CONSTRUCTION OF AN OPERATIONAL VIEW OF FRACTIONS:

A CASE STUDY OF A MIDDLE SCHOOL TEACHER

by

HYUNG SOOK LEE

(Under the Direction of Leslie P. Steffe)

ABSTRACT

The purpose of this study was to investigate a middle school teacher's multiplicative fractional reasoning. One middle school teacher participated in a fifteen-week-long teaching experiment, one session per week, in two ways. The first way was observing videotaped excerpts of a 7th-Grade student solving fractional tasks, and the second way was engaging in fractional problems of the same general types that the student solved. Each teaching episode lasted fifty minutes and was videotaped. By observing the student's ways of thinking and by engaging in fractional problems, the teacher developed insight into the fractional operations of dividing and repeating. An ability to produce fractional amounts of quantities using these operations enabled the teacher to relate two quantities multiplicatively and to establish three levels of units. However, the fractional operations were not sufficient to develop a fractional multiplicative scheme. Differentiating between the operations and the anticipated results of operating was critical in establishing a fractional multiplicative relationship between two quantities. Based on these results, an operational view of fractions was defined as a reflective level of fractional reasoning that permits differentiating the anticipated results of fractional operations from the operations.

INDEX WORDS: An Operational View of Fractions, Equi-Partitioning, Fractions as Operations, Fractions as Anticipated Results of Operating, Fractional Multiplicative Reasoning, Fractional Reasoning, Fractional Schemes, Multiplicative Reasoning, Radical Constructivism, Reciprocal Reasoning, Teaching Experiment, Teacher Learning Through Student Mathematics, Two Levels of Units, Three Levels of Units, Units-Coordinating Scheme, Splitting Operation

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DEDICATION

To my husband, parents and my two daughters with deepest love

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First of all, I would like to express my deep gratitude to my participants, Ashley, Mia and Ken (pseudonyms). Thanks to their voluntary dedicated help, I was able to start implementing my ideas. I appreciate all their favors and efforts in this study.

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destined to flourish. I am so happy that I fulfilled your belief. I remember your excitement when I first unfolded my study abroad plan to you. You had no doubt about my plan and blessed my future while saying that you knew that some day my passion about learning would drive me to seek new opportunities. Reflecting on my life with you, you always encouraged me to step forward and see farther, securing my dreams. You have never pushed me, waiting until I was ready to act. Your prayers were always with me. I appreciate your lifelong support.

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fundamentals. It was so painful that I sometimes wished to run away from such a shameful situation. Do you remember what you said when I first confessed my struggle? I confessed my despair about not understanding the ways students think about fractions. You said that understanding students’ ways of thinking is always challenging to everybody, even to you. At that time, I realized that my anguish came from vanity, so I first needed to admit how much I do not know. That was the first illumination in my long journey. I then noticed you never blamed one for lack of knowledge or inability to do something; rather, you seemed curious about others’ thought processes and eager to develop your understanding of others to help them advance their knowledge. Over a long period of time, I realized that is the attitude I wish to possess. Even though I was enthusiastic about the philosophy of constructing one’s own experiential world, it took me a long time to recognize that understanding one’s experiential world is compatible with creating a whole new world. I truly appreciate you for waking me up to such a curious world and showing me that understanding such worlds is a precious responsibility.

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TABLE OF CONTENTS

Page

CHAPTER 1

INTRODUCTION

> Children may produce a solution that makes good sense to them, but fails to be recognized as a solution by the teacher who has a different problem in mind. In such cases, much is lost if the teacher does not attempt to find out why and how the children operated. (von Glasersfeld, 1993, p. 240)

Emergence of the Study

The constructivist view of learning has affected the view of teaching in the reform ideas about the role of the teacher (NCTM, 1991; NCTM, 2000). In particular, radical constructivism has been a major force in the current reform in the teaching and learning of school mathematics, by providing psychological and epistemological foundations with respect to ways of knowing (Steffe & Kieran, 1994). However, Simon (2000) presents how demanding applying a radical constructivist view of knowing to instruction is: many teachers participating in current reforms in mathematics education interpreted the goal of the reform as being students experiencing mathematics in the same way that teachers experience it. As long as teachers want to have their students experience mathematics in a certain way, they will try to learn the ways that specific mathematical concepts are effectively learned and the mathematics that results from such ways of learning. That is, teachers' own knowledge and their role in guiding students to the teacher's knowledge would be primarily emphasized.

Thompson and Thompson (1994) showed that a teacher who has a specific strong mathematical concept tended to interpret his child's explanation in light of his own mathematical understanding and misjudged what his students were thinking. The teacher's case supports

Begle's (1979) result that the more a teacher knows is not always reflected in the effectiveness of the teacher. Concerning what teachers know, there has been a great body of research to investigate how teachers' knowledge impacts students' learning and instruction (Ball, 1991; Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, & Perlwitz, 1991; Even, 1999; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson 1996; Fernandez, 1997; Lehrer & Franke, 1992; Monk, 1994; Putnam & Leinhardt, 1986; Sowder, Philipp, Armstrong, & Schappelle, 1998).

I consider teachers' knowledge from the radical constructivist view of knowledge (von Glasersfeld, 1985; Steffe, 1990), which means it is to be built by teachers based on their experience. I also view students' mathematics as something that teachers should construct at the conceptual level. Therefore, teachers should be able to use their constructs of students' mathematics in order to conceive a situation in which their students do mathematics. Such a notion of students' mathematics made me interested in seeing the relationship between teachers' knowledge and students' learning differently. Rather than investigating how teachers' knowledge impacts students' learning, I am interested in investigating how teachers' experience of students' mathematics impacts teachers' learning. The importance of students' mathematics should be emphasized in order to improve teachers' knowledge for teaching in that understanding students' mathematics is significant in building interactions with students' mathematical experiential worlds.

Problem Statement

There is no question that teachers should understand how to help their students learn mathematics with conceptual understanding. Given the complexity and the dynamic nature of knowledge, it seemed infeasible to point out a specific set of components or characteristics of knowledge that teachers need to acquire to impact students' learning. However, many researchers share a strong belief in the importance of mathematical knowledge that teachers have and use in classrooms (Fennema & Franke, 1992); "teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17). Depending on one's perspectives of mathematics and knowledge, there could be various ways to describe the mathematical knowledge required for teachers as well as the mathematics they are supposed to teach.

Using the radical constructivist perspective (von Glasersfeld, 1985; Steffe, 1990), I would like to view teachers' mathematical knowledge in terms of the conceptual operations that they use to construct their mathematical experiential worlds. This perspective encourages me to consider teachers' mathematical knowledge and how they relate it to their knowledge of students' mathematics. Student's mathematics should be one of the most critical factors that affect how teachers teach. In addition, the view of that teachers construct students' mathematics grounds my belief that knowledge of students' ways of thinking influences teachers' mathematical knowledge for teaching. I am not surprised to read that "research has provided little support for a direct relationship between teachers' knowledge of mathematics and student learning" (Fennema & Franke, 1992, p. 148) because the examined research might have considered teachers' mathematical knowledge as separate from teachers' knowledge of students. Viewing teachers' mathematical knowledge and teachers' knowledge of students as interconnected sheds doubt on the finding that "there is little evidence indicating whether or not this knowledge [knowledge about learners] is useful to teachers in making decisions about teaching" (Fennema & Franke, 1992, p. 154) because the authors assumed that "these principles [how students acquire knowledge and develop it] were applicable to all students . . . if teachers knew them [the principles], they would transfer their knowledge as they planned . . . " (ibid, p. 154).

Based on the above perspective of teachers' mathematical knowledge, teachers' attempts to apply a learning theory or implement a research result concerning students' ways of thinking in the classroom needs to be carefully considered. Rather than thoughtlessly implementing the findings of others' research, teachers need to understand or take into account the experiences of their own students. That is, the constructivist viewpoint is needed for teachers' learning as well as students' learning. However, I suspect that insufficient attention has been given to a constructivist approach to learning students' mathematics even though there is a wide consensus that teachers as well as students should be exposed to constructivist learning environments when learning mathematics. It is important to be aware that teachers' mathematical knowledge in classrooms involves their sensitivity to students' ways of thinking, and teachers should attempt to see both their own and the students' actions from the students' viewpoint.

Given teachers' mathematical knowledge related to students' mathematics, the concept of second order knowledge can be highlighted with respect to the mathematics that teachers know or learn. This concept asserts that teachers' first-order perspective on mathematics is inadequate for understanding their students' learning. Therefore, teachers' second-order perspective on the mathematics they are supposed to teach is necessary to understand what students learn as well as how they learn. In regard to the notions of first order and second order knowledge, von Glasersfeld's principles of radical constructivism (Steffe, 1995) are valuable. The first principle—knowledge is actively constructed by the cognizing subject—admits the subject's first order knowledge, which is the knowledge to comprehend and control his or her experience. The second principle—the function of cognition is adaptive and serves in the organization of the experiential world—directs the notion of second order knowledge, which is the knowledge to explain his or her observations or experience of the observed subject's activities, by applying the principle of relativity.

One way to think of teachers based on radical constructivism is to view them as learners who can actively construct their experiential world. Students' mathematics is one of the crucial factors that affect teachers' construction of a mathematical experiential world. So, the view of teachers as learners transforms interacting with students into an opportunity for teachers to learn. Two questions arise related to the perspective of teachers as learners: What do they learn through the interaction with students' mathematics? And, how do they learn what they learn?

Steffe (1990) made a comment based on his observation that "the teachers, being sensitive to their own recent mathematical experience, see links between their experience and the possible experience of their students" (p.177). According to the comment, it seems desirable that teachers learn to be sensitive to their mathematical experience. It also seems probable that one way they would learn such sensitivity is through seeing links, or being aware of interaction, between their experiences and their experiences of their students. Teachers' making an effort to see the links involves their realization of the notion of the second order perspective on the mathematics that they experience. Teachers who are aware of the interaction of their experiential world with their students' will maintain their role as a second order observer[1].

Based on the idea of teachers' sensitivity to students' ways of thinking, I was interested in teachers' construction of mathematical knowledge through interacting with students' mathematics. Even (2003) asserted that knowledge and practices of learning and knowing are

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[1] Steffe (1996) describes a second order observer with the following characteristics of a teacher: being aware of how she interacts, the consequences of interacting in a particular way as opposed to some other way, her own interactional routines, the roles she and the children play in contributing to the constitution of the pattern of interaction, and the interpretations each makes of the other's contributions to the interaction.

inseparable. Therefore, to investigate what mathematics teachers know, researchers must create a situation in which teachers can be learners. Since it is my belief that teachers construct mathematical knowledge by means of interacting with students, I provided a constructivist learning environment for a teacher to construct a student's ways of thinking as well as specific mathematical concepts.

Research Questions

The purpose of this study was to investigate how a teacher develops her mathematical knowledge as she builds a conceptual sense of a student's mathematics[2]. The mathematical contexts of the study were fractions, and multiplicative reasoning was to be probed within these contexts. The overarching research question for the study was:

1. How does a teacher develop her mathematical knowledge through experiencing the mathematics a student constructs?

The following two questions were asked to specify the above research question:

2. How does a teacher develop multiplicative reasoning by engaging in a student's and her own fractional reasoning?
3. How is fractional reasoning related to multiplicative reasoning?

Rationale

This study has two main purposes, the first is to investigate teachers' knowledge based on students' mathematics and the other is to explore multiplicative reasoning through fractional reasoning. I will first describe how my study can add to the field of teacher education and then illustrate how the study can impact development of multiplicative reasoning in school mathematics through fractions.

[2] According to Steffe's distinction (in press), "children's mathematics" means whatever constitutes children's first-order mathematical knowledge; "mathematics of children" means our second-order models of children's mathematics; and "mathematics for children" means mathematical concepts and operations that children might learn.

I started this study with a concern about teachers' knowledge of students' ways of thinking. In particular, I was wondering about the existing literature that investigates the impact of teachers' mathematical knowledge on students' learning or performance. Examining that research, I began to question whether teachers' mathematical knowledge has been defined relative to students' mathematics because I thought teachers' mathematical knowledge should encompass students' thinking in order for researchers to discuss how one affects the other. Teachers' knowledge was mostly defined separately from students' mathematics (Ball, 1991; Fernandez, 1997; Monk, 1994; Putnam & Leinhardt, 1986; Sowder et al., 1998) even though students' thinking has been considered as one of the components constituting teachers' knowledge and even though many researchers have advocated the idea that teachers need to know about how students acquire and develop knowledge (Fennema & Franke, 1992). On the other hand, some research explored teachers' knowledge grounded on students' thinking (Cobb et al., 1991; Even, 1999; Fennema et al., 1996; Lehrer & Franke, 1992) and used a research-based model of students' ways of thinking to inform teachers.

This study is distinct from other studies which address teachers' knowledge built on students' thinking in two ways. First, I did not provide a research-based model of students' thinking for my participant, a practicing teacher, to master or learn before exploring a student's mathematics because it is my belief that students' mathematics should be considered as an object that teachers need to construct rather than an object to which they apply a research-based model. In other words, since I would like to see how a teacher develops her mathematical knowledge based on her construction of a student's mathematics, it was my intention not to provide an established model of students' thinking beforehand. Rather, I let the teacher adapt and develop her mathematical knowledge as she understood a student's way of thinking. Second, I designed the study to find the value of teachers' experiencing students' mathematics. Instead of seeking the impact of teachers' mathematical knowledge on students' mathematics, I wanted to examine its opposite direction, the effect of a student's way of thinking on a teacher's mathematical knowledge. Because the latter would support the former, we need to have a picture of the latter case.

Next, related to the mathematical topic of this study, multiplicative reasoning through fractional reasoning, I believe fractions provide a rich context to improve multiplicative reasoning. However, many researchers seem to suppose that fractional reasoning requires multiplicative reasoning or to emphasize a relation between two quantities as the nature of fractions without mentioning a multiplicative relationship between them. For example, Smith (2002) differentiated fractions from ratios in terms of the multiplicative nature of ratios He considered ratios as involving a multiplicative nature and fractions as involving a divided quantity.

Vergnaud (1988) pointed out that multiplicative reasoning has various dimensions and fractions should be one of them. However, I don't think he means fractional reasoning is just a part of multiplicative reasoning, just as the sub-constructs of rational numbers are not just separate pieces of rational numbers. Several investigators reported five sub-constructs of rational numbers—part-whole, measure, quotient, ratio, and operator (Behr, Harel, Post, & Lesh, 1993; Kieran, 1988), but understanding each of the sub-constructs does not guarantee conceptual understanding of rational numbers. Along this line, I view multiplicative reasoning through fractional reasoning as a way to develop multiplicative reasoning.

There is a body of research that investigated multiplicative schemes within fractional contexts (Olive, 1999; Thompson & Saldanha, 2003; Tzur, 1999). Olive (1999) hypothesized the rational numbers of arithmetic (RNA) as a scheme that specifies multiplying schemes involved in fractional reasoning. A view of fractions as measures was regarded as necessary for the RNA. Thompson and Saldanha (2003) placed “fraction reasoning squarely within multiplicative reasoning as a core set of conceptual operations” (p. 100) and asserted that “for someone to understand multiplication multiplicatively, he or she must also understand fractions as entailing a proportion” (p. 104). Based on that assertion, they based fractional schemes on students’ conceptualizing reciprocal relationships of relative size. Focusing on an iterative aspect of a unit fraction, Tzur (1999) considered a multiplicative relation in terms of an invariant relation between the size of a unit fraction and the reference whole. He also commented that “the iterative fractional scheme involves a rudimentary form of multiplicative reasoning about fractions” (p. 393).

The idea of constructing multiplicative reasoning through fractional reasoning attempts to understand how people operate fractionally and how the operating is involved in constructing an advanced way of operating, multiplicative reasoning. My fundamental stance is consistent with my perspective on additive reasoning: additive reasoning is what we develop as we construct whole numbers and whole number structures rather than what we have to develop prior to constructing whole numbers. In that sense, this study is distinctive because many studies investigated multiplicative reasoning as a way to ground fractional reasoning or vice versa. However, the present study is related to the work of the above authors in that it also considers multiplying schemes of whole numbers (Olive’s study), a multiplicative relationship between two quantities which involves a reciprocal relationship (Thompson and Saldanha’s study) and an iterative fractional scheme for producing improper fractions (Tzur’s study).

CHAPTER 2

THEORETICAL AND CONCEPTUAL FRAMWORK

Theoretical Framework

From the constructivist perspective, knowledge refers to a conceptual activity that is actively built up by the knower on the basis of her or his experience. In order to understand knowledge along the above sense, two questions can be asked: “What constitutes a conceptual activity?” and “Why does knowledge need to be based on individuals’ experiences?”

Knowledge as Schemes

Related to the question “What constitutes a conceptual activity?” constructivists have introduced the concept of scheme. Piaget (1970) identified a scheme as an activity that can be repeatable and generalized through application to new situations. Olive and Steffe (2002) argue that some components other than an activity should be involved in order to explain a scheme. These components include a recognition template to elaborate how individuals recognize a new situation to activate records of operations used in past situations. The components also include a result of and a generated goal for the activity in order to explain how individuals relate their activity to the records of operations. Given the above question concerning conceptual activity, a scheme’s activity constitutes a conceptual activity. Further, an individual’s knowledge is considered as the schemes that are functioning reliably and effectively.

Regarding the question “Why does knowledge need to be based on individuals’ experiences?” we need to ask what reality means for individuals. Constructivists argue that for individuals, reality is not a mirror of an ontological world; rather, reality is constructed through actions and reflections on actions (Steffe & Kieran, 1994). This means that the construction of reality is based on individuals' experience and what is constructed is referred to as experiential reality. Since knowledge is considered as schemes, the relationship between knowledge and experience is analogous to the one between a scheme and experience. In order to produce a scheme, individuals not only need to assimilate a situation they encounter to what they have already experienced, but they also need to create an activity through which the new situation can be solved in a productive and creative way. The activity is not just of an assimilatory nature because it is produced by a continuous coordination of its results and goals being generated. As such an activity is interiorized through a cognitive regulation system, it affects a conceptual structure that has already been constructed, thereby being accommodated to another new situation. The accommodated activity provides an assimilatory situation for the operations for lateral or vertical learning. Such activities constitute an individuals' experience and illustrate how the construction of knowledge is based on individuals' experiences.

The teachers' knowledge that teachers bring forth in their teaching can also be considered from the perspective of knowledge as schemes. On the consideration of teachers' experience through working with their students, Shulman (1986) mentioned case knowledge, which is not the report of events or a sequence of events, but the knowledge that makes the events cases[3]. He rationalized the need for case knowledge in teacher education by noting the effectiveness of the case method in legal education. Teaching practice can provide a rich context for teachers to describe what their students are doing while engaging in and engendering a mathematical situation. Once teachers are prepared to see students' activities as cases through careful observation and interaction with student's activities, they can identify an event as a case on the

[3] In the sentence, I interpret events by whatever situation that students produce and cases by situations that engender schemes.

basis of their interpretation of the event. The interpretation relies on how the teachers conceive of the event that their students produce, and it also relies on how they have interacted with their students, that is, their experiential reality through working with students. In the case students' ways of thinking and operations are considered as a case, teachers would use them to build up their knowledge of students' mathematics. At this time, referring to Shulman's (1986) question about how the knowledge of cases becomes case knowledge, teachers need to consider what they should do in order to construct case knowledge for their teaching based on the cases they have encountered while teaching. Since knowledge as a scheme requires a cognizing subject to make her action interiorized, that is, more than internalized, teachers should be able to abstract as well as reflect on the cases students produce in order to elevate them into case knowledge.

Knowledge Based on Human Needs

From the radical constructivist perspective, knowledge is "actively built up by the cognizing subject and the function of cognition is adaptive and serves in the organization of the experiential world" (von Glasersfeld, 1989, p. 162). Two questions arise from these assumptions: "Why are human beings willing to construct knowledge? and "What encourages them to construct knowledge?" According to Dewey's instrumentalism (Eldridge, 1998), the human need to transform a problematic situation into one that is more satisfying engages human beings in activities, whose aim is to restructure the needs or reformulate the problematic situation. Regarding human needs, he articulated that the needs do not come from any standard outside individuals' experiential world: rather, human needs are generated within individuals' experiential reality. In addition, he argues that human needs should be understood based on the idea of a continuum between means and ends. The needs will be solved as human beings are interactively juggling ends and means, in which each one alternatively becomes the other one, rather than separately considering them, in which means and ends have distinct roles. His argument on human needs is based on the perspective of truth, where he denies purely objective truth and pursues a truth based on individual human needs; human needs rely on subjectivity so they are not absolute. Therefore, various forms of human activities become instruments for human beings to solve their problems, and knowledge is used as a tool for useful prediction without reference to its possible truth.

From the view of knowledge based on Dewey’s instrumentalism, the following questions related to teachers’ knowledge are raised: What kind of knowledge do the teachers need in order to understand their students’ mathematics? In what sense is the knowledge useful for teachers? Teachers have a chance to recognize problematic situations primarily as they are trying to understand what their students are doing. In addition, teachers’ interpretation of their students’ need for their learning affects their decisions for interacting with the students (Sztajn, 2003). If teachers’ mathematical experiential world is based on interactions with their students, students’ activities and operations provide problematic situations for which teachers realize they need some knowledge for locating mathematical contexts that might be viable in their students’ experiential world.

For more elaboration of teachers’ needs and the ways to satisfy them, let’s focus on the following three situations: teachers might wonder why some mathematics problems they presented while teaching did not work for students, what a student’s specific response meant, and what the next question could be following the student’s response. Teachers’ needs arising from such situations can be figured out in terms of how they recognize the problematic situations: a mathematics problem does not work for their students; a student’s response does not make sense to them; they feel difficulty in responding appropriately to a student’s question or response.

Along with the above three problematic situations, the following needs can be considered correspondingly: the need to know how they should transform the mathematics problem in order to make it work for their students; the need to know what elicited a student's response that did not make sense to them; the need to know what the student can do based on his or her response.

Let's think about the three kinds of needs one by one. I will first discuss the need to transform the mathematics problem that does not work for students. That a problem being posed does not work for students means that the problem is beyond what the students can deal with using the mathematics available to them. Therefore, teachers first of all should try to figure out what mathematics is available to the students while attentively listening to what they are saying and intensively looking at what they are doing and how they are responding to specific mathematics. Based on such knowledge, teachers will be able to transform the problem in order to make it work for their students or restructure their needs by posing other problems appropriate to investigate what mathematics is available to the students. The second need, knowing where a student's unexpected response came from, goes along with figuring out the mathematics available to the student. The third need, inferring a student's way of thinking, makes teachers realize that they should have a conceptual sense of the student's mathematics. Therefore, the second and third needs are closely relevant to the first need in that consideration of the mathematics available to their students becomes the basis of their needs.

Teachers' needs are raised by how they conceive of the situations that they encounter. The needs contribute to encouraging teachers to seek a useful solution: teachers' needs are closely related to their desire to understand their students' mathematics and become a crucial motivation to build useful knowledge in the sense that they can satisfy their desire using such knowledge. Therefore, teachers' insights into the ways to transform problematic situations into ones that are satisfying become an essential component to construct the knowledge that they need in order to have their students engage in doing mathematics. Along with the context, when thinking about the usefulness of knowledge, we do not refer to a more or less absolute end where teachers' needs are satisfied. Rather, the teachers' needs are satisfied only momentarily because a satisfaction of a need can lead to new needs being opened. Knowledge becomes active as a means for teachers to do something while referring an end being developed in the context of action. In addition, teachers' knowledge based on their needs is useful in that such knowledge encourages teachers to engage in promising teaching[4].

Knowledge Based on Viability

From a constructivist view of interaction, "a construction always involves interaction of some type" (Steffe, 1996, p. 85), an individual is viewed as a socially interactive being as well as a self-organizing and self-producing being. It can be also argued that interaction is indispensable to knowledge construction in two ways. The first is based on Piaget's concern that a cognizing subject's construction is the consequence of the interactions between the subject and object (Piaget, 1970). Although he presupposes that objects are distinct from the subject when talking about interaction, the subject interacts with objects constructed in prior interactions. Piaget articulated that construction involves more than physical experience and empirical information: it also involves logical-mathematical experience and the subject's coordinations of her actions. The second way to consider interaction as being indispensable in constructing knowledge is on the basis of the intersubjective aspect of knowledge construction. In radical constructivism, the notion of viability replaces the traditional conception of truth (von Glasersfeld, 1995a). Since knowledge does not reflect an ontological reality but an organization of the world based on our

[4] Promising teaching means constructivist teaching: understanding students' mathematical thinking, interpreting what schemes and operations are available to the students, and anticipating what the students can do (Steffe & D'Ambrosio, 1995).

experience (von Glasersfeld, 1984), the knowledge we construct is only viable in our experiential world. However, that does not mean that the viability of knowledge excludes others' experiential realities because viability includes recognizing our experiential realities as reaching an intersubjective level. In addition, realization of others' experiential reality through viability, which is the crucial feature of knowledge, leads to the notion of second-order of viability[5]. The realization is very important in that "it opens the way to formulating models of the mathematical or scientific realities of others" (Steffe, 2000, p.281).

From the above two perspectives on the necessity of interaction in knowledge construction, two corresponding types of interaction are drawn: within-subject interaction and individual-environment interaction (Steffe, 1996). Within-subject interaction refers to the interaction among available constructs in the course of re-presentation or other operations that involve previously constructed constructs. Individual-environment interaction means interaction in the subject's environment including her social and cultural environment as well as physical environment. However, according to Piaget's argument (Steffe, 1996) that the objects [observer's objects] in one's environment are those that should be assimilated by the subject and that tools for assimilation are endogenic, within-subject interaction may engender individual-environment interaction, which in turn modifies the following interaction of constructs previously constructed.

Since practicing teachers cannot be engaged in teaching without any kind of interaction with their students, interaction becomes a primary issue for them. Before going further, I would like to clarify that by interacting with students, I mean interacting with students' mathematics. From the perspective of teachers' interaction with students, I raise the following questions: How

[5] A second-order of viability means that the viability of a subject's knowledge reaches others' experiential world beyond that of the subject, and second-order viability plays an important role in stabilizing and solidifying the subject's experiential world (Glasersfeld, 1995b)

should they interact to improve their students' learning? What should they do while interacting with their students? and What do they come to know about students' mathematical reality through interaction? According to the above discussion about two kinds of interaction, practicing teachers interact not only with their students through the operations and activities that the students bring forth or get engaged in, but they also engage in within-subject interactions, which are active during the course of the between-subject interactions. Teachers also use the constructs they have previously constructed about students in order to understand their current interactions with students. Therefore, students' first order experience and teachers' second order experience are central grounds for interaction from teachers' perspective. Interaction between a teacher and students means more than the relationships among what the teacher and students bring forth. Interaction primarily requires the teacher to have a model of students' mathematical concepts because without such a model the teacher is likely to pay attention only to her own mathematical reality, which is independent of the students' mathematical reality. Interaction also requires the teacher to reflect on her mathematical experience of the concept, regardless of whether the experience is related to students or not, because such reflection allows the interaction to be viable in the teacher's experiential world. Therefore, from teachers' perspectives, an essential part of interaction is for teachers to be aware of students' operations and activities through a model of students' mathematics they have been building.

An Operational View of Fractions

By implementing dividing into 7 parts and repeating one of the parts 4 times I mean considering 4/7 as an operation. Such an interiorized action for a fraction is essential to understanding an operational view of fractions, but an operational view of fractions I attempt to define is more than that. "Operational" in "an operational view of fractions" is comparable to a meaning of operations describing operational theory: “Operations as grouping themselves of necessity into complex systems . . . these . . . are mobile and reversible . . .” (Piaget, 1950, p. 18). Taking the quote into account, I define an operational view of fractions as an integrated but specified concept within fractional contexts along the following two ideas: reflection (Piaget, 1950) and image (Thompson, 1996). In order to clarify the concept of an operational view of fraction, I discuss each idea while relating it with a fractional context.

First, I consider Piaget’s (1950) three conditions for transition to reflective level from sensori-motor level: an increase in speed, an awareness, and an increase in distance.

> Firstly, an increase in speed allowing the knowledge of the successive phases of an action to be moulded into one simultaneous whole. Next, an awareness, not simple of the desired results of action, but its actual mechanisms, thus enabling the search for the solution to be combined with a consciousness of its nature. Finally, an increase in distances, enabling actions affecting real entities to be extended by symbolic actions affecting symbolic representations and thus going beyond the limits of near space and time. (p. 134)

Fractional reasoning is addressed in terms of schemes, which involve operations associated with an activity that is part of an enacted scheme. When children attempt to solve a fraction problem conceptually, many, at least one, fractional schemes and operations are involved. Considering a fraction in terms of dividing and repeating is not enough to solve a quite large number of fraction problems successfully because the interiorized action, dividing and repeating, needs to be well coordinated and integrated with other schemes and operations. That is, considering a fraction as an operation should be developed toward a reflective level so it can serve to recognize a situation for an advanced fractional operation.

An operational view of a fraction is mentioned when a fraction is considered on a reflective level in a way that the operations carried out are differentiated from anticipated results of the operations. In other words, an operational view of fractions enables one to differentiate operations for producing a fraction from a result produced by operating. Such a view of fractions needs to be independent of the contexts because for a concept to reach a reflective level implies it is abstracted and generalizable to a new context. However, since there are some situations where the differentiation is not critical or barely noticed, people might think an operational view of fractions is dependent on the context. Suppose that a student was asked to make two-thirds of a candy bar. Responding to the question, he might divide the bar into three parts and take two parts for producing two-thirds of the bar. The student would not need to differentiate the operation implemented from the result—two parts—of his operating, because dividing into three parts would give him a one-third part. Let's consider another context in which differentiating operation from a result of the operation becomes critical. Suppose a student was asked to make a bar so that a given bar is two-thirds of the original bar, and that she divided the bar into two parts and repeated one part three times. To conceive of this situation using fractional reasoning, she would be required to differentiate her operating, dividing into two parts, from a result of the operation, one-third of the original bar. In other words, her articulation of that three-halves of the given bar is the desired bar implies her development of an operational view of fractions in that she differentiated her operating, one-half, from a result of the operation, one-third.

An operational view of fractions is grounded in the idea that fractions essentially consist of mental operations and their organization and/or reorganization. The essence of an operational view of fractions is an ability to differentiate operating from an anticipated result of operating. Let's closely investigate a situation that the differentiation between fractions as operations and results is critical in advancing fractional reasoning. Suppose that a child produced a stick so that a given unmarked stick is three-fourths of the stick by implementing dividing and repeating, but he was not able to produce a fraction four-thirds in reference to the given stick. The fact that he produced the desired stick indicates a reversible fractional scheme was available to him and it was anticipatory. However, the fact that he had difficulty producing one-third with respect to dividing the unmarked bar into three parts implies he had yet to develop an operational view of fractions. I consider two possible paths involving an operational view of fractions in order to explain the inability to produce four-thirds.

First, the child would have difficulty differentiating dividing into three parts from making three parts with respect to four parts. Considering fractions as operations requires the child to produce one-third for dividing into three parts. On the other hand, constructing fractions based on anticipated results of operating requires him to create one-fourth for the dividing activity because his dividing was to make three parts belonging to four parts. So, the differentiation between the fractions based on his operating and the anticipated results of operating implies an ability to create an iterable unit fraction independent of a given whole. It also means his ability to advance his partitive fractional scheme to an iterative fractional scheme by constructing an iterable unit fraction in a generative sense. Therefore, an operational view of fractions requires being at a reflective level in that the child has to be aware of his operating and simultaneously abstract the operating from its result.

Second, the child might be engaging in splitting when dividing. However, it is certain that his splitting remained within a whole, which means he was restricted by a partitive fractional scheme. So, even though he would be able to construct three-thirds by one part produced by the splitting, he would not able to go beyond three parts because thinking of one more part—one-third—would constrain him by the other whole four-fourths. Therefore, to conceive of the original bar in terms of four-thirds, he has to take four parts as a unit containing three-thirds and one more third, which is a unit of units of units. At this point, he becomes able to differentiate his operation, four-thirds, from the result of operating, four-fourths. This is compatible with Piaget's reflective level because the child is aware of operating and correlates operating with its result.

As another idea to relate an operational view of fractions, I consider the concept of image Thompson (1996) conceptualized. Asserting "By 'image' I mean much more than a mental picture" (p. 267), he related Piaget's idea of image to mental operations. Focusing on the dynamics of mental operations, he distinguished three types of images: images associated with the creation of objects, primitive forms of thought experiments, and thought experiments with reasoning by way of quantitative relationships. The first type of image, associated with the creation of objects, is "internalized act of imitation . . . the motor response required to bring action to bear on an object . . . a *schema* of action" (Piaget, 1967, p. 294). I associate this type of image with considering a fraction as a perceptual result in terms of counting and matching. For example, a child who is in the INS[6] stage of the counting scheme can learn to act in such a way to fragment [fragmentation does not involve equal sized parts as a necessity] a piece of candy into four parts to share it, take three parts and give one part to each person. In the case, a perceptual result suppresses an action performed because when each child has one part, the action may well be forgotten. So, even though the child divided into parts, took some of them and repeated working with one part as many times as needed, she or he would not notice the activities performed involves a fraction. That is, the child created an image of an adult's activity by re-presenting the parts as a plurality, but the parts would have no relation to the actions.

[6] With an initial number sequence (INS), children can count on when more objects are given for counting because they have internalized counting acts (Olive, 2001).

The second type of image, primitive forms of thought experiments, is described as follows: “in place of merely representing the object itself . . . this image expresses a phase or an outcome of the action performed on the object . . . [but] the image cannot keep pace with the actions because . . . such actions are not coordinated one with the other” (Piaget, 1967, pp. 295-296). Given this notion of image, a fraction is considered as a sequence of actions as well as a result of the sequence. So, a child who has this type of image of a fraction would have an image of what he did and what he produced previously for the fraction, such as a sharing process or a situation comparing two quantities. The child would consider a fraction in terms of a result of operating as well as an operation while using an of-statement emphasizing an activity rather than a result. However, considering a fraction as an operation would be isolated from considering a fraction as a result.

For instance, imagine a child who has constructed only a partitive fractional scheme. The child can engage in equi-partitioning, create a part that can be used to comprise a fractional whole by repeating it, and produce a fraction in terms of the number of parts being considered. Suppose that given a problem “Make one-third of a 2-inch bar,” the child divided each inch into three parts and repeated one part two times and noticed it is two-thirds of one inch by looking at a 1-inch part divided into three parts with his teacher’s guide. However, the child would not know how one-third of the 2-inch bar is related to two-thirds as operations. First of all, we need to think about his partitioning. That he partitioned each inch into three parts to produce one-third of a 2-inch bar shows he was engaged in recursive operations. However, his partitioning might not involve an operational view of fractions in that the partitioning activity would not reflect a view of one-third in the given problem as an operation. That is, partitioning each inch into three parts would be to produce six parts and repeating one part two times to generate two parts would be based on the relationship of two to six and one to three. In addition, noticing two parts as two-thirds would be relying on his awareness of three parts being one inch so that one part should be one-third of one inch. So, it seems certain that his notion of two-thirds is based on one-third as an operation, but it seems unclear that two parts he produced is considered one-third as an operation. Such unclearness would be affected by his lack of differentiating a fraction as an operation from a result of operating. Also, it would lead him to fail to understand one-third times two equals two-thirds in terms of his constructs because he would have thought a result of the product should be one-third. So, I hypothesize that an operational view of fractions would be necessary to implement recursive partitioning for distributive reasoning.

The third type of image, thought experiments with reasoning by way of quantitative relationships, implies that "An image conjured up at a particular moment is shaped by the mental operations one performs, and operations applied within the image are tested for consistency with the scheme of which the operation is part" (Thompson, 1996, pp. 268-269). This type of image is compatible with my definition of an operational view of fraction in that it considers an operation being coordinated with other possible schemes of which the operation is part. Suppose that a child had this type of image of a fraction in the above problem context where he divided each inch into three parts and repeated one part two times. He would explicitly realize that his dividing each inch into three parts is to produce one-third of the given bar. That implies recursive operation associated with distributive reasoning along his operational view of fractions. In addition, by distinguishing one-third as an operation from an anticipated result of operating, two parts, she would be able to consider one-third as an operation for the result, two one-thirds of one inch. In that an operational view of fractions involves various levels of units and is related with distributive reasoning through recursive operations, an operation view of fractions is critical in developing multiplicative reasoning.

Multiplicative Reasoning in Fractional Contexts

In this section, I review four studies involving multiplicative reasoning related to fractions. First, Steffe (2002a) grounded his research on the reorganization hypothesis, which states that fractional schemes can be constructed as accommodations of numerical counting schemes. The equi-partitioning scheme is presented as evidence of the hypothesis under Steffe's concern that we should "explain how children might come to use their number sequences, not to produce a composite unit of counted items, but to use this composite unit as a template for partitioning a continuous unit item" (p. 270). That is, he considered partitioning as an operation to bridge whole number knowledge with fractional knowledge. He also introduced the concept of connected numbers as a mental image of whole numbers for continuous items and found three ways to construct the connected numbers: partitioning, segmenting, and iterating and uniting. Constructing a connected number implies constructing a composite unit that consists of the parts produced by partitioning, segmenting or iterating and uniting.

Steffe argued that constructing a composite unit fraction, such as one-third of 15 items, involves multiplying schemes because in order to construct a composite unit fraction (5/15 as 1/5), children are required to establish a multiplicative relationship between the quantities involved in the construction and develop an insight that they produced commensurate fractions. For example, recognizing a 5-stick as a third of a 15-stick not only requires an awareness of a relation between five-to-fifteen and one-to-three, but it also promotes establishing one-third and five-fifteenths as commensurate. In addition, he commented on conceptualizing the composite unit fraction one-third such that "the children would need to maintain an awareness of the structure of the unit containing three units of five and operate on the units of this given structure as well as on the units of one that the three units of five contained" (Steffe, 2002a, p. 278-279). This comment implies that constructing a composite unit fraction requires an ability to coordinate the units involved: a unit of three and a unit of five. Therefore, multiplicative reasoning emerges in the construction of a composite unit fraction through a units-coordinating scheme.

In addition, we need to consider how a units-coordinating scheme works for a situation that requires partitioning a connected number. For instance, to produce one-seventh of a 5-inch bar, the given bar needs to be first considered as a connected number produced by iterating a one-inch bar five times. Children then need to conceive of the situation of making one-seventh of the 5-inch bar while considering how to make seven equal parts out of the given bar and reconstitute the given bar by repeating one of the parts seven times. Such a conception of one-seventh would provoke envisioning another composite unit of seven, and the unit of seven would be used as a result of partitioning the unit of five. Furthermore, partitioning the 5-inch bar to produce seven equal parts should imply coordination between the unit of five and the unit of seven—partitioning each inch into seven parts and selecting one part from each inch to produce 1/7 of 5 inches as 5/7 of 1 inch, thereby constructing the composite unit fraction one-seventh of the 5-inch bar.

Steffe (1992) regarded a units-coordinating scheme as necessary to establish a situation as multiplicative. With such a notion of multiplication, construction of a composite unit fraction involving units-coordinating addresses how closely fractional reasoning is related to multiplicative reasoning. This perspective shapes my study. That is, my study is grounded on the concern of linking fractional reasoning with multiplicative reasoning through constructing composite unit fractions based on a units-coordinating scheme. The concept of interiorized composite units (Steffe, 1994) can be developed toward iterable units along with an operational view of fractions. Such a consideration of an iterable composite unit in fractional contexts supports my basic idea of developing multiplicative reasoning through fractional reasoning.

Second, Olive (1999) developed a model of children's rational number concepts and operations, called the rational numbers of arithmetic (RNA), by observing two elementary children in a 3-year project. From the perspective of whole-number knowledge as a foundation of fraction knowledge, he regarded RNA as "a scheme in which fractions have become abstracted operations" (p. 281) and specified the multiplicative operations that generate the RNA.

He found four distinctive modifications of fractional schemes for constructing RNA: a common partitioning fractional scheme, a fractional composition scheme, a measurement fractional scheme and a whole number dividing scheme as the inversion of a whole number multiplying scheme. A common partitioning fractional scheme is a scheme that enables children to find a common partition for two fractions and further to construct a unit fractional amount using another unit fractional amount (e.g., make 1/5 of a unit bar using 1/7 of a unit bar). A fractional composition scheme is a scheme that explains the construction of a fraction of any other fraction (e.g., make 1/3 of 2/5 of a unit bar); reversible partitioning operations and distributive reasoning are essential for the construction of the scheme. In order to make 1/3 of 2/5 of a unit bar, two parts representing 2/5 need to be equally partitioned into three parts; that situation requires using distributive reasoning. Through distributive reasoning, small parts are produced by dividing each part into three small parts, and two small parts are considered as 1/3 of 2/5 of a unit bar. Then, to deduce the magnitude of the two small parts—1/3 of 2/5 of a unit bar, reversible partitioning is used because constructing a unit bar requires an ability to reverse the operations implemented for 2/5, dividing the unit bar into five parts and taking two parts or repeating one part two times. Through reversible partitioning, three of one small part is deduced as one part, and five of three small parts are considered as constituting a unit bar. A measurement fractional scheme explains the construction of a fraction from any other fraction, thereby making division of fractions meaningful (e.g., make 2/3 of a unit bar using 3/5 of a unit bar). As the last component of RNA, Olive mentioned reorganizing whole number division into multiplication by its reciprocal. The reorganization promotes recursive division by reinteriorizing the generalized number sequence inwards. For example, suppose a student was asked how many 1/432 of a given bar is in 1/4 of the bar given that the student has already made the 1/432-bar using the 1/4-bar by making 1/4 of the 1/4-bar and 1/3 of the resultant amounts followed. The student would reason it out as follows using the reorganization of whole number division into multiplication by its reciprocal: 432 divided by 3 equals 144, 144 divided by 3 equals 48, 48 divided by 3 equals 16, and 16 divided by 4 equals 4. Using the divisions, he would answer the question as follows: three 1/432 makes 1/144, three 1/144 makes 1/48, three 1/48 makes 1/16, finally four 1/16 makes 1/4; so, I need 108 of 1/432 to make 1/4.

My study has common ground with Olive's study (1999) in the following aspects: 1) he was interested in a scheme in which fractions are considered as operations; 2) he grounded multiplication in fractional contexts on a units-coordination scheme (e.g., make a share for one person if you share a 3-foot long sub sandwich with your four friends), which is whole number reasoning about multiplicative structures, and on particular types of units[7] such as iterable units, composite units and iterable composite units; and 3) he considered the concept of fractions constituting a mathematical field through multiplicative structures. Based on these shared

[7] Steffe (1992) claimed that iterable units, composite units and iterable composite units are necessary for multiplicative reasoning.

interests, my basic inquiry is consistent with his view of fractions: “the construction of fractions as iterable unit items and as operations” (p. 311). My study investigates an operational view of fractions and a construction of a fraction based on the view, and then relates such construction of fractions based on an operational view of fractions to multiplicative reasoning, which is about iterable items. However, concerning the approach to multiplicative reasoning in fractional contexts, there is a basic difference between my study and Olive’s. He focused on reorganization or modification of whole-number multiplicative operations or an iterative fractional scheme, whereas I am concerned with conceiving of multiplicative structures through the emergence of fractional reasoning based on equi-partitioning.

As the third study involving multiplicative reasoning related to fractions, Tzur (1999) mentioned the multiplicative relation between a unit fraction and a reference whole while distinguishing an iterative fractional scheme from a partitive fractional scheme. He defined a term *the partitive fractional scheme* as “characterized by the child’s ability to iterate and operate numerically on unit and nonunit fractions as long as those fractions do not *exceed* the partitioned whole” (p. 393) and described how the iterative fractional scheme is constructed as follows: “In constructing the iterative fractional scheme, the child abstracts the invariant, multiplicative relation between the sizes of the unit fraction and the reference whole” (p. 410). He considered iteration as a basic activity to reorganize whole number knowledge for fractions: it is “a means to both reconstruct the unit and divide it equally” (p. 392). According to Tzur’s definition, iteration is very similar to Steffe’s concept of equi-partitioning on which this study is grounded in that iteration concerns both producing equal parts and reconstituting a whole of the parts. In particular, iteration was considered as an operation to generate a composite unit structure for fractional reasoning and was further developed toward an iterative fractional scheme. That is, by attributing an interiorized concept of an iterable unit to a unit or non-unit fraction, Tzur characterized a generative unit fraction independent of its reference whole.

As Tzur observed, iterating a unit fraction is not enough to produce an improper fraction. He distinguished iterating for producing a numerosity of parts constituting a whole from conceiving of a size of a part relative to the reference whole. For example, he argued an ability to construct 1/7 of a whole by iterating a part seven times and producing a whole through the iterating activity does not imply establishing a relationship between one part and the whole or abstracting the anticipatory structure of the whole. That is, students in his study were not able to think about a way to produce a fraction 7/6. Conceiving of a size of a part relative to the reference whole highlights an invariant relation between the sizes of a part and a reference whole. Through such an idea of an invariant relation, he explained an activity-result relationship and further elaborated a multiplicative insight necessary for constructing an improper fraction as grounded on the relationship. So, although his articulation of the invariant relationship for an iterative fractional scheme was explained as a quantitative or numerical relationship of a unit fraction to the reference whole, I infer he meant a multiplicative aspect of an iterative fractional unit. However, by viewing the invariant relationships as conceptual structures, independent of how they were produced, he seemed to focus more on a proportional approach to a multiplicative structure in fractions than on an operational approach. He argued:

> The word *one sixth* and the numeral 1/6 symbolize the size of a unit fraction that maintains a 1-to-6 relation to the whole regardless of how this unit was produced (e.g., dividing a whole into six parts) or which operations were performed on it (e.g., iterating 1/6 seven times or adding 5/6 + 4/6). (p. 410)

That is, he developed an insight into the invariant relationship between a unit fraction and a reference whole and separated it from the "iteration" he mentioned as a basic activity for creating fractions. In addition, he argued that a concept of fractions becomes powerful when a fraction is conceived of as a single quantity involving an abstracted invariant relationship, and its size can be expressed as a numerical relation with respect to the whole.

Compared to Tzur's study, my study proposes another perspective of iteration-based conceptions of fractions. Tzur considered a concept of an iterable unit fraction as a construction of a relation based on a proportion between the sizes of a unit fraction and a reference whole. On the other hand, I view the concept as an interiorized operation relying on an operational view of fractions and the invariant relation promoting a multiplicative insight into the iteration as an anticipated result of the operation.

Finally, Thompson and Saldanha (2003) investigated how people reason when they understand fractions and suggested multiplicative reasoning in the investigation. While placing "fractional reasoning squarely within multiplicative reasoning," they articulated "Coherent fractional reasoning develops by interrelating several conceptual schemes often not associated with fractions" (p. 100). Four schemes were characterized for that assertion: division schemes, multiplication schemes, measurement schemes, and fraction schemes. Each scheme was examined around the main idea of reciprocal relationships of relative size. In particular, for fraction schemes, the idea of reciprocal relationship was emphasized as a comparison rather than an inclusion relationship such as "out of," and the researchers argued "students gain considerable mathematical power by coming to understand fractions through a scheme of operations that express themselves in reciprocal relationships of relative size" (p. 107-108).

Concerning four schemes being considered, Thompson and Saldanha elaborated the mathematical importance of reciprocal relationships of relative size in multiplicative reasoning as follows: (1) two division schemes, partitive and quotitive ways of thinking of division, are highly related from the perspective of operative imagery[8], so “any measure of a quantity induces a partition of it” (p. 107) and any partition of a quantity produces a measure of it; 2) for multiplicative comparison, a measuring unit should be imagined apart from what is measured, and measurement is an image of ratio relationship; 3) by “conceiving multiplication as about quantifying something made of identical copies of some quantity” (p. 104), envisioning a multiplicity of identical objects becomes critical in conceptualizing multiplication.

I read Thompson and Saldanha’s research (2003) as opening a probable way for developing multiplicative reasoning through operating fractionally, which is the basic idea of my study. But the basic idea is presented consistently but differently in the two studies. They proposed to understand fractions through “conceiving two quantities as being in a reciprocal relationship of relative size” (p. 107). According to the perspective, “‘Amount *A* is 1/*n* the size of amount *B*’ means that amount *B* is *n* times as large as amount *A*” (p. 107) and they emphasize that 1/n should be based on comparison rather than inclusion because a relationship indicates one of relative size in their context. I attempt to consider their notion of a reciprocal relationship from the concept of equi-partitioning which captures two reciprocal activities, dividing a whole and iterating a part. However, equi-partitioning is not sufficient for producing reciprocal relationships in that dividing a whole and iterating a part does not necessarily imply reconstituting a whole in terms of a part. So, in order to conceive of reciprocal relationships through partitioning, the activity of partitioning needs to be developed toward the splitting

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[8] The ability to envision the result of acting prior to acting and to suppress attention to the process by which one obtains those results (Thompson & Saldanha, 2003, p. 106)

operation. Furthermore, reciprocal reasoning with respect to a non-unit fraction requires a reversible fractional scheme with the splitting operation. Therefore, I conclude their notion of reciprocal relationships is not based on partitioning; rather it is based on the idea of proportions.

Multiplicative Reasoning Through Fractional Reasoning

This study proposes to investigate conceiving of fractions as a way of developing multiplicative reasoning. In regard to a relation between fractions and multiplicative structures, I mention three main perspectives. First, Vergnaud (1988) defined fractions as one of the concepts constituting the conceptual field of multiplicative structures. Specifically, he considered a concept of fractions as grounded on multiplicative structures: “Mathematical concepts are rooted in situations” (1988, p.142); “A conceptual field is a set of situations” (1996, p. 225); the main situations for the multiplicative conceptual field consist of dimensional relationships in simple and multiple proportions (1994). Second, Davis, Hunting, and Pearn (1993) did not presuppose a multiplicative structure on fractions as they considered fractions as operators. For them, fractions are “operators that are logically and psychologically anterior to ratio operators” (p. 66). Third, through the reorganization hypothesis (2002a), Steffe opened a way to investigate multiplicative structures based on fractional reasoning: multiplying schemes are developed based on the growth of number sequence structure (1994); the hypothesis is about reorganizing numerical counting scheme into fractional schemes.

I am interested in the last perspective because it is my belief that fractional reasoning provides a rich context to develop multiplicative reasoning rather than “requires multiplicative reasoning” (Vanhille & Baroody, 2002, p.224). Kieran (1994) questioned “What actions, action schemes, and mental actions or operations lead to the concepts and understanding that an observer sees as a person’s multiplicative structures” (p.389). Responding to the question and according to my concern of the relationship between fractions and multiplicative reasoning, I highlight an operational view of fractions in order to specify multiplicative structure that emerges from fractional reasoning.

A perspective of fractions as operations is differentiated from an operator meaning of fractions in that an operator meaning emphasizes first-order mathematical meaning and fractions as operations emphasizes second-order mathematical meaning. Through the reorganization hypothesis, Steffe (2002a) argues that children develop fractional schemes as they use their numerical counting schemes in a positive way and partitioning and iterating provide a recognition template for developing fractional schemes. Under the consideration of second-order meaning, by being aware of such a recognition template I mean an operational view of fractions. Therefore, in scheme theory, an operational view of fractions will be considered as an assimilatory structure in a sense that it is "used to assemble a recognition template in creating an experiential situation that may have been experienced before" (Steffe, 2002a, p.268).

I am using the term *equi-partitioning* in a particular way as "a psychological structure that includes both operations of breaking a continuous unit into equal sized parts and iterating any of the parts to reconstitute the whole" (Steffe, 2002a, p. 272). However, if I use the term in an inclusive way, it can be used for "partitioning" as explained by Pothier and Sawada (1983), Kieran (1976), and Mack (1990). In those contexts, partitioning was identified as informal knowledge that students bring to make a meaning of fractions. In particular, Mack (1990) anticipated a development of rational number based on partitioning relying on the result of her study that students were able to solve a variety of fraction problems as they utilized partitioning for symbolic representations. Proposing an operational view of fractions as a part of a fractional scheme underlying multiplicative structures can be a promising way to support her idea, constructing rational numbers based on partitioning. Of course, it will be probable when we specifically develop a meaning of partitioning such as used in Mack's context toward the abovementioned Steffe's term. I am maintaining the term "fractions" even under the consideration of multiplicative structures, instead of changing it to "rational numbers" because I think multiplicative structures are developed through an advanced fractional reasoning. So, in my terms fractions constitute a mathematical field if we expand the concept of fractions to signed number systems.

Concerning multiplicative structures, I am considering two perspectives: Confrey's splitting and Steffe's units-coordinating scheme. Kieran (1994) differentiated them as "The former [splitting] approach observes multiplicative actions as independent of addition ideas, whereas the latter [Steffe's idea of multiplicative structures[9]] appears to consider at least early multiplicative acts as making natural use of counting-based mental structures" (p. 380). However, I view the difference between them in terms of an issue of units: creating simultaneously multiples of multiples of a unit vs. producing recursively a unit of units of units. I define the number of levels of units by the number of different units contributing to a unitized item to account for the resultant items. According to the definition, a basic activity to create multiple levels of units is maintaining a prior unit of one for the next anticipatory operation and abstracting each unit of one involved in the implemented operations to an iterable unit. For example, in the case of 3 times 4 where the operation pertains to splitting each of three equal segments into four parts, the idea of splitting produces 12 items by producing 4 items from each of 3 items simultaneously. In the situation, each of the resultant 12 items is considered as equally quantified like in the unit of one used for 3 and 4. So, the unit of one used for 3 and 4 transfers to

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[9] Kieran (1994) identified Steffe's idea of multiplicative structure as multiplicative acts based on the number sequence structure, which is different from my identification of it as a units-coordinating scheme.

a unit for the resultant, whereby there is no more concern about a unit of one that was used to conceive of 3 or 4. In that sense, splitting is concerned with one level of unit. On the other hand, a units-coordination produces 12 items by distributing partitioning into four parts across each element of a unit of 3. Each of the 12 items involves a notion of the unit of 3 as well as the unit of 4. That is, a unit item for the resultant 12 is defined by an element of the unit of 4 and an element of the unit of 3. So, a units-coordinating scheme produces multiple levels of units.

Confrey (1994) defined splitting as “an action of creating simultaneously multiple versions of an original” (p. 292) and “a primitive operation that requires only recognition of the type of split and the requirement that the parts are equal” (p. 300). She further identified multiplication and division as follows: “Multiplication is established when the whole is defined in relation to the objects after the split, and division is defined when the whole is not reinitialized after the split” (p. 300). Such a definition and view of mathematical operations show that she appeared to consider only a one dimensional unit presupposing an operation necessary to conceive of multiples from a unitary item. In other words, her saying “multiple versions of an original [a whole]” and “… the whole is defined … after the split” seems to imply shifting an initial dimension of operating to another dimension rather than relating a dimension of “before split” with one of “after split.” In addition, to clarify how she figured out the numerosity of multiples or parts as differentiated from an additive situation, which she explained as “identifying a unit and then counting consecutively instances of that unit” (1994, p.292), she mentioned “one-to-many” action. Through the action, she seemed to consider creating a composite unit but did not explain how to conceive of multiples of “one-to-many.” If she intended to differentiate multiplication by splitting from addition in terms of the type of counting such as consecutive counting vs. simultaneous counting, it is necessary to consider creating a unit of units of units, which is multiples of multiplies of an original in her terms.

On the other hand, Steffe (1994) considered a units-coordinating scheme as an assimilatory structure in producing multiplying schemes. Units-coordinating means “the mental operation of distributing a composite unit across the elements of another composite unit” (Steffe, 2002a, p. 279). He argued that coordinating two composite units in this way is necessary to establish a situation as multiplicative. According to his idea, multiplicative structures are attained when a units-coordinating scheme includes an iterable composite unit. A concept of iterating should be differentiated from an act of repeating in such a way that *iterating* involves an interiorized concept of a unit whereas *repeating* expresses a large range of abstraction of a unit, from a figuratively conceived unit to an interiorized unit. So, if a student repeats a composite unit so many times under the consideration of an iterable unit, this action produces a result that can be abstracted as a multiplicative structure.

I am using a units-coordinating scheme to account for the construction of multiplicative reasoning in fractional reasoning. An operational view of fractions is considered as a part of a multiplicative fractional scheme in the sense that constructing a fraction based on operations and differentiating the operations from an anticipated result of operating involves an insight into establishing a multiplicative relationship between two quantities. Although traditional fraction instruction emphasized the activities of counting and matching for a concept of fractions as they focus on cardinality of a set of elements (Carraher, 1996), fractional schemes are developed along with a notion of partitioning and iterating. That is, depending on how students conceptualize partitioning and iterating for fractional reasoning, we consider various schemes and operations: equi-partitioning scheme, partitive fractional scheme, iterative fractional scheme, recursive operation, inverse operation, and so on (Steffe, 2002a). Of course, what fractional schemes and operations are available will affect a student's view of fractions as operations and also impact on how to conceive of a result produced through his/her operations. For the second part of a multiplicative fractional scheme, I am interested in how multiplicative reasoning emerges as an operational view of fractions develops. How to establish a multiplicative relationship between two quantities will be an advanced activity of a multiplicative fractional scheme because, when established, the multiplicative relationship will belong to the first part of the scheme and the implementation of the operations will appear in activity. There are some cases where recognizing a fraction based on an operation implies producing a result without actually engaging in an activity. However, to investigate the first part of the scheme, I focus on the activity and its result.

Teachers' Learning Involving Students' Mathematics

Most people agree that "the success of the student depends most of all on the quality of the teacher" (American Council on Education, 1999, p. 5), and teachers' knowledge is a crucial factor in mathematics instruction and learning (Fennema et al., 1996). However, a lot of controversial arguments prevail about which characteristics of the teacher determine the quality of the teacher (Begle, 1979; Darling-Hammond, 1999; Monk, 1994). The notion that teachers' mathematical background is not sufficiently equivalent to the mathematical knowledge needed for teaching led to the investigation of the knowledge that would matter in teaching: knowledge of students' mathematical thinking (Fennema & Franke, 1992; Fennema et al., 1996); teachers' responses to unexpected answers that students bring forth (Fernandez, 1997); knowledge package: a network of procedural and conceptual topics around the topic a teacher is teaching (Ma, 1999); pedagogical content knowledge (Shulman, 1986); second order knowledge of the mathematics of students (Steffe & Wiegel, 1996); and image of understanding a mathematical concept (Thompson & Thompson, 1996). Teachers not only need a strong mathematical knowledge to be effective teachers (Thompson & Thompson, 1994), but they also need experience of students' mathematics (von Glasersfeld, 1995a).

In what ways does the knowledge that teachers would use in teaching link with and become grounded in students' way of thinking? Students' mathematical knowledge is viewed as perturbations for the teachers (Steffe & D'Ambrosio, 1995). Analogous to von Glasersfeld's view (1985) that radical constructivism is "one viable model for thinking about the cognitive operations and results" (p. 100), teachers' knowledge for teaching mathematics can be considered as stemming from an appreciation of students' mathematics based on the students' experiential world. Teachers' knowledge can be developed toward creating situations of learning in which teachers might bring forth the mathematics of students. Further, a radical constructivist argument that knowledge is seen as instrumental (von Glasersfeld, 1985) supports a functional aspect of teachers' knowledge which creates the situations that can provoke the schemes available to the students and engender accommodation in the students' conceptual structures rather than imposing problems or tasks leading to disequilibrium. As teachers are engaged in creating such situations, they are expected to be able to investigate their students' possible ways of thinking as well as their current ways of thinking. Teachers' knowledge can be considered as teachers' ways of thinking to restructure mathematical situations based on students' ways of thinking. Therefore, knowledge of students' mathematical thinking is essential in order for teachers to teach for students' learning with understanding (Fennema et al., 1996).

Teachers' Knowledge Based on Students' Mathematics

There is a body of research on teachers' knowledge based on students' mathematics. The literature varies in terms of how students' mathematical thinking was used for the purpose of developing teachers' knowledge of the mathematics of students. First, Cognitively Guided Instruction (CGI) programs provided the teachers with specific research-based models of children's mathematical thinking to help them understand their students' mathematical thinking. They asserted that the knowledge needed for changing their instruction along with current reform recommendations is related to teachers' understanding of the development of their students' mathematical thinking (Fennema et al., 1996). Given the assertion, CGI studies indicated that "teachers can attend to individual students when they have appropriate and well-organized knowledge" (Fennema & Franke, 1992, p. 156) and CGI teachers' practice affected student achievement (Fennema et al., 1996). Such findings have a critical impact on the research of teachers' learning and the teaching of mathematics in terms of students' mathematics because some researchers have doubted that teachers remember and use knowledge of each child's ways of thinking to make instructional decisions. So, they concluded that knowledge of students' mathematics is not important in expert teaching (Putnam & Leinhardt, 1986). However, I am concerned that if teachers were provided specific and well-organized knowledge which they would implement in their classroom, they would still have difficulty developing their own models of students' mathematics and mathematical topics. My study differs from CGI programs in that any research-based model of students' mathematics or mathematical topics were not given to the eighth grade teacher Ashley in my study. Along with her own learning experience from a constructivist perspective, she had an opportunity to develop a conceptual sense of a student's mathematics. I argue that such an experience provided her an opportunity to construct students' mathematics from a constructive perspective. The following study reflects my concern.

Cobb, Wood, and Yackel (1990) provided an opportunity for teachers to directly investigate students' thought processes by engaging them in planned instructional activities and having them teach according to the activities. The original purpose of the study was to analyze children's mathematical learning in a constructivist learning environment. However, interestingly, such an environment for children became a learning opportunity for a teacher in the sense that she had to develop models of her students' mathematical understandings in order to discuss her interpretations of the activities with the students. The researchers developed instructional activities based on research-based models of children's mathematical experiences, so she was able to pose problematic situations, experientially-based learning opportunities, for her students. The teacher selected for the study first learned research-based models developed by Steffe as she discussed and watched video excerpts of her students doing mathematics that were conducted by the researchers. However, such learning was not interiorized in her mind, so she could not bring it forth to her classroom because learning research-based models should have been preceded by her understanding of the students. Therefore, in order to recognize the situations that the students created, she needed to develop her own models of the students' mathematics before she could use the research-based models that she learned in her classroom teaching.

This finding is consistent with a constructivist perspective of learning and a perspective of knowledge as schemes if we consider the teacher as a learner in the research setting because a repertoire of her students' mathematics needed to be constructed and developed by her instead of her using a research-based model. She finally elaborated on her learning as follows: listening to students' mathematical thinking will make her teaching rich; self-satisfaction is more important for students to engage in productive mathematical activities than teacher reinforcement or rewards; and she should be a facilitator instead of a decision maker with all the answers. Most importantly the study indicated that the major surprise for the teacher was her realization of far more sophistication of the second graders' ways of thinking.

Cobb et al.'s study has common ground with my research in that we share an interest in teachers' own understandings of students' mathematics. However, the focus of each study was different. Their study emphasized how the teacher initiated and guided the development of the social norms in order to solve some conflict between her previous belief in mathematical instruction and her mathematical sense-making of a constructivist learning environment. On the other hand, my study was concerned with a teacher's development of mathematical knowledge through interaction with a student's mathematics that relied on my belief that students' mathematics should be constructed by a teacher. So, I gave attention to the teacher's schemes along with her understanding of a student's ways of thinking.

While Cobb et al's study gave teachers the developed instructional activities, Even (1999) provided teachers with an opportunity to conduct a mini-study with their students referring to research articles about students' learning and understanding of various mathematical topics. The three-year professional development program was intended to have the teachers experience students who do not necessarily construct mathematical knowledge in the instructed way. For that purpose, the teachers first engaged in reading and discussing common inquiries into the learning of mathematics expressed in the last three decades of research. Then they were encouraged to examine research-based knowledge acquired from the first stage by conducting a mini-study with their students. So, they were expected to integrate research-based knowledge and practice. She reported what the teachers learned in each stage. Through reading and discussion, the teachers learned a general view of constructivist learning— students construct their knowledge. Through conducting a mini-study, they became able to specify and solidify the research ideas for their students. As a result, the teachers had learning experiences that challenged their preconceived conceptions of student learning and their classroom practices.

My study differs from Even's study in two ways even though its fundamental idea f providing a learning experience of students' mathematics to teachers coincides. First, while Even provided research-based ideas of some mathematical topics for teachers' development of theoretical knowledge, I conducted a teaching experiment with a teacher so that she could construct the mathematical concepts that research (Olive, 1999; Steffe, 2002a; Tzur, 1999) identified as relevant for understanding students' mathematics. In other words, Even seemed to consider research ideas of mathematics of students as separate from teachers' own knowledge. I attempted to investigate the research-based concepts through the teacher's ways of thinking because I believe the teacher needs to abstract and interiorize them for observing the student doing mathematics. So, the teachers in Even's study came to appreciate that students construct their knowledge, and the teacher in my study developed a conceptual structure in which she could reflect her observation of the student's mathematics from the constructivist perspective.

For another element of differentiation, we should consider the concept of experience-based knowledge. This issue might be subordinate to the perspective of research-based ideas. Even (2003) discussed experience-based knowledge separate from theoretical knowledge, so she considered research-based knowledge as the object to study. A mini-study was planned to provide the teachers with an opportunity to apply theoretical knowledge within their practical knowledge. On the other hand, I intended to connect research-based ideas with teachers' knowledge by engaging the teacher in constructing and developing them within the teacher's conceptual structure. Such an intention is based on the idea that those research-based mathematical constructs are also from students, which means that the teacher would have her own path to construct them like we hypothesize for children. So, the teacher's watching of a student was considered as integrating her mathematical knowledge based on the mathematics of the student, not the constructs by research, with her observing of the student.

Finally, some researchers (Simon, 1995; Simon & Shifter, 1991) provided teachers with an opportunity to experience new ideas about mathematics as they considered teachers' knowledge of the mathematics of students through the development of teachers' mathematical knowledge and constructivist pedagogy. Such an idea was reflected in my study as I probed the teacher's mathematical concepts by challenging problems that she would have not faced before. However, the studies differ from mine in that they did not involve students or students' mathematics.

I am considering that teachers should construct students' mathematics from a constructivist perspective. So, differently from the above mentioned studies, I engaged a teacher in investigating a student's way of thinking without a research-based model of students' mathematics to which she could refer. That is, this study proposes that a teacher develop a sense of students' mathematics by experiencing a student's way of thinking and engaging in the construction of mathematical concepts in the same way the student did. For this purpose, the study encouraged the teacher to develop her own understanding of the student's mathematics and participate in a constructivist learning environment. Thereby, the teacher was expected to experience the student's mathematics while being aware of what she was doing and construct mathematical concepts with a notion that her mathematical experiential world would be different from the student's.

Teachers' Knowledge Based on Conceptual Teaching

One promising way to investigate teachers' knowledge is to consider it as related to conceptual teaching based on a constructivist view of learning—an investigation of the teacher's knowledge of mathematics under the consideration of the knowledge of learning (Thompson & Thompson, 1996). The question, "What are the teachers' images[10] of understanding a mathematical idea conceptually and how are they expressed while teaching?" motivated the researchers to focus on teachers' images of students' learning and their language to capture the images. The researchers concluded that the images and language crucially influence what teachers do and what they teach; they also found that how teachers teach was influenced by how they understand what is to be learned. In addition, teachers' instructional actions relied on their own understanding of the image-based reasoning they hoped a student would build. The researchers also found that teachers' mathematical understanding critically affected teachers' pedagogical orientations and decisions such that how they pose questions, select tasks, assess students' understanding and make curricular choices. In addition, the research argued that teachers should possess schemes needed for conceptual understanding by building such schemes in school or in teacher preparation programs. Further, such schemes should encourage teachers to strive for conceptual coherence in their pedagogical actions and students' conceptions. However, a great deal of research has reported that practicing or pre-service teachers commonly had not yet developed the schemes that researchers hoped their students would build for conceptual

[10] The loose ensemble of actions, operations, and ways of thinking that come to mind unawarely (Thompson & Thompson, 1996, p.16)

understanding (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Even, 1993; Post, Harel, Behr, & Lesh, 1991; Putnam, 1992; Simon & Blume, 1994).

This study provided an eighth grade teacher with an opportunity to build mathematical schemes, so she could bring them forth in order to understand what a student is doing. Such experience is crucial because it would serve as a guide when the teacher attempts to interpret the schemes and operations available to the student. According to von Glasersfeld (Olive & Steffe, 2002), a scheme consists of an experiential situation, a specific activity associated with the situation and a result of the activity. From the perspective of knowledge as schemes, I investigated how she developed fractional schemes and operations as she observed the seventh grader's mathematical activities. In addition, she was encouraged to reflect on her activity and the student's activities while implicitly or explicitly considering the parts that constitute a scheme and the relationship among them.

Creating a mathematical situation where students can be actively engaged is a primary responsibility of the teachers (Steffe & D'Ambrosio, 1995). As teachers create such situations, they are encouraged to continuously change their understanding of the schemes available to their students. The students are also encouraged to produce a situation generated by the schemes. As a result, through the situations generated by the students, the teachers deduce what to do while interacting with students' mathematics. It is another primary responsibility of the teachers, under the notion of schemes, to have the students pay attention to their mathematical activities. As the teachers encourage their students to make their operations explicit, the students actively reflect on the results of what they are doing and think about the relationship between their operations and the results. This reflection engenders a situation that can activate their prior knowledge. Therefore, by encouraging students to make their operations explicit, teachers can interpret what their students are doing and can further anticipate what the students can do. According to the above discussion of teachers' knowledge, the knowledge that teachers expect to gain is based on students' mathematical knowledge rather than on their own mathematical knowledge.

This study has some limitations in creating a situation for the investigation of the student's mathematics because it was designed for the teacher's participation only through viewing the videotaped student. She could not directly interact with the student. However, by providing the teacher with an opportunity to construct specific mathematical concepts, which the student might have constructed or struggled with, I helped her explicitly elaborate her understanding of the student's mathematical activities and anticipate the schemes available to the student. Therefore, I finally guided the teacher to develop her mathematical knowledge based on the student's mathematics.

Related to my consideration of teachers' mathematical knowledge based on students' mathematics, one question is raised along with the notion of knowledge as schemes: Is it possible to consider the knowledge that teachers might bring while interacting with their students within the idea of schemes? I think such knowledge can be considered as schemes from the teacher's perspective; it can be described in terms of the parts of a scheme by observing what teachers construct as they interpret and anticipate students' schemes or operations while interacting with them. In such a case, a scheme's situation refers to how the teachers have their students activate schemes or operations and interpret the schemes available to them. The scheme's activity means the teacher's operations on her/his understanding of the students' mathematics, and the teacher's working hypotheses of the students' activities or operations become the scheme's result. An important concern in the above discussion is that a scheme's situation is crucial in that it activates teachers' activities and operations on their understanding of the students' mathematics as well as students' activities and operations.

Teachers' Learning

Comparing teachers' learning to teach to students' learning of mathematics, Simon (1994) developed six learning cycles[11] of teachers while noticing the recursive relationship between learning mathematics and learning to teach mathematics. The cycles not only describe the interconnection of different domains of teachers' knowledge, but they also reflect the concerns about what might be an appropriate sequence of mathematics teachers' experiences and how they might be connected. Responding to the need of researchers in mathematics teacher education, which is generating specific models of teachers' learning based on particular components of mathematics teachers' knowledge, the learning cycles provide useful conceptions of the ways that mathematics teachers learn to teach mathematics.

What motivates teachers' learning? Simon (1994) argued that teachers' own mathematical experiences are considered as a fundamental basis for the purpose of their positive learning experience; Cobb et al. (1990) reported that teachers' realization of their teaching as being problematic became a starting point to engage actively in learning to teach mathematics. The notion of how children construct mathematical knowledge gives rise to interests in teaching as well as learning mathematics by having teachers think about whether or not their teaching is viable in their students' construction. In this regard, teachers' interaction with students' mathematics becomes an important and fundamental source for teachers to realize that their teaching is problematic; that is, interaction engenders a positive learning environment for

[11] The six cycles refer to teachers' own mathematical experience, learning about the nature of mathematics, learning about how people learn mathematics, learning about students' learning of specific mathematics content, learning about planning mathematics instruction, and learning about the aspects of teaching that involve interacting with students (Simon, 1994).

teachers as well as their students in that teachers try to develop their own understanding of the mathematics of students rather than apply knowledge of students that has been informed by formal cognitive models to their teaching (Cobb et al.[12], 1990).

From the teachers' perspective, the nature of the classroom as a learning environment instead of a teaching environment might make teaching practices uncertain and unpredictable unless teachers are willing to adopt their students' mathematical thinking into their ways of thinking because prescribed ways of thinking in the teachers' minds would no longer be effective in students' emerging ways of thinking. However, once teachers recognize that their students' mathematical thinking is far more sophisticated than they have previously assumed, their initial concern about the uncertain or unpredictable nature of emerging mathematics would disappear because attentive listening to and appreciation of students' way of thinking would help teachers to be able to predict students' way of thinking.

What teachers learn while working with students might be different than what researchers expect them to learn through constructivist teaching. Compared with academic way of knowing based on research-based models, Cobb et al. (1990) described teachers' way of knowing as follows:

> Formal models are a product of a series of abstractions and formalizations made by researchers who operate in the context of academic reasoning and attempt to satisfy the current standards of their research community. In contrast, teachers operate in the context of pragmatic pedagogical problem solving in which they have to make on the spot decisions as they interact with their students in specific situations. (p. 132)

[12] Although they focus on the classroom as a learning environment, the study context pays attention to interaction occurring while teaching in order to avoid the space restriction.

Based on the above insights, Cobb et al. concluded that researchers should make an effort to "help teachers develop forms of pedagogical practice that improve the quality of their students' mathematical education" (p. 145), not "to transform the teachers into constructivists" (p. 145).

My fundamental interest is what teachers learn through their experience of students' mathematics. So, considering that teachers' learning may be very relevant to their sense of students' mathematics, this study tried to answer the question "How does a practicing teacher develop her mathematical knowledge while constructing a mathematical experiential world involving students' mathematics?" I expected the question would help me answer my two initial questions: What and how does a teacher learn through interacting with students' mathematics? Although there are some research-based models of student learning, as Cobb and his colleague noted, the research-based models might be different from the models that teachers are building while interacting with their students. I hope this study presents a teacher's a conceptual sense of students' mathematics along her mathematical knowledge development.

CHAPTER 3

METHODOLOGY

> Only a protracted effort of observation, interaction, and further observation can supply the intensity and the continuity of experience that are indispensable if substantive results are to be obtained. (Steffe, 1994, p. 34)

This study is a case study of one middle school teacher, Ashley. The purpose of the study was to investigate what a teacher learns as she engages in attempting to understand a student's mathematics and how the teacher's development of specific mathematical knowledge is related to her experience of the student's mathematics. I designed the study to provide an opportunity for the teacher to watch and interpret videotaped excerpts of a seventh grader, Mike, engaging in fractional reasoning and to construct fractional schemes as she engaged in solving fractional tasks.

Since I view knowledge from the radical constructivist perspective (von Glasersfeld, 1985; Steffe, 1990), I needed to experience the teacher's mathematical language and actions in the context of teaching episodes (Steffe, 1988, 1990). Toward this end, I used teaching experiment methodology (Steffe & Thompson, 2000), purposes of which are "for researchers to experience, firsthand, students' mathematical learning and reasoning" (p. 267) and for researchers to "construct models of students' mathematical knowledge" (Steffe, 1988, p. 137). So, my goal for the teaching experiment was to construct a model of the teacher's way of thinking so I could specify the operations she carried out.

A teaching experiment consists of a sequence of teaching episodes. It is an exploratory tool derived from Piaget's clinical interview to explore a cognizing subject's mathematics. But it is more than a clinical interview "because it involves experimentation with the ways and means of influencing students' mathematical knowledge" (Steffe & Thompson, 2000, p. 273). In other words, a teaching experiment aims to understanding the progress of one's mathematical knowledge as well as understanding her/his current knowledge.

In the teaching episodes I conducted, there are two different types of sessions: problem solving sessions and video watching sessions. In the problem solving sessions, the teacher, Ashley, was provided an opportunity to construct fractional schemes from a constructivist perspective. In the video watching sessions, the teacher was engaged in observing Mike solve fractional tasks. During the video sessions, I engaged the teacher in situations in which she could use her constructs of the student's independent ways of thinking (Steffe & Thompson, 2000) and build a conceptual structure of the student's mathematics.

The meaning of "teaching" in a teaching experiment is found in the context of a researcher's progressive attitude toward interaction with a learner, which may range from a responsive and intuitive interaction to an analytic interaction. Interacting in a responsive and intuitive way means the researcher becomes the student and attempts to think as a student does. An analytic interaction becomes possible when the researcher has insight into the student' mental operations so an itinerary of what she/he might learn and how she/he might learn it is available to the researcher. That means the researcher comes to have a goal of where to take the student and modifies the goal until the student's schemes seem to be well articulated; in this, teaching necessarily appears in a teaching experiment.

This study involved two phases. The first phase—Learning student mathematics—allowed me to experience the student's mathematics and the second phase—Working with a teacher—allowed me to experience the teacher's mathematical experience and her experience of the student's mathematics. For qualitative research methods, Ezzy (2002) raises two issues: Does research writing represent participants' experience? How do researchers use and report their own role on their research? Since the purpose of this study was to understand how a teacher builds a conceptual structure through a student's way of thinking, the study required me to consider how I could experience the teacher's mathematics and the student's mathematics. Because there would be no way to directly access the teacher's mathematics and what she constructs of the student's mathematics (von Glasersfeld, 1995b), I needed to experience the student's mathematics in a way that closely mirrors what the teacher might experience. Based on this need, I designed the first phase of the study. Conceiving of others is plausible only by interpreting their language and actions using our own subjectiveness and attributing our conceptual constructs to them. In terms of such an attribution, others will be thought of as possessing cognitive structures and ways of operating that are similar to our own (von Glasersfeld, 1985). Thus, it was necessary and crucial for me to have an experience of building a model of a student's way of thinking before I worked with the teacher, which is the second phase.

Research on qualitative methods argues that researchers' personal experience becomes an important part in doing research because it provides data, ideas for theory building, and contacts for selecting research subjects (Ezzy, 2002) In light of the concern of personal experience, the first phase—Learning student mathematics—played an effective role in conducting the second phase—Working with a teacher—in that it encouraged me to be aware that meanings and themes emerging from the second phase would rely on my personal experience of mathematics and

Mike's mathematics, and that awareness encouraged me to coordinate my understanding and the teacher's understanding of the student's ways of thinking.

While conducting the second phase of the study design, I considered grounded theory (Charmaz, 2000) because theoretical sampling (Ezzy, 2002), in which emerging analysis guides the collection of further data, would be necessary to explore the teacher's ways of thinking to understand the student's mathematics. According to grounded theory, what to collect is not defined prior to conducting the research but as the theoretical dimensions emerge during the research (Ezzy, 2002). The fundamental idea of grounded theory is consistent with the theoretical basis for the study, "we can define the meaning of *to exist* only within the realm of our experiential world and not ontologically" (von Glasersfeld, 1995b). Therefore, I recognize that my mathematical reality is distinct from hers so all that I can do is to interpret her learning and experience on the basis of my interaction with her ways of thinking. However, since interpretations are not found but are actively constructed through interaction (Ezzy, 2002), what I collected, asked, and pursued throughout the teaching experiments needed to be based on my ongoing interpretation of the research being developed, which was the most crucial feature in collecting data in the second phase.

Participants

One teacher and one student were involved in this study. The teacher selected for this study was a practicing teacher, Ashley. She was teaching pre-Algebra and Algebra I for 8^{th} and 9^{th} grade students at a public magnet school for the fine and performing arts in east Georgia, which draws students from grades six through twelve. The year she was in the study was her first year in the school and the ninth year in her teaching career. She taught mathematics every year and her favorate topic was Algebra. She mostly taught middle grade Algebra, so this was the first time she had taught ninth graders. She majored in middle grades education with a concentration on language arts. However, she wanted to teach mathematics because she was so impressed by her student teaching experience of teaching mathematics. So, she took six mathematics courses such as college Algegra, pre-Calculus, and Calculus I and II, which was more than required.

Ashley appreciated the students' Ah-ha moments and pursued understanding-oriented teaching as she was willing to listen to the students' ideas and encourged the students to explore and discuss the topics they were supposed to learn. When asked to describe a teaching episode to me, she elaborated the following:

> . . . when we're working with like in my pre-algebra classes, right now, the eighth grade math we're working with the idea of slope but we're not calling it slope. We're calling it rate of change, which is slope. So we're talking about things like races when two people are in a race, and constant change. If this person walks one meter per second and if this person walks two meters per second, who's gonna win in a ten-meter race?, that kind of thing. I'm seeing it now. So ohhhh, this is going to be linear because there's the same change every time. He walks one meter every single second and um . . . a couple of days ago, someone said, "oh this one wouldn't be linear because it started out fast and then they slowed down." So they're realizing the visual concept of linear graph verses things that aren't linear and so . . . this happened just the other day. (Interview on March 16, 2006)

She devoted four months to this study. I met her during lunch time every Thursday between classes, but she was never late and she never hurried to finish the sessions. She always did her best and showed her curiosity about Mike's way of thinking. So, before the day's session began she sometimes told me how long she had reflected on the previous session and how her experience of the study affected her current teaching. For example, she explained that she was trying to carefully think about what the students' responses meant and find out what was behind the students' reasoning.

Another participant, Mike, was one of the seven seventh graders who participated in a three-year project. I selected his videotaped work to have the practicing teacher Ashley observe because he displayed significant aspects of the schemes in which I was interested. The time I observed him was his second year in the project, and he was accustomed to using JavaBars (Biddlecomb & Olive, 2000), computer softerware that he was supposed to use while solving mathematical problems.

Mike developed a partitive fractional scheme based on construction of a part that can be repeated within a given whole. When asked to make a sandwich so that a 2-foot sub sandwich is three times as long as the original sandwich, he produced six parts by dividing each foot into three parts, took the third and fourth ones out of the six, and verbalized one-third three times moving them along the six parts. Such an activity shows that the two parts have been considered as one-third depending on the given 2-foot bar. The unit of two parts was an iterable unit but constrained by the six parts because even though he divided each foot into three parts he seemed to focus on producing six parts rather than employing distributive reasoning. So, during his engaging in the problem, he had difficulty developing an iterative fractional scheme. When asked to produce a fraction of the two parts in feet, he said two-thirds referring to one foot divided into three parts. However, the two-thirds seemed not based on an iterable unit fraction one-third of one foot because he was not able to relate it with one-third of two feet. He tended to construct a fraction based on the number of parts constituting a whole while considering a fractional whole as presupposed. So, one part was considered as one-third of one foot and separately as one-sixth of two feet. Further, he had yet to develop a units-coordinating scheme even though he divided each foot into three parts because his dividing each foot into parts seemed to aim at using the equivalent relation between two-to-six and one-to-three.

Procedure of Data Collection

This study was designed in terms of two phases which are distinct from each other but closely interrelated (Figure 3.1). Phase 1—Learning student mathematics—was to collect data from a student, which would be shown to a teacher. It consists of sixteen teaching episodes with a seventh grader, Mike, for one semester and analyzing them. Phase 2—Working with a teacher—was to conduct a teaching experiment with a teacher, Ashley. It consists of two components. Phase 2A represents the problem solving sessions where Ashley engaged in solving fractional tasks. Phase 2B represents the video watching sessions in which she observed Mike engage in fractional reasoning.

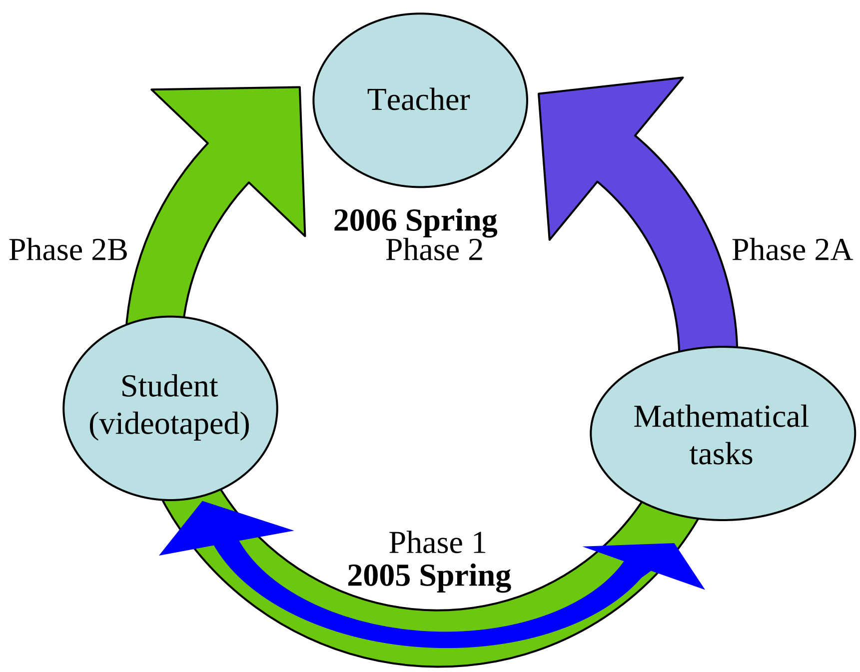

Figure 3.1, Schematic of the study design

During Phase 1, I worked in two ways. First, I observed middle school students engaged in a teaching experiment (Steffe & Thompson, 2000) between November 10, 2005 and May 2, 2005. The purpose of the experiment was to “specify a school algebra that is squarely based in the productive and creative thinking of students [of middle school age]” (Steffe, 2002b, p. 1). One professor and four doctoral students, including me, were involved in data collection. Seven seventh graders at a rural middle school in north Georgia participated in the experiment and they were taught twice a week for twenty to thirty minutes by one of the researchers. Every teaching episode was videotaped using two video cameras. My observation was more focused on one of the students, Mike, because his way of thinking displayed the schemes in which I was interested. Mike worked with three researchers during this period. In addition, he worked with another seventh grader until the middle of February, and after that, he worked alone. I taught him five times and participated in all the weekly meetings to plan the next teaching. On the occasions when I was not his teacher, I usually took detailed field notes on what he was doing and how he responded to the questions asked by his teacher.

Second, I analyzed the data produced between January 19, 2005 and May 2, 2005 by Mike. While analyzing, I listed every question posed to Mike and his corresponding responses and produced a list of the topics that emerged from each of the episodes. Since the teaching experiment was well organized according to themes, I selected most of the first nine episodes for the next phase. I expected that these videotaped excerpts would motivate the teacher to wonder about the student’s way of thinking as well as encourage her to reconsider what she already has known related to emerging mathematical topics. On the basis of the analysis, I then developed two kinds of materials. I first produced a question set for video watching sessions, where she was supposed to observe Mike’s ways of thinking. The purpose of developing a question set was to investigate how the teacher (1) becomes aware of the student's mathematics, (2) relates her current mathematical concepts to her understanding of the students' way of thinking, and (3) develops her conceptual structures of fractions based on her understanding of the student's mathematics. For each of the problems posed to Mike, I wrote down my analysis of Mike' responses and made questions to probe Ashley's interpretation of the response and insight into the response. The questions were modified as the study proceeded based on her previous knowledge of Mike's mathematics and her own mathematical knowledge.

Next, I developed a problem set for problem solving sessions, which were to engage the teacher in solving problems. The purpose of developing a problem set was (1) to help Ashley build a repertoire of fractional reasoning so that she could draw on it when elaborating or interpreting Mike's way of thinking, and (2) to help me investigate her current fractional schemes. The problems created for each problem solving session were later modified based on her performance on the previous sessions.

For Phase 2, I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) where I worked with Ashley at a public magnet school in east Georgia from February 2, 2006 to May 18, 2006. The purpose of the teaching experiment was to investigate the teacher's learning through student mathematics and the relationship between her development of mathematical concepts and her experience of student mathematics. I also conducted one thirty-minute semi-structured interview (Patton, 2002) to gain background information about the participant such as her mathematical preparation, teaching experience and her perspectives on students, mathematics, and teaching.

Once a week over fifteen weeks, I visited the school where she was teaching; I did not visit the school during spring break, but I visited her twice the next week to make up for the missed session. Thus, I collected the data from fifteen episodes, each of which lasted fifty minutes and was videotaped using two video cameras; one for capturing the interaction between Ashley and me, and the other for capturing the computer screen on which she was either solving mathematics problems or watching Mike.

There were two different types of teaching episodes, problem solving sessions and video watching sessions, and they were alternated in the following order: 2 problem solving sessions (First Block), 3 video watching sessions (Second Block), 3 problem solving sessions (Third Block), 6 video watching sessions (Fourth Block) and 1 final problem solving session. I usually began a problem solving session by posing a problem, and Ashley worked on it using JavaBars (Biddlecomb & Olive, 2000). I tried not to interrupt her while she was solving a problem. When she easily answered a problem without elaboration, I challenged her by asking fundamental questions leading to conceptual understanding. When she faced difficulty, I guided her by asking probing questions. For the video watching sessions, I prepared the videos of Mike in chronological order. She was allowed to manipulate the videotape so she could stop it whenever she wanted to go back to a part that she wanted to watch again. She always watched it from the beginning but did not have time to go through the entire recordings. She usually went through about a half of Mike's work during one video session. She usually stopped at the point when one problem ended and elaborated her understanding of Mike's way of thinking by herself or in response to my questions. Most of the time, I probed her elaboration based on her observation or understanding of the problem or on Mike's way of thinking.

During the First Block, I proposed problems that would be related to problems that would be presented to Mike during the Second Block. During the Third Block, the teacher was asked to solve the problems that were intended to elucidate changes from the First Block through the

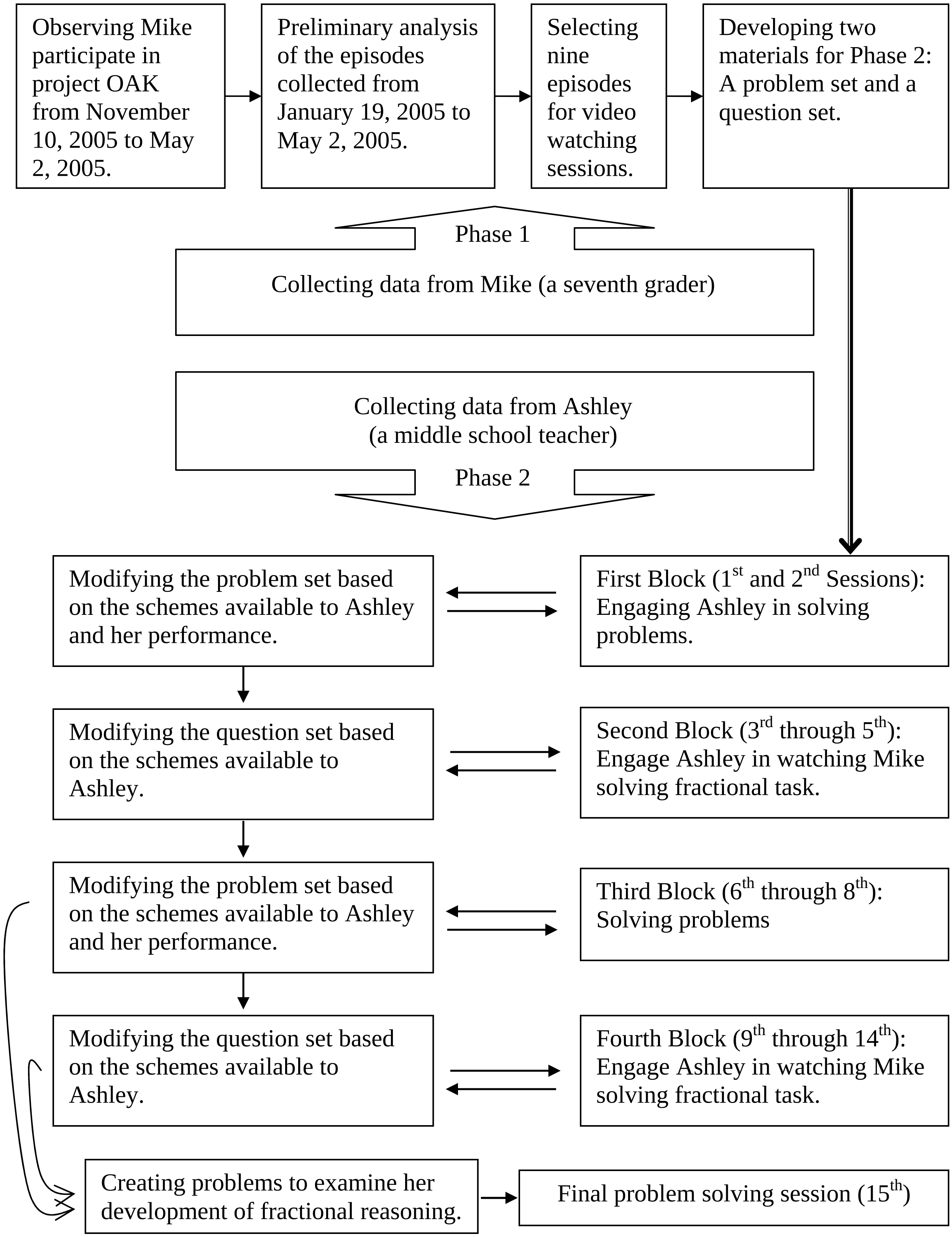

Figure 3.2, Flow of the procedure of data collection

Second Block or to provide an opportunity for her to construct schemes that she might use while engaging in Mike's mathematics during the Fourth Block. The problems posed during the First and Third Blocks were selected from a problem set that I developed in the first phase and modified based on her performance on the prior sessions. Through the problem solving sessions, I expected that she would develop a repertoire that could be used to explain Mike's reasoning and became aware of what she was doing to understand mathematics in a meaningful way. The video watching sessions were conducted to provide her with an opportunity to integrate her mathematical knowledge with her knowledge of student mathematics. Figure 3.2 summarizes the procedure of data collection.

Procedure of Data Analysis

The data I analyzed involved sixteen 20 to 25-minute-long videotaped teaching episodes from the student, Mike, and fifteen 50-minute-long videotaped episodes from the teacher, Ashley. My analysis began with Mike's data in order to select nine episodes for video sessions among the sixteen videotapes collected from January 19, 2005 to May 2, 2005. For each of the episodes, I first developed a list of the problems and questions posed to Mike and his responses with a title for each episode. Since the sixteen episodes have been well sequenced around a theme, which was multiplicative reasoning through fractional reasoning, I selected most of the first nine episodes[13] . In order to produce questions for a video watching session, I wrote "memos" (Charmaz, 2002; Strauss & Corbin, 1990) including Mike's response to a posed problem, my interpretation of his response and questions for Ashley. The following excerpt shows a part of a typical memo I usually brought to a video watching session.

[13] The selected episodes are as follows: Jan. 19, Jan. 26, Jan. 31, Feb. 2, Feb. 16, Feb. 28, Mar. 2, Mar. 7, and Mar. 9.

02/02/05 --- Mike revealed limitations of a part-whole concept of fractions: not yet developed an iterable fractional unit.

What he did

For the question, your dog is ¾ of 48 pounds dog,

- He made 36 pieces and explained that dividing 48 by 4, getting 12 then multiplying it by 3. How much is 12 of the 48 pound? One-third, finally he answered one-fourth (by probing). How did you get? 12 over 48 and then simplify or divide 12 into 48--- *for him, one-fourth is not an internalized action but just a quantity by procedural knowledge. 12 is a quantity to compare to the whole 48, so 12 doesn't need to be repeated to get the whole, i.e. 12 is not yet an iterable unit, even a unit, for him.* How much is 12 of 36? --- *He couldn't answer immediately; his first answer was one-fourth. He then corrected it to one-third because 12 and another 12 and another 12: this is the evidence of his part-whole concept of fractions.* How much is 36 of 48? --- *he first answered four-thirds, then he corrected it to three-fourths*.
- My interpretation: he is not aware of his activity, dividing and taking, for producing a fraction. His mentioning a fraction is not based on his action or reflection on the action but the amount of the parts embedded in the whole in his mind.

Q1. How do you think Mike understands one-third, one-fourth? What do you think makes him confused when he was saying a fraction corresponding to 12 pieces? (I think he knows how to simplify a fraction but for the question, how much is 12 of 36? he seemed unsure and couldn't make an answer for a while.) What do you think is important for him to figure out what fraction 12 is of a quantity? --- focusing on his action on the quantity.

Q2. (08:15) The teacher asked him if he could tell her more about four-thirds using the bars on the screen, but he didn't talk more anything about that. What would you ask him to encourage him to explain four-thirds he briefly mentioned at the beginning if you were the teacher?

Figure 3.3, A sample of "memos" for a video session

When each of the video watching sessions ended, I briefly transcribed and analyzed the recorded videotape. Based on the ongoing analysis, I prepared for the next session by revising the planned questions and confirming that the selected videotape would be appropriate for the theme of the next session. In addition, the video sessions conducted during the Second Block affected my revision of the scheduled problems and creation of a new problem for the following problem solving sessions.

Before I began the teaching experiment, based on preliminary analysis of the selected nine episodes, I had produced a tentative problem set for five problem solving sessions according to the following four topics: (1) Conceptualizing fraction multiplication based on a concept of an iterable fractional unit; (2) Simultaneous units-coordination and multiplicative reasoning based on the coordination; (3) Coordinating two units through their unit fractions; (4) Fraction division. I used the problem set for the first session. For the second session, I revised the problem set considerably taking into account the mathematical schemes currently available to her. Appendix A shows the problems I actually brought to the research site.

After I finished data collection, I implemented my analysis following four stages. During the first stage, I watched the collected 15 videotapes sequentially while jotting down memos (Charmaz, 2002; Strauss & Corbin, 1990) in order to find a theme that emerged throughout the teaching experiment. During the second stage, I watched the videotapes again and this time wrote detailed notes. During the third stage, I transcribed every session. As I did so, I wrote my comments in separate MS-Word files to describe important moments. During the last stage, I developed a set of protocols that would be included in my dissertation and wrote detailed descriptions of what was occurring during those protocols. Then I identified themes for subsets of the selected protocols. Appendix B shows the titles of the protocols selected from the problem solving sessions and the themes that emerged through the protocols; for the video watching sessions, refer to Appendix C. In the following Chapters 4 and 5, I presented in detail how I analyzed the teacher's data along with the student's data based on the procedure I described above.

Table 3.1, Flow of Data Analysis

Analysis	Artifacts
Preliminary Analysis Analyzing Mike's work collected from January 19, 2005 to May 2, 2005.	List Producing a list of the problems and questions posed to Mike and his responses with a title of each episode.
Analyzing the Selected Nine Episodes 1) Selecting nine episodes based on preliminary analysis. 2) Transcribing and analyzing the selected episodes.	Detailed Notes and a Set of Questions 1) Producing detailed notes including my interpretation of Mike's responses to the posed problems. 2) Producing a set of questions for video sessions on the basis of the analysis.
Ongoing Analysis 1) Watching the videotaped episode before conducting the next session. 2) Briefly analyzing the data collected to revise or confirm the planned questions or problems.	A Revised Problem Set and Question Set Revising the planned problems or questions for the next session on the basis of the ongoing analysis.
Analyzing 15 videotapes from Ashley 1) First Stage 2) Second Stage 3) Third Stage 4) Fourth Stage	1) Memos 2) Detailed Notes 3) Transcriptions and MS-files describing important moments 4) A Set of Protocols and Detailed Description of the protocols

CHAPTER 4

ASHLEY'S INITIAL CONCEPTS

Ashley's Initial Concepts of Fractions (Day 1-2)

A Concept of Fractions

At the beginning of her participation in this study, Ashley expressed a view of fractions that relied on part-whole comparisons as she used statements including "out of." She also thought that understanding fractions is a matter of dealing with two numerals comprising a fraction, numerator and denominator, and how to deal with them should be based on basic number sense such that "3/4 is less than one" and "4/3 is more than one." She paid attention to distinguishing the numerator and the denominator in the problem, "Given a bar, make three-fourths of the bar." She responded that three-fourths is less than one, so the bar she is going to make should be shorter than the given bar, and she should divide the given bar into four rather than three parts because she can make a shorter bar by taking three of the four parts produced. In addition, she used "out of" in the case of choosing a larger number of parts from a smaller number of parts as follows (A stands for Ashley):

Protocol 1.1: Making a 7/7-bar from a 9/7-bar.

A: Mine is nine-sevenths of his. That means his bar is seven-sevenths, one whole bar. Mine is nine-sevenths, *which is little bit more,* so divide into 9 pieces because I have 9 pieces *out of* 7. And his, I made 7 pieces out of 7, that same size, which made one whole bar. (02/02/06)

Such a way of understanding fractions led to difficulties with fractions in two ways. The first one is her changing the fractional whole in the case of improper fractions.

Protocol 1.2: Explaining nine-sevenths using a 9-bar[14].

A: I have 9 out of 7 and that means one whole and two extras, which is whole there (pointing at 7 parts in a 9-bar) and the 7, two more ninths. (02/02/06)

Even though she was thinking of her bar as 9 parts while considering 7 parts and noticed that 7 parts would make one whole, her concept of fractions based on part-whole comparisons led her to refer to 9 parts as one whole and two-ninths because two extras are actually embedded in 9 parts. Secondly, by relying heavily on a measuring out strategy when using a part-whole comparison, she had trouble finding a relationship of a larger amount to a smaller amount. To avoid such trouble, she tended to change a problem asking her to make a smaller amount from a larger amount into its inverse and measure out the larger one using the smaller one. Such tendency ironically made her struggle with thinking of part-whole comparisons at the beginning of the problem situation provided. Three excerpts showing the tendency are as follows (I: interviewer, A: Ashley):

Protocol 1.3: Employing a measuring out strategy, then using a part-whole comparison.

I: How much is my candy bar (a 7-bar) of yours (a 3-bar)?

A: Two and one-third, mine is three partitions. Your candy bar, I can divide it into two groups of three with one left over, one out of three, left over. (02/02/06)

Protocol 1.4: Changing a question to its inverse to use a measuring out strategy.

[14] A bar partitioned into 9 parts

I: What would you have to multiply mine (a 17-bar representing seventeen-ninths of a yard) by to get yours (an 11-bar representing eleven-ninths of a yard)?

A: So seventeen-ninths, what would I have to multiply that by to get eleven-ninths? (pause) One, two, three, four, five, six (saying to herself). One and six-ninths, one and two-thirds (whispering). Mine times one and two-thirds will give yours. (02/09/06)

Protocol 1.5: Employing a measuring out strategy for a question leading to a proper fraction.

I: How much is this (a 5-bar) of your string (an 11-bar)?

A: Two and one-eleventh.

I: How much is this string (the 5-bar) of the bottom string (the 11-bar)?

A: (Sits quietly for a while.) Well, this (the 11-bar) is two and one-eleventh of this one (the 5-bar).

I: But my question was--

A: I know it’s backwards.

I: Yeah, backwards. How much is this one (the 5-bar) of your string (the 11-bar)?

A: (pause) five-elevenths.

I: Because this one little piece (pointing at one part in the 11-bar) is--

A: one-eleventh.

I: of your string. (02/09/06)

The above protocols bring other issues related to her concept of fractions. She seemed yet to construct a fractional number and a partitive fractional scheme. Regarding her ability of constructing a fractional number, Protocol 1.4 shows that for Ashley, a fraction was closer to a name than a number; a name is given to an object by looking at its definite property whereas a number is constructed by attributing an individual's operation to an object. She was considering each part as a ninth, but that seemed not based on her operation, partitioning a yard into nine parts and taking one of them. Rather, she seemed to use "a ninth" as a name indicating a definite amount of one part. Thus, counting six parts resulted in six-ninths even after she identified one with eleven parts.

Related to a partitive fractional scheme, Protocol 1.5 shows that she was not able to construct an iterable unit to produce a fraction. She first looked at a 5-bar to measure an 11-bar, and then figured out that there are two of the 5-bar and one remaining to fit into the 11-bar. She called the remaining one-eleventh. That means one part was considered based on the 11-bar. However, she was not relating one part with the 5-bar as consisting of five one parts: five parts was not resulting from iterating one part five times or collecting five of one part. The "one-eleventh" that one part is indicating did not serve as a unit that can be iterated to generate a fraction corresponding to more than one part. Her inability of generating a fraction by a unit fraction produces doubt that she would use "one-eleventh" as a fraction that reflects an activity as well as a definite size of a quantity.

A Concept of a Unit Fraction

Ashley adopted the idea of "out of" as referring to part-whole comparisons to understand improper fractions. Extending the concept of fractions based on part-whole comparisons to improper fractions encouraged her to conceive of a fractional unit in an abstract manner, but not necessary an iterative manner.

Protocol 1.6: Reasoning that a 7-bar is seven-thirds of a 3-bar.

I: You told me seven-thirds of yours (a 3-bar) is mine (a 7-bar). How did you get it?

A: Mine is divided into 3 pieces. Three-thirds makes one whole candy bar. So, seven-thirds because there are 7 pieces, 7 of these pieces (pointing at parts in the 3-bar) makes 7 out of each little thing being one-third.

I: So, what you did, divided and repeated, how is that related to seven-thirds?

A: Each one of one-thirds (pointing at the end of the 3-bar), each one of the same lengths, size and shape would be another third. So, seven of them together make seven-thirds. (02/02/06)

There were two bars, a 3-bar and a 7-bar, in her perceptual field. She was asked how she made the 7-bar using the 3-bar. By focusing on the parts in the 3-bar, she first identified the size of one part as being a third. She then used it to give a name for seven parts already there rather than explaining how to produce the 7-bar. However, counting seven parts was not sufficient for naming the 7-bar. She needed to specify the size of one part, and that led her to reflect that the parts in the 3-bar can contribute to producing the 7-bar in an individual manner. Such insight is very critical in conceptualizing one part as an abstract unit, or constructing a partitive fractional scheme, in a sense that she began to think of several parts in terms of individuals being equal. This is different from her previous conception of fractions based on part-whole comparisons. However, she seemed yet to develop an iterable unit fraction because by saying “seven of them together” she showed that her way of figuring out the 7-bar was based on how many parts were collected rather than how she generated it using the abstracted unit fraction she constructed.

For proper fractions, the way she was thinking about a unit fraction seemed implicit, not explicit in that counting the number of parts considered dominated paying attention to an activity to produce individual parts contributing to generating a fraction. In addition, her part-whole concept of fractions led her to focus on comparing the numbers of parts between an amount considered and a whole. For example, she wouldn’t primarily think of three-fourths in terms of one-fourth because four, the number of parts in a whole, involves counting activity of an amount considered, three, and so she could get the fraction three-fourths through the counting activity without paying attention to the size of each part, one-fourth of the whole. Of course, she could also think of three-fourths using a unit fraction such as three one-fourths; however, such reasoning seemed more dependent on the number of parts in a whole than on an insight into an abstracted unit because she sometimes showed that she got a fraction from direct counting by saying “five means fifths.”

When trying to use the idea of “out of” for improper fractions, however, she had to reflect on the size of each part because seven parts were not contained in three parts. She conceived of the situation as seven of one-third because she knew she had to make seven parts to make the marked candy bar and each part was one-third. At the same time, by mentioning, “seven of these pieces makes seven out of each little thing being one-third,” she explicitly expressed she was paying attention to the size of each part for a fraction being produced. Although she sometimes used “out of” statement for improper fractions like “nine out of seven,” she was actually not just comparing the numbers of parts but also paying attention to a unit fraction comprising a whole. So, I would argue that she reified a concept of unit fractions when trying to conceive of improper fractions. She saw seven parts in a bar representing seven-thirds and simultaneously thought that the size of each part is a third of a whole and also each of those parts has same length, size and shape. However, her way of understanding seven parts seemed more collective than iterative. By mentioning “another third” for a same sized and shaped part and “seven out of each little thing,” she seemed to present seven equivalent parts being collected rather than seven identical parts being iterated one part seven times. That is, she was activating a partitive fractional scheme and an iterative fractional scheme when conceiving of improper fractions but yet to construct an iterative fractional scheme in an explicit way.

A Concept of a Whole

Although Ashley sometimes changed a whole within a problem context, she seemed to think that a whole should be constant throughout the context considered so there was only one thing she called a whole. Related to the problem of making Mr. V's bar so that a given bar is to be nine-sevenths of his bar, the episode[15] is as follows:

> Protocol 1.7: Finding a reciprocal relationship between a bar and nine-sevenths of the bar based on her construction of the 9/7-bar.
>
> A: Mine is nine-sevenths of his. That means *his bar (a 7-bar) is seven-sevenths, one whole bar*. Mine (a 9-bar) is nine-sevenths, which is little bit more, so divide into 9 pieces because I have 9 pieces out of 7. And *his*, I made 7 pieces out of 7, that same size, which made *one whole bar*.
>
> I: Each little piece, how much is each little piece of yours?
>
> A: One-seventh.
>
> I: How much is one little piece of yours?
>
> A: (Sits quietly without answering.)
>
> I: One-seventh?
>
> A: Oh, okay, I was, here is *one-seventh* (pointing at one of the parts in the 7-bar). Okay, um, <u>one-ninth</u> (pointing at one part in the 9-bar).
>
> I: So, is that the same, one-ninth of yours is the same as one-seventh of his bar?

---

[15] The statements with *Italic* indicate the moments she mentioned his bar as a whole, and the underlined statements indicate that she was referring to hers as a whole.

A: (Starts verbalizing after a while) Yes, if you are talking about the whole bar. Yes, one-ninth of my entire bar is equal to *one-seventh of his entire bar*.

I: What would you multiply your bar by to get his bar?

A: I know you want me to use these [bars] (laughing). What would I multiply mine (pointing at the 9-bar) by to get his bar (pointing at the 7-bar)? One (placing her cursor at the end of 7-bar which is right below the seventh part in the 9-bar and whispering) nine-sevenths (writing 9/7 on the screen using cursor) times what equals (pause) seven-ninths.

I: Seven-ninths, could you?

A: Because there is *one whole bar, his* (pointing at the 7-bar).

I: Could you explain what you did using this bar [these bars, the 7-bar and the 9-bar]?

A: I did it by reciprocals (without confidence, her voice was getting smaller).

I: What do you mean by reciprocals?

A: Um, not this bar as I think, if I wanted to equal one whole, nine-sevenths times x equals one whole, and x is (flipping her hand to express that she got the answer by flipping 9/7).

I: Yeah, we usually do that. But this time …

A: I know. I am trying, so ask me again, I am sorry.

I: What do you have to multiply, what do you multiply your bar by to get his bar?

A: (Sits quietly without answering.)

I: What did you do to get his bar from yours?

A: Well, because mine was divided into nine equal pieces.

I: When you divided into nine, what is the number to represent your dividing activity, your dividing activity represents what?

A: Well, I mean, I am thinking along the line, the reason I divided it into nine is because I said I have 9 out of 7, so that means I have *one whole* and two extras, which is where these are the seven (pointing at 7 parts in the 9-bar) and then two more makes them nine-sevenths.

I: After dividing into 9, what did you do?

A: After I divided it into nine to get his candy bar, I took each one of <u>my ninths</u> and put seven of them together. Because one whole candy bar is, if mine was 9 out of 7, *he has to be a whole*, has to be *7 pieces out of 7 to make one whole*.

I: When you divided your bar into nine, the piece you got is one-seventh of his bar, right?

A: Yes.

I: So, you repeated it seven times to get his bar.

A: Yes.

I: Right?

A: I did.

I: So, your answer, my question is seven times, seven . . . ninths.

A: 7 (pointing at the 7-bar) out of my <u>ninths</u> (pointing at the 9-bar).

I: Yeah, you gave me the answer, seven-ninths. So, your dividing into 9 and repeating 7 times, how is it related to seven-ninths?

A: Because <u>mine is divided into ninths</u> and I repeated that 7 times to make 7 . . . 7 of my pieces, in which each is one-ninth.

I: Divided into 9 is the same as times one-ninth?

A: Yes.

I: Then 7 times one-ninth, what is that?

A: Seven-ninths. (02/02/06)

For Ashley, the notion of one-ninth was based on the nine parts. That implies one-ninth is considered more as a resultant amount than as an operation: after I get nine parts, I can see one of them as one-ninth vs. if I divide a bar into nine parts, I will get one-ninth of it because there will be nine of them. Until she realized there were two whole bars she could refer to for a part, a 9-bar was considered as 9 parts consisting of 7 parts corresponding to a whole she was looking for and 2 parts remaining after making the whole: 7 parts as the same as the 7-bar, that is, seven-sevenths, and 2 parts as two-ninths. However, by saying "if you are talking about the whole bar," she began to pay attention to how many parts there are in each bar and could say one part in terms of her bar as well as Mr. V's bar: "one-ninth of my entire bar is equal to one-seventh of his entire bar." In other words, rather than generating Mr. V's bar, the 7-bar, using one-ninth of her bar, the 9-bar, she was looking at seven parts and nine parts in her perceptual field and named one part by relying on the number of parts: a part was embedded in where it belongs, and not yet independent from where it comes from. Such reasoning shows that she only developed a partitive fractional scheme because she was able to properly produce a fraction after setting a whole she would refer to. However, it is clear that her reasoning was not based on an iterative fractional scheme because she couldn't produce nine-sevenths in terms of a seventh and that means a seventh was never involved in generating a fraction.

She also seemed yet to develop a unit fractional scheme until she realized one part in the 9-bar is one-ninth of it. When she said "one," it seemed to indicate his bar, which was what she was considering as a whole throughout the problem context. She noticed that she would get one if she multiplied nine-sevenths by something and used her procedural knowledge, multiplying reciprocals arrives at one, when finding a number that gives one, which represents his bar. In addition, she explicitly expressed that her dividing activity was to get seven parts for his bar, and it resulted in two remaining. So, even though she explained how to get 7 parts for his bar while saying “I took each one of my ninths and put seven of them together,” seven parts were not yet seven-ninths but seven-sevenths because it should be a whole, one. I tried to call an attention to her repeating activity, and some changes were noticed in her conceptions through “seven out of my ninths” not “seven out of nine parts.” This phrase is very critical in that she began to consider a part as a unit fraction, not just as one part comprising what she is looking for.

How she was conceptualizing fractions seemed closely related to how to conceive of a whole. With only a partitive fractional scheme, it was very hard for her to think of a part in terms of two referents, two kinds of wholes because for a part to be a fraction, it had to be embedded in a whole. So, even when she said one-ninth of the 9-bar is equal to one-seventh of the 7-bar, I doubt she was thinking of a relationship between the 9-bar and the 7-bar based on one part. In addition, constructing a fraction without a notion of operating led to her confusion that the 7-bar should be a whole as well as seven-ninths. She was missing that she got a whole by means of operating with seven-ninths on the 9-bar. That is, seven-ninths was an operation and the result of her operating, seven-ninths of the 9-bar, was a whole.

A Concept of Multiplication

Ashley had some trouble in replacing an of-statement with a times-statement. However, she had no problem with replacing an of-statement with a times-statement when a fraction followed a whole number. She could easily transform “eight *of* one-fifth” into “eight *times* one-fifth.” But it is not clear whether she used “of” in a collective manner or an iterative manner.

She also had no trouble interpreting a product when a mixed number followed “times”: she said, “this 5-bar (a bar divided into 5 parts to represent five-ninths) times two and one-fifth gives you eleven-ninths” when asked to elaborate “five-ninths times two and one-fifth equals eleven-ninths.”

For Ashley, an of-statement with fractions kept its “times” meaning in a collective or iterative manner by focusing on a unit fraction. For example, she interpreted “eight-fifths times five equals eight” as “an 8-inch bar is eight-fifths of a 5-inch bar” with some guide. She considered eight-fifths as getting one-fifth of the 5-inch bar and then repeating it eight times, that is, she viewed a fraction 8/5 as an operation in the situation. However, by dominantly conceiving of “times” as “repeating” within whole number contexts and a fraction as a resultant amount rather than an operation related to repeating, she failed to produce a “times” even after she successfully arrived at a fraction she was looking for.

> Protocol 1.8: Producing a multiplication using a relationship between a 5/9-bar and an 11/9-bar.
>
> A: The bottom bar (a 5/9-bar consisting of 5 parts; representing five-ninths of a yard) is five-elevenths of the top (an 11/9-bar consisting of 11 parts; representing eleven-ninths of a yard).
>
> I: Five-elevenths times eleven-ninths gives you what?
>
> A: (Without saying, points at the 5/9-bar and the 11/9-bar.) (02/09/06)

Given a times-question involving two fractions, she seemed to interpret both fractions as resultant amounts indicating two bars in her perceptual field even though she used one of the fractions to relate the two bars. She was able to correctly name the two bars in yards such that the 5/9-bar is five-ninths of a yard and the 11/9-bar is eleven-ninths of a yard, and she could also verbalize a relationship between them using a fraction like the 5/9-bar is five-elevenths of the 11/9-bar. However, it was very challenging to her to interpret the times-question implying the relationship she just produced in a meaningful way. The times-question, five-elevenths times eleven-ninths, requests her to view one fraction as an operation while considering the other as a resultant amount resulted from her operating. Therefore, if she had gained an explicit insight into distinguishing two distinct perspectives of fractions, she would have dealt with the fractions involved distinctively.

According to her current perspective of fractions, she was yet to develop considering fractions as operations. Thus, it must have been very demanding for her to consider the 5/9-bar from the perspective of fractions as operations. Five-elevenths specifies the 5/9-bar without any concern about the size of one part comprising the 5/9-bar and the 11/9-bar. However, as I described the above, she considered a fraction as an operation when a whole number was followed by a times-question. She had some sense of the view of fractions as operations, but it seemed to be activated in certain types of quantitative situations such as whole number contexts. That means the view was not yet interiorized. As a result, the phrase “five-elevenths times eleven-ninths” led her to a completely different context from her current understanding that the 5/9-bar, five-ninths of a yard, is five-elevenths of the 11/9-bar, eleven-ninths of a yard. In addition, the of-statement that she used to elaborate a relationship was not the same as the times-statement for her in that the times-statement was not implying any relationship between two given quantities. The following protocol was continued.

Protocol 1.8: (Cont.)

A: I see that it’s been 5, and I understand what’s five-elevenths and I understand why it’s also five-ninths of a yard (pause). I see I can take this one (the 5/9-bar) goes into my string, two wholes times with one left over and I see that. But I am not seeing backwards.

I: Then this way (pointing my finger toward the 11/9-bar from the 5/9-bar), could you give me a multiplication starting with this one (the 5/9-bar) to get this one (the 11/9-bar)? Because you said the top string (the 11/9-bar) is gonna be two and one something [of my string]?

A: Two and one-eleventh.

I: One-eleventh?

A: (After a short pause) No, fifth, one-fifth. So, two whole ones and one-fifth left over. This (the 5/9-bar) times 2 and one-fifth give you . . . eleven-ninths.

I: Could you make a multiplication using the numbers?

A: Five-ninths times two and one-fifth equals eleven-ninths.

I: Instead of two and one-fifth, could you use a fraction?

A: Five-ninths times eleven-fifths equals eleven-ninths. (02/09/06)

In the first part of Protocol 1.8, she tried to figure out five-elevenths times eleven-ninths through the relationship that the 5/9-bar is five-elevenths of the 11/9-bar. But she failed doing it. Now, by saying "I understand what's five-elevenths and I understand why it's also five-ninths of a yard," it became certain that she was trying to figure out the 5/9-bar using two different fractional representations. However, we need to carefully examine her way of saying, "what's five-elevenths": five-elevenths seemed to be considered as a result of her activity, and she seemed to activate a partitive fractional scheme on the 11/9-bar when producing five-elevenths. At the same time, she reminded of the size of the 5/9-bar, five-ninths of a yard. However, she could not relate the two fractions in a meaningful way. To figure out the given times-statement five-elevenths times eleven-ninths, she first focused on the number of parts in two bars and employed measuring out strategy, which is to produce a larger amount from a smaller amount by figuring out of how many the smaller ones is needed to make the larger one. She finally produced a multiplication leading to eleven-ninths as corresponding to a larger amount: five-ninths times two and one-fifth equals eleven-ninths. The measuring out strategy produces a mixed number to represent a relationship between two quantities, and it seemed easy for her to consider the produced number, two and one-fifth, while relating it with her activity. But I was uncertain that she conceived of the improper fraction, eleven-fifths, based on what she was doing like she did the mixed number.

Through engaging in producing a multiplication, she expressed that she has two kinds of views of fractions: fractions as operations and resultant amounts. Which view she used depended on the problem context: multiplication involving a whole number is more likely to let her conceive of fractions as operations, whereas multiplication of fractions led her to view both fractions as resultant amounts.

Current Understanding of Fractions as Operations and Unit Fractions as Iterable Units

There is another issue that emerges other than her preferring the measuring out strategy when Ashley tried to produce a multiplication starting with a larger amount: considering fractions as operations.

Protocol 1.8: (Second cont.)

I: Could you do that this way? [Can you produce a multiplication going toward the 5/9-bar from the 11/9-bar?]

A: I will try (laugh). Eleven-ninths times (long pause; her cursor was pointing at the 11/9-bar).

I: How much is a little piece of the top string (the 11/9-bar)?

A: Oh, of the string? One-eleventh.

I: One-eleventh. You repeated 5 times. [I wanted to remind her of what she had done.]

A: Yes, five-elevenths.

I: Five-elevenths of this one (the 11/9-bar).

A: Yes.

I: So, starting with this eleven-ninths.

A: Eleven-ninths times five-elevenths, no, because I am trying to get five-elevenths, right? Eleven-ninths times something equals five-elevenths. (02/09/06)

She tried to produce a multiplication to arrive at the 5/9-bar from the 11/9-bar, but it was not clear how she would conceive of the 5/9-bar at the beginning of the episode. I probed her to help reflect on what she has done to get the 5/9-bar using the 11/9-bar, and it ended up with five-elevenths. Let's closely look at how she constructed five-elevenths. She was first encouraged to focus on the size of one part comprising the 11/9-bar and then reflect on how many parts are needed to make the 5/9-bar. She verbalized five-elevenths to describe the 5/9-bar with respect to the 11/9-bar while referring to her activity—repeating one part being one-eleventh five times. At that time, she seemed to rely on a partitive fractional scheme to figure out the fraction 5/11 in a sense that 1) one-eleventh was considered as one of the 11 parts comprising the 11/9-bar, all of which were already locating in her perceptual field, rather than as one part to make the 11/9-bar; 2) five-elevenths was considered as five parts embedded in the 11/9-bar rather than as being generated by five independent parts regardless of whether each part is equivalent or identical. Her later saying "I am trying to get five-elevenths" provides the indication that she was considering five-elevenths as the 5/9-bar itself, a result of her operating rather than an operation. She was missing a meaning of five-elevenths as an operation.

However, she expressed she was kind of considering fractions as operations by saying, “one-seventeenth times seventeen-ninths equals one-ninth because one of these (17 parts in a 17/9-bar) is a ninth of a yard. So, that (one part in the 17/9-bar) is seventeen-ninths times one of these pieces (parts in the 17/9-bar). I am trying to picture, is that only one-seventeenth, so only seventeenth of it. So, that made it one-ninth of a yard.” While trying to connect an of-statement with a times-statement, she seemed to develop a view of fractions as operations: she interpreted “one-seventeenth times seventeen-ninths” with “one of these (17 parts in the 17/9-bar).” She also transformed a multiplication “one-seventeenth times seventeen-ninths” into “seventeen-ninths times one of these pieces (17 parts in the 17/9-bar).” That shows she tried to make sense of one-seventeenth as an operation. However, she did not explicitly distinguish a meaning of one-seventeenth as an operation and as a result of the operation: “[one part in the 17/9-bar] is that only one-seventeenth, so only a seventeenth of it.” She seemed to be in the process of developing an operational meaning of fractions. The following protocol shows she interpreted a times-operation while relying on a view of fractions as results of her operating.

Protocol 1.9: Finding a product 17/9 times 4/17.

I: How about seventeen-ninths times four-seventeenths?

A: Four-ninths. 17, each one is one-ninth. Seventeen-ninths times four-seventeenths, one two three four seventeenths, it’s broken into four, 17 equals, four of these equals to four-ninths. (02/09/06)

Ashley considered seventeen-ninths in terms of seventeen parts, each of which is one-ninth. For “four-seventeenths,” she kept four parts each being one-seventeenth in her mind. At this point, it is clear that one-ninth was referring to a yard, but it is not clear what a seventeenth was referring to. However, by saying, “it’s broken into four,” she revealed she was thinking of a bar divided into four. By “17 equals,” she seemed to use the equation, seventeenth times seventeen equals one, and then ended up with four parts. Since she thought of each part as being one-ninth, she finally got four-ninths. In terms of this episode, one-seventeenth seemed not yet an 30operation and four-seventeenths looked more like a collection of the 1/17-bars. That means she has an abstract level of a concept of a unit fraction but had not yet developed an iterative aspect of a unit fraction.

Summary

In this section, I described Ashley’s initial concepts of fractions by investigating her concept of fractions, a unit fraction, a whole and multiplication, and her current understanding of fractions as operations and unit fractions as iterable units.

Ashley thought that understanding fractions is a matter of dealing with two numerals comprising a fraction, numerator and denominator, and a part-whole concept based on an “out of”-statement helps decide how many parts are needed for a whole. Even for improper fractions, she used an “out of”-statement: “9 pieces out of 7.” Such a way of understanding generated two kinds of difficulties. First, she had difficulty converting improper fractions to mixed numbers based on her construction: “9 out of 7 . . . that means one whole and two extras . . . two more ninths” [cf. Protocol 1.2]. Second, she had trouble relating a smaller amount with a larger amount because a part-whole comparison forces her to have an embedded amount or measure a larger amount using a smaller one. For example, responding to “How much is this (a 5-bar) of your string (an 11-bar)?, she answered “Two and one-eleventh” because she failed to disembed the 5-bar from the 11-bar and identify one part in terms of the redefined whole [cf. Protocol 1.5]. As a result, she considered a fraction as a resultant amount without relating it with her activity—dividing into parts and taking some parts or repeating a part as many times as needed. In addition, she had yet to construct a unit fraction that can be repeated to produce a fraction, an iterable unit fraction.

A fractional context involving improper fractions provided her with an opportunity to construct a unit fractional scheme: “Each one of one-thirds (pointing at the end of the 3-bar), each one of the same length, size and shape would be another third. So, seven of them together make seven-thirds” [cf. Protocol 1.6]. However, she had yet to construct an iterable unit fraction in a generative manner because she seemed more interested in collecting the parts needed rather than generating the needed parts by an abstracted unit.

Related to a concept of a whole, she tended to maintain one kind of a whole throughout a problem context. She tried not to use the term “whole” for a quantity other than what she set as a whole at the start. So, given a problem, “produce a fraction to make a 7-bar from a 9-bar”, even though she explained “I took each one of my ninths and put seven of them together,” the seven parts was considered as a whole seven-sevenths because the 7-bar was set a whole at the start [cf. Protocol 1.7]. Thus, she was confused with the fact that the 7-bar could be a whole as well as seven-ninths. In addition, she missed that the whole was produced by means of operating with seven-ninths on the 9-bar: seven-ninths was an operation and the result of her operating, seven-ninths of the 9-bar, was a whole. Therefore, I argue a concept of a whole in an absolute manner closely related to a partitive fractional scheme as well as a view of fractions as resultant amounts.

Ashley had difficulty reasoning out a fraction multiplication based on her construction. She considered fractions mostly as resultant amounts. Such a view of fractions led her to struggle with using a fraction to relate two fractional amounts for conceiving fraction multiplication. In addition, she considered times as repetition, which might cause a problem in fractional contexts. Thus, even after she arrived at a correct answer with respect to a question, when she established a relationship between two fractional amounts, she was confused with deducing fraction multiplication: she said, “The bottom bar (a 5/9-bar) is five-elevenths of the top (an 11/9-bar),” but she sat quietly without answering in response to the question “Five-elevenths times eleven-ninths gives you what?” [cf. Protocol 1.8]. However, she had no problem with producing a multiplication involving fractions by using a mixed number: “Five-ninths times two and one-fifth equals eleven-ninths” as she referred to the 5/9-bar and the 11/9-bar [cf. Protocol 1.8 (Cont.)].

Throughout this block, Ashley showed a lack of understanding fractions in terms of an activity due to her level of fractional reasoning—a partitive fractional scheme. For example, she stated with some guide that a 5/9-bar is five-elevenths of an 11/9-bar based on her activity, pulling one part out of the 11/9-bar and repeating the part five times. However, she failed to produce a product based on her constructs such as five-ninths, eleven-ninths, and five-elevenths by saying “I am trying to get five-elevenths” [cf. Protocol 1.8 (Second cont.)].

However, while trying to connect an of-statement with a times-statement, she seemed to develop a view of fractions as operations. She transformed the statement “one-seventeenth times seventeen-ninths” into “seventeen-ninths times one of these pieces (17 parts in the 17/9-bar).” She tried to make sense of one-seventeenth as an operation. However, she did not explicitly distinguish the meaning of one-seventeenth as an operation and as a result of the operation: “[one part in the 17/9-bar] is that only one-seventeenth, so only a seventeenth of it" [cf. the text immediately preceding Protocol 1.9].

Ashley's Initial Concept of Mike's Mathematics (Day 3-5)

A Relationship Between Mixed Numbers and Improper Fractions

Ashley watched a video clip where Mike was working on the question, "You have 7 inch-long candy bars which is three times the amount of candy I have. Can you make how much I have?" The protocol of Mike's work follows (T: teacher, M: Mike).

> Protocol M2.1: Making an amount so that seven 1-inch bars are three times the amount.
>
> (Mike made one bar and copied it six times to represent seven 1-inch bars. He then divided each into three parts. To make the teacher's amount, he pulled one part out from a bar and repeated it three times so he got a 1-inch bar consisting of three parts. He then copied it and pulled one part out so finally there were two bars and one part.)
> T: You are saying how much is this (pointing at two bars and one part) of your portion?
> M: One-third.
> T: Okay, one-third. How do you know that that's one-third?
> M: Because you add (moving the top bar to overlap the second bar, then making one part overlap the bars) these together and take it (one part) away from this one (the third bar among seven bars).
> T: How long is your collection of candy bars?
> M: Um. Seven inches.
> T: Seven inches? So, okay, how long is my collection?
> M: Two and . . . one-third.
> T: Um, two and one-third. How many, how many, how much is this (one part)?
> M: One-third.
> T: Um. One-third. How many thirds of an inch do I have?
> M: Seven.
> T: Oh, so how much do I have again?
> M: Seven . . . thirds.
> T: Oh, seven-thirds. So, that is the same as two and one-third?
> M: Yes.
> T: Oh, I see, okay, Wow! That's good. That's really neat, Mike. What would you have to do, what would you multiply mine by to make yours?
> M: (After a short pause) three.
> T: Um, So, I want to know what seven-thirds times three is.
> M: (Produces a 7/3-bar by joining seven parts and then copies it two times so that three 7/3-bars are located vertically.)
> T: Seven-thirds inches times three, do you know how much that is?
> M: (Moves the bottom 7/3-bar next to the top 7/3-bar back and forth.)

T: How much is that?
M: Should be … twenty-one, twenty-one, twenty-one-thirds.
T: Oh, twenty-one-thirds.
M: Which is (inaudible) seven.
(On the upper left part of the screen, there are seven 1-inch bars, each of them was divided into three; on the lower right part of the screen, there are three 7/3-bars.)
M: (Moves each 1-inch bar towards a 7/3-bar. He then makes them overlap the 7/3-bars while showing that they have the same number of parts.)
T: Huh? That's pretty neat how that works. You said, you just show me if you do seven-thirds times three, you get, how many inches?
M: You get … seven inches.
T: Cool. Seven inches, okay, here is a challenging question. I wonder what would you multiply seven inches by to get my collection, seven-thirds?
M: (25-second pause)
T: Can you think about what number multiplies seven inches by to make seven-thirds?
M: Make seven-thirds?
T: Yeah.
M: Let me see.
T: Seven times something.
M: Okay, (3-second pause) Makes seven-thirds (saying to himself). (10-second pause) one-third?
T: Oh, do you think that works?
M: Yeah, sure.
T: How come?
M: Because you are using multiply cross. One times (inaudible but I guess: seven equals seven, three times one equals three).

Mike constructed a fraction 1/3 by producing a mixed number two and one-third through the relationship that three of two bars and one part make 7 bars. However, the fraction 1/3 for relating two bars and one part with 7 bars seemed to have no connection with one-third of one bar. Let's investigate what he meant by one-third. He first made a 1-inch bar and then produced seven of it. According to the given relationship that seven bars represent three times the teacher's amount, he produced two bars and one part, three of which comprise the 1-inch bar, for the teacher's amount. He said that the teacher's amount would make a third of seven bars as he used a matching and taking away strategy, that is, if he took away two bars and one part three times from seven bars, there would be nothing remaining. At this point, two bars and one part served as a unit to construct the seven bars; he actually showed there are three of two bars and one part in the seven bars. Such activity indicates that he seemed to be activating a unit fractional scheme to figure out one-third. In addition, it implies that Mike was relating two and one-third with seven. However, noticing that two 1-inch bars and one-third of a 1-inch bar becomes a third of seven 1-inch bars does not necessarily mean conceiving of a third of 7 inches in terms of a third of an inch. Through the episode, he mentioned two kinds of a third, one referring to one bar and the other one referring to seven bars, but did not seem explicitly aware of that. For him, only one bar seemed to play a role of a whole in a traditional way.

His saying seven-thirds did not seem to have a meaning related to seven bars, i.e. a third of seven inches. Let's closely look at how he got a sense of seven-thirds. The teacher led him to pay attention to the size of one part, and he named it one-third and noticed there are seven of a third of one bar comprising two bars and one part. Since he was able to think of seven of a third of one bar he made as "seven," he could say seven-thirds. Thus, his sense of seven-thirds was only coming from the notion of how many parts are in his perceptual field rather than how he produced seven parts while considering seven bars. So, the next question "what is seven-thirds times three?" led him to a completely deferent context from what he has been engaged until then.

Mike created a 7/3-bar by breaking two bars into three parts each and adding one extra part to them. He then made two copies of what he had made and then placed three 7/3-bars together as he said "should be . . . twenty-one-thirds, which is . . . seven." However, he did not seem to have an insight that seven bars each of which has three parts are the same as three 7/3-bars each of which has seven parts. Rather, he seemed to use his calculation. That is, without any consideration of seven bars, he arrived at the answer twenty-one-thirds by only relying on the total number of parts being one-third and confirmed that it should be the same as 7 by matching the number of parts. All the things that have been done were on the basis of one part being a third of one bar without a notion of seven bars as a whole. Therefore, when asked “what number multiplies seven inches by to make seven-thirds?” some doubt about the question seemed to arise. He could not figure out the question using what he has done because a 7/3-bar was constructed separate from seven bars.

Ashley’s response: Two and one-third might not be equivalent to seven-thirds (02/16/06).

When Mike could not answer the question, what he has to multiply 7 by to get seven-thirds, Ashley began to doubt that it was certain for Mike that two and one-third is equivalent to seven-thirds. The following protocol shows how she responded to the incident (I: interviewer, A: Ashley).

Protocol A2.1: Comments on Protocol M2.1.

A: He did not see the reverse steps. He didn’t see.

I: What is the reverse step?

A: I mean doing the same problem backward, kind of thinking of it, like um it was basically the same problem.

I: Um, basically the same problem?

A: Right because, but he couldn’t think about the other way. He didn’t realize that he was thinking that seven-thirds is the same thing as two and one-third at this point. He wasn’t thinking about what he had before where he had two and one-third and he showed that two and one-third three times is equivalent to 7. . . Yeah, two and one-third three times is equivalent to 7 so he is not doing the reverse of that.

I: Reverse means seven . . . .

A: Seven times what gives you two and one-third, or seven-thirds. He wasn’t at the end, wasn’t thinking two and one-third is the same as seven-thirds.

I: In terms of your explanation, if the teacher asked him what you have to multiply 7 by to get two and one-third, and then do you think Mike could answer?

A: Based on what he did it at the beginning, yes. He might have an easier time with that because that was his first thought when he made hers, what he has is three times as much as what she has. So he made her amount two and one-third, so that's the way he thought of it as a mixed number two wholes and a piece. So I think maybe if she asked that, it maybe helped some for him to get it more quickly with the manipulatives rather than the calculation.

Ashley first responded to the question "What number multiplies seven inches by to make seven-thirds?" as if it was basically same problem as "What would you multiply mine (2 bars and one-third of one bar) by to make yours (7 bars)?" even though one question is the inverse of the other. For her, being able to answer one question implied being capable of responding to the other. Therefore, she anticipated there should be no problem with answering the former question because there was no problem for him to say three for the latter question. However, Mike had difficulty answering the former question. Mike's struggle prompted her to think about some possible interpretations of Mike's way of thinking: 1) Mike might not develop the equivalence relationship between two and one-third and seven-thirds at a conceptual level; 2) the former question might not be the same as the latter one.

Related to the second interpretation, she pointed out that Mike could not think about the inverse of the question asking how to get seven-thirds from 7, and that meant he did not recall that he produced 7 bars by repeating the two and one-third bar three times. Her interpretation indicates that she was assuming an equivalent relationship between seven-thirds and two and one-third. Based on the assumption, she further reasoned that thinking that 7 is three times two and one-third would not guarantee knowing that two and one-third is a third of 7: thinking reversibly is not taken for granted.

Regarding the first interpretation, she closely related it with the second interpretation while considering that he might know the inverse. In other words, he might know that two and one-third is a third of seven but not realize one-third for the question “What number multiplies seven inches by to make seven-thirds?” because the question includes seven-thirds instead of two and one-third, and two and one-third is not the same as seven-thirds in his mind. She then thought if the teacher asked him the same question using two and one-third rather than seven-thirds he might be able to answer the question like he did when asked what he would need to multiply seven by to get two and one-third.

However, she never made a meaning of seven-thirds by relating it to 7 bars. She seemed to consider Mike’s concept of seven-thirds only in terms of the perceptual amount of two bars and one part for a total of seven parts, without wondering about how seven parts are related to seven bars based on the given relationship, seven bars is three times the teacher’s amount. At the beginning of this session, she anticipated he would know that two and one-third is equivalent to seven-thirds because for two bars and one of three parts comprising one bar, he was able to say seven-thirds as well as two and one-third by paying attention to each part. That is, she thought that since he named two and one-third and seven-thirds using the same amount of bars, he must have known their equivalence. In addition, when Mike made a segment to represent seven-thirds as he broke two bars into three parts each and joined them with another part in a row for the question “what seven-thirds times three is,” Ashley commented that such activity tells that Mike was considering that seven-thirds is equivalent to two and one-third because the amount referring to seven-thirds resulted from two bars and one part.

Figure 4.1 summarizes how Ashley understood Mike's way of thinking about two and one-third, seven-thirds and seven, and their relationships. The thin red arrows represent her initial interpretation of Mike's mathematics. At the beginning, she had no doubt about his conception of the equivalence between two and one-third and seven-thirds. She also thought that he understood the relationship between two and one-third and seven in both ways; seven is three times two and one-third, and two and one-third is a third of seven. In addition, since two and one-third would be equivalent to seven-thirds in his mind, she thought he would know that seven-thirds times three equals seven at a conceptual level. The thick blue arrow indicates the incident that she began to question about his conception of seven-thirds with respect to seven. She watched Mike's struggle related to the question asking about a number to multiply seven by to make seven-thirds. Based on her concern about such struggle, she inferred that he might not have been conceptualizing that seven-thirds should be the same as two and one-third, and two and one-third is a third of seven. The black arrows represent such inference.

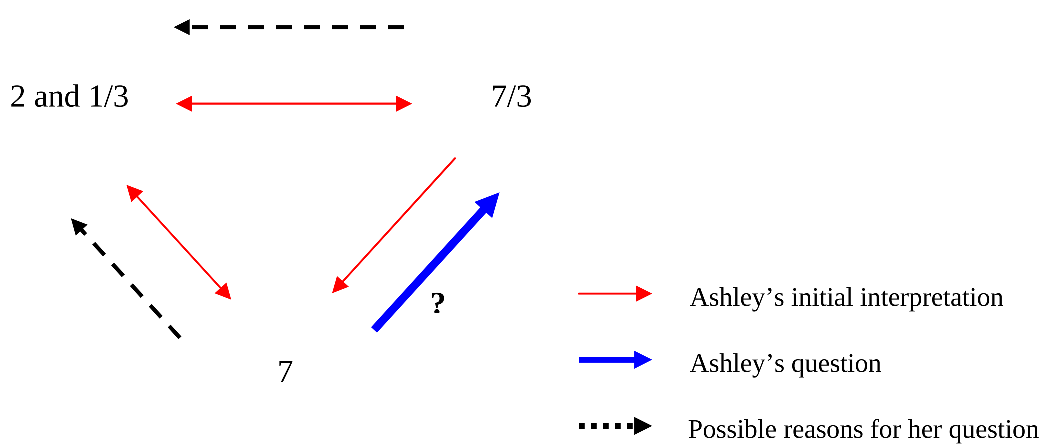

Figure 4.1, Ashley's interpretation of Mike's way of thinking to relate 2 1/3, 7/3 and 7

A Meaning of Fractions Through Conceptualizing a Whole in Fractional Contexts

Ashley watched a video clip where Mike was engaged in the question, “You have a 2-foot long sub sandwich. That’s three times longer than mine. So, I wanna know how long mine is.” Because in Protocol M2.1 Mike seemed to deal with only one level of units, the teacher would want to investigate how many levels of units he could work with by posing the above question (T: teacher, M: Mike, O: observer).

> Protocol M2.2: Making a sandwich so that a 2-foot long sub sandwich is three times longer than it.
>
> (Mike was asked to make the teacher’s amount so that his sub sandwich, 2-foot long, is three times longer than hers. He drew a bar divided into two parts to represent 2 feet.)
> T: Okay, so, there is your sandwich, okay. What do you think you will do to make mine?
> M: Make yours?
> T: Yeah.
> M: Divide each side into three and then take two of them from both sides.
> T: Oh! (To the observer) Do you have a question about that?
> O: Can you give me a multiplication problem for that?
> M: Oh, a multiplication problem? Let’s see . . . one-third? (Looking at the observer) . . . one-third times one-half?
> T: One-third times one-half? Do you think so? (Looking at the observer) Okay. Let’s try that, let’s try what you are doing. Let’s see if that multiplication works. Okay?
> M: (Divides each foot of the 2-foot long bar into three parts, and pulls out two parts in the middle of the partitioned bar. He joins them and measures the 2-foot bar using the two-part bar he made as he says aloud.) One-third, one-third, one-third.
> T: Uh, uh, uh, (keeping nodding) let’s see, how long is my sandwich?
> M: Your sandwich is two-sixths.
> T: Is it?
> M: Yeah.
> T: Two-sixths of what?
> M: Of . . . six-sixths.
> T: Oh, two-sixths of six-sixths. Here now, oh, wait, how long is yours again?
> M: Two feet.
> T: Two feet. Oh, okay. So, I wanna how many feet I have.
> M: Okay. Let’s see, yeah, two-thirds of a foot.
> T: Oh, two-thirds of a foot.
> M: Then I will have a multiplication problem.
> T What is that? What is the multiplication problem?
> M: Two-thirds times one-half. Which equals that . . .
> T: Two-thirds times one-half? What would that equal?
> M: That equals sixth, two-sixths.

T: . . . Oh? Oh?
M: (inaudible) which is of my entire bar.
T: Oh, oh. What about, could you, is there any multiplication problem you can start with your sub sandwich to make mine? Two times something would make this (pointing at the 2-part bar).
M: (Sits quietly for a while) two-thirds.
T: You are gonna end up two-thirds. Right?
M: Yeah.
T: That's how long mine is, two-thirds of a foot. You get to start with two feet, multiply two by something to get two-thirds of a foot. Huh?
O: Before, you, Mike, said two-thirds of a half, right? That is not too far off.
T: No, that is not.
O: How many times (inaudible) what is the half of?
M: What is the half of? The half of . . . two feet.
O: What is that to be?
M: One foot (long pause) heh, heh.
T: Heh, heh. That is so hard of a question, isn't it?
M: (inaudible)
T: What would you think to take, to make my sandwich into yours? What do you have to do to two-thirds to make it into two feet?
M: Multiply by three.
T: Oh, multiply by three. That's quite easy. All right. So you are gonna multiply two-thirds times three, you are gonna get two feet. All right, but the other way is kind of hard. Multiply by two to make two-thirds. That's kind a challenging.
M: I'd think divide two into three.
T: Yeah, divided into three, yeah. That's a good idea, and that works. Doesn't it? Yeah, yeah. We are gonna have to think a multiplication problem, some more. Let's try a different one.

Mike constructed the teacher's sub by pulling two middle small parts out of the 2-foot bar consisting of six small parts, which he produced by dividing each part representing one foot into three parts. He verbalized he would take each small part from each foot and confirmed his construction while loudly speaking "one-third" three times as he moved it along the 2-foot bar. However, it is not clear if he was thinking of a third of the 2-foot bar in terms of a third of one foot or whether he got a sense of a third on the basis of a concept of "two out of six." During his construction, he seemed to rely more on a partitive fractional scheme rather than distributive reasoning in that his first response was two-sixths of six-sixths when asked about the size of two small parts he just got. In addition, his saying two-thirds of a foot with respect to two small parts does not seem to imply that he could coordinate two levels of units involved in the problem—two feet and one foot—because he could produce the fraction without thinking of a third of two feet.

Mike tried to produce a multiplicative phrase representing his construction. His first response was "one-third times one-half." He seemed to make it without relating one-third with one-half. That is, each of the fractions was referring to the two-foot sub. There is, of course, a possibility that he would think of a third while reflecting on his dividing each part, one foot, into three parts. However, he never mentioned how he would relate "one-third times one-half" to two small parts. As soon as he realized he created two small parts indicating two-thirds of a foot, he suggested another multiplication "two-thirds times one-half." This response reveals that he began to consider fractions as operations because he seemed to reflect on dividing each foot long bar into three parts and taking two of the parts, which means two-thirds, and then dividing the 2-foot bar into two parts, which means one-half. His comments also indicate that he was implicitly thinking of two levels of units: the 2-foot bar as a fractional whole for one-half and the 1-foot part as another whole for two-thirds; however, his awareness of two levels of units was not explicit because he never mentioned that one small part can be one-sixth as well as one-third.

The teacher encouraged him to produce a multiplication starting with two and ending with two small parts, then led him to realize the two small parts should represent two-thirds of a foot. In addition, the observer helped him to clarify that one foot resulted from one-half of the 2-foot bar. However, he could not figure out what "two-thirds times one-half" he just produced means. On the other hand, it was very easy for him to go to the other direction, which is two-thirds times three equals two. Reflecting on the opposite way, he finally produced the idea of dividing two into three parts. I expect that if the teacher had allowed him to further think about that, he would have said that the 2-small part bar is a third of the 2-foot bar. However, I doubt that he could construct the relationship that a third of the 2-foot bar is equivalent to two-thirds of a 1-foot part because while mentioning the fractions two-thirds and one-half, he never explained how he arrived at the fractions relying on his construction.

Ashley's response 1: Two-thirds of a foot is right but that's not two-thirds of the whole thing (02/16/06).

Ashley was watching Mike engaging in making a bar so that a 2-foot long bar becomes three times as much as the bar. She thought that Mike said he would take two small parts from each part of the bar divided into two parts and that was right. However, when he created only two small parts instead of four and said it is two-thirds of a foot, she commented that his comment was right but his actions were not correct: "two-thirds of a foot is right but that's not two-thirds of the whole thing (pointing at the 2-foot long bar), which is what the beginning problem was about." I asked her to state the original problem posed and she said it correctly. Therefore, based on her comment, it is obvious that she first arrived at the answer two-thirds procedurally, like two divided by three, without any concern about its referent, then referred to the 2-foot bar because she considered the 2-foot bar as a fractional whole. After a while, she realized his taking two small parts was right, but I am not sure how she figured it out.

She seemed to have no concern about the two kinds of units involved in his activity. Through the problem solving sessions, she showed that she was yet to develop a view of fractions as operations. She tended to view fractions as resultant amounts by presupposing a fractional whole. So, it is natural for her that any concerns about referents came later than, not simultaneously with, activity. In addition, since her perspective of fractions as resultant amounts was closely related with part-whole comparisons, she would not need to consider two kinds of units or to deal with two different fractional wholes when given a problem context; rather, she would want to keep one kind of whole to compare the parts she would get. However, she realized there was some confusion about the referents of the answer she procedurally produced, whether it should be two feet or one foot, but seemed not to consider how two feet and one foot were being related when she tried to resolve the confusion. Experiencing such confusion might have helped her to describe Mike's way of thinking. However, it was not reflected in her explanation of his activity: a third of two feet results in two-thirds of one foot, not two feet.

> Protocol A2.2 (1): Ashley's comments on Mike's construction of a bar so that a 2-foot long bar is three times longer than the bar.
>
> A: He divided his 2-foot sub sandwich into two equal pieces, each is a foot long, then divided each foot into three equal pieces and took one piece from left half of it and one piece from right half to make *two-thirds of one foot*.

The realization that Mike should create two-thirds of one foot as an answer affected her way of interpreting his activity. Even though he verbalized "a third" for the result, two small parts, to confirm his construction, she did not pay attention to such an activity and focused on finding the size of the result in feet. In addition, she just described how he produced two small parts without any comment. For her, it seemed most important whether he made an exact amount corresponding to the answer, two-thirds of one foot. So, it was not a goal for her to see how to construct one-third of the 2-foot long bar, which was what the original question was about; rather it was a goal to see how to make two-thirds of a foot, which was a later concern about his construction.

Regarding the last sentence Mike made in Protocol M2.2, "I'd think divide two into three," Ashley did not make any comment on it. Mike considered dividing the 2-foot long bar into three parts as a possible way to solve the problem he was working, producing a multiplication to get two small parts from the bar. At that moment, it would be critical whether he could reason relating the idea of dividing with multiplying. If he was able to do that, his idea of dividing could provide a good ground to consider the 2-foot long bar as a fractional whole when producing a fraction one-third and coordinate it with his answer two-thirds of one foot. If Ashley had concerned about two levels of units involved in the question, his mentioning about dividing would have prompted her to deeply think about the 2-foot long bar as a fractional whole and the relationship of the answer two-thirds with it. Based on her response with respect to the two small parts produced, she was not considering one-third other than two-thirds referring to one-foot.

Ashley's response 2: Saying "of" instead of "times" helped make sense of a multiplication statement (02/16/06).

Ashley was watching Mike engaging in producing a product referring to his construction: taking two small parts, one from each part divided into three small parts. Mike produced *two-thirds times one-half* after he recognized that the size of two small parts in feet is two-thirds of one foot. She elaborated his activity as follows.

> Protocol A2.2 (2): Comments on "two-thirds times one-half" Mike produced in Protocol M2.2.
>
> A: He divided his sandwich (the 2-foot bar) into two equal one-foot pieces so one-third (pointing at one small part in the 2-part bar Mike produced), two-thirds (pointing at the other small part) of this half (three out of the six small parts comprising the 2-foot bar).
>
> I: Oh, two-thirds of this part (a foot part divided into three small parts).
>
> A: A half, which is where he came up with two-thirds times one-half, one-half of the whole.

I: Ah, two-thirds times this one-half (pointing at a foot part). So, that makes two-sixths. How can you change from this (pointing at a foot part in the 2-foot bar) to this (pointing at the 2-foot bar)? When he said two-thirds, he referred to this one (a foot part in the 2-foot bar) but when he said half, half of two feet, he referred to the top long bar (the 2-foot bar).

A: So, he was talking about half of this whole bar (the 2-foot bar).

I: Then he came up with two-sixths. So, two-sixths referred to the top long bar (the 2-foot bar). . . . How can you explain his change of referents?

A: I think the “of,” saying the “of” helped because instead of “times,” because it’s two-thirds “of” this half.

I: Ah, two-thirds of half.

A: He can see that, because this (the segment distinguishing the third and the fourth part in the six small parts) is half of his entire thing. So two-thirds of, visually shows up, if the bar is broken right there (pointing at the segment) he’s got two-thirds of this half (a foot part consisting of three small parts). And two-thirds of a half.

I: would make two-sixths.

A: Two-sixths, and he can see there are pieces out of his entire 2-foot thing, which is broken into six pieces.

The protocol shows that she was reconceptualizing fraction multiplication in a meaningful way by replacing a times-statement with an of-statement. In addition, it shows she was getting used to specifying a referent when verbalizing a fraction without just depending on a measurement unit throughout the context: two-thirds of a half of the entire thing as well as two-thirds of a foot. According to her initial concept of fractions, given a “a fraction times a fraction” problem, she tended to think of both fractions as resultant amounts even though she showed some sense of a fraction as an operation in case of “a fraction times a whole number.” However, by using an of-statement for a fraction product, she seemed to notice that a fraction can be constructed while related with an activity such as dividing and taking or repeating, which is a way of developing a view of fractions as operations.

Developing Initial Concepts of Fractions Through Mike’s Mathematics

A concept of a fraction based on a unit fraction. Ashley watched a video clip where Mike was working on the question, “Can you make three-fourths of that candy bar (a given bar)?” (T: teacher, M: Mike, J: Jenny who is another seventh grader working in pairs with Mike).

> Protocol M2.3: Making ¾ of a given amount and distinguishing it from another amount, three-fourths of which is the given amount.
>
> (When asked to make ¾ of a given bar, Mike produced a 4/3-bar by dividing the given bar into three parts and pulled one part out and repeated it until he got four parts.)
> T: That’s interesting. Can you explain to me what you did?
> M: At first, I divided this one into three pieces and then pulled one out so I moved it down here, added three more to it, which is actually three-fourths because that (the given bar) is three.
> T: Three-thirds?
> M: Yeah, three-thirds.
> T: Uh huh. I see, okay, so let’s see, my original question was to tell the candy bar to make three-fourths of the candy bar. Did you make ¾ of the, of the original (pointing at the given bar divided into three) candy bar?
> M: Oh, oops. I think you said four-thirds.
> T: Okay, I like what you did, but I think different answer, different question. Let’s go ahead and change that.
> M: (Clears the original bar and dividing it into four.)
> T: Don’t erase this (the 4/3-bar) because we are gonna keep there. Okay?
> M: (Moves the 4/3-bar to the left side of the screen from below the original bar. He then produces a 3/4-bar by dividing the original bar into four parts, pulling one part out and repeating it until he got three parts.)
> T: Oh, okay, all right. So, why don’t you explain what you did?
> M: Okay, this time I divided it into four, so that’s (the original bar) 4/4 and I took one off and add two to it. So, that’s three-fourths.

T: Three-fourths, okay, all right, good. This (pointing at the 4/3-bar), the first thing you did is little bit different, right? Do you know what question you solved when you do that one?
M: Yeah.
T: Could you state the problem for that this (the 4/3-bar) is the solution? How do you think about it, Jenny, you can think about the problem where that (the 4/3-bar) was the solution, what would you do the first time?
J: I know one.
T: Do you think you know one? Okay.
M: (Sits quietly without answering.)
T: ¾ was involved (inaudible) but differently.
M: Yeah, it was just switched.
T: Yeah, they switch, uh-huh? How would you, Jenny, what would you say, his, what question did Mike solve when he did this one (the 4/3-bar), the first time?
J: That bar (the original bar) is three-fourths of what's the original bar.
T: Of another bar or the original bar? (Nodding) yeah.
J: (Nodding)
T: Okay, say that one more time?
J: That (the original bar) is three-fourths of that bar (the 4/3-bar) just to make the big bar.
T: Ah, yeah (keeps nodding). Do you see the difference, Mike? (To Mike)
M: (Moves his head left and right.)
T: No, okay, why don't you, what is the problem you solved? (pointing at the original bar and 3/4-bar that Mike secondly produced) Can you say the problem, I mean, second time when you did this?
M: I can't remember what (inaudible).
T: How, what did you make?
M: What I make?
T: Yeah.
M: Three-fourths.
T: Of the candy bar, right? Of the original candy bar. And Jenny, you said the other one when he solved first time.
J: I don't know.
T: But you just said. When he made this solution (the 4/3-bar)
J: That—(after a short while) what?
T: Well, you just said you knew a problem Mike solved (pointing at the 4/3-bar) when he made this one, this bar.
J: That bar up there, one of these (pointing at the original bar and 3/4-bar) is three-fourths, make thirds (inaudible).
T: Uh-huh. In the first case, Mike, this bar (pointing the original bar) was three-fourths of another bar.
M: (Seems unsure.)
T: In the second case, you made three-fourths of that bar (pointing at the original bar).
M: Hold on (pause).
T: Do they seem different?
M: (Sits quietly without answering)
T: Sort of?

M: Yeah.
T: A little bit, maybe. But they are different, right? One time you end up with a shorter bar, one time you end up with a longer bar, right?
M: (Moves the 3/4-bar to the right side of the screen to overlap with the 4/3-bar)

For the question, make three-fourths of a given bar, Mike created four-thirds of it. According to his saying "Added three more to it (one part pulled out), which is actually three-fourths," he seemed to think that the given bar should be three parts compared to four. That is, he seemed to think of a ratio three-to-four between the given bar and a desirable bar. So, he paid attention to the number of parts of the bars rather than producing a fractional amount based on an iterable unit fraction, one-fourth of the given bar. Reflecting on his self-correction of the activity, producing four parts, by "I think you said four-thirds," his activity might be a mistake; however, we need to carefully investigate his activity to see if this kind of mistaking a problem statement was because he just thought of the problem statement differently or he had any other conceptual reason to do that.

After Mike made a correct amount, the 3/4-bar, for the original question, the teacher asked how differently three-fourths was involved in his construction of two bars, the 3/4-bar and the 4/3-bar. He called the 4/3-bar four-thirds but couldn't make any statement explaining a relationship between the original bar and the 4/3-bar while using three-fourths. The original bar was only considered three-thirds, which implies he was considering only one kind of a fractional whole throughout the context. Based on such reasoning, there was no way to think of one part in terms of any other unit fraction than one-third because one part had to be identified relying on the three parts where it was originated from. Therefore, he had no problem with producing a fraction once he defined a whole, but he had difficulty identifying a unit fraction as he constructed a whole. That means he was yet to develop an iterative fractional scheme.

Dividing a bar into four equal parts can be a way to get one-fifth or other unit fraction of another bar as well as one-fourth of the bar. To reason like that, he should develop an insight into distinguishing fractions as operations and as anticipated results. Such an insight is related with a concept of an iterable unit in a generative manner because dividing a quantity into parts results in not just getting one of the parts comprising it but also generating a fraction based on another quantity different from it. Constructing proper fractions given a fractional whole would not require such a differentiation between an operation and a result of operating. So, he had no problem with making three-fourths of a given bar and explaining a relationship between the given bar and the 3/4-bar using a fraction 3/4 because dividing into four parts implied four-fourths so he could produce three-fourths without any concern about a generative perspective of one-fourth.

Ashley's response: Asking how many thirds are in a 4/3-bar would help figure out one part in the bar is one-fourth of itself (02/23/06).

When he made the 4/3-bar by dividing an original bar into three parts and repeating one of them four times, she commented that such confusion might come from his incorrect selection of a number for dividing the given bar. In case that she had to choose a number to partition a given bar, she tended to use her number sense like since 3/4 is less than a whole so she should get a bar less than a given, she has to divide the given bar into four parts and take three of them. Relying on such knowledge of fractions, she attributed Mike's confusion, producing four-thirds of a given amount rather than three-fourths of it, to his incorrect guess involved in deciding whether to use 3 or 4, the numerals comprising the fraction 3/4. She also pointed out even though he could not explain how three-fourths can be used to figure out a relationship between the original bar and the 4/3-bar, he was able to express their relationship using the fraction 4/3 and commented that “he understands this (pointing at the original bar and the 3/4-bar), he understands this (pointing at the original bar and the 4/3-bar) but he didn’t understand all together.” To explain Mike’s difficulty about looking at all three bars together, she paid attention to a figurative aspect of the bars such that they were not lining up but never mentioned about a conceptual aspect. When Jenny, another seventh grader next to Mike, verbalized that the original bar is three-fourths of the 4/3-bar, Ashley confessed that she did not even think about that. She has been thinking of one part in the original bar divided into three parts only as one-third; that evidences her concept of a unit fraction was based on a part-whole comparison and not based on a generative manner.

When asked to pose a question to help Mike figure out how a relationship between the original bar and four-thirds of it can be represented using a fraction 3/4, Ashley said:

> Protocol A2.3: A question to help Mike figure out a relationship between the original bar and four-thirds of it.
>
> A: I might go back to what he said, here (the 4/3-bar of the original bar) about this being four-thirds, and just ask him if this is four-thirds, what, I would line them up and would say then what is this (the original bar) compared to this (the 4/3-bar), this is four-thirds, how many thirds is this (the original bar)? So he would hope go back to what he said three-thirds, the teacher said, three-thirds, and then reverse it, say if this one (the 4/3-bar) is four-thirds of this (the original bar) how much is this one (pointing at one part in the original) of the original, is it of this (the 4/3-bar)? I think he would struggle with that, but I hope that will lead him to three-fourths, this one (the original bar) is three-fourths of this (the 4/3-bar).

Through the above excerpt, we can see that Ashley's concept of fractions was changing from comparing the numbers of parts toward paying attention to a fractional unit to generate a fraction. She seemed to express an idea of comparison by saying "what is this (the original bar) compared to this." However, such her mentioning was based on the number of thirds, not just the number of parts because as she said "how many thirds is this?" she showed her focus was to construct a fraction using thirds. At that point, she was conceptualizing four-thirds as four one-thirds rather than four parts compared to three parts. However, I was wondering about two aspects of her current concept of fractions: 1) Did she consider a fractional unit, one-third or one-fourth, relating it with an activity such that it is one-fourth because it was being produced by dividing into four parts or repeating it four times would make a whole? 2) Was she explicitly aware that one part is one-third of the original bar as well as one-fourth of the 4/3-bar?

A concept of a fraction based on activity. Ashley watched Mike engaging in making two bars, one that is three-fourths of a 12-inch candy bar and the other one, three-fourths of which becomes the 12-inch bar (cf. Protocol M2.4; T: teacher, M: Mike, J: Jenny). In Protocol M2.3, Mike showed a difficulty of producing a fraction while relying on a part-whole relationship. Since such difficulty is related with his lack of an operational view of fractions, the teacher posed a question referring to a whole number context in order to help him reflect on his activity.

Protocol M2.4: Making a bar that is three-fourths of a 12-inch bar.

T: Okay, you guys, think to yourself. I have a candy bar that is three-fourths as long as that one (a 12-inch bar). Think about what it would be. How long mine would be.
J: Three-fourths of that one?
T: Uh-hum. Three-fourths of that one, three-fourths as long.
M: Got it.
T: You got it? Okay.
M: I think I do.
T: Do you think you do, all right? Mike, what do you think?
M: I think there is supposed to, our copy, this one three more times.

T: Oh, okay. So mine is three-fourths as much as that. Who, which is more, mine, which is longer, that candy or mine?
M: More? Right? [His response means the teacher's amount would be more]
T: Yeah. That's twelve inches, right? My candy is three-fourths of the twelve.
M: Oh, yours is only three-fourths?
T: Uh-hum. I have only.
M: I thought that the other way. The longer would be.
T: Oh, you think the other way. Okay. So, mine is three-fourths of that twelve inches.
M: You have a shorter one plus I have it again.
T: Oh, okay. I have a shorter one?
M: Yeah.

To make three-fourths of a 12-bar representing a 12-inch bar, Mike first wanted to repeat the original 12-bar three more times and produce four 12-bars. Only focusing on making one-fourth, he paid attention to creating a bar being made up of four parts and for that, he considered repeating the 12-bar four times. However, it is not clear how he was figuring out the fraction 3/4 he was looking for based on such activity. There is a possibility that he might think of the question differently like the 12-bar is three-fourths of the teacher's amount, but even such possibility does not explain his activity. He never tried to divide the original 12-bar even though he used to divide a solid bar into parts to get a fraction of it. In result, his activity shows that three-fourths has not yet been considered three one-fourths.

Responding to the teacher's continued probing, he realized he did something wrong as he said, "I thought that the other way": he should have produced a shorter bar than the 12-bar. He thought that he created an amount following the other way, which means thinking of the 12-bar as three-fourths of the bar he was supposed to make. However, his construction was not for the other way he was talking. I was wondering what his concept of 3/4 was at that point as well as how he was relating his activity with the fraction 3/4 he was supposed to produce.

Ashley's response: Mike is mixing up how many parts with how many times (02/23/06).

When Mike said he would make a copy of the 12-inch bar three more times to produce three-fourths of it, Ashley commented, "He is doing the reciprocal of it. Instead of three-fourths, he is saying four-thirds, which is what he is showing, or what he is telling her." As soon as she realized he wanted to copy the given bar, not a part of it, three more times, she said, "I didn't know if it's the whole thing."

> Protocol A2.4: Comments on Mike producing four 12-bars for three-fourths of the 12-bar.
>
> A: He is mixing up how many groups he needs to think about, the original in. So, he is thinking of, this (pointing at the 12-inch bar) is one, so he has gotten to do that three more times to make three out of four. … The other time, he knew how to do with three or how to do with four but he is again getting those two mixed up about how many pieces he needs to think about and repeats it how many times.

Ashley interpreted Mike's activity like he would try to figure out a meaning of three-fourths through the number of repetition he needs to make a whole. However, she seemed not concerned with a part to be repeated, a fractional unit for generating a fraction; rather, she paid attention to how he was dealing with the numerals comprising a fraction 3/4 and whether he was able to connect them with the number of groups or the number of times.

In addition, she compared the above problem involving a 12-inch bar with the previous one (Protocol M2.3) in which one solid bar was involved. She was concerned that for three-fourths, he figured out how to use three and four in a solid bar context but seemed to have no idea of three and four as numbers of groups while dealing with 12 parts. In other word, she doubted Mike's ability to construct a composite fractional unit. However, we need to clarify the way she was addressing compared to a scheme to construct a unit fraction of a composite unit. She thought that in terms of the number of groups, the size of a group is determined as a unit, whereas a scheme to construct a composite unit fraction describes that the size of a group determines what fractional unit it would be on the basis of the number of the groups to make up a whole considered.

She anticipated that he would do better if he dealt with manipulatives like what he did with pencils previously. When asked about how many pencils two-fifths of ten pencils are, he divided ten pencils into two equal groups and took two out of each five pencils. I asked Ashley to predict his way of solving to make three-fourths of 12 pencils. She said, "Based on what he did before [in the pencil problem], he would divide [12] into groups of four and then takes three groups of four." Interestingly, the reasoning she was addressing with 12 pencils is not what Mike used for the original pencil problem, distributive reasoning. If she had wanted to use distributive reasoning for 12 pencils, she should have grouped them so that each group has four pencils then taken three from each group. Although she wanted to base her reasoning on distributive reasoning, she mentioned making four groups using 12 pencils and taking three groups out of the four groups. I was wondering if her construction was based on a fractional unit. Her confusion about two different kinds of reasoning leads me to think about a partitive way of thinking compared to a quotitive way of thinking in fractional contexts.

There are two possible ways to solve a problem about making three-fourths of some amount: making four groups vs. considering each group with four elements. Making four groups is considered as a partitive way of thinking in whole number sense. Related to fractional reasoning, the idea of making four groups can be used to develop either a partitive fractional scheme or an iterative fractional scheme. If she has a concern of one group as a composite fractional unit, her partitive reasoning is more likely toward an iterative fractional scheme. If she is focusing on the numerosity of the groups she produced, her partitive reasoning is probably toward a partitive fractional scheme. Compared to making four groups, the idea of each group with four elements is considered as a quotitive way of thinking in whole number sense. In that the idea intends to take three out of each group and collect all the threes, the quotitive reasoning is cooperating with distributive property and a partitive fractional scheme. While interpreting Mike's solution of the ten-pencil problem, she said, "fifth means five." It reveals that she was not seeking for a unit fraction but looking for the amount to use a part-whole concept of fractions, and that led her to employ a partitive reasoning.

Protocol M2.5: Making a bar so that a 12-inch bar is three-fourths of the bar.

T: Now this time, same candy bar, twelve inches, that's three-fourths of the candy Mr. Hope has.
M: He has more or less?
T: Oh, this (pointing at a 12-bar) is three-fourths of his candy bar. So who--
M: Oh, okay, he has longer one.
T: He has longer one? Oh, okay.
M: That was that I did before.
T: Oh, let's see you can make that one.
M: Okay.
T: Let's see you can make Mr. Hope's candy bar.
M: Do you want me to make it?
T: Yeah.
M: (Copies the 12-bar three times so finally there were four 12-bars.) That's his.
T: Oh, that's his! Wow he has a lot of candy. He is really happy to have you give that more candy. Are you sure he has that much candy?
M: Hold on. Three-fourths (moving his cursor over the original 12-bar).
T: This (the 12-bar) is three-fourths of his, right? Twelve inches is three-fourths of what he has.
M: (Moves one of the copied bar beside another copied bar and puts the other bar on the joined bar.)
T: So this is, this (the 12-bar) is three-fourths of what he has, right? Could you tell me what's one-fourth of what he has? Do you know how many inches that would be?
M: (After a short pause) Oh, okay. I have three inches (inaudible) right there.
T: Remember this is three-fourths of what he has, right?
M: That's only three-fourths?

T: Uh-huh. That's only three-fourths, that's not a whole thing. That's just three-fourths of what he has.
M: Oh, four inches.
T: Why would that be?
M: Because you divide twelve into four, I mean, divide twelve into three, and then four.
T: Why don't you pull that out?
M: (Pulls one part in the 12-bar out and repeats it three more times.)
T: Uh-huh. So, how much is that you just made for his?
M: How much is it?
T: Uh-huh.
M: It's three-fourths of the entire candy, twelve inches.
T: Well, this (pointing at the original 12-bar) is three-fourths of his, right?
M: Yeah.
T: So, how much is this (pointing at four parts he just made)?
M: Three-fourths, that's one-fourth.
T: Oh, could you use that then?
M: (Repeats the four parts pulled out three more times.)

As expected through the previous Protocol M2.4, for the question about making a bar so that a 12-inch bar is three-fourths of the bar, Mike created four 12-bars as he said, "That was that I did before." The teacher prompted him to think about what one-fourth of Mr. Hope's bar would be by repeatedly stating the problem statement: the 12 inch bar is just three-fourths of what Mr. Hope has. He finally answered four inches and made it, but interestingly it was identified as three-fourths of the 12-bar. That means that he did not construct one-fourth at his conceptual level as well as three-fourths in terms of one-fourth.

He did not seem to consider three-fourths as a quantity being generated by one-fourth because he never tried to partition an amount given as three-fourths in order to produce one-fourth. In addition, even though he made a 4-inch bar by "12 divide by 3" using the information that the 12-bar is three-fourths of Mr. Hope's bar, the 4-inch bar did not mean either one-fourth or one-third, rather it was still considered as three-fourths. This kind of difficulty is related with a part-whole comparison lacking an explicit concern about a fractional unit. Within such conception, partitioning is more likely considered as a way to locate an amount than to generate a unit amount that can be used for another amount: partitioning does not necessarily mean splitting. Such insight informs that lack of an explicit notion of a fractional unit would prevent him from reversing the operation of iterating. As a result, he was yet to develop a splitting operation as well as an iterative fractional scheme.

Ashley's response: Having the idea of *one group* before he figures out all the others could be helpful (02/23/06).

Ashley watched the process that Mike produced a 16-bar as the bar asked, three-fourths of which becomes a 12-inch bar. He first copied the 12-bar three more times and made four 12-bars but eventually considered a 4-bar as one-fourth of the bar that he was supposed to make and created the 16-bar by repeating it four times.

Protocol A2.5: Comments on Protocol M2.5.

A: He knows he has to do with 3s and 4s, groups of 3s and groups of 4s. But he is not just visualizing how to deal with those groups of 3s and 4s.

I: For him, what's the groups of 3 and what's the groups of 4?

A: I think he was trying to do before, three of these whole things (three copies of the original 12-inch bar) out of the four total (the original bar and three copies of it), that's what he was saying. Three repeats out of four the original.

I: Ah-ha, he kind of, he kind of had an idea of three-fourths as three repeated one-fourth [repeating one-fourth three times].

A: Right.

I: So, he repeated three times to get--

A: To get three-fourths. But,

I: But each one is one-fourth of the total.

A: Right.

. . . .

I: Could you see some connection [of this problem] with the beginning problem [making three-fourths of a given amount]?

A: Yes, he did the same, he had the same basic idea to repeat it three times to get three out of four original.

I: Basically, his idea is based on the “out of,” “groups of.”

A: I think so.

I: Three-fourths is still three out of four, but kind of he understands three means repeats [repeating] three times, but he doesn’t know what he has to repeat.

A: Right. He knows he has to do with three and four, but not sure how many times or what group to repeat that many times.

I: For him, in your explanation, just my understanding of your explanation, maybe he has some problem to find out the fractional unit, one-fourth or one-third.

A: The part they need to be repeated.

I: Yes, the part he needs to repeat.

A: So, instead of asking him a question, straight out, what is three-fourths of something, you might say to help him what’s one-fourth of this, then once he figures that out he may better be able to visualize three-fourths of something.

I: Could you show me how he would do with the problem, could you pretend you are Mike, then click here (edge of the computer monitor) so we can see the bars.

A: (Opens JavaBars.)

I: Make a 12-inch long bar.

A: (Makes a bar and divides it into 12 parts so he produces a 12-bar.)

I: Let's start from here. You can be a teacher, and you can be Mike. Please ask some questions to Mike.

A: To get him to do the right thing or--

I: Just your idea to--, at first what's the question you'd like to ask him first?

A: If it was me teaching, I wanted to get him to figure out the concept of three-fourths of this (the 12-bar), I'd ask him what is the one-fourth of this (the 12-bar). Going along with what he was thinking before, he might say this (the 12-bar) is one-fourth. But I think with little leading he would say, well, this is divided into 12 pieces so I have to take groups of three (clicking the first three parts to make them ready for pulling out). And then pull out (pulling out three parts), this (the 3-bar pulled out) is one-fourth of it.

I: You think he could see this one (the 3-bar pulled out) is one-fourth of this one (the 12-bar).

A: Yes, I think he could get that.

I: How do you think he could get this one (the 3-bar) is one-fourth of this one (the 12-bar)?

A: Because he is already, I think, he is thinking of in a group of 12, this is 12 inches, if you want one-fourth, that means I think he wanted to divide into four groups.

I: Four groups? How would he make four groups? Just pulling out three little pieces?

A: Well, if he counted them 1 2 3, 4 5 6, 7 8 9, 10 11 12, so he could divide, 1 2 3, okay, that makes one group, 4 5 6 another group. I think he can count to figure out the fourth. So, this (the 3 bar) is one-fourth, which happens to be three pieces, one-fourth. And then after we got this point, I would say, okay, this is one-fourth of what? One-fourth of the original, so what would three-fourths of mine be? And he would I hope repeat this (the 3-bar) three times.

I: Yeah, he already knows --

A: This repeating.

I: So, maybe this is the key question (pointing at the 3-bar and the 12-bar) to help him to get.

A: To lead him to a higher fraction. So, any of them I would say him, not just three-fourths one, but if you said this is a stick 45 inches long, I wanna know one-ninth of it or two-ninths of it, you might say divided into ninths, one-ninths, give me one-ninth, and then after he figures out the one-ninth, you could ask him for tow-ninths, or eight-ninths but having the key that one group before he figures out all the others could be helpful.

At the beginning, Ashley focused on how he dealt with the numerals, 3 and 4, comprising 3/4 because she thought that working with three-fourths is a matter of how to manage three and four. She pointed out that Mike had a notion of parts to produce an amount corresponding to three-fourths but was unsure of a unit part and the number of repetitions of it. This insight into his difficulty led her to realize an importance of a unit fraction when constructing a non-unit fraction. This is very important in the process of the development of her knowledge of fractions in that such insight would help her develop a concept of fractions towards a generative perspective from a definite perspective relying on a given whole. However, her description of how to get a unit amount shows that her grouping idea is more based on a quotitive way of thinking rather than multiplicative reasoning that is central to developing an abstracted or iterative unit.

In the last sentence of the above protocol, she introduced the idea of "one group" for figuring out all the fractions that she would produce. This was the first time for her to explicitly express how to generate a fraction using a specific quantity. Previously, her grouping idea in fractional contexts has never functioned to connect it with a fractional unit, rather seemed to solidify a view of fractions as a matter of dealing with two numerals comprising a fraction. For example, for three-fourths, she considered either four groups to take three of them or groups of four to take three out of each group of four then collect the threes taken. However, this protocol shows that she began to use a grouping idea to figure out a unit to be repeated and consider an operation of repeating "one group."

A concept of a fraction based on awareness of the units involved. Ashley watched Mike engaging in making an estimate of a bar, two-fifths of which is a 20-centimeter bar, without looking at the 20-bar. The task would provide a better chance to investigate the operations he was employing while helping avoid a figurative way of thinking (T: teacher, M: Mike).

Protocol M2.6: Making a bar so that a 20-centimeter bar is two-fifths of the bar.

T: That's underneath 20-centimeter bar, that's two-fifths of the bar Jenny has (pointing at the cover, all right? Okay? Again, let's make an estimate, this time Mike, you can make it because Jenny made the other one. Make and draw the bar, that's an estimate, how long Jenny's bar, it's gonna be?
(Mike tries one but erases it.)
T: You can just make one big bar. You don't need to make a perfect one, we just make an estimate. Okay, that's not perfect.
M: (Makes an estimate, which goes beyond the screen.)
T: Oh, okay.
M: I think its overlap.
T: Do you think you made too long?
M: Yes, it's gonna be longer.
T: Longer than the string coming out here?
M: Yes.
T: All right, okay, that's okay. Let's see, how do you know that's gonna be that long?
M: Because this (moving his mouse over the cover) is an only two-fifths of what she has. So that has to be longer.
T: Uh um. I see, so how could we check to see if the estimate is a good estimate?

M: Divide it into five and twenty two, twenty inside each one.
T: Okay, okay, you said divide it (pointing at the estimate Mike just made) into five, and put twenty into each one?
M: Or something.
T: Something? Let's think about it, let's think about it carefully. So, would you do it?
M: (Looks at the screen.)
T: We wanna think about it, how we check if this estimate (pointing her finger at the estimate) is accurate.
J: (inaudible)
T: Well, okay, let's pretend his estimate is here because I realized you guys think Mike's estimate goes on little bit longer. But we will just say, we will estimate it here. What would you do that to check and see it if that was an estimate, that was.
M: (Moves the estimate.)
J: You can check one, what was that again? One-fifth, two-fifths.
T: Okay, you can check one-fifth or, Mike, you said divide into five, right? All right? Why don't you do that so you will see that it will help us?
M: (Divides the estimate into five parts.)
T: Okay.
M: That piece is right there, there is one (moving his mouse over the cover) of those, so that's two, that's hers, no, wait. Okay, that's (moving his mouse only around the first part of the five parts in the estimate) two-fifths.
T: Okay, this is two-fifths (pointing at two parts among five parts in the estimate)?
M: No, I want to divide this one (the first part of the estimate) into two (pretending to subdivide each part in the estimate into two parts).
T: Okay, let's see. Her (pause), how long is this bar (a bar underneath the cover)?
M: Two-fifths.
T: There was two-fifths. How many centimeters?
M: Twenty two.
T: Twenty.
M: Yes.
T: Okay, all right? Twenty centimeters. Yeah, so twenty centimeters is two-fifths of Jenny's bar, all right. So, (long pause) tell me again how you use this to check (pointing at the estimate).
M: (24-second pause) (Pretends to divide each part into two parts.)
T: Do you think it's a lot more centimeters or a lot less centimeters than twenty?
J: More.
T: More! Okay. You guys are saying a lot more. You say still a lot more, Mike because you think it goes maybe off the screen.
M: I think I divided it, here (moving the mouse over the middle of the first part of the estimate divided into five parts).
T: Okay. Here is divided?
M: (Dials to 2 and clicks the first part of the estimate, so the first part was divided into two parts.) and they go to two-fifths (moving his cursor only around the first part among the five parts in the estimate).
T: Oh, wait a minute. I thought this was, this (pointing at the estimate) was her whole bar, right?

M: Yeah.
T: So, what would be two-fifths of her bar?
M: (Clears the estimate and divides it into five parts again to cancel his mess-up.)

This episode shows that for figuring out a fraction 2/5, Mike was heavily relying on a partitive fractional scheme without a notion of an iterative fractional unit. When asked to make an estimate of a bar, two-fifths of which is a given amount, without looking at it, he first drew a big bar, divided it into five parts and tried to have each part corresponded to the given amount. Such activity encourages me to deeply think about two important things related to constructing a fraction: a meaning of partitioning and a fractional unit. Regarding partitioning, there are two ways for it to contribute to constructing a fraction. When partitioning only purposes to collect parts needed, it is more likely to lead a view of fractions as resultant amounts being collected. It also seems to reinforce a perception that partitioning a whole amount is reasonable whereas partitioning an amount resulted from partitioning is unreasonable because partitioning would be supposed to result in a set of parts or a part in a collective sense rather than parts themselves. Compared to the above view of partitioning, there is another case: partitioning implies repeating (we call it splitting), where each part resulting from partitioning is considered as what can be used to create a fraction.

Regarding a fractional unit, there are two possible ways to have the notion of it: one from a partitive fractional scheme and the other from an iterative fractional scheme. Let’s think about the first case, having a notion of a fractional unit from a partitive fractional scheme. In this case, a fractional unit is perceived only through a notion of “one part” out of the parts that comprises a whole. That means one part can be considered only if there are parts making up a whole, so this view would make a trouble with a hypothetical whole because it requires an opposite way of thinking, creating a whole from a part. There is another case that a student considers a fractional unit with a concern about iterating it to make a whole. This case opens a possibility to generate a fraction on the basis of a fractional unit because this view does not restrict its limit to a whole.

According to his way of checking to see if the estimate produced is accurate, he never tried to partition the 20-centimeter bar, which should be two-fifths of the estimate. That shows he used partitioning as a means of collecting, not a way of generating. In addition, he divided the estimate into five parts and associated the 20-centimeter bar to each part. That implies his conception of a fractional unit was based on a partitive fractional scheme because for Mike, one part to generate a fraction seemed to have to come from the parts making up of a whole so he needed five parts. In result, for a fractional unit, his view of partitioning as a way of creating a set of parts cooperated with a partitive fractional scheme, and it led him confused.

Ashley's response: Mike was thinking of a 20-centimeter bar in terms of two-fifths, but not 20 one-centimeters (03/02/06).

Ashley was watching Mike trying to match a 20-centimeter bar, a 20-bar, with each of the five parts comprising the estimate he produced. As she paid attention to how he dealt with the 20-bar for the question, making a bar so that the 20-centimeter bar is two-fifths of the bar, she commented as follows.

> Protocol A2.6: Ashley's comments on Mike's dealing with a 20-bar as two-fifths of some amount.
>
> A: He was thinking in terms of this (the 20-bar) is two-fifths of the girl's bar, so he was not thinking about the small pieces at all until (inaudible) . . . . He was not thinking of this bar (the 20-bar) in terms of the twenty. He was thinking of it is two-fifths.

She pointed out that Mike was considering a given amount only in terms of a fractional whole, not based on a measurement unit, centimeters. He had no notion of a centimeter unit even though the 20-bar consists of twenty parts. That is, she noticed there were two kinds of units considered in the problem context but he focused only on one kind of unit that two-fifths is referring to. She then elaborated Mike's difficulty as she compared his perception of the 20-bar with Jenny's as follows:

Protocol A2.6: (Cont.)

A: She (Jenny) took the 20 centimeters and the idea is that the 20 centimeters is two-fifths. She used both of those amounts. She was thinking of it in terms of what can I do with the 20 centimeters to make it two-fifths, so I can use individual centimeters . . . . She said this is 20 centimeters and if I divided it half each fifth is 10 centimeters . . . . He was not, not like that, he was still thinking about that this (the 20-bar) is one piece rather than 20 pieces.

Ashley indicated that unlike Jenny, Mike could not coordinate two kinds of units, a centimeter unit and a fractional whole he was looking for. Such indication raises two concerns: concerns about constructing a composite fractional unit and coordinating two kinds of units. Based on the above comment, she was thinking that thinking of two-fifths based on 20 parts affects considering one-fifth using two-fifths through centimeter units. That is, an ability to coordinate a fractional part with a measurement unit would influence making a composite fractional unit. This is very critical in her developing a concept of units in fraction contexts in that she became explicitly aware of the importance of coordinating various units involved.

Summary

Ashley's watching Mike engage in fraction problems provided many opportunities to reconsider what she has taken for granted related to fractional reasoning. First, thinking reversibly should not be taken for granted. She observed “What number multiplies seven inches by to make seven-thirds?” was not the same for Mike as “What would you multiply mine (two bars and one-third of one bar) by to make yours (7 bars)?” [cf. Protocol M2.1]. Mike quickly answered the latter question correctly. Since she thought those two questions were same, she anticipated he would have no problem with the former question, but he had difficulty answering it. Second, constructing improper fractions requires more than procedural or perceptual understanding of converting mixed numbers to improper fractions. Through her observation of Mike, she doubted his ability to relate seven-thirds with seven. He knew that two and one-third is one-third of seven pictorially, but he did not know seven-thirds is one-third of seven. Related to Mike’s difficulty conceptualizing seven-thirds, she did not pay attention to how seven 1/3-parts of one bar relate to seven bars. She focused on how he produced a 7/3-bar representing seven-thirds—breaking two bars into three parts each and joining one 1/3-part to them, thereby confirming that he was able to create the 7/3-bar using two bars and one-third of one bar. Therefore, her doubt remained unresolved with no further comment on his difficulty with the former question, “What would you multiply seven by to make seven-thirds?” As a result, she had yet to develop a units-coordinating scheme in fractional contexts.

As she watched Mike’s way of thinking, Ashley revealed some lack of understanding of fractions. She observed Mike produced two 1/3-parts by dividing each inch into three parts and pulling two parts in response to a problem, “Make a sandwich so that a 2-foot long sub sandwich is three times longer than it.” At that time, she was confused with a fractional whole. Ashley said “[The fact that the two 1/3-parts is] two-thirds of a foot is right but that’s not two-thirds of the whole thing (pointing at the 2-foot long bar), which is what the beginning problem was about” [cf. the text immediately following Ashley’s response to Protocol M2.2]. She must have arrived at two-thirds procedurally, like two divided by three, and then looked for its referent. Her answer two-thirds did not involve any concern about two levels of units and coordinating them. She seemed not to consider how two feet and one foot related to each other when she tried to resolve the confusion. In addition, she did not attend Mike's activity to produce the two 1/3-parts from the 2-foot bar. Her interpretation of his construction of two 1/3-parts shows her goal was to make two-thirds of one foot rather than one-third of the 2-foot long bar: "He . . . took one piece from the left half of it (the 2-foot bar) and one piece from the right half to make two-thirds of one foot" [cf. Protocol A.2.2 (1)]. She even disregarded Mike's saying one-third for the two 1/3-parts. As a result, she did not consider two-thirds of one foot with respect to two feet.

Ashley tended to consider fractions as resultant amounts, so she would not have needed to consider various levels of units in a fractional context. By using an of-statement for a fraction multiplication, she specified referents of fractions as she interpreted Mike's activities: "two-thirds of a half of the entire thing as well as two-thirds of a foot." Her awareness of referents is critical in her fractional reasoning in that it would help her notice various units involved in a fractional context and further develop a view of fractions as operations. However, her concept of fractions was still heavily based on a part-whole comparison and a partitive fractional scheme. So, to create a unit fraction, she intended to find the numerosity of parts comprising a given or presupposed fractional whole rather than create a whole that is generated by a part that she was considering. Given an unpartitioned bar, she had no problem with producing three-fourths or four-thirds of it, but she had difficulty creating a bar so that the unpartitioned bar is three-fourths of the bar because she thought one part produced by dividing a bar into three parts must imply one-third, one out of three, and the part could not be any fraction other than one-third.

As she interpreted Mike's way of thinking and engaged the questions to probe her conceptual development, Ashley showed some progress in her initial concepts of fractions. First, when producing a fraction, she focused on the number of a unit fraction rather than the number of parts in her perceptual field: four-thirds was considered as four one-thirds rather than four parts compared to three parts. In addition, by noticing the importance of a unit part and the number of repetitions of it, she developed a concept of fractions towards a generative perspective from a definite perspective relying on a given whole. In particular, she began to use a grouping idea to figure out a unit to be repeated and consider an operation of repeating "one group." Her grouping idea in fractional contexts previously had been used to solidify a view of fractions as a matter of dealing with two numerals comprising a fraction.

Second, Ashley realized the importance of various levels of units involved in a fractional context. She watched Mike construct an estimate two-fifths of which is a 20-centimeter bar. To check to see if the estimate is accurate, he made a bar, divided it into five parts, and matched the 20-centimeter bar with each part. Related to his construction, she pointed out Mike considered the given amount only in terms of a fractional whole, not based on a measurement unit, centimeters. Thus, even though the 20-centimeter bar consists of twenty parts, he had no notion of a centimeter unit. Her insight into two levels of units indicates Mike's inability to coordinate them: he focused only on one kind of unit that two-fifths is referring to.

CHAPTER 5

ASHLEY'S CONSTRUCTION OF FRACTIONAL SCHEMES

Some Changes in Ashley's Initial Concepts of Fractions Through Engaging in a Constructivist Learning Environment and Mike's Mathematics (Day 6-8)

Developing a Concept of a Unit Fraction

Through two problem solving sessions and three video watching sessions, Ashley developed a concept of fractions in terms of a unit fraction as well as a part-whole comparison. So, for her, three-eighths did not just mean three out of eight parts constituting a whole, but it also implied three one-eighths. In addition, as she related an insight into a unit fraction with her experience of Mike's difficulty coordinating two kinds of units—a measurement unit and a fractional whole—she became concerned with the various units involved in a problem context.

For example, while trying to create three-eighths of a 24-centimeter bar, she reasoned about the fraction three-eighths like "if you divide the big bar (a 24-bar) into eighths, each one is (inaudible) three, so one-eighth, two-eighths, and three-eighths (as she moves her cursor over the three parts) " and "I divided the whole 24-bar into eighths so that's eight groups of three centimeters, so if I wanted three-eighths, I did one group of three, two groups of three and three groups of three." Her saying "dividing the big bar into eighths" shows that she was considering a dividing activity as a way of creating a unit fraction rather than getting the parts to correspond to the numerals comprising a fraction. Also, by mentioning that dividing into eighths resulted in eight groups of three parts each, she connected a grouping idea with a unit fraction and bridged the idea with the concern of two kinds of units: three parts becomes one-eighth.

However, it was challenging for her to expand a grouping idea based on the number of groups comprising a whole toward generating a whole using the group. It indicates a difficulty in constructing an iterative fractional scheme from a partitive fractional scheme. The following protocol shows her struggle with a concept of a unit fraction in a generative manner (I: interviewer, A: Ashley) (03/09/06).

Protocol 3.1: Using the construction of a 9-bar as three-eighths of a 24-bar, finding a fraction for the 24-bar in terms of the 9-bar.

I: How much is your candy bar (a 24-bar) of mine (a 9-bar)?

A: (After a short pause) yours is three-eighths of mine, so mine is eight-thirds.

I: How did you get it?

A: (After a short pause) Well, this (the 9-bar) is, I was thinking of this the (9-bar), well yours, which is three-eighths, if I repeated your whole bar three times it would make mine.

I: Ah, this one (the 9-bar)?

A: This (pointing at the 9-bar) is a whole bar.

I: This whole bar (the 9-bar), if you repeat three times, then you are gonna get yours, is it right?

A: I think-- no, no, it’s gonna be too big. Three, eight-thirds. I know that’s the answer but no, I said there, I was incorrect. (After a short while) Eight-thirds, I know.

I: And then, when you made this 9-centimeter whole candy bar, you just repeated three times. What did you repeat? [I tried to help her recall what she did to get the 9-bar]

A: One-eighth.

I: One-eighth.

A: Three times.

I: Could you color [the parts for one-eighth]?

A: (Colors three parts light blue.)

I: I think you just repeated this much (pointing at the colored parts)?

A: Yes, because that's one-eighth, three times.

I: One-eighth, three times, you repeated and then you could get this [9-bar].

A: Area.

I: Yeah, three-eighths.

A: Three-eighths.

I: How many this three, this (pointing at three colored parts) is one-eighth of the 24-inch long candy bar.

A: There are eight of those.

I: Yeah, there are eight of those. So, is that helpful for you to elaborate how much this one (the 24-bar) is of this one (the 9-bar)?

A: Yes, because this (three colored parts) is one-third, and there are eight groups of three in here (pointing at the 24-bar) so they have to be, there has to be eight of the one-third, so eight-thirds.

Prior to the above episode, Ashley was asked to construct three-eighths of a 24-bar and produced a 9-bar. Then, she was engaged in a question to have her think about reversing the way of the construction: "How much is the 24-bar of the 9-bar?" Regarding the question, she noticed she needed to make the 24-bar using the 9-bar but did not figure out how it would be used. She had used 3 parts when producing the 9-bar, but to answer the question she did not consider them, so she never tried to divide the 9-bar to make 3 parts. Rather, she intended to measure the 24-bar using the 9-bar as a divisor. She wanted to use one of the strategies she usually employs when producing a fraction: a measuring out strategy, which is a way to find a relationship between two quantities by measuring a larger amount using a smaller amount. However, she did not apply it successfully to the question. Using the strategy, she happened to produce a mixed number by first examining how many times a smaller amount can be contained in a larger amount then using a part-whole comparison between the left over and the smaller amount. However, this time she unfortunately gave up using the strategy before figuring out the question by using it.

In order to answer the question, “How much is the 24-bar of the 9-bar?” she first tried to use knowledge of reciprocals she procedurally formulated so she could make a quick answer, eight-thirds. However, she did not find a way to explain how to produce eight-thirds in terms of an activity. One of the possibilities causing such difficulty would be her concept of a whole. For her, it was a whole that should be partitioned, and the 24-bar was only a whole. Thus, she would neither attempt to partition the 9-bar to make the 24-bar nor try to reverse the operations she performed to produce three-eighths. When I tried to remind her of a repeating activity using the 3-part, she realized she was, at that time, paying attention to finding one-eighth, which consists of three parts among 24 total, and considering three-eighths as three one-eighths. The awareness of the 3-part as a composite fractional unit prompted her to think about them based on the 9-bar, a third of it. She finally arrived at eight-thirds using the unit as she said, “this (the 3-bar, three colored parts) is one-third, and there are eight groups of three in here (pointing at the 24-bar) so they have to be, there has to be eight of the one-third, so eight-thirds.”

It is very interesting that she operated in a similar way to how Mike operated in the previous session she observed on February 23, even though she had pointed out his not trying to divide a 12-bar to find three-fourths of it. At that time, she identified his difficulty as lack of a concept of partitioning a 12-bar into parts for the purpose of repeating a part to make a whole. According to my analysis of the session, however, her comment on finding a part for repeating was not related to iterating a part. She said that Mike's difficulty constructing three-fourths of a 12-bar was because he did not have any sense of a part comprising the 12-bar. But the part she was talking about implied one of the groups corresponding to the denominator in the notation "3/4" rather than to generating an amount based on it. Such a conceptual status of a unit fraction affected her struggle with the question asked in Protocol 3.1. In other words, until she developed three-eighths as three one-eighths with a generative perspective of a unit fraction, she would have had trouble establishing that one-eighth of the 24-bar would be a third of the 9-bar she produced by finding three-eighths of the 24-bar.

Developing Distributive Reasoning Through a Grouping Idea

Given a division situation, she distinguished a doling out strategy from a measuring out one. When asked to share a 36-centimeter sub sandwich (a 36-bar) among four people, she first wondered "Do you want me to say how I would do it or like it, I think students would do it or may do it?" then elaborated her way of thinking of the question while contrasting it with one that her students would do it: "I'd count off groups of four . . . but I think students would dole it out, 1,2,3,4, 2,2,3,4, 3,2,3,4, like that." By the time she clearly expressed two different ways of division, she had been able to construct a unit fraction of a composite unit as follows (03/09/06):

Protocol 3.2: Creating one-fourth of a 36-bar.

I: How much is this (a 9-bar) of this one (a 36-bar)?

A: One-fourth.

I: One-fourth, how do you know that this (the 9-bar) is one-fourth of this one (the 36-bar)?

A: Because four of this (the 9-bar) together would create one whole bar if this is repeated four times.

Extending the idea of doling out to a fractional context was very challenging to her because distributive partitioning is necessary for doling out in fractional contexts. It also requires conceptualizing two levels of units, but she tended to consider partitioning only based on a given fractional whole. For example, when asked to divide a 5-centimeter bar into six parts, she said, “if it was the case, a 5-centimeter bar and you wanna divide it into six, you can divide it into half, and then each half into thirds, so you might not have the unit amount but you might have the pieces. That’s what I would think of it.” Based on my observation, it is evident that she did not use centimeter units in the 5-centimeter bar in her dividing activity. Previously, there were some incidents that encouraged Ashley to think about two kinds of units–a measurement unit and a fractional whole (Mar. 2nd), but at that time, she never explained how to coordinate them even though she was concerned about Mike’s difficulty with two kinds of units. The following protocol shows how she began to construct a units-coordinating scheme while relating it with a doling out strategy she already developed, thereby developing distributive reasoning (03/09/06).

Protocol 3.3: Sharing a 3-foot sub sandwich among five people.

I: Share this (a 3-bar representing a 3-foot sub sandwich) among five people. Could you make a share for one person?

A: Yes (laughing and staring at the 3-bar for a moment). I know I need to divide it (the 3-bar) into fifteen pieces, but I try to figure out why or how to show that, or why that I know that (pause). Because that’s the common denominator (laughing) but I am trying, you know once I get to explain that--, well, again, if these (two segments to represent the 3-bar being 3 feet) wouldn't be there, I would just divide it into five equal pieces and go from there, um, but it's gotta be fifteen. I wanna divide it, let's see if this works in my head, I wanna divide this sub between five people, so I am gonna take each third and divide it into five pieces, and that makes fifteen total.

I: When you say this a third (pointing at one part in the 3-bar), each a third divided into five, when you said like that, in your mind, just [did] you think of the common denominator, fifteen or?

A: Well, I think the explanation helps, I mean I got, because I knew fifteen was the common denominator, that's how I arrived at that but it makes sense to me now because there is one-third, since I have to divide each third, all three-thirds between five people, I might as well divide each third into five pieces so I can, one two three four five, one two [three four five].

I: Ah, yeah. So, your way, your idea is not just to make fifteen, a common denominator.

A: That's what I started with, (laughing) so, yeah, I take. How do I do those straight up and down only in the section?

I: Yeah, you can.

A: (Dials to five and divides each part in the 3-bar into five smaller parts. So, the 3-bar was converted to a 15-bar.)

I: Could you make a share for one person?

A: Un-hum, it was formerly divided into three pieces, and I wanna give it, I wanna make it between five people, okay. So, I would say (pulling one small part out and repeating it two more times so she produced a small 3-bar consisting of three small parts).

I: You just repeated it three times, how do you know you need only three pieces?

A: Because the idea that I would take one piece like we are talking about before dole it out, one two three four five, one two three four five, one two three four five, then I am done because I have only three groups of five.

I: So each little piece is coming from, came from (indicating five small parts using two fingers and moving them along the 15-bar).

A: This group, this group, this group (moving her cursor along the 15-bar).

I: How much is this (the small 3-bar) of this one (the 15-bar)?

A: It's one-fifth.

I: Yeah, it's one-fifth of this one (the 15-bar).

A: Yes because this little piece is one-fifteenth.

I: So, in terms of one-fifteenth, how can you name this one (the small 3-bar)?

A: Three-fifteenths.

I: So, three-fifteenths is same as one-fifth?

A: One-fifth, yes.

I: Please tell me why the equivalent relationship [is true].

A: Because we would divide it between 5 people and so 3 out of the 15 make a share for one person, which is one-fifth of the sandwich. So, three-fifteenths is equivalent or equals to one-fifth of the sandwich.

I: To check this (the small 3-bar) is one-fifth of this one (the15-bar), what could you do with this one (the small 3-bar)?

A: What could I?

I: How could you—

A: I could repeat this (the small 3-bar) five times, then it would be the same length (pointing at the 15-bar) as the original.

I: Repeating it five times is gonna make this one (the 15-bar), and that means this one (the small 3-bar) is one-fifth?

A: (After a short pause, she nods.) Yes (seemingly unconfident).

Ashley noticed she needed fifteen parts to get a share for one person using a procedure for finding a common denominator. Her mentioning "common denominator" rather than "common multiplier" implies she was simultaneously thinking about multiplying two numbers, three and five, and getting smaller parts than a given amount; partitioning was implicitly involved in her mind. It is very interesting that she mentioned "denominator" for fifteen even though she seemed not to think of one-third or one-fifth at that moment.

Keeping her attention to how to make fifteen parts, she coordinated two levels of units, but constructing three levels of units through the coordination was not explicit. She thought if the 3-bar was unpartitioned she would divide it into five parts because there were five people to share the sub sandwich (the 3-bar). At that moment, she was considering only one kind of unit, the 3-foot bar, as a fractional whole for partitioning. She thought of partitioning the unpartitioned whole bar into three parts and, separately, partitioning it into five parts. She then realized dividing each of three parts into five would make fifteen small parts. It was here that she began to coordinate the two partitioning activities in that she began to consider one-foot part other than the 3-foot bar for partitioning. She then completed the partitioning and used a doling out strategy to produce three small parts for a share for one person. She reasoned that three small parts results in one-fifth of the 3-foot sub and produced an equivalent relationship between three-fifteenths and one-fifth: three out of fifteen parts is a share for one person and there are five people to share the given amount. However, her coordination of two levels of units was yet to lead her to reason with three levels of units because she seemed unaware of one small part as one-fifth of one foot or one-fifth of one-third of the 3-foot bar. In other words, it was unclear that she was considering one-fifth in the question asked when dividing each foot into five parts. Rather, relying on a result of her partitioning activity which led to 15 small parts, she seemed to perceive each small part only in terms of one of the fifteen parts comprising the bar she started with.

Ashley implemented distributive property while elaborating three small parts becomes one-fifth of the 3-foot bar but did not seem to complete distributive reasoning for fractions. Distributive reasoning requires constructing three levels of units on the basis of units-coordination between a total quantity and each quantity considered for distribution. In addition, an ability to construct a unit fraction of a composite unit is necessary to complete distributive reasoning. Let's consider her difficulty in completing distributive reasoning along the above two aspects of it. Related to the construction of three levels of units, I already talked that she worked at both levels of units considered, a unit of one foot and a unit of 3 feet, but it was not explicit whether she was able to construct one-fifth of one foot while considering a fraction one-fifth of 3 feet being asked. Regarding constructing a unit fraction of a composite unit, there is evidence that she was able to construct a composite fractional unit within whole number contexts [cf. Protocol 3.2]. However, fractional contexts are different from whole number contexts in that partitioning is involved for creating a composite fractional unit. Such need of partitioning might cause her to conceive of a unit fraction only relying on partitive reasoning without any concern of its iterative aspect like what she did in Protocol 3.3.

Protocol 3.3: (Cont.)

I: This (the 15-bar) is a 3-foot long sub sandwich, so 3 divided by 5 represents this (pointing at the bars on the screen, the small 3-bar and the 15-bar), do you think so? [I wanted her to recall the problem she was engaged in right before; she elaborated her partitive approach to division compared to a quotitive way of thinking.]

A: (Laughing) yeah (seemingly unconfident; after a short pause) Yes. By the same idea, I mean I see what you are saying.

I: What is 3 divided by 5?

A: (inaudible) three-fifths.

I: 3 divided by 5 is three-fifths. How much is this one (the small 3-bar)?

A: One-fifth.

I: One-fifth of this one (the 15-bar). How much is it in inches?

A: Oh, (pause) 3.

I: Three?

A: (After a short pause) do you want it in inches?

I: This one (the 15-bar) is three inches, not one inch. No, three feet, this is 3 feet, sorry.

A: Okay, so how many feet is in, okay, three feet. Um. Well--, okay, so this is saying that, I see what you are saying because this, one two three four five of them (counting the small parts in 15-bar), this is one foot, divided into five equal pieces, so that's, each one of those is one-fifth of a foot. So this (one small part) is one-fifth plus another one-fifth (pointing at the second small part in the small 3-bar) plus another one-fifth so this (the small 3-bar) is 3/5 of a foot.

I: Yeah, 3/5 of a foot. But this is, how much is this (the small 3-bar) of 3 feet?

A: (Sits quietly without answering.)

I: How much is this (the small 3-bar) of 3 feet?

A: (pause)

I: You just told me.

A: (Laughing) (long pause)

I: Yeah, this (the small 3-bar) is 3/5 of one foot.

A: Yes.

I: Right. But this one (the small 3-bar) is something of,

A: Of the whole.

I: Yeah, of the whole, how much is it?

A: Nine-fifths, no, (pause). No, (pause) I don't know.

I: How many times do you have to repeat to get this one (the 15-bar)?

A: Five.

I: Yeah, so how much is this (the small 3-bar) of this one (the 15-bar)?

A: One-fifth.

I: Yeah, one-fifth, one-fifth of?

A: Of the whole.

I: So, one-fifth of three equals?

A: Three (laughing) three-fifths.

I probed how Ashley related two kinds of units. She seemed not to understand that a 3-foot sub shared among five people can be thought of as division, 3 divided by 5. Even if she had considered division for the sharing problem, she might have hesitated to accept the division proposed as another way to think about the problem because she noticed that 3 divided by 5 would not give her one-fifth that she had already produced for a share for one person. Responding to the size of three small parts, she first considered them only as one-fifth referring to the 3-foot bar. But when asked about it in feet, she realized one small part was produced by dividing one foot into five parts, so it represents one-fifth of a foot and three of them makes three-fifths of one foot. However, interestingly, she could not correlate this with her previous answer, one-fifth of the whole, after figuring out three small parts indicates three-fifths of one foot. This indicates that she was not able to reason with three levels of units.

There are two critical conceptions for going beyond two levels of units, that is, toward three levels of units. One is considering three small parts as three-fifths of one foot as well as one-fifth of three feet; the other one is conceptual understanding of their equivalency. Ashley did not have any trouble in considering three small parts in terms of one foot and, separately, three feet. However, since she was not explicitly aware that she dealt with three kinds of units for making a bar that is one-fifth of a 3-foot bar, she was so confused when she produced two different fractions for the bar. In other words, she was yet to construct three levels of units at her conceptual level. Such lack of reasoning with three levels of units prevented her from relating two fractions produced and figuring out an equivalent relationship between them. While two levels of units requires her to think of a fraction based on a unit where she operated, three levels of units requests to consider a fraction referring to a unit of three feet in terms of a unit of one foot. That implies an iterative fractional scheme is necessary for constructing three levels of units.

Let's closely look at how she produced a fraction one-fifth as three small parts. Reflecting on her saying "because this little piece (one part in the 15-bar) is one-fifteenth" and "[three of them is] three-fifteenths," we can see that she produced one-fifth by means of producing three-fifteenths. That means she based the collected amount as a fractional part of the 3-foot bar: she was working on a unit of three feet. However, I infer that at that time she was not explicitly aware of a fractional whole because later she could not answer the question "how much is this (the small 3-bar) of three feet?" which specifically asked about the fractional whole she had been referring to. When asked about the small 3-bar in feet, she reflected on how she produced small parts and verbalized one small part is one-fifth of one foot. That means she based one small part on one foot, a unit comprising the fractional whole she started with, and was working on a unit of one foot, which is another level of the units involved. At that point, she shifted her referent from the 3-foot bar to one foot part. However, since she maintained her initial concept of a whole which is conceiving of a whole in an absolute manner, with respect to three small parts she produced one-fifth in terms of three-fifteenths while disregarding three levels of units. That is, she was yet to develop a concept of a whole based on an iterative aspect of a unit fraction, so she tended to remain one kind of a whole throughout the problem contexts. Such a tendency is consistent with her lack of an ability to construct an iterable unit fraction, and she had further difficulty reasoning with three levels of units.

An Operational View of Fractions as a Part of a Fractional Multiplication Scheme

By implementing dividing a unit into seven parts and repeating one of the parts four times I mean considering four-sevenths as an operation. Such an interiorized action for a fraction is essential to understanding an operational view of fractions, but the operational view of fractions I attempt to define is more than that. An operational view of fractions is defined as a scheme involved in differentiating fractions as operations from anticipated results of the operations. A view of fraction as an operation, the splitting operation, and establishing a multiplicative fractional relationship between two quantities constitute the scheme. In this section, I investigate situations where an ability to differentiate fractions as operations from results becomes crucial.

In the following Protocol 3.4, Ashley began to distinguish an operational meaning of fractions and the meaning of fractions as results of operating. Such a distinction is critical because, as the distinction becomes explicit, it is more likely for her to take the fractional operations she carries out as a situation for other fractional operations. In the following protocol, by using the operations she carried out as a new situation, she activated a recursive scheme and engaged in distributive reasoning. In addition, by using an “of-statement” to elaborate her operating, she started relating fractions with her activity and it provided her with an opportunity to consider simultaneously and distinctively two perspectives of fractions. In Protocol 3.4, “one-seventh” was considered as an operation to find one-seventh of one-third as well as a result of her operating. Further, such a conception of one-seventh was used to produce a result of taking one-seventh of one pound by operating recursively and distributiveley (03/16/06).

Protocol 3.4: Making one-seventh of one pound using one-third of one pound.

(Ashley drew a bar. Since the bar represents a third of one pound, I will call it a 1/3-bar.)

I: Let’s suppose this (the 1/3-bar) is a third of one pound. Using this bar, could you make a seventh of a one-pound bar?

A: A seventh of one pound? This is a third of a pound. (She dials up to seven, stares at the screen for a long time and divides the 1/3-bar into seven parts. She then pulls one part out of the bar divided into seven parts and repeats it three times. Thus, she produces a 1/7-bar).

I: How would you know this is a seventh?

A: Well, I know that one-seventh is smaller than one-third, just conceptually. And this (the 1/3-bar) is one-third and you wanted one-seventh so I broke the third into seven pieces and I took one of those seven pieces right here, well, that is one-seventh of one-third, so I did that two more times because this whole thing (the 1/3-bar) was one-third so again this whole bar again would make two-thirds in my whole bar, again will make three-thirds. And I took one- seventh from each third.

I: I think that's one way to check that this is, these little pieces are a seventh of one pound.

A: A seventh of one pound, yes.

I: Is there another way to check?

A: Well, how to check if I was right?

I: Yeah.

A: Well, this, my original bar three times should equal one pound. So I can repeat my bar three times and then this (the 1/7-bar), my new bar, one-seventh seven times should equal one pound. So far, I repeated this (the 1/7-bar) seven times, it should be the same length as this (the 1/3-bar), three times.

When asked to make one-seventh of one pound using one-third of a pound, Ashley first commented "I know that one-seventh is smaller than one-third." Then, this time, instead of focusing on a certain amount for a specific fraction, she reasoned it out.

Ashley said "a seventh of a third" with respect to one of the parts that resulted from dividing a given amount, one-third of one pound, into seven parts. Such a description is very important because it indicates she began to consider her partitioning activity as a way of creating a unit fraction. However, I wonder how deeply she was relating "a seventh" from the dividing activity with “a seventh” of the whole bar at a conceptual level. After describing one part as one-seventh of one-third, she repeated it three times while saying she would consider three of one-third for completing her construction. This indicates that she produced one-seventh of one pound based on a view of fractions as operations because she recognized the given context as the situation that requires partitioning and iterating and used fractional language based on the implemented operations. In addition, her way of construction of one-seventh of one pound indicates that considering fractions as operations promoted recursive reasoning because if she had perceived the given bar without a notion of one-third as an operation, she would have needed the fractional whole present before starting her construction and collected three of the “1/7s of a third of one pound.” Furthermore, the fact that she used the given bar as indicating the other thirds means that she conceived of one-third in terms of two perspectives of fractions: one-third as an operation and as a result of activity. It also means that the 1/3-bar was identical to the other thirds of the bar. So, one-seventh of the given 1/3-bar could be distributed over the other parts through the operation of “a seventh of a third.” That is, considering fractions as operations was interconnected with distributive reasoning.

Since I am interested in how conceiving of a fraction as an operations is related to establishing a multiplicative relationship between two fractions, I now turn to investigating whether Ashley was aware of the relationship between the 1/3-bar and the 1/7-bar.

Protocol 3.4: (Cont.)

I: How much of this one (the 1/3-bar divided into seven parts) is this one (the 1/7-bar divided into three parts)?

A: (pause) oh, three-sevenths.

I: Three-sevenths, could you elaborate why?

A: Well, each single small piece is one-seventh of one-third because I broke the one-third into seven pieces, so that's one-seventh and I repeated it three times. So one-seventh plus one-seventh plus one-seventh (pointing her finger at each part in the 1/7-bar) would be three-sevenths.

Ashley seemed to have no problem establishing a fraction as referring to the results of her activity. It seemed obvious to her that one-seventh meant breaking an amount into seven parts and three-sevenths meant repeating one-seventh three times. However, we need to carefully examine how she was conceiving of three-sevenths: Did three-sevenths refer to a multiplicative relationship between the 1/3-bar and the 1/7-bar? Although she answered "three-sevenths" when responding to the question that asked about a multiplicative relationship between the two bars, such an answer might not have implied what the question intended. She might have produced three-sevenths while focusing on the number of parts in her perceptual field even though she clearly mentioned one-seventh of one-third as referring to one small part in the 1/3-bar divided into seven parts. As soon as she deduced that one small part is one-seventh of one-third, if this was involved in producing a multiplicative relationship between one-third and one-seventh, she would know what times the 1/3-bar produced the 1/7-bar. So, I present the following protocol to further investigate whether her answer three-sevenths constituted a multiplicative relationship between the 1/3-bar and the 1/7-bar.

Protocol 3.4: (Second cont.)

I: Could you give me a multiplication starting with this one (the 1/3-bar) and then ending up with this one (the 1/7-bar)?

A: Okay, one-third times (After a short pause, she moves her cursor toward the 1/7-bar from the 1/3-bar.) gives one-seventh.

I: You just said this (the 1/7-bar) is three-sevenths of this one (the 1/3-bar).

A: Yes.

I: Yeah, could you just make a multiplication for that?

A: (Sits quietly for a while.)

I: This one (the 1/7-bar)

A: is three-sevenths of the first one (the 1/3-bar). One-third times—(after a while), Oh, man (laughing).

I: How much is this (the 1/7-bar)?

A: Of this one (the 1/3-bar)? Three-sevenths.

I: In pounds.

A: One-seventh.

I: Yeah, this (the 1/7-bar) is one-seventh, this (the 1/3-bar) is one-third. So, one seventh equals (moving a finger toward the 1/7-bar from the 1/3-bar).

A: One-seventh equals--

I: How much is this one (the 1/7-bar) of this one (the 1/3-bar)?

A: Oh, equals three-sevenths of this one (pointing her cursor at the 1/3-bar). Okay, one-seventh equals three-sevenths of one-third.

I: Instead of “of”?

A: One-seventh equals three-seventh *times* one-third.

I: Yeah, exactly (both laughing).

When asked to produce a multiplication to obtain the 1/7-bar using the 1/3-bar, Ashley seemed not to look back to her operations, which was dividing the 1/3-bar into 7 parts and repeating one of the parts three times. That implies that she did not consider three-sevenths as an operation that related one-third and one-seventh in the situation. In addition, she seemed confused that one-seventh represented the 1/7-bar while three-sevenths referred to the operations she performed to produce the1/7-bar using the 1/3-bar. That means she thought of three-sevenths as a result of operating rather than as those operations that produced the 1/7-bar. So, even though she knew that three-sevenths referred to what she did to make the 1/7- bar, she was yet to abstract three-sevenths as the operations that related the 1/3–bar to the 1/7-bar. If such an abstraction had been made, I would refer to 3/7 as an operator, or to what Steffe calls a rational number of arithmetic.

When she paused after mentioning "one-seventh equals," I reminded her of a question leading to three-sevenths in order to help her think of one-seventh while relating it with three-sevenths. The intervention prompted her to notice a relational meaning of three-sevenths. She said "Oh, [the 1/7-bar] equals three-sevenths of this one (pointing her cursor at the 1/3-bar)." She seemed to begin to consider three-sevenths as a relation between two quantities, the 1/3-bar and the 1/7-bar. However, her activity to relate two quantities was not yet interiorized because when asked to produce a multiplication in the inverse way, she made it quickly but confessed, "I am not picturing it yet."

According to my argument that a mature fractional multiplication scheme entails having a relational insight between two fractional quantities prior to activity, I carefully investigated whether Ashley could produce such a relational insight. The following protocol shows that she was yet to establish fraction multiplication in that sense (03/23/06).

Protocol 3.5: Making a bar so that a 5-inch bar is four-sevenths of the original bar.

(Ashley drew a bar divided into five parts to represent five inches.)

I: This 5-inch bar is four-sevenths of mine. Could you make mine?

A: Okay, this is four-sevenths of yours. First, just let me tell you what I think. I'm thinking that yours is going to be close to double the size of mine. Because four-sevenths, 4 is little more than half of the seven. Just trying to think it's gonna be close to double the size (pause).

I: That kind of reasoning is related to your, the reason you chose this 4 (pointing at the number Ashley dialed PARTS to)?

A: Well, that's what I'm thinking about four-sevenths. I was thinking (pause) if I divided, what I was thinking I am trying to work it out to see if that works, if I divide each piece into 4, (pause) but I don't know about that. (Dialing at 7) I was trying to decide, okay, do I divide it into seven pieces? But I don't think that's it. (Dialing at 4) I think 4 but I can't, I am not dealing why. Okay, if I did it (dividing each piece into four).

I: Do you think you have two choices?

A: Yes.

I: 4 or 7?

A: Yes. That's what I am thinking of, and I think it's 4 because that gives me 20 pieces total. And I want to end up with 35 pieces total.

I: How do you know that?

A: Because, how do I know that (whispering), I wanted to be, (pause) how do I know that (whispering)?

The question posed requests Ashley to consider a fraction four-sevenths as a relationship between a 5-inch bar and a bar that she was asked to make. Further, it implicitly asks her to establish a new relationship, the inverse relationship, to make the new bar using the given relationship, four-sevenths. Yet, she did not use four-sevenths as a relationship between two quantities even though she seemed to engage in inverse reasoning in some sense; her inverse reasoning did not involve four-sevenths, but one half, which she used to make an estimate for four-sevenths.

She first used her number sense with respect to four-sevenths to deduce how the 5-inch bar would be related with what she would make: “Yours is going to be close to double the size of mine (the 5-inch bar) because four-sevenths, 4 is little more than half of the seven.” This indicates that she did conceive of the situation—the other bar—and produced a relation between the unknown bar and the 5-inch bar. In other words, her decision concerning whether she should use 4 or 7 to partition each inch seemed to be based in the necessity to engage in inverse reasoning to produce the unknown bar. But she seemed unable to engage in the operations that would be necessary to use four-sevenths to produce the unknown bar. Even though she verbalized four-sevenths in terms of two separate numerals, according to her then current concept of fractions, she would have been thinking of “4” in a relation to seven, which was the number of unit parts comprising a whole. Concerning such a conception of fractions, her elaboration of four-sevenths would not only mean that she was relying on four parts out of seven, but it also would include the concept of four-sevenths as being repeated four times. Further, her elaboration of four-sevenths seemed to involve a relational insight. However, the insight seemed to rely on a quantitative comparison between four-sevenths and seven-sevenths without accompanying a partitioning activity on the given 5-inch bar. Therefore, in this context, saying that four-sevenths involved a relational insight does not acknowledge all of the operations in finding one-seventh of a unit bar using one-third of a unit bar.

Although it is not certain that she regarded the 5-inch bar as a result of making four-sevenths of some bar, she seemed to understand thinking of the 5-inch bar in terms of four-sevenths, and her goal seemed to be to make seven-sevenths: she said “thirty-five.” However, she was not able to consider dividing the 5-inch bar into four parts as a way to produce one-seventh of the desired bar. That is, she failed to construct one-seventh as a composite unit fraction that could be repeated four times to produce the 5-inch bar. When trying to explain why dividing each inch into four parts works for making a new bar, she could not connect her operation with her reasoning about “4” implying a repetition: “If I divide each piece into 4 (pause) but I don’t know about that.” This indicates she was not explicitly assimilating the problem context involving four-sevenths as a situation for partitioning into four parts and iterating one part seven times. This makes it possible for me to infer that she considered four-sevenths as an operation without inverse reasoning.

I also consider the issue of a composite unit fraction as a possible reason that contributed to her difficulty in using four-sevenths to transform the 5-inch bar into four-sevenths of another bar. She had previously constructed a composite unit fraction such as 1/7 of 21 items, but was yet to use it as input for further operating. So, given the 5-inch bar, she was yet to restructure the bar as four-sevenths of another bar even though she divided each inch into four parts. It is plausible that the reason she made this partition was because she knew that four-sevenths is equal to twenty-thirty fifths and that she could produce 20 parts by dividing each inch into four parts. That means the partitioning of each inch into four parts was not connected to conceiving one of four groups as one-seventh of another bar. So, whether she had constructed generalized operations to engage in inverse fractional reasoning is problematic. The following continuation of Protocol 3.5 is a first step in the construction of those operations (03/23/06).

Protocol 3.5: (Cont.)

I: This (the 5-inch bar) is four-sevenths of mine. And then [in] this four-sevenths, how many one-seventh of mine do you have?

A: Four, so I need seven-sevenths.

I: Yeah, you need seven-sevenths to get, in order to make…

A: To make your whole bar, I have four-sevenths. I need seven-sevenths.

I: How could you get one-seventh of mine from your bar?

A: Divide, I guess each one into sevenths instead of fourths.

I: Why do you think so?

A: If I divided each inch into sevenths, then I can take one from each inch, and then (pause).

I: How many one-seventh of mine do you have?

A: Four, I know that.

I: Four, so if you make one-seventh of mine?

A: Right, I should be able to this (the 5-inch bar; each inch of which was divided into 4.) into four equal pieces.

I: Yeah, right? How do you know that?

A: Because mine is one-sevenths repeated four times, so one-sevenths, one-sevenths, one-sevenths, one-sevenths. So I should be able to divide this into four. So if I take one from each part, that's one-fourth of this one (the 5-inch bar), no one-seventh of this one (the 5-inch bar).

I: One-seventh of this one (the 5-inch bar)?

A: Yes, no, one-seventh of the whole.

I: Yeah, one-seventh of my bar.

A: Because I have four-sevenths. Okay, so I need to pull out (pulling one part at the right end of the 5-inch bar consisting of 20 small parts out and repeating it, but hesitating to continue to do that when she repeated it 3 times) it was 5, is it 5?

I: Yes, 5 inches.

A: (Making a 5-part bar consisting of five small parts) so this was one-seventh because I repeated it, this (the 5-inch bar) is four-sevenths of yours, so I have to figure out what one-seventh was, so if I repeated this group of 5 (the 5-part bar) four times I would get mine. Because there are 20 pieces, 5 times 4 equals 20. So to get yours, I am gonna repeat this group of 5, that one-seventh 7 times, 1,2,3,4,5,6,7.

At the beginning Ashley had no sense of four-sevenths as a relationship between two quantities, a given 5-inch bar and a hypothetical amount. Rather, she seemed to consider four-sevenths as an absolute quantity. In other words, she seemed to consider four-sevenths as a result of operation—a 4-part bar where each part was one-seventh—in the situation requiring inverse reasoning, thereby experiencing a cognitive conflict between the given quantity 5 and a resultant quantity four-sevenths. So, regarding the question that a 5-inch bar is four-sevenths of another amount, she first wanted to make a 7/7-bar but had difficulty finding a way to use the 4/7-bar (the 5-inch bar) to make a 7/7-bar (a hypothetical bar). Two possibilities arise that are related to her difficulty. First, she might not have attributed four-sevenths to the given 5-inch bar and, second, she might not have been able to use the composite unit, the given 5-inch bar, to engage in fractional reasoning concerning four-sevenths. The second possibility is related to recursive operating and inverse reasoning. Recursive operation in this context means considering the operations for four-sevenths based on each inch, and inverse reasoning means an ability to reverse the operations for four-sevenths while considering fractions in terms of operating. I infer her lack of differentiating fractions as operations from anticipated results of operating affected her difficulty in implementing recursive operation and inverse reasoning.

The query “If you make one-seventh of mine (a hypothetical bar)?” provoked her to think about how four-sevenths can be established using the given 5-inch bar: “I should be able to divide this (the 5-inch bar) into four equal pieces.” That means she finally began to consider four-sevenths as operations to relate the 5-inch bar with another bar that she would like to call a whole. So, four-sevenths was no longer considered as an identical object to the 5-inch bar, and the 5-inch bar was regarded as a product of finding four-sevenths of another quantity. She then tried to determine what one of the four equal parts should be and relate it to the 5-inch bar and to a bar she was looking for: “That’s (the 5-part bar) one-fourth of this one (the 5-inch bar), no one-seventh of this one (the 5-inch bar),” “one-seventh of the whole.” The statement is very crucial in that it indicates that she was developing a fraction as a multiplicative operation. In addition, verbalizing “one-fourth” and “one-seventh” for the 5-part bar shows she was in the process of differentiating a fraction as an operation from an anticipated result of operating: one-fourth referring to her operating vs. one-seventh referring to an anticipated result of the operating. However, immediately after verbalizing one-fourth, she gave up using one-fourth for the 5-part bar and adhered to producing one-seventh of the bar she was supposed to make. That means the differentiation was not strong enough to implement the splitting operation.

Reversing the operations involved in four-sevenths prompted recursive operating by encouraging her awareness of considering fractions as operations. Ashley's comments that, "If I repeated this group of 5 four times I would get mine," "I am gonna repeat this group of 5, that one-seventh 7 times," do indicate that she constructed four-sevenths as an ensemble of operations. Further, these operations were involved in her producing the hypothetical bar as a 7/7-bar, where a 1/7-bar consisted of five-fourths inches. However, she had yet to develop inverse reasoning. She neither verbalized seven-fourths nor mentioned anything about the operations related to seven-fourths.

An Impact of Establishing a Multiplicative Fractional Relationship Between Two Quantities on Conceptualizing Fractional Multiplication

At the beginning of her engagement in this study, Ashley regarded both of the fractions involved in a fraction product as the resultant amounts. In Protocol 3.5, she began to establish a fraction as a multiplicative relationship between two quantities. The following protocols show how she established a multiplicative fractional relationship between two quantities and conceived of fractional multiplication based on the relationship (03/16/06).

> Protocol 3.6: Making a string so that a fifth of one decameter is four-sevenths of the string.
>
> (Ashley draws a string.)
>
> I: Let's say this string is a fifth of one decameter long. You have another string, but this one-fifth of one decameter string is just four-sevenths of your string.
>
> A: This (the string she drew) is a fifth of one decameter, and it is the same as four-sevenths. We want to get a whole decameter, so we would repeat that five times. But it's only four-sevenths of the length of my string (dialing to 4 and dividing the string into four parts, producing a 4/7-bar). So, this (the 4/7-bar) is four-sevenths of my string, so to

get my string, I would repeat this (one part of the 4/7-bar) seven times to get seven-sevenths (pulling one part out and repeating it seven times to produce a 7/7-bar). This one (the 7/7-bar) is mine.

I: How long is it (the 7/7-bar)?

A: This (the 4/7-bar representing a fifth of one decameter) is four-sevenths so this (the 7/7-bar) is seven-sevenths … of … one-fifth (looking at me)?

I: Of one-fifth, please say it again in decameters.

A: Seven-sevenths of a fifth or seven-sevenths times a fifth of a decameter so (pause).

I: How long is it one little piece in decameters?

A: Um, twenty, one-twentieth.

I: Then how much is it (the 7/7-bar)?

A: One two three four (counting each part moving her cursor along the 7-bar), seven-twentieths.

I: Yeah, seven-twentieths, seven-twentieths of one decameter.

A: Yes.

Given the problem "Make your string so a fifth of one decameter is four-sevenths of your string," Ashley first wanted to figure out the fractions, a fifth and four-sevenths, by thinking of five-fifths and seven-sevenths. However, she seemed unaware of what five-fifths and seven-sevenths, respectively, indicated. Since she focused on making a bar representing seven-sevenths using the 4/7-bar, she produced a 7/7-bar by dividing a one-fifth of one decameter bar into four parts and repeating one of them seven times, which corroborates that her inverse operations in the previous protocol were more or less permanent. But the operations were not recursive operations in the sense that she could take their results as input for operating on the hypothetical unit, one decameter, which was not visually present in the situation.

While elaborating her construction of the 7/7-bar, she seemed to consider four-sevenths in terms of operations, but did not differentiate the operations she used to make the 7/7-bar from the result of the operations, which is “seven-sevenths.” That is, she did not notice one of the parts constituting the 4/7-bar could be one-fourth of the bar even though she produced the 4/7-bar by dividing the given bar into four parts. She had difficulty implementing the splitting operation and inverse reasoning. By closely examining her ability of inverse reasoning, I discuss my argument that differentiating fractions as operations from anticipated results of operating is critical in the splitting operation.

I consider three components as involved in inverse fractional reasoning: an anticipatory fractional scheme, a reversible fractional scheme, and the splitting operation. First, related to an anticipatory fractional scheme, Ashley’s dividing the given unmarked bar into four parts shows there was a hypothetical bar in her mind to which the given bar is related. She produced a result of an anticipatory scheme. Next, regarding a reversible fractional scheme, her dividing into four parts and repeating one part seven times clearly indicates she reversed the operations implemented for four-sevenths. Inverse operations were available to her. However, she was not engaged in the splitting operation because dividing into four parts was mentioned only as a way to produce one-seventh. It means she would transform the concept of four-sevenths as operations into a result of the operations in the situation requiring inverse reasoning. Thus, recognizing her activity of dividing a bar into four parts was suppressed by the fact that four-sevenths is a result of the operations performed for the 4/7-bar. In conclusion, her lack of differentiating fractions as operations from anticipated results of operating affected her difficulty of engaging in the splitting operation. If she had been able to differentiate an operation from its anticipated result, she should have considered the 7/7-bar as a result of dividing the one-fifth of one decameter bar into four parts and repeating one of the parts seven times.

The following continuation of Protocol 3.6 shows how she complemented her lack of differentiation between operations and results of the operations, thereby conceiving of fraction multiplication based on such a differentiation (03/16/06).

Protocol 3.6: (Cont.) Finding a product 7/4 times 1/5.

I: How much is this (the 7/7-bar) of this one (the 4/7-bar)?

A: Seven- fourths.

I: Seven- fourths. So, 7/4 times 1/5, what's 7/4 times 1/5?

A: (Sits quietly for a long time without answering.) Seven-twentieths.

I: Could you elaborate this multiplication based on your construction?

A: Seven- fourths, because each one of these (pointing her cursor at the 7/7-bar), each small piece is a fourth of this fifth (the 4/7-bar), of this piece. So, I have seven of them. So, seven, 1,2,3,4,5,6,7, 7/4 times 1/5, this whole thing is one-fifth (pointing her two fingers at the 4/7-bar), it's four-twentieths, so each piece is one-twentieth (pointing her finger at one part in the 4/7-bar), and I have seven, 1,2,3,4,5,6,7 (pointing her index finger at each part in the 7/7-bar).

Before the above protocol, Ashley never showed she was engaging in any operation related to seven-fourths while reasoning about four-sevenths given the context that one-fifth of a decameter is four-sevenths of a string. However, when asked about the relationship between the 7/7-bar and 4/7-bar, she quickly answered seven-fourths. So, I wonder whether her verbalizing seven-fourths is related to four-sevenths in a reciprocal way through fractional reasoning because there is a possibility that she might have been referring to the bars without a notion of their fractional amounts.

She produced another representation of the 7/7-bar, seven-fourths. Her elaboration of seven-fourths, “Each small piece is a fourth of this fifth (the 4/7-bar), of this piece. So, I have seven of them,” shows that she constructed seven-fourths by employing the splitting operation in that her dividing the 4/7-bar into four parts implied repeating four times. At this point, we need to investigate how she was differentiating two views of fractions for the splitting operation. Consider her statement concerning 7/4 times 1/5: “7/4 times 1/5, this whole thing (the 4/7-bar) is one-fifth, it’s four-twentieths, so each piece is one-twentieth, and I have seven, 1,2,3,4,5,6,7.” She first considered the size of the given amount 4/7-bar [one-fifth of one decameter], and then found an equivalent fraction of one-fifth, four-twentieths, through operating recursively. By confirming that four-twentieths indicates the 4/7-bar, she identified the size of one part as one-twentieth, and counted the number of parts in the 7/7-bar that she had already constructed as seven-fourths of one-fifth. Such an activity indicates that she considered the 7/7-bar as a result of seven-fourths, and that makes it possible for me to infer that she interpreted “7/4 times 1/5” based on seven-fourths as the operations performed. So, in order to answer the question “What’s 7/4 times 1/5?” she referred to the result of the operations for seven-fourths, which is the 7/7-bar. In conclusion, seven-fourths played a role relating the given one-fifth of one decameter bar to the 7/7-bar multiplicatively. That is, she established a multiplicative relationship between two fractional quantities through seven-fourths.

However, some doubt still remains that seven-fourths might not be related with four-sevenths in a reciprocal manner. For “seven-fourths,” she clearly knew that she needed seven of an amount but seemed unaware of why she used that amount. In other words, she produced the 7/7-bar by relying on the statement that one-fifth of one decameter (the 4/7-bar) is four-sevenths of a desired bar, but "seven-fourths" did not involve a notion of "four-sevenths" in her construction. The way she associated seven-fourths and four-sevenths was through the 7/7-bar. That is, the 7/7-bar was constructed as a hypothetical whole of the 4/7-bar and considered as a result of the operations performed for seven-fourths. However, the two relationships were not simultaneous, and I infer that such an inability to consider them simultaneously would prevent her from establishing seven-fourths as a reciprocal of four-sevenths in the problem context.

An Impact of Conceptualizing a Composite Unit as a Unit for Fractional Operating on Establishing a Multiplicative Relationship

Considering fractions as operations is related to an iterative aspect of a unit fraction, which is a fundamental element to produce a multiplicative relationship. Constructing an iterable unit fraction implies an ability to produce a fraction by iterating the unit fraction while noticing that the fraction produced is a unit being iterated. So, it is critical to recognize that a fractional whole is determined by a unit fraction through iterating. An operational view of fractions is grounded on the idea that equi-partitioning is a basic operation to conceive of a fraction. In that equi-partitioning involves both dividing into equal parts and reconstituting a whole, equi-partitioning engages students in implicitly or explicitly dealing with a quantity to be partitioned as a unit being iterated as many times as being divided. At the same time, equi-partitioning requires to recognize another quantity prior to activity because it has an anticipatory nature. Thus, considering fractions as operations implies recognizing a quantity as a reconstituted whole through equi-partitioning, and such recognition would prompt an insight into an iterative aspect of a unit fraction. Given that consideration of fractions as operations and an iterable unit fraction, I am interested in investigating how a view of fractions as operations contributes to constructing an iterable unit fraction.

In the previous section, I showed Ashley established a multiplicative relationship using a fraction on the basis of her ability to differentiate fractions as operations from fractions as anticipated results of the operations. However, she did not relate the multiplicative relationship to its reciprocal while multiplying fractions. So, we can say an operational view of fractions promotes establishing a multiplicative relationship but does not guarantee inverse reasoning. The following protocol shows that conceptualizing a composite unit as a unit for partitioning was also critical in establishing a multiplicative relationship and it corroborated an operational view of fractions (03/23/06).

Protocol 3.7: Making four-sevenths of a 5-inch bar.

(When asked to make a bar that is four-sevenths of a 5-inch bar, Ashley divided each inch into 7 parts, pulled one part out and repeated it four times. By doing that, she produced a 4-part bar that represents four-sevenths of one inch. She verbalized that she wanted to repeat the 4-part bar seven times. When she arrived at the sixth, she realized there was something wrong with her construction.)

I: Could you tell me what you wanted to do?

A: Okay, I had 5 inches, right? And so I knew that I wanted, you wanted four-sevenths, yours was four-sevenths. So, I thought to myself, just number sense that’s a little more than half, I knew that. So what I did, each inch, I divided into seven parts.

I: How come?

A: Because I was thinking four-sevenths. But I, I was gonna, what I was thinking wrongly, but I was thinking I was gonna take 4 from, out of the seven but that’s not working.

I: You repeated six times, so this is not mine?

A: Right. I was going for 7 then I know, No, that’s not right.

I: Why did you want to repeat seven?

A: I was thinking it was four-sevenths but it should be 4 times not 7 times.

. . . .

I: Could you tell me again why you wanted to repeat four times?

A: Because four-sevenths.

For the problem “Make four-sevenths of a 5-inch bar,” Ashley first considered one inch for dividing and produced 35 small parts by dividing each inch into seven parts. That she divided each inch into seven parts means that she was implicitly considering the 5-inch bar as a whole “seven-sevenths” compared to “four-sevenths” she was supposed to produce. She then created a 4-part bar consisting of four small parts as an amount to repeat. The 4-part bar was considered as a unit amount to repeat. However, by attempting to repeat the unit amount seven times, she revealed she was not engaging in recursive operating. In addition, her repeating it seven times to produce four-sevenths shows that 4 and 7 must be all the numbers available to her for operating. That is, she disregarded she was dealing with a composite unit of 5. That indicates she did not consider the 5-inch bar as a unit she has to deal with and further a unit for fractional operating. So, even though she implicitly considered the 5-inch bar as seven-sevenths compared to four-sevenths, she never tried to make an amount that can be the 5-inch bar by repeating it 7 times.

She did not consider four-sevenths as multiplicatively relating the 5-inch bar with a bar she was supposed to make.

It is certain that four-sevenths was considered in terms of operations: “it should be 4 times not 7 times. . . . because four-sevenths.” However, I wonder about what she was thinking of as one-seventh for four-sevenths. She never seemed to doubt that the 4-part bar is to be repeated 4 or 7 times. So, I raise an issue of her ability to construct a unit fraction of a composite unit. Although she produced four-sevenths of one inch based on one-seventh of one inch, she appeared to have no idea of a unit fraction of the 5-inch bar because: 1) she did not consider one-seventh as referring to the given 5-inch bar and 2) there was no connection between “four-sevenths” of one inch and “one-fifth” of the given 5-inch bar. In regard to the first aspect, we need to remind that she considered four-sevenths as operations, dividing and repeating. So, if she had viewed the 5-inch bar as a unit for operating, she would have tried to find a way to divide the 5-inch bar into 7 parts and then consider four as the number of repetitions. However, her view of fractions as operations seemed to only work for a unit of one. So, I infer she had difficulty conceptualizing the 5-inch bar as a unit of 5 units of one. Related to the second aspect, if she had explicitly conceived of one inch as a fifth of the 5-inch bar, she would have repeated the 4-part bar selected for repetition five times, not seven times: the 4-part bar as 4/7 of one-fifth of the 5-inch bar. That is, she had difficulty engaging in recursive operations and distributive reasoning. She did not know how to relate four-sevenths of one inch to four-sevenths of 5 inches.

In Protocol 3.7, she had difficulty producing a fraction of a composite unit due to her lack of ability to conceive of a composite unit as a unit for fractional operating. Thus, to call her attention to the given 5-inch bar as a unit for fractional operating, I asked her to color one-seventh of the 5-inch bar and she colored five small parts blue while reasoning that “because I took one-seventh from each group and there are five groups and I divided each group into seven, so one-seventh from each group, that ends up being five of the small pieces, 5 out of 35." This elaboration indicates constructing a composite unit as a unit for fractional operating prompted recursive operations and, possibly, distributive reasoning. In addition, an awareness of a composite unit as a unit for operating helped her corroborate a view of fractions as operations. She verbalized one-seventh for dividing one inch into seven parts and mentioned one-seventh for a group of five parts. As soon as she got the sense of a unit fraction of the composite unit of the 5-inch bar, she quickly figured out what she should do for the original question, making four-sevenths of a 5-inch bar: "I need to repeat this blue part (five one-sevenths of one inch parts) four times because this (the blue part) is one-seventh." As a result, construction of a unit fraction of a composite unit while considering fractions as operations promoted constructing an iterable unit fraction and developing a multiplicative insight into a fraction four-sevenths.

Generating an improper fraction using a unit fraction seemed challenging to Ashley even though she constructed a fraction or produced a fractional amount using an iterable composite unit based on a view of fractions as operations. When asked about a size of a bar consisting of twenty small parts, which was produced by repeating five one-sevenths of one inch four times, she first measured out the bar using one inch consisting of seven small parts and answered it in two ways: "two and six-sevenths" and "20 out of 35 of the original (the 5-inch bar)." She never tried to answer it in terms of one-seventh of one inch that she used when figuring out one-seventh of the 5-inch bar. It took her a long time to produce a size of the 20-bar in inches using a fraction because she considered each part only as one of the 35 parts and also tended to locate a fraction only for a remaining part. In that sense, I conclude that even though she produced "four-sevenths" based on "one-seventh" of the 5-inch bar, the "one-seventh" was not interiorized as an iterable composite unit based on the one-seventh of one inch, which is another level of an iterable unit fraction.

I have discussed Ashley's development of fractional reasoning centering on an operational view of fractions. The following protocol clearly shows her progression of multiplicative reasoning through this development (03/23/06).

Protocol 3.8: A reciprocal of 4/7 of a 5-inch bar.

(This protocol is the continuation of the Protocol 3.7 where she engaged in a problem "Making four-sevenths of a 5-inch bar." After the Protocol 3.7, I asked her to color one-seventh of the 5-inch bar blue and she colored five out of 35 small parts produced by dividing each inch into seven parts. On the computer monitor there were three bars she produced: 1) a 5-inch bar consisting of 35 small parts produced by dividing each inch into 7 parts, 2) a 1/7-bar comprised of 5 small blue parts, and 3) a 4/7-bar consisting of 20 small parts produced by repeating the blue 1/7-bar 4 times.)

I: The blue little pieces, one-seventh of what?

A: One-seventh of the whole bar (pointing her cursor at the 5-inch bar).

I: Yeah, of the whole bar. So, how many of the 1/7s are in the whole bar?

A: Seven-sevenths.

I: Yeah, here (pointing at the 5-inch bar) how many of the blue parts are in here?

A: Seven.

I: How many the blue parts are in the bottom bar (the 4/7-bar)?

A: Four.

I: Four. So, how much is the top bar (the 5-inch bar) of the bottom bar (the 4/7-bar)?

A: (Sits quietly for a long time without answering) I keep want to say seven-sevenths, but that's not right.

I: What's the--

A: Seven-fourths.

I: Ah, seven-fourths.

A: Yeah (laughing).

I: How did you get it?

A: (pause) I was thinking about reciprocals.

I: Please tell me more about that.

A: Well, four, this (the 4/7-bar) was four-sevenths of the original (the 5-inch bar).

I: How do you know that?

A: Because that's what I made. That was the, it was one-seventh four times (pointing at the 4/7-bar while indicating four intervals by holding her thumb and index finger at a fixed distance and moving this interval four times), so four-sevenths. And … if I, … if I divided this one (the 4/7-bar) into four pieces, it could be repeated seven times to get this (the 5-inch bar).

I: Why did you divide it by 4?

A: Because I was trying to divide it into 4 equal pieces since it was 7, no it was four-sevenths originally.

I: Ah, this (the 4/7-bar) was originally four-sevenths.

A: Four-sevenths of mine, so I was thinking, if I divide it into fourths, each fourth (pointing her fingers at five small parts in the 4/7-bar) is one-seventh of that (the 5-inch bar).

I: Wow, so when you divided this one (the 4/7-bar) into 4 and then you are gonna get

A: 5 pieces.

I: 5, but 5 pieces is the one

A: fourth of this one (the 4/7-bar).

I: But

A: One-seventh of this one (the 5-inch bar).

I: So, you need how many one-fourth in here (the 5-inch bar)?

A: Seven

I: Seven, so that's why

A: That's why seven-fourths of this one (the 4/7-bar).

Ashley tried to answer the reciprocal question "This 5-inch bar is how much of this bar (the 4/7-bar that is four-sevenths of the 5-inch bar)?" using an one-seventh of the 5-inch bar, which was colored blue and used to construct four-sevenths of the 5-inch bar. At the beginning of this protocol, by saying "I keep want to say seven-sevenths, but that's not right," Ashley showed she was above a partitive fractional scheme. Based on such an advanced conception of fractions, she figured out the 5-inch bar is seven-fourths of the 4/7-bar through her knowledge of reciprocals. Differently from what she did in the previous incidents involving reciprocals, she began to reason it out while reflecting on her construction of four-sevenths of the 5-inch bar and verbalizing that the 4/7-bar is four-sevenths of the 5-inch bar because "it is one-seventh four times." The expression indicates that she constructed four-sevenths as an ensemble of operations performed and the construction was based on an iterable unit fraction. In addition, her saying "If I divided this one (the 4/7-bar) into four pieces, it could be repeated seven times to get this (the

5-inch bar)" clearly shows how an iterative fractional scheme based on an operational view of fractions has been developed toward reciprocal reasoning.

Constructing a fraction relying on an operational view of fractions promoted constructing an iterative fractional scheme and further inverse reasoning. Let's closely look at how she acted to support the claim. First, to elaborate her producing a reciprocal of four-sevenths, she reflected on her construction of four-sevenths based on operational view: "one-seventh four times." This is very critical in showing that an operational view of fractions was used as a recognition template to the reciprocal question that asked how much the 5-inch bar is of the 4/7-bar. I infer that such an operational view of fractions would have provoked her insight into a multiplicative relationship between the 4/7-bar and the given 5-inch bar through her activity "four times." Even though the 4/7-bar and the 5-inch bar were the materials she had to deal with, she would have been playing with the 1/7-bar, one-seventh of the 5-inch bar, at her conceptual level because the materials were being conceived by the 1/7-bar: "one-seventh four times," "It (the blue 1/7-bar) could be repeated seven times to get this (the 5-inch bar)." In addition, by differentiating seven-fourths as an ensemble of operations from an anticipated result of the operations, she was able to engage in the splitting operation and verbalized "one-fourth" of the 4/7-bar for the 1/7-bar. This is where she constructed an iterable unit fraction with respect to a composite unit. She clearly established a multiplicative relationship of the 1/7-bar with both the 4/7-bar and the 5-inch bar: "Each fourth [of the 4/7-bar] is 1/7 of that (the 5-inch bar)." She accomplished inverse reasoning by establishing a multiplicative relationship based on an operational view of fractions.

Summary

In this section, I elaborated on how Ashley developed mathematical concepts related to fractions on the basis of her initial concepts of fractions and conceptions of a seventh grader's fractional reasoning. As she engaged in fraction problems, she made some progress in her conceptual level, and I investigated her progress by focusing on a composite unit as a unit for fractional operating, a fraction as an operation vs. a result of operating, and a multiplicative fractional relationship between two quantities.

A grouping idea played a fundamental role for Ashley when constructing a unit fraction of a composite unit, and she explicitly used the unit fractional amount to produce a proper fraction: “I divided the whole 24-bar into eighths so that’s eight groups of three centimeters, so if I wanted three-eighths, I did one group of three, two groups of three and three groups of three” [cf. the text immediately preceding Protocol 3.1]. However, she had difficulty expanding the grouping idea toward developing an iterative unit fractional scheme. When asked how much the 24-bar is of the 9-bar she never considered the 9-bar as three groups of three parts each even though she generated the 9-bar using the group of three parts. In other words, she did not consider the result of making three-eighths as a quantity for partitioning. As soon as she realized that the 9-bar represents “three” one-eighths, she figured out that the three parts used for one-eighth is “one-third” of the 9-bar and since the 24-bar is eight groups of the three parts, the 24-bar is eight-thirds of the 9-bar. However, her realization of the 9-bar as “three” groups was not an insight made independently of the researcher’s suggestions.

By using the doling-out strategy developed in whole number contexts in a fractional context, she began to engage in a units-coordination activity when partitioning and developed distributive fractional reasoning. Responding to the problem “Make a share for one person if a 3-foot sub sandwich is shared among five people,” she verbalized “if these (two segments to represent the 3-bar being 3 feet) wouldn’t be there, I would just divide it into five equal pieces” and “there is one-third (indicating one foot), since I have to divide each third, all three-thirds between five people, I might as well divide each third into five pieces so I can, one two three four five, one two [three four five].” Her reasoning involves two composite units: a unit of three units of one foot and another unit of five units of a share for one person. In addition, that she produced the other unit of five units of small parts [one-fifth of one foot] indicates that she coordinated the unit of three feet and the unit of five one-shares. Responding to the question how much the share for one person is of the 3-foot sub sandwich, she answered one-fifth and reasoned it out as follows: “Because we would divide it between 5 people and so 3 out of the 15 make a share for one person, which is one-fifth of the sandwich. So, three-fifteenths is equivalent or equals to one-fifth of the sandwich” [cf. Protocol 3.3]. She established the equivalent relationship between two fractions, three-fifteenths and one-fifth. By saying “3 out of the 15” followed by “we would divided it between 5 people,” she showed her out-of statement is a result of coordinating two levels of units. In addition, her elaboration “which is one-fifth of the sandwich” indicates she revisited the unit of 5 units of a share for one person. However, she seemed not to understand that a 3-foot sub shared among five people can be considered as division, 3 divided by 5. She separately found one-fifth of the 3-foot sub and three-fifteenths of one foot, and never correlated them.

Ashley distinguished an operational meaning of fractions and the meaning of a fraction as a result of operating, and she used the result of operating as a new situation. Such an ability to distinguish two perspectives of fractions, as operations vs. results of operating, helped her develop a recursive fractional scheme and engage in distributive reasoning. In particular, using an “of-statement” provided her with an opportunity to consider simultaneously and distinctively two perspectives of fractions. For instance, she made one-seventh of one pound (a 1/7-bar) using one-third of one pound (a 1/3-bar) by dividing the 1/3-bar into seven parts and repeating one part three times. She considered one-third in terms of her operating. That is, without making a whole using the given 1/3-bar, she thought three of one-third using the 1/3-bar partitioned into seven parts. Such a view of one-third enabled her to use recursive partitioning. Her verbalizing “a seventh of a third” with respect to partitioning the 1/3-bar into seven parts indicates she related partitioning to creating a unit fraction. In addition, her construction of “one-seventh” of one pound indicates that she used “one-seventh” in two ways: one-seventh as a result of making one-seventh of one pound and as partitioning the 1/3-bar into seven parts. Through her verbalization to justify her construction of one-seventh of one pound, she explicitly showed she engaged in distributive reasoning: “this (the 1/7-bar), my new bar, one-seventh seven times should equal one pound. So far, I repeated this (the 1/7-bar) seven times, it should be the same length as this (the 1/3-bar), three times.”

She produced a fraction while referring to the activities she carried out. So, she made the statement, one-seventh of one pound is “three-sevenths” of one-third of one pound, as she referred to dividing a 1/3-bar into seven parts and repeating one part three times. However, producing a fraction based on operating did not mean that she established a multiplicative fractional relationship between two quantities. When asked how much the produced 1/7-bar is of the 1/3-bar, she never considered she produced the 1/7-bar by means of three-sevenths of the 1/3-bar and was confused that the 1/7-bar is represented “one-seventh” of one pound as well as “three-sevenths” of the 1/3-bar. That is, even though she knew that three-sevenths referred to what she did to make the 1/7- bar, she was yet to abstract three-sevenths as the operations that related the 1/3-bar to the 1/7-bar.

Ashley had difficulty conceiving of a composite unit as a unit for partitioning even though she related the composite unit with an unknown bar through a fraction. When asked to make a bar so that a 5-inch bar is four-sevenths of the original bar, she figured out that the 5-inch bar is a result of repeating some amount four times. However, she never engaged in partitioning leading to the operations that would be necessary to use four-sevenths to produce the unknown bar. She did not adjust her operational view of 4/7 so that she would first partition into four parts and then iterate that part seven times in a situation that involved inverse reasoning to solve. Further, she seemed confused with differentiating fractions as operations from anticipated results of operating. Given the problem statement "a 5-inch bar is four-sevenths of the original bar," the given quantity 5 seemed to be in conflict with another quantity [four-sevenths] that indicates a bar partitioned into four parts. As a result, even though she divided each inch into four parts, she engaged in neither recursive operations nor inverse reasoning. So, she was yet to restructure the 5-inch bar as four-sevenths of another bar.

As soon as she explicitly realized a fraction as an operation to relate two quantities, the fraction was no longer considered as identical to an absolute object. Regarding the problem "Make a bar so that a 5-inch bar is four-sevenths of the original bar," she finally figured out that four-sevenths related the 5-inch bar with the original bar, thereby reconstructing the 5-inch bar by implementing recursive operations and inverse reasoning: "I should be able to divide this (the 5-inch bar) into four equal pieces," "That (five 1/4 of one inch parts)'s one-fourth of this one (the 5-inch bar), . . ., one-seventh of the whole." In addition, establishing a multiplicative fractional relationship between two bars helped her explicitly differentiate a fraction as an operation from an anticipated result of operating. For a multiplicative relationship between the 5-inch bar and a hypothetical bar, she produced one-fourth and one-seventh, respectively, referring to her operation of dividing and an anticipated result of the operating. However, immediately after verbalizing one-fourth, she gave up using one-fourth for the five 1/4 of one inch parts and adhered to producing one-seventh of the bar she was supposed to make. That means the differentiation was not strong enough to implement the splitting operation and complete inverse reasoning based on the operation.

Given a situation that requires operating recursively and inverse reasoning, Ashley showed a critical confusion between two perspectives of fractions. She engaged the problem “Make your string so a fifth of one decameter is four-sevenths of your string” by producing a 7/7-bar. To produce the 7/7-bar, she divided the one-fifth of one decameter bar into four parts and repeated one part seven times. That is, she used the operations to construct four-sevenths when producing the 7/7-bar. However, she did not differentiate the operations she used from a result of the operations, which is “seven-sevenths.” As she tried to reverse the operations carried out for producing the 7/7-bar, her operational view of four-sevenths seemed to be transformed into another view of four-sevenths as a result of operating. So, dividing one-fifth of one decameter into four parts meant making one-seventh. This implies she had difficulty implementing the splitting operation and inverse reasoning because her dividing did not mean repeating in a simultaneous manner necessary for constructing fractions from an operational view. In other words, she had no problem with an anticipatory fractional scheme and a reversible fractional scheme (her dividing one-fifth of one decameter into four parts and her repeating one part seven times show that the schemes were anticipatory and reversible) but had difficulty with the splitting operation. Recognizing her activity of dividing a bar into four parts as an operation to produce a fraction was suppressed by the fact that four-sevenths is a result of the operations performed to make the 4/7-bar. In conclusion, lack of differentiating fractions as operations from anticipated results of operating affected her difficulty of engaging in the splitting operation. Therefore, I infer that the splitting operation will contribute to developing an iterative perspective of a unit fraction by differentiating one-fourth as operating from one-seventh as a result of the operating, thereby enabling her to engage in inverse reasoning.

Although Ashley never showed she engaged in any operation related to seven-fourths while reasoning about four-sevenths, she produced a reciprocal of four-sevenths by looking at the given bar 4/7-bar and the produced 7/7-bar: "Each small piece is a fourth of this fifth (the 4/7-bar), of this piece. So, I have seven of them." However, whether her finding seven-fourths in this way is related to reciprocal fractional reasoning is doubtful because she established a part-whole relationship rather than engage in reciprocal reasoning. Responding to a fraction product 7/4 times 1/5 from the above problem context, she said, "7/4 times 1/5, this whole thing (the 4/7-bar) is one-fifth, it's four-twentieths, so each piece is one-twentieth, and I have seven, 1,2,3,4,5,6,7." She considered seven-sevenths as a result of making a bar so that one-fifth of one decameter is four-sevenths of the bar and interpreted "7/4 times 1/5" on the basis of the operations performed. She first verbalized four-twentieths, which implies she implemented dividing recursively over a whole to which the given fifth bar refers and conceived of dividing into four parts as an operational meaning. Then, she focused on that she had seven of the parts in order to produce the product 7/4 times 1/5. Thus, I conclude the fraction 7/4 involved in 7/4 times 1/5 was conceived of as operating. Further, she previously mentioned seven-sevenths for the 7/7-bar she constructed in response to the problem "Make a string so that a fifth of one decameter is four-sevenths of your string." That is, for Ashley, seven-sevenths was a result of making the 7/7-bar from a one-fifth of one decameter bar and seven-fourths was an operation to produce the 7/7-bar. Such a differentiation between fractions as operations and results of operating would help her establish a multiplicative relationship between the fifth of one decameter and the 7/7-bar. However, the two relationships—the 7/7-bar and seven-sevenths, and the 7/7-bar and seven-fourths—were not established simultaneously, so she could not establish seven-fourths based on reciprocal fractional reasoning with four-sevenths in the problem context.

Conceptualizing a composite unit as a unit for partitioning was critical in establishing a multiplicative relationship, and the conceptualization corroborated a view of fractions as operations. Given the problem “Make four-sevenths of a 5-inch bar,” she created a 4/7-bar by dividing each inch into seven parts and repeating one part four times and considered it as an amount to repeat for making the desired bar. She first attempted to repeat the 4/7-bar seven times and then four times for producing four-sevenths of a 5-inch bar. That means she neither engaged in recursive operating nor considered the 5-inch bar as a unit she has to deal with. That is, she never tried to make an amount that could be used to produce the 5-inch bar by repeating it 7 times. She did not consider four-sevenths as multiplicatively relating the 5-inch bar with a bar she was supposed to make. Since she considered four-sevenths as an operation, she thought “it (the number of repetitions) should be 4 times not 7 times. . . . because four-sevenths.” She never seemed to doubt about the 4/7-bar as a unit for repeating four or seven times; she never referred to the 5-inch bar as a fractional whole for the 4/7-bar.

Constructing a composite unit as a unit for fractional operating also prompted recursive operations and distributive reasoning. When asked to color one-seventh of a 5-inch bar blue, she divided each inch into seven parts and colored five parts blue while reasoning that “because I took one-seventh from each group and there are five groups and I divided each group into seven, so one-seventh from each group, that ends up being five of the small pieces, 5 out of 35.” In addition, an awareness of a composite unit as a unit for operating helped her corroborate a view of fractions as operations. She verbalized one-seventh for dividing one inch into seven parts and mentioned one-seventh for a group of five parts. After producing one-seventh of the 5-inch bar, she verbalized that a bar consisting of 20 parts is four-sevenths of the 5-inch bar because "it is one-seventh four times" and "If I divided this one (the bar consisting of 20 parts) into four pieces, it could be repeated seven times to get this (the 5-inch bar)." That shows construction of a unit fraction of a composite unit based on an operational view of fractions promoted constructing an iterable unit fraction and developing a multiplicative insight into a fraction four-sevenths, which make it possible for her to engage in inverse operation.

Ashley's Fractional Knowledge Development Through Observing Mike (Day 9-14)

In this section, I investigate how Ashley developed a multiplicative fractional scheme as she conceived of Mike's engaging in solving fractional tasks. As important aspects related to the scheme, four topics are considered: an iterative unit fraction scheme, a units-coordinating scheme, fraction multiplication, and improper fractions.

Developing an Insight Into an Iterative Aspect of a Unit Fraction by Considering Fractions as Operations

Ashley watched Mike make an estimate of three-fourths of 48 without looking at a bar representing 48 pounds. Previously, given a similar problem, "Make a bar that is three-fourths of a 12-inch long bar" [cf. Protocol M2.4], he first drew a solid bar representing a 12-inch bar, and then repeated the bar three times, so he produced four of the bar in total. He called the 12-inch bar three-fourths. That indicates he had yet to consider three-fourths as related to partitioning in a situation involving a composite unit, because he focused only on the number of repetitions. In light of his saying, "I thought that the other way," he meant to make a bar so that the 12-inch bar is three-fourth of that bar. That shows he also had difficulty engaging in inverse reasoning. The following protocol shows that he began to implement partitioning but did not use partitioning to conceive of a fraction nor did he develop a concept of a unit fraction of a composite unit. In the protocol, “T’ stands for “teacher,” “M” for “Mike,” and “O” for “Observer.”

Protocol M4.1: Establishing a multiplicative relationship between a 36-pound bar and a 48-pound bