might get tired and slow down or change their speeds throughout the race. Based upon responses such as this, I infer that John conceptualized the covariation of distance and time as a dynamic process of accumulation. Thus, having collected a single specific measurement from the race provided no guarantee that the accumulated time and distance occurred as a result of traveling at a constant speed the entire trip.

Similar to his comparison of Flash and Speedy's races, I claim that John considered the covariation dynamically as the race progressed rather than only focusing on the completed changes indicated by the measurements. My warrant for this statement comes from considering the alternative—if John had not considered the covariation as a dynamic process, he would have no basis for his awareness of the possibility of a non-constant speed. This conception of covariation as a dynamic process is related to considering the instantaneous rate of change over the course of the given interval.

Accounting for John's conception of the dynamic process of accumulation.

I hypothesize that John's ability to conceive of the covariation dynamically as the race progressed both supports, and is supported by, the quantitative operations he used to make sense of constant speeds. The reasoning he used to solve a sequence of tasks during the January 28, 2014, teaching session helps to illustrate this aspect of his mathematics. Because John had expressed concern over whether or not the inch worms traveled at constant speeds, I developed tasks that would allow me to investigate how he would use his available quantitative operations to identify a situation as indicating a constant speed. I provided an animation for Abby's race that included variable measures that tracked Abby's total duration and distance. John had the ability to start, stop, and restart the race as he saw fit to help him solve the tasks.

John leverages his iterable units to decide that Abby traveled at a constant speed.

Protocol 6.15: John makes sense of Abby's time trial race.

D: And what I'd like you to do is take some measurements, whatever measurements that you would need, so that you could decide, do you think that Abby is going at a constant speed throughout the race?

John: Okay. [Starts the race and stops it after Abby traveled roughly 1 centimeter. The actual measurement is 1.02 centimeters and 2.55 seconds.]

D: Now we might need to do a little bit of rounding like last time. So since this is a little over one, a little over two and a half—so maybe we call that like 1 centimeter and 2 ½ seconds.

John: Okay. [Records that information on his paper.]

D: Kind of like with the stopwatch. If we stop it a little bit after, we round it a little bit.

John: [He then repeats the process of restarting the race and collects three additional measurements. See Figure 6.14.] Okay.

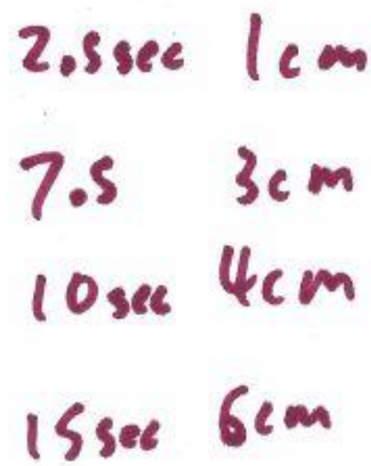

Figure 6.14. The measurements John collected to decide if Abby was traveling a constant speed.

D: So what do you think?

John: It's going pretty much a constant speed.

D: So how did you decide?

John: Well, I times that by 3 [points at "2.5 sec"], you get 7.5. And if I go there [points at "10 sec"], which is only 1 centimeter adding 2.5, that's 10 seconds. So it's going up pretty much 2.5 for 1 centimeters. Or close to around there. [Thinks for 15 seconds.]

D: Okay. Do you have more that you were thinking about there?

John: Hmm, no.

D: So how can you tell by looking at this? Back here we had this measurement at 4 centimeters was 10 seconds. How can you tell that that's the same speed as this one here [points at "2.5 sec, 1 cm"] or any of these other ones?

John: Because it matches up with this [points at "10 sec, 4 cm"].

D: What do you mean by it matches up?

John: Like, at 10 seconds they both stop at 4 centimeters. So it means they're pretty much going the same speed.

I claim that John's construction of iterable units, combined with his ability to conceive of the covariation of the quantities as the race progressed, accounts for his decision that Abby traveled at a constant speed. For instance, notice that John first measured the duration of time it took Abby to travel a distance of 1 centimeter. This is consistent with John's assimilation of

Flash's and Speedy's race in which he identified finding the number of seconds per centimeter as a strategy for measuring crawling speed.

In addition, recall that in both the swimming pool and the Adopt-A-Highway contexts John demonstrated the ability to construct the extensive quantities in his unit ratios as iterable composite units that he could take as given in further reasoning. For example, in the swimming pool context John constructed the unit ratio 0.6 inches per minute. Then, as part of his reasoning in Protocol 6.4, he assimilated 4 minutes as a duration 4 times as long as 1 minute and, thus, decided to iterate 0.6 inches four times to produce the desired result of 2.4 inches per 4 minutes. In doing so, John provided evidence that he had constructed 0.6 as a quantity that characterized the covariational relationship more generally. For the present discussion, I refer to ratios such as John's 0.6 inches per minute as *iterable unit ratios* to highlight the fact that the iterability of the extensive quantities guided John's use of the unit ratio in further reasoning.

In Protocol 6.15, I infer that John similarly constructed 2.5 seconds per cm as an iterable unit ratio. As such, John could anticipate the race continuing beyond 1 centimeter with iterations of the extensive quantities in his unit ratio defining the accruals of the quantities as the race progressed. John's explanation supports this inference. For example, he decided that Abby maintained a constant speed by verifying that three iterations of this unit ratio (i.e., "I times that by 3") matched the measurement he collected. Likewise, uniting this result with another iteration of the unit ratio matched his 4 centimeter measurement. As a result, when John said, "At 10 seconds they both stop at 4 centimeters," I infer that the *both* he had in mind was imagining the quantities continuing to accumulate at a rate of 2.5 seconds per cm compared to actually running the animation for 4 centimeters.

***John reconstitutes his iterable unit ratio in terms of new measurement units*.**

Following this task, I decided to further investigate John's understanding of constant covariational relationships. John's use of 2.5 seconds per centimeter as an iterable unit ratio in Protocol 6.15 only involved imagining the race progressing beyond 1 centimeter and accumulating 1 centimeter at a time. Thus, it remained unclear how John might use his conception of a constant speed to reason about distances, and changes in distance, smaller than 1 centimeter. Thus, I asked John to predict how long it would take Abby to crawl one-tenth of a centimeter. The protocol continues with John's solution to this task and his interpretation of the result.

Protocol 6.15: First continuation.

John: [On his paper her computes the long division of 2.5 divided by 10.] 0.25 seconds.
D: 0.25 seconds?
John: I think.
D: Okay. So she went for 0.25 seconds to go 0.1 centimeters.
John: Um hmm.
D: Could she still say that she's going at the same speed of 2.5 seconds per centimeter?
John: Yeah. Because it's just a smaller version of this. [Points at the "2.5 sec, 1 cm" measurement on his paper.] You're just making this even smaller.

I claim that John employed the same reasoning and used the same quantitative operations both here and in the previous excerpt with one difference—in this case he transformed the measurement unit. For example, notice that John assimilated finding the time for one-tenth of a centimeter as a situation of division. In terms of my model for John's division, dividing by 10 would accomplish his goal of splitting 2.5 seconds into 10 parts. Further, because the result accomplished a splitting goal, John would anticipate that the result could be iterated 10 times to reconstitute the 2.5 seconds. Thus, as a situation of what I have named John's unit ratio division scheme, carrying out the long division activity produced a "unit" ratio where the unit in this case is per one-tenth of a centimeter rather than per centimeter.

I include a brief digression here to clarify my use of notation and terminology. When using the notation unit ratio I am referring to situations in which a quantity is measured *per one unit* of the concomitant quantity. In contrast, I use the notation, “unit” ratio, to indicate that I inferred that the same operations one uses to produce unit ratios were used to produce a new ratio in which a quantity is measured *per something other than one unit* of the concomitant quantity. Because they are produced by the same quantitative operations, I consider a “unit” ratio to have the same properties as a unit ratio but with the defining characteristic of having a measurement unit other than one. I introduce this language in an attempt to better convey my intentions.

Returning to the first continuation of protocol 6.15, my analysis of this excerpt can account for why John considered traveling for 0.25 seconds to go 0.1 centimeters as still representing a speed of 2.5 seconds per centimeter. To justify the constancy of the speed John stated, “Because it’s just a smaller version of this” and pointed to the “2.5 sec, 1 cm” measurement on his paper. I infer that having constructed a new iterable “unit” ratio with a smaller measurement unit, the same operations John used to verify that 2.5 seconds per centimeters produced the same total accumulations as 10 seconds per 4 centimeters enabled him to anticipate that 0.25 seconds per 0.1 centimeters would produce the same total accumulations as 2.5 seconds per centimeter. Hence, they represented the same speed.

To test these hypotheses about John’s reasoning, I decided to ask John to consider measuring Abby’s speed in a different unit. Rather than measuring seconds per centimeter, as we had been doing, I asked John if he could measure her crawling speed in centimeters per second.

Protocol 6.15: Second continuation.

D: What would be her speed in centimeters per second?
John: Okay. One second.

D: Take your time. You can write anything you need to or if you want to make any drawings to think about it.

John: Let's see… [Thinks for 18 seconds.] Wait, it would be 0.4 centimeters.

D: Do you want to describe what you were thinking about?

John: Let me see. One second. 0.4 centimeters. [Writes those on his paper while talking.] Because it's pretty much the same thing like that [points to his long division of 2.5 divided by 10.] Because that's 0.1 [centimeter], which that's for going one-fourth of a second. I just timesd [*sic*] it by 4, like the distance.

D: Oh, okay. So you took this one over here which you knew was 0.25 seconds per one-tenth of a centimeter.

John: Um hmm. And then timesd [*sic*] it by 4.

D: Okay. [Points to the long division.] What if you hadn't done this before? Could you figure that out from here? [Points to the measurement of 10 seconds and 4 centimeters.]

John: Well…let's see. [Set's up and completes the long division of 4 divided by 10.] It's still 0.4.

John's two strategies for quantifying Abby's crawling speed in centimeters per second provide additional evidence to support my claims. First, John multiplied both extensive quantities of his previous iterable "unit" ratio, 0.25 seconds per 0.1 centimeters, by 4 to determine that Abby could crawl 0.4 centimeters in 1 second. I infer this multiplication symbolized uniting four iterations of his "unit" ratio. Thus, this activity supports my inference that 0.25 seconds per 0.1 centimeters was constructed as an iterable unit ratio. Hence, John could anticipate that iterating 0.25 seconds per 0.1 centimeters would produce the same total accumulations as 2.5 seconds per centimeter and as 0.4 centimeters per second. John's ability to construct iterable composite units enables him to produce new iterable "unit" ratios with larger measurement units. In addition, John's second strategy provides another indication that John could assimilate the measurements he collected from the animation as situations of his unit ratio division scheme. As a result, John could also produce new iterable "unit" ratios with smaller measurement units.

Reconsidering my hypothesis in light of John's activity.

Before moving on to consider John's activity in the remaining teaching sessions, I return to discuss my hypothesis about John's conception of covariation in relation to the entirety of his activity in Protocol 6.15 and its continuations. Earlier I claimed that John's ability to conceive of the covariation dynamically as the race progressed both supports, and is supported by, the quantitative operations he used to make sense of constant speeds. Based upon my analyses of protocol 6.15 and similar interactions I had with John in related tasks, I infer that John leveraged iterable composite units and his unit ratio division scheme to make sense of constant covariational situations. In particular, by assimilating the measurements of the extensive quantities as iterable units, John constructed his unit ratios as iterable unit ratios. This enabled him to conceive of the accumulation of the quantities as the covariation progressed beyond the measurement unit.

Further, using his available conceptual operations, John could quantify new iterable "unit" ratios with either bigger or smaller measurement units. Thus, John's conception of the accumulation of the quantities was not constrained to any particular measurement unit. I infer that this aspect of his reasoning supported his ability to conceive of the covariation dynamically as the race progressed. Reciprocally, being able to conceive of the covariation in progress supported John's awareness that within a particular duration, the speed may not remain constant as the duration elapses. In general, the question of which is primary, the quantitative operations or the ability to conceive of the quantitative covariation dynamically as the race/trip progresses, remains an open question for me.[21] My hypothesis is that they co-emerge and open new constructive pathways for an individual.

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[21] I use "quantitative" covariation to refer to the ability to not only conceive of the covariation but also to operate flexibly with measures for the values of the quantities as the covariation progresses.

John's Mathematical Activity With Graphical Representations

I found the above characterizations of John's reasoning useful when accounting for his ability to construct and reason with graphical representations of speed. John's responses during the February 6, 2014, and February 25, 2014, teaching sessions were characteristic of his graphing activity during the teaching experiment. For one of the tasks on February 6th I told John that an inch worm named Al had a crawling speed of seven-thirds seconds per centimeter and asked him to make a graph that represented Al's speed. I anticipated he would make a graph explicitly about the quantities time and distance, and I was interested in how John would construct and interpret a graph that represented the covariational relationship between these two quantities (i.e., the inch worm's crawling speed).

John creates a graphical representation of a crawling speed.

Protocol 6.16: John creates a graph that stands for a speed of seven-thirds seconds per centimeter.

D: [Hands John a blank set of axes.] I want you to think about using Al's speed. So he's got the speed of seven-thirds seconds per centimeter, and I'd like you to try to make a graph that would stand for that speed.

John: Okay.

D: So a graph that would stand for Al's speed.

John: Let's see. [Moves his marker along the horizontal axis.] Distance…and this could be time. [Moves his marker towards the vertical axis. Then labels the vertical axis as "Time" and the horizontal axis as "Distance".]

D: How'd you decide to put the time over there [on the vertical axis] and the distance on the bottom?

John: Well, actually you could do it any way. I think you could use any…like you could change it. But I probably just might like the time going up because I'm going to need more [space] to put seconds. Like 1 second, 2 seconds, 3 seconds, 4 seconds…

D: Okay. Well why don't you go ahead and start making it and tell me how you're thinking about it.

John: [Creates a scale on both axes by placing equally spaced tick marks. Then labels each mark on the Time axis with consecutive numerals up to 14.] Hmm, on this you could use any kind of number you want. Pretty much. I'll just go with ones because it's easier. [Labels the marks on his horizontal axis with consecutive numerals up to 11.]

D: Okay.

John: Let's see. [Places his marker at a point on the graph that is roughly (1 centimeter, 7/3 seconds) and then connects from that point to the origin with a linear segment and makes a point at the end.] [Inaudible] One. And let's see… [Places another point on the graph at roughly (2 centimeters, 14/3 seconds) and connects from that point to his first with another linear segment.] And it's just pretty much going to go in a straight line. [Motions with his marker as if to continue the line in the same direction.]

D: Okay. Do you want to draw that?

John: Draw it?

D: Or just try to approximate that?

John: Hmm…probably like that. [Extends the line, corrects this line to make it go straighter, and then puts an arrow on the end. See Figure 6.15.]

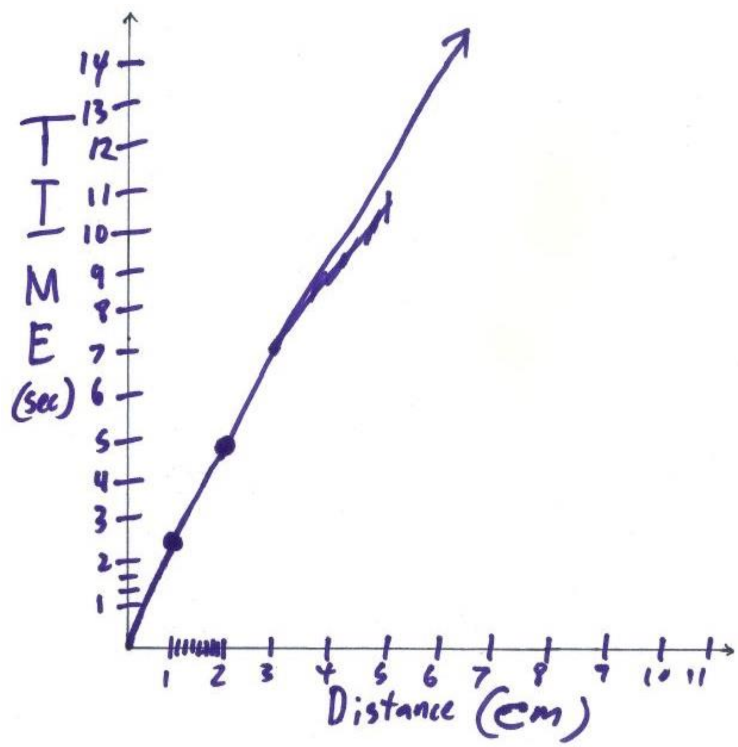

Figure 6.15. The graph John created to stand for a speed of seven-thirds seconds per centimeter.

Several aspects of John's activity in this excerpt stand out. First, John's comment, "Actually you could do it any way," suggests that he viewed the placement of the quantities, time and distance, on the axes as arbitrary. Further, the fact that he placed time on the vertical axis and distance on the horizontal axis suggests that the graph he produced was not simply an enactment of routines he learned in school where time is most commonly represented on the horizontal axis. Second, the manner in which John produced the graph is consistent with my model of his iterable unit ratios. John plotted points for one and two iterations of the unit ratio before connecting them with a straight line. Lastly, when choosing a scale for the horizontal distance axis John said, "Hmm, on this you could use any kind of number you want. Pretty much. I'll just go with ones because it's easier." To me this indicates that even though John constructed his graph by accumulating the quantities in iterations of the given unit ratio seven-thirds seconds per centimeter, he was aware that this was not the only way to construct the graph. Unfortunately, this comment did not stand out to me during the teaching session, and so I never investigated what John meant by it. However, I infer that had John wanted to, he could have constructed a new "unit" ratio and used iterations of it to construct the same graph.

Accounting for John's intuition to represent the constant speed as a linear graph.

It is also noteworthy that John knew that he should connect the points he plotted with straight lines. However, the source of that intuition is not clear from the excerpt above. In order to develop a model of the reasoning that supports this intuition, I present two excerpts back-to-back and then include my analysis of both protocols afterwards. The first interaction followed immediately after John produced the graph above.

Protocol 6.16: Continuation.

D: Why do you think it's going to be a straight line?
John: Because it's a constant speed.
D: Okay.
John: Constant speed is a straight line.
D: So if someone else looked at this, how would they know that this graph was standing for that speed [of seven-thirds seconds per centimeter]?

John: Well they could look at this right here. [Darkens the first two points he made at 1 and 2 centimeters while constructing the graph.] Like the points where I put them at and the distance.

D: Okay.

W: How'd you make that first point on the graph?

John: Well, because this is seconds [Moves his marker up vertically.] I'd go 1, 2, and one-third of it and then just go over.

D: Um hmm. So you could look at this point and use it to figure out that this was a speed of seven-thirds seconds per centimeter?

John: Basically just keep going up.

The second relevant interaction occurred during John's next teaching session on February 25, 2014. For this task, I asked John to try to make sense of Jackie's crawling speed based upon interpreting a prepared graph of her race measurements (see Figure 6.16). Prior to this excerpt, he had compared changes in distance and time to justify that the second, fourth, and sixth linear segments (reading the graph from left to right) indicated faster crawling speeds than the other linear sections.

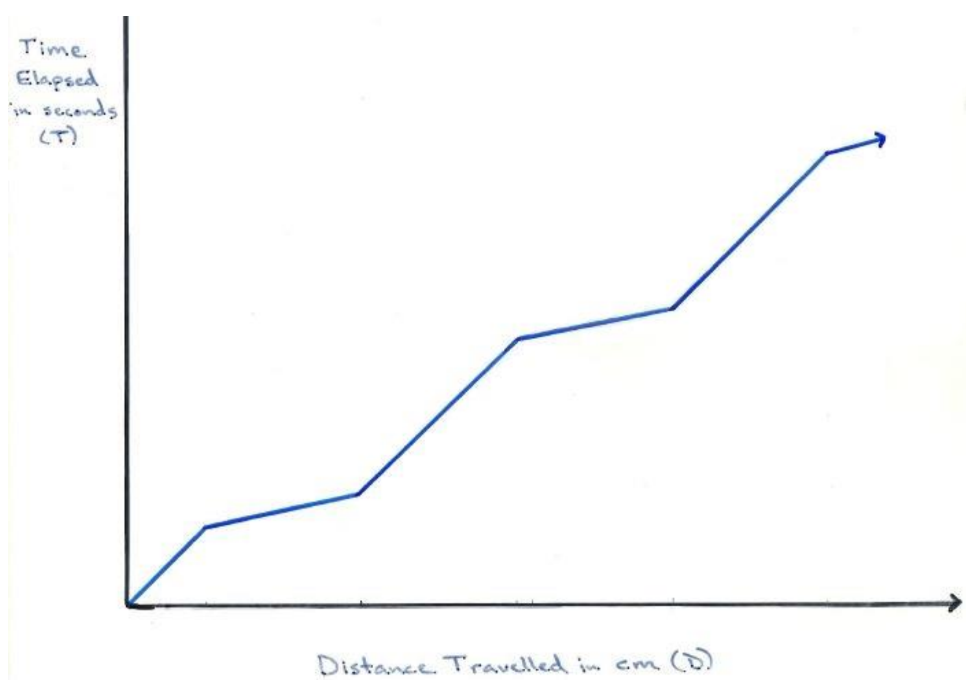

Figure 6.16. The graph of Jackie's race measurements.

Protocol 6.17: John explains his rationale for why a linear graph represented a constant speed.

D: So point to one of the sections that you think is where Jackie's going faster than the others. [Points to the 2nd and 4th linear segments.] Okay. So say this second section here. So it's faster than the first section, but is Jackie going at a constant speed or is her speed changing in this section?

John: Well…like if you just think you, well—like in science. It's like a straight line is a constant speed when we're doing the speed, time, distance. And when it's curving [traces a curve with his finger that curves upward], it's acceleration. And when it's curving [traces a curve with his finger that curves downward], it's deceleration.

D: So when you look at this…

John: I kind of match it up to science.

D: So we can think about it comparing it to, like, a science graph.

John: Yeah.

D: Can you just tell from looking at the graph though? Like, could you make sense and justify it for yourself based on this graph without having to compare it to science class?

John: Like…how is it going constant speed?

D: Yeah, like how do the people in science know that a straight line indicates constant speed?

John: Hmm. Because…let's see. [Thinks for 7 seconds.] It's like, just like, graphing. Like, how do you say? Like up two and there. [With his finger traces up two imagined units vertically and then over horizontally some amount. Then repeats this motion.] And, like, you know how, like the slope. [Traces his finger up and over again.]

My inference is that John's intuition that he should draw a linear graph for Al's speed of seven-thirds seconds per centimeter stemmed from a combination of his quantitative reasoning and things he had learned from school experiences. Both excerpts suggest that John's decision to make his graph of Al's speed a linear graph was because "constant speed is a straight line." John's activity suggests two reasons for his confidence in this knowledge. First, in Protocol 6.17 John discussed how he had learned in science class that straight lines indicated constant speed. However, considering John's production of linear graphs as only arising from learned associations between shapes of graphs and particular types of speed cannot fully account for his activity. For instance, he explained that in science class he learned that curving upward indicated acceleration. Yet, he had no problem identifying the second linear segment, which is visually flatter, as indicating a faster speed than the visually steeper linear segments. Thus, I infer that John's reasoning with the graphs draws upon a combination of his learning from school experiences and his quantitative reasoning.

Second, I also infer that the iterability of John's unit ratios accounts for his intuition that constant speeds indicate linear graphs. In both the continuation of Protocol 6.16 and Protocol 6.17, John uses a *rise over run* type explanation to justify the shape of the graph. This way of thinking likely also arises from a combination of his learning from school experiences and his quantitative reasoning. Because his unit ratios are iterable quantities, John knows that each iteration will produce the same accruals of each quantity. Further, given John's awareness that he did not have to scale the distance axis with units of one and his ability to quantify "unit" ratios, I infer that John understood that any ratio he created to characterize the quantities would similarly result in constant accruals of each quantity with each iteration of the chosen "unit" ratio.

John's Mathematical Activity With Variable Quantities

I also find it useful to characterize John's quantitative operations as mutually supportive with his ability to conceive of covariation dynamically as the race progressed in order to better understand his attempts to reason about variable quantities. For example, consider the following excerpt from the teaching session on February 25, 2014. For this task, I provided John with an interactive graph of Sam's race measurements (see Figure 6.17). The graph contained a variable point on the line with its time and distance measurements displayed on the screen. Prior to the interaction in Protocol 6.18, I moved the variable point back and forth along the line and stopped it at a measurement of 10 centimeters and 6 seconds. Then I asked John how he could figure out Sam's speed using the measurements of the variable point. The protocol includes his solution and the ensuing discussion of his reasoning. I use the symbol, […] to indicate places in which I have removed sections of transcript where the conversation became repetitive.

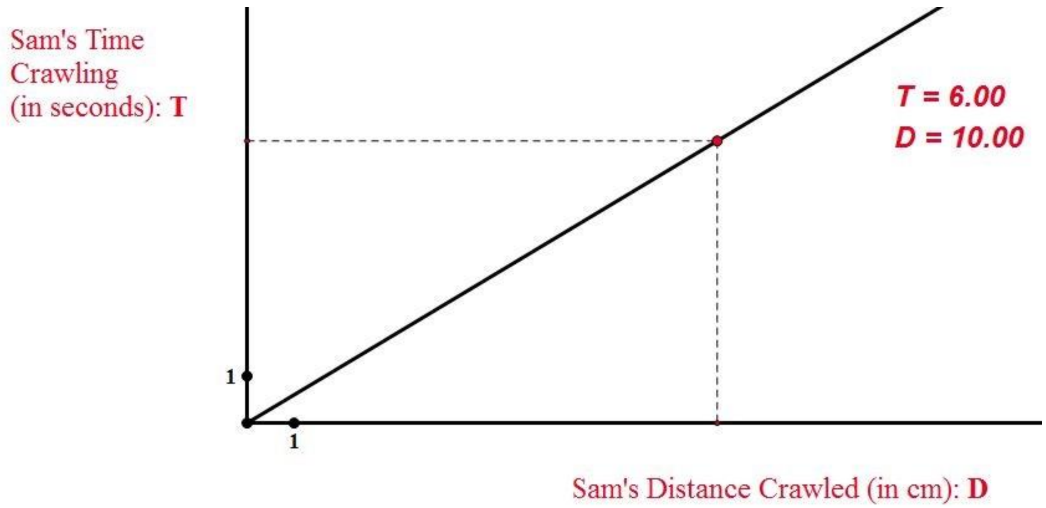

Figure 6.17. A screen shot of the interactive graph of Sam's race measurements.

John constructs a unit rate and uses it to reason about variable quantities.

Protocol 6.18: John determines Sam's speed using variable point measurements.

John: [Sets up the long division of 10 divided by 6, computes that using paper and pencil, and gets 1.66.] That's going to be one point six six six six forever.
D: Okay. So let's talk about that for a little bit. What does that mean? 1.66 what?
John: Distance.
D: Say a little bit more about that.
John: One second per distance. Like if you times that by 6 it's going to give you 10. Which it should give you 10 but it's a forever number so it's going to get you an answer close to 10 because it's not a whole number.
[…]
D: Okay. So does this [the 1.66] tell us our distance, our time, or something else?
John: It just tells her…what is it? It doesn't tell her the exact distance. But…
D: Like does this number tell us…
John: It just tells us how much she travels in 1 second and you just times it by any number and you get any distance on the line you want.

D: Um hmm. So would it be right to say that this is just a distance?
John: No not really.
D: Would it be right to say this is just a time?
John: Hmmm, no.
D: It's kind of…so what would you say it is then?
John: It's kind of like a number that you could control. Like you could times it by any number to get anything you want on the line. So if you want to times it by like 100 you could get the distance in 100 seconds.
D: And this is, and then what would be the unit on this? So this is, umm, like a measure for speed then. Right?
John: Um hmm.
D: It's not measuring just the distance or the time. So that would be measuring her speed.
John: Yeah. Her speed.
D: And so, the way you divided it. What unit would you put on that?
John: Umm…I'm not sure. Probably…her speed per…her speed per centimeters per second.
D: In centimeters per second?
John: Yeah, centimeters per second I think.

This interaction indicates that John could reason in terms of variable measures of the quantities time and distance. At the beginning of the protocol, I infer that he assimilated the specific measurements for the variable point as a situation of his unit ratio division scheme. Thus, he formed a goal of splitting the 10 centimeters into six equal parts and used long division to quantify this result. However, the critical aspect of this protocol is John's explanation for the meaning of the result 1.66 in terms of the constant covariational situation. First, the iterability of John's unit ratio supported his awareness that six iterations of his result would reconstitute the original measurements. More importantly, John leveraged this iterability to reason in terms of variable measures of the time and distance.

For instance, consider the following aspects of John's reply. He stated, "Like you could times it by any number to get anything you want on the line. So if you want to times it by like 100 you could get the distance in 100 seconds." Based upon this explanation, I infer that he assimilated the measure for time as any but no particular value. Yet, whatever the value was, say 100, that value defined the number of iterations of the unit ratio that would be needed. Then, John truncated the iterating activity by using multiplication to symbolize the 100 iterations. Thus, the same operations that account for his construction of unit ratios as indications of the covariation in progress also support his ability to reason with variable measures of the quantities.

While John could imagine varying the measures of the quantities and reasoning with any but no particular instantiation of those measures, this excerpt also suggests that he was in the process of developing the mathematical language to complement his ways of reasoning. For example, when I first asked John what the 1.66 meant, he said "distance." However, his subsequent replies and the manner in which he used 1.66 to explain finding an unknown distance for some particular chosen time measurement demonstrate that 1.66 represented more than simply a distance for John. Thus, as it became clearer that John operated with the 1.66 as a speed, I introduced the language of calling this a speed in hopes this would help John to more clearly distinguish between the extensive quantities, distance and time, and the quantity 1.66 centimeters per seconds that I infer John had constructed as an intensive quantity symbolizing the relationship of distance and time.

John struggles to create an algebraic equation to relate speed, time, and distance.

Because the interaction in Protocol 6.18 occurred right at the end of the teaching session, I could not ask John about how he might represent the relationship among the quantities with an equation. Thus, I decided to return to this task during the next teaching session on February 27, 2014. His replies indicate that adopting my use of "speed" as a word to describe the quantity John used as a relationship was not completely meaningful.

Before addressing a possible equation on February 27th, I first asked John to determine Sam's speed measured in seconds per centimeter and to give examples of how he could use that result to make predictions. My goal in changing the measurement unit to seconds per centimeter was twofold. First, I wanted to make sure that John's quantitative operations were active in his thinking as a result of recent operating, rather than simply recalling the results from the previous teaching session without necessarily having the operations that produced the results activated. Second, I wanted to obtain confirmation of my inference that John would be able to use this unit ratio in conjunction with any but no particular measurements of the variable point.

John provided indications of both of these things prior to the following excerpt. He quantified the unit ratio by dividing 6 by 10 to determine that Sam needed 0.6 seconds to travel 1 centimeter. And despite some initial struggles, once John began to use this unit ratio as an iterable unit, he began to successfully make predictions by using the quantity 0.6 seconds per centimeter to determine the length of time needed to travel various distances. The protocol starts just before I asked John about writing an equation.

Protocol 6.19: John attempts to write an equation to relate speed, time, and distance.

D: So if I gave you any—if you wanted to predict for anything. What if we wanted to know how long it would take to do the whole race—25 centimeters? [Sets the prediction value for the animation to 25 centimeters.] You can just tell me what you would do, you don't actually have to do it.

John: Well I just did 25 times 0.6.

D: Okay. And that would give you the time. Right? You don't have to actually do it.

John: I think 15 seconds only.

D: [Starts the race.]

John: It should be 15 seconds. Or, if I did my multiplication wrong…

D: [While the animation is progressing from a distance of zero to 25 centimeters, I continued talking with John.] Now as the race goes, they keep changing right. [Referring to the time and distance values on the animation screen.] So we could pick any distance on here and we should be able to find a time. Right?

John: Yeah.

D: [The race finishes and it has a time value of 15 seconds for the completed 25 centimeters.] 15.

John: Okay.

D: Could you write an equation that would describe that?

John: Well…

D: So for an equation, let's use these variables. [Points to the screen which has *D* and *T* used for the variable total distance and the variable total time, respectively.]

John: Total distance…this is probably the…this would be the seconds per centimeter times the distance equals the… I forgot. This is probably speed equals *T* times *D*. [Writes "*S* = *TD*" on his paper.]

Thus, despite being able to reason with variable measures of the time and distance, John struggled to produce an equation using algebraic notation to symbolize the relationships. In the end, John settled upon "*S* = *TD*" to represent his statement "speed equals *T* times *D*." John experienced similar constraints when attempting to write an equation in other tasks as well. In each case, he could operate fluidly with the speed and variable measures of distance and time yet was unable to generate an algebraic equation to symbolize those relationships.

I infer that the issue with John's struggle to create an algebraic equation did not lie with his quantitative operations. For example, to make his prediction for the time required to travel 25 centimeters, John multiplied his unit ratio (0.6 seconds per cm) by the total number of centimeters (25 cm) to find the unknown value of time. He initially used the same reasoning when asked to try to write an equation and stated, "This would be the seconds per centimeter times the distance equals the…" Thus, Jack's initial conception of what the equation should mean and accomplish matched his earlier quantitative reasoning with the 25 centimeters.

My explanation for John's struggle to use algebraic notation to appropriately symbolize the relationships is that John's equation actually matched what he intended, but his use of language and algebraic notation caused a conflict between his understanding of the quantities and his understanding of the algebraic symbols.[22] Protocol 6.18 showed that John did not independently refer to 0.6 seconds per centimeter as a speed, even though from my perspective he operated with it as such. Thus, while his reasoning with the quantities themselves was anticipatory and coherent, he seemed unsure what language and algebraic notation would be

[22] I use appropriately in the sense that mathematically knowledgeable others could assimilate the written equation as indicating the same relationships as it did for John.

appropriate for each quantity. In this case, I infer that John used T in place of what he actually understood as the unit rate for the speed. Perhaps because the speed was a *number of seconds* per centimeter he used T to symbolize the unit rate. Then, knowing that he had previously multiplied this number of seconds by the number of centimeters, he wrote $T \cdot D$.

Even though John never wrote an appropriate algebraic equation, I infer that doing so fell well within his zone of potential construction. Further, John's body language and uncertain descriptions of his equation suggest that he was unsatisfied with the equation he produced. This makes sense for a couple of reasons. First, while John seemed comfortable using $T \cdot D$ to represent multiplying the time for one centimeter by the desired value for the variable distance, he seemed less sure how to use the symbol, S, and the language *speed*. In his previous operating with 25 centimeters, John knew that multiplying 0.6 seconds per centimeter by 25 centimeters produced the value of time needed to travel the specific, but arbitrarily chosen, distance. Thus, I infer that the equation John had in mind was really $T = T \cdot D$ where the first T would refer to the variable time measurement and the second T would refer to the time needed to travel one centimeter. I hypothesize that first developing a language to distinguish these quantities and then establishing an algebraic notation to symbolize that language would enable John to generate mathematically appropriate equations that communicated his intentions to others.

Three Important Aspects of John's Reasoning During His Final Teaching Session

Lastly, I conclude my analysis of John's mathematics with three tasks that occurred during his final teaching session on March 6, 2014. Each of these tasks revealed an aspect of John's reasoning that contributed to my model of his mathematics in important ways. The first two tasks helped me to better understand John's reasoning in previous teaching sessions. The final task revealed an aspect of constructing and reasoning with intensive quantities that I did not explore at great depth in this study but which retrospectively I would consider to be a vital direction for future research into students' intensive quantitative operations.

John reasons about speed using both fraction and decimal quantities.

The first task involved reasoning about an inch worm named Cassandra. To begin, I showed John an animation of her race that stopped after 5 centimeters with an elapsed time of 1 second. After having John take a moment to imagine what her entire race would be like if she continued crawling at that same pace, I gave John a diagram that used one unmarked horizontal segment to represent one second. The interaction is somewhat long, but I have included the entire transcript because I find the interplay between John's reasoning with fractions and decimals as the important aspect of this exchange.

Protocol 6.20: John reasons about speed with both fractions and decimals.

D: Could you figure out how much that would be per each centimeter?
John: Hmm, 0.2. Or 0.2 seconds per 1 centimeter.
D: How'd you get the 0.2?
John: Well I divided 1 by 5. Basically.
D: Okay. So if you had to convince someone that, like, "I know it's 0.2 seconds per centimeter," how would you convince someone? Like how would you show them "Here's how I know it's true"?
John: Well, I could just start the race and stop the race at 0.2 seconds.
D: Okay. What if you didn't have the race handy? Is there a way that you could, kind of, convince someone that it's got to be 0.2 seconds per centimeter based upon what we know here?
John: Because if you add up 0.2, it equals 1 and you get 5 centimeters. And if you use a different number it would be over. […] [Writes "0.2 · 5 = 1.0" on his paper.] It equals 1 second.
D: Sure. Good. Yeah. That makes a lot of sense. So, can you show me on this diagram? So if this represents 1 second, what could you do to this diagram to also kind of show me, like, to also think about her speed in seconds per centimeter?
John: Hmmm, cut it into five.
D: Um hmm. Do you want to go ahead and do that?
John: [Partitions the 1 second segment into five parts.] Five.
D: And I'm just wondering, could you use this diagram that you've made and now you've added some marks to it—could you use that as another way to think about, or figure out, what her speed in seconds per centimeter would be?
John: Well I will just label it first. [Labels each of his marks: 0.2, 0.4, 0.6, 0.8, and 1. See Figure 6.18.]

Figure 6.18. John's partitioned and labeled 1 second diagram.

D: Okay. And as a fraction?
John: Well it's a fraction or decimal.
D: Okay. So as a fraction what would this 0.2 be?
John: Hmm…let's see. [Thinks for 7 seconds.] One-fifth.
D: How'd you get the one-fifth?
John: Well I put it over 10. [Points to the 0.2 he has labeled for his 1st partition.] And divided—and simplified it. Simplest form.
D: Okay. So does it make sense looking at your diagram why it's one-fifth?
John: One-fifth, two-fifths, three-fifths, four-fifths, and five-fifths. [Taps his marker on each labeled mark while saying the fractions.]
D: And then five-fifths. Yeah. So you could say it's 0.2 seconds per centimeter or one-fifth of a second per centimeter.
John: Um hmm.
D: So if you wanted to know how long it took her to travel, like, 17 centimeters, what would you do?
John: 17 centimeters.
D: Um hmm.
John: I would probably do 17 times 0.2 and that would be… [Uses paper and pencil to compute "$0.2 \cdot 17 = 3.4$".]
D: Um hmm. Okay. So 3.4…?
John: Seconds.
D: 3.4 seconds. Okay. And if you did it as a fraction. You wouldn't necessarily have to reduce it or simplify it—but what would that be as a fraction?
John: Umm. [Looks at his multiplication of $0.2 \cdot 17 = 3.4$ and starts to "write" something in the air above his paper.]
D: Oh. Well let me rephrase just a moment. So you could convert this to a fraction. [Points to the 3.4.] And we talked about before you could either use a decimal or a fraction.
John: Um hmm.
D: As a fraction what was her speed?
John: Umm one-fifth per centimeters.
D: One-fifth seconds per centimeter.
John: Yeah. One-fifth seconds per centimeter.
D: Yep. And then, so could you think about that and then just say, "Well, if we wanted to know how long it took to go 17 centimeters." Could you use that one-fifth of a second per centimeter?
John: You could use it for 17 centimeters.

D: So how would you do that?
John: You would just add it up 17 times.
D: Okay. And say you did that. What would you get?
John: 3.4.
D: [Chuckles.] As a decimal. What about as a fraction?
John: As a fraction…let's see. [Thinks for 11 seconds.] Probably 17 over 5.
D: Sure. How did you think about that? That's good.
John: Well, since it's 3 times more than 5 I just put it. So I just did 5 times 3 is 15 and I just added like these 2 [Points to the 0.2 and the 0.4 marks on his diagram.] plus 15. That's 17 and then they're over 5.
D: Sure. And before you also told me you could think about adding them up as you go.
John: Um hmm.
D: So if you went 1 centimeter it would be one-fifth. You know 2 centimeters would be [what]?
John: Two-fifths.
D: And 3 centimeters would be?
John: Three-fifths.
D: Four centimeters?
John: Four-fifths.
D: Five centimeters?
John: And five-fifths.
D: What about 6 centimeters?
John: Six. That would be six over fifths.
D: And 7 centimeters?
John: Seven over 5.
D: And then for 17 centimeters you could do seventeen-fifths.
John: Um hmm.

On one level, I find this exchange important because it demonstrates ways in which similar quantitative operations support John's reasoning with fractions and decimals. For example, John quantified 0.2 using his unit ratio division scheme. Hence, he viewed 0.2 as the value that accomplished the goal of splitting 1 second into five parts. Similarly, John split the provided diagram into five parts to indicate the time required to travel 1 of 5 centimeters. Further, the fact that both 0.2 seconds per centimeter and one-fifth seconds per centimeter were constructed as iterable unit ratios played an important role in his reasoning. It allowed him to use five iterations of the fraction and decimal to justify that these unit ratios adequately described the given scenario of traveling 5 centimeters in 1 second. In addition, he leveraged these operations to imagine continuing at that speed for 17 centimeters and could quantify the unknown time both as 3.4 and seventeen-fifths seconds. As a result, I consider this interaction important because John demonstrates the quantitative operations one can use to reconcile fraction and decimal quantities.

However, I also find this excerpt important because it suggests a subtle difference in the way John assimilates fractions, decimals, and requests to use one or the other. I have chosen to include this protocol as an example of something I noticed regularly throughout my interactions with John—that while he can convert between fractions and decimals, the equivalence often stems from using a computational procedure for division rather than his quantitative operations.

To exemplify this, consider the following aspects of John's reasoning during Protocol 6.20. Toward the beginning of the protocol, John justified that Cassandra's speed was 0.2 seconds per centimeter because five iterations of it would reconstitute the initial measurement. Further, I infer that his multiplication statement, "$0.2 \cdot 5 = 1.0$" symbolized this iteration. Immediately after this, John used his splitting scheme to create five equal partitions in the one second segment, one for each centimeter. Both of these would seemingly support an awareness that Cassandra had a speed of one-fifth seconds per centimeter on the basis of the unit iteration involved in each. Yet, John explained that 0.2 was equivalent to one-fifth because "I put it over 10. [Points to the 0.2 he has labeled for his first partition.] And divided—and simplified it. Simplest form." Thus, rather than leveraging his activated iterable units, John converted 0.2 to the fraction two-tenths and then used a mental procedure for simplifying fractions to obtain one-fifth.

Ultimately, John demonstrated the ability to switch between fraction and decimal representations while correctly responding to every question I posed. This indicates the flexibility of his ways of reasoning. However, I included this protocol to emphasize that despite this, John's procedures and computations at times interfered with the understanding he could construct using quantitative operations alone.

John experiences and overcomes a constraint to establishing a unit ratio.

Following the exchange in Protocol 6.20, I posed John the task of allocating 7 pounds of water among 9 cups and asked him to determine a value for the quantity *number of pounds per cup*. Similar to the previous task, I provided a diagram that used one unmarked horizontal segment to represent 1 pound (see Figure 6.19). Rather than providing a diagram with 7 segments for the 7 pounds, we devised this diagram intentionally to represent only 1 of the 7 pounds to encourage the students to form a mental image of the remaining pounds and to operate upon this mental image rather than simply carrying out activity on the diagram.

Figure 6.19. The diagram provided for the task of allocating 7 pounds of water to 9 cups.

Protocol 6.21: John overcomes a constraint to quantifying the number of pounds per cup.

John: Well, I was thinking of dividing 7 by 9 because there's 9 cups. And since there's 7 pounds I'd try to divide it.
D: Okay. So tell me about how you're thinking about dividing it. So when you say dividing, like without doing the calculations, is there a way to think about what the fraction would be?
John: Well…not really because it will be inaccurate but I know that it cannot be bigger than 1 because there's 9 cups.
D: Okay. Sure, so it's got to be something less than 1.
John: Um hmm.

D: Let's see if we can figure out how much it would be. I mean, could you use this diagram to help you figure out the fraction. Kind of like last time you figured out the fraction.

John: Hmm…

D: Like, looking back to this one. [Sets his diagram from the previous task in front of John. See Figure 6.18.] How did you figure out the fraction from the diagram?

John: It had the distance which helped. [I infer he was talking about having already labeled the first partition as 0.2, which he used to identify the fraction as one-fifth in the previous protocol.]

D: Okay. And I remember, so when you worked with the diagram you made the marks and then you said this was 0.2, 0.4… But then you also said, "Yeah, but this would also be one-fifth." Why does this mark represent one-fifth of a second?

John: Hmm…because it's…how is it. How do you say? This is basically a second divided by 5 and it's the distance per centimeters.

D: So yeah, when you took that second and divided it into the five parts, umm, and then that first one is one-fifth of the second. Right?

John: Yep.

D: So this is a little bit of a different scenario because I didn't give you quite the same kind of information.

John: Uh huh.

D: But if we think about this as a pound [points to the 1 pound segment] and we had 7 pounds that we were allocating to the 9 cups.

John: Seven.

D: And we're trying to spread it out into the nine different cups. Right?

John: Um hmm.

D: Would you be able to make some marks on this diagram that would help you to split that up—to think about how to split that up, that weight up?

John: Well I…I could mark right about here I think. Right about here. [Puts a mark roughly a little to right of the middle of the segment.] Or somewhere close right here.

D: Okay. Tell me why you were thinking about that spot.

John: Because I was trying to make it add up add up add up and see if this equals like, 2 of these. Like if you add up all 9 cups. Which this is 1 cup.

D: Um hmm.

John: And I want 9. And the extra ones, they add up to another 2 pounds.

D: Okay. So you're trying to think about if you could have this and then do 9 of those then it would end up equaling 7 of these segments?

John: Um hmm.

I infer that John initially assimilated this task as a situation of his unit ratio division scheme. Had I let him, I infer he would have used long division to compute 7 divided by 9 to establish the decimal value of the desired unit ratio. Further, because this division would not result in a "nice" decimal, unlike previous situations where John could at times recall the decimal value for a particular division from memory, I do not think he had any decimal value in mind for 7 divided by 9. I stopped him before carrying out the division and asked him to try to think about it as a fraction because I wanted to see how he might achieve his goal of splitting the 7 pounds into 9 parts using his quantitative operations. The fact that he did not recognize that dividing 7 by 9 would imply the fraction seven-ninths underscores the significance of the point I was trying to make with protocol 6.20.

After I encouraged John to use fractions, I infer that his approach was essentially the same as that which he used for the initial tasks in the Adopt-A-Highway context (cf. Protocol 6.10). For example, he knew it was less than one but greater than one-half and indicated an estimate on the segment. This, along with his description of adding up all 9 cups, is consistent with his strategy of constructing the share for each cup as a composite unit of unknown size. In this case, nine iterations of this unit should comprise all 7 pounds. When describing his thinking John also said, “And the extra ones, they add up to another 2 pounds.” I interpret this as indicating that John envisioned cutting off a composite share from each of the 7 pounds, leaving a remainder segment from each of the 7 pounds. Then, those remaining segments should be able to be reconstituted to form an additional two shares. Thus, I infer that “another 2 pounds” actually referred to another two shares containing some number of pounds.

I believe that attempting to quantify the unit ratio as a fractional amount of 1 pound per cup placed John in a state of perturbation. His responses indicated that he had formed a clear goal of splitting the 7 pounds into 9 composite shares, and he could even estimate a share as greater than one-half pound but less than 1 pound. Yet, in the moment I was surprised that John could not quantify this fractional amount. For example, by the end of the Adopt-A-Highway task John could quantify the fractional number of miles per organization within a few seconds after starting the tasks. While I will save my discussion of the implications of this for the next chapter, suffice it to say that my hypothesis is that John did not assimilate this task as a situation of the pattern-based reasoning I inferred that he used to quantify unit ratios in the Adopt-A-Highway context. Thus, he did not intuitively recognize this as seven-ninths of a pound per cup and would need to use some other quantitative operations to construct this understanding in activity.

To try to help John resolve this perturbation, I introduced a modification to the task and asked John to consider only having 1 pound to allocate to 9 cups.

Protocol 6.21: Continuation.

D: How about this question. What if you just had 1 pound of water and you were trying to pour it evenly into your 9 cups? As a fraction, how much in 1 cup?
John: One-ninth.
D: How'd you get that?
John: Well there's basically 9, and 9 cups and you're dividing. Which is one-ninth as a fraction.
D: Um hmm. And how would you mark this diagram to show that?
John: Well, let's see. [Creates nine equals segments in the 1 pound segment and labels them 1 through 9.]
D: So if you had 1 pound of water and these 9 cups that you wanted to pour it into you could think about making these marks and splitting it up, and then you'd get one-ninth of a pound for each cup. Right?
John: Um hmm.
D: So that's pretty similar to this scenario we have, except we have 7 pounds of water for 9 cups. So you have to kind of imagine the other pounds. Right. So this just would stand for *1* of the pounds. But we really have 7 pounds of water that we're pouring into these 9 cups. So what do you think in terms of the fraction?
John: Seven-ninths for each cup.
D: Yeah, great! How'd you think about that?
John: Well, I just did like this part right here [points to the first of the nine partitions he made in the 1 pound segment], times by 7 which give me… Wait, times by…yeah 7. Which give me 7 over 9.
D: So seven-ninths of…what does that mean then? Seven-ninths what?
John: It equals 1 pound for each cup because you're just basically like putting it…like you're just drawing all of these lines. [Traces his finger on the page as if making 6 more segments to represent the rest of the pounds.] Like these same lines except 1, 2, 3, 4, 5 pounds. And then you just putting these parts in there. One [motions with his hands as if removing one-ninth of each of the imagined 7 one-pound segments.] And you get seven total parts.
D: Sure. So you'd have seven-ninths of a pound for 1 cup.

John: Um hmm.
D: Good!

My suggestion to first consider allocating only 1 pound to 9 cups proved very productive for John. I infer that John overcame his previous perturbation by recognizing that he could use distributive partitioning operations to quantify the desired unit ratio. After deciding that splitting 1 pound among 9 cups would be one-ninth of a pound per cup, John reconsidered the task with 7 pounds and immediately recognized the unit ratio as seven-ninths of a pound for each cup. Further, John's activity suggests that he accomplished this by envisioning having additional segments for the additional pounds and taking one-ninth of each pound. I infer that because John's composite units were iterable quantities, envisioning additional copies of the one-pound segment for the additional pounds implied these envisioned segments were likewise partitioned into nine parts by virtue of the partition on the unit John used to produce the copies. This supported John's recognition that taking one-ninth from each of the 7 one-pound segments would enable him to produce nine equal shares while also exhausting all 7 pounds.

In comparison to John's earlier activity, I infer that his realization that first partitioning each individual unit could also accomplish partitioning the total number of pounds evenly allowed him to overcome his perturbation. To understand what might account for this realization, consider how John conceived of measured quantities. The beginning of Protocol 6.21 indicates that he initially assimilated the task as a situation of his unit ratio division scheme. In general, John used this scheme to achieve a goal of splitting a composite measured quantity into a given number of parts. Thus, I infer that John initially assimilated the 7 pounds as a single composite whole. By reconstituting the single composite whole into a sequence of 7 individual units during the continuation of the protocol, John was able to implement distributive partitioning operations to quantify the unit ratio. As a result, I believe that my suggestion proved helpful because it focused John's attention on the individual unit of 1 pound rather than the composite unit of 7 pounds.

John demonstrates a limitation to his ways of reasoning with constant speeds.

I turn now to the final task of the teaching experiment with John. To introduce this task, I showed John the graph of Jackie's race (see Figure 6.16), and I asked him to explain what he remembered about the situation from our earlier conversations about the graph during the February 25th teaching session. He explained that the second, fourth, and sixth linear segments indicated faster speeds than the first, third, and fifth segments. In addition, John described how even though Jackie's speed changed, her speed remained constant within any particular section of the graph as if she alternated between jogging and running, but maintained a constant pace while doing each. I designed this final task to investigate John's ability to quantify the constant speed that Jackie maintained during the fourth linear segment of the graph.

Protocol 6.22: John attempts to quantify a constant speed for one interval on a graph.

D: Okay. So say you wanted to know, to calculate a speed for just this part of the race—like how fast she was running on this section of the race. [Uses his fingers to span the length of the fourth linear section on the graph.] Umm, you know, if I gave you a point on that graph like right here [puts marker at a point on that section] would that be enough information for you to figure out her speed on this section?

John: Umm no. Because there's no distance and no time and you're also there's no other point to compare it. [The graph included no scales.]

D: So say I gave you a point with a distance and time. That wouldn't be enough or that would be enough?

John: That wouldn't. Because you might need another one because it's going constant speed.

D: Okay. So we can get those points actually. Let's take a look. [Opens up the interactive graph of Jackie's race which has two variable points on the fourth linear segment. See Figure 6.20.] So actually I've got two point's measurements. So measurement one is right at the very beginning and her distance is 10 centimeters and 21 seconds. And measurement two right at the end of that section is 14 centimeters and 24 seconds. And I'm just wondering if you could use that information to think about on this section of graph, what's her speed in seconds per centimeter?

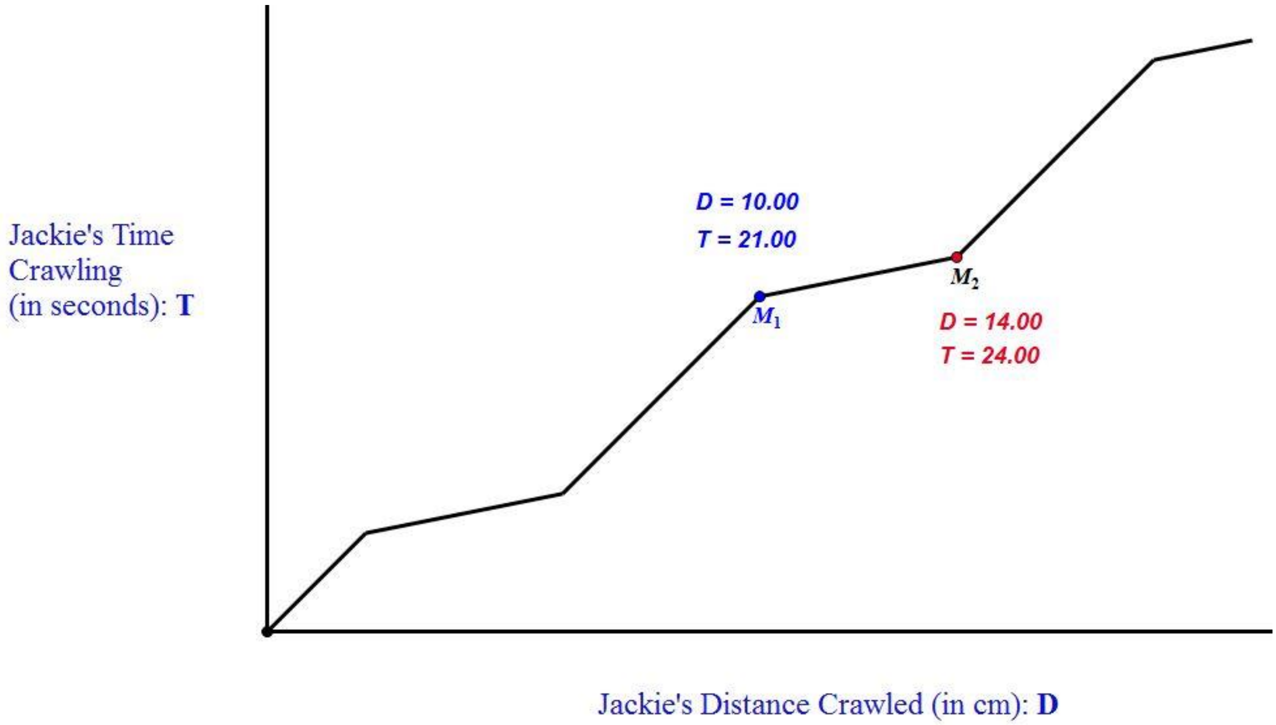

Figure 6.20. A screen shot of the interactive graph of Jackie's race, which included two variable points.

John: Seconds per centimeter. Okay. Hmm… Seconds per centimeters… [Thinks for 25 seconds.]
D: What are you thinking about?
John: Well, just dividing the time and the distance to get the 1 second and then. No no, to get the distance for one centimeters. Like 21 divide 10. [Does the long division of 10 divided by 21 on his paper and gets .4.]
D: So 21 divided by 10. So you're thinking about that with this first point?
John: Yeah. […] And I could just add up the time by 3 seconds and see if I get the distance.
D: […] You mentioned that you would need two points so that you could compare.
John: Um hmm.
D: So how were you thinking about comparing?
John: Like…they should be going the same speed if it's going a little bit faster and then you could tell that it's either you did something wrong in your problem or he's not going a constant speed at all.
D: Okay. So you're not sure if she's going a constant speed?
John: I'm sure she's going a constant speed because it's a straight line. But sometimes you might put the number wrong.
D: Okay. So you would kind of—just to make sure I understood what you meant. You would take what you did here [referring to the result of his long division] and then

> you were saying you would go 3 more seconds at that speed and see if it got to this measurement [points to the measurement of the second variable point]. Is that what you were picturing?

John: Yeah.

Initially, John divided the distance by the time to quantify a unit ratio. Then, I interpret John's explanation as indicating the following: a) that the result of his division indicated the number of centimeters traveled in 1 second; b) that he envisioned starting with the measurements of the first variable point and accumulating the time and distance by iterating the unit ratio 3 times, once for each second between 21 and 24 seconds; c) and he would evaluate the suitability of his unit ratio by comparing the accumulated time and distance to the measurements of the second variable point. John's reasoning here is consistent with his previous efforts to reason with constant speeds. However, because the graph did not indicate a single constant speed, this approach did not accurately quantify Jackie's speed during that interval of time.

As a result, John's activity in this task alerted me to an aspect of intensive quantitative reasoning that I had not explicitly considered during the teaching experiment—the importance of the reference point for measurements and constructing intervals of change. The vast majority of tasks that involved quantifying unit ratios involved measurements of quantities. Technically speaking, a measurement defines an interval from zero up to the measurement value. However, in working with the students, the reference point of zero almost always remained implicit; the students reasoned with the numerosity of the measurement without explicitly paying attention to the reference point of zero. However, this task highlights the importance of including opportunities to actively construct intervals of change between two measurements rather than always operating from zero to a measured value.

Given his ability to construct and reason with an interval of 45–49 minutes in the swimming pool context (cf. Protocol 6.4), I hypothesize that learning to operate successfully with tasks such as this lay within John's zone of potential construction at the time. In fact, had I had more time to continue working with John, this would have become my goal for his future learning—becoming explicitly aware of the intervals upon which he operated. My goal in doing so would be to bring forth all of the ways of operating he used in the teaching experiment within the context of constructed intervals of change so that his reasoning would not remain constrained to an implicit reference point of zero.

CHAPTER 7

CONCLUSIONS AND IMPLICATIONS

To this point, I have focused on elaborating second-order models of quantitative reasoning that account for my observations of the students' mathematical activity. These case studies of the mathematics of Jack and John highlight how each student used his available quantitative operations to make sense of a range of situations involving constant covariational relationships. In developing these analyses, each student's mathematical activity and explanations provided the context for identifying his characteristic ways of reasoning and for making inferences about his goals, assimilating structures, and the nature of the quantities he had constructed in each situation. While the progression of each student's reasoning throughout the teaching experiment was unique, comparing their case studies and the constraints and affordances of their individual ways of reasoning revealed several common themes.

With this final chapter, I return to the original questions that framed this study and consider the cases of Jack and John within a broader context. In doing so, I pursue three primary objectives. First, I use the analyses presented in the previous chapters to consider plausible answers to the three research questions. Second, while these conclusions represent ends for this study, they also raise several new questions that could serve as starting points for future research. Thus, I briefly discuss potential avenues for inquiry that could extend this work in productive ways. Finally, I conclude by considering the implications of this work in relation to school mathematics and supporting students' construction of intensive quantities and flexible algebraic reasoning.

Research Questions Revisited

In the first chapter, I posed three research questions that guided the design and conduct of this study into students' construction of intensive quantity. I turn now to considering each of these questions in light of my model of the mathematics that Jack and John used to make sense of the tasks and contexts throughout the teaching experiment.

Research Question 1

> What conceptual constructs, including extensive quantitative schemes and operations, can explain each student's assimilation, as well as any changes in that student's assimilation, of quantitative situations involving intensive quantity?

My analyses throughout the previous two chapters represent my response to this first research question. In particular, those chapters describe my model of the conceptual constructs that can account for each student's assimilation of the tasks throughout the duration of the teaching experiment. Lacking direct access to the students' ways of reasoning, I do not claim that these models match the students' conceptual structures directly. However, I do put forth the characterizations I have presented as one plausible way of accounting for the students' mathematical activity. Further, while I have considered alternative explanations for the students' activity in each task, those alternative explanations could not account for the students' activity across tasks. Thus, the models of the students' reasoning I have presented in the previous chapters are those that remained viable throughout my retrospective analysis of all of the teaching session interactions.

Research Question 2

> What aspects of the mathematics of each participant, including extensive quantitative schemes and operations, impede or facilitate that participant's ability to work with quantitative situations involving intensive quantities?

In response to this question, I summarize important aspects of my analyses from the previous chapters that I infer characterize pivotal aspects of each student's reasoning that account for his ability to make sense of the various tasks and covariational relationships. In doing so, I take a step back from the task-by-task progression of the previous chapters to highlight specific aspects of each student's reasoning that had the greatest explanatory power in accounting for his successes and challenges across tasks. Often times, the things that impeded and the things that facilitated each student's quantitative reasoning were complementary; identifying something that facilitated a student's success also revealed its non-use as an impediment to his making sense of the covariational relationships. Thus, I consider these in tandem as they relate to each student's reasoning throughout the teaching experiment.

In addition, I borrow a phrase from Hackenberg (2005) and refer to these important aspects of the students' reasoning as *constructive resources*. In the present context, I use the term constructive resources to refer to particular conceptual constructs or types of quantitative reasoning that facilitated the students' abilities to construct intensive quantities and to make sense of constant covariational relationships. It is important to note that by identifying an aspect of the students' reasoning as a constructive resource I am not necessarily claiming that the particular conceptual construct or way of reasoning was available to the student throughout the entire teaching experiment. It may have been, but it might also be the case that constructing a particular understanding or way of reasoning in activity created new possibilities for the student's reasoning, hence, revealing that construct as an important constructive resource for making sense of the types of tasks that I posed.

Lastly, even though Jack and John employed their available operations in unique ways and demonstrated differing degrees of success across the three contexts, I found that several of the same conceptual constructs could account for the reasoning of both students. In some instances, I infer that each student leveraged these constructs in similar ways and to similar effects. However, in other cases the importance of these constructs revealed themselves differently for each student as they demonstrated differing abilities to make sense of the various tasks. Thus, I organize my discussion of this research question by first briefly characterizing some common aspects of their reasoning before focusing on what I learned from each student's unique progression of reasoning throughout the teaching experiment.

Constructive resources common to the mathematics of both Jack and John.

Constructive Resources 1 and 2: Three levels of units and a strategy of coordinated partitioning/iterating.[23]

The first two constructive resources represent critical aspects of Jack and John's mathematics that have remained largely implicit throughout the analyses of the previous two chapters—their abilities to take three levels of units as given in assimilation and operating and their use of a coordinated partitioning/iterating strategy for assimilating and quantifying changes in measures of the extensive quantities. I first identified these as features of each student's reasoning during the initial interviews. However, even though I have not always explicitly pointed out their role in supporting the students' activity, both Jack and John leveraged these aspects of their reasoning in combination with their available quantitative operations to coordinate changes in covarying quantities throughout the teaching experiment. Without these two fundamental ways of reasoning, the students would not have been able to reason as they did.

While critical, the conceptual resource of reasoning with three levels of units gains explanatory power when one also considers the particular quantitative operations that one incorporates to make use of this conceptual tool. For instance, three levels of units are necessary for constructing a recursive partitioning scheme that one could use to carry out a fraction

[23] I use numbering here as a naming device. Thus, Constructive Resource 1 is not necessarily more important, nor necessarily constructed before, Constructive Resource 2.

composition such as finding one-fifth of one-seventh (Steffe, 2010g). Establishing the result of one thirty-fifth certainly requires reasoning with three levels of units, and the recursive partitioning scheme represents a specific model for how one leverages those three levels of units to operate upon the quantities.

Similarly, the conceptual resource of a strategy of coordinated partitioning/iterating becomes more useful when one also considers how the coordination gets carried out. For example, students construct a wide range of partitioning and iterating operations that each support operating with quantities in particular ways (Steffe & Olive, 2010). I intend that the constructive resource of coordinated partitioning/iterating captures this range. Then, identifying a specific partitioning or iterating schemes adds important detail that explains the mechanisms that account for one's actual process of coordination. For example, a student who operates with a coordinated splitting scheme could take as given in assimilation the understandings that a student operating with a strategy of coordinated equi-partitioning would need to construct in activity. Thus, more than simply "doing the same thing to both quantities," I use a strategy of coordinated partitioning/iterating to refer to both the coordination between the quantities and the particular type of quantitative schemes students use to carry out the coordination.

Thus, I consider the first two constructive resources as broad categories of reasoning that underpin the construction of intensive quantities. However, not all ways of reasoning with three levels of units, nor all partitioning and iterating schemes, support the same levels of sophistication in one's reasoning. As a result, I wanted to highlight these broadly defined constructive resources before continuing on to consider more specific constructive resources that leverage three levels of units and a strategy of coordinated iterating partitioning.

To that end, I infer that a splitting scheme and iterable composite units represent two of the primary constructive resources that the students used to make sense of the constant covariational relationships and to coordinate changes in the extensive quantities. Both of these require reasoning with three levels of units (cf. Chapter 2). Thus, every claim involving these two constructive resources is inherently also a claim that reasoning with three levels of units was involved. I characterize the role each of these played before moving on to consider the unique aspects of each student's reasoning.

Constructive Resource 3: A splitting scheme.

A splitting scheme served two primary purposes in the students' thinking. First, both students' explanations suggest that they assimilated changes in the extensive quantities as situations of their splitting scheme. Second, they formed goals of splitting the concomitant quantity into the same number of parts in order to accomplish their goal of coordinating changes in the two quantities. Thus, the splitting scheme both accounts for the students' assimilation of changes in the quantities and characterizes their goals for their subsequent activity.

For example, consider how each student replied to the task of finding how much the pool level would rise in 1 minute given that it had risen 3 inches in 5 minutes. In Protocol 5.1 Jack commented, "I was trying to split the 3 [inches]…equally into each minute." Similarly, John explained that he decided to divide 3 by 5 because "5 was the time…so I was trying to make 5 into 1" and he wanted to "divide it [the 3 inches] into even sections" (Protocols 6.1 and 6.3). These comments help to clarify how the splitting scheme served as an integral component of the students' coordinated partitioning/iterating strategy. I infer that both students assimilated the change from 5 minutes to 1 minute as a situation of their splitting schemes and, then, decided to similarly split the concomitant quantity into the same number of parts to identify the corresponding change in depth for 1 minute. This dual use of their splitting schemes remained a characteristic way of operating for both Jack and John across all of the tasks and contexts.

The significance of using one's splitting scheme in these two ways is that doing so constructs the relationship between the original and transformed states of the quantities as a multiplicative relation. Recall that I refer to the splitting scheme as an extensive quantitative scheme because the sequential partitioning and iterating activities of the equi-partitioning scheme become available simultaneously during assimilation with the splitting scheme (cf. Chapter 2). Consider this in relation to the example in the previous paragraph. As a situation of the students' splitting scheme, 1 minute could be recognized as one-fifth of 5 minutes because five iterations of 1 minute would be equivalent to a duration of 5 minutes. Similarly, the students knew that five iterations of the unknown depth per minute would produce the same total changes in depth and duration as the original ratio 3 inches per 5 minutes.

Determining this unknown depth per minute involves using quantitative schemes that enable one to quantify the result of using one's fractions as operators. Because 1 minute is one-fifth of 5 minutes, the students' activity and explanations indicate their goal of finding one-fifth of 3 inches. The quantification of these intensive quantitative unknowns is one area in which the students' reasoning differed. However, regardless of the process used, the fact remains that for both Jack and John quantifying the unknown depth accomplished a splitting goal.

Constructive Resource 4: Iterable composite units.

Much like the splitting scheme, iterable composite units also served a dual purpose in the students' reasoning. First, they account for the ability to recognize situations as involving a constant covariational relationship. Considered generally, any two ratios could be considered the same speed if iterating the first ratio would produce the same total accumulations of the quantities as given in the second ratio. Second, the role of iterable composite units in establishing a situation as a constant covariational relationship becomes especially apparent when considered with respect to constructing unit ratios. Because iterating is one of the assimilating operations of a splitting scheme, the results obtained from splitting a quantity are inherently constructed as an iterable unit. In terms of coordinating their splitting activity across two quantities, the implication is that after quantifying a unit ratio, both Jack and John could anticipate that the appropriate number of iterations of the unit ratio would result in the same total accumulation of each quantity as the given ratio. Thus, iterable composite units account for the students' awareness that a constructed unit ratio represented the same speed as the given ratio or that two given ratios represented the same constant speed.

Examples of this reasoning can be found throughout the previous two chapters. For instance, in Protocol 6.15 John determined that Abby crawled at a constant speed because he could envision that iterating 2.5 seconds per 1 centimeter would result in the same total accumulation of each quantity as the other time and distance measurements he had collected. Likewise, Jack used similar reasoning in Protocol 5.15 and determined that 0.6 seconds per one-half centimeter and 1.2 seconds per cm indicated the same speed "because once he gets to that 1 [centimeter] it's the same time every time." Similarly, when using their splitting scheme to conceive of unit ratios, the anticipated result represented a measure of the quantity for which the appropriate number of iterations would reproduce the same initial measure of the quantity. Issues of quantifying unit ratios aside for the moment, both Jack and John exemplified this way of reasoning when thinking about the number of incher per minute that would be equivalent to 3 inches per 5 minutes (cf. Protocols 5.1 and 6.1).

However, more than supporting the recognition of situations as instances of constant covariational relationships, iterable composite units also provide a mechanism for conceptualizing the continuation of those relationships. For instance, suppose that one has assimilated a given ratio as an iterable ratio. Then, the ratio inherits its iterability from the iterability of the extensive quantities it relates. Hence, constructing an iterable ratio would support anticipating the covariation continuing beyond the given measurements of the quantities with iterations of the given ratio defining the covariational process.

This way of reasoning with ratios became part of my model of both students' mathematics and accounts for their construction of a wide range of understandings. This enabled both students to quantify various intensive quantitative unknowns. For instance, during Jack's first teaching session, iterations of the given 3 inches per 5 minutes ratio underpinned Jack's reasoning about accumulations of the respective quantities while he imagined continuations of the variation. This allowed him to identify the unknown depth after 10 or 25 minutes of pumping water and to construct a new ratio, 2 feet per 40 minutes, that he used as an iterable ratio to help identify the amount of time required to raise the pool level 111 inches. Similarly, after reflecting upon his earlier activity, John used his constructed unit ratio of 0.6 inches per minute to determine several intensive quantitative unknowns.

Furthermore, I conclude that constructing iterable ratios supported both the students' awareness that linear graphs represented constant covariational relationships as well as their ability to reason with variable quantities. In each of these cases, I inferred that the students operated with a sense that the extensive quantities could continue to vary and that the covariation would continue to accumulate according to any but no particular number of iterations of the given ratio.

Before continuing, it is important that I digress briefly to point out several features of how I think about iterable composite units that are relevant for the current discussion. First, I find it important to distinguish between using a composite unit as an iterating unit in activity compared to having iterable composite units available in assimilation. With the former, one can constructs his/her understandings of the relationship *in activity* as a result of carrying out the iterating. While this could enable students to construct similar understandings to those described in the previous paragraph, those understandings are inherently tied to the specific iterating activity that he/she carried out. John's activity in Protocols 6.1–6.2 indicate that this iterating activity may not support reasoning with the covariational relationship beyond the bounds of the initial measurement.

However, having iterable units available in assimilation means that one can anticipate the results and meaning of any but no particular iteration prior to actually carrying out any activity. For example, in my analysis of Protocol 6.13, I inferred that John's conviction in his unit ratio eight-elevenths of a mile per organization stemmed from the fact that he had constructed this quantity as an iterable unit ratio. It was enough for John to know that the iteration could be carried out for him to have a sense of certainty regarding his result and what it meant in terms of the context. Thus, having iterable composite units available in assimilation limits the cognitive demand required for a student to coordinate and reason about changes in the extensive quantities. This speaks to the benefit of abstracting one's ways of operating to the point that they become available in assimilation as ways of structuring one's conception of future experiential situations.

I do not claim that the students automatically assimilated all ratios as iterable ratios throughout the entirety of the teaching experiment. On the contrary, there is evidence in the previous two chapters to suggest this was not always the case. Rather, the claim is that both students could construct ratios as iterable ratios and when they did, the ways of reasoning with constant covariational relationships described above became available to them.

Another important feature of how I think about iterable composite units is that the units that I am describing can be considered composite in two senses. For example, consider the unit ratio three-fifths of an inch per minute. If this is constructed as an iterable unit ratio, I consider this an iterable *composite* unit because the number of inches is a composite unit. One could use this to determine that in 12 minutes the depth of the pool would increase by $12 \cdot 3/5$ inches. However, even though three-fifths can itself be constructed as three iterations of one-fifth, the previous example with 12 minutes involves iterating three-fifths of an inch rather than one-fifth. Hence, I consider the three-fifths inches in the quantity three-fifths inches per minute to represent a composite unit. Furthermore, the same analysis holds if the unit ratio in question includes decimal values of the quantities such as 0.6 inches per minute.

The second sense of three-fifths inches per minute as composite stems from considering this ratio in conjunction with the students' coordinated partitioning/iterating strategy. Using this strategy, Jack and John transferred any transformations carried out on one quantity to the concomitant quantity as well. Thus, I also consider three-fifths of an inch per minute an iterable *composite* unit because both quantities, the three-fifths inches and the 1 minute, were iterable quantities, and the students understood that iterations of either quantity implied an equivalent number of iterations of the other quantity. I intend that referring to three-fifths inches per minute as an *iterable unit ratio* will capture both senses of the composite nature of this quantity.

Constructive Resource 5: A process for quantifying unit ratios.

The construction of a process for quantifying unit ratios is essential as it provides a pathway for leveraging one's splitting scheme and iterable composite units within the context of a constant covariational situation. In particular, this fifth constructive resource entails constructing a process for accomplishing the splitting goals that one forms while trying to transform a given ratio into a unit ratio. I refer to this constructive resource as a process for quantifying unit *ratios* rather than unit *rates* because as John demonstrated in the inch worm context, there is no guarantee that the ratio will characterize a relationship beyond the initial measurements from which it was constructed. However, if the unit ratio one produces is assimilated using iterable operations, as characterized in the previous section, the unit ratio can be iterated to construct meaning for this result within the context of the covariation of the two extensive quantities. In that case, I would consider the result of one's quantification process a unit rate.

For example, consider the situation of transforming the ratio 3 inches per 5 minutes into a unit ratio three-fifths of an inch per minute. Both Jack and John described wanting to split the 3 inches into five equal parts. Further, their activity and explanations in Protocols 5.1 and 6.1, respectively, indicate that each student anticipated that five iterations of the result of that split should equal 3 inches. Yet, John successfully enacted a strategy for quantifying that result while Jack did not. Thus, the construction of a process for quantifying unit ratios is necessary for actually accomplishing one's goal of transforming a ratio.

However, the actual process each student constructed for quantifying unit ratios varied and depended both upon the task and the quantitative operations he had available. For instance, comparing Jack's and John's activity on any given task, their characteristic ways of operating were often quite different. This was especially evident in the swimming pool context. John typically used his unit ratio division scheme to coordinate changes in the extensive quantities in that context. In contrast, Jack initially struggled to quantify the result of splitting 3 inches into five parts. Thus, he drew upon his whole number multiplicative reasoning and his reversible fraction schemes to coordinate the quantities with varying degrees of success depending upon the particular coordination required for a task. Later, his use of distributive partitioning operations enabled him to overcome his earlier constraints and find unit ratios even in situations where whole number reasoning proved insufficient. The essential component to each student's successful operating within the constant covariational situations was the fact that he had a quantification process available for a given task, not necessarily which particular process he used.

These differences in the students' processes for quantifying unit ratios speaks to the fact that even though I have identified many similarities in their ways of reasoning, several significant differences existed as well. For example, the fact that Jack initially had no quantification process available in the swimming pool context highlighted its importance as a constructive resource to his later success. However, the ways in which Jack used his quantitative operations to overcome that constraint revealed a constructive resource that was not evident in John's solutions to the same tasks. Similarly, considering the quantitative reasoning John employed throughout the teaching experiment helped me to identify another constructive resource that was not initially a part of my model of Jack's mathematics. Thus, in the following sections I characterize two additional constructive resources that I abstracted from my retrospective analyses of each individual student.

Constructive resources abstracted from the mathematics of Jack.

Constructive Resource 6: A simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units.

This sixth constructive resource proved critical in the progression of Jack's reasoning during the teaching experiment. Recall that he initially experienced significant constraints quantifying unit ratios in situations that required splitting a composite quantity such as transforming 4 inches per 5 minutes into a unit ratio. In situations such as this he knew what goal he wanted to accomplish (to split 4 inches into five equal parts) and he anticipated what the result would mean in terms of the covariation of the quantities (the number of inches per minute). However, for the better part of three teaching episodes he remained uncertain of how to use his available quantitative operations to quantify unit ratios such as four-fifths of an inch per minute.

Constructing a simultaneous awareness of measured quantities as a single composite whole and as a sequence of individual units provided a pathway for Jack to overcome his initial constraints and resolve his perturbation. Several features of his construction of this awareness stand out. First, Jack's characteristic way of reasoning involved assimilating a measured quantity, such as 4 inches, as a single composite whole and using his splitting scheme to operate upon the composite quantity. However, he also demonstrated that he had distributive partitioning operations available with which he could operate upon a sequence of individual units. The process of unifying these to construct a simultaneous awareness of both conceptions is similar to Piaget's (1970) account of a child's construction of an interval of distance as an abstraction from alternating his/her centration between two focal points. In this case, I infer that alternating between these two conceptions of the measured quantity during Protocol 5.6 provided a foundation from which Jack could coordinate the implications of each conception and construct a simultaneous awareness of both conceptions of 4 inches. While Jack could use his available quantitative operations quite powerfully using either of these perspectives individually, his reasoning became most flexible when he demonstrated a simultaneous awareness of both conceptions and could switch between them at will.

Constructing a simultaneous awareness of both conceptions of measured quantities opened up new constructive pathways for Jack's quantitative reasoning. For instance, doing so enabled Jack to use his fractions as operators to quantify one-m^{th} of a composite n units as n/m of one unit. This involved coordinating the results of using his splitting scheme to act upon the single composite whole with the results of using his distributive partitioning operations to act upon the sequence of individual units. In addition, having this simultaneous awareness available allowed Jack's reasoning to become much more anticipatory. For example, after reconstructing the simultaneous awareness in the context of allocating various amount of highway to different numbers of volunteer organizations, Jack began to produce solutions to new tasks before I could even finish asking the questions (cf. Protocol 5.9). Thus, Jack began to truncate his actual partitioning activity and carried out his operations hypothetically.

Considering Constructive Resource 6 with respect to John's mathematics.

Even though I first identified Constructive Resource 6 during my retrospective analysis of Jack's mathematics, I also found it informative to use it as a lens through which to consider John's reasoning. Looking across his activity in all three contexts, John almost always operated within the conception of a measured quantity as a single composite whole. Yet, John reasoned quite powerfully whenever he did operate upon both conceptions of measured quantities.

One interaction in particular underscores the importance of Constructive Resource 6 and demonstrates the potential role it could play in John's reasoning as well. Protocol 6.21 focused on the task of allocating 7 pounds of water among 9 cups and determining the number of pounds per cup. Using his characteristic way of reasoning, John initially assimilated the 7 pounds as a single composite whole and formed a goal of splitting the 7 pounds into nine equal parts, one for each cup. Accordingly, his unit ratio division scheme supported his intuition that he could divide 7 by 9 to quantify this measure. However, I find it significant that John had little intuition regarding the measure of the quantity as a fraction of 1 pound. In other tasks involving more common fractions, such as one-fifth, John could flexibly switch between using fractions and decimals to answer my questions. But in this case, short of carrying out the long division computation, John had little insight regarding the fractional number of pounds per cup. Significantly, his strategy for overcoming this constraint involved distributive partitioning and operating upon the 7 pounds as a sequence of 7 individual one-pound units. As soon as he recognized he could accomplish his goal of splitting the total number of pounds by splitting each individual pound, he instantly recognized the result as seven-ninths of a pound per cup.

Could John have solved the task and made sense of his result had I allowed him to carry out a long division computation? Most likely. But that is not the point. Rather, I included this protocol to highlight how stepping outside the bounds of his initial conception of 7 pounds as a single composite whole supported John's use of distributive partitioning operations and his intuition of the result as seven-ninths. I hypothesize that this type of activity could support John's construction of an equivalence between his division operations and his fractional numbers more generally. If both sets of operations were available in assimilation, I infer that John could conceive of the result as 7 divided by 9 and as seven-ninths on the basis of the quantitative reasoning he could anticipate carrying out with each conception of the 7 pounds.

Constructive resources abstracted from the mathematics of John.

Constructive Resource 7: The ability to flexibly change the measurement unit of both quantities in a given ratio.

This seventh, and final, constructive resource arose as a way of accounting for my observation of the flexibility with which John could conceive of the constant covariation of two quantities. Previously, I have claimed that John perceived quantitative covariation as a dynamic process in which he could conceive of the changes in progress and actively monitor the accumulation of the extensive quantities. The clearest examples of the reasoning that led to this characterization occurred in Protocols 6.9 and 6.15. The remarkable aspect of John's reasoning in these protocols was his ability to construct new ratios and justify why each characterized the same pumping rate or crawling speed. His activity indicated that he had constructed each newly quantified ratio as an iterable unit ratio. This enabled him to anticipate that each ratio characterized the same intensive quantity because each would result in the same total accumulations of the quantities if he actually carried out the iterations. Thus, John's ability to flexibly change the measurement unit of either extensive quantity in a given ratio meant that his ability to conceive of the covariation as it progressed was not constrained to any particular measurement unit.

To exemplify the power in this way of reasoning, consider the ratios that John created in Protocol 6.15. During that protocol, John used the unit ratio 2.5 seconds per 1 cm to justify that a series of time/distance measurements all indicated the same crawling speed. However, more than that, he also determined that 0.25 seconds per 0.1 cm or 1 second per 0.4 cm represented equivalent crawling speeds. In each case, John anticipated that iterating any of these ratios would result in the same accumulations of the extensive quantities as the covariation progressed. As a result, rather than being constrained to thinking in terms of the 1 cm intervals defined by his first ratio, John could conceive of the quantities accumulating through iterations of any size unit he desired. Not only did this support his awareness that each of these ratios represented the same crawling speed, but it also facilitated his awareness that constant covariational relationships would be represented graphically as linear relationships.

In addition, John's ability to flexibly change the measurement units indicates that he had constructed the ability to use his processes for quantifying unit ratios recursively. This means that the output of his unit ratio division scheme could be taken as input for the same scheme to produce any other equivalent "unit" ratio. Further, John can use these "unit" ratios such as 0.25 seconds per 0.1 cm just as he would the unit ratio 2.5 seconds per 1 cm because both are constructed as iterable unit ratios. The only difference is the change in the measurement unit and his corresponding awareness of the covariation progressing in either 0.1 cm or 1 cm intervals.

Because John had constructed his ratios as iterable ratios, he could also construct "unit" ratios with larger measurement units. The iterable nature of his ratios enabled him to imagine the covariation continuing through several iterations of a given ratio. Then, using that result, he formed a new "unit" ratio with which to conceive of the covariational process. For instance, following the interaction in Protocol 6.15 John used his multiplicative reasoning to quantify 24 cm per 1 minute as an equivalent crawling speed. Consequently, the quantitative operations John had available in conjunction with his ability to use those operations recursively supported his ability to reconstitute the covariation in terms of any measurement unit he desired.

This finding confirms one of the hypotheses we had formed prior to conducting the teaching experiment. In particular, we hypothesized that, “What is needed is a scheme of *recursive* distributive partitioning operations” (Steffe, Liss II, et al., 2014, p. 59). Essentially, by this we meant that one would need to construct the ability to take a result of their distributive partitioning operations, which is a unit ratio, as input for the same distributive partitioning operations to produce any other but no particular equivalent ratio. John’s ability to flexibly change the measurement unit achieves the thrust of this hypothesis: He could use his available quantitative operations to transform any but no particular given ratio into any other equivalent ratio. Thus, using one’s distributive partitioning operations recursively would support the same ability to reconstitute any but no particular ratio in terms of any but no particular measurement unit.

My findings also suggest a need for a slight revision to this hypothesis. In particular, the results of this teaching experiment indicate that the operations that accomplish the desired changes in the measurement unit may not be distributive partitioning operations. In John’s case, I infer that he used iterable units, his splitting operation, and his unit ratio division scheme to support his ability to flexibly change the measurement unit. As a result, I would rephrase our initial hypothesis as follows: What is needed is the ability to use one’s quantitative operations recursively so that any but no particular ratio can be reconstituted in terms of any but no particular measurement unit. These “quantitative operations” would include the hypothesized distributive partitioning operations, which could be used to reconstitute a ratio in terms of a smaller measurement unit, as well as any other quantitative operations a student could use to flexibly change the measurement unit.

Considering Constructive Resource 7 with respect to Jack's mathematics.

After identifying this important aspect of John's reasoning, I reconsidered Jack's activity and found that Constructive Resource 7 provided a way to better understand his characteristic ways of reasoning as well. For instance, previously I described Jack's conception of ratios as indicating completed change. This characterization arose from observing that, with few exceptions, Jack did not demonstrate the same flexibility in changing the measurement units as did John. Thus, his reasoning often remained constrained to the given measurement unit.

Jack's reasoning with the crawling speed of 1.2 seconds per cm helps to clarify why I have characterized his reasoning in this way (cf. Protocol 5.15 and its continuation). Prior to the protocol, he had constructed a new unit ratio of 2.4 seconds per 2 cm and indicated he could produce as many such ratios as he wanted. This indicated he had constructed the given ratio as an iterable unit ratio and could leverage this construction to envision the covariation progressing in successive intervals of 1 cm beyond the given 1.2 seconds per 1 cm ratio. I have no doubt that Jack could conceive of time increments of less than 1 second. However, Protocol 5.15 exemplifies that he had some uncertainty as to whether or not one could still be considered to have the same crawling speed if he/she hasn't traveled 1 full centimeter. Jack's construction of the new "unit" ratio 0.6 seconds per 1/2 cm during the continuation of the protocol alleviated this uncertainty and enabled him to reconstitute his image of the covariational process in terms of 1/2 cm intervals. Thus, using his quantitative operations recursively to transform the unit ratio to an equivalent ratio with a new measurement unit accounts for the increased sophistication of his reasoning.

I believe that the fact that this type of reasoning (i.e., changing the measurement unit) was not typical of Jack's activity helps to account for the difficulty he had constructing a process for quantifying unit ratios. Recall that Jack struggled to transform a given ratio into a unit ratio within the swimming pool context. I inferred that at that point in the teaching experiment, Jack's quantitative operations did not support quantifying the result of splitting a composite unit. Doing so would have required constructing a new measurement unit. For example, quantifying a unit ratio for the given ratio of 3 inches per 5 minutes would require reconstituting the 3 inches as 5 iterations of three-fifths of an inch. Later, he overcome this constraint and developed the ability to use his distributive partitioning operations to quantify unit ratios such as this. Yet, carrying out this process only changed the measurement unit for one of the extensive quantities while maintaining the measurement unit for the concomitant extensive quantity. In the case of the unit ratio three-fifths inches per minute, this meant that Jack's subsequent operating remained constrained to whole numbers of minutes. Thus, even though he constructed the ability to quantify a unit ratio, it was only through using these operations recursively and constructing the ability to change the measurement for *both* extensive quantities that Jack's reasoning became the most flexible.

Lastly, I find it important to reiterate that I am not claiming that Jack *could not* change the given measurement unit to construct new "unit" ratios. In fact, interactions such as that in Protocol 5.15 indicate that he did at times reconstitute ratios in terms of new measurement units for both quantities. Rather, the point I am trying to make is that he often *did not* engage in this type of reasoning. Consequently, Jack primarily operated within the given measurement unit (e.g., whole numbers of centimeters or seconds) while John could more freely transform the measurement unit to suit his needs at the time (e.g., partial amounts of centimeters or seconds). Further, in the few instances in which Jack did change the measurement units for both quantities, he appeared to construct these understandings in activity. Thus, my conclusion is that John could leverage Constructive Resource 7 in assimilation while Jack had to reconstruct this awareness anew within each covariational situation. This difference accounts for my characterization of Jack's image of covariation as assimilating ratios as indications of completed change and John's image of covariation as a dynamic change in progress.

Research Question 3

> What conceptual constructs, including extensive quantitative schemes and operations, are involved in the construction of intensive quantitative schemes and operations?

In my response to the second research question, I described seven constructive resources that I abstracted from my retrospective analysis of the students' mathematical activity. I outline these constructive resources here for clarity. Then, for my response to this research question I consider possible relationships among these constructive resources regarding the construction of intensive quantities and types of reasoning they support.

Seven constructive resources that support the construction of intensive quantities:

1. Three levels of units.
2. A strategy of coordinated partitioning/iterating.
3. A splitting scheme.
4. Iterable composite units.
5. A process for quantifying unit ratios.
6. A simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units.
7. The ability to flexibly change the measurement unit of both quantities in a given ratio.

I chose to identify the important aspects of Jack's and John's reasoning as constructive resources as opposed to defining them as particular operative schemes for a particular reason—I believe these represent conceptual tools one can use in various combinations to construct intensive quantitative schemes that serve different purposes with different types of units. The most sophisticated intensive quantitative reasoning that I observed drew upon all seven of the constructive resources.

It is imperative that I also emphasize that the availability of all seven constructive resources in not a prerequisite for intensive quantitative operating. In fact, I would anticipate that students' initial forms of intensive quantitative reasoning would likely involve first constructing these conceptual resources in activity prior to abstracting them as available in assimilation. Further, students often do not need to draw upon all seven constructive resources at the same time to solve a given task. For example, using a given ratio in iteration to produce equivalent ratios whose values are multiples of the given ratio qualifies as intensive quantitative operating. In this case, a student might only need to draw upon, or possibly construct in activity, Constructive Resources 1–4 to reason successfully.

Thus, identifying the seven important aspects of the students' reasoning individually allows one to consider how a student incorporates these constructive resources in relation to the nature of the task a students is solving and the type of units upon which he/she is operating. I hypothesize that these seven constructive resources are sufficient for constructing the three understandings of intensive quantity outlined in Chapter 2: An intensive quantitative unknown, an intensive quantitative variable, and a basic rate scheme. In the sections that follow, I consider the implications of leveraging various combinations of these seven constructive resources and describe the intensive quantitative reasoning they support.

Necessary constructive resources.

I regard the constructive resources of reasoning with three levels of units and using a strategy of coordinated partitioning/iterating as necessary but insufficient for the construction of intensive quantity. In particular, the underlying use of these conceptual resources makes it reasonable to consider the role that the remaining five constructive resources play in constructing intensive quantity. Further, at a minimum one must be able to construct these resources in activity. However, constructing a more general concept of an intensive quantity and intensive quantitative schemes and operations for using that concept in further operating requires having Constructive Resources 1 and 2 available in assimilation.

First, consider the necessity of reasoning with three levels of units. Suppose that the ratio 7 seconds per 1 cm characterizes a constant covariational relationship. In order to consider this ratio an intensive quantity for an individual, it must characterize more than a single instance. In other words, it must represent a relationship between the quantities as they covary. Using this relationship to quantify measurements for different instances of the covariational process involves assimilating changes in the quantities as multiplicative changes. For example, finding the time required to travel 5 cm would involve iterating the composite unit 7 seconds and monitoring the number of iterations as one carries out his/her progressive integration activity. Thus, achieving the proportional comparison of 35 seconds per 5 cm involves taking a composite unit as a countable item. Consequently, even making a proportional comparison that only requires the availability of whole number operations requires at least constructing three levels of units in activity. Accordingly, my hypothesis is that having three levels of units available in assimilation is required for assimilating a quantity as intensive prior to carrying out any activity.

Similarly, maintaining the multiplicative relationship among the quantities while their values covary requires using a strategy of coordinated partitioning/iterating. In the previous example, identifying 35 seconds per 5 cm as a second instantiation of the intensive quantity 7 seconds per 1 cm involves at least constructing the given measurements as iterating units in activity. In general, using one's partitioning and iterating operations to carry out equivalent transformations on both quantities inherently maintains the multiplicative relationship between the quantities. In contrast, I have observed students who assimilate changes in quantities additively and use a strategy of increasing the value of each quantity by the same amount. However, unlike a strategy of coordinated partitioning/iterating, such a strategy of coordinated increases does not maintain the multiplicative relationship.

Constructive resources that generate a class of intensive quantitative schemes.

Considered together, Constructive Resources 3, 4, and 5 can be thought of as forming a class of intensive quantitative schemes. The situations of these schemes would be a given ratio of two extensive quantities and a goal of transforming the ratio in such a way that preserves a constant covariational relationship. The activity of the schemes would be the quantitative operations one carries out to achieve that goal. Lastly, the result of these schemes would be a transformed ratio that represents a successful proportional comparison.

To demonstrate one of the intensive quantitative schemes in this class, consider the following example. Suppose one wanted to transform the ratio 3 inches per 5 minutes into a unit ratio. Assimilating 1 minute as one-fifth of 5 minutes accounts for forming a goal of splitting 3 inches into five parts. This, in conjunction with iterable composite units, allows one to anticipate that accomplishing this splitting goal will produce a measured quantity such that five iterations will reconstitute the initial ratio. Lastly, one's process for quantifying unit ratios, such as distributive partitioning, represents the activity that accomplishes the splitting goal and allows one to quantify the unit ratio as three-fifths inches per minute. In this example, the situation was a given ratio and a goal of transforming this to establish the unknown change in depth for 1 minute. Then, a splitting scheme acts as the assimilating structure, a process for quantifying unit ratios represents the activity, and iterable composite units accounts for the ability to interpret the result in relation to one's expectations. Further, this particular scheme results in the production of a unit ratio that preserves the constant covariational relationship.

Suppose instead that one wanted to find the unit ratio as a means of determining the unknown depth at 37 minutes. In this case, the situation of the scheme would be a ratio of 3 inches per 5 minutes with a goal of establishing the value of an intensive quantitative unknown. Then, in addition to the reasoning described in the previous example, the iterability of the unit ratio would support identifying the intensive quantitative unknown as $3/5 \cdot 37$ inches. While similar to the previous example, one uses his/her iterable units in a different way to accomplish a different goal. In the unit ratio example, one's goal was to establish the change in depth for 1 minute and the iterable units allowed one to test the suitability of the result of their activity. However, in the second example, one's goal was to establish the change in depth for some number of minutes by reasoning with the unit ratio. Thus, in the second example iterating the unit ratio comprised the activity that allowed one to accomplish his/her goal.

Lastly, consider a third example. Suppose that one wanted to establish the change in duration needed for the water level to rise 60 inches. In this case, the iterability of one's composite units could act as the assimilating structure to establish the desired 60 inches as 20 times as large as the given 3 inches. Alternatively, one's splitting scheme could also act as the assimilating structure to establish the given 3 inches as one-twentieth of the desired 60 inches.

As a result of one's splitting scheme, this would similarly support knowing that 20 iterations of the given 3 inches would result in a change in depth of 60 inches. In either case, the activity of the scheme that would enable one to accomplish this goal would be to carry out 20 iterations of 5 minutes to establish the result of 60 inches per 100 minutes.

Returning to the class of intensive quantitative schemes more generally, I consider this a class of schemes for two reasons. First, depending upon the relationship between the values in the given ratio and the value of the desired transformed result, these constructive resources can serve different roles in the scheme. For example, one might assimilate the desired transformation using splitting or iterating operations. Similarly, one's process for establishing a unit ratio might serve as the activity in some cases, while iterating composite units might comprise the activity in other cases. Second, I also consider this a class of intensive quantitative schemes rather than a single scheme because Constructive Resource 5, a process for quantifying unit ratios, does not represent a single way of operating. Thus, different students might construct different schemes that accomplish the same goals. The merits of any particular quantification process can be debated. However, one of the vital component to the students' successful operating during the teaching experiment was that they each had constructed a process for quantifying unit ratios, not necessarily which process they used.

The role of constructive resources in the construction of a reversible distributive partitioning scheme and distributive reasoning.

A splitting scheme (Constructive Resource 3) and iterable composite units (Constructive Resource 4) support the abstraction of a simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units (Constructive Resource 6) as well as the construction of a reversible distributive partitioning scheme and distributive reasoning.

Prior to developing Constructive Resource 6, students can alternate between operating with a measured quantity as a single composite whole and as a sequence of individual units. Each conception when considered with respect to a splitting scheme affords a different understanding of the fractional relationships.

For instance, suppose one wanted to quantify the result of splitting five units into eight parts. First, consider the implication of conceiving of the five units as a single composite whole. Splitting the composite whole into eight parts produces a part that exists in a 1:8 multiplicative relationship with the whole. Hence, a splitting scheme accounts for one's awareness that this can appropriately be called one-eighth of the whole (see Figure 7.1). In this sense, he/she remains at least implicitly aware that the original unit contains five parts, but operates upon the five units as if they were a single quantity. The dashed lines in the figure are intended to capture this implicit awareness of the five units. However, within this conception, there is no way to quantify the fractional size of one part with respect to a single unit.

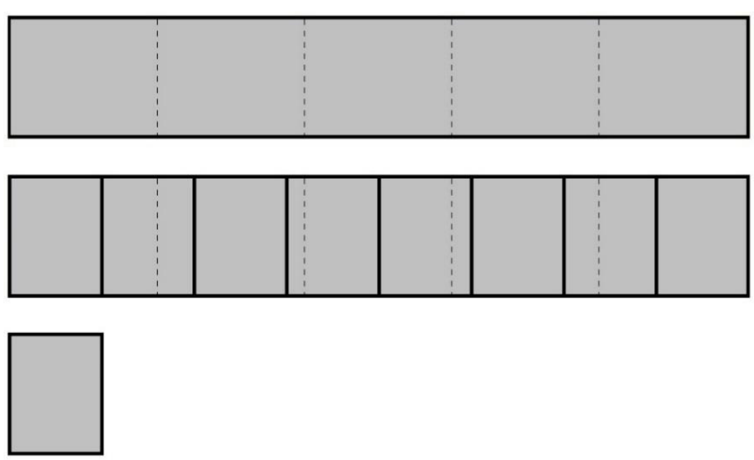

Figure 7.1. Splitting a composite five units.

Alternatively, consider the implication of assimilating the five units as a sequence of individual units. Operating within this conception, one can achieve splitting the five units by splitting each individual unit into eight parts. Then, taking one part from each unit he/she can recognize the desired result as five-eighths of one unit. Further, because splitting each unit produces a total of 40 parts, he/she can also call the result five-fortieths of the entire five units (see Figure 7.2).

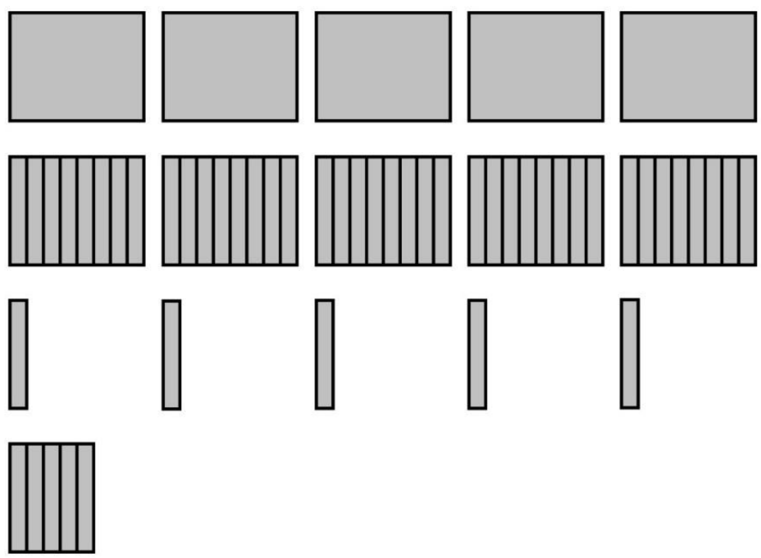

Figure 7.2. Splitting a sequence of five individual units.

Depending upon one's reason for wanting to quantify the result of splitting five units into eight parts, either one of these conceptions may be sufficient on its own. However, simply being aware of both conceptions provides no justification as to why the fractional results from each should be equivalent. Yet, as the case of Jack shows, constructing a simultaneous awareness of both views and being able to switch between them at will provides one with the greatest flexibility in his/her reasoning.

Iterable composite units (Constructive Resource 4) provide a way to account for the process by which one reconciles the alternating views to construct the simultaneous awareness that defines Constructive Resource 6. This process involves constructing five-eighths as an iterable unit. If this is accomplished, one can reconstitute the forty-eighths produced by splitting each unit as eight iterations of five-eighths. This supports the awareness that five-eighths and forty-eighths exist in a 1:8 multiplicative relationship, providing a basis for calling five-eighths of one unit one-eighth of all five units. This is precisely the coordination that Jack carried out in Protocol 5.8. The achievement of this coordination signifies the construction of what I have previously referred to as the *reversible distributive partitioning scheme* (Liss II, 2014).

Furthermore, these ways of operating account for the construction of distributive reasoning with fractional quantities. For example, using a reversible distributive partitioning scheme one could split n continuous units into m parts by splitting each of the n units into m parts. Then, one share could be constructed as n/m of one unit and also as $1/m$ of all n units. Thus, reasoning with a reversible distributive partitioning scheme supports a constructing distributive reasoning with fractional quantities: It entails both the understanding that one-m^{th} of a composite whole can be found by taking one-m^{th} of each part (i.e., $\frac{1}{m}(a+b)=\frac{1}{m}a+\frac{1}{m}b$) and that taking one-m^{th} of multiple individual units and combining those parts produces a new composite unit that is itself one-m^{th} of the totality of the multiple units (i.e., $\frac{1}{m}a+\frac{1}{m}b=\frac{1}{m}(a+b)$). As a result, Constructive Resources 3, 4, and 6 support the construction of a reversible distributive partitioning scheme as well as distributive reasoning more generally.

Constructive Resources 6 as a potential pathway to Constructive Resource 7.

Constructing a simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units (Constructive Resource 6) also creates a potential pathway to using one's operations recursively and flexibly changing the measurement units of both quantities (Constructive Resource 7). I abstracted Constructive Resource 6 from my observations of Jack's ways of operating with *whole number* measures of quantities. For instance, Jack could consider 4 inches as a single composite whole or as a sequence of four individual units of 1. Reconciling these views, he quantified the unit ratio four-fifths inches per minute. However, suppose that this result of four-fifths inches per minute was taken as input for further operating to change the measurement unit to one-third minutes. Then, using Constructive Resource 6 recursively the *fractional number* four-fifths could be simultaneously considered as a single composite whole and as a sequence of four individual units of one-fifth. Splitting each quantity into thirds, one could produce 1/3 of 4/5 as 4/15 and, hence, quantify the unit ratio 4/15 inches per 1/3 minute.

I never observed the students use distributive partitioning operations recursively in the context of covarying quantities. However, during the pilot study Jack carried out this type of reasoning to mentally compute fraction composition tasks such as 3/7 of 4/9. Thus, I infer that using Constructive Resource 6 recursively in conjunction with a reversible distributive partitioning scheme creates a pathway for abstracting Constructive Resource 7.

The significance of constructing a simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units.

At various times, I have found it tempting to think that focusing on division as the primary process of quantifying unit rations would alleviate the need for Constructive Resource 6.

The fact that John's success throughout the teaching experiment almost never relied upon the awareness afforded by Constructive Resource 6 makes a plausible case for this position. Using his unit ratio division scheme alleviated the need to consider operating upon each unit individually to solve the tasks we presented him.[24]

However, I believe that Constructive Resource 6 is vital because it provides a way to reconcile the numeric operation of division and the decimal quantities it produces with one's fraction operations. For instance, there were numerous instances throughout the teaching experiment where both Jack and John had formed a goal of splitting a composite unit but did not recognize an equivalence between division and fractions. For example, in Protocol 6.21 John recognized the task as a situation of his unit ratio division scheme and formed a goal of carrying out the computation 7 ÷ 9. While he could have carried out long division to quantify this result as a decimal quantity, he initially had no intuition that 7 divided by 9 and the fraction seven-ninths were equivalent. Yet, by reconceiving of the 7 pounds as a sequence of 7 one-pound units that could be split individually, John constructed this intuition in activity. As soon as he imagined using his distributive partitioning operations to split each pound individually, he became aware of the result as seven-ninths. Thus, abstracting the simultaneous awareness of both conceptions of a measured quantity would support conceiving of division as implying a fractional quantity and vice versa.

Speaking more generally, Constructive Resource 6 supports developing anticipation regarding the outcome of quantifying a unit ratio on the basis of one's quantitative operations. For example, consider the pattern-based intuition John constructed in the Adopt-A-Highway context. Recall that John initially assimilated the task of allocating the highway to various

[24] The alleviation here is intended with respect to my thinking as the observer, not John's.

numbers of organizations as a situation of his unit ratio division scheme and said he would divide 4 by 7 to allocate 4 miles among seven organizations. However, when asked to determine the number of miles per organization as a fractional quantity, it took the better part of an entire teaching session for John to quantify the result as four-sevenths of a mile. My analysis of his activity in subsequent teaching sessions was that he had abstracted a pattern, *number of miles / number of organizations*, that he used to quantify unit ratios throughout the remainder of the tasks in the Adopt-A-Highway context. He leveraged this pattern to great effect and could intuitively solve every task I presented him within that context.

Next, consider the intuition that John developed once he used Constructive Resource 6 during Protocol 6.21. Like the Adopt-A-Highway context, John initially assimilated the task as a situation of his unit ratio division scheme. Thus, I infer that he had formed a goal of splitting a composite 7 pounds into 9 parts. Then, after mentally reconstituting the composite 7 pounds as 7 individual one-pound units Jack intuitively recognized the result as seven-ninths of a pound per cup on the basis of his distributive partitioning operations.

Comparing both cases, I see two drawbacks to John's use of pattern-based reasoning as the source of his intuition. First, his pattern-based reasoning did not provide a mechanism for him to reconcile his numeric operation of division with the fractional quantities he produced with his pattern. Essentially there were two separate ways of reasoning for John with no inherent link between the two—divide to get a decimal or use the pattern to get a fraction. Second, the intuition John developed from abstracting this pattern remained constrained to the Adopt-A-Highway context. John never recognized tasks in the inch worm context or the task of allocating 7 pounds of water among 9 cups as situations of the pattern even though an observer might consider these as structurally equivalent tasks. Thus, John did not transfer his pattern-based reasoning to other contexts. I infer that he could have constructed a similar pattern in the inch worm context and reasoned powerfully with that as well. However, relying upon pattern-based reasoning meant that John would first need to construct a pattern for each new context before he could develop intuition within that context.

In contrast, I believe that having Constructive Resource 6 available in assimilation enables one to construct intuition that overcomes these two limitations of pattern-based reasoning. First, I have already described how Constructive Resource 6 made it possible for John to reconcile division and fractional quantities in the context of Protocol 6.21. Second, John's intuition that the result would be seven-ninths was related to the goal he formed from his initial assimilation of the task. Wanting to split 7 pounds into 9 parts, (i.e., $7 \div 9$), he realized he could split each pound into 9 parts (i.e., seven-ninths). Thus, the intuition for the fractional result relates to his splitting goal. Because this goal is not specific to any particular context, I infer that this intuition would be transferable to any context in which John formed a goal of splitting a composite unit. Thus, Constructive Resource 6 provides a way to account for the construction of the intuition that splitting a composite a units into b parts would result in the fractional amount a/b units.

Lastly, I find it important to step back for a moment and include a few remarks about how I think about John's reasoning. Throughout Chapter 6, I pointed out numerous instances in which John experienced a conflict between his reasoning with division and decimal quantities compared to his reasoning with fraction quantities. In fact, both students demonstrated this distinction between the ways they thought about division/decimals compared to fraction operations/fractional numbers. I did this to highlight that these represented two distinct ways of conceptualizing the quantities that arose from different quantitative operations.

However, I did not point out these differences to disparage the students' reasoning, nor to suggest that decimals and division should be avoided. In fact, my purposes were actually the opposite. The fact that the students could adapt to my questioning and construct ways of reasoning with either fractions or decimals indicates the sophistication of the quantitative operations that each had available. Further, I recognize that using division to produce decimal values for quantities certainly can and should play a role in students' mathematical reasoning.

Rather, my underlying purpose was to present these differences because I believe that the constructive resources presented here would enable one to reason with division/decimals and fraction operations/fractional numbers on the basis of the same quantitative operations. Briefly consider John's activity in Protocol 6.20. In that case, he used long division to compute $1 \div 5$ to get 0.2 seconds per cm, converted 0.2 to the fraction two-tenths, then used a mental procedure for reducing fractions to convert two-tenths to one-fifth, and finally arrived at the equivalent unit ratio one-fifth seconds per centimeter. In contrast, the availability of Constructive Resource 6 in assimilation could support the awareness that $1 \div 5$ was equivalent to one-fifth on the basis of one's goal of splitting the quantities. Thus, fraction and decimal quantities could be unified as two ways of characterizing the results of the same quantitative operations.

Implications for Future Research

While the results of this study provide insight into students' construction of intensive quantity, they also raise several new questions that could serve as starting points for future research. In particular, I focus on three possible directions: a) investigating the quantitative reasoning that both precedes and follows the constructive resources identified in this study; b) investigating the generalizability of these second-order models; and c) investigating the process of constructing algebraic notational systems that arise as abstractions of students' quantitative reasoning.

One possible line of inquiry would involve studying the types of quantitative reasoning that students use to make sense of intensive quantitative relationships but which precede those identified in this study. For instance, this exploratory study revealed seven constructive resources that facilitated the participants' abilities to make sense of the tasks and contexts. However, Jack and John started the study having already constructed rather sophisticated quantitative schemes and operations. The choice of these two students for my case studies was intentional as it allowed me to explore questions such as how they would use those operations to make sense of constant covariational situations and to investigate what new ways of operating they might construct while doing so. However, this approach leaves several questions unanswered. For instance, how might students who reason with one or two levels of units respond to similar tasks? What constructive resources do they use to make sense of the situations and how do these compare and relate to the constructive resources identified here?

Another similar line of inquiry would involve studying the types of quantitative reasoning that follow on from those identified in this study. For example, the final protocol in my analysis of John's mathematics revealed that intervals of change may have remained implicit during much of the students' reasoning. Thus, how might these constructive resources be brought forth in the context of intervals that do not originate from zero? Similarly, what role do these constructive resources play in making sense of situations involving non-constant rates of change?

A second avenue for future inquiry could involve investigating the generalizability of the second-order models that I constructed. In terms of modeling students' conceptual constructs, questions of generalizability pertain to the operating of the individual using the models. In particular, “It is not a matter of generalizing the results in a hypothetical way, but of the results being useful in organizing and guiding our experience of students doing mathematics” (Steffe & Thompson, 2000b, p. 300). For instance, suppose that I worked with other students who had constructed similar quantitative operations as Jack and John demonstrated during their initial interviews, and I found that those students’ reasoning developed similarly to what I observed with Jack and John. Then my models would have been useful in organizing new experiences and, hence, could be considered a more generalized concept in my thinking. However, it would also be important to pursue new activities and different types of questions than I did in this study in order to try to develop superseding models that provide more explanatory power than the current ones and better help me to organize my future experiences.

In pursuing this type of generalizability of the results of this dissertation, one could pursue questions such as the following: Would the results of this study prove useful in helping one to organize his/her experiences in the context of other students’ mathematics? What other constructive resources might better account for students’ reasoning with intensive quantitative relationships? How might these models of the students’ mathematics need to be modified to account for posing different kinds of questions in different task contexts? What other superseding models might be constructed?

An additional area for future research relates to developing a better understanding of the process by which students come to symbolize their quantitative reasoning. I consider Jack’s and John’s reasoning in this study to be algebraic. For example, both students demonstrated the ability to imagine the values of the quantities changing and to quantify the values of intensive quantitative unknowns. However, both students also experienced significant constraints using symbolic notation to characterize the relationships with which they could reason. I inferred that part of the students' difficulty stemmed from previous experiences using symbolic notation to represent unknowns to find rather than as potential measures for quantities that could vary. In addition, the particular terminology the students' used to describe their thinking also seemed to interfere at times with their efforts to symbolize their reasoning. I chose not to focus specifically on this symbolization process because I wanted to better understand the types of reasoning the students could construct.

Focusing on characterizing the types of reasoning the students could use leaves open a variety of questions regarding the symbolization of the reasoning that I identified. For instance, how might students construct an algebraic notational system that symbolizes, and develops as an abstraction from, their quantitative reasoning? However, this question is not limited to variable notation and symbols like *x* and *y*, but rather a question of symbolizing one's quantitative reasoning more generally. For example, how might students construct a notational system so that written inscriptions and spoken language are truly symbolic of the quantitative operations they can carry out rather than only symbolizing computations and procedures? Do students like Jack and John recognize $1/5 \cdot 3$ as indicating a goal of splitting three into five parts? If not, how might they construct that awareness?

Implications for School Mathematics

> The predictive power of epistemic mathematical students provides teachers and researchers with means for purposefully designing situations of learning likely to provoke students in productive ways. (Steffe & Norton, 2014, p. 321).

I find this comment orienting when considering the implications of this study in relation to school mathematics. Whether or not the models I have developed represent aspects of the reasoning one attributes to an epistemic algebraic student can only be determined over time as myself and others investigate the extent to which these models represent viable and productive ways of organizing one's future efforts to support students' learning. As they have achieved a second-order of viability in organizing my own experiences with students, I put them forth in hopes that others might also use them profitably in their efforts to support students' construction of intensive quantities and flexible algebraic reasoning.

The seven constructive resources I have elaborated provide a lens through which to reconsider standards for school mathematics and to define alternative standards in terms of the mathematics of students. As the authors of the Common Core State Standards for Mathematics state, "These Standards define what students should understand and be able to do in their study of mathematics" (NGACBP & CSSO, 2010, p. 4). Thus, standards such as the Common Core State Standards for Mathematics focus on the question of *what* mathematical knowledge a teacher should emphasize. In contrast, the constructive resources defined above highlight particular ways of reasoning abstracted from my experiences with actual students. Thus, they provide insight as to *how* a student might come to reason in ways that achieve the goals of the standards. This allows one to recast the standard in terms of the ways of reasoning a teacher should try to engender when working with his/her students. I consider this change of perspective within the context of two standards for school mathematics related to the construction of intensive quantities.

> <u>*Current Standard 1*</u>*: Students should be able to understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. (NGACBP & CSSO, 2010, p. 42)*

Achieving this standard involves the ability to reconstitute a given ratio as a unit rate and developing appropriate language to symbolize the relationships those quantities entail. Jack and John demonstrated that the construction of a reversible distributive partitioning scheme would support one's recognition that the ratio *a:b* could be reconstituted as the unit ratio *a*/*b*. Further, their reasoning highlighted the importance of assimilating the measures of the extensive quantities that comprise the unit ratio as iterable units. Doing so enabled Jack and John to establish the unit ratios they produced as unit rates. Lastly, my attempts to help the students develop a language they could use productively often times fell short of my goals. In retrospect, this is likely due to the fact that the terminology I suggested (e.g., "speed" and "rate") foregrounded my concepts as opposed to the students' ways of reasoning quantitatively. As a result, I suggest the following alternative to Current Standard 1:

Alternative Standard 1:

1a) Teachers should support students' construction of a reversible distributive partitioning scheme that they can use to transform any given ratio into an equivalent unit ratio. This involves

- engendering students' constructions of a simultaneous awareness of a measured quantity as a single composite whole and as a sequence of individual units.
- supporting students' efforts to coordinate the results of splitting each unit with the results of splitting the total quantity.

1b) Teachers should support students' efforts to use their constructed unit ratios in further activity as iterable unit ratios they can use to make sense of constant covariational relationships.

1c) Teachers should support students' construction of mathematically appropriate terminology and language by

- encouraging the students to creatively develop terminology and language that the students feel describes the quantities they use in reasoning.

- negotiating the meanings of the students' terminology with them to eventually establish the conventional terminology as a product of the students' constructive activity.

<u>*Current Standard 2*</u>*: Students should be able to choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. (NGACBP & CSSO, 2010, p. 64)*

In terms of intensive quantities, this standard could relate to producing equivalent forms of a given ratio as a means of coming to understand the properties of the constant covariational relationship the ratio describes. Jack and John have demonstrated that accomplishing this standard required the availability of quantitative operations that they could use to transform a given ratio and the ability to use these operations recursively to reconstitute the ratio in terms of new measurement units. Thus, I suggest the following alternative to Current Standard 2:

<u>*Alternative Standard 2*</u>*:*

2a) Teachers should support students' efforts to construct ways of reasoning quantitatively that enable them to transform any given ratio into an equivalent ratio with a different measurement unit.

2b) Teachers should support students' efforts to use their quantitative operations recursively in order to flexibly change the measurement unit of both quantities of the given ratio to any but no particular measurement units.

2c) Teachers should provide opportunities for students to use the ways of reasoning involved in Revised Standard 1 and Revised Standard 2 to make sense of constant covariational relationships.

The seven constructive resources that I have elaborated are not things one can to teach to students in a direct sense. Rather, they characterize specific ways of reasoning quantitatively that one might try to engender in the course of students' creative mathematical activity. Toward this goal, teachers play a vital role in the constructive trajectories of their students by virtue of the opportunities they provide for students' creative reasoning, the kinds of questions they ask, and the things that they attend to during interactive mathematical communication with students.

Each of the alternative standards that I have posed encapsulates rather complex forms of quantitative reasoning. Thus, it would be imperative that as mathematics educators we elaborate additional standards to characterize the nature of students' reasoning that would both precede and follow the alternative standards that I have suggested. However, reconsidering the standards in terms of the mathematics of students in this fashion offers teachers a vision of the constructive pathways students might traverse in order to construct sophisticated and flexible mathematical understandings.

Closing Remarks

I began this study with a goal of elaborating the role that particular numeric and fractional understandings play in the development of algebraic reasoning and the construction of intensive quantities. The results of this exploratory study elaborate seven constructive resources that help to clarify these roles and represent a first step toward this goal. I have found these conceptual resources personally useful in organizing my experiences with students.

Considering the next step, I draw inspiration from the words of Ernst von Glasersfeld. Drawing upon the writings of Alexander Bogdanov, an early 20th century forerunner of cybernetics, von Glasersfeld (1995b) stated, "Knowledge, Bogdanov says, functions as a tool. How good a tool is, or how much better it could be, comes out when a group of people work together at the same task" (p. 121). In this case, the task is coming to understand the mathematics of students and their construction of intensive quantity. I am excited about the prospects of how I and other mathematics education researchers and teachers might use the results of this study to produce increasingly more viable models of students' algebraic reasoning that serve as useful tools for organizing one's experiences with students.

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APPENDIX A

INITIAL INTERVIEW TASKS

1. *Equi-partitioning:* [Display a piece of waxed string.] Let's pretend that this string is like a big sub sandwich and imagine you are going to share this with five people. Cut off one person's share.

    a. [Witness-researcher come and whisper in the interviewer's ear.] [Witness-researcher's name] thought that your piece was too small, how would you prove that your share is the right amount?

    b. What fraction of the original sub would one person receive?

2. *Recursive Partitioning:* [Display a piece of string.] Let's pretend that this string is a piece of licorice and that you want to share this piece of licorice among three people. Can you cut off the piece of licorice that one person would get? [Let student partition and cut the piece off of the original.] Now, imagine this piece you cut off is your share and you want to share it with four of your friends. Can you cut off one of those shares? [Let student partition and cut the piece off of the first cut piece.] How much is this piece of the whole piece of licorice?

3. *Splitting:* I have a piece of string here. I want you to make a piece of string so that my string is five times longer than yours. How would you make your string?

    a. [If the student is unsure or struggles to answer appropriately, then have a roll of string available and ask the following question. Even if they answered the question well verbally, still ask them to make it with the piece of string and ask the following question:] Here's some more string. Can you use this to make your string?

    b. [If the student does not explain how/why they made their string a certain size, then ask the following:] How would you prove that my string [point to the initial piece of string we presented to them] is five times longer than the one you described/made?

    c. [If they describe a string that is longer than the one given them:] Whose string would be longer? [Also, depending upon what they say we might follow-up with "What did I ask you?" and we might need to restate if they do not remember.]

4. *Distributive Partitioning Task I:* [Present the student with two same flavor, play-doh cakes of the same sizes.] Suppose these are two (flavor) cakes, and that the two cakes are the same size. [Cover the cakes with a cover.] Now I want you to imagine that you're going to cut up the cakes.

    a. Can you tell me how you might share all the cake fairly among three people?

i. [If the student can operate mentally, then skip carrying out the cutting activity.]

ii. [If the student cannot operate mentally, show the two cakes and ask them to carry out the sharing and use sharing language:] What would you do to share these fairly among three people?

iii. [If the student puts the cake altogether, after they are finished, give a context where they can't put the two cakes together. Ask them:] Can you find a way to share the cake among three people without combining the two cakes?

b. What amount of all the cake do you have?

i. How do you know what you've found is 1/3 of the cake? How would you check if it is 1/3 of the cake?

c. What amount of one cake would one person get?

5. *Distributive Partitioning Task II:* [Present the student with two same flavor, play-doh cakes of different sizes.] Suppose these are two (flavor) cakes, but this time the cakes are different sizes. [Cover the cakes with a cover.] Now, I want you to imagine that you're going to cut up the cakes.

a. Can you tell me how you might share all the cake fairly among three people?

i. [If the student can operate mentally, then skip carrying out the cutting activity.]

ii. [If the student cannot operate mentally, show the two cakes and ask them to carry out the sharing and use sharing language:] What would you do to share these fairly among three people?

iii. [If the student puts the cake altogether, after they are finished, give a context where they can't put the two cakes together. Ask them:] Can you find a way to find 1/3 without combining the two cakes?

b. What amount of all the cake do you have?

i. If the student says "two out of six pieces" (2/6), I could follow-up with:

1. Are the pieces the same size?

2. (Pull out two small pieces and two big pieces) If we think about the amount of the cake, would that be a fair share? What amount of all the cake do you have?

3. If this is 2/6, then what would this (Pick one piece up) be? 1/6?

4. How would you check if that is 1/6 of the cake? Can you use it to make all the cake? Do you get the cake you had at the beginning?

5. I could also recognize their thinking and agree they have two out of the six pieces. However, point out that's not what I'm asking about. I'm wondering what amount of all the cake one person would have.

ii. How do you know what you've found is 1/3 of the cake? How would you check if it is 1/3 of the cake?

6. *Lemonade Mixture Problems:* Suppose that you're making lemonade for your classmates. When you mix up a pitcher of lemonade, it takes 2 ounces of lemon concentrate to make 3 cups of lemonade. But that's not enough lemonade for everyone.

   a. How many ounces of lemon concentrate would you need to make 6 cups of lemonade?

   b. What if instead you wanted 15 cups of lemonade? How much lemon concentrate would you need?

   c. How much lemonade can be made with 1 ounce of lemon concentrate?

   d. How much lemon concentrate would you use in order to make 1 cup of lemonade?

   e. How much lemon concentrate would you need in order to make 7 cups of lemonade?

      i. First have the students try to answer these verbally.

      ii. If they struggle to answer these verbally, then have paper and pencil available.

      iii. If they setup a proportion to find the value, then follow-up with something like: "That's very interesting. Can you figure it out in a different way?

7. *Units-Coordinating:* Next we're going to talk about measuring some lengths. We could measure in inches, feet, or yards. How many inches does it take to make a foot? How many feet does it take to make a yard?

   a. Please measure this table using the foot ruler [give the student the 1 foot ruler.] What did you get? [Give the student the 1 inch ruler.] How many of these would it take to measure the table?

   b. A google is 2 feet. How many googles are in 4 yards and 6 inches?

8. *Disembedding:* The tasks for equi-partitioning and recursive partitioning should allow us to determine if they are able to conceptually disembed parts from wholes without destroying the wholes. This is why we ask the students to cut the pieces off the string rather than cutting a new string and leaving the whole intact.