EXPLORING MIDDLE GRADE TEACHERS' KNOWLEDGE

OF PARTITIVE AND QUOTITIVE FRACTION DIVISION

by

SOO JIN LEE

(Under the Direction of Denise S. Mewborn)

ABSTRACT

The purpose of the present qualitative study was to investigate middle grades (Grade 5-7) mathematics teachers' knowledge of partitive and quotitive fraction division. Existing research has documented extensively that preservice and inservice teachers lack adequate preparation in the mathematics they teach (e.g., Ball, 1990, 1993). Especially, research on teachers' understanding of fraction division (e.g., Ball, 1990; Borko, 1992; Ma, 1999; Simon, 1993) has demonstrated that one or more pieces of an ideal *knowledge package* (Ma, 1999) for fraction division is missing. Although previous studies (e.g., Ball, 1990; Borko, 1992; Ma, 1999; Simon, 1993; Tirosh & Graeber, 1989) have stressed errors and constraints on teachers' knowledge of fraction division, few studies have been conducted to explore teachers' knowledge of fraction division at a *fine-grained* level (Izsák, 2008). Thus, I concentrated on teachers' operations and flexibilities with conceptual units in partitive and quotitive fraction division situations.

Specifically, I attempted to develop a model of teachers' ways of knowing fraction division by observing their performance through a sequence of division problems in which the mathematical relationship between the dividend and the divisor became increasingly complex.

This is my first step toward building a learning trajectory of teachers' ways of thinking, which can be extremely useful for thinking about how to build an effective professional development program and a teacher education program. The theoretical frame that I developed for this study emerged through analyses of teachers' participation in the professional development program where they were encouraged to reason with/attend to quantitative units using various drawings such as length and area models. As part of the larger research project, I observed all sessions of the professional development project, which met 14 times for 42 hours in a large, urban, Southern school district. The data collected for this qualitative study included videotaped lessons, reflections, and lesson graphs of the five relevant instructional meetings, and pre-, post-, and delayed-post assessment interviews and interview graphs for eight teachers.

INDEX WORDS: Teacher Knowledge, Partitive Division, Quotitive Division, Fractions

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DEDICATION

To the companion in my whole life, Jaehong, and my family for their love and support

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Background

The National Council of Teachers of Mathematics' standards for teaching and learning (NCTM, 2000) require teachers to have a richer understanding of mathematics than traditionally required. Teachers need to have a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess in order to support students' ways of knowing mathematics. Despite that, it is not as clear how different and how much richer their understanding of mathematics needs to be. A number of studies have demonstrated that supporting teachers to meet the visions of mathematics reform is difficult (e.g., Borko et al., 1992; Jaworski, 1994; Kazemi & Franke, 2004; Shifter, 1998).

Historically, the emphasis in teacher research has focused on observable behaviors rather than on teachers' cognition (e.g., process-product research framework from Good & Grouws, 1979). Researchers operating in the process-product paradigm believed that teachers' cognition could not be observed or accessed, so their research focused on determining the relationship between inputs, such as an experimental condition or a curricular treatment, and outputs, such students' performance. Since Shulman (1986) announced the 'missing paradigm' in educational research by emphasizing the importance of teachers' subject matter knowledge, research on teacher knowledge parsed teacher knowledge into subject matter content knowledge, pedagogical knowledge, pedagogical content knowledge, and curricular knowledge, and many studies of mathematics teaching have investigated teacher knowledge under this frame (e.g., Ball, 1991; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Borko & Putnam, 1996; Ma, 1999). There are some variations in the way various components of teacher knowledge are described and delineated in these studies, but most researchers agree that teachers' content knowledge and their understanding of students' learning and thinking are critical aspects of *good* teaching. In mathematics education, Ball and colleagues (Ball, Thames, & Phelps, 2008) have used the phrase *mathematical knowledge for teaching* that is related to pedagogical content knowledge to emphasize knowledge teachers use when solving problems that arise in practice—for instance, using curricular materials judiciously, choosing and using representations, skillfully interpreting and responding to students' work, and designing assessments. While research on teacher knowledge (e.g., Hill, Schilling, & Ball, 2004) in this line proved that there is a positive correlation between teacher knowledge and student achievement, research is needed to elaborate mathematical knowledge for teaching particular topics (Izsák, 2008)

Many teachers learned mathematics as inactive participants where mathematics was presented them as dualistic (either right or wrong) and consisting of routine procedures. Thus, many teachers view mathematics teaching as the careful presentation of formal, symbolic rules or procedures and often have a limited interpretation of conceptual understanding in mathematics. As reform-oriented curricula have been implemented in schools, teachers have begun to think about the importance of conceptual understanding and of engaging students as active participants in mathematics learning. However, engaging in meaningful mathematics teaching is difficult for them when they have not engaged in mathematical explorations that help them see the unifying concepts in the mathematics that they teach. Without such experience, teachers' perspectives on conceptual understanding can be limited by superficially assimilating basic principles of constructivism and believing that the term *concept* is used to refer to mathematics topics in the curriculum as mind independent entities and *understanding* to refer to computational fluency with those topics.

To mathematics educators, conceptual understanding refers to an integrated and functional grasp of mathematical ideas (Kilpatrick, Swafford, & Findell, 2001). To elaborate I will adopt von Glasersfeld's (1995) notion of "Conceptualized understanding" (p. 106), which involves awareness of the characteristics inherent in the concept or whatever one is re-presenting to oneself, which constitutes a higher level of mental functioning. For instance, teachers may be able calculate fraction division problems such as $\frac{2}{3} \div \frac{1}{4}$, but not many teachers would claim to have a "conceptualized understanding" of the sequence of operations that is involved in finding the fractional number which, when multiplied by $\frac{1}{4}$, is equal to $\frac{2}{3}$.

Conceptual knowledge cannot be transferred ready-made from one person to another but must be built up by knower on the basis of his or her own experience. As Steffe (1990) said,

> Attempts to influence the world view (believing that mathematics is a human activity and that mathematical meaning is constructed as a result of such activity) of mathematics teachers are most appropriately made in the context of their ongoing mathematical activity. Rather than select novel subject matter, however, I place the teachers in possible mathematical environments that are seemingly familiar to them. In doing so, it is one of my intentions that the teachers deepen their meaning of the mathematical concepts that they teach. (p. 168)

That is to say, to foster the development of students' mathematical meaning, teachers need to experience mathematics as active learners and realize for themselves how they can generate more powerful knowledge when their goal in learning is not limited to the contents of the curriculum.

Professional development programs can be ideal environments to influence teachers' world views because the teachers would be able to share their reasoning with other teachers. Moreover, professional development programs are great contexts for individuals to reorganize their ways of knowing by interacting with other teachers. *Egocentric* (Steffe, 1990) thought cannot support the construction of intersubjective knowledge because "It [sharing alternative ideas] sensitizes them [teachers] to the reasoning of others and encourages *decentering*–the attempt to imagine one's experience from another perspective–from their own thought processes in an attempt to understand the thinking of others" (p. 179). In other words, as teachers become aware of other points of view and consciously reflect upon their knowledge by progressively decentering, they will reorganize their previous knowledge and generate more viable knowledge.

In spite of a growing number of professional development opportunities for teachers, scholars have criticized that many of them that are currently available for teachers are woefully inadequate. They are not formulated based on the studies that inform how teachers learn, contents of the program are superficial, and their forms are fragmented (Ball & Cohen, 1999; Hill, 2007). Therefore, it is critical for mathematics educators to formulate effective professional development opportunities for teachers, yet how to design such a program is still a big question. As we expect teachers to design their lessons based on their students' knowledge, mathematics educators should design a professional development program based on their understanding of teachers' knowledge. Thus, for the present study I used data from a professional development program to examine teachers' partitive and quotitive division knowledge with fractions. The professional development course was designed to support teachers in learning about the mathematics necessary for teaching the new standards. The course also utilized various technologies such as the Fraction Bar software and Wikispaces, which supported teachers' decentering process.

While studies (e.g., Borko et al., 1997; Campbell & White, 1997; Cobb, Wood, & Yackel, 1990; Schifter, 1998; Simon & Schifter, 1991, 1993) abound in mathematics education that describe the relationship between teachers' knowledge and professional development programs, few studies were conducted to explain the differences in teachers' learning during the programs. Understanding teachers' assimilatory structures, which consists of operations that activate the key knowledge constructs to solve various mathematical problems, is important in order to understand teacher learning. If we understand how teachers construct their knowledge, we can use it as a guideline to provide more meaningful professional development programs. Unfortunately, while researchers have provided considerable insight into students' conceptual understanding of rational numbers, the literature has yet to provide comparable insight into teachers' knowledge of these concepts. The lack of research in this area was problematic for finding a theoretical framework for the present study, which was one reason that I used the ideas from the literature about children's learning.

Specifically, in the present study I investigated middle grade (Grade 5-7) teachers' capacities to reason with fractional quantities in partitive and quotitive division situations during a professional development program. Studies have emphasized that fraction concepts lay the foundation for the study of other advanced mathematics topics such as proportion and algebra (NCTM, 2004). In spite of the significance of these concepts, they are among the most complicated and abstract concepts for many elementary and middle school students. In fact, state and national assessments have shown that students fail to grasp the conceptual ideas of fractions.

Whereas some researchers (e.g., the Rational Numbers Project) attribute children's errors with fractions to their deep-rooted whole-number knowledge, others, with whom I agree, have argued that the curricula that we used for the children and the way in which we have taught the curricula, are problematic. Researchers subscribing to this point of view (Olive, 1999; Olive & Steffe, 2002; Pithethly & Hunting, 1996; Steffe, 2002, 2004; Tzur, 1999, 2000, 2004) asserted from their studies that children's previous whole-number knowledge sometimes interferes with students' learning of rational number concepts. These researchers pointed out how we (as adults) may interfere with children's ways of constructing mathematical concepts by infusing our language and concepts, which, for some children, make little sense. Believing that children's mathematical knowledge should correspond to and be explained by conventional mathematical concepts and operations, mathematics educators traditionally have regarded the content of children's mathematical knowledge as fixed and a priori.

The *Fractions Project* (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2004; Steffe & Olive, 2010; Tzur, 1999, 2000, 2004) built upon the findings from the intensive longitudinal clinical interviews that Piaget, Inhelder, and Szeminska (1960) conducted with children. Based on the Fractions Project study, it is reasonable to help children use whole number knowledge in constructing fraction knowledge because children's fractional reasoning involves operations that are also involved in whole-number reasoning. At any rate, whole-number knowledge is a main body of prior that students bring to the study of rational numbers. I believe those operations that generate children's fraction knowledge should be part of teachers' mathematical knowledge for teaching. However, as we cannot expect students to assimilate exactly what teachers tell them, teachers will learn about children's mathematics meaningfully when they experience themselves operations embedded in various, so called, algorithms.

Traditionally, division of fractions has often been taught by emphasizing the algorithmic procedure “invert and multiply” without considering students’ ways of constructing fraction division knowledge. While teachers do see the value of students’ invented algorithms, they could not guide children to expand the construction because they did not understand the conceptual underpinnings of fraction division themselves.

Although previous studies (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999) have stressed errors and constraints on teachers’ knowledge of fraction division, they have not considered teachers’ capacities to reason in a sequence of fraction division situations. For instance, we know partitioning and units structure (including conceptual units) are fundamental knowledge elements for constructing fractional knowledge, but teachers may be able to use other, perhaps more sophisticated, methods to solve problems. Further, even though the studies have revealed various knowledge components that are critical in developing teachers’ knowledge of fraction, very few studies analyzed teachers’ knowledge at a finer grain size. Thus, I focused on one approach for supporting teachers’ development of mathematical knowledge for teaching fraction division by emphasizing the relationship between the units and operations associated with partitive and quotitive fraction division. The research questions that guided this study were:

- What operations and conceptual units did teachers use to make sense of partitive fraction division problems?
- What operations and conceptual units did teachers use in reasoning about quotitive fraction division problems, and how did they modify their initial conceptions of quotitive divsion?

I attempted to develop a model of the fractional knowledge of elementary and middle school teachers to see where the model concurs with or deviates from existing models of children's fractional knowledge. This is the first step toward building a learning trajectory of teachers' ways of thinking, which can be extremely useful for thinking about how to build an effective professional development program and a teacher education program. I adapted the meaning of building a learning trajectory of teachers' knowledge from Simon's hypothetical learning trajectory (1995). A *hypothetical learning trajectory* (HLT) is the sequential development and enrichment offered to learners as a result of working on a sequence of tasks and interacting with their teacher. It is an anticipated possible path of concept development and technique improvement. According to Simon (1995), HLT is similar to the zone of potential construction from von Glasersfeld (1995) or the zone of proximal development from Vygotsky (1978). Thus, a HLT is a second-order model because is a model that the observer constructs of the observed, and constructing second-order models of the knowledge of learners is an important goal in education (Steffe, 1995). In my study, I adapted the constructivists' use of learning trajectory and models of knowledge to the study of teachers; hence whenever I use the terms, they are my construction of teachers' knowledge of partitive and quotitive fraction division.

CHAPTER II

LITERATURE REVIEW AND THEORETICAL ORIENTATIONS

Literature Review

In this chapter, I reviewed literature relevant to the present study and the theoretical orientations by which the present study was guided. The review of the literature is organized into three parts: 1) Teachers' difficulties with division of fractions; 2) Teachers' primitive knowledge of multiplication and division; 3) Teachers' insufficient problem-solving strategies; 4) Teachers' difficulties reasoning with conceptual units.

Teachers' Difficulties with Division of Fractions

So far, research on teachers' knowledge of fractions has revealed more of what teachers do *not* know about fraction division and multiplication than what they know or how they know what they know. Ball (1990) asked U.S elementary and secondary prospective teachers to represent the problem $1\frac{3}{4} \div \frac{1}{2}$, and only 5 out of 19 teachers generate appropriate word problems or situations even though most of them could calculate the solution using the "invert and multiply" method. She found that teachers were likely to confound division by $\frac{1}{2}$ with division by 2 or multiplication by a half when they were asked to represent $1\frac{3}{4} \div \frac{1}{2}$. While she claimed that this struggle came from a dependence on partitive division, I think this is mainly due to teachers' failure to establish a quantitative relationship between the two quantities.

Simon (1993) adapted the problem from Ball's (1990) study and asked prospective elementary teachers to write a story problem for $\frac{3}{4} \div \frac{1}{4}$. Twenty-three out of 33 teachers could not create an appropriate word problem. Twelve of those 23 teachers used "invert and multiply" and represented the problem as multiplication thereby creating a problem situation for multiplication rather than for division. Borko et al. (1992) also examined prospective teachers' knowledge as exhibited during their student teaching and reported the case study of a middle school teacher, Ms. Daniels. When asked by a child to explain why the invert-and-multiply algorithm worked, Ms. Daniels attempted to explain by using an area model but failed to show fraction division; instead she modeled fraction multiplication, despite having completed a fair number of mathematics courses in her undergraduate program and being able to compute accurate answers with the invert and multiply method. The middle grades teachers that Armstrong and Bezuk (1995) investigated during a professional development program not only conflated division and multiplication situations when fractions were involved but also showed evidence of inflexibility with regard to the referent unit concept for fraction multiplication problems. Armstrong and Bezuk revealed that teachers were extremely challenged to revisit multiplication and division of fractions. They claimed that teachers' experiences with operating with fractions are symbolically oriented and algorithmic in nature.

In her study of 23 U.S. teachers and 72 Chinese teachers, Ma (1999) introduced the idea of a *knowledge package*, which refers to pieces of knowledge consisting of numerous subtopics that are related to one another and support later, more advanced learning. For fraction division, she suggested that teachers' knowledge package should include whole-number multiplication, the concept of division, the concept of division as the inverse of multiplication, the meaning of multiplication with fractions, and the concept of unit. Ma's comparison between Chinese teachers and U.S teachers revealed that one or more pieces of the ideal teachers' knowledge packages were relatively weak for U.S teachers. She illustrated how U.S teachers were likely to confound division by $\frac{1}{2}$ with division by 2 or multiplication by $\frac{1}{2}$ when they were asked to explain $1\frac{3}{4} \div \frac{1}{2}$ using a word problem. For example, for $1\frac{3}{4} \div \frac{1}{2}$ one teacher Ms. Francine, reasoned as follows,

> So some kind of food, graham cracker maybe, because it has the four sections. You have one whole, four fourths, and then break off a quarter, we only have one and three fourths, and then we want, how are we going to divide this up so that let us say we have two people and we want to give half to one, half to the other,…, would we get three and one half, did I do it right? Let us see one, two, three, yes, that is right, one, two, three. They would each get three quarters and then one half of the other quarter…. (p. 68)

Even though she knew that $3\frac{1}{2}$ quarters (the quantity that she derived from the word problem) and $3\frac{1}{2}$ wholes (an answer that she got using the invert and multiply algorithm) are two different answers, the latter did not trigger her current operations for fraction division. This instance illustrates what I call inflexibility with the referent unit, which was stressed by Ma as one important knowledge piece for sufficient fraction division knowledge.

Teachers' Primitive Models of Multiplication and Division

Studies have also reported constraints of preservice teachers extending their knowledge of whole-number divisions to fractional contexts. Several studies were conducted to understand teachers' knowledge of fraction division using the primitive model framework of Fischbein, Deri, Nello, and Marino (1985). Tirosh and Graeber (1989) found that developing an understanding of the two models of division (partitive and quotitive) affected the choice of operations, multiplication or division. Simon (1993) revealed that inflexible and implicit use of the two models restricted prospective elementary teachers' success with fraction division problems. Behr et al. (1994) reported constraints of preservice teachers extending their knowledge of whole number division to fractional contexts. Harel and Behr (1995) interviewed 32 inservice teachers to identify and classify their strategies in solving rational number multiplication and division problems, looking for violations of basic intuitive models that were identified by Fischbein et al. (1985). It is not surprising that teachers had major difficulties in reasoning with division when they learned fractions without any conceptual linkage with whole numbers. Similarly, Graber and Tirosh (1988) revealed that teachers invert division (i.e., to divide a larger number by a smaller number) for problems like the following: “Twelve friends together bought 5 pounds of cookies. How many pounds did each friend get if they each got the same amount?”

Teachers’ Insufficient Problem Solving Strategies

In order to understand teachers’ mathematical reasoning, it is imperative to observe their problem-solving strategies because knowledge only emerges out of interpretation of one’s experience. However, most of the studies were conducted to reveal teachers’ insufficient knowledge; consequently most of the research concerning teachers’ knowledge of fractions was done to investigate their unsophisticated problem-solving strategies rather than their reorganization of their previous numerical schemes in the construction of fractional concepts and operations. Seaman and Szydlik (2007) suggested that the problem solving strategies or mathematical reasoning displayed by the participant elementary teachers differed from that of practicing mathematicians in that the teachers’ reasoning was mathematically unsophisticated and impoverished. They identified some fundamental norms of the community of mathematicians and demonstrated how such norms could help researchers understand why many preservice teachers find mathematics difficult. In the study, they interviewed 11 preservice elementary teachers and asked them to interpret a word problem involving fraction multiplication: "Brooke has a $\frac{3}{4}$ pound bag of M & M's. If she gives $\frac{1}{3}$ of the bag to Taylor, what fraction of a pound does Taylor receive?" Six out of 11 teachers used the subtraction operation because they focused on key words from the problem "give,…, to" rather than making sense of the stated situation. Finding key words can be a part of an effective problem solving strategy, but in this case it was not sophisticated enough to allow the preservice teachers to reason about the mathematical relationship in the situation. Seaman and Szydlik suggested that 3 out of 11 teachers who could successfully identify the problem as fraction multiplication used approaches similar to mathematicians:

> A mathematician faced with an unfamiliar or confusing story would attend very carefully to the language in the problem, making certain that she was able to make sense of the situation. She might draw a picture or create another model for the problem. Finally, she would consider whether her answer made sense in the context of her model. (p. 178)

To summarize, teachers confused situations calling for division by a faction with those calling for division by a whole number or multiplication by a fraction. Moreover, teachers showed a lack of capacity to devise word problems or to choose the word problems that appropriately represent the problem situations. Furthermore, as with children, teachers also appeared to be influenced by primitive models of division, which scholars blamed on teachers' whole number knowledge. Finally, the third group of scholars reported a variety of unsophisticated problem solving strategies that teachers employed to make sense of the problem situations, that were very different from the ways practicing mathematicians did. I am building on these earlier studies by considering teachers' knowledge of partitive and quotitive fraction division. While all of the studies measured teachers' understanding in a non-traditional way, they seemed limited in that they mostly used teachers' abilities to interpret or devise appropriate word problems as an alternative way to measure teachers' knowledge. I analyzed teachers' knowledge not only across various contexts and also using various forms such as devising and interpreting division problems and drawing and interpreting drawn representations.

In the following section, I will discuss a few studies that investigated teachers' knowledge at a finer grain size by emphasizing cognitive elements that are involved in context-sensitive ways. They are different from the previous studies on teacher knowledge in that they did not look at teachers' mathematical knowledge in broad categories such as common content knowledge, specialized content knowledge, etc. Particularly, the studies are relevant to the current research in that they did not simply document constraints on teachers' knowledge of fraction division but tried to explain where the constraints come from by analyzing teachers' capacities to reason with conceptual units.

Teachers' Difficulties Reasoning with Conceptual Units

Behr, Khoury, Harel, Post, and Lesh (1997) interviewed 30 preservice elementary teachers to explore their strategies for working with tasks that focused on one of the rational number subconstructs (see Kieran, 1976), namely rational number as operator. Behr et al. further decomposed the operator subconstruct into duplicator and partition-reducer, stretcher and shrinker, and multiplier and divisor. The Bundles of Sticks problem, which basically asks how many piles of sticks are in three-fourths of eight bundles of four sticks ($\frac{3}{4} \times 8(4)$), was designed by Behr et al. to allow for a variety of partitive number-exchange or quotitive size-exchange strategies to be used for the problem solution.

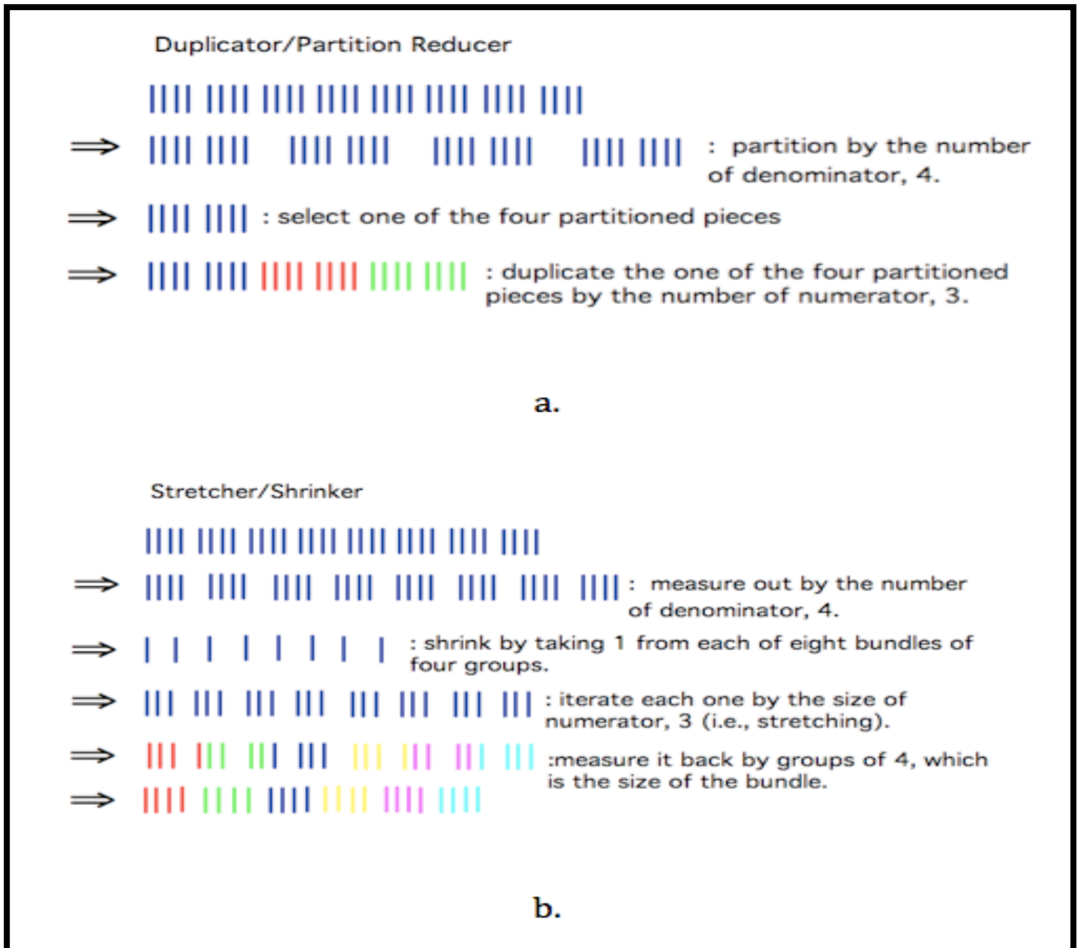

Figure 1. Determining 3/4 of 8. a. Using the duplicator and partition-reducer subconstruct. b. Using the stretcher and shrinker subconstruct[1].

In using the duplicator/partition-reducer (DPR) subconstruct (Figure 1a.), teachers partitioned 8 by the quantity of the denominator (4) and selected one of the four partitioned pieces. Then they iterate one of these pieces 3 times (numerator). On the other hand, in applying the stretcher/shrinker (SS) subconstruct (Figure 1b.), the operator 3/4 has a quotitive division effect. Teachers measured out four sticks in eight bundles by the quantity of the denominator, four, and shrank by taking 1 from each of eight bundles of four groups, and iterated each one by the size of the numerator 3 and then measured it back by groups of 4, which resulted in the final answer of 6 bundles of 4 sticks (i.e., stretching).

[1] This model is a reproduction of the model in Behr et al. (1997).

For both strategies, order is not important (i.e., teachers may duplicate first then partition and reduce or stretch and then shrink). DPR differs from SS in that DPR operates on the "number" of embedded units while SS operates on the "size" of embedded units. Moreover SS requires distributive reasoning. Behr et al. found that teachers were most likely to use DPR and SS rather than other strategies to interpret the Bundles of Sticks problem. (See Behr et al., 1997 for a complete discussion.) They found that teachers who attended to the number of bundles of sticks tried to operate on the number of units in a unit of units. However, teachers who attended to the number of sticks in bundles tried to operate on the sizes of the units in a unit of units.

Izsák et al. (Izsák, 2008; Izsák, Tillema, & Tunç-Pekkan, 2008) also provided an account of middle school teachers' mathematical knowledge for teaching fraction multiplication and addition by focusing on their use of quantitative units. In brief, Izsák et al. found that teachers' abilities to make sense of students' reasoning differed based on the teacher's conceptual unit structure. By observing teachers' recursive partitioning processes, Izsák (2008) revealed that teachers reasoning with three levels of units and explicit attention to distributive reasoning are fundamental for mathematical knowledge for teaching in the case of fraction multiplication. For instance, Ms. Archer used unsophisticated reasoning to understand a student's use of the area model of fraction multiplication for the problem $\frac{3}{4} \times \frac{1}{2}$ because she was only attending to two levels of units. The student's area model of $\frac{3}{4} \times \frac{1}{2}$ was based on cross partitioning using a fraction equivalent to $\frac{1}{2}$, namely $\frac{2}{4}$. He partitioned the area of a rectangle vertically into four parts and shaded the left most two pieces. Then he partitioned the model horizontally into four parts and shaded the top three parts of pieces that had already been shaded.

Rather than analyzing relations among nested levels of units to see if the student got the correct answer using the drawing, Ms. Archer counted the number of double shaded parts and the total number of parts, six of sixteen, and reduced the numerical answer $\frac{6}{16}$ to the one she already knew to be correct, $\frac{3}{8}$. In other words, the teacher could not make sense of her child's reasoning because she used a strategy, counting the number of double shaded parts for fraction multiplication, and this strategy was based on her use of two levels of units instead of three because the referent unit for both $\frac{1}{2}$ and $\frac{3}{4}$ was one whole. On the other hand, the child was using three levels of units in that the child was shading $\frac{3}{4}$, not of an entire whole, but of $\frac{1}{2}$ of a whole and he knew how much of the whole the shaded part comprised.

In contrast to the former study that analyzed teachers' unit structures, Izsák (2008) examined middle school teachers' knowledge of fraction addition by looking at their repeating and dividing operations while using area models and number line models. For instance, Ms. Archer, who was a sixth grade teacher, was asked in an interview to make sense of a drawing made by her student Emily to represent of $\frac{2}{5}$ of $\frac{3}{4}$. Emily divided a rectangle into four pieces vertically and shaded three pieces, and then she subdivided, again vertically, each of the three pieces into five pieces, and shaded two in each section. When Ms. Archer was asked to evaluate Emily's drawing, she looked for the answer $\frac{6}{20}$ in the drawing. Izsák noted that she had trouble describing the drawing appropriately: "Her comments focused on whether Emily might have arrived at the correct answer, not on whether Emily's drawing would allow her to relate multiple levels of nested units" (p. 128). In addition to Ms. Archer, Ms. Reese concentrated more on the final image showing the size of an “amount” than on processes for partitioning units that she perceived to be fixed (Izsák et al. 2008). For example, when she used a number line to represent fractions such as 3/5, she first marked ‘0’ and ‘1’ on each edge of the number line and always drew tick marks from left to right with equal spacing and circled the location where 3/5 lied As a result of her pattern of drawing fractions, she often changed the location of the fixed whole whenever her final point, say 5/5, was a little off the original whole that she first marked. Her strategy confused one of her students, who marked 5/5 beside 1, and marked 6/5 at 1. Izsák (2008) stated that the unit structures teachers employed shaped the purposes for which they used drawings when teaching fraction multiplication. A teacher who reasoned primarily with just two levels of units used a computed answer as a guide when developing an interpretation of her drawn representation, whereas a teacher who evidenced more consistent attention to three levels of units structures used drawings to infer a computation method (e.g., using the overlapping strategy in fraction multiplication context). He further stated that teachers would require more than an explicit attention to three levels of units structure in order to respond to students’ thinking by “inferring students’ understandings of the whole, parts of the whole, and parts of parts of the whole, and attending to the variety of ways that student might begin assembling three level unit structures as evidenced by their explanations and drawings” (p.107).

The three studies that I mentioned above are different from the previous studies on teacher knowledge in that they did not simply state that teachers have trouble explaining fraction multiplication when fractions are understood as quantitative units but attempted to examine the underlying knowledge that could account for such trouble. The studies tried to explain where the constraints come from by analyzing teachers’ capacities to reason with conceptual units. Research on multiplicative reasoning has emphasized the central role played by reasoning with conceptual units, which involves attending to conceptual units of various types and having flexibilities with multi-levels of unit structures.) While several studies (e.g., Behr, Harel, Post, & Lesh 1992; Steffe, 1994, 2001, 2003) emphasized the role of reasoning with multi-level unit structures in developing multiplicative reasoning, most of the research on this area has concentrated on students (e.g., Izsák, 2005; Smith, 1995; Steffe, 1988, 1992, 1994, 2001, 2003, 2004). A few researchers (Behr, Khoury, Harel, Post, & Lesh, 1997; Izsák, 2008; Izsál et al., 2008) have extended this area by studying teachers' capacities to reason with multi-levels of units structures when reasoning about situations that call for multiplication of fractions, but no research is yet published in the area of fraction division on either students or teachers. Although Ma (1999) identified the knowledge of units as one piece of knowledge that is essential for teachers' knowledge of fraction division, there have been no specific studies conducted that used the term referent unit to analyze teachers' knowledge of fraction division. Thus, the current study is contributing to the field of teacher knowledge research by investigating teachers' knowledge of fraction division in a finer grain size and explaining how teachers modify their operations in the sequence of division situations. Moreover, the present study is conducted under the professional development setting where teachers become active learners, which makes the current research differ from the previous studies.

Theoretical Orientations

Grounded in a constructivist perspective on ways of learning, in many respects, the perspective of knowledge in which I situated the present study is compatible with others found in the literature on cognition in mathematical domains (cf., Izsák, 2005, 2008; Piaget, 1970; Smith, 1995; Steffe, 1988, 1994, 2001). I will elaborate on the perspective of knowledge using the three characteristics Izsák (2008) used to frame his study of teachers' mathematical knowledge for teaching fraction multiplication, which is the study that I just described under the literature review section. Izsák stated, “Knowledge elements are often more fine grained than statements found in textbooks” (p.105). He highlighted the statement by comparing Ma (1999)’s knowledge package for fraction division[2] with the research by Steffe (1988, 1994, 2001, 2003, 2004) and by members of the Rational Number Project (Behr, Harel, Post, Lesh, 1994; Behr et al. 1997). Izsák (2008) compared the research by Steffe and the members of the Rational Number Project who examined knowledge at a finer grain size with Ma’s knowledge package for fraction division because the former two studies further broke down such concepts into “numerous smaller cognitive structures” (p.105). As I already described in the literature review, Izsák examined knowledge at a grain size similar to that analyzed by Steffe and members of the Rational Number Project by attending to teachers’ nested unit structures. I am building on those studies by attending to not only the nested unit structures but also various partitioning operations and teachers’ flexibilities with referent unit concepts.

The second characteristic Izsák (2008) delineated was “Knowledge elements were diverse, which means that a person’s knowledge elements related to a particular domain are not only numerous but also of different types” (p. 105) Izsák applied this perspective by finding the relationship among teachers’ nested levels of units, their pedagogical purposes for using drawings[3], and numerical aspects of multiplication (e.g., a fraction times a number is the same as a fraction of the number). By attending to the relationship between teachers’ partitioning operations and their capacities to reason with levels of units, I examined teacher knowledge of two different types.

[2] Knowledge package for fraction division is described under the literature review.
[3] Briefly explained under the literature review.

Finally, the third characteristic was “Knowledge is context sensitive and often tacit” (p. 105) Izsák (2008) went on to say that:

> A person who uses a given piece of knowledge when reasoning about one situation may not use that same piece again in further situations where, from an observer’s perspective, it might be applied productively. Furthermore, a person may only evidence some knowledge elements by responding to situations through combinations of talk, gesture, and inscription. (p. 105)

In the present study, I observed teachers’ knowledge not only across partitive and quotitive fraction division situations but also across four sequences of quotitive fraction division situations. In the following paragraphs, I will explain the four sequences that I used to analyze teachers’ knowledge of quotitive fraction division and explain the terms that are critical for understanding the current study. The theoretical frame I developed emerged through analyses of teachers’ participation in the professional development program where they were encouraged to reason with/attend to quantitative units using various drawings such as length and area models.

From my initial analysis of teachers’ knowledge of partitive division, I revealed that teachers’ operations were different in terms of four knowledge components: (1) partitioning operations (common partitioning, cross partitioning, and distributive partitioning), (2) knowledge of units (levels of units, and referent units), (3) interpretations of the problems, and (4) use of distributive reasoning (only among teachers whose reasoning seemed more sophisticated). For my analysis of teachers’ knowledge of quotitive division, I considered teachers’ measurement division knowledge across different operations and levels of units and across four different types of division situations: (1) When the divisor partitions the dividend evenly (Sequence 1[4]); (2) When the divisor does not partition the dividend evenly (Sequence 2); 3) When the denominator of the dividend and the divisor is relatively prime (Sequence 3); and 4) When the divisor is

[4] From now on, I will use Sequence 1 for the sequence when the divisor quantity evenly measures out the dividend quantity, Sequence 2 for the sequence when the divisor quantity does not evenly measure out the dividend quantity, and so forth.

bigger than the dividend (Sequence 4). Within each of these situations, I considered a sequence of tasks in which the divisor and dividend were combinations of whole numbers and fractions. The sequence of situations is not necessary hierarchical. This was meaningful because I was able to see teachers modifying their operations and attending to more complex levels of units as the sequence progressed. In the following section, I will discuss some theories relevant to understanding the current study and describe which theories guided the present study.

Definition of Referent Unit

I used the concept of *referent unit* similar to the term *adjectival quantities* that Schwartz (1988) defined. According to Schwartz, "All quantities that arise in the course of counting or measuring or in the subsequent computation with counted and/or measured quantities have referents and will be referred to as adjectival quantities." (p. 41) He further stated that all quantities have referents and that the "composing of two mathematical quantities to yield a third derived quantity can take either of two forms, referent preserving composition or referent transforming composition." (p. 41). In other words, the referent unit of the third quantity remains the same in addition and subtraction, but it is different from either of the two original quantities in multiplication and division. For instance, the referent unit of the third quantity for "3 children have 4 apples each. How many apples do they have altogether?" is the total number of apples, which is different from the referent units of the multiplier (the number of children) and of the multiplicand (the number of apples per child). However, in the following addition context, the referent unit of the quotient for "4 apples and 3 oranges are in the basket. How many fruit are in the basket?" is the total number of fruit, which is same as the referent unit of the two addends.

The Study of Schemes[5] *and Operations That are Relevant to My Study*

Table 1

Operations and Conceptual Units Emerged from the Present Study

	Partitive Fraction Division	**Quotitive Fraction Division**
Operations	Partitioning operation to coordinate a unit of divisor with a unit of dividend	Unit-Segmenting Operation
	Distributive partitioning operation	Recursive Partitioning Operation
		Common Partitioning Operation
		Cross Partitioning Operation
Conceptual Units	Measurement Units Co-Measurement Units Referent Units Levels of Units	

The Table 1 displays the knowledge components used by teachers in the present study. The two partitioning operations under partitive fraction division served as key operations when the task called for sharing, and the unit-segmenting operation was the key operation (along with the other three partitioning operations) when the task called for measurement division. Before I discuss how I adapted the terms in Table 1 for the present study, I will briefly explain how they are explained in the literatures.

According to Steffe (1992), the first dividing scheme is the *unit-segmenting scheme* (typically known as quotitive or measurement division). This entails the operation of segmenting the dividend by the divisor (i.e., unit-segmenting operation). In his teaching experiment, Steffe observed that a child needed to reason with at least two composite units in the unit-segmenting scheme: one composite unit to be segmented and the other composite unit to be used in segmenting. For example, consider how many times a child would count if dividing a pile of

[5] Although I am not using schemes in the present study, I am including some of the terms because it is impossible to describe operations that were involved in fractions (or fractions operations) without schemes from Steffe's and Olive's study. However, I will mainly focus on the dividing schemes and the schemes that entailed the operations that I used for the current study.

fifteen books by three. That child would use three as a segmenting unit to divide the composite unit fifteen. Modification of the unit-segmenting scheme in an attempt to solve situations known as partitive division engenders an equi-portioning scheme and a distributive partitioning scheme. For instance, from a box of 24 candies, if the children decide to distribute the candies evenly among four children where the candies are completely used and no candies are left over, there would be no segmenting unit available unless the children are given a trial unit or the children use four as a segmenting unit. If the children select a trial portion of the 24 candies and then iterate that portion four times to examine if the selected portion exhausts the entire set of candies, the children are referred to as using an *equi-portioning scheme* (typically known as sharing or partitive division). In contrast, for the same problem, if the children distribute one candy to each of the four children and then keep track of counting by four to find how many candies each child would get if all 24 of the candies were distributed in that way, the children are referred to as using a *distributive partitioning scheme* (more advanced partitive division knowledge)[6].

Steffe and Olive (2010) observed how two children, who were at the time assumed to have constructed an Explicitly Nested Number Sequence (i.e., children can form three levels of units structure as a result of operating because they have constructed iterable unit), used the equi-portioning scheme. After positing the portion, both of them used *progressive integration operations*[7], which is the activity of unit-coordinating scheme, to progressively integrate the portion so many times in their attempt to produce a composite unit of numerosity. In other words, if children can use an iterable unit of one as a given, they can construct the equi-portioning scheme. However, the distributive partitioning scheme, which is a critical scheme in

[6] Description of distributive partitioning operation is provided later in this chapter.

[7] According to Steffe (Steffe & Olive, 2010), this is call an *iterating units-coordinating scheme*. It is lower powered than using iterable units-coordinating scheme in that it is still additive in nature and the children can not produce a unit containing a sequence of composite units prior to engaging in the activity of their schemes.

constructing a more advanced number sequence seems to require more advanced operations because ENS children could not construct the scheme. According to Olive and Steffe (2010),

> The operations of distributing one partition over the elements of another partition to make a distributive partitioning are more involved than recursive partitioning. They entail partitioning the whole of, say, four bars into six parts by partitioning each bar into six subparts and assembling one of the six parts by taking one of the subparts from each bar. It would seem as if the operations that produce three levels of units, recursive partitioning operations, and splitting operations would be sufficient to engage in distributive reasoning. (p. 30)

The reason that children could not construct the scheme seems to be due to the fact that they were not able to re-present a composite unit containing a sequence of composite units prior to engaging in the activity of their schemes. In whole number contexts, Steffe (1992) observed children reasoning with two composite units when engaging in unit-segmenting: one composite unit to be segmented and the other composite unit to be used in segmenting. The context in which he observed the children working is similar to Sequence 1 of my analysis in that the children were only exposed to situations when the unit to be used in segmenting clearly segments the unit to be segmented. However there are two differences between my work and Steffe's work: First, the teachers in my study chose to use a unit-segmenting operation to determine a quotient for numerical expressions of division using drawn representations, whereas children in Steffe's study were given word problems where they needed to use unit-segmenting operations. Secondly, I observed teachers reasoning with unit-segmenting in fraction contexts (because they chose to use unit-segmenting only if fractions were involved), whereas Steffe observed children reasoning with unit-segmenting operations in whole number contexts (because they were given the problems to use this operations). When teachers needed to find the quotient for a division problem where the divisor quantity clearly measures out the dividend quantity, a units-segmenting operation was activated, but reasoning with only two composite units was not enough because fractions were involved. Hence, I will elaborate some of the operations that emerged in my study in the following paragraphs.

According to the Fractions Project, the ideas of *iterating* and *partitioning* have been determined to be fundamental for fraction knowledge development. This is the notion that a unit whole can be divided into any number of pieces (partitioning) and that any one of these pieces may be iterated to reconstruct the unit. Steffe (2003) defined *the unit fractional composition scheme* (known as fraction multiplication for unit fractions) as the operations that are involved in finding how much, say, 1/3 of 1/4 of a quantity is of the whole quantity. He found *recursive partitioning* as an essential component in establishing the goal to find how much three fourths of one fourth of a segment was of the segment, which is a non-partitioning goal. It is important to point out that the goal that activates recursive partitioning is, ironically, a non-partitioning goal. Producing a recursive partitioning implies that, “a child can engage in the operations that produce a unit of units of units, but in the reverse direction” (p. 240). To produce a unit of units of units, children first produce a composite unit, make multiple copies of the composite unit, and then unite the copies into a unit of units of units. However, recursive partitioning is the inverse operation of the former because it requires learners to start with the three levels of units and operate on and distribute composite units on that structure. The learner uses the structure in the service of a new goal. This operation is not only fundamental in the production of the unit fractional composition scheme but also a commensurate fractional scheme. Two fractions are *commensurate* if one of the two fractions is produced by a quantity preserving transformation of the other.

Common partitioning fractional scheme refers to a child’s ability to coordinate and compare his two number sequences involving two composite units until a common number is found. The child who had common partitioning fractional schemes could keep track of how many of each composite he had used to get to the common multiple of both composite units. In other words, it requires units-coordination at three levels to produce a coordination of two iterable composite units. To illustrate, when the child was asked to partition a bar that would allow him to pull out both one-third and one-fifth of the same bar, his procedure was to count by 3s and 5s until he found a common number in the two sequences. The common partitioning fractional scheme was a building block for the children's construction of a *measurement fractional scheme*. With fractions as *measurement units,* division of fractions becomes meaningful in that children can now answer questions like "How many thirds are in two-ninths?" When children have constructed fractions as measurement units, they can then find a *co-measurement unit* for the two fractions by constructing any fraction from any other. A co-measurement unit is defined as a measurement unit for commensurable segments; that is, segments that can be divided by a common unit without remainder. For example, one child could make one-ninth of a unit stick using one-twelfth of the stick by finding one-thirty sixth as a co-measurement unit for both one-ninth and one-twelfth.

In addition to the common partitioning operation children used to find the common partition for two fractions, cross-partitioning using an area model also emerged in Olive's teaching experiment. For instance, the children used the *cross partitioning* (distributive reasoning) operation to solve $\frac{1}{3} \times \frac{1}{5}$ by partitioning a bar vertically into three parts and horizontally into five parts to get 15 pieces of one-fifteenth automatically. In his observations of Nathan, Olive (2000) found the use of cross-partitioning operations in a fraction multiplication context differed from the common partitioning operation in that the former provides for a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts. Both the common partitioning operation and cross partitioning operation are fundamental operations to produce a *fraction composition scheme* (Steffe, 2004), which is multiplying scheme for fractions.

According to Steffe and Olive (2010), coordinating the basic units of two number sequences, such as the unit of three of a sequence of such units and a unit of four of a sequence of such units opens new possibilities in coordinating two recursive partitionings like common partitioning. Thus, common partitioning entails recursive partitioning operations. In Steffe's teaching experiment, Nathan tried to find a common partition of two bars of the same size, one partitioned into three parts and one into five parts, by iterating a unit of three and a unit of five until producing fifteen as the comon partition. Steffe said that the activity of coordinating the two composite units was carried out for the purpose of producing a fractional unit of which both one-third and one-fifth were multiples. This goal was in turn predicated on Nathan's goal of recursively partitioning each part of the 3/3-bar into a sufficient number of subparts and each part of the 5/5-bar into a sufficient number of subparts so that the entirety of both bars would be partitioned into an equal number of these subparts.

While Steffe's and Olive's research provided me with a tool to describe teachers' knowledge of partitive and quotitive fraction division, Steffe and Olive used the teaching experiment methodology (Steffe & Thompson, 2000) and constructed a trajectory of children's learning by describing their mental operations or schemes. I had no intention to make a model of each teacher's construction of knowledge because of the nature of the data collection afforded by the larger DiW project. Moreover, while Steffe and Olive made hypotheses about operations that children might need to construct fraction division knowledge, they did not have an opportunity to test these hypotheses with children. Hence, my study contributes not only to the research on teachers' knowledge but also to the broader research on learning fractions. In the following, I will explain how I apply the terms to analyze teachers' knowledge of partitive and quotitive fraction division. First of all, I will start by explaining what partitive and quotitive division is.

Defining Partitive Division and Quotitive Division

Division is typically thought of as having two different cases - quotitive and partitive. Quotitive (measurement) division is used when one wants to identify the number of groups that can be made of a particular size by measuring out a dividend. It usually answers questions such as "how many groups of the divisor are in its dividend?" (e.g., if there are eight cupcakes and each person gets two, how many people can share the cupcakes?) On the other hand, partitive (sharing) division is grounded in one's knowledge of fair-share. In partitive division, the divided is shared into groups the size of the divisor to figure out how much or how many one person or one thing gets (e.g., If four people want to share eight cupcakes, how many cupcakes would each person get?) While some studies (e.g., Simon, 1993) argue that prospective teachers' knowledge of the quotitive and partitive division supports their understanding of fraction division, others (Harel, Behr, Post, & Lesh, 1994) have explored the difficulties in extending both quotitive and partitive division to fractional contexts. In partitive fraction division, it is intuitively unfamiliar to share something by a fractional amount, and quotitive division entails greater constraint even in the whole number system because measuring out by the divisor when it is larger than the dividend is intuitively awkward for teachers. For instance, it is easy to figure out how many groups of 2 fit into 6 for $6 \div 2$, but it gets difficult to think about how many groups of 7 fit into 6 for $6 \div 7$. Given these two ways to think about fraction division and difficulties associated with them, it seems necessary to discuss operations and conceptual units teachers use to solve problems that can be modeled by partitive and quotitive division.

Before I embark upon an analysis of teachers' knowledge, I will explain some of the terms that I used to analyze the data. The premise that underpins my study is that teachers can reorganize their mathematical operations for a sharing situation through experiencing more complicated problems so that the ways could be applicable to more sharing problem situations and generalizable to a broader spectrum of problems. As teachers reorganize their ways of solving sharing problems, I assumed they would associate more knowledge elements, and my focus was to identify these knowledge elements and to describe to what extent the teachers coordinated the elements. It is by no means my assumption that teachers have never used the knowledge elements previously in their lives. Teachers may have already constructed such mathematical knowledge but may begin to be aware of it through revisiting mathematics with quantitative reasoning that has often been neglected by the curriculum. For instance, we use the unit of one as a referent for all whole number operations (e.g., why is 2 the answer to $6 \div 3$?), but many people do not recognize it until they realize the importance of referent units by facing situations that involve fractions.

In the following, I will elaborate the terms that I used for Part 1 of my analysis, which focuses on partitioning operations to produce a common multiple, cross partitioning operations, and distributive partitioning operations. Then, I will briefly explain the terms that I used to analyze teachers' knowledge of quotitive fraction division (Part II), which involves unit-segmenting operations, recursive partitioning operations, common partitioning operations, cross partitioning operations, commensurate fractions, co-measurement units.

Partitioning Operations to Produce a Common Multiple

In terms of partitioning operations, the problem situation in which the divisor and the dividend are relatively prime entails more complex operations than the situation in which the divisor or the dividend is a factor of another. To illustrate, it is easier to solve the gummy bear problem where one is asked to share six gummy bears equally among three people than to solve the candy bar problem where one is asked to share two candy bars equally among five people. To determine the answer to the gummy bear problem one might start by drawing a unit interval subdivided into six parts as shown in Figure 2a. This requires attention to just two levels of units–the whole and the six units of one. To determine how many gummy bears one person gets, one may form a three-level unit structure as in Figure 2b – the whole, six units of one, and three units of two – using one's whole number multiplication knowledge (e.g., 6 = 2 x 3). Forming the three-level structure allows one to find the number of gummy bears for one-third of the three people.

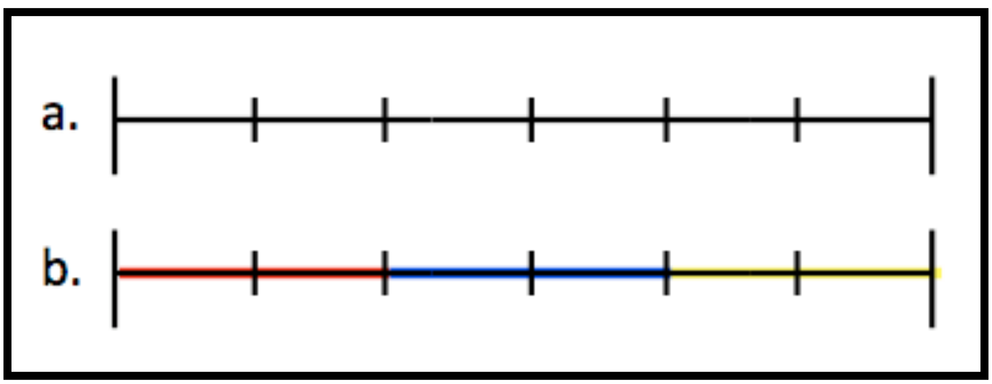

Figure 2. Determining 6 ÷ 3. a. A two-level structure for 6. b. A three-level structure for 6.

Now consider the more complicated problem of sharing two candy bars among five people. Here, using length quantities to determine the answer requires coordinating two three-level structures. One can start again by constructing a two-level structure for 2 as in Figure 3a. The new challenge is that fifths do not partition two units of one evenly. Thus, one needs to anticipate a finer partition that simultaneously subdivides twos and fifths. One way to accomplish this goal is to use one's knowledge of whole-number factor-product combinations – 2 and 5 are factors of 10. Figure 3a shows 2 as two units of five-fifths, and Figure 3b shows 2 as five units of two-fifths. One now has constructed two three-level structures from which to find one-fifth of each candy bar, that is two-fifths of one. In other words, one's initial three-level unit structure is coordinated with the second three-level unit structure of 2 in which the mid-level unit of the first three-level, two units of one, is in the background and the second mid-level unit of five units of 2/5 is in the foreground. I called this a *partitioning operation to produce a common multiple*, which is similar to a *common partitioning operation* (which is discussed later) in that both partitioning operations produce two three-level structures. However, they are different because the former operation produces common multiples[8] while the latter operation produces a common denominator between the dividend and the divisor quantities. Thus, the smallest unit from the partitioning operation for producing a common multiple is ten units of one-fifth.

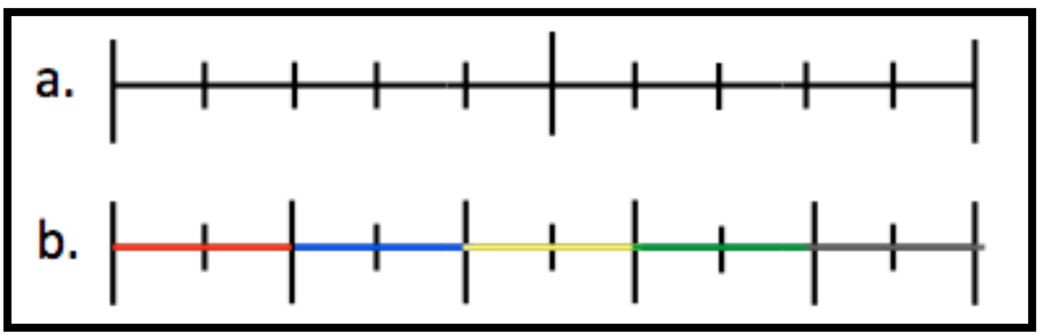

Figure 3. Determining 2 ÷ 5. a. The first three-level structure for 2. b. The second three-level structure for 2.

Whether one answers the question "How much candy bar does one person get?" or "How much of a candy bar does one person get?" could determine one's choice of the answer. Answering the former question does not necessary require one to reason with three levels of units because one can simply pull out one-fifth of two as in Figure 4a as opposed to finding how much the one-fifth of two is (i.e., two-fifths of one) in Figure 4b.

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[8] I am using plural because any common multiple between the 2 and 5 can be produced. For instance, one could subdivide each of the 2-part bars into 10 parts instead of 5 parts.

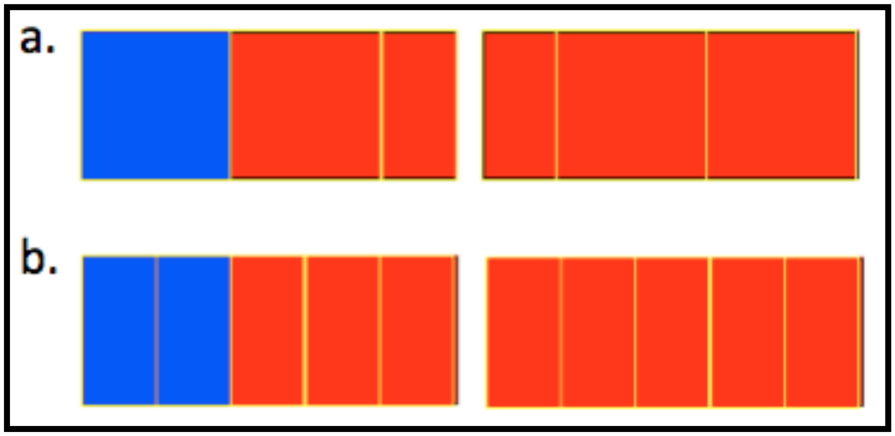

Figure 4. Determining 2 ÷ 5. a. Reasoning with two-level structures b. Reasoning with three-level structures.

One can pull out one-fifth of the two candy bars as in Figure 4a, and refer to the quantity as one person's share, but it does not tell you how much the portion is. However, when one reasons with a three-level structure, one can answer how much of a candy bar one gets, that is, two-fifths of one. In addition, when one starts with two separate candy bars (as in Figure 4) instead of one continuous bar (as in Figure 3), one needs to be aware that one piece from each bar in Figure 4b is same as two-fifths of one (i.e., distributive reasoning.)

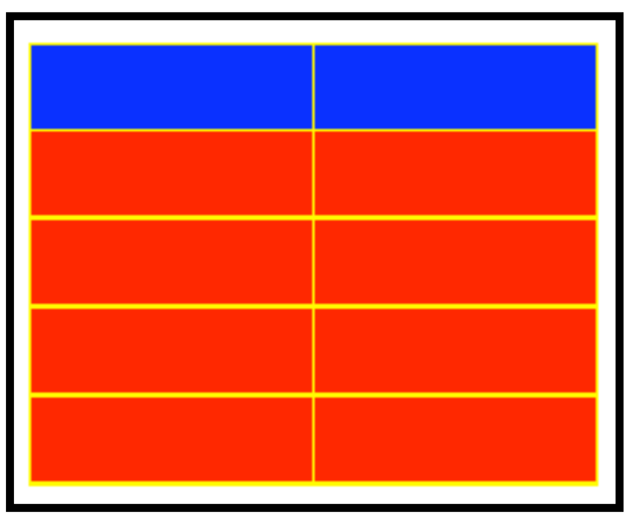

Figure 5. Determining 2 ÷ 5 using a cross partitioning.

One may use a *cross partitioning operation* to determine the answer to the candy bar problem by partitioning a bar vertically and horizontally by coordinating two composite units of

2 and 5 as in Figure 5. The cross partitioning operation is different from the common partitioning operation in that the former provides one with a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts (Olive, 1999). If one is aware of intervals of 1, then the one is reasoning with a cross partitioning operation. In other words, if one is not aware of the initial mid-level unit (two units of 1), one is not using a cross partitioning operation but a cross partitioning procedural strategy. Thus, using a cross partitioning operation implies one's reasoning with two three-level structures, whereas a cross partitioning strategy merely entails one's reasoning with two-level structures.

Distributive Partitioning Operation

According to Olive and Steffe (2010),

> The operations of distributing one partition over the elements of another partition to make a distributive partitioning are more involved than recursive partitioning. They entail partitioning the whole of, say, four bars into six parts by partitioning each bar into six subparts and assembling one of the six parts by taking one of the subparts from each bar. It would seem as if the operations that produce three levels of units, recursive partitioning operations, and splitting operations would be sufficient to engage in distributive reasoning. (p. 30)

Similar to how Steffe and Olive defined the term, a *distributive partitioning operation* is used when one partitions the whole quantity into the number of pieces in the divisor by partitioning each bar into the number of pieces in the divisor and assembles one piece from each bar (See Figure 6 for an example). It requires one's flexibility to forming three-level structures but does not necessary entail distributive reasoning.

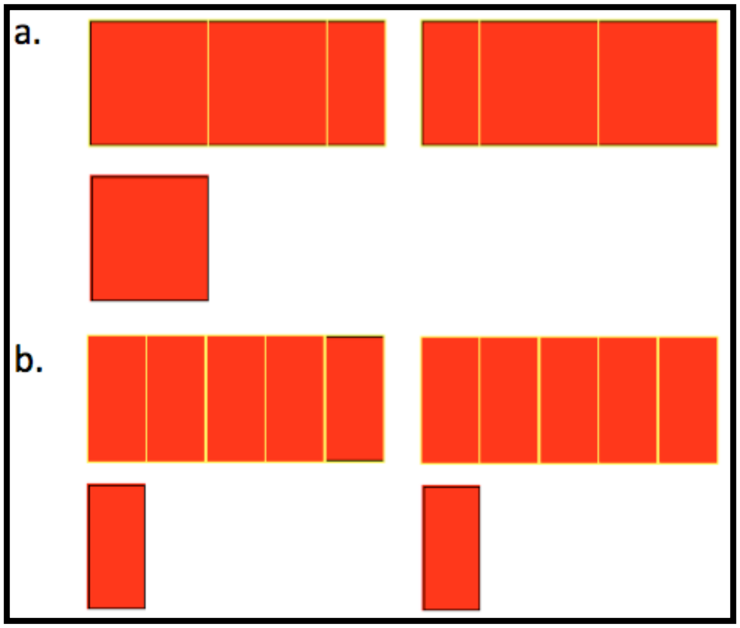

Figure 6. Determining 2 ÷ 5 pulling one of. a. Two 5-part bars. b. 5-subpart from two bars.

If a distributive partitioning operation is used with distributive reasoning, when one partitions each bar into five parts, one knows that the result of pulling one part of the bar shown in Figure 6a. is the same quantity as taking one of the subparts from each bar as shown in Figure 6b. If one cannot use the result of recursive partitioning as given material, in other words, if one cannot conceive fractions as iterable units, one may not think of 1 of the five subparts in each bar as one-fifth of 1 and assemble two of those results into two-fifths of 1. Had one's distributive reasoning supported his distributive partitioning operation, one is likely to lose track of the whole and focus on the number of pieces (10 pieces for the candy bar problem) without considering the fact that increasing the number of pieces in the whole decreases the size of the pieces. As a result, one may get 2/10 as an answer to the candy bar problem.

In quotitive fraction division, my analysis was based on teachers' using their drawings of quantities that were interpretations of numerical expressions, their interpretations of drawings of quantities that could model a measurement interpretation of division, and their statements of reasonable measurement division word problems. In measurement fraction division problem in the form of a/b ÷ c/d, one may ask oneself "How many groups of *c/d* is *a/b*?" The fundamental operation teachers used to find an answer to the problem was a *unit-segmenting operation.* This entails the operation of segmenting the dividend by the divisor. Steffe (1992) claimed that one needed to reason with at least two composite units to use a unit-segmenting operation: one composite unit to be segmented and the other composite unit to be used in segmenting. To elaborate, consider the problem of grouping a pile of fifteen books by three. One could use three as a segmenting unit to divide the composite unit fifteen and measure the fifteen by threes. While it is enough for one to reason with only two levels of units in such a simple problem situation, my claim is that one who has flexibilities with forming and transforming three-level structures could better adapt his reasoning to more complex problem situations, especially, those where one needs to reason with fractions. The simplest measurement fraction division situation that I considered in the present study was when the dividend was evenly divisible by the divisor (Sequence 1).

In sequence 1, which is typified by a problem such a 2 ÷ 1/4, one could mentally calculate the answer to the problem and say that there are eight one-fourths in two; hence, the answer is eight. On the other hand, when one needs to reason with drawn quantities, one cannot simply use mental mathematics but must *re-present*[9] one's reasoning. Similarly to the candy bar problem, one can start by constructing a two-level structure of 2 as in Figure 7a. This requires attention to just two levels of units, the whole and the two units of one. To determine how many groups of one-fourth in two, one needs to use a unit-segmenting operation and segment the unit of two by the measurement unit one-fourth. Thus, one needs to transform the two-level structure

[9] "Re-present" is used to emphasize that one cannot replicate an objective reality (i.e., representation) in the cognitive domain of organisms but can merely re-present one's experiential reality. This is compatible with Piaget's sense that one can reconstruct one's past experience that is not necessarily identical to the experience but is a re-presentation of it.

so that one can use the unit-segmenting operation. One may use a *recursive partitioning operation*. Steffe (2003, 2004) defined recursive partitioning to be taking a partition of a partition in the service of a non-partitioning goal. Consider a task that asks for the result of how many groups of 1/4 in 2 using lengths.

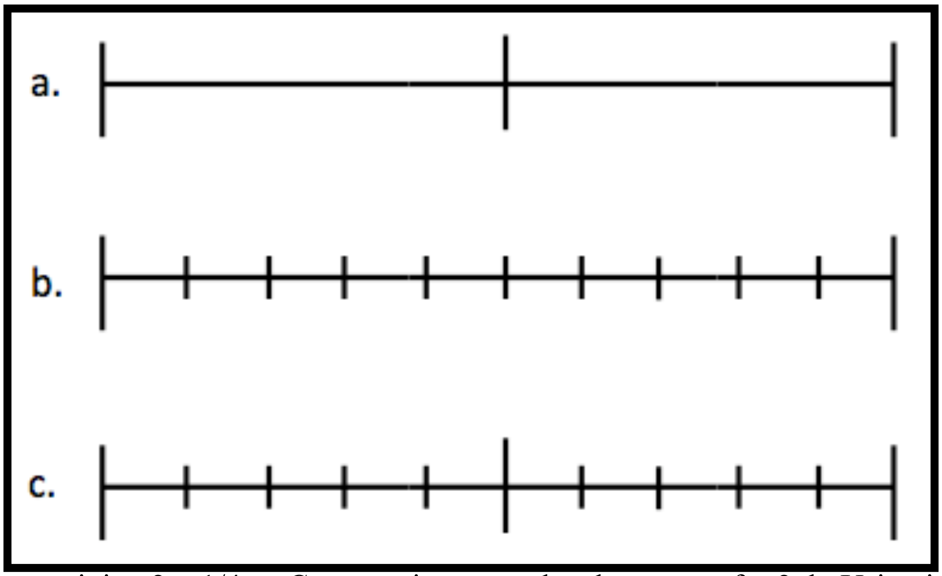

Figure 7. Determining 2 ÷ 1/4. a. Constructing a two-level structure for 2. b. Using iteration and two levels of units. c. Using recursive partitioning and three levels of units.

One may construct the requested quantity by partitioning one bar into four pieces (reasoning with two levels of units) and another bar into four pieces (reasoning with two levels of units a second time). Determining the number of 1/4 in 2 is a non-partitioning goal, and one could establish this in more than one way. One might simply iterate the smallest unit and count to see that eight copies exhaust the original unit (Figure 7b). This solution involves reasoning with two levels of units a third time (one unit containing eight fourths). Alternatively, one might recursively partitioning by subdividing two ones into four pieces (Figure 7c). In contrast to the first solution, recursive partitioning involves applying a distributing the partitioning operation across the parts of another partition (Steffe & Olive, 2010). In other words, it involves reasoning with three levels of units in which four of the smallest units are nested within each mid-level unit (two units of one), and so there must be 2 x 4 = 8 units in the whole unit.

I use the term *common partitioning operations* to refer to the partitioning operations that one uses to coordinate and compare two iterable composite units until a common number to be used in partitioning is found. In the present study, teachers used this operation when denominators of the divisor and the dividend were relatively prime (e.g., 2/3 ÷ 1/7). It requires units-coordination at three levels of units–that is, a coordination of two composite units. Some teachers used recursive partitioning operations while others used a common denominator strategy when using common partitioning operations. When teachers used common partitioning operations, they could find the commensurate fractions for the dividend and the divisor quantities using the co-measurement unit. A *co-measurement unit* is defined as a measurement unit for commensurable segments, that is, segments that can be divided by a common unit without remainder (Olive, 1999). For example, in 2/3 ÷ 1/7, one can start with a two-level structure of 2/3 and 1/7 as in Figure 8a and 8b. If one tries to use the two two-level structures of 2/3 and 1/7, it is very complicated to find how many groups of 1/7 are in 2/3 as in Figure 8c.

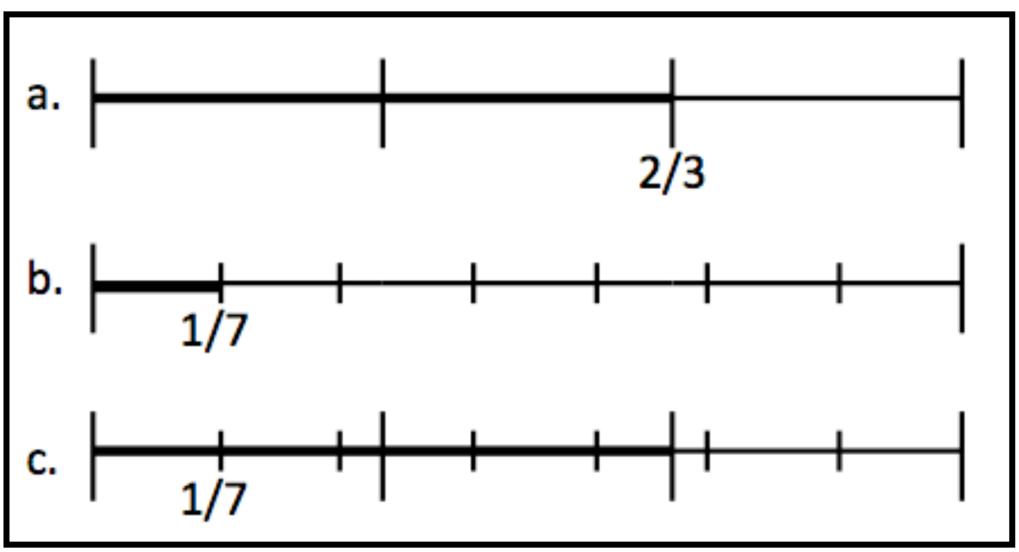

Figure 8. Determining 2/3 ÷ 1/7. a. A two-level structure for 2/3. b. A Three-level structure for 1/7. c. Using two of two-level structures of 2/3 and 1/7.

When one uses common partitioning operations, one can find two-thirds of a unit stick using one-seventh of the stick by finding 1/21 as a co-measurement unit for both one-third and one-seventh. Using the co-measurement unit as a base, one could find *commensurate fractions* for three-fourths and one-third as nine-twelfths and four-twelfths as in Figures 9a and 9b.

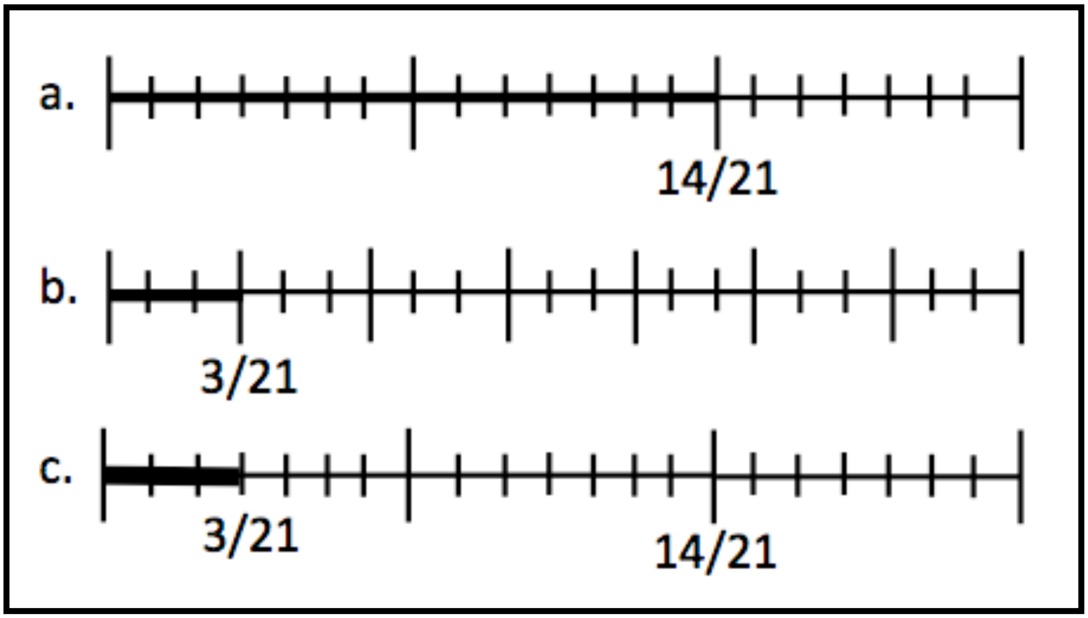

Figure 9. Determining 2/3 ÷ 1/7. a. A three-level structure of 2/3. b. A three-level structure of 1/7. c. Two three-level structures of 2/3 and 1/7.

Finding commensurate fractions entails reasoning with three-level structures. I use the term commensurate fractions (Steffe & Olive, 2010) to describe when one uses drawings of quantities to figure out, in conventional terms, equivalent fractions. Then, one who could use the co-measurement unit as an iterable unit can use a whole number three-level structure to find the quotient. In other words, for this person, solving 2/3 ÷ 1/7 is the same as solving 14 ÷ 3 because 1/21 is an abstracted entity that is not restricted to partitive fraction scheme (i.e., one has constructed iterative fraction scheme) and it will ease one's complexity in solving the problem. One may also attend to three levels of units to determine how much of 3/21 is 14/21. At any rate, one coordinates two three-level structures to determine the quotient for 2/3 ÷ 1/7 as in Figure 9c.

It is hard to tell whether a teacher constructed a unit fraction as an iterable unit because none of the division problems we gave included improper fractions. One may also calculate a common denominator between the divisor and the dividend quantities and then re-present one's reasoning using drawn quantities. As long as one can clearly explain the co-measurement unit and commensurate fractions for the divisor and the dividend quantities, I include calculating a common denominator beforehand as part of common partitioning operations. In these cases I stated that the person made a conceptual association between procedural knowledge and common partitioning operations. One can also use a cross partitioning operation to determine the answer to 2/3 ÷ 1/7 using the measurement interpretation of division by partitioning a bar vertically into three parts and horizontally into seven parts as in Figure 10. Similarly to using a common partitioning operation, this requires coordinating two composite units of the divisor and the dividend.

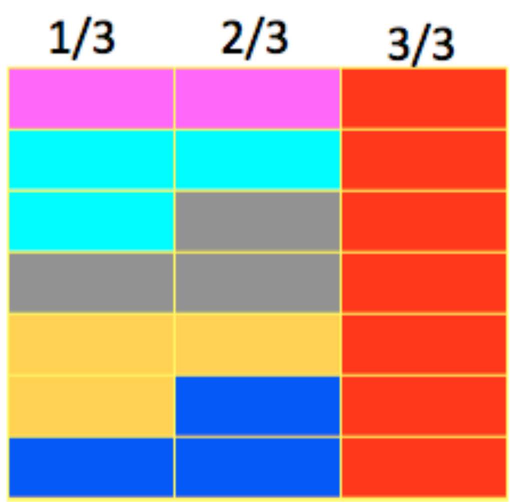

Figure 10. Reasoning with cross partitioning operation to determine 2/3 ÷ 1/7.

CHAPTER III

METHODOLOGY

The present study was conducted within the activities of an ongoing project, *Does it Work?: Building Methods for Understanding Effects of Professional Development* (DiW), funded by the National Science Foundation (NSF). The aim of DiW was to understand teachers' learning of rational numbers during professional development and to evaluate whether increases in teacher knowledge lead to increases in student achievement. DiW funded numerous graduate assistantships, and the project was my assistantship for Fall 2007 – Fall 2009. The DiW research team consisted of three professors and graduate students from three different departments (instructional technology, mathematics education, and educational measurement.)

The project offered professional development programs to middle school teachers in urban districts at four sites. This research was conducted in large, urban district. The professional development program was based on a 42-hour professional development course that focused on multiplication and division of fractions, decimals, and proportions and ratios. The course was an *InterMath* course developed for the DiW research and was designed by the DiW team to support teachers in learning about the mathematics necessary for teaching the new standards in Georgia, known as Georgia Performance Standards (GPS), which are similar to standards developed by the National Council of Teachers of Mathematics (2000). The state standards address content areas, such as number and operations, and processes of mathematical thinking, such as representing and solving problems. To meet the new state standards, teachers need not only to compute efficiently and accurately with fractions, decimals, and proportions, but also to reason about fractions, decimals, and proportions embedded in problem situations. The InterMath course provided teachers with opportunities to develop their content knowledge of multiplication and division of fractions/decimals and to explore direct and inverse proportions by engaging them in solving technology-enhanced, task-based investigations and by exploring a variety of drawn representations.

The content of the course was directly related to the state standards for fifth, sixth, and seventh grade mathematics. The InterMath course focused on the three themes for the purpose of DiW research: referent unit, drawn representations, and proportionality. Thus, most of the InterMath tasks somehow incorporated one or two of the three themes. Teachers were not only asked to come up with various drawn representations, such as length model and area model, but also needed to interpret various drawn representations. The computer software, Fraction Bars (Orrill, undated), opened the possibility for the participating teachers to create and enact various operations (i.e., partitioning, disembedding, iterating, pullout, break, etc.) on various geometric figures such as rectangles and squares. The Fraction Bars helped the participating teachers to easily represent their mathematical thinking in perceptual material so that they could not only develop their own mathematical knowledge but also communicate their mathematical reasoning with other teachers.

Context

The InterMath class met 3 hours per week for 14 weeks from September 2008 to December. 2008.The facilitator of the InterMath[10] class typically began each session by posing a warm-up problem or a task that contained the topic (See Table 1) of the week. She always gave teachers small group discussion time to let teachers to explore the problems before she began the

[10] From now on, for the convenience, I will refer to the course as 'InterMath' rather than the InterMath Augusta course.

whole group discussion. In small group discussion, she facilitated teachers' use of various drawn representations and always tried to encourage teachers to work together. After the whole group discussions of approximately two tasks, each week teachers were given the tasks to write-up and post on the InterMath course website, which was a Wiki space. Finally, teachers needed to complete a reflection question before they left class at the end of every class meeting except for the last meeting.

The DiW project administered a multiple-choice pre-assessment (August. 2008), post-assessment (December 2008), and delayed-post-assessment (May 2009) that was developed by modifying (with permission) the middle grades Learning Mathematics for Teaching measure of the Mathematical Knowledge for Teaching project (Hill, 2007). In addition, follow-up interviews were conducted with eight teachers after each assessment (see Izsák, Orrill, Cohen, Brown, 2009 for further information on the assessment.)

Table 2

Syllabus of the InterMath Course

Weeks	**Dates in 2008**	**Content of InterMath**
Week 0	August 26th	Pre-assessment
Week 1	September 9th	Parts
Week 2	September 16th	More Parts
Week 3	September 23rd	Parts of Parts (Fraction Multiplication)
Week 4	September 30th	Pesky Parts I (Fraction Division)
Week 5	October 7th	Pesky Parts II
Week 6	October 14th	Decimal Multiplication
Week 7	October 21st	Decimal Division
Week 8	October 28th	Recap
Week 9	November 4th	Ratios
Week 10	November 18th	Proportions
Week 11	December 2nd	Inverse Proportions
Week 12	December 9th	Mathematical Connections
Week 13	December 16th	Post-assessment

Participants

The course participants included 12 sixth and seventh grade mathematics teachers, 1 fifth grade teacher, and a technology specialist based at the board of education building who was not a mathematics teacher but had been a mathematics tutor for a long time. Half of the teachers (except for the technology specialist) had teaching experience of more than 10 years. Among the 12 (grade 6-7) teachers, two were special education teachers at the middle school level. Ten participants had Master's degrees, and one had an Education Specialist degree. Teachers came from seven schools in the district. Four of the 12 teachers came from the same middle school, and 2 schools had their sixth and seventh grade teachers participate in the InterMath course together. A few of the teachers seemed to know each other from a workshop they took together in the summer of 2008.

Data Collection

Under this section, I will describe the data that I used to answer my research questions:

1. What are the operations and conceptual units that teachers used to make sense of partitive fraction division problems?
2. What are the operations and conceptual units used by teachers in reasoning with Quotitive fraction division problems, and how do they modify their initial conceptions of quotitive division knowledge?

Table 3

Data Collected for the Present Study

Data Collected	**First Question**		**Second Question**	
Videotaped Lessons & Lesson Graphs	Week 1, Week 2, Week 12		Week 4, Week 5	
Tasks	Week 1	Candy Bar problem	Week 4	Exploring Division With Fractions
	Week 2	Licorice problem	Week 5	What Does 3 ÷ 1/2

				look like?
	Week 12	Candy Bar revisit		Juice Container Problem
Reflections	Not applicable		Reflections of Week 4 and 5	
Pre-/Post-/Delayed-Post- Assessment Interviews	Not applicable		Three Problems	

Videotaped Lessons

All class sessions were recorded with two cameras; one recorded the class view, which included the facilitator and overall classroom discussion, and another camera recorded the written work view, which captured a much closer look at participants' facial expressions, hand gestures, and their working with paper or computers. Specifically, using a video recording of the work view contributed to analyzing hand gestures, which is central in understanding teachers' actions, such as shading areas, dividing, segmenting, and circling. DiW also used four microphones; two were attached to the first camera and the other two were set near teachers' desks or the instructor, depending on where interesting conversations were occurring. The two tapes were combined into a restored view (Hall, 2000) so that both the class view and written work view could be analyzed simultaneously. As Table 2 indicates, I use the Week 1, 2, and 12 data to answer the first research question and the Week 4 and 5 data to answer the second research question.

Lesson Graphs

Following the video mixing, the facilitator created a lesson graph (Izsák, 2008), which is a document parsed into key segments of the episodes, each week. A lesson graph consisted of the timeline, description, and comments. A DiW co-principal investigator who was not present when the class was taught but who watched each video each week reviewed the lesson graphs. I am only using lesson graphs of Week 1, 2, 4, 5, and 12.

Reflections

Researchers on the DiW team collected the written work produced by the teachers during their participation in the InterMath course, and this was used to supplement my analysis of teachers' knowledge from the InterMath class. At the end of every class, the instructor handed out a reflection question sheet to be completed before the participants left for the evening. A reflection question usually asked participants to reflect on what they had discussed in the InterMath classroom that day. Teachers could work alone or together in resolving the reflection questions. The following two reflections were the ones that I used for my analysis.

Assessment Interviews.

Of the 14 teachers in the InterMath course, eight were selected to participate in in-depth interviews about the three assessments given as part of the research effort. In the pre-assessment interview, the interviewers started with the questions about teachers' experience with drawn representations in other professional development programs or workshops and asked if they used them in teaching fractions. We also asked about specific items on the written assessment, asking the teachers to explain what they thought a given stem was asking, how they chose their answer, and why they did not choose other answers.

In the 14th week of the InterMath course, after they had an hour of class, teachers took the first post-test, which included similar kinds of questions[11] to the pre-test, and the DiW team conducted all post-test interviews the day after teachers took the post-test. The delayed-post-assessment interview was conducted five months after the end of the InterMath course. For both interviews, the DiW team chose 7-10 items that required teachers to reason with quantitative units. These videotaped interviews were approximately 45-60 minutes long. Each was recorded using two video cameras – one to capture the participants' facial expressions and the other to

[11] Some of the questions were the same as on the pretest.

capture their written work and hand gestures. These two sources are then mixed to create a *restored view* of the event (Hall, 2000). The videotapes were transcribed verbatim.

I used these data as additional support to answer the second research question to understand teachers' measurement fraction division knowledge. Hence, I only analyzed eight teachers' responses to three items[12] from the pre, post, and post-post assessments. In one item, teachers were to interpret how an area model could be showing 2/3 shaded and 1/2 of 2/3 unshaded or 3/2 in all. In the second item, teachers were to select an area model that best displayed $1/3 \div 1/9 = 3$, and teachers were to select a number line model that implied partitive division in the third item.

Tasks

With regard to exploring middle-grade (Grade 5 – 7) teachers' partitive fraction division knowledge, I limited myself to examining their ways of solving two problems that the teachers approached with a sharing goal. In other words, I paid more attention to teachers' operations for solving these two problems than to understand their operations for solving problems that most teachers conceived as proportion problems by setting up the proportional relationships between the two quantities of a divisor and a dividend. One was the Candy Bar problem in which teachers were asked to share two candy bars equally among five people. They were to answer, "How much of one candy bar does one person get?" The other was the licorice problem in which teachers were to share 11 inches of licorice equally among 12 people, and to answer, "How much licorice is there for one person?" All of the teachers had their own computer to construct bars (they could also use paper and pencil to draw), and for each problem the facilitator gave the

[12] Because our assessment items are secure, all descriptions depict similar items to those actually used.

teachers 10 or 15 minutes to work individually or to work in small groups prior to the whole-group discussion.

The two problems were initially given to the teachers as an attempt to get them acquainted with the various functions in the Fraction Bar software but ended up being discussed two more times after the first week's discussion. The problems were adapted from L. P. Steffe's Problem Set I: Revisiting Fractional Operations that was used in the course *Curriculum in Mathematics Education* at the University of Georgia. Asking the two problems not only helped the teachers to utilize various functions of the Fractions Bar software but also provided me with a good deal of data to answer my research question "What are teachers' operations and units that are involved in teachers' knowledge of partitive fraction division?" With regard to the second research question about teachers' knowledge of quotitive fraction division, I focused on two InterMath sessions (Week 4 and Week 5). Particularly, I analyzed the three tasks in which teachers needed to reason with fraction divisions.

Data Analysis

Data analysis occurred in stages. The first stage was ongoing analysis throughout the implementation of the professional development course. The DiW principal investigator, the facilitator, and I debriefed at the end of each session. Our discussion focused on how the participating teachers were making sense of the content and planning for future sessions. Immediately after each session, the facilitator created annotated timelines of each session using a lesson graph format. These summaries provided not only a written description of teachers' mathematical activities and interactions with the instructor but also emerging key points in teachers' reasoning that were taken into account for the next session. To answer my research questions, initially I focused on teachers' partitioning operations and flexibilities with referent units and disregarded the levels of units for two reasons: 1) teachers talked about referent units many times in the InterMath course as it was one of the InterMath themes; 2) I observed that teachers used various partitioning operations.

From the lesson graph that the facilitator created, I added another column in each of the lesson graphs[13] and wrote down my comments by focusing on teachers' partitioning operations and referent unit concepts. I highlighted the time line and descriptions whenever I saw instances where teachers used the two knowledge components. Comments that were made by both the facilitator and the co-principle investigator were fruitful. Because teachers' knowledge of partitive and of quotitive fraction division was also an interest of the co-principle investigator, he provided me with critical comments about teachers' operations in the division situations, and I was able to compare my ideas with his comments.

The second stage of analysis was retrospective analysis. I worked from our initial lesson graphs that outlined the key incidents of the class meetings to identify key episodes and developed the lesson graphs that only contained the episodes related to my research interests (which are described under data collection). From the retrospective analysis, I found that teachers used three different types of partitioning operations in the candy bar and the licorice bar problems (related to the first research question) and these were common partitioning operations, cross partitioning operations, and distributive partitioning operations. From there, I used the three partitioning operations as codes analyzing the differences in teachers' reasoning with the two partitive fraction division problems. However, I did not restrict my focus to only the three partitioning operations.

[13] As I stated under data collection, I have only focused on the five tasks (two: partitive, three: quotitive) in the lesson graphs of week 1, 2, 4, 5, and 12.

In the third stage of analysis, I generated hypotheses about the possible differences that might reside in each of the three partitioning operations. One hypothesis was about the issue of referent units. While some teachers had no difficulties finding the quotients for one person's share using quantitative units, others seemed to confound the referent unit. From there, I used the levels of units as a conceptual tool to explain the differences between the two groups. Various hypotheses that came from this stage were united into a comprehensive account of teachers' knowledge of partitive division, which is described in Chapter IV Part I.

With respect to the analysis of teachers' knowledge of quotitive division knowledge, in the second stage of analysis, I found that teachers' mathematical reasoning was different in the three situations: (1) When the divisor measures out the dividend evenly; 2) When the divisor does not measure out the dividend evenly; 3) When the divisor is a bigger than the dividend. I used the three sequences as the coding schemes to answer my second question. I generated a hypothesis that teachers seemed to use more sophisticated operations as the sequence increased. Then, I went back and forth between the video data and my lesson graphs several times and found that teachers' initial conceptions of unit-segmenting operations were reorganized in each sequence by using more sophisticated partitioning operations and by attending to the referent unit. From the second analysis, I came up with the following sequence (Figure 11).

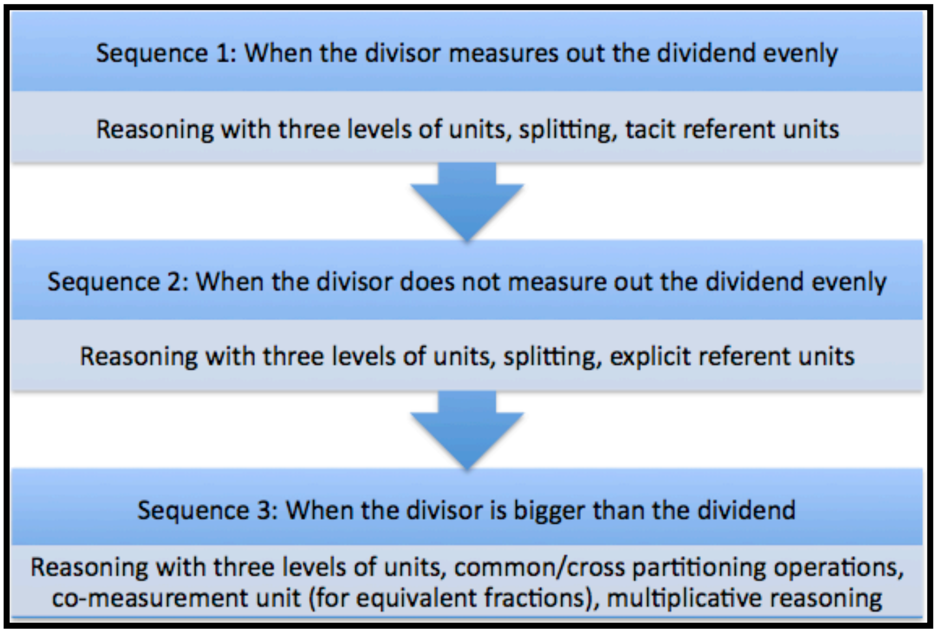

Figure 11. Sequences of quotitive fraction division and associated knowledge components that were initially developed.

During the third round of analysis I realized errors in my analysis. I re-watched the videos and made the third lesson graph of the week 4 and 5 InterMath lessons. Then I realized (from conversing with the co-principle investigator) that some teachers could reason with only two levels of units throughout the sequence. This was a breakthrough for me and I concentrated on the two things: (1) the differences between teachers reasoning with two levels of units and three levels of units in using unit-segmenting operations; (2) the operations the teachers who reasoned with three levels of units used to activate the unit-segmenting operations. I also added one more sequence *When the denominators of two fractions were relatively prime*[14] in between the second and the third sequence. Moreover, at this stage, I realized that teachers thought that partitive model of division should be used when the divisor was a whole number and quotitive model of division should be used when the divisor was a fraction. These additions motivated me to look at the reflections and assessment interview data.

[14] I was not until the fourth analysis that I realized that it was unnecessary to add the third sequence.

In the fourth round of analysis, I analyzed all eight teachers' assessment interview data. From the restored views, I made interview graphs, which had the same format as the classroom lesson graphs. The interview graphs were produced to describe how the interviews proceeded along with snapshots of teachers' work. A timeline and my brief analytical notes were also included. While I only lesson graphed the classes that I used for data in my dissertation, I created interview graphs of every item the DiW team had asked during interviews. I was able to realize from creating the interview graphs that most of our teachers drew a measurement division model to show the numerical expressions in the partitive division item. This also supported my hypothesis that teachers used a partitive model of division when the divisor was a whole number and quotitive model of division when the divisor was a fraction. This hypothesis was also supported by my analysis of the reflections week 4 and 5. Following is the table that I developed after the fourth analysis.

Table 4

Summary of Operations That Teachers Used or Might Use in Each Sequence

<table>
<tr><th></th><th>Whole number dividend</th><th>Fraction dividend</th></tr>
<tr><td>Sequence 1</td><td colspan="2">• Unit-segmenting operation using two levels of units
• Recursive partitioning operation for unit-segmenting operation</td></tr>
<tr><td rowspan="2">Sequence 2</td><td>• Unit-segmenting operation using two levels of units
• Recursive partitioning operation for unit-segmenting operation</td><td>• Unit-segmenting operation using two levels of units
• Common/Cross partitioning operations for unit-segmenting operation (co-measurement unit and commensurate fractions)</td></tr>
<tr><td colspan="2">• Establishing a part-whole relationship between the leftover quantity and the divisor quantity (novel situation) was a key knowledge.</td></tr>
<tr><td>Sequence 3</td><td>Not applicable</td><td>• Sequence 2
• Sequence 4</td></tr>
</table>

Sequence 4	• Establishing a part-whole relationship between the dividend and the divisor quantity using two levels of units. • Recursive partitioning operation for establishing a part-whole relationship between the dividend quantity and the divisor quantity.	• Establishing a part-whole relationship between the dividend and the divisor quantity using two levels of units. • Common/Cross partitioning operations for establishing a part-whole relationship between the dividend quantity and the divisor quantity.

In qualitative research, notions of reliability, validity, or viability hinge on the quality of thinking and internal consistency of the witness-researcher, because a researcher doing qualitative research is, in a significant sense, the research instrument (cf. Peshkin, 1988; Richardson, 2000; Weis & Fine, 2000). However, von Glasersfeld (1995) emphasized the role others play in developing one's own thoughts and characteristics. He speaks of intersubjective knowledge as the most reliable knowledge in experiential reality[15]. Hence, I took advice and shared my thoughts with other witness-researchers of the InterMath class (the principal investigator and one of the co-principal investigators), with the chair of my dissertation committee, and with one of the graduate students who is writing his dissertation on students' constructions of fraction division knowledge under the guidance of Steffe and Olive.

According to Stake (2005), triangulation of data includes redundancy of data gathering and ensuring ways of seeing the case from different perceptions. Combining dissimilar data sources, in this case lesson/interview graphs, reflections, and videos, helped me to triangulate the data. In addition, I gathered multiple data types to support any conclusions or hypotheses I drew. I analyzed teachers' knowledge across various problem situations. Specifically, my analysis was based on teachers using their drawings of quantities that were interpretations of numerical

[15] Intersubjective knowledge is a knowledge that a person finds to be viable in her own experiential reality as well as in that of another's experiential reality (von Glasersfeld, 1995, p. 120).

expressions, their interpretations of drawings of quantities that could model a measurement interpretation of division, and their statements of reasonable measurement division word problems.

CHAPTER IV

RESULTS OF DATA ANALYSIS

Part I: Teachers' Knowledge Of Partitive Fraction Division

Overview

With regard to exploring middle grade (Grade 5 – 7) teachers' partitive fraction division knowledge, I will limit myself to examining their ways of solving two problems that the teachers approached with the sharing goal. One was the candy bar problem where teachers were asked to share two candy bars equally among five people. They were to answer, "How much of one candy bar does one person get?" The other was the licorice problem where teachers were to share 11 inches of licorice equally among 12 people, and to answer, "How much licorice is there for one person?" All of the teachers had their own computer to construct bars (they could also use paper and pencil to draw), and for each problem the facilitator gave the teachers 10 or 15 minutes to work individually or to work in small groups prior to the whole group discussion. The two problems were given to the teachers to acquaint them with the Fraction Bar software. The problems were adopted from L. P. Steffe's Problem Set I: Revisiting Fractional Operations that was used in the course *Curriculum in Mathematics Education* at the University of Georgia. Asking the two problems not only helped the teachers to utilize various functions of the Fractions Bar software but also provided a good deal of data for answering my research question "What are teachers' operations and conceptual units that are involved in teachers' knowledge of partitive fraction division?"

Teachers'[16] capacities to deal with the two similar types of sharing division problems differed in terms of four knowledge components[17]: (1) partitioning operations (partitioning operations for a common multiple and distributive partitioning), (2) Conceptual units (units structure, levels of units, and referent units), (3) interpretations of problems (includes interpreting the word problem itself and how they conceived of the divisor), and (4) use of distributive reasoning. These four knowledge components function interactively to accomplish teachers' goals for solving the problem.

Part I was organized in the following way. First, I explained how teachers interpreted the two problems and how teachers' use of distributive reasoning supported them to see the difference between the first two interpretations of the problem (which is discussed in the following paragraph). In the second section, I discussed teachers who reasoned with three levels of units versus those teachers who reasoned with only two levels of units for the candy bar and the licorice problems. While I compared the two groups, I elaborated their use of partitioning operations, conceptual units, and distributive reasoning. When the teachers' goal was to find one person's portion of the entire bar as opposed to the portion of one (or one inch) bar, it was enough for them to reason with only two levels of units. However, it was imperative for teachers to reason with three levels of units if their goal was to find how much of one (or one inch) bar. It does not necessary mean that teachers who reasoned with three levels of units all set up the goal of finding how much of one (or one inch) bar one would get.

[16] I would like to emphasize that my analysis is only based on those teachers who were captured on camera. By no means am I saying that all teachers could or did use the four knowledge components.
[17] These knowledge components are explained in Chapter 3 with examples.

Interpretations of the Candy Bar Problem and the Licorice Problem

As I stated under the theoretical framework, teachers' interpretation of the word problems influenced their ways of solving the problem. In presenting the primary operations that the IM teachers used for dividing whole numbers that are relatively prime, I will begin with three different ways that teachers interpreted the same candy bar problem, and then I will explain how it determined teachers' strategies for solving the candy bar problem. As a project team, we did not expect that these teachers would interpret the problem in multiple ways.

Some teachers shared two *identical*[18] candy bars equally among five people and determined how much of one candy bar one person gets. Others discussed the possibility for sharing two *different* candy bars among five people and figured out one person's portion of two candy bars or of each candy bar. The others insisted on sharing only one candy bar among five people and disregarded the second candy bar. The third interpretation came about because the second sentence of the question asked how much *of one candy bar* does one person get rather than asking how much *of two candy bars*. They thought that there was too much information in the question and assumed the first sentence was there to trick them.

The candy bar problem was first[19] presented during the first class of InterMath when the teachers only had about 20 minutes of practice with the Fraction Bar software, so they were still learning how to split a bar, how to make a copy, etc. In fact, one of our purposes for giving them this task was to provide teachers with an opportunity to practice key functions of the Fraction Bar software. Thus, we did not expect we would spend an hour discussing this problem.

The discussion of whether the two candy bars were the same size or not arose in the whole group discussion when the instructor asked why it was important to figure out how to

[18] I used an *identical* candy bar to refer to the candy bar having the same size and characteristics to the other one, not necessarily the self same candy bar.

[19] InterMath had two more class discussions of the candy bar problem after this day.

copy the bar to solve the problem. After giving them a moment to use the Fraction Bar software to solve the candy bar problem by themselves, Rachael (the instructor) provided teachers with the two identical candy bars constructed using the Fraction Bar software and initiated the discussion by asking why it was important to copy the first bar to have two bars instead of making another bar. I guess Rachael tried to emphasize the importance of using the identical bars because often teachers do not see the importance of keeping the whole and drawing the bars that are different in sizes[20].

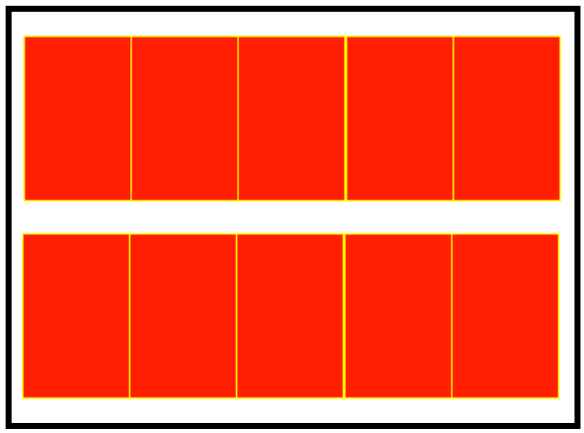

Figure 12. Reconstruction of the two 5-part[21] bars.

Protocol 1: Discussion of "Are the candy bars equal or not?"

RA[22] (Rachael): So the question that I have for you is why do we care about copying the bar? Why not just make two bars?
RO: Because they are exactly the same.
WI: You take one and split it into five equal parts and then you copy it, then you got two of them split up into five parts.
CL: So they are both the same size. Basically that is the reason.
CA: I took one big one and made it in half and pretended that it was two.
DO: That's not two. That's one.
MI: That's what I did.

[20] This is important part of mathematical knowledge for teaching because students learn from their teachers.
[21] N-part refers to one bar partitioned into n parts (e.g., 5-part bar means a bar partitioned into five parts)
[22] For convenience, I will write the first two letters of speaker's name (e.g., Rachael = RA, Rose = RO). All names are pseudonyms.

RA: (The class all laugh)[23] We have other people doing it the same way. Don't worry. So one way is what we see up there (Figure 12) which is two separate bars, and I hear somebody say it is important to figure out how to copy so that um
WI: They are the equal.
RA: Rose, I think it was you what did you say you want?
RO: I wanted to copy mine exactly so they would be exactly the same since it says, 'two bars five equal.'[24]
RA: Okay.
KE: *Couldn't they be different sizes?*[25]
RO: But I thought she said equal.
RA: So you thought in the direction it said
CA: Two candy bars among five friends I would assume that it was two of the same candy bars.
DO: It could be two different candy bars.
KE: It could be Hershey Bar and Kit-Kat.
WI: It could be.

Most of the teachers started with two separate candy bars and had no question when Rachael initiated the discussion with two separate bars until Carrie commented that she had started with one bar and separated it into two subparts. Though Carrie said she thought of her bar that was separated into two subparts as two bars, it later prevented her from forming another three-level structure, and as a result, she conflated ten units of one-fifth with ten units of one-tenth and stated that one person got one-fifth of a whole even though she said she had attended to the question, "How much *of one candy bar* does one person get." Donna argued that Carrie was changing the whole from two to one. It was surprising for me that she did not think Carrie's model appropriate because Donna started with one bar and partitioned it into 11 subparts to indicate 11 inches of licorice for the licorice problem.

Rachael wanted to get to her point of addressing the importance of drawing the same sized bars; hence, she said it was okay whether teachers started with one bar or two separated

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[23] I will use parenthesis (….) to describe what teachers and Rachael did, and brackets [….] to describe my inferences.
[24] I will use quotation marks when I directly quote or when I think teachers or Rachael directly quoted the statements.
[25] I will italicize the statement when it seems very important.

bars, and Keith suddenly broke into the conversation and asked if the bars needed to be the same bars, which brought up the second interpretation of the problem. Some of them thought that it was stated in the problem the two candy bars were of equal size, and the others just assumed that they were the same until Keith's question. Although teachers seemed a little confused at first, some teachers seemed to realize Keith's point and agreed with him when Keith provided them with the contextual example of Hershey's and Kit-Kat chocolate bars. If the candy bars are the same, the answer to the question "How much of one candy bar does one person get?", is different than if the candy bars are different unless the question asked "How much of each candy bar does one person get?". Rachael noticed the difference and asked the whole class their opinion.

Protocol 2: Does the answer change if you have different candy bars?

RA: So if they aren't the same candy bars, how do you answer the question "How much of one candy bar does one person get?" Does that change?
PA (Pascal): No, because you still.
CA: Yes.
KE: It would be one-fifth of each candy bar.
PA: Yes. So that's one-fifth of a whole.
KE: You get one-fifth of each candy bar.
WI: Or you get one-fifth of total candy.
CL: Then how much is one-fifth of total is the question.
CL: It would be two-fifths of one bar if assuming they are the same size, but if you have different sizes, you need to have one-fifth of each bar if you want it to be equal.
WI: right.
RA: Let's hear that again, Claire. I am brainstorming. Once more time, please.
CL: Okay. If the bars are same sizes, it would be two-fifths of one bar.
RA: Okay. Pause there. Does everybody agree with that?
RO: You could do that.
MI: Yeah. That's good.
RA: Okay. Keep going.
CL: If the bars are different sizes, in order to keep it equal, you would have to say one-fifth of each of the bars.
RA: In order to keep what equal?
CL: The amount of candy each person got.
RA: Okay.
WA: (looks at Claire) Excuse me. But that is still one-fifth of all the candy.
CL: They get one-fifth of large one [bar] and one-fifth of small one [bar].
RA: It is still one-fifth of all the candy, which is what you guys were saying earlier.

CL: Right.
RA: So one-fifth of all the candy is equivalent to one-fifth of each candy bar if I have two different bars or it [one-fifth of all the candy] is equivalent to two-fifths of one candy bar if the candy bars are the same?
MI: Yes.

When Rachael asked teachers if the question “How much of one candy bar does one person get?” needed to be changed when having two different candy bars, Keith, who had brought up the issue, said the answer would be one-fifth of each candy bar, and Walt added that the answer could also be one-fifth of the total candy bars. Claire summarized the teachers’ and Rachael’s argument and stated that the answer to the candy bar problem would be two fifths of one bar assuming they are the same size and one-fifth of each bar assuming they are different. After Claire explicitly stated the difference between two situations, a lot of teachers agreed with her point, and Rachael even re-stated the difference to the whole class. In other words, these teachers were aware that the question “How much of one candy bar does one person get?” could not be answered as it is, but needed to be changed into “How much of each candy bar or how much of the whole does one person get?” Teachers discussed that answers could be different depending on which referent units they attended.

Understanding the difference was driven by some of our teachers’ abilities to reason distributively. As the protocol shows, Claire explicitly stated that one-fifth of a whole bar was equal to two-fifths of a bar if two bars were identical and to one-fifth of each of the two bars if two bars were different, which is evidence to support her use of distributive reasoning. Walt emphasized that the answer was, at any rate, one-fifth of a whole bar regardless that two answers (two-fifths of a bar and one-fifth of each bar) were given rise by the characteristics of the candy bar, and Claire agreed with him. Both Claire and Walt showed a strong sense of distributive reasoning, and the fact that some of the other teachers (Rose, Mike, Will) agreed with them provides evidence that some of our teachers could reason distributively from the very first InterMath class.

So far, we saw how two different interpretations of the candy bar problem led to two different answers. In addition to the two that were mentioned here, a few teachers brought up the third interpretation of the problem when Rachael asked teachers to assume that the candy bars were the same so that they could discuss the problem with one interpretation. After she made this assumption, she asked teachers again how much of one candy bar would one person get, and most teachers immediately said “two-fifths of one candy bar” except for those few teachers who said “one-fifth of one candy bar.” Rachael specified her question by asking teachers how much she would get if she was one of the five people to share two identical candy bars.

Protocol 3: The third interpretation: You don’t even need to see the second one!

RA: So I am one of the people let’s say. I am one of the people, what do I get out of these pieces (she points at the Figure 1)? What do I get?
WI: (almost immediately) One-fifth of one candy bar.
CA: (speaks almost simultaneously to Will) Of one candy bar, you have (pause); It’s asking you about one. You don’t even need to see the second one. It’s not asking you about both candy bars. It says, ‘How much of one candy bar does one person get?’ So if I have a candy bar and I have five people, I have a piece, a piece, a piece, a piece, a piece. That takes care of all.
DA (Darcy): Really, you don’t even need the second bar.
CA: You don’t need to know anything about two candy bars. *You need to know that there are five people in one candy bar.*
DA*: So it’s almost like there is too much information.*
CA: Uhum [agrees her].
WI: You have to have the first sentence because the first sentence says equally and the second sentence does not say equally.
CA: Okay. Then you can change it to say, “How much five people share one candy equally how much does each person get?” [Carrie modified the candy problem into five people share one candy bar problem, but the problem she devised was grammatically incorrect. I guess she was saying, “If five people share one candy bar equally, how much does each person get?”].
WI: But that’s not what it says.
CA: No. It is not. But (paused).

As the protocol 3 shows, when Rachael asked teachers how much she would get if she was one of the five people to share two candy bars, Will, Carrie, and Darcy all thought that Rachael would get one-fifth of one candy bar because they interpreted “How much of one candy bar does one person get” as sharing only one of two candy bars among five people. These teachers thought that the question entailed too much information and needed some revision. Note that Carrie started with one 2-part bar (See Protocol 1) and she even mentioned that she considered the two bars in the candy bar problem the same; thus she made the two bars at the beginning. However, as she said, she only shared one candy bar because she thought that the second part of the problem asked her to do so. Actually, both Carrie and Will changed their interpretation of the problem to the first one, which was to share two identical candy bars among five people.

I have discussed the three ways teachers interpreted the candy bar problem. It is interesting that one problem can result in three interpretations depending on how one conceives of the problem situation. In the following section, I will discuss teachers’ solution methods for the candy bar problem when they revisited the problem twice after the initial class. To avoid multiple interpretations, the first part of the candy bar problem was modified into “Share two identical candy bars equally among five people” after the first class. While teachers interpreted the candy bar problem in three ways, their goal was to find how much of one candy bar one person would get. On the other hand, the teachers I discussed in Part I formed two goals in solving the licorice problem mainly due to the way the licorice problem was written. I did not see the distinction between the candy bar and the licorice bar problems until I saw Diane’s reasoning. In the candy bar problem, the referent unit was one candy bar because it explicitly asked, “How much of *one candy bar* does one person get?” but for the licorice bar problem “How much licorice is there for one person?" was not specific enough and brought up two goals. Some teachers (e.g, Keith, Sharlene, and Diane) answered "How much of a whole piece of licorice (11 inches) does one person get?" while others (e.g., Walt, Claire, and Donna) answered "How much of a one-inch piece of licorice does one person get?" The answer to the former question is 1/12 whereas the answer to the latter question is 11/12.

Teachers Who Reasoned with Three Levels of Units for the Candy Bar Problem

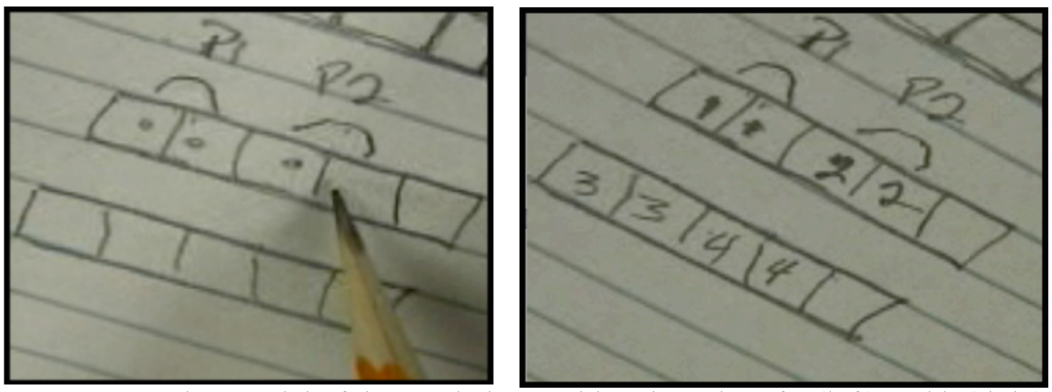

Figure 13. Walt's model of the candy bar problem in order of a (left) and b (right).

In order to solve the candy bar problem (share two candy bars among five people),Walt partitioned each of the two bars into five parts using partitioning for a common multiple as in Figure 13a. He constructed two 5-part bars to group them into five parts in an attempt to construct a structure that he could partition into five parts. First, he put dots after he connected two boxes (two-fifths of one) with an inscription P_1, and when he was about to put the second dot in boxes that he connected with P_2 (See Figure 13a), he replaced dots and P_1 & P_2 with numerals (Figure 13b). Putting numerals in each box was definitely a more efficient choice that he made than to use two inscriptions. From that, he produced the second three levels of units of the two 5-part bars as one unit of two, five units of two fifths, and ten units of one fifth. Although he did not put the numeral five in the rightmost box of both candy bars, he knew one from each 5-part bar was equal to two-fifths of one candy bar as he explicitly said one person got two-fifths of one candy bar as he was talking to his partner Will. He was not confused by the two pieces in the rightmost side of each candy bar; thus, his common partitioning operation and distributive reasoning enabled him to identify two pieces as two-fifths of one bar rather than one-fifth of each bar.

Figure 14. Mike's coordination of two three-level structures.

Mike was another teacher who used common partitioning. While Walt used various symbols (dots, numerals, and P_1 and P_2) to indicate each of the five people, Mike used five colors as a tool to coordinate the second three-level structure. When Rachael asked him if Figure 14 was what he was thinking to solve the candy bar problem, he responded to her:

> So let's say four of the people get two-fifths of one candy bar. The other guy gets one-fifth of each [candy bar]. So that's a matter of how you divide it. Or you can have everybody gets one-fifth of one candy bar, and they get from each one. So it's a matter of how you divide them up.

By coloring two 5-part bars in two fifths using five different colors, he was able to conceive the whole as one unit of two, five units of two fifths, and ten units of one fifth (See Figure 14). Even though Mike's drawing in Figure 14 shows his use of a partitioning for a common multiple, he was also aware that the problem could be solved with a distributive partitioning operation. The fact that he was aware that two-fifths of one candy bar was equal to pulling one-fifth of each candy bar indicates that he was aware of the distribution of drawn quantities embedded in his model.

Teachers Who Reasoned with Three Levels of Units for the Licorice Problem

Figure 15. Claire’s model of the licorice problem[26].

Protocol 4: Claire explaining to Carrie how she came up with Figure 15 during whole-group Discussion.

CL: I used each bar to represent an inch and I put up eleven bars, so that is an eleven inches and I split each of the twelve equal parts, and so each person would get 11/12 of an inch and then the twelfth person would get the last little pieces of everybody [each bar], which will be eleven twelfths total, so you have twelve people getting 12 equal shares of 11 inches.
CA: Say it one more time Claire please?
CL: Okay, there is 11 bar that represents 11 inches.
CA: Eleven bars going down.
CL: No across.
CA: Okay.
CL: Eleven horizontal bars each one represents one inch.
CA: Okay.
CL: and I split each bar into 12 equal parts. So the first person would get 11/12 [of the first bar], second person would get 11/12 of the second bar, etc, and then the twelfth person would get the blue pieces [purple in the screen] that ends the every little bar which comes out to 11/12 again.

[26] Note that Claire commented that the 11 bars were supposed to be equal in size. Because it was only the second InterMath class, she was not accustomed to the Fraction Bar software.

Claire used a partitioning operation for a common multiple to solve the licorice problem. Claire conceived of the 11-inch licorice bar as 11 groups of twelve one inch bars and laid each bar horizontally as in Figure 15 (i.e., Claire made 11 12-part bars). Then she colored the 11 12-part bars to show the distribution to 12 people. She colored eleven-twelfths of each 12-part bar in green to show a portion for 11 people and then colored the remaining one-twelfth of each 12-part bar in purple to show a portion of the 12th person. Thus, her partitioning operation was supported by distributive reasoning in that she knew that the rightmost column in Figure 7 was not just one-twelfth of each 12-part bar but eleven-twelfths of one bar. She was also explicitly aware of the two three-level structures with different mid-level units (one unit of 11 [a unit of units], 11 units of one [singleton units], 12 units of 11/12 [unit of units of units], and 132 units of 1/12 [unit of units]) by coordinating the two three level units structures. In other words, when she coordinated the second three level units structure (one unit of 11, 12 units of 11/12, and 132 units of 1/12) with the first three level units structure (one unit of 11, 11 units one, 132 units of 1/12), she was aware of the initial mid-level unit (11 units of one) and got 11/12 of one as an answer to the licorice bar problem.

Claire's model was difficult for Sharlene to understand because Sharlene had reasoned with two levels of units to solve the problem. Sharlene made a bar and broke it into 12 parts; then she pulled out one-twelfth of the 11-inch bar to figure out one person's portion. (She used a similar method to Keith's reasoning, which I describe later in this section.) After Claire explained her model (Figure 15 & Protocol 4), Sharlene asked Claire why she separated the bar into 11 pieces.

Protocol 5: Claire explains to Sharlene where 11 units of one are.

CL: Second person would get 11/12 of the second bar, and etc., and then twelfth person would get blue[27] pieces [the purple pieces in the screen] that ends of every little bar which comes out to an 11/12 again.
RA: Sharlene had a question back here, sorry.
CL: Oh, okay.
SH: I am sorry. I just didn't understand why you separated it into 11 pieces. That just threw me off.
CL: Because we have 11 inches (Sharlene frowns her face) so I am using each bar as one inch and I had to magnify to see it, and there was 12 people so that's why I made each bar into twelve parts.
SH: So each person gets a [she was keep thinking].
CL: 11/12 of one whole.
SH: 11/12.
CL: 11/12 of one inch
SH: 11/12 of one inch. Right [Sharlene still looks very confuse].

Even though Claire explained to Sharlene where the 11 pieces came from, the fact that Sharlene kept frowning shows that she was still not sure why Claire had to start with 11 separate pieces. It seems that she had a more fundamental problem than her inattention to the initial mid-level unit; that is, failure to attend to units themselves. As the protocol shows, she was not attending to the referent unit when she said, "11/12" for the portion that each person got while Claire said, "11/12 of one whole" right before, and Claire reaffirmed that the referent unit for 11/12 was one inch. Note that Claire was the first person to demonstrate her model for the Licorice Bar problem during the whole class discussion; hence it seems fair to assume that Sharlene was not able to understand the subsequent teachers' models because they started with separate bars (like Claire) or a partitioned bar (like Donna and Diane). It is not surprising that Sharlene was confused by Claire initiating the model with the 11 separated bars if we consider Sharlene's model for the Licorice Bar problem.

[27] Often the colors of the bar from the screen appear a little different from the teachers' computer monitors. So Claire might have used blue for the last column, but it appeared on the screen as purple.

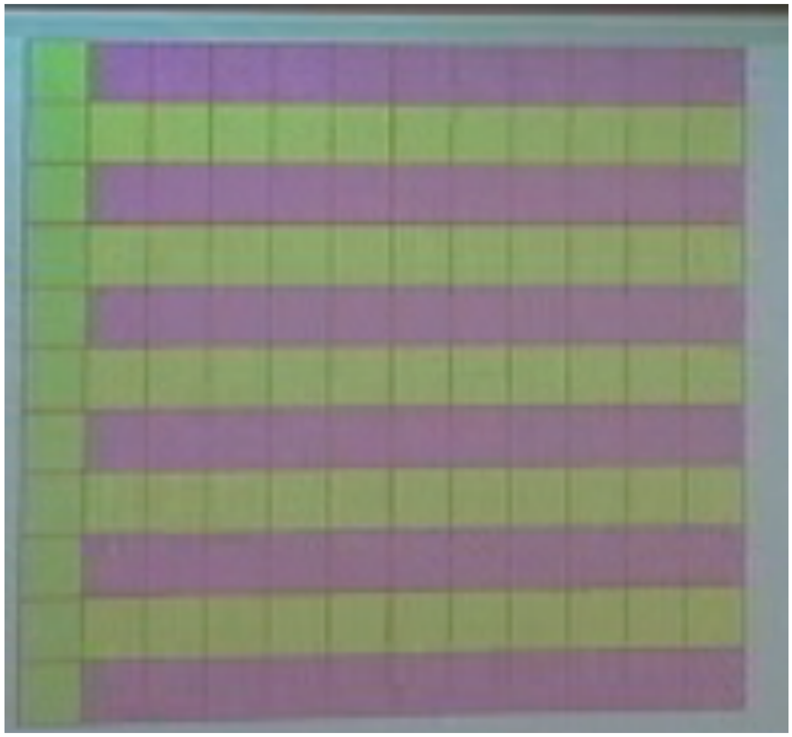

Figure 16. Diane's model of the licorice problem.

Protocol 6: Diane focuses on the number of pieces to solve the licorice bar problem.

DI: I actually did the exact same thing as she (Claire) did. I just instead of making individual bars, I split it up into 11 pieces, the whole bar into 11 pieces
RA: So hang on, you just made one big bar
DI: square.
RA: Okay.
DI: and then I split it into 11 and that is 11 inches, and here is my 11 inches going this way (she points each inch bar from the bottom in Figure 16). So basically I think mine is reversed from your [Claire] model. I think you [Claire] did eleven this way (moves her hand from left to right)? And then I took each bar of one inch and divided it [each bar] into 12, and I was going to give each person one piece [1/12 of 1] from all eleven bars (she slides her hand from the bottom to the top) so that is one whole piece [the leftmost column] for everybody they get 12 [whole piece]. *Two, four, six, eleven (she counts the pieces in the leftmost column) pieces out of [each inch] divided into 12.* So that's what I did. It's the same concept [as Claire], just a little different.
RA: How did you know I mean (pause) so how much does one person get?
DI: for mine, they get 1/12 because they get um one out of twelve pieces that were divided up.
RA: so you are saying this (the leftmost column) is one piece?
DI: that's *out of* all 132 pieces. They get 11 pieces *out of* 132.
CL: It works.
RA: 11 pieces *out of* 132. (Silent for about 2 minutes) so you are saying one person, if you answer the question one person, are you saying one person gets 1/12? What's the referent unit for that?

DI: Eleven inches whereas her [referent unit to the Claire's answer] was one inch and she had 11/12.
RA: Okay, *so you got a different answer depending on what your referent unit* is kind of like the Sam and Morgan [the problem discussed in the previous week] right? Which one are you saying is the whole? And then when you say 11 out of
DI: 132?
RA: What is the referent unit there?
DI: Um that was *eleven pieces divided up into twelve*. 11-inch is divided up into twelve each so each inch is divided up into twelve pieces. So the referent unit is the whole thing all eleven inches.
RA: Okay.

Diane was another teacher who reasoned with three levels of units in making sense of the Licorice Bar problem. While the former three teachers used a partitioning operation to find a common multiple, Diane used a distributive partitioning operation. However, her partitioning was limited in the sense that it lacked distributive reasoning. After Diane made a bar, she said she partitioned the bar horizontally into 11 parts and then she subdivided each inch bar into 12 parts so that she could share the inch bar among 12 people as in Figure 16. Then she said one person got one piece from all eleven bars, which amounted to the leftmost column that she had colored in green[28]. While Claire and Donna used common partitioning by coordinating two three levels of units, Diane's partitioning was based on a distributive partitioning operation in that her partitioning operation was motivated by her goal to share each of the 11 inches of bar among 12 people instead of sharing an entire bar. Her attention to 11 units of one inch was induced by her goal to share each inch bar among 12 people.

Diane said she subdivided each inch bar into 12 parts because it was easier for her to share each inch among 12 people. Diane was reasoning flexibly with her whole number three levels of units. Diane said each person gets one piece from each of the 11-inch bars that were divided into 12 parts, and that these pieces summed up to 11 out of 132, which was 1/12. She

[28] Note that each of the even rows is colored in yellow not green.

conceived of 132 pieces as 11 units of 12 and 12 units of 11 and was aware of the equivalent relationship between 11/132 and 1/12. Such flexibility requires having at least constructed a Generalized Nested Number Sequence, which accompanies working with three levels of units. The smallest unit that she used was 1/132, and 11 of those units constituted each column in Figure 13 hence, each person gets 1/12 and the referent unit for her answer is the 11-inch bar. She said her answer was different from Claire's because their referent units were different. She further stated that Claire used one inch as the referent unit for her answer 11/12 whereas she used 11-inches as the referent unit for her answer of 1/12. Diane used her whole number three levels of units to solve the problem correctly. Also, it allowed her to see the distinction between hers and Claire's answers to the problem. However, her distributive partitioning was not based on distributive reasoning because she conceived the smallest unit as 1/132 as opposed to 1/12. Her meaning of 1/12 switched during the discussion of the licorice problem in the class.

To illustrate, whenever Rachael asked Diane how much one person's portion was, Diane said it was either 1/12 of each inch bar that was divided up into 12 pieces or 11 out of 132. Her meaning of 1/12 changed. She thought that 11/132 was 1/12 of 11 and thus each person would get each of the twelve columns. Even though she said the referent unit for 1/12 was 11-inches (or sometimes she used the term 'whole' or 11 inches), she was not thinking of the whole 11-inches as 12 units of 11/12 but as 12 units of 11. For her, it was unnecessary to view the smallest unit as 1/12 because her goal was different from Claire's. She was answering the question "How much of a whole (11-inch) licorice bar would one person get?" as opposed to "How much of a one-inch licorice bar one person would get?"

Nevertheless, I still claim that her meaning of 1/12 switched a few times. Her switching does not necessarily mean that she was inflexible with referent units but more likely means that her interpretation of the referent unit was different from others. To elaborate, when Rachael asked Diane what the referent was when Diane said each person got "11 out of 132", Diane responded to Rachael's question that the referent unit for "11 out of 132" was 11 inches where each inch was composed of twelve pieces. Furthermore she said, "So the referent unit is the whole thing–all eleven inches." The fact that she used pieces to indicate twelve sub-parts and inches to indicate each inch and 11 inches or the 11-inch bar shows her conflation of the unit. I did not hear her say that one of the 12 sub-divided units was worth 1/12-inch. Diane was aware that the whole was the 11-inch bar and 11 units of one inch comprised the whole; however, the third-level unit for her was 132 units of 1/132, while the third-level unit for the 11-inch bar is 132 units of 1/12. At first glance, I thought that her distributive partitioning was based on distributive reasoning in that she seemed to pull out 1/12 of each bar as she knew pulling out 1/12 from each bar was the same as 1/12 of the 11-inch bar. In other words, even though the problem was devised for one to share the bar, had one set the goal to pull out 1/12 of each bar as one knows a priori that 1/12 of 11 equals (1/12 of 1) 11 times, namely, had one used distributive reasoning, one might have solved the problem with coordination of two three levels of units structures without any conflation of units.

11/12 of 11-inch bar = 1/12? When Rachael asked "What is 11/12 of the 11-inch bar?" Diana quickly responded to her that 11/12 of the 11-inch bar was 1/12, whereas Keith used multiplication and got 121/12 and Claire used distributive reasoning to get 10 and 1/12. Claire even indicated from her model (See Figure 15) that 11/12 of the 11-inch bar was the green ones in the picture, which were all but one column. Even though Rachael explained to the whole class how Claire's reasoning made sense, most of the teachers were confused, and Diane was also confused considering that she had repeatedly mentioned that 11/12 of 11-inch was 1/12.

When Keith explained to the whole class that he could not understand why the teachers shared 11 inches of bars when they could just pull out 1/12 of the 11-inch bar, Diane said

> It is because my unit is one inch and your unit is eleven inches so I have to show that there are 11 units of mine divided into 12 because I was using one inch as my unit as opposed to using the whole eleven inches as my unit. So I think that's why there is a difference.

If 'my unit' was referring to the referent unit, she was definitely conflating as protocol 9 shows that she had repeatedly stated that the referent unit for her answer was 11-inches of the whole bar. One might think that "my unit" referred to the smallest unit; still she was conflating because it clearly shows that she was not attending to the 1/12 unit, which is related to my argument of her unit conflation that she was not conceiving 132 units of 1/12 but 132 units of 1/132. I hypothesize that if one uses a distributive partitioning operation that is given rise by one's distributive reasoning, one could have accomplished the ideal coordination of two three levels of units in which the units are not conflated. It would entail one's awareness of the fact that the mid-level unit from the first three-level structure was shifted when it was coordinated with the second three-level structure and the iterability of the smallest level of unit. The two three levels of units in the coordination are as follows:

First three-level structure:
One unit of 11, 11 units of one, and 132 units of 1/12
Second three-level structure:
One unit of 11, 12 units of 11/12, and 132 units of 1/12

As I have mentioned before, this coordination is supported by one's construction of the Generalized Number Sequence so that he can recognize the multiplicative relationship between 11 units of 12 and 12 units of 11. One's distributive reasoning is also important. Diane seemed to have coordinated the following structure instead of the former two three-level structures to find an answer to the licorice problem:

First three-level structure:
One unit of 11, 11 units of one, and 132 units of 1/132
Second three-level sturcture:
One Unit of 11, 12 units of 11, and 132 units of 1/132

Because she used her whole-number knowledge to count 11 pieces in the first column and saw that it was 1/12 of an entire rectangle instead of using 1/12 as an iterable unit. She colored in the entire first column green and said the green column was 1/12 of ‘12 by 11’ [12 x 11 = 132] pieces of licorice bar; thus one person’s portion was 11/132. As I stated, for her goal, reasoning with such unit structures was enough.

Iterability of a unit fraction. When one’s notion of fractions is restricted to part-to-whole relationships, one might conceive the smallest bar in Figure 16 as one out of twelve but might not be aware that iterating the smallest bar twelve times constitutes one. The importance of understanding the iterability of unit fractions was emphasized in the studies by Piaget and his colleagues and the Fractions Project, but it has not been acknowledged as much as the role of partitioning in reasoning with fractions in the fractions research literature. One reason may be that it is harder to confirm whether the participants have an iterability of unit fractions or not than to confirm what partitioning operations they use because iterating usually occurs mentally whereas partitioning can be seen through their partitioning actions using representations. Even though one knows that iterating the unit fraction, say $1/n$, n times constitutes one, and that iterating 1/n more than n times constitutes improper fractions, one might not use 1/n as an *iterable unit*. When one can use unit fractions as iterable units, the unit fraction can be conceived as a fractional entity itself without having to compare the part to the whole to figure out the quantity of the unit fraction. One can use the fraction as a countable unit to iterate more than one and know how much the fraction was iterated as soon as one iterated (or counted).

For example, when Rachael asked Claire to talk through her picture (See Protocol 5), Claire could use the smallest piece (1/12) as an iterable unit and knew that 11 pieces of size 1/12 resulted in 11/12 of 1. On the other hand, Diane (See Protocol 6) considered the smallest piece as one rather than 1/12 and to determine how much the eleven pieces were worth in terms of the whole bar, she counted the number of 1/12 pieces in all eleven inches of Licorice Bar, which was 11 x 12 pieces, and determined that the eleven pieces of the smallest bar were worth 11/(11 x 12), that is 11/132. When Diane was asked what the referent unit of 11/132 was, she said the referent unit she chose was different from Claire's referent unit because she used the entire bar (11 inches) as the referent unit while Claire used one as a referent unit. Similar to Claire, Diane used a partitioning operation to find a common multiple, but she did not need to use distributive reasoning with her model as in Figure 16.

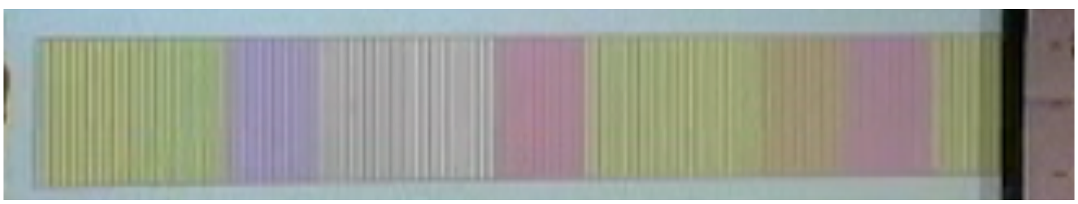

Figure 17. Donna's model of the licorice problem.

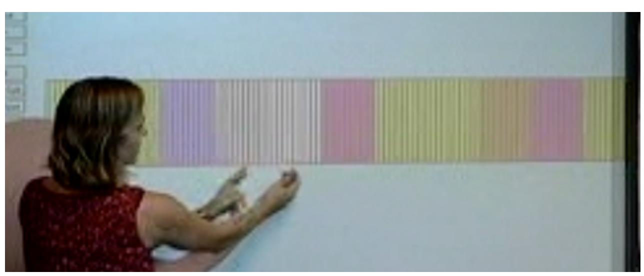

Figure 18. Rachael bracketing 11/12 inches from Donna's bar.

Protocol 7: Donna explains her strategy to the whole class.

DO: I was just literally thinking of an 11-inch strip of licorice. So I made this [Figure 17] long strip and divided it into 11 inches and then I divided each inch into 12 pieces and shaded 11 of each 12 [pieces].
RA: So this [11/12 of the leftmost portion in Figure 17] is 11 little pieces [of 1/12] that are yellow.

DO: Uhuh [yes]. So I have each person's portion together instead of the little left over [like what Claire did].
RA: So there should be 12 colored blocks if you went [all up to the 11 inches] (she points her fingers to the right in Figure 18) this is grey [Figure 18], this is white, it went all the way to the end.
CL: Okay.

Donna first divided the bar into 11 inches, then subdivided each inch into twelve pieces, and it gave the initial three level unit structure where one unit of 11 is the biggest unit, the length of one is the mid-level unit, and the length of 1/12 is the smallest unit. After that, she colored in 11 consecutive pieces to show each person's portion together. In other words, her initial three level unit structure was coordinated with the second three level unit structure of 11 in which the mid-level unit of the first three level unit structure (11 groups of one) was in the background and the second mid-level unit of the length of 11/12 was in the foreground. Hence, when one uses partitioning operations to find a common multiple to figure out one person's share in sharing division situations, it is significant to strategically shift mid-level units when forming three-level structures- the first level unit (a whole), two (initial and the second) mid-level units (intervals of one and intervals of 11/12), and the third level unit (intervals of 1/12). And it seems plausible that a piece of knowledge that enables this ability is acknowledging a relationship between the two mid-level units. The relationship that I am referring to is knowing that the initial mid-level unit structure of 11 groups of one is actually 11 groups of 12(1/12) and the second mid-level unit of 12 groups of 11/12 are 12 groups of 11(1/12). In other words, the quantity remains the same whether you iterate the quantity 11 twelve times or the quantity 12 eleven times. Note that the whole is not 132 but still 11 because the smallest unit that one needs to attend to is not one but 1/12.

Embedded units of one in Donna's model. I cannot emphasize enough the importance of attending to the initial mid-level unit when one uses common partitioning or distributive partitioning to get an answer to a partitive whole number division problem.

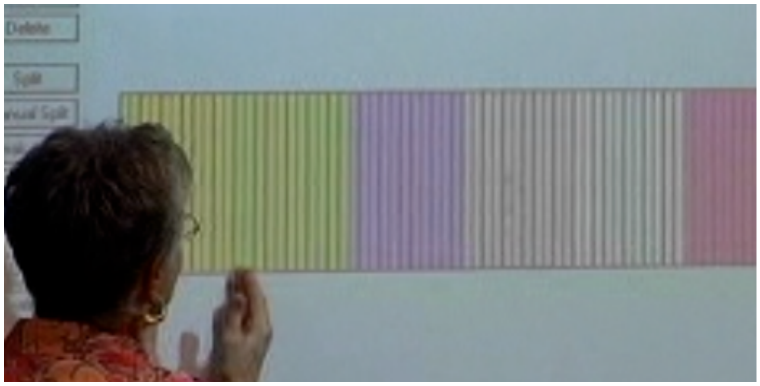

Figure 19. Donna explaining where 11/12 of one is in her model.

Protocol 8: Where Are Inches?

WA: Excuse me. I am still having trouble seeing where the inch is [in Donna's model].
RA: Okay.
CL: Twelve pieces is an inch and eleven she colored.
WA: I know but I mean just seeing it is hard to see it from the sense.
RA: Donna, can you talk us through how you did it? You first made your bar and what did you do? Can you talk us through like how you split it and that kind of stuff?
DO: (She walks to the smart board and explains how she models) Yeah. I first split the bar into 11. So it was representing 11 separate inches and then I divided each inch into 12 pieces so that [Figure 19] is 11/12 and that is one person's portion. There is another eleven twelfths.
RA: So how many of those little things [1/12] you would need for one inch? Is that what you are asking Walt?
WA: Yes.
WI: It will be 12 of them [1/12].
DO: 12 of them [1/12].
RA: 12 of the little skinny thing.
WA: Yeah, so they are going to run into the different colors. If you set it up by inches they are not going to stay true to the colors.
DO: Right. I have to go back and mess with my colors.

Walt did not want to accept Donna's model in Figure 19 because it did not explicitly display the initial mid-level unit (11 units of one). As the protocol shows, Walt was not sure where intervals of one inch were in Donna's model; hence, he did not think the model helped him to see the problem situation. When Claire (instead of Donna) answered Walt that 11 pieces represent the inches, Walt said he knew but it was hard to see the inches in the bar. Note that Donna already explained to the whole class (See Protocol 7) that she considered the 11 inch licorice bar as the whole and that she divided the whole into 11 parts, subdivided each bar into 12 parts, and consecutively colored 11/12 twelve times. Her explanation clearly showed that she was strategically switching the mid-level unit by coordinating two three-level unit structures. Thus, Walt seemed to have a chance to understand where the one inch was in Donna's model.

It might be that Walt just did not want to accept the model because he often emphasized the importance of representation throughout the InterMath course. Walt's conception of representation was different from that of the course organizers because he commented once how his students had developed understanding of the cross-multiplication algorithm (e.g., 2/3 = x/5) by learning the algorithm with the "heart method." The heart method is a meaningless strategy where one draws a heart shape to remember to multiply 2 and 5 and x with 3 and then solve the resulting equation $10 = 3x$ to get the value of x. He said the heart method was one example of applying visual representation in classroom. Therefore, if he understood Donna's model but just did not like it because it did not clearly display all the units, he might have thought of validating Donna's model by considering whether he could use it with his students in class.

Some of our teachers (Keith, Sharlene, Carrie, Linda) reasoned with only two levels of units while others like (Claire, Donna, Diane, Mike, Walt) reasoned with three levels of units. Teachers who reasoned with only two levels of units were limited in that the two-level structure they used did not show how much of one (candy bar or inch licorice bar) one person would get. In other words, when teachers did not have the goal of measuring how much of one candy bar one person gets for the candy bar problem (share two candy bars among five people) or how much of one inch does one person get for the licorice problem (share 11 inches of licorice among 12 people).

Teachers Who Reasoned with Two levels Of Units for the Candy Bar Problem

Figure 20. Carrie's use of cross partitioning *strategy* as opposed to *operation*.

Even though Carrie initially supported the third interpretation of the candy bar problem, she tried to apply the first interpretation (i.e., Share two identical candy bars among five people) when the class revisited the problem a week after the first class discussion of the candy bar problem. She made a bar and split it into 2 parts vertically and 5 parts horizontally. When Rachael asked her what the answer to the candy bar problem was she said, "If they [five people] split the first bar, each gets 1/5, and if they [five people] split the total (two candy bars) each gets 2/10, which is the same." It shows that she switched between two two-level structures. The first two-level structure (a whole and five-fifths) was formed to answer the question "Share one candy bar among five people (the third interpretation)," and the second two-level structure (a whole and ten-tenths) was formed to answer the question "Share two identical candy bars among five people (the first interpretation)."

She used a cross partitioning strategy as opposed to a cross partitioning operation because she was not aware of the unit of one from her model. When she split the bar into 10 parts using a cross partitioning strategy, she conceived each of the smallest unit as one-tenth instead of one-fifth (i.e., reasoning with two-level structure, ten units of one-tenth) Carrie had a hard time making sense of two pieces as two fifths of one (Figure 20) because she only reasoned with two levels of units – the whole and ten units of one-tenth. For her, when she grouped the ten pieces into five parts, she conceived the two candy bars as a referent unit and concluded that one person's portion was 2/10 or 1/5 of the whole. Forming the first two-level structure was enough to answer her question from the third interpretation but was not enough to answer her question from the first interpretation. When she was trying to figure out one person's portion for sharing two candy bars among five people, she insisted on her answer 2/10 and could not understand the reason that other teachers (e.g., Claire, Diane, Keith, Mike, etc) mentioned 2/5 of one candy bar was one person's portion for the Candy Bar problem.

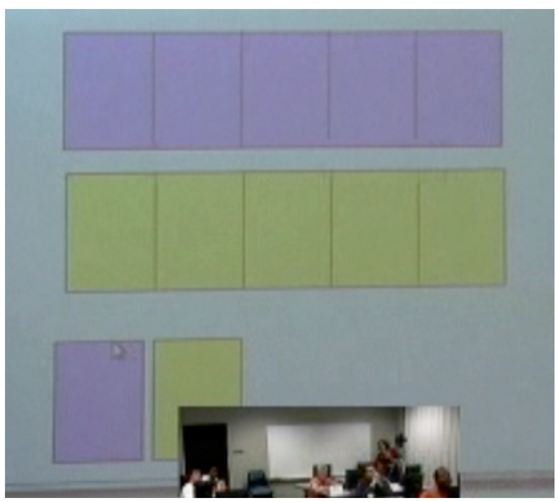

Figure 21. Two 5-part bars constructed by Rachael to help Carrie see 2/5 of 1.

While Carrie was confused when reasoning with the area model, she was able to see that one person got two-fifths of one candy bar when Rachael started with two separated 5-part bars as in Figure 21) instead of one bar that was cross partitioned into 2 by 5 pieces as in Figure 20. After Rachael made two 5-part bars, she pulled out one from each 5-part bar and asked Carrie, "How much of one does one person get?" Carrie immediately responded to Rachael that each person got two-fifths of one candy bar, and she was not aware that she changed her answer until other teachers pointed that out. Carrie might have seen the unit of one because she was given two separate candy bars as opposed to one bar. Starting with the two separate bars helped her to reason with yet another two-level unit structure – the whole and five-fifths – and she saw two of the one-fifths as two-fifths of one candy bar.

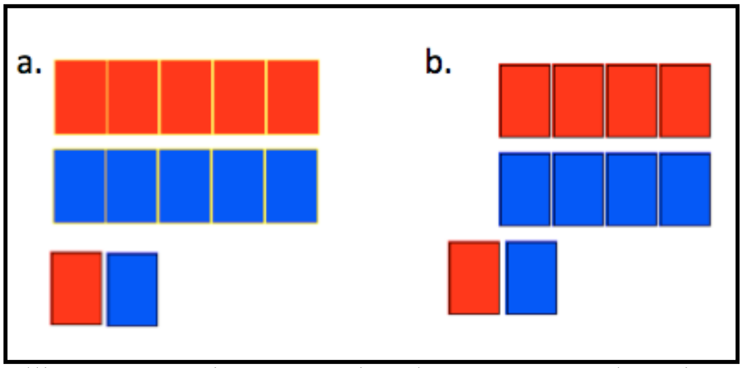

Figure 22. Pulling out two pieces. a. Using the PULLOUT. b. Using the BREAK.

In my opinion, there are two things that need to be highlighted: One is the use of two separate bars and the other is the use of the 'PULLOUT' function of the Fraction Bar software. First, I think Carrie definitely benefited from reasoning with two separate candy bars. When she needed to reason with one bar as in Figure 20, she was not aware of two units of one (i.e., initial mid-level unit) when she split the one 2-part bar horizontally into five parts and said one person got two-tenths or one-fifth of a whole. Secondly, the 'PULLOUT' function in the Fraction Bar software helped her to see that two pieces that were pulled out by Rachael were two-fifths of one bar because she was able to compare physically the two quantities of two 5-part bars and one from each 5-part bar. Pulling out pieces using the 'PULL OUT' function differs from pulling out pieces using the 'BREAK' function. When teachers pull out pieces using the 'PULLOUT' function, the whole is visually unchanged (Figure 22a), and it often supports teachers to compare the quantities between the whole and the parts that they pulled out. When teachers pull out pieces using the 'BREAK' function, the whole is visually destroyed in that the pulled-out pieces are eliminated from the whole by breaking the bonds between each piece (Figure 22b).

Teachers Who Reasoned with Two Levels of Units for the Licorice Problem

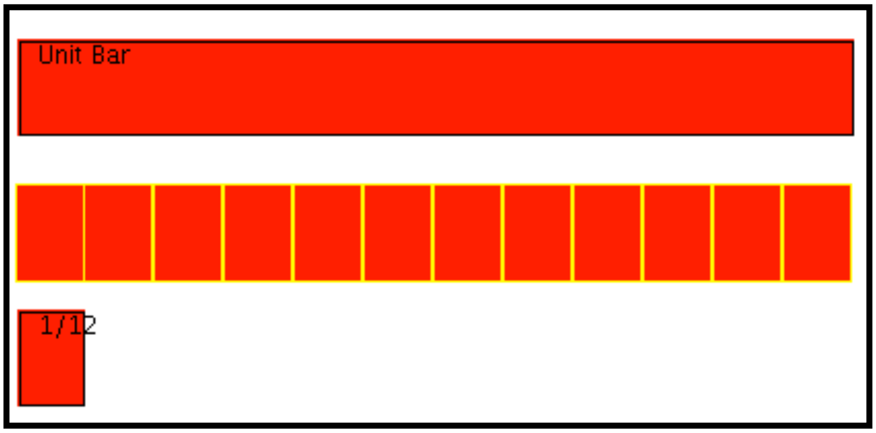

Figure 23. Reconstruction of Keith' model.

Unlike teachers who reasoned with three levels of units, Keith's reasoning was based on two levels of units to solve the licorice problem – the whole and 12 units of 1/12. When Rachael asked him how much one person gets, he pulled out one of the 12 pieces and told her it was one person's portion like in Figure 23. As Rachael continued to ask him how much the portion was, he clicked 'MEASURE' and told her it was 1/12 of the 11-inch bar but did not mention how much it was in terms of one. Even though his solution method was based on reasoning with two levels of units, he was able to understand other teachers' models that required him to reason with three levels of units because he explicitly stated that he knew 11/12 of one was equal to 1/12 of 11. He even said he understood other's (Claire, Donna, Diane, Mike) models and he was attending to the three levels of units in solving the Candy Bar problem. He and Sharlene said they just could not understand why the others like Claire, Donna, Diane, and Mike needed to break a part into finer pieces. This seems to have an implication to teachers' mathematical knowledge for teaching. It is hard for teachers to reason with quantitative units when they have already abstracted mathematics by teaching mathematics for several years. In particular, Keith was a high school teacher for a long time and it was his first teaching in the middle school.

The following protocol contains more evidence to support the importance of reasoning with the partitioned 11-inch bar instead of the 11-inch bar with no partition for the sake of students' learning. After Claire, Diane, Donna, and Mike explained their strategies to the whole class, Rachael asked Keith to share his strategy because she knew Keith used a different but simple strategy.

> Protocol 9: Keith and Sharlene question the purpose of starting with 11 inches instead of 11-inch Licorice Bar
>
> KE: I am seeing how most people were dividing whatever piece they have in the eleven pieces and then taking twelve and then dividing that each of those eleven pieces in the twelve little pieces. Well I personally did it in a totally different way [compares to Claire, Diane, Donna, and Mike]. I just took the 11 inches as a whole, and then I cut it into 12 pieces. So I am not giving you a little piece, you a little piece, you a little piece, and then doing that 12 times [he is explaining distributive partitioning]. I am just saying cut here is your portion cut here is your portion.
> SH: I did mine the same way, so that's why when I see all these representations I am like (she shakes her head side by side to express no) why so many itzie bitzie pieces when you can just take that 1/12 of it [11 whole].
> RO: Well, the itzie bitzie pieces are showing that they are equally divided I think more so.
> WA: Well, no, the simple cut of 12 cuts is easy to see, but the problem is you can't (stopped as Diane was speaking simultaneously).
> DI: I was saying that it is because my [referent] unit is one inch and your [referent] unit is eleven inches so I have to show that there are 11 units of mine divided into 12 because I was using one inch as my [referent] unit as opposed to using the whole eleven inches as my [referent] unit. So I think that's why there is a difference.
> RO: But then how do you know it is originally eleven units if you don't represent …
> DI: I think that's what they are saying that it is not eleven units the whole eleven inches is one unit and they are dividing it into twelfths whereas we are saying that each inch is one unit.
> WA: the only problem is *you cannot answer exactly what it [1/12 of 11 whole] is compared to an inch something a little less than an inch.* (Claire also nodded and agreed with his thinking.)
> CA: Okay whatever this is just very annoying.

Keith asked why some teachers attempted to start with 11 inches of bar or partition the bar into 11 inches prior to dividing each of them into 12 parts when they could just divide the whole 11-inch bar into 12 parts and take 1/12 of the whole bar. Sharlene supported Keith's point, which explains why she was so confused by Claire's model. Keith and Sharlene both emphasized that they understood other teachers' models of partitioning into finer pieces, but they thought that their methodology was more efficient in terms of the number of pieces they needed to deal with. Rose sort of touched the surface but did not explicitly get to the point of addressing the importance of working with multi-level unit structure. If one do not have the goal of measuring how much of one inch one person's portion is, reasoning with two levels of units is enough. But one needs to form three-level structure to accomplish the goal.

Walt's comment, "You cannot answer what it [1/12 of 11-inch bar] is compared to an inch something a little less than an inch," in response to Keith's question shows that he was aware of a drawback the model of Keith and Sharlene could have. Also, Walt was aware of the intervals of one in the 11-inch bar, which facilitated him to deduce the fact that 1/12 of an 11-inch bar was a little less than an inch. Despite the fact that Walt could see such a relationship, it may not be an easy task for students (assuming that Keith uses the model in teaching) to think of 1/12 of 11 as a little less than an inch. The model used by Keith merely displays the amount (that was not yet measured) one person receives from the 11-inch bar. He also stated during the small group discussion that the answer was 1/12 of the 11-inch bar, but the model did not show but nor did he answer the question "How much of one candy bar does one person get?" Unfortunately, Walt's comment did not provoke them to restate the answer in this way. When Walt commented on the drawback of Keith's method, only a few teachers, such as Claire, agreed with his point, while some of them, such as Carrie, were more confused. Unfortunately, Walt's comment got lost in the discussion because of the emphasis other teachers were placing on the referent unit issue. Rachael wrapped up the discussion of the problem by emphasizing that the answer to the Licorice Bar problem depended on the referent unit one chose.

Working with pre-partitioned bar versus non-partitioned bar. Though some teachers understood each other's models and discussed their solutions and how the solutions could be different based on which referent unit one chose, others were confused and discussed their difficulties solving the problem. One teacher, Joyce, said she did not think they had trouble with the math but with the modeling and talked about the challenge she faced whenever Rachael asked her to model the problems. Then teachers began to discuss how they would like to present the problem to their students so that students could make sense of the licorice bar problem.

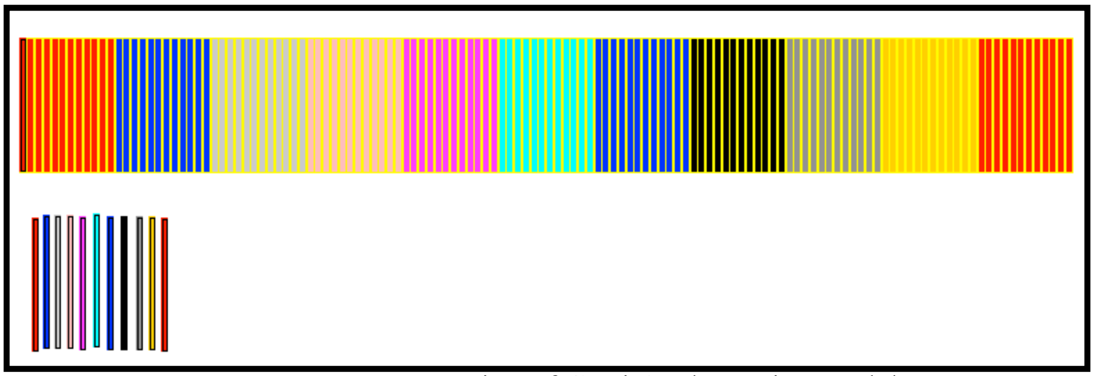

Figure 24. Reconstruction of Carrie's alternative model.

Protocol 10: Carrie introduces an alternative way to solve the licorice problem.

CA: Instead of having 12 pieces of a little one inch block but I am thinking of a bigger scale of a foot block and have eleven feet blocks and then twelve inch blocks for each foot block.
CL: yeah, that would work.
WI: Okay.
CA: And then you can show them [students] "you get a piece, you get a piece, you get a piece, you get a piece, etc"
WA: Um (pause) okay (repeats two more times.)
CA: Yeah, so I mean I could do it this way but I could think of using that manipulative with that.
WI: Yeah, that seems like the best way to do get them know the concept.
CA: Right.
RA: You know what you can break these [Fraction Bars] apart and then pull a piece, pull a piece, I mean what you are just saying you can do what you just said in the Fraction Bar.
CA: I tried doing that but I massed up like three times and I said forget it and then you know I can pull out my blocks out of my box and say "Okay, here we go. This is what we are doing."

DI: You can even do that with the sheet of paper. You can hand a sheet of paper to eleven kids "Here is your inch of licorice, your inch of licorice, … now we got to share this with twelve people. How would you do that? Here is your scissors and rulers."

Carrie preferred to work with foot blocks that she had in her classroom because it allowed her to use her knowledge that one foot = 12 inches. She said she would put 12 inch blocks in each of eleven foot blocks and take one inch block from each of the eleven foot blocks like in Figure 24. Even though she did not mention how much licorice one person would get in her problem, I have no doubt that she could say 11 inches for her problem. I guess this was another reason that she preferred to reason with foot blocks. When she used her foot blocks, she did not need to reason with fractions because she could use inch blocks in place of fractional amounts of foot blocks. While the teachers whom I described so far used their knowledge of whole-number factor product combinations to guide their partitioning, Linda used a trial-and-error strategy to guide her partitioning.

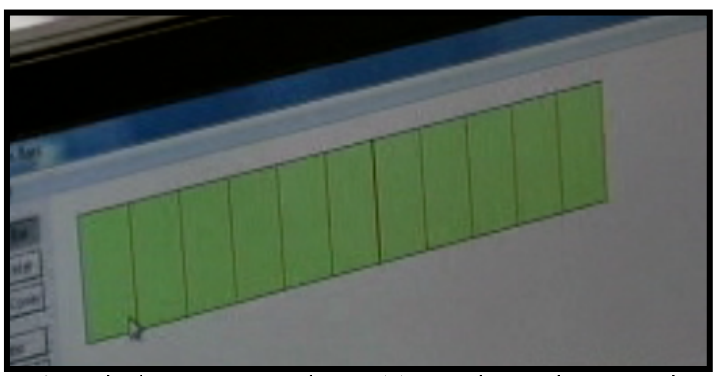

Figure 25. Linda constructed one 11-part bar using Fraction Bar.

Protocol 11: Linda tries to find the one person's portion for the Licorice Bar problem.

LI: I divided it [the whole bar] into 11 inches.
RA: So you have 11 pieces right there [Figure 25]? So you are saying each piece [one eleventh] represents how much?
LI: (She did not answer to the RA's question) okay, that's going to be 11 inches, and then I need to divide this [11 inches bar] among 12 equal groups.
RA: So what does this entire bar represents?
LI: that's all I got. One, two, …, eleven (She counted from one to eleven twice) okay, that's my 11 inches
RA: 11 inches of licorice?

LI: Uhuh [yes]. Then I divide this [the entire bar] among 12 people.
RA: So do you want to divide them among 12 people? If you gave each person a piece, what would happen?
LI: If I give each of these [11 inches of licorice bar] to 12 people, someone would not get a piece, so I got to figure out how to divide these [referring to the entire candy bar] into twelve [parts]…
RA: So that you can give one piece to everybody.
LI: Right.
RA: So how many pieces do you think you should split it into?
LI: I got 12 people so, but this one, eleven, I split it each one into half that will give me twenty two, then will end up in trouble…can I divide these [11 inches of candy bar] into two parts?
RA: Yeah, however you want to divide it! So what is the problem with splitting it into two?
LI: Because I am going to end up with more than 12 if I split each of these into halves because there will be twenty two [smallest bars].

It seems meaningful to analyze Linda's strategy because she was not one of the people often involved in whole group discussion and she was one of a few teachers whose mathematical knowledge was pretty weak from the beginning of the InterMath course. In order for Linda to partition the 11-inch bar into 12 parts, she tried to group the bar into 12 parts. As she realized that she could not partition the 11-part bar into 12 parts, she tried to partition each inch bar into two parts. However, splitting each inch bar in half did not work because 22 pieces could not be grouped by 12. It shows that her partitioning operation was not guided by whole-number factor product combination (e.g., 11x 12 = 132).

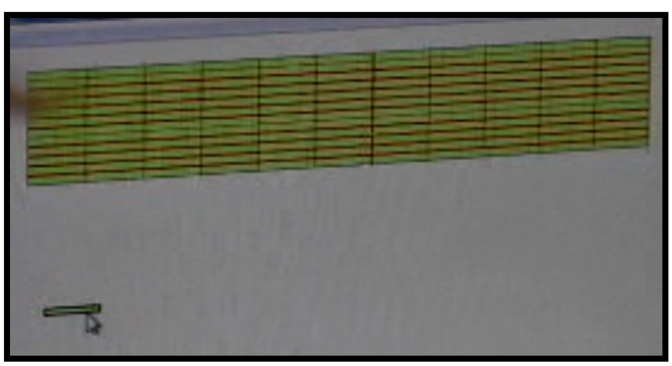

Figure 26. Linda's model of the licorice problem.

After she realized that splitting each bar [one inch] into two parts did not work, she horizontally partitioned the entire bar (which is split vertically into 11 parts) into 12 parts (as in Figure 26), and then she made sure each bar was split into twelve pieces by counting. When Rachael asked how much one person got when sharing 11 inches of licorice, Linda said "Everybody is getting a little piece of each one (pointing at each of the pieces in the 12th row in Figure 26)," and pulled out one piece from the first column under Rachael's direction (Figure 26), but she could not tell how much the piece was until Rachael pointed to the first column and asked her how much the piece was out of one column.

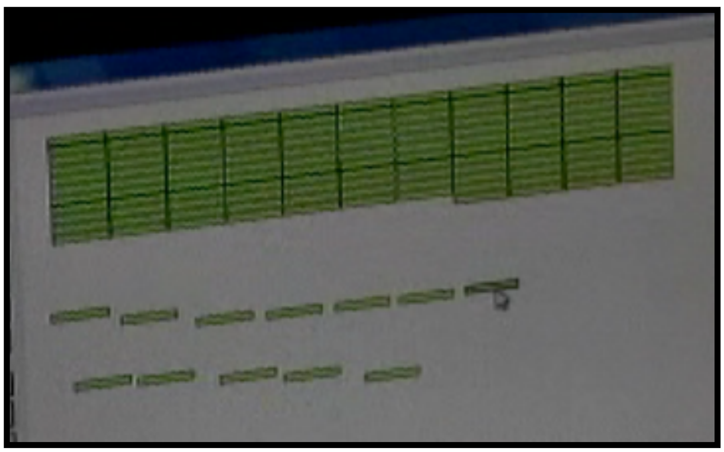

Figure 27. Linda pulls out one from each 12-subpart column.

Rachael continued to ask Linda how much each person would get, and Linda could not answer to Rachael's question. She pulled out each of the smallest piece from each inch bar (or each column). First, she used the PULLOUT function and pulled out one twelfth from the first five columns (from left to right). Then she did not like to use the PULLOUT function because she expected to see the piece that she pulled out eliminated from the whole. Thus, she stopped using the PULLOUT function and began to use the BREAK function and then removed one piece from each inch bar (The difference between the two functions was already mentioned in this chapter.) As soon as Linda pulled out all 11 pieces (See Figure 27), she told Rachael 11 pieces were what each person got. Rachael continued to question her,

> RA: How much is that [11 pieces] is what we got to think about.
> LI: Okay, so that is eleven out of the…(counting each of the eleven pieces in Figure 20 and swiftly moved the cursor through the entire bar) eleven out of one, two, three, …, eleven (she counted the number of one twelfth in the 11th column) I think I am going to keep the whole candy bar (laugh).

She knew that each person got 11 pieces, but when Rachael asked her to find how much the 11 pieces were, she tried to count each piece in the whole bar but gave up after counting pieces in one column. She could not use whole number three levels of units because she was not aware of the relationship between 11 units of 12 and 12 units of 11 structures.

Summary

Teachers' capacities to deal with the two similar types of sharing division problems differed in terms of four knowledge components: (1) partitioning operations (partitioning operations to produce a common multiple, cross partitioning, and distributive partitioning), (2) Conceptual units (units structure, levels of units, and referent units), (3) interpretations of problem (includes interpreting the word problem itself and how they conceived of the divisor), and (4) use of distributive reasoning (only among teachers whose reasoning seemed more sophisticated). These diverse knowledge elements were interrelated to support teachers' ways to solve the two problems.

When teachers needed to deal with partitive division situations where the dividend and the divisor were relatively prime, some teachers reasoned with two levels of units and others reasoned with three levels of units. Teachers who reasoned with only two levels of units were limited in that the two-level structure they used did not show how much of one (candy bar or inch licorice bar) one person would get. In other words, when teachers did not have the goal of measuring how much of one candy bar one person gets for the candy bar problem (Share two candy bars among five people) or how much of one inch one person gets for the licorice problem (Share 11 inches of licorice among 12 people), it was enough for them to reason with only two-level structures. Moreover, in order to determine the answer to the candy bar problem, teachers needed to strategically shift the mid-level units between the two three-level unit structures.

The notion of referent units may sometimes be used as one's defensive strategy if one is not accustomed to reasoning quantitatively. This is a hypothesis that I developed after I revisited the Candy Bar problem as part of a partitive division lesson in a course for prospective elementary teachers. Most of my students who had little experience reasoning quantitatively with fractions could not conceive that the quantity of 1/5 of two (candy bars) was equal to the quantity of 2/5 of one (candy bar). Similar to the teachers, until one of my students recognized that the problem asked "How much *of one candy bar* does one person get?" and discussed it in the whole class discussion, they were struggling with which one to choose as the correct answer to the Candy Bar problem. After the discussion, I asked them if 1/5 of two is different from 2/5 of one, and they looked very confused. They knew that the answer to the Candy Bar problem was 2/5 because the question specifically stated how much "of one candy bar" one person got, but they were not attending to the fact that the two quantities are equal until we used the Java Bars software and pulled out the pieces and compared the sizes of the two quantities.

Part II: Teachers' Knowledge Of Quotitive Fraction Division

Overview

I considered teachers' measurement fraction division knowledge across four different types of division situations: 1) When the divisor quantity measures out the dividend quantity evenly (Sequence 1[29]); 2) When the divisor quantity does not measure out the dividend quantity evenly (Sequence 2); 3) When denominators of the dividend and the divisor quantities are relatively prime (Sequence 3); 4) When the divisor quantity is larger than the dividend quantity (Sequence 4). Within each of these situations, I considered a sequence of tasks in which the divisor and dividend were combinations of whole numbers and fractions. The sequence of situations is not necessary hierarchical.

The unit-segmenting operation, that is the operation of segmenting the dividend by the divisor, was a fundamental operation for teachers' measurement division knowledge, but teachers' capacities to deal with the problems in each sequence were differed in terms of not only partitioning operations but also levels of units that were available to them. In whole-number contexts, Steffe (1992) observed children reasoning with two composite units in the unit-segmenting scheme: one composite unit to be segmented and the other composite unit to be used in segmenting. The context, in which he observed the children working, is similar to Sequence 1 of my analysis in that the children were only exposed to situations when the unit to be segmented clearly segments the unit to be segmented evenly. However there are two differences between my work and Steffe's work. First, the teachers in my study chose to use a unit-segmenting operation to determine a quotient for numerical expressions of division using drawn representations while children in Steffe's study were given word problems where they needed to use unit-segmenting operations. Secondly, I observed teachers reasoning with unit-segmenting in fraction contexts (because they chose to use unit-segmenting only if fractions were involved) whereas Steffe observed children reasoning with unit-segmenting operations in whole number contexts (because they were given the problems to use the operations).

[29] From now on, I will use Sequence 1 for the sequence when the divisor quantity clearly measures out the dividend quantity, Sequence 2 for the sequence when the divisor quantity does not clearly measure out the dividend quantity evenly, and so forth.

In addition to the sequence of four situations described above, there is one more situation– W*hen the divisor quantity is a whole number*–that I did not include as part of the sequence because teachers did not use a measurement model of division in this situation. Even though teachers used a measurement interpretation of division for other problems, they were hesitant to use the interpretation of division when the divisor was a whole number. Likewise, teachers were hesitant to use a partitive division when the divisor was a fraction. They found it awkward to share something into fractional quantities. Thus, I will write about this sequence here before I initiate discussing the four sequences.

When the Divisor is a Whole Number Versus When it is a Fraction[30]

In general, the way teachers interpreted the division problems guided their use of drawn quantities. While teachers preferred to use a quotitive interpretation of division when fractions were involved in the division, teachers used a partitive interpretation of division when the divisor quantity was a whole number unless we deliberately asked teachers to use the other model.

When the divisor was a whole number. Even if the mathematical relationship between the divisor and the dividend could have fallen under one of the four sequences described above (i.e., $10 \div 2$ in Sequence 1, $2 \div 3$ in Sequence 3), the word problems or models that teachers came up with reflected sharing situations.

[30] I am not using “fractions” as a synonym for “Rational Numbers”.

Without using an algorithm, consider the quotients for the following problems:

$$2 \div 3 \qquad \frac{1}{3} \div 3$$

$$2 \div \frac{1}{4} \qquad \frac{2}{3} \div 3$$

$$2 \div \frac{3}{4} \qquad \frac{2}{3} \div \frac{1}{4}$$

$$\frac{2}{3} \div \frac{3}{4} \qquad \frac{2}{3} \div 4$$

What do you notice about the similarities and differences in the different division problems?

What relationships do you see in these problems?

What real-world problem could you use for each of these problems?

Figure 28. 'Exploring division with fractions' in week 4 (Task 1).

When we asked teachers to find the pattern among all eight numerical expressions for division in Task 1 (See Figure 28), Claire immediately came up with word problems describing sharing situations (i.e., partitive division) for 2÷3 and 1/3 ÷ 3. She said, "Two divided by three, you got two cookies divided among three children and one third of the cake divided among three people." Moreover, Walt explained to Will that the quotient for 1/3 ÷ 3 was one-ninth because one third was divided into three parts and one of the three parts was one-third of one-third (i.e., Walt seemed to use a recursive partitioning operation for sharing because he immediately knew one-third of one-third was one-ninth.) Will continued to say that he could use the same partitive model to show 2 ÷ 1/4, but he quickly reversed his thinking and said he was talking about 1/4 ÷ 2 not 2 ÷ 1/4, and Walt agreed. Even though we deliberately asked teachers to come up with word problems for 7 ÷ 3 that modeled measurement division in Reflection 2 (Figure 29), only four (Debbie, Donna, Keith, and Mike) out of 12[31] teachers came up with appropriate measurement interpretation word problems. The remaining teachers either came up with word problems that modeled partitive division or skipped the problem.

Consider the following two expressions:

a. $7 \div 3$

b. $7 \div \frac{3}{5}$

1. Write two word problems for each expression – one that is partitive and one that is quotative (measurement).
2. For expression b, use drawings to model the partitive and quotative approaches.
3. Do you think your students struggle more with partitive or quotative fraction division? Why?

Figure 29. 'Pesky parts II' in week 5 (Reflection 2).

While teachers demonstrated the quotient of 2/3 ÷ 3/4 using area models, Mike commented that they needed, "To develop more strategies as the numbers got more complex." In order to encourage deeper class discussion of Mike's comment, Rachael immediately asked teachers how they would model 2 ÷ 3 and wrote down the numbers on chart paper. I think she started with 2 ÷ 3 because she thought of it as a problem with easy numbers. However, teachers did not interpret 2 ÷ 3 using measurement division and emphasized that different questions were asked by Rachael for problems like 2 ÷ 3.

[31]Claire had to leave the class early in the Week 5; hence, we do not have her reflection. Salihah was in the class through the end on Week 5 but she did not turn in her reflection.

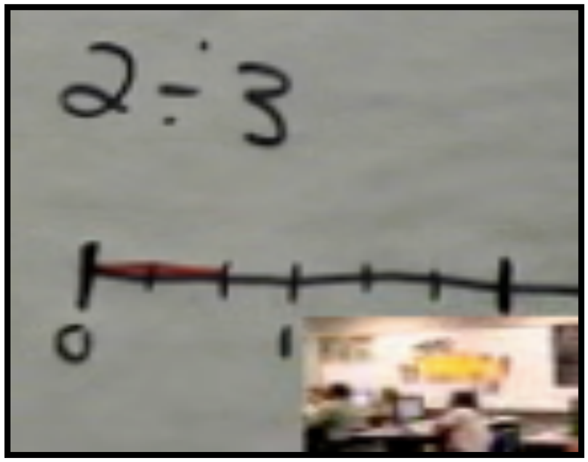

Figure 30. Teachers' number line model of 2 ÷ 3.[32]

Protocol 12: Which division models to use to solve 2 ÷ 3?

KE: Well, we ask different questions for whole number division. Well, we would just take whatever the first number [two] was and actually divide it into three, so we had two and we were just divide it into three parts.
JO: Our whole unit was two, and then divide that [two] into thirds.
WI: So two out of each one.
JO: It is like our candy bar problem.
KE: So that [the red line in Figure 30] is one of your thirds, which is actually two-thirds [of one].
CA: We took two wholes and divide it into three parts.

As the protocol 12 shows, Rachael drew what teachers explained to her as in Figure 30, and they clearly used a distributive partitioning operation was motivated by the goal of partitioning the dividend quantity into three units. Teachers also reminded themselves of the candy bar problem, which was a sharing problem. This is indeed surprising because division in the whole numbers does not extend to situations where the dividend is not a multiple of the divisor.

When the divisor was a fraction. Even though we asked teachers to come up with word problems for 7 ÷ 3/5 using a partitive approach in Reflection 2 (Figure 29), only Diane came up with a reasonable partitive division word problem by stating "I have a ribbon. 7 inches is 3/5 of

[32] Rachael drew the number line for the teachers.

the ribbon. How long is the entire ribbon?" The remaining 11[33] teachers either skipped the problem or wrote word problems that were not intended. Teachers generally used the measurement model of division when the divisor quantities were fractions. They thought that it was awkward to share the dividend into a fractional amount.

During the small group discussion of the Task 1 in Week 4 (Figure 28) while Will and Walt used the partitive model of division to solve 2 ÷ 3 and 1/3 ÷ 3 and knew that they could not use the same method to solve 2 ÷ 1/4, Will was not sure how to come up with a measurement division situation when the divisor was a fraction. Walt showed Will how to use quotitive division to show 2 ÷ 1/4 = 8. As a matter of fact, it is true that one cannot solve 2 ÷ 1/4 using the partitive model if one wants to use exactly the same operations used to solve the Candy Bar and the Licorice problems. Not only had teachers not considered division with whole number divisors as quotitive division, they also dismissed the double number line model of c) in Task 2 as representing 3 ÷ ½ (See Figure 31).

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[33] Please see footnote 31 for information on why we have only 12 teachers' reflections.

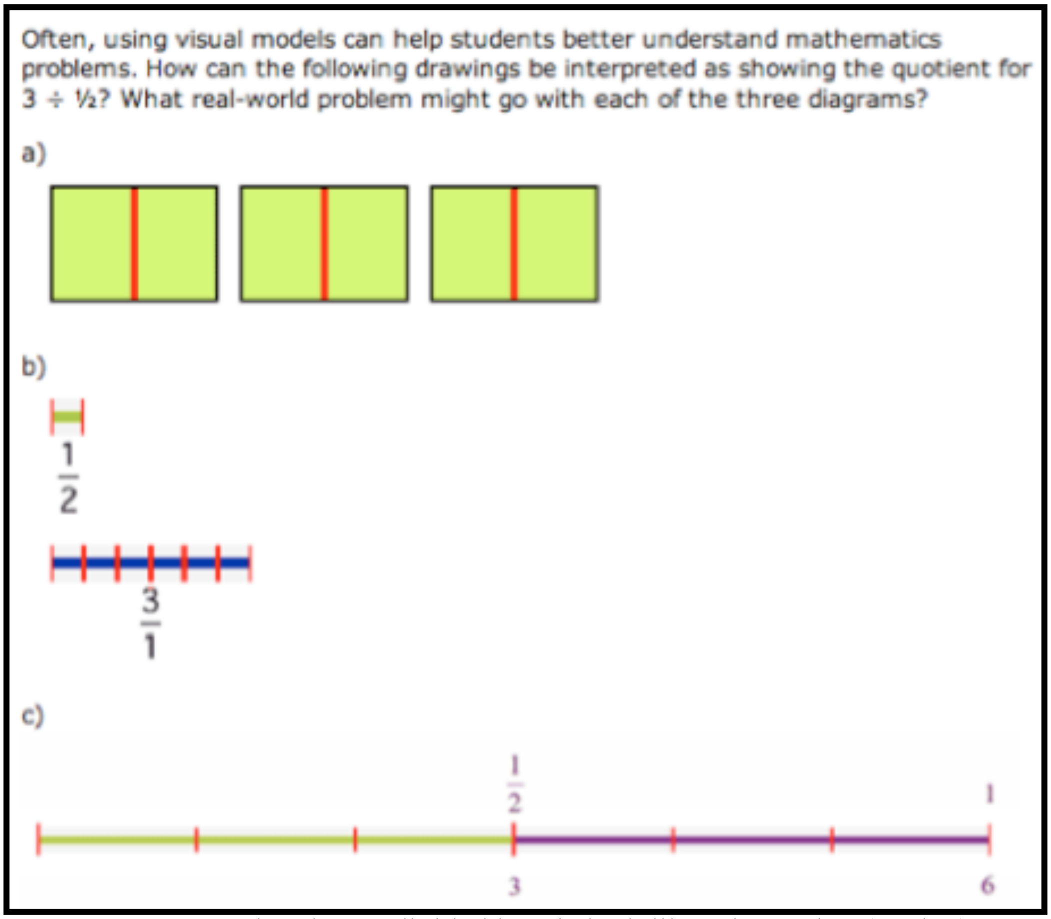


Figure 31. 'What does 3 divided by 1/2 look like?' in week 5 (Task 2).

While most teachers liked and thought that their students would show models a) and b) for representing 3 ÷ 1/2, only a few (e.g., Claire) understood how model c) could represent 3 ÷ 1/2.

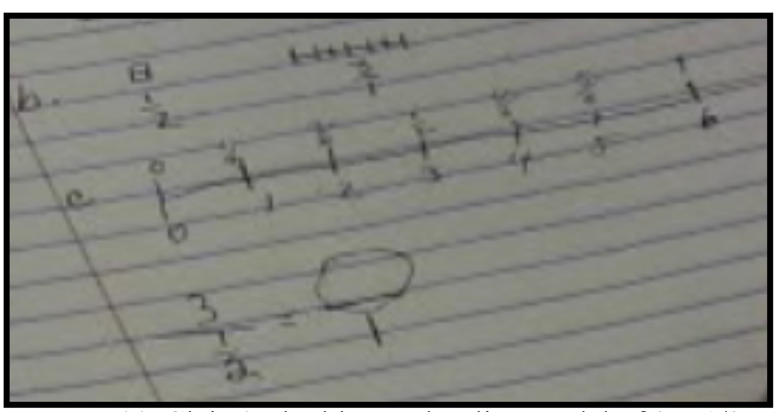


Figure 32. Claire's double number line model of 3 ÷ 1/2.

In the small group discussion among Claire, Donna, and Salihah, Claire was aware of the double number line model in part c) in Task 2 (Figure 31) from a previous class and tried to explain to Donna how to think of the model as division using ratio. Claire told them that the scales in the upper and the lower parts of the number line were different, but Donna and Salihah had no idea what she was taking about. Donna modified the number line so that she could measure three with halves. Claire asked them, “If half of something equals three, how many would be for one? 3 ÷ 1/2, they are using ratio thing?” and showed the double number line model with the proportion expression as in Figure 32. Regardless of Claire’s effort, Donna said she could understand Claire’s idea, but it was not a method that she would use with her children and Claire also agreed with Donna. In the whole-group discussion of the same problem, Rachael asked Donna to talk through model c) in Task 2, and Donna said she wanted to think about modeling 3 ÷ 1/2 with a) and b). She continued to say that she did not see how model c) showed 3 ÷ 1/2 because the double number line was supposed to be used to model proportional situations, not division. Mike agreed with Donna that model c) was very confusing and did not think of it as a good model. A lot of teachers commented that model c) might be appropriate for proportion but not for division. Teachers were not connecting proportion with division.

When Rachael asked teachers if they could think about connecting the proportion with division problems, none of the teachers answered her question. Actually, Rachael went back to the discussion after she asked teachers to do Task 3, which entailed various word problems that could be interpreted as partitive and quotitive division. The class discussed the difference between problems like a, c, e in Task 3 and b, d, and f in Task 3 (See Figure 33 for the task).

Pretend not to know any algorithms for computing with fractions and explain how to use a model to answer the following questions.

a. If 7 containers hold 4 gallons of juice, how many gallons does 1 container hold?
b. I have 7 gallons of juice. If a container holds 4 gallons, how many containers can I fill?
c. If 7 containers hold $\frac{1}{4}$ gallons of juice, how many gallons does 1 container hold?
d. I have 7 gallons of juice. If a container holds $\frac{1}{4}$ gallon, how many containers can I fill?
e. If 7 containers hold $\frac{3}{4}$ gallons of juice, how many gallons does 1 container hold?
f. I have 7 gallons of juice. If a container holds $\frac{3}{4}$ gallon, how many containers can I fill?

What are the similarities and differences between the problems?

Figure 33. 'Juice containers' in week 5 (Task 3).

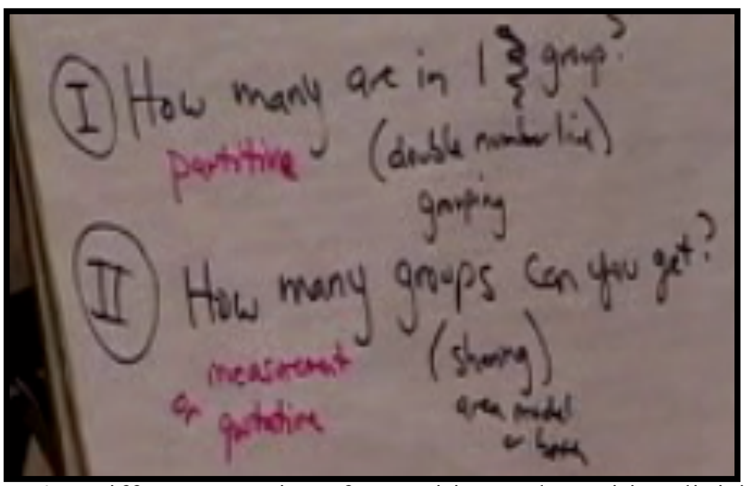


Figure 34. Different questions for partitive and quotitive divisions.

Moreover, Rachael wrote down two different questions that were answered using partitive and quotitive division situations on chart paper as in Figure 34). Furthermore, Rachael addressed the fact that referent units for each of the three quantities (i.e., divisor, dividend, and quotient) were different in partitive and quotitive division after the teachers discussed Task 2 (Figure 31) and 3. Regardless of her efforts, the class was extremely quiet, and teachers expressed their thinking and stated that there seemed to exist a big gap between sharing and proportion problems.

Sequence 1: When the Divisor Partitions the Dividend Evenly

I have labeled as Sequence 1 the case in which the divisor quantity evenly measures out the dividend quantity as in 2 ÷ 1/4. I analyzed teachers' drawings of quantities that were interpretations of numerical expressions, their interpretations of drawings of quantities that could model a measurement interpretation of division, and their statements of reasonable measurement division word problems. While a few teachers (e.g., Claire and Donna) had been using a quotitive approach to division as evidenced by the pre-assessment interview, the class as a whole did not talk about it until the discussion of Task 1 (See Figure 28 for the problem). In Task 1, teachers were expected to find the quotients without using algorithms and to find the patterns among the expressions. This class was the fourth InterMath content class, and the three previous classes were spent mainly on discussing problems that entailed referent units, the partitive model of division, or a partition of partition activity (e.g., the candy bar and licorice problems), and fraction multiplication, especially when fractions were conceived as an operator. Rachael encouraged teachers to use various drawn representations. They had computers to show their reasoning with Fraction Bar software. The teachers also drew their models on paper.

I will just give a brief note on how Task 1 was introduced to teachers and why it was included as part of the tasks for the DiW project. Rachael expected teachers to generalize the pattern by finding similarities and differences in numerical expressions with their quotients. The pattern that she hoped teachers would generalize out of solving Task 1 was the effect of the divisor quantity on its quotient: The denominator of the divisor increases the size of the dividend, and the numerator of the divisor decreases the size of the dividend. I know this because the task was included after she and the principal investigator of the DiW project were informed of the strategy by Dr. Harel, who is a member of the Rational Numbers Project, during the advisory board meeting of the Does it Work project. For instance, 2 ÷ 3/4 is 2(2/3) because four (the denominator of the divisor) increases the size of the dividend quantity two into eight (2 × 4) and three (the numerator of the divisor) decreases the size of the eight into 8/3; hence, the quotient is 2(2/3). The order does not matter; one may start by decreasing the size of the dividend with the numerator of the divisor and then increase the decreased amount with the denominator of the divisor. This operation is an inverse operation of the Duplicator/partition-reduction [DPR] subconstruct[34] in a fraction multiplication context in that the divisor operates on the dividend as the partition-reduction/duplicator.

Teachers' interpretations of quotitive division. Regardless of Rachael's initial intent, teachers did not find patterns such as just discussed (even though they understood her intent later in the class). Rather, they generated various drawn representations that modeled the numerical expressions. First of all, how teachers interpreted the numerical expressions of division framed their choices of division models (partitive or quotitive). In general, except for the cases when the divisor was a whole number, teachers used unit-segmenting operations and used an interpretation of either "How many times one can subtract the divisor from the dividend (repeated subtraction)" or "How many groups of the divisor quantity fit in its dividend."

Even though teachers accepted both as valid quotitive approaches to the division, there was a difference in the teachers' abilities to reason in the sequence of division situations using drawn representations and to choose the word problems that depicted measurement division. For instance, Donna said she did not accept the word problem that modeled the decimal

[34] More discussion of DPR subconstruct from the Rational Numbers Project research is under literature review chapter 2.

measurement division situation because the divisor quantity was bigger than the dividend quantity; hence, a repeated subtraction strategy could not be used. This is understandable considering her operations to determine a quotient of 2/3 ÷ 3/4 using a bar model in Sequence 4[35].

With the whole-number dividend. When the dividend quantity was a whole number in Sequence 1, teachers could activate recursive partitioning operations for unit-segmenting operations or use unit-segmenting operations with two levels of units. For instance, Claire knew the quotient of 2 ÷ 1/4 was eight before she even showed us her model. During the small group discussion period of Task 1 (Figure 28), Rachael came by Claire and Claire explained to her, "Well, there is eight-fourths in there. Two divided by three gets two cookies divided them among three people (she pointed at her numerical expression of 2 ÷ 3 = 2/3 on her notebook), then (she laughed)."

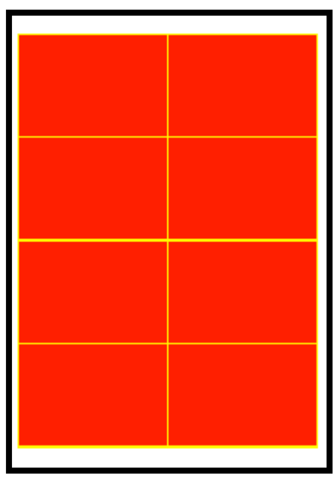


Figure 35. Reconstruction of Claire's model of 2 ÷ 1/4.

Protocol 13: Claire explaining why 2 ÷1/4 was eight units of one-fourth.

RA: So two divided by three is kind of fluff?
CL: Yeah.
RA: So what were you saying about two divided by one fourth?

[35] More discussion on this is under Sequence 4.

CL: One fourth, there are eight-fourths in two.
RA: Eight-fourths in two.
CL: Eight-fourths, (she counts the rectangles) one, two, three, four, five, six, seven, eight.
RA: So this [left] bar is one and this [right] bar is one and that is four pieces (she counts rectangles in the left side bar in Figure 35)? And you counted them up “one, two, three, four, five, six, seven, eight?” (She counts each rectangle in the 2-part bar in Figure 35).
CL: Uhum [Yes].
RA: Does that make sense to you Rose?
RO: Yes.
RA: Okay.

When Rachael asked Claire what the answer to 2 ÷ 1/4 was, Claire immediately said the answer was eight. She did not need to draw an area model to figure out how many one fourths were in two. She might have mentally calculated the answer or quickly knew that two was composed of eight units of one fourth. Rachael asked Claire to show her reasoning with drawn representations, and Claire used an area model (like in Figure 35) as usual. In brief, to generate a situation that illustrates 2 ÷ 1/4, she used the interpretation, “How many groups of one-fourth are in 2?” and generated a three level unit structure in which the length of 2 was the largest unit, the length between 0 to 1 was the mid-level unit, and the length of 1/4 was the smallest unit. In order words, she conceived the whole 2-part bar as one unit of two, two units of one, and eight units of one fourth. If one uses a unit-segmenting operation with two levels of units for the same problem, one may construct one bar and split it into four parts and know that there are four-fourths in one. One may know that there are another four-fourths in one by re-presenting a bar that is split into four parts or by copying the 4-part bar beside the original 4-part bar.

When the dividend was a fraction[36]. In Sequence 1, it did not seem to make a big difference in teachers' reasoning if the dividend was a fraction. Similar to the situation when the dividend was a whole number, teachers could use unit-segmenting operations with two levels of units, or they could activate their recursive partitioning operations for their unit-segmenting operations. For instance, in order to generate a model to solve 1/3 ÷ 1/9, one may start with a bar that is partitioned into three parts to answer "How many groups of 1/9 fit in 1/3?" Knowing that one third is one third of a whole requires reasoning with two levels of units sequentially. In order to measure 1/3 in terms of 1/9, one can draw another same size whole bar below the first bar and then partition it into nine parts (i.e., two levels of units) as in Figure 36. Because the two bars were lined up so that one could clearly see that three ninths in the second bar fit in the third in the first bar, he may say the quotient for 1/3 ÷ 1/9 is 3.

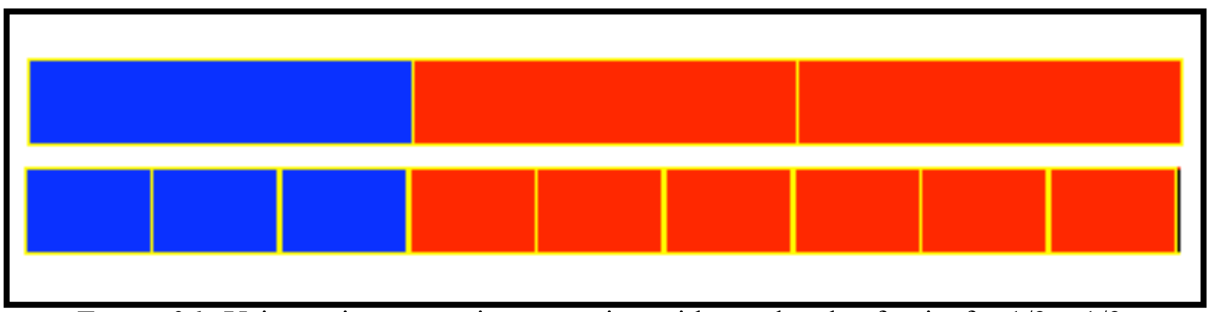


Figure 36. Using unit-segmenting operation with two levels of units for 1/3 ÷ 1/9.

In contrast, when one uses three levels of units to find the quotient, one does not need to line up the two bars as in Figure 36 to find out how many divisors fit in its dividend but could start with the third bar and split each third into three units knowing one ninth is one third of one third; hence, there are three ninths in one third. In brief, the person took the result of recursive partitioning for granted in that knowing one ninth as one third of one third requires the person's

[36] The discussion under this section is mostly my hypothesis based on my analysis of teachers' reasoning for the assessment item, which asked teachers to choose the correct area model that might represent 1/3 ÷ 1/9.

attention to the three levels of units, that is to view one as one unit of one, three units of thirds, and nine units of one ninth.

As part of Pre-, Post-, and Delayed-Post assessment interviews, we asked teachers to choose the model that best depicted the quotient 3 for 1/3 ÷ 1/9. It was clearly stated in the problem that a person started by drawing one third (the gray colored part in Figure 37a. and 37b.). In addition, it was stated that each diagram showed the original shaded 1/3 (gray colored part of the leftmost column) and three more regions shown as the following:

This prevented teachers from choosing a model because they could see the quotient 'three' in it. Figure 37b. was the common incorrect answer to the problem even after the InterMath class was over.

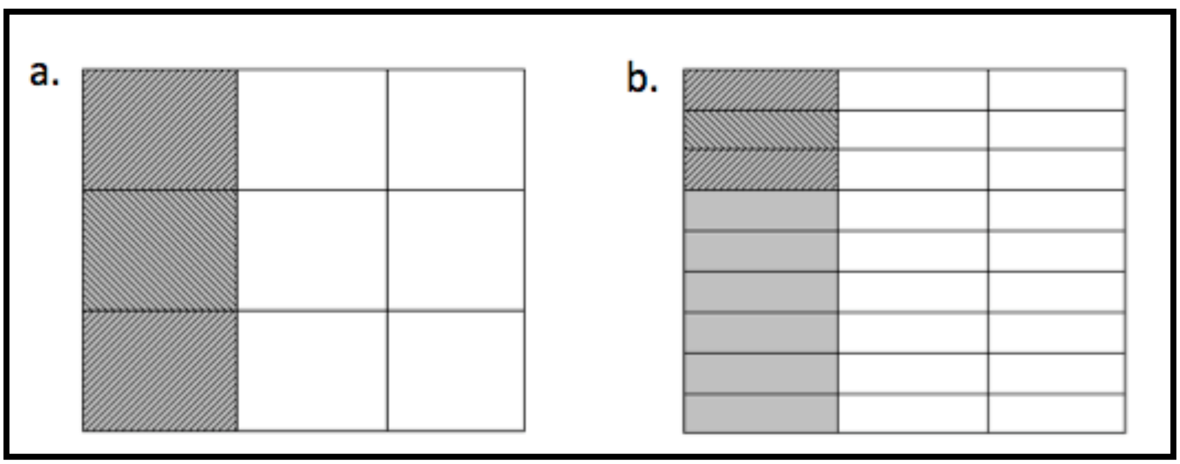
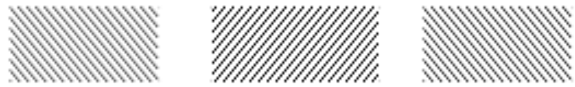


Figure 37. a. One possible model for showing 1/3 ÷ 1/9 using three levels of units. b. Teachers' incorrect choice for showing 1/3 ÷ 1/9[37].

Only three (Keith, Claire, Donna) out of eight teachers who were interviewed could choose the correct model for 1/3 ÷ 1/9 (See Figure 37a.) from the pre-assessment. All three teachers clearly stated that they chose a. in Figure 37 because it showed three ninths in one third.

[37] As I stated before, since the assessment items are all secured, I modified the problem and reconstructed the models that were similar to the models in the assessment.

Walt did not choose a. but b. as the answer in both pre- and post- assessments, but he changed his answer to a. during the post-assessment interview by attending to the referent unit for the quotient 3. There were two other interviewed teachers (Will and Diane) who chose b. as a correct model for showing 1/3 ÷ 1/9. Despite the fact that the teachers tried to answer how many ninths were in the third, they saw the three  in Figure 37b as three ninths because they did not attend to the three levels of units of one ninth, one third, and a whole simultaneously. Because it was given in the problem, the teachers might have seen the leftmost column as one third, but they might have conflated the unit while they were finding the number of ninths that fit in the third. One of the teachers, Diane, said she disliked model a. because she could not see the unit of one ninth (i.e., she could not attend to the three levels of units). She said,

> I don't know where the ninth came in, unless you are talking about suddenly making nine pieces which is not, it is dividing it by three or cutting it into a third. I want to get what I should get out the drawing.

Capacity to devise word problems for measurement division in Sequence 1. Considering growing attention to the task-based mathematics classroom, teachers' capacities to devise word problems that represent problem situations seems an important part of mathematical knowledge for teaching. In Sequence 1, teachers needed to explicitly address the divisor and the dividend quantities with the appropriate referent units to devise word problems that could be judged as quotitive division problems. As I stated, teachers were to complete the reflection task in each InterMath class, and teachers turned in their answers to Reflection 1 (See Figure 38) after they finished the discussion of Task 1 (Figure 28).

a) Write a word problem for 5 divided by 1/3 and illustrate the computation with a model.
b) Write a word problem for 5 divided by 2/3 and illustrate the computation with a model.
c) What questions could you ask students to help them understand the proportional relationship between the problem in part a and the problem in part b?

Figure 38. ‘Pesky parts 1’ in week 4 (Reflection 1[38]).

All 14 teachers turned in their answers, and only three (Keith, Claire, and Carrie) devised proper word problems. Except for one teacher (Salihah), who consistently used the same method as she used to model the 1/3 ÷ 3 during small group discussion of Task 1 (Figure 28), most teachers drew a model that showed 15 units of thirds in five. However, they did not devise the correct word problems; some changed the problem situation that one did not need to partition the partitioned piece, while others devised the word problem without attending to the referent units or without including both divisor and dividend quantities.

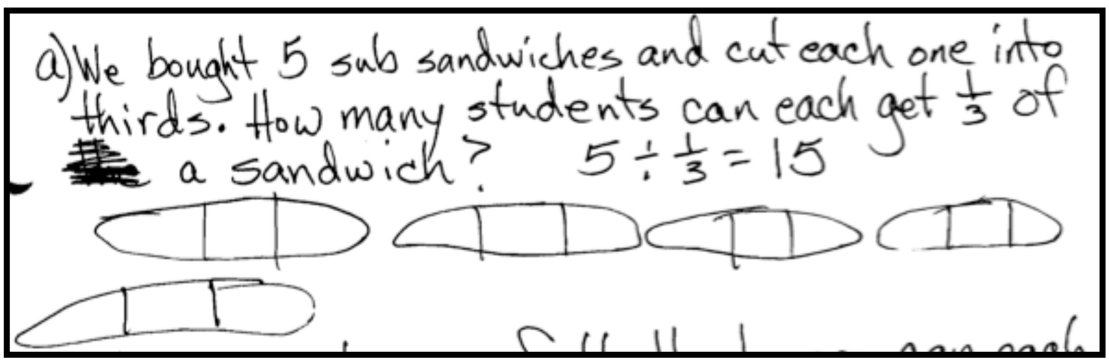


Figure 39. Donna’s word problem for 5 ÷ 1/3 with which partition of partition is already included as part of the problem.

Revealing crucial operations for solution in problem. First, Donna was clearly attending to the referent units of the dividend (5 sub sandwiches) and the divisor (1/3 of a sandwich) in devising her word problem as Figure 39 shows. However, Donna revealed a crucial operation for solving the problem, which is partitioning operations, by stating that each sandwich was cut into three parts in the problem. She might have devised the word problem for herself or for her

[38] As I stated in the methods section, each teacher was to complete the reflection before she left each InterMath class. Reflection tasks were related to the content discussed that day of the InterMath class.

students because she was asked to write questions to help students to see the proportional relationship between a. and b. in Reflection 1 (Figure 38). Considering that she was a teacher, providing her students with such a problem may help the students to get a correct answer to the problem but will prevent her from assessing her students' capacities to use or to develop critical operations in divisions. The students will get a correct answer to the problem as long as they could keep track of the number of one-third in five 3-part bars with her word problem, whereas stating the problem like "We bought 5 sub sandwiches. How many students can each get a third of a sandwich?" will require students to think of themselves that they need to use partitioning operations.

Donna's model reminded me of Carrie's comment about changing units in the Licorice problem (share 11-inch licorice bar among 12 people) in Part I. Carrie wanted to modify the licorice problem from 11 inches of licorice divided among 12 people into 11 feet of licorice divided among 12 people so that she could use inch block manipulatives to share among 12 people. She said it made it easier for her to distribute one from each foot bar by inserting 12 inch blocks in each foot. This provided more evidence of her lack of capacity to partition a partitioned bar. While it was extremely hard for her to partition the 11 inches of licorice bar, it was much easier for her to partition the 11 feet of licorice bar into 12 units using her knowledge of one foot = 12 inches. Similarly, Donna might have come up with the word problem by revealing partitioning operations in the problem because it was much easier for her to conceptualize the problem situation or she thought it would make more sense for her students in class.

As a matter of fact, there is much evidence to support Donna's lack of partitioning operations throughout her performances in each of the four sequences of the measurement fraction divisions. Common partitioning operations or recursive partitioning operations were not available to her in her attempt to use unit-segmenting operations, so most of the time, she reasoned with two levels of units. It might be one reason that she would want to devise a word problem with which her students or she did not need to partition.

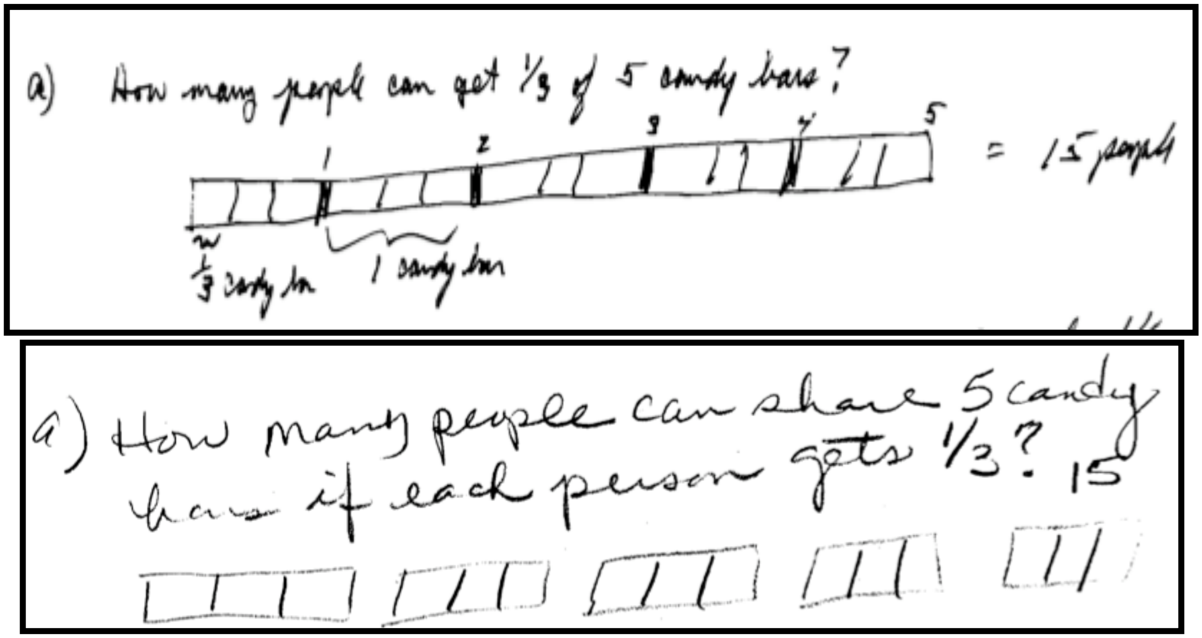


Figure 40. Teachers' inattention to the referent unit in illustrating a word problem for 5 ÷ 1/3 (Top: Mike's writing; Bottom: Linda's writing).

Inattention to referent units in devising word problems. On the other hand, Mike's incorrect word problem (as in Figure 40 top) displays his inflexibility with referent units. Because he had partitioned each section of the 5-part bar into three units and noted 15 people beside the model, he might have thought of measurement division or he simply might have not expressed himself clearly in the word problem. At any rate, his intent was to devise a word problem showing the measurement model of division, he did not clearly indicate the referent unit of 1/3 that he wrote down under the leftmost piece in the bar, and he also devised the word problem with a whole (5 candy bars) as a referent unit for the third. One-third of five candy bars is five-thirds of a candy bar, not 15 people; hence, his word problem is misleading. Furthermore, Mike did not clearly state the dividend quantity in the word problem. Using his word problem, any number of candy bars can be the whole to start with. While Linda clearly stated the dividend and the divisor quantities as in Figure 41 at the bottom, she was also not attending to or failed to include the referent unit by stating the divisor quantity as one third instead of one third of a candy bar, which could also mislead students.

Sequence 2. When the Divisor Does Not Partition the Dividend Evenly

When the divisor quantity did not partition the dividend quantity evenly, it was not enough for teachers to use unit-segmenting operations. They also needed to establish a part-to-whole relationship between the leftover[39] quantity and the divisor quantity.

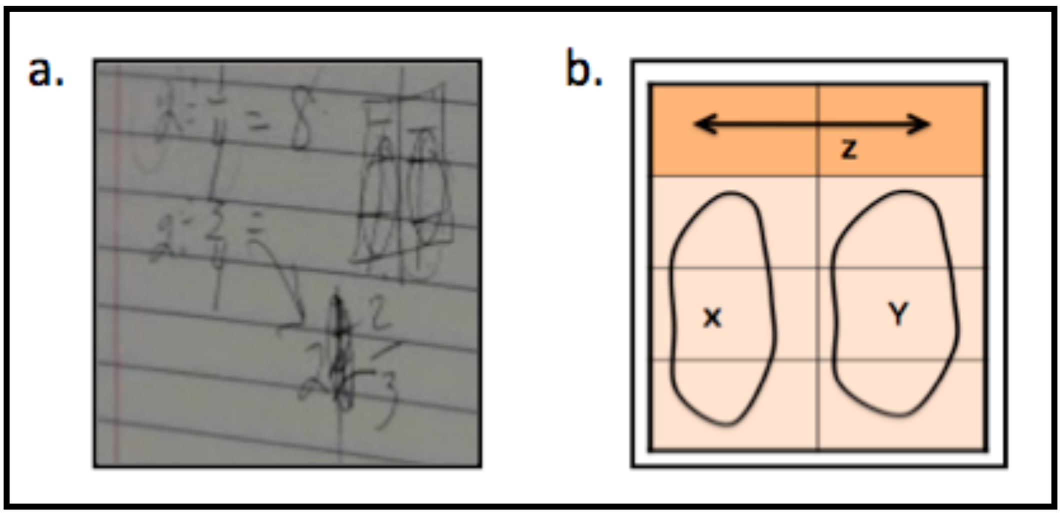


Figure 41. a. Claire's actual writing and drawing for 2 ÷ 1/4 and 2 ÷ 3/4. b. Reconstruction of Claire's model of 2 ÷ 3/4.

When Claire sensed that both Rachael and Rose (who was sitting beside her) were satisfied with her method to find the quotient for 2 ÷ 1/4 (see Protocol 13 and Figure 35), she started to show how she would solve 2 ÷ 3/4 using the same area model.

Protocol 14: Claire's limited attention to the referent unit of the leftovers.

CL: 2 divided by 3/4 is three-fourths [X in Figure 41b], three-fourths [Y in Figure 41b], two and (she writes the number 2 in the paper as in Figure 41a], I have 2 and (she writes 2/4 first then changes it into 1/2 as in Figure 41a). That one [2 1/2].

---

[39] I will use 'the leftover' to denote the quantity that was leftover from measuring out the dividend by the divisor.

RA: So that one [2 ÷ 3/4] is two groups of three-fourths and then you have two leftovers (She points at the two leftover pieces)?
CL: Two-fourths of [Pause] Oh! The two [z in Figure 41b] would be two-thirds because yeah [it takes] *three to make a group*, wouldn't it?
RA: Hang on. So we went from two-fourths to two-thirds?
CL: Oh, well, it is two-fourths I was right the first time, wasn't I? *Those [z in Figure 41b] are fourths*.
RA: (Laughter) Hang on. Hang on. You are going too quickly. So two fourths or two thirds, let's think about it (She tries to replace the 1/2 with 2/3 as in Figure 41a). So you are saying these [z in Figure 41b] are fourths, so there are two [one-fourth] of them. Now when we talk about two-fourths, we are talking about what referent unit? When you say a fourth (halted).
CL: The fourth is of one whole, *but new [referent] units [that I use] are* three-fourths [X and Y in Figure 41b], so they (she swiftly moves two pieces in z of Figure 41b) would be two thirds.
RA: So the referent unit we really want is the three-fourths
CL: Three-fourths is one whole.
RA: And you said you have two out of three (points at three that Claire circled).
CL: Yeah, three.
RA: Does that make sense [Rose]?
CL: Two and Two-thirds.
RA: So when you think of division, Claire, you are asking "How many of these [three fourths] would fit that [2]?
CL: Right, that is what I am asking.
RA: So how many groups of these [three-fourths] would fit that [two]?
CL: Yes.

Note that Claire used a recursive partitioning operation to use a unit-segmenting operation to show 2 ÷ 1/4 when she used a length quantity (Protocol 13). Her recursive partitioning operations were in her assimilating situations of her unit-segmenting scheme, and it allowed her to find the quotient 8 immediately without having to segment the unit 2 by 1/4. On the other hand, in order to find the quotient for 2 ÷ 3/4, she needed to actually segment the bar as described in Protocol 14. Because she used a recursive partitioning operation to show 2 ÷ 1/4 and that she knew she could use the same bar, which was a product of her recursive partitioning operation, suggests that she was attending to the unit of two at three levels: one unit of two, two units of one, and eight units of one-fourth. Using the unit-segmenting operation, she counted the number of three-fourths by circling (X and Y in Figure 41b.) the three pieces from each column and wrote down 2 2/4 on the paper. The result of activating the unit-segmenting operation was two and two-fourths, but there was no perturbation for encountering the unit that could not be clearly segmented by the segmenting unit. It displays that she conflated the unit to be used in segmenting (i.e., a measurement unit) with the mid-level unit of the three levels of units-structure (the unit of one) to which she was attending by evoking a recursive partitioning operation, and determined the leftover quantity as two-fourths.

Rachael noticed Claire's conflation of referent units in measuring the leftover quantity and asked her, "So that one [2 ÷ 3/4] is two groups of three-fourths and then you have leftovers?" by pointing at the leftover pieces on her picture, Claire saw three pieces and the two pieces on her model and established the part-to-whole relationship by visually comparing the unit of two and the unit of three and changed her answer to 2 2/3. She realized that it took three of the smallest pieces to make a group by looking at her drawing and found that the two leftover pieces would be two-thirds because it took "Three to make a group." Nevertheless, she switched the answer back to 2 2/4 because knowing that each piece was one-fourth was also prominent. Given that Claire had two points of view on one situation, it is not surprising that she was a bit unstable. Even though Claire established the part-whole relationship by visually comparing the leftover three pieces and the measurement unit in her drawing, Claire conflated the units again when she reflected that each piece was one-fourth. It seems no surprise for Claire to a priori conceive the smallest piece as one-fourth knowing that she used a recursive partitioning operation (i.e., a partitioning operation, which produces three levels of units-structure) to use a unit-segmenting operation when she explained her reasoning to Rachael. When she showed Rachael why the quotient to the 2 ÷1/4 was eight, she pointed at each of the eight pieces in the bar of Figure 35 to show that there were eight-fourths in two as in Protocol 13. Claire corrected her incorrect answer when Rachael asked her what the referent unit was when the leftover two pieces were “Two-fourths”. Claire realized that she was supposed to use the measurement unit [three-fourths] as the referent unit to the leftover quantity. Using the measurement unit as the referent unit to the leftover quantity, she established a part-to-whole relationship using a visual support from her drawing.

My hypothesis is that if Claire had iterability of a unit fraction [one-fourth] as a given operation in assimilating this problem situation, she could have conceived the three-fourths as three units of one-fourth each of which can be iterated three times to make three-fourths and also can be iterated eight times [2=8/4] to make eight-fourths prior to using unit-coordinating operation. In addition, with an iterability of a unit fraction, *any number* of which can be disembedded from (pulled out of) the three levels of unit of two without destroying the unit two. In other words, one-fourth can be used as an abstract entity that is not restricted to part of whole notion for fractions (i.e., one-fourth is one-fourth of one or one out of four) but can be used as a unit that can be iterated. Having the iterability of a unit fraction as an operation may have helped her to assimilate the problem 2 ÷ 3/4 as the same as a situation for solving “8 ÷ 3.”

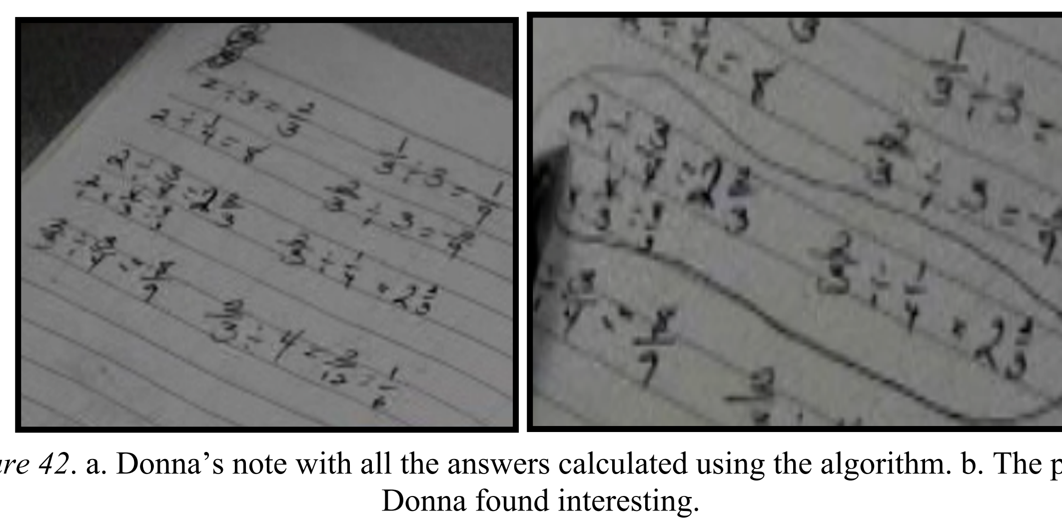


Figure 42. a. Donna’s note with all the answers calculated using the algorithm. b. The pattern Donna found interesting.

Donna had already calculated the answers to each expression, and it was not until Rachael came to her table that she started to use models. Donna tried to explain to Rachael the pattern that she found. Donna circled 2 ÷ 3/4 = 2 2/3 and 2/3 ÷ 1/4 = 2 2/3 as in Figure 42b, and told Rachael that she thought those two division statements were interesting because the answers were the same. When Rachael asked her why she thought the answers were same. She said, "I am not sure how to answer it, but you are right; um well because we multiply by the reciprocal."

Donna decided to use the Fraction Bar software to show Rachael why 2 ÷ 3/4 and 2/3 ÷ 1/4 share the same quotient as in Figure 43. While the quotients for the two division number statements were the same, they are different quantities because the referent unit of the quotient 2 2/3 from 2÷3/4 is 3/4 whereas the referent unit of the quotient 2 2/3 from 2/3 ÷ 1/4 is 1/4.

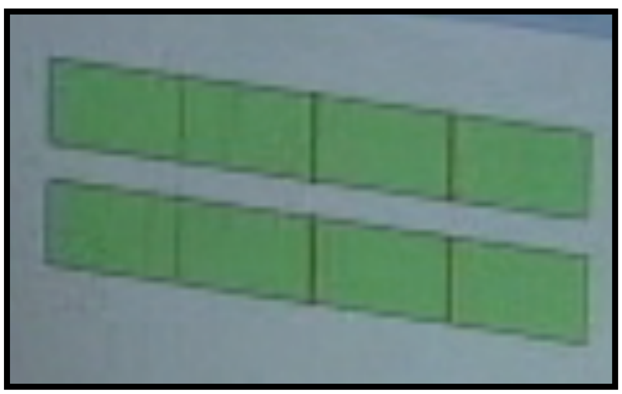


Figure 43. Donna's bar model for 2 ÷ 3/4.

Protocol 15: Donna making a bar model to show 2 ÷ 3/4.

RA: So we got two, so we want to divide it by three fourths so *what would you do to each bar? How* do you want to *split it?*
DO: I think I came out with I think I need to split it into fourths.
RA: Why would you split it into fourths?
DO: So that I can, okay so 2 ÷ ¾, so that I can pull out three fourths.
DO: Does that make sense?
RA: Cause you said you think of division as what again?
DO: Repeated subtraction.
RA: Okay.
DO: Repeated subtraction.
RA: Okay. So you are going to try to figure out how many three fourths you can subtract off.

DO: Uhuh.
RA: Okay. So yeah…Let's try it.

Donna started by drawing one bar, and she contemplated for a while whether to copy another bar under the original bar. Because Donna could not continue after she drew two bars, Rachael gave her a hint by asking how she would split each bar. Donna split the first bar into fourths and repeated the same process for the second bar. She said she split each bar into four bars so that she could take three fourths. Donna used repeated subtraction because she said, "When we are dividing [a whole] by a fraction, we are saying how many times we subtract three fourths from two; it is just the way how I think about division–how many times you subtract."

The fact that Donna started to split each one into four units after Rachael asked her "How do you want to split it?" may indicate that recursive partitioning operation was not a part of her assimilating operations to initiate unit-segmenting operations. It seems meaningless to compare the two teachers' reasoning with drawn quantities because Claire solved 2 ÷ 3/4 after she showed Rachael her justification why the quotient to 2 ÷ 1/4 was eight while Donna started from 2 ÷ 3/4. However, I guess Donna was able to use the operation that produced three levels of units, but the three levels of unit structure was not given from the beginning as in Claire's solution. Donna produced the two bars, each of which was partitioned into four parts with several contemplations and with Rachael's guiding questions; hence, at best it seems that she used the operation that produced a three-level unit structure.

The video did not capture anymore of Donna's work because Rachael moved to different tables, but the data at least shows the difference between Claire's and Donna's capacities to use drawings of quantities. Nonetheless, Donna was one of a few teachers who devised the correct measurement division word problem (see Figure 44) with the correct bar model in Reflection 1 task[40]. She was clearly measuring the leftover quantity 1/3 using the measurement unit 2/3, and found the quotient 7 1/2 as indicated below when more than half of the teachers determined the quotient as 7 1/3. Those teachers were confined to the mid-level unit of one and conceived the leftover piece as one third.

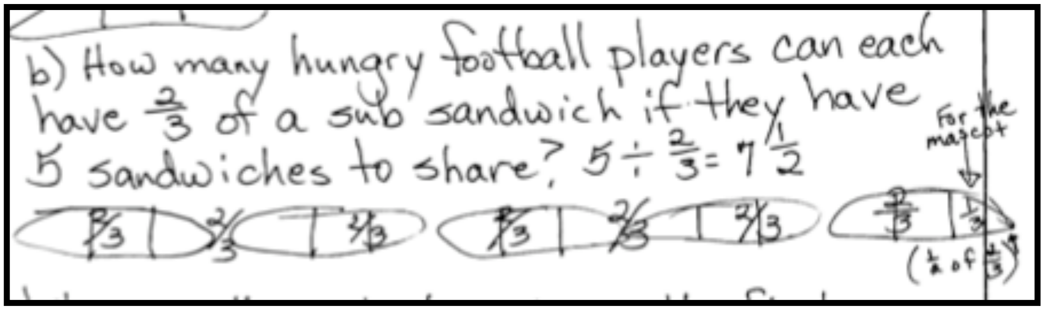


Figure 44. Donna ‘s word problem of 5 ÷ 2/3.

While it was inevitable for most teachers to calculate the numerical answer to each division task, it did not necessary mean that they did not know what the quotients stood for. Walt also calculated all the answers before he came up with the models, but he knew how 2 2/3 could come from using a measurement approach to the division problem of 2 ÷ 3/4. Note that the class did not yet discuss a measurement approach, so it was Walt’s own idea. After Walt explained to Will how they could model 2 ÷ 1/4 using measurement division, he began to explain to Will how he could model 2 ÷ 3/4 = 2 2/3 without using an algorithm[41]:

> Here is what I (Walt) am saying. I am going to divide these two by three fourths. Here is one fourth, two fourths, and three fourths in one. Here is another one fourth, two fourths, and three fourths in the second one. So that is the two and two fourths left over. Why is it two thirds? Oh, because one fourth is third of three fourths.

While Walt had to spend some time to figure out why the answer was two and two-thirds rather than two and two-fourths, he soon realized that one-fourth of one was one-third of three

[40] As I stated, Reflection 1 was asked to reflect the class discussion of Task 1.

[41] I could not tell which model Walt used to show Will for solving 2 ÷ 3/4 because the two cameras and one microphone were all capturing Rachel, Pascal and Carrie discussing on drawing 2/3 ÷ 3/4 using area models. The analysis of Walt and Will discussing of 2 ÷ 3, 1/3 ÷ 3, 2 ÷ 1/4, and 2 ÷ 3/4 was conducted from the audio data that was coming from one microphone.

fourths; hence, the quotient to 2 ÷ 3/4 was 2 2/3. Even if having the answer calculated guided him to correct his error so quickly, he would not have corrected the answer without knowing that the quotient 2 2/3 stands for two and two thirds groups of three fourths in two. He knew the relationship among the divided, divisor, and quotient quantities. Furthermore, Walt was one of the two teachers (Claire was the other teacher) out of eight interviewed teachers who thought that the following model in Figure 45 could represent the area model of 1 ÷ 2/3 with showing 2/3 shaded and 1/2 of 2/3 unshaded or 3/2 in all.

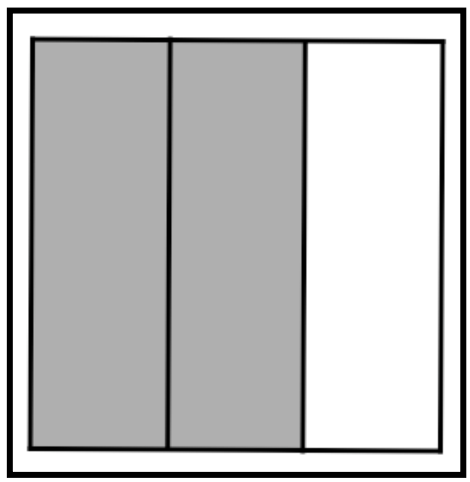


Figure 45. Area model that can be interpreted as showing 2/3 shaded ; 1/2 of 2/3 unshaded; or 3/2 in all.

Even though Walt explained to Will how he used measurement division to find the quotient for 2 ÷ 1/4 and 2 ÷ 3/4 during the small group discussion and Will said he got it, Will continued to show inflexibility with referent units. During the whole group discussion, he commented that the answer to 2 ÷ 3/4 was 2 1/2 because:

> You get three quarters out of the first one [in Figure 45] and three quarters out of the second one [in Figure 45] right? So you got a quarter left from the first one and the quarter left from the second one, right? So you got one quarter in the first one unit, you got one full three quarter unit out of the first one, and you got one full three quarter unit out of the second one. Now you got two one-quarter units, one left in the first and one quarter…you got two plus two fourths, wrote 2 1/2.

Sequence 3: When Denominators of the Dividend and the Divisor are Relatively Prime

When the denominators of the dividend and the divisor were relatively prime, common partitioning operations and cross partitioning operations for unit-segmenting operations were used, which overlaps with the Sequence 2 with fraction dividends and with Sequence 4 with fraction dividends. Some teachers used recursive partitioning operations while others used a common denominator strategy in the situation of using common partitioning operations. When teachers used common partitioning operations, they could find the commensurate fractions for the dividend and the divisor quantities using the co-measurement unit. My analysis of teachers' knowledge in Sequence 3 is based on the teachers' discussion of finding the quotient for 2/3 ÷ 3/4 (which also falls under Sequence 4) and their drawn representations for the model 2/3 ÷ 1/7[42] (which also falls under Sequence 2) in the assessment. When teachers faced the problems in Sequence 3, some teachers needed to use either common partitioning or a cross partitioning operation. Most of those teachers used the term 'common denominator' whenever they had to find a common partition to transform two fractions into fractions with like denominators. At first I thought that the teachers might have used an algorithm, but finding the common denominator is not the algorithm for fraction division; rather the invert and multiply strategy is the algorithm.

The teachers' use of common denominator method was more than procedural. They did not use it to calculate the numerical answers to the division problems but to clearly show commensurate fractions for the dividend and the divisor quantities. Thus, their common denominator method was associated with common partitioning operations. In the following, I

[42] As I stated under method section, teachers were asked to find the number line model with a word problem that best showed the division 2/3 ÷ 1/7, but all eight teachers did not think there was an appropriate model displaying 2/3 ÷ 1/7 because their fraction division knowledge was confined to measurement approach to division. Most of the teachers attempted to show us their models of 2/3 ÷ 1/7 during interviews, thus I was able to understand their operations for solving 2/3 ÷ 1/7. This was actually the one and only problem that only fell under the Sequence 3.

illustrate teachers' common partitioning operations by comparing the number line model of Claire (Figure 46b) and Walt (Figure 47b).

As I stated in the method section, we had one fraction division item where we asked teachers to choose the number line model that could show 2/3 ÷ 1/7 = 14/3[43], and most of the teachers (including Claire and Walt) disliked the choices in the problem because none of the models could be understood if one uses a measurement interpretation of division. While Claire was asked to justify the number line model for 2/3 ÷ 1/7 = 14/3 that she had already drawn (like a. in Figure 47) during the post-assessment interview and Walt was asked to draw a model showing 2/3 ÷ 1/7 = 14/3. I will start by discussing Claire's number line model for 2/3 ÷ 1/7. Her common partitioning operation was activated to use her unit-segmenting operation.

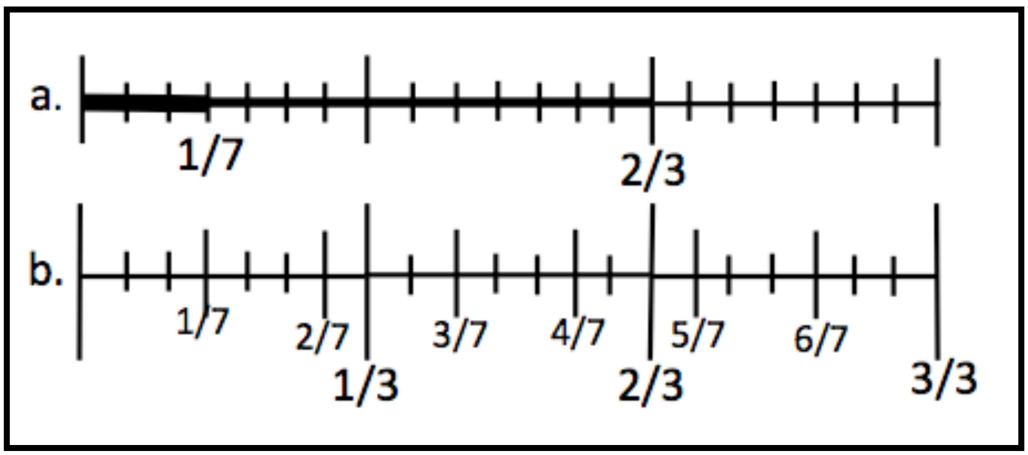


Figure 46. a. A typical representation of common partitioning operation for 2/3 ÷ 1/7. b. Reconstruction of Claire's common partitioning operation for 2/3 ÷ 1/7.

Protocol 16: Claire justifying her number line model of 2/3 ÷ 1/7 in the interview.

Interviewer[44]: Would you explain me your model you draw on the bottom [of the page]?
CL: (She already has her number line model like b. in Figure 46) I got the whole and then I divided it into sevenths and then I divided each seventh into thirds which came out each little lines twenty firsts which is your common partition and then um (she reads the

[43] Since items are all secured for the purpose of Does it Work project research, I changed the numbers. Moreover, I reconstructed the number line models of Claire and Walt so that they met the 2/3 ÷ 1/7. The original division item was fell under the Sequence 2 and 3.
[44] I was an interviewer.

> numerical expression 2/3 ÷ 1/7 from the problem) it says how many one sevenths are contained in two thirds so two thirds is fourteen twenty-firsts and one seventh is three twenty-firsts. So it comes out to be four and two thirds. So you got one, two, three, and two little pieces leftover here.

According to her statement in Protocol 16 and her number line model in Figure 46b, Claire started with a number line and partitioned it into seven parts and labeled each seventh. Then she recursively partitioned each seventh into three parts and produced the unit of twenty firsts. Finally, she drew over each third with the vertically longer line exactly like in b. in Figure 46. She was even aware that she had produced a “common partition” between the dividend and the divisor quantities. As usual, Claire reminded herself of the question “How many one sevenths are contained in two thirds?” and found *commensurate fractions* for two thirds and for one seventh using a co-measurement unit one twenty-first. I use commensurate fractions (Steffe & Olive, 2010) when teachers used drawings of quantities to figure out, in conventional terms, equivalent fractions.

Claire knew that a commensurate fraction for two thirds was fourteen twenty-firsts and for one seventh was three twenty-firsts from her number line model. When she determined the quotient, she knew four one-sevenths fit clearly in the two thirds, and the two pieces were leftovers. Labeling each seventh and third and drawing vertically different lengths of sevenths and thirds helped her measure the size of the two leftover pieces as two out of three. While her model might not be a typical common partitioning operation for 2/3 ÷ 1/7 (like a. in Figure 46) in that she started with a 7-part bar instead of a 3-part, she used the common partitioning operation because she coordinated two three levels of units of fractions 2/3 and 1/7 and produced a co-measurement unit of 1/21. Walt was another teacher who used a number line to model 2/3 ÷ 1/7, but his common partitioning operation was different from Claire’s in that he first calculated a common denominator between the divisor and the dividend quantities.

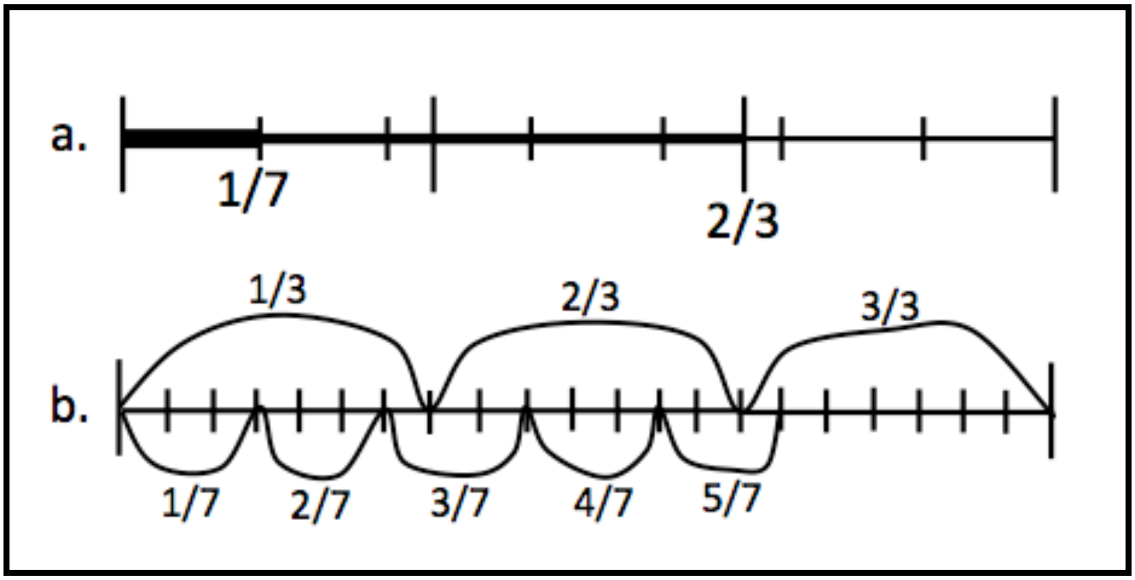


Figure 47. Reconstruction of Walt's model for 2/3 ÷ 1/7 in order of a. and b.

He drew a number line and divided it into three parts, and he divided the same number line into sevenths by estimating the size of one seventh as in Figure 47b. He did not partition the bar into seven parts by comparing the size of sevenths with thirds. In other words, he was not attending to the unit of third when he split the bar into seven parts. He started with two levels of units of 2/3 and 1/7 in one number line, but it did not take him long to realize that his model would not work. As soon as he constructed a number line like the one shown in b. Figure 47, he said:

> Actually, to properly model that [2/3 ÷ 1/7], I would have to model tenths no thirds um well [contemplated for a while] I would get, okay, what is the common denominator [of 2/3 and 1/7] 21, yeah, I would have to model 21 that is what I need to do. So (he draws another number line below the one like a. in Figure 47) assuming there is 21 equal parts (he puts 21 marks on the number line like b.), one third is equal to 7/21, so one, two, three, … , seven, that is one third (he puts an arrow from 0 to 1/3), two thirds is another 7/21, so, one,…, seven, that is two thirds (he puts another arrow from 1/3 to 2/3), and that (the remaining seven pieces) would be your three thirds (he puts the third arrow from 2/3 to 1, and labels each arrow as 1/3, 2/3, and 3/3 from the left). And inside that [each third], the seventh will jump like 3/21, so it will be one seventh, two sevenths, three sevenths, four sevenths, and two of three will be two thirds, which is fourteen thirds or four and two thirds.

Walt used a common denominator strategy and calculated the common denominator between the dividend and the divisor quantities to partitioned the bar into 21 parts. He found commensurate fractions for a third (i.e., 7/21) and a seventh (i.e., 3/21), and used them to label each third and five sevenths. He labeled fractions after he connected each third and each of the five sevenths with the curved lines. Note that he drew the curved line only up to five sevenths by iterating three twenty-firsts five times. Similar to Claire, iterating the fifth seventh helped him to measure the leftover quantity as two thirds because he could clearly see two out of three pieces were leftovers. While Claire explained 2/3 ÷ 1/7 using length quantities in post assessment interview (Figure 46 and Protocol 16), she used a cross partitioning operation by partitioning a bar vertically and horizontally by coordinating two composite units of the divisor and the dividend in delayed post assessment interview (Figure 48).

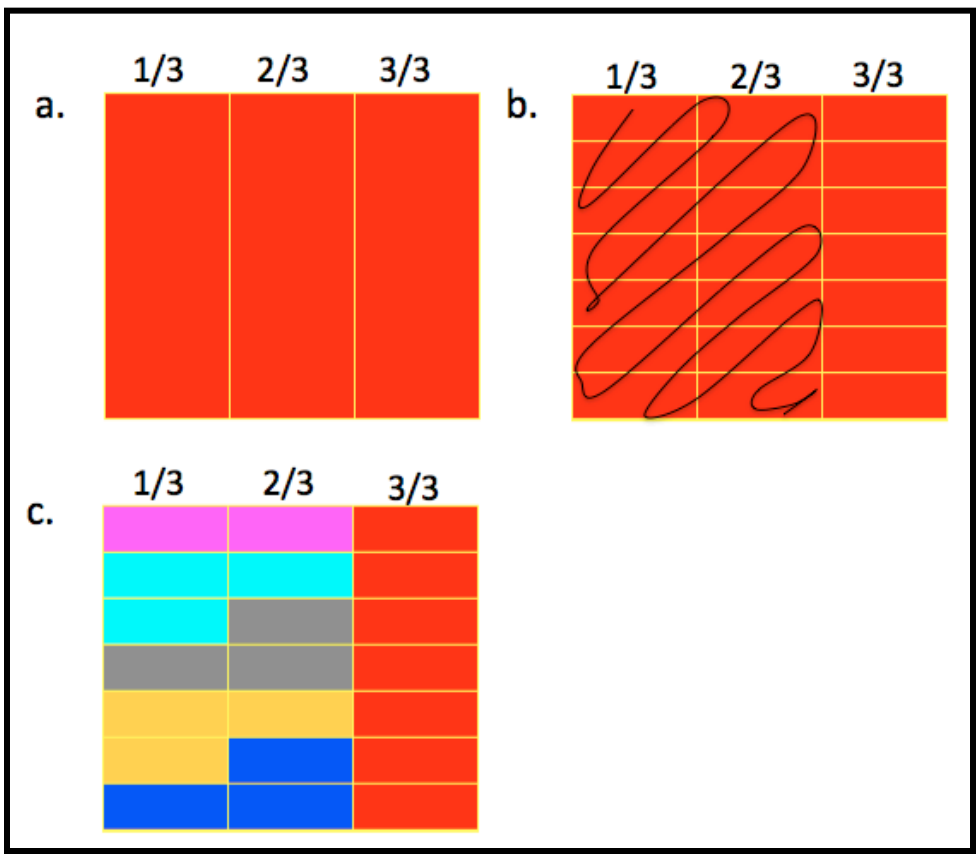


Figure 48. Claire’s area model to demonstrate 2/3 ÷ 1/7 in order of a., b., c.

Her cross partitioning operation was different from her common partitioning operation in that the former provided her with a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts. When the interviewer asked Claire to explain how she would like to model 2/3 ÷ 1/7, she drew an area model like in Figure 48a and vertically split a bar into three parts and labeled each column 1/3, 2/3, and 3/3. Then she horizontally split the 3-part bar into seven parts and shaded in two columns to indicate the dividend two-thirds like in Figure 48b. Then she reflected on the problem and asked "How many sevenths fit in two thirds?" which was the interpretation of fraction division that she usually used. Then she continued to say,

> One, two, three (she shades three blue pieces in Figure 48c), so three of those little sections is the seventh, one, two, three (counted yellow pieces in c.) two [sevenths], one, two, three (counted gray pieces in c.), three [sevenths] and one, two, three, four [sevenths] (counted sky blue piece in c.), so it would be four and two thirds (wrote down 4 2/3) besides the model).

After she counted four units of three from the area model she drew like in Figure 48c, she used the unit of three to measure the two leftover pieces and correctly stated the quotient for 2/3 ÷ 1/7 as 4 2/3. Using a cross partitioning operation allowed her to find the quotient to the division problem without explicitly using commensurate fractions. Note that she said one seventh was equal to three pieces while she used the commensurate fractions of 3/21 and 14/21 in her number line model (Figure 46b.). Moreover, she iterated (counted by) three pieces to find how many groups of three in 14. I guess that she might not know each piece was 1/21, but she could use whole-number three-level structure (i.e., a whole, the length of three, and the length of 14) when she used a cross partitioning operation with area quantities. As a matter of fact, in the class, she used a cross partitioning operation to show the whole class how she found the quotient for 2/3 ÷ 3/4[45] and she knew 2/3 was 8/12 and 3/4 was 9/12. But, again, she did not need to explicitly attend to the co-measurement unit 1/12 to determine the quotient for 2/3 ÷ 3/4. She used whole-number three levels of units: a whole, the length of eighth, and the length of nine[46].

Sequence 4: When the Divisor Is Bigger Than the Dividend

In Sequence 4, when the divisor quantity was bigger than the dividend quantity, teachers needed to find an alternative way to measure the dividend quantity with the divisor quantity. They could not iterate the divisor or count the number of divisor in its dividend to find the quotient because the divisor quantity was bigger than the dividend quantity. Teachers determined the size of the dividend quantity by setting up the part-to-whole relationship between the two quantities (e.g., 8/12 = 8/9 of 9/12 for 2/3 ÷ 3/4). The operation is not novel in that they used it in Sequence 2 when the divisor quantity did not evenly measure out the dividend quantity. For example, to determine a quotient for 2 ÷ 3/4 (Sequence 2) without using the algorithm, Claire generated a 2-part bar and recursively partitioned each part into four parts. After she counted three fourths from each part, she measured the two parts that were leftovers with the three fourths as the measurement unit. In order to measure, she was attending to the part-whole relationship in which the leftover quantity was the part and the measurement unit was the whole and found that two fourths was two thirds of three fourths. In other words, the situation in which teachers needed to measure one quantity with the bigger quantity (i.e., Sequence 4) already appeared in Sequence 2 when teachers needed to measure the leftover quantity with the divisor quantity.

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[45] During the whole-group discussion of Task 1 in InterMath class, teachers discussed how they could find the quotient for 2/3 ÷ 3/4. Like 2/3 ÷ 1/7 overlaps with the Sequence 2, this problem also overlaps with the Sequence 4, and more discussion of teachers' operations for solving 2/3 ÷ 3/4 will be included in the Sequence 4.

[46] How Claire found the quotient using her whole-number three levels of units is under Sequence 4.

While Keith, Claire, and Donna established the part-whole relationship between the dividend and the divisor quantities to determine a quotient for 2/3 ÷ 3/4, none of them seemed to recognize that it was a similar situation to the one where they had to deal with the leftovers in Sequence 2. As a matter of fact, teachers thought that the situation was very different from others like 2 ÷ 1/4 (Sequence 1) or 2 ÷ 3/4 (Sequence 2) because they could not use a unit-segmenting operation. After Rachael gave teachers small group discussion time for about thirty minutes for Task 1 (the task that teachers needed to find patters for eight fraction division expressions, see Figure 28 for the problem), she initiated the whole group discussion and chose Keith and Claire to introduce their models. Keith and Claire did not yet know which model they needed to draw, but they were chosen because Keith said he used a number line model and Claire said she used an area model. Because one of the foci of the InterMath class was to get teachers acquainted with various drawn representations, Rachael wanted the whole class to experience different ways to model the division problem, so she asked the whole class what division expression they wanted Keith and Claire to explain. Both Keith and Claire started to show 2/3 ÷ 3/4 because it was the division problem Carrie wanted the whole class to discuss[47]. I will first explain Keith's number line model of 2/3 ÷ 3/4 and then Claire's model.

[47] Carrie was one of the InterMath teacher participants who struggled a lot to model the division situations using measurement approach. I will demonstrate her struggle of using measurement division after this sequence under the subtitle *Teachers Who Did Not Use Units-Segmenting Operation*.

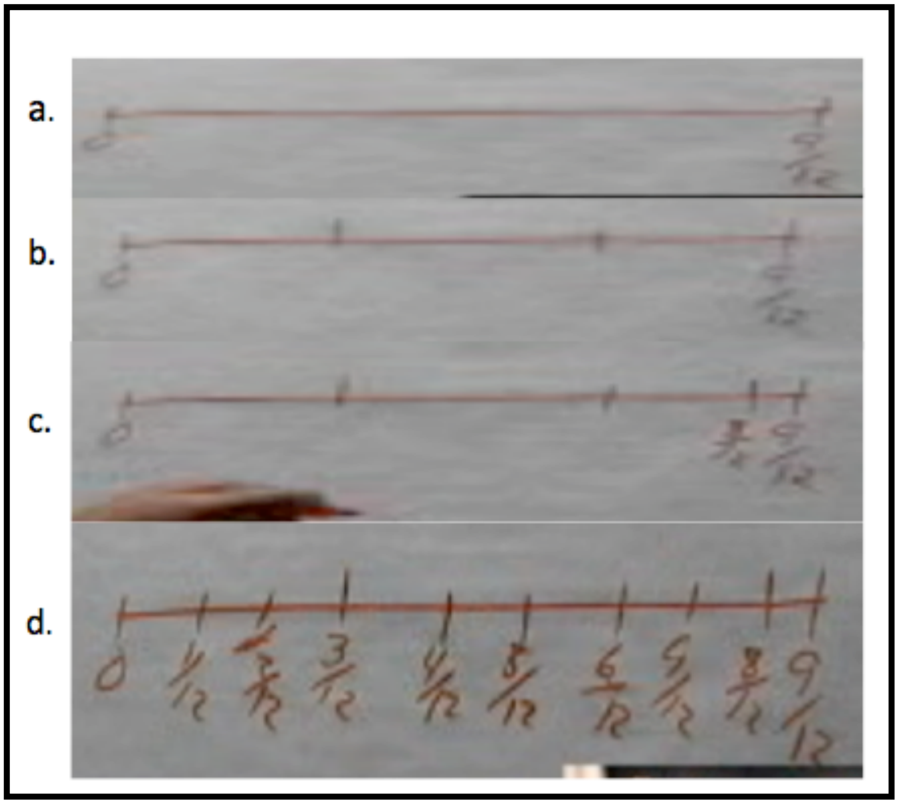


Figure 49. Keith's construction of a number line model of 2/3 ÷ 3/4 in order of a., b., c., d.

Protocol 17: Keith explaining his model to the whole class.

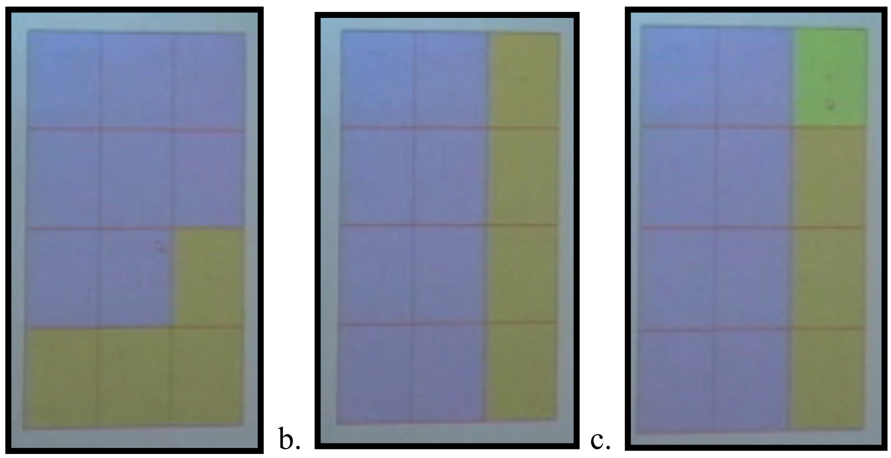
KE: This is the one that you really need to work because three fourths is bigger than two thirds, but! (He draws a number line as in Figure 49a) That's going to be nine- twelfths.
CA: Where did you get nine-twelfths?
KE: Three fourths (He divides 9/12 into three parts as in Figure 49b).
CA: Okay.
RA: Hang on. Why did you get to 9/12?
KE: I am going to get to the 8/12 in the second (He labels 8/12, then 7/12, …, 1/12 by recursively partitioning each of the three fourths into three parts as in Figure 49d).

Usually when teachers drew models using the measurement interpretation of division, they started with the dividend quantity because it is the unit to be segmented using the divisor quantity. However, Keith not only started with the divisor quantity but also in its commensurate fraction form using the co-measurement unit between one third and one fourth as in Figure 49a, and it confused Carrie. Keith skipped so many steps (because he had already thought of it in small group discussion time), which might have confused a lot of our teachers. Rachael asked him “Why did you get to nine-twelfths?” with an intention to understand how Keith got a common partition, but Keith seemed to misinterpret her question and thought that she was asking why he did not start from two thirds (or eight-twelfths).

However, Keith did use a common partitioning operation by coordinating the two iterable composite units of three and of four. Later in the class, Rachael asked Keith why he chose nine twelfths among all the other equivalent fractions of three fourths, and Keith said, “Because it gives a common denominator so that I can do eight twelfths and show eight twelfth is eight ninths of nine twelfths.” Keith chose nine twelfths among all equivalent fractions for three fourths because one twelfth was the co-measurement unit of both nine twelfths and eight twelfths. Moreover, despite the fact that he started with the length of nine-twelfths number line instead of one, he was attending to the one whole because he only had nine parts. Otherwise he would have gotten twelve parts in nine-twelfths. He was able to construct one whole [12/12] in re-presentation so that he could pull out nine parts from the whole for three-fourths and also eight parts for two-thirds. That is, his 9-part number line stood for nine-twelfths of the re-presented whole. This was possible by his coordination of two iterable composite units [three and four] to find the commensurate fractions of two-thirds and three-fourths.

> Protocol 18: Keith Continues…
> KE: Well. This is two-thirds (points at 8/12 in Figure 49d.) and this is three-fourths (he draw an arc from 0 to 9/12), and you can’t quite get three-fourths into two-thirds. Right? Like I was, we were dividing, our question, if it was a fraction, we were asking ourselves how many three fourths can we put into two thirds. Right?
> CA: Yeah.
> KE: Well you can’t put three fourths into two thirds not if it is bigger than two thirds, but you can get, one, two, three, …., eight –ninths of three-fourths into two-thirds
> CA: I don’t see that at all. I am so lost it is unreal.

Keith realized that he could not use the interpretation that he had been using to solve other measurement division problems in which the divisor was less than the dividend quantity when the divisor was bigger than the dividend quantity. In short, he could not use his unit-segmenting operation in this situation. He set up the part-whole relationship between the eight-twelfths (part) and the nine-twelfths (whole), and knew that eight-ninths of three-fourths was two-thirds. While Keith used part-whole relationship, he was using three levels of units of the fractions three-fourths and two-thirds because he used a common partitioning operation.



a. b. c.

Figure 50. Claire's use of area quantities to demonstrate 2/3 ÷ 3/4 in order of a, b, c.

After Keith demonstrated his number line model to the class, Claire used area quantities to illustrate her operations for solving 2/3 ÷ 3/4 as in Figure 50. Claire partitioned the bar into 12 pieces by cross partitioning it into thirds vertically and fourths horizontally. Then she colored in the three pieces of the bottommost row and one more piece from the third row in yellow as in Figure 50a and said "The blue is the two thirds [which is] eight-twelfths."

The fact that she was attending to the commensurate fraction of two-thirds shows that she was using a cross partitioning operation with the co-measurement unit one-twelfth. Moreover, it seems that she knew eight-twelfths was one-twelfth less than nine-twelfths because she colored in the bottommost row by thinking of three fourths (See blue pieces instead of yellow pieces in Figure 50a.) and colored in one piece, which would leave her with eight-twelfths. However, she soon moved the two blue pieces from the rightmost column to the bottommost row because she did not think the model in Figure 50a would clearly show two thirds and three fourths.

Instead of removing one-twelfth from three-fourths to get two-thirds, she made another area model that showed adding one-twelfth to two-thirds was the same as three-fourths as in Figure 50b. She said, “The blue was two-thirds and let’s put one green that represents (colors in one piece) blue plus green is three fourths. Okay? So how many three-fourths are in two-thirds? Eight of them. Eight-ninths of them.” After she made the area model of Figure 50b showing both two-thirds and three-fourths, she set up the part-whole relationship between the shaded pieces and figured out that eight-ninths of three-fourths were in two thirds. She modified the model not because she needed to have a visually clear representation of two-thirds and three-fourths but because she knew most teachers could not see them in Figure 50a.

Figure 51. Donna’s establishing a part-whole relationship using two levels of units.

While Keith and Claire coordinated two three level units of fractions eight-twelfths and nine-twelfths using a number line or area model to determine the quotient of 2/3 ÷ 3/4 as in Figure 49 and Figure 50, Donna set up the part-whole relationship between the divisor and the dividend quantities without using three levels of units as in Figure 51. After Claire demonstrated her area model of 2/3 ÷ 3/4, Donna said she copied Claire's area model, but she was confused because she did not know where nine pieces (in the bottom bar of Figure 51) came from. She continued to say, "Your [Claire's] ninths, I am not sure how to explain it in my model where the ninths came from." The fact that she partitioned only the three-fourths into thirds and that she did not understand where the ninths came from suggests that she was not re-presenting three parts in the last fourth; hence, she was not using common partitioning. Rachael asked Donna to demonstrate her strategy to the whole class.

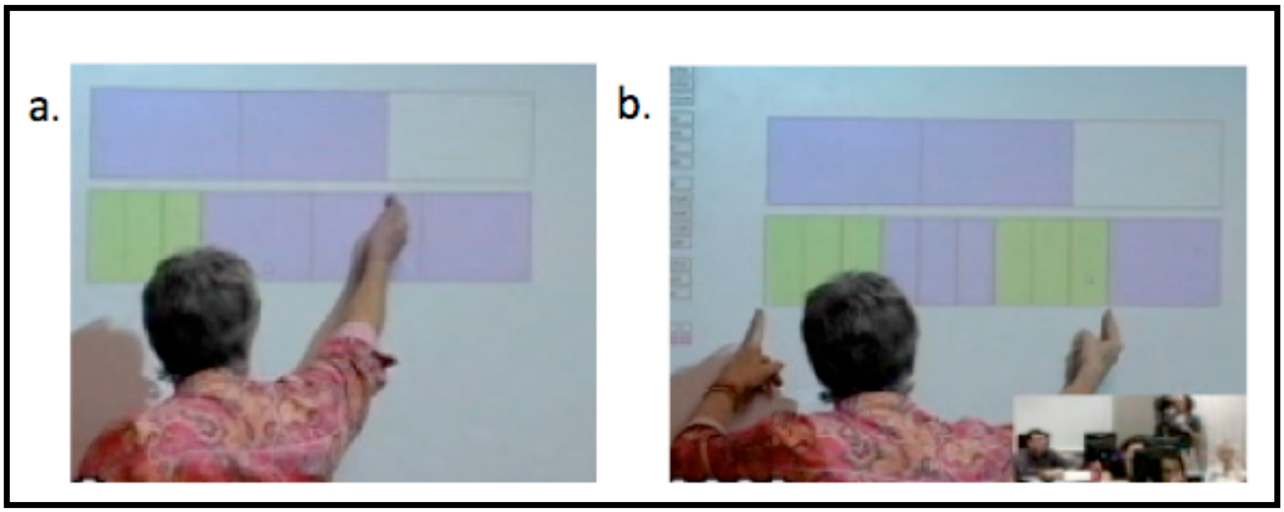


Figure 52. a. Donna points at where 2/3 of the top bar meets on the bottom bar. b. Donna brackets the referent unit of the quotient 8/9.

Using the Fraction Bar software, she made one bar and copied another bar below as in Figure 52a. Then she split the first bar into three parts and the second one into four parts. She colored the rightmost piece in the top bar white so that she could clearly see two thirds as in Figure 52a, then she said:

> How many times three fourths would fit into two thirds, so I have got two models to compare (she points at two bars in Figure 52a). There [3-part bar] is two thirds, and this one [4-part bar] can show three fourths, one, two, three, fourths, so what I am looking for is a way to name this point right here [where two thirds in 3-part bar meets in 4-part, see Figure 52a]. So I realized that then I had to split these pieces (pointed at each fourth) into smaller pieces in order to give it a name. Is that what I mean (Claire answered yes)?

Then she partitioned other two parts of 4-part bar in Figure 52a into three parts. She partitioned each of the three fourths into three parts but not as part of her common partitioning operation because she was not re-presenting three parts in the last fourth. In other words, she was anticipating three levels of units but was unsure how to do so to solve the problem. She partitioned in such a way because it produced the nine parts that she saw from Claire's model.

Protocol 19: Donna continues …

RA (Rachael): Everybody following that? Show again which thing you are trying to locate.
DO: This right here [Figure 52a.]
RA: Are you trying to locate it on the bottom bar?
DO: The two-thirds [of the 3-part bar] ends on this place [Figure 52b.] in the three fourths. (She partitions each of the three fourths in the 4-part bar into three pieces.) So that would work. So what is the name of this place? It is eight-ninths.

Even though she partitioned each of the three fourths in the 4-part bar into three pieces, she was not aware that the smallest piece was one-twelfth and three-fourths was nine-twelfths and two-thirds was eight-twelfths. She was only attending to the part-whole relationship with three-fourths being the whole, and knew that two thirds was eight out of nine pieces, namely, eight-ninths in three-fourths. Rachael continued to ask Donna why she partitioned each of the three-fourths into three parts, and Donna said she knew it because it gave her nine pieces, which is another indication to support that she was not re-presenting the 12/12-part bar. Donna split the last fourth into three parts when Mike and Claire told her that partitioning the last fourth into three parts would give her the common partition and the commensurate fractions of three fourths and two thirds, Donna did not seem to fully get it. As a matter of fact, Donna could not use either common partitioning or cross partitioning operations to model $2/3 \div 1/7$ using a measurement approach to division in any of three assessment interviews. She drew a line as shown in Figure 52a.

Summary

I considered teachers' measurement fraction division knowledge across four different types of division situations: 1) when the divisor quantity partitions the dividend quantity evenly (Sequence 1[48]); 2) when the divisor quantity does not partition the dividend quantity evenly (Sequence 2); 3) when denominators of the divisor and the dividend are relatively prime (Sequence 3); 4) when the divisor quantity is bigger than the dividend quantity (Sequence 4). As the mathematical relationship between the divisor and the dividend got more complex (i.e., the sequence increased), teachers needed to reorganize their initial conception of measurement division. In addition, teachers' knowledge was different when the dividend was a whole number in Sequence 1. They employed new partitioning operations and units, more complex units structures, and their initial interpretations of quotitive model of division were modified (See Table 4 under Chapter III for the summary of operations and units associated in each sequence).

First, how teachers interpret numerical expressions of division framed their choice of division models (partitive or quotitive). In general, teachers came up with word problems that were based upon partitive interpretations of division when the divisor and dividend in numerical expressions were whole numbers. On the other hand, they asked themselves questions that were based on quotitive interpretation of division when fractions were involved in the numerical expressions. In one of the whole-group discussions, they agreed among themselves that they asked different questions in the former and the latter situations. In general, teachers used the interpretation of either "how many times one can subtract the divisor from the dividend (repeated subtraction)" or "how many groups of the divisor quantity fit into dividend," and those teachers could generate a drawn representation that modeled measurement division using their unit-

[48] From now on, I will use Sequence 1 for the sequence when the divisor quantity clearly measures out the dividend quantity, Sequence 2 for the sequence when the divisor quantity does not clearly measure out the dividend quantity evenly, and so forth.

segmenting operations. Even though the teachers accepted both as valid interpretations of quotitive division, there was a difference not only in the teachers' capacities to reason in the sequence of division situations where the mathematical relationship between the divisor and the dividend become more complex, but also there was a difference in those teachers' capacities to accept the word problems that depicted measurement division. Even though the word problem depicted the measurement model of division, one teacher did not accept it as showing the measurement model because the divisor quantity was bigger than the dividend quantity in the post-assessment. It was an interesting observation because she was one of the teachers who used a measurement model of division from the pre-assessment and had been involved in an extensive whole-class discussion of how $2/3 \div 3/4$ (Sequence 3 & 4) could be modeled with drawn representations using the measurement interpretation of division. Despite the fact that it took her a long time to model $2/3 \div 3/4$ using a repeated subtraction interpretation and she needed a help from other teachers, she was able to construct the model with valid reasoning.

Teachers' operations were different in terms of the levels of units afforded to them. While teachers could reason with only two levels of units if their goal was to find the answers to the division problems using unit-segmenting operations in all four sequences, it was not as sophisticated as the reasoning of teachers who used three levels of units structures. I will briefly explain in the following key operations and conceptual units that the teachers who used three levels of units structures used to activate their unit-segmenting operations. In Sequence 1, the teachers had a tacit notion of a measurement unit. In other words, they were using the divisor to count the number of it that fit in the dividend but had little awareness that they were comparing the size of the dividend in terms of the divisor. I would say the teachers had an explicit conception of a measurement unit when they measured the size of the dividend or the size of the leftover quantity in terms of the divisor and established a part-whole relationship between the two quantities. While recursive partitioning operations activated their unit-segmenting operations in Sequence 1 when the dividend was a whole number, and teachers' common or cross partitioning operations activated their unit-segmenting operations in the Sequence 2 when the dividend was a fraction. Moreover, teachers' capacities to establish a part-whole relationship between the leftover quantity and the divisor quantity by explicitly attending to the divisor quantity as the referent unit of the leftover quantity helped them to eliminate their perturbation when facing a novel situation.

In Sequence 3, when the denominators of the dividend and the divisor quantities were relatively prime, teachers could not simply use the divisor as the measurement unit to measure the quantity of the dividend. In addition, in order to continue to use unit-segmenting operations, teachers used common partitioning or cross partitioning operations to generate two three levels of units structures. Whether they used cross partitioning or common partitioning operations, they needed to use a co-measurement unit. Teachers could find commensurate fractions for the divisor and the dividend quantities using the co-measurement units. Sequence 3 overlaps with Sequence 2 & 4 with fraction dividends.

In Sequence 4, when the divisor quantity was greater than the dividend quantity, teachers needed to find an alternative way to measure the dividend with the divisor. It was not easy for them to use a unit-segmenting operation because they could not iterate the divisor or count the number of the divisors in its dividend to find the quotient. Thus, they began to compare the size of the two quantities using the divisor as the referent unit. While none of the teachers realized it, the situation they faced in sequence 4 was not novel in that they already faced a similar situation in Sequence 2 when they had to determine the size of the leftover quantity. While teachers might have not realized it, they actually used the same method and determined the answer by establishing a part-whole relationship between the divisor and the dividend quantities.

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary

I investigated middle grades (grade 5-7) teachers' capacities to reason with fractional quantities in the context of partitive and quotitive division situations during a professional development course. The course was designed to prepare teachers for new curriculum standards in their state, which were similar to the standards developed by the National Council of Teachers of Mathematics (2000). Specifically, the course focused exclusively on rational number concepts and emphasized the use and interpretation of drawn representations. The course was designed and offered through a National Science Foundation-funded project called *Does it Work? Building Methods for Understanding Effects of Professional Development*. The aim of the Does it Work project was to understand teachers' learning of rational numbers during the professional development program and evaluate whether increases in the teachers' knowledge lead to increases in their students' learning. The present study explored two research questions:

- What operations and conceptual units do teachers use in reasoning about partitive division problems?
- What operations and conceptual units do teachers use in reasoning about quotitive division problems, and how do they modify or reorganize their quotitive division conceptions as the relationship between the dividend and divisor changes?

The purpose of the present qualitative study was to investigate middle grades (Grade 5-7) mathematics teachers' knowledge of partitive and quotitive fraction divisions. Improving teachers' knowledge of mathematics is crucial for improving the quality of instruction (Ball, Lubienski, & Mewborn, 2001; Ma, 1999; Mewborn, 2004; Sherin, 1996), and efforts to improve the quality of classroom instruction have led to increased attention to promoting the development of teachers' mathematical knowledge for teaching. Nevertheless, existing research has documented extensively that preservice and inservice teachers lack adequate preparation in the mathematics they teach (e.g., Ball, 1990, 1993). Many teachers view mathematics teaching as the careful presentation of formal, symbolic rules or procedures and often have limited *conceptualized understanding* (von Glasersfeld, 1995) in mathematics.

Especially, research on teachers' understanding of fraction division (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999) has demonstrated that one or more pieces of an ideal *knowledge package* (Ma, 1999) for fraction division is missing. Although previous studies (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999; Tirosh & Graeber, 1989) have stressed errors and constraints on teachers' knowledge of fraction division, few studies have been conducted to explore teachers' knowledge of fraction division at a *fine-grained* level (Izsák, 2008). Thus, I concentrated on teachers' operations and flexibilities with conceptual units in partitive and quotitive fraction division situations. Specifically, I attempted to develop a learning trajectory of teachers' ways of knowing fraction division by observing their performance through a sequence of division problems in which the mathematical relationship between the dividend and the divisor became increasingly complex. This is my first step toward building a learning trajectory of teachers' ways of thinking, which can be extremely useful for thinking about how to build an effective professional development program and a teacher education program.

Because little research has been done to conduct fine-grained analysis of teacher knowledge, I adapted ideas that appeared from the Fractions Project, which has studied children, to study teachers' knowledge of fraction division. In contrast to most research on teacher knowledge, this allowed me to study teacher knowledge at a fine-grained size and, thus, to consider learning trajectories for teachers' knowledge. The theoretical frame that I developed for this study emerged through analyses of teachers' participation in the professional development program where they were encouraged to reason with/attend to quantitative units using various drawings such as length and area models. As part of the larger research project, I observed all InterMath professional development sessions, which met 14 times for 42 hours in a large, urban, Southern school district. The data collected for this qualitative study included videotaped lessons, reflections, and lesson graphs of the 5 relevant instructional meetings, and pre-, post-, and post-post assessment interviews and interview graphs for eight teachers. In the rest of this chapter, I will present conclusions from this study, implications of my findings, and suggestions for future research.

Conclusions

Some teachers could reorganize their fraction division knowledge by refining their partitioning operations and units through the sequence of problem situations in which the mathematical relationship between the dividend and the divisor became increasingly complex. The teachers' ways of solving two similar partitive division problems were different in terms of four knowledge components: partitioning operations, conceptual units, interpretations of problems, and use of distributive reasoning. In measurement division problems, the fundamental operation that the teachers used was the unit-segmenting operation. Similar to the partitive division case, common or cross partitioning operations were the operations that provided teachers with two three levels of units coordinated together.

While unit-segmenting operations were key operations when the divisor was smaller than the dividend, establishing part-whole reasoning between the dividend and the divisor was the key operation when the divisor was bigger than the dividend. Despite the fact that few teachers were aware of it, the teachers had previously established a part-whole relationship between the two quantities when they had to quantifying the leftover quantity by the measurement unit. However, the teachers established this quantitative relationship between the two quantities by using their unit-segmenting operations.

Moreover, the teachers did not want to think about the situations that they could not easily conceptualize with their existing conceptions of divisions. In general, those teachers preferred to use a quotitive division model when fractions were involved as divisors and a partitive division model when whole numbers were involved as divisors. Their preference for one division model over another model reflected their ability to devise valid word problems for both division models. Regardless of the fact that the question explicitly asked the teachers to come up with both models, only 3 out of 14 teachers came up with reasonable word problems and others did not offer a partitive word problem when a divisor was a fraction.

In addition, when the teachers devised word problems for numerical expressions involving division, some teachers revealed important operations to solve the division problems. To illustrate, some teachers devised a word problem that might have evoked their students'[49] recursive partitioning operations. Furthermore, the teachers did not clearly indicate referent units for the quantities they included in word problems–not necessarily because they did not really know what the referent units for the quantities were but because they were not accustomed to devising word problems or they did not identify referent units systematically.

[49] In the reflection, we asked teachers to devise the word problems as if they were asking their students.

In addition, the teachers' coordination of two three levels of units structures activated more sophisticated[50] partitioning operations and supported teachers making sense of more complex fraction division situations. In partitive division, those teachers could keep track of the referent unit of the quotient by using partitioning operations for a common multiple or distributive partitioning operations. Particularly, in partitive division, those teachers who reasoned with distributive reasoning were those who could attend to the two three levels of units structures. Reasoning with three-level structures was necessary if the teachers' goal was to *measure* how much one person's portion was. Similarly, in measurement fraction division, while the teachers could use two levels of units across each measurement fraction division sequence to determine the quotient using drawings of quantities they chose, those teachers could not use more sophisticated[51] partitioning operations (i.e., recursive partitioning operations and common partitioning operations) to activate unit-segmenting operations. In addition, when the teachers established part-whole reasoning between the two quantities, their ability to use the measurement unit as the referent unit was critical, and it was impossible without the teachers' coordination of two three levels of units structure.

Above all, the present study indicated that the knowledge components found in the previous research literature about children's fractional knowledge appeared in the participating teachers' mathematical activities with fraction problems and further turned out to be essential for their mathematical thinking in the context of division problems. Although there are some compatibilities between children's and teachers' ways of knowing so that applying the results from research with children could be a viable way to start, I began to realize that the

[50] Sophisticated in the sense that they are the partitioning operations, which produce multi-level unit structures. Recursive partitioning operations produce one three-level unit structure and common partitioning operations produce two three-level unit structures.

development of teachers' knowledge differs from that observed in children because the teachers are already well equipped with procedural knowledge and they are likely to have more sophisticated number sequences already developed.

To elaborate, some participating teachers' common partitioning operations were evoked by their strategy of finding the common denominator between the two fractions. While the teachers brought forth common partitioning operations by themselves, not only in partitive division but also in quotitive division, they believed they used an algorithm. They were referring to the algorithm in finding a common denominator for two fractions, which was a procedural strategy that they usually used in fraction addition or subtraction problems. Even though the common denominator algorithm was associated with common partitioning operations for the teachers, some of them thought that they used the algorithm. It is plausible that some of the teachers could have already been equipped with procedural knowledge that was associated with (mental) operations because they have revisited the content over and over[52]. I think this may cause serious problems when the teachers go back to their classrooms because they may teach common partitioning operations as an algorithm to their students. This conclusion also has an implication for designing effective professional development program in which teachers could explicitly become aware of the associations they make between the operations and the procedural algorithms. Having such awareness will constitute the "Conceptualized Understanding" (von Glasersfeld, 1995) of mathematics that teachers need to have as a part of mathematical knowledge for teaching.

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[52] I do not think it matters to teachers whether the content was based on the traditional curriculum instead of the reformed one to associate their procedural knowledge with operations.

Implications

Although past research has stressed errors in and constraints on teachers' knowledge of partitive and quotitive fraction division, relatively few studies have been devoted to investigations of teachers' knowledge in terms of their mathematical operations and especially little has been done in examining teachers' reasoning about fractional quantities in terms of conceptual units. Teachers' abilities to conceptualize quantities in word problems or in drawn representations into units turned out to be one of the important knowledge components. Thus, an implication of this study is that teacher education activities should place deliberate emphasis on quantitative reasoning, interpreting and writing word problems, and creating and interpreting drawn representations. To elaborate, measuring teachers' knowledge across various contexts (InterMath course and assessment interviews) and forms (devising/interpreting word problems and drawings of quantities, assessments, and reflections) was useful not only because I was able to examine the teachers' partitioning operations in detail but also because it provided teachers with opportunities to develop their fractional knowledge that they were not likely to be aware of by establishing quantitative relationship between the quantities.

Because teachers were good at finding numerical answers to fraction arithmetic operations it initially appeared that they all had similar levels of mathematical knowledge. However, the data analysis revealed that their available mathematical operations were very different one from another, which provides clues about how to differentiate teachers' *mathematics knowledge for teaching* (MKT). While some teachers reasoned with three levels of units throughout partitive and quotitive division sequences, others could go through each sequence reasoning with only two levels of units by assimilating the reasoning of the teachers who reasoned with three levels of units. The latter group of teachers could use more sophisticate partitioning operations to develop partitive division knowledge and to activate the unit-segmenting operations in quotitive division situations.

The computer software, Fraction Bars, opened the possibility for the participating teachers to create and enact various operations (i.e., partitioning, disembedding, iterating, etc.) on geometric figures such as rectangle and squares. Through the professional development session I was able to observe that the appropriate use of the technology benefitted the participating teachers because it allowed them to easily represent their mathematical thinking into perceptual material so that they could not only develop their own mathematical knowledge but also communicate their mathematical reasoning to other teachers. Educators should develop more technologies like the Fraction Bars software in different content areas to engage learners to develop their knowledge.

The InterMath course also provided teachers with tasks that were challenging enough to bring forth fruitful discussions. The participating teachers had an opportunity to look inward and become aware of their own mathematical knowledge and use that knowledge to reorganize what they teach. This is critical because it can allow teachers to become the authority for what gets taught in their classrooms rather than pointing to a textbook as the primary source of the mathematical knowledge that they teach. Viewing school mathematics as not being a priori can be exciting as well as challenging. Sharing in the responsibility of choosing the mathematics that they teach can lead teachers to once again become mathematically active-an activity that can be facilitated in educational environments where they seriously re-construct the mathematical knowledge they thought they understood. This will not only strengthen teachers' knowledge that they would use when solving problems that arise in teaching practice but will prepare teachers to become inquiring, reflective thinkers who do not place authority in textbooks or more politically significant personnel.

Future Research

Future research should explore teachers' knowledge within and beyond the measurement division sequences that I have identified in this study. In terms of within the sequences I identified, even though we did not get to discuss during the InterMath course the connections between sequence 4 and sequence 2, teachers who could attend to the measurement unit using common and cross partitioning operations in sequence 2 seemed to deal with the sequence 4 problem with little perturbation as opposed to those teachers who used unit-segmenting operations with two levels of units in sequence 2. Because we only had one problem that fell in sequence 4, it is hard to establish this conjecture in general. Thus, this conjecture is worthy of further study. Moreover, future studies need to reveal operations that are available for teachers to engage in unit-segmenting operations. By uncovering how teachers' fraction division knowledge is limited by their available operations, teacher educators will be better able to foster growth.

In terms of studies that go beyond the sequences I identified, future studies need to expand on teachers' knowledge of fraction division by including whole numbers, decimals. The studies should particularly look into teachers' operations for solving partitive division problems and investigate their operations for solving linear equations in algebra. My conjecture is that teachers who could use common partitioning operations to solve the candy bar problem may use such operations to make sense of problems such as $5x = 2$. While many of our teachers paid too much attention to the semantics of the candy bar and the licorice problems, which caused them to have a hard time conceptualizing the quantities as units, the problem teachers could pose themselves to solve the candy bar problem may be "five of what will give me two?" and this is actually the same problem situation for $5x = 2$.

While the present study explored teachers' partitive fraction division knowledge in two problems, studies should construct teachers' knowledge of fraction division across various sequences of tasks. Based on the results of the present study, I would like to suggest a possible sequence for partitive division. The first level is when the dividend is a multiple of the divisor (e.g., Share six gummy bears equally among three people). The second level is when the divisor is larger than the dividend and they are relatively prime (e.g., Share four candy bars equally among three people). The third level is when the dividend is a factor of the divisor (e.g., Share two candy bars equally among four people). And the fourth level is when the dividend is smaller than the divisor and they are relatively prime (e.g., Share two candy bars equally among five people). The levels of the sequence may not necessarily be hierarchical because of the relationship[53] between the second and the fourth levels, but it seems meaningful to help teachers to conceptualize partitive division situations in which the divisor is bigger than the dividend.

As to my inferences on the effect of iterability of unit fractions in both partitive and quotitive parts that I made in Chapter IV, I had no strong evidence to support my inferences because there was no task in the InterMath involving improper fractions. In the near future, I want to conduct teaching experiments with a pair of teachers or a pair of students to understand the role of iterability of unit fractions in developing various operations with fractions. Due to the complexity of the problem situations, I will choose participants who can make units-coordinations prior to activity (i.e., constructed Generalized Number Sequence) and investigate

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[53] For instance, say a student wants to share four candy bars equally among three persons, the student may first distribute one candy bar to each person, and may try to share the leftover [one candy bar] among three persons. This is exactly the same situation as the fourth level in that the dividend (one candy bar) is smaller than the divisor (three persons) and they are relatively prime.

the mathematical activities in which they could become involved. Particularly, Steffe and Olive (2010) conjectured that students with GNS could see the *commutative* solution between the partitive and the quotitive interpretations of division. When a person can see the commutative solution, the person can answer partitive division problems like “six of what will give me 24” by finding how many sixes are in 24.

In a future study I want to model the development of teachers’ knowledge of students’ mathematical knowledge as well as the teachers’ MKT in the classroom setting by coordinating the students’ and the teachers’ knowledge. A similar study has already been conducted in the area of fraction multiplication (Izsák, 2008), but it has not yet been done in the area of fraction division. While studying teachers’ mathematical knowledge during a professional development course is valuable, teachers might not use the knowledge that they used in the course to teach their students in the classroom because of other issues. In the InterMath course, as part of the Does it Work data collection process, I observed several teachers teaching their students fractions, and I was surprised by what I saw. One of the teachers who performed well in the InterMath class and interviews in terms of making sense of the various division situations, understanding other teachers’ solution methods, and elaborating his thinking did not utilize what he experienced in InterMath as much as other teachers whose performance in class and interviews was weaker because he did not believe his students could reason with drawings of quantities. On the other hand, another teacher tried to utilize what she had learned through the InterMath course in her teaching, but her weak content knowledge prevented her from teaching effectively. Hence, future studies need to explore the relationship between teachers’ knowledge when they are acting as teachers in school as opposed to the situations when they are acting as students in a professional development program.

Final Comments

Given the significant issues in the United States with the preparation and professional development of mathematics teachers, understanding teachers' knowledge has been considered imperative for designing efficient learning programs for teachers. Although past studies have documented that teachers lack sufficient fraction knowledge, my analysis suggests that teachers can reorganize the knowledge of fraction division that they have, when provided with a professional learning experience in which they can reason in terms of drawn quantities. Because little research was done in the field of teacher knowledge at a small grain size, I adapted the ideas that emerged from the Fractions Project (which studied children's fractional knowledge) to the study of teachers' knowledge of fractions division with an expectation of suggesting a learning trajectory for teachers' fractional knowledge. I investigated the fractional thinking of in-service middle school mathematics teachers to construct a *mathematics of teachers* that is analogous to the *mathematics of children*[54]. One might think that the mathematics of teachers and the mathematics of children are poles apart, but I have shown in this study that is not the case at all. Operations that emerged when teachers dealt with fraction division problems (e.g., various partitioning operations and unit-segmenting operations) were the operations that were found in children's mathematical operations in the previous research literature. However, the mathematics of middle school teachers is not simply recapitulated mathematics of children, which might be misjudged as somehow inferior to the mathematical knowledge of secondary school mathematics teachers. Rather, a model of the fractional knowledge of middle school teachers will *supersede* a model of children's fractional knowledge (Personal communication with Dr. Leslie P. Steffe, February, 2010). What Steffe means by "supersede" is that the later

[54] Mathematics of children' means researchers' models of children's mathematics, that is, their schemes of children's mathematical knowledge (cf. Steffe & Olive, 2010). Therefore, mathematics of teacher's can be regarded as a researcher's models of teachers' mathematics.

model solves all of the problems solved by the earlier model but solves them better and solves problems that the later model does not solve. The idea is somewhat similar to including the natural numbers in the rational numbers where the natural numbers take on new meaning within the new structure in which they are included. My work is best understood as a research program whose goal is to construct a model of the fractional knowledge of elementary and middle school teachers that supersedes the models of children's fractional knowledge. This goal is a crucial goal not only because of the well-known problem that the teaching of fractions represents, but also because it represents the construction of ways of thinking that can be extremely useful in the further education of mathematics teachers.

Children's early experience with multiplication and division, both in and out of school, is grounded in behavior largely limited to simple situations involving discrete objects and mathematically restricted within the positive integer domain. When it becomes necessary to broaden the domain beyond the positive integers and to deal with new classes of situations, problems are caused because fractions were taught in a way that has little relationship to students' prior knowledge. Teachers need to know that reconceptualization of division beyond the positive integer domain is difficult because fractions were taught in a way that makes little sense to students. By providing teachers with opportunities to revisit division problems without using conventional algorithms, teachers may begin to be aware of various operations and conceptual units that serve important roles in (fraction) division. That is not to say that students will reason the same way as do their teachers, but by being aware of their own knowledge the teachers could use it as a tool to understand students' knowledge by comparing with theirs.

While I adapted ideas that emerged from the Fractions Project, which has studied children, in order to study teachers' knowledge of fraction division with the aim of establishing learning trajectories for teachers' knowledge, the trajectories of the teachers' knowledge could be used as an input to understand children's knowledge. After all, if we are serious about regarding students as mathematicians we should value their ways and means of doing mathematics to foster the growth of their knowledge. When teachers are aware of operations that were used in their solving processes of division problems, they may use it as a tool to understand their students' mathematical operations.

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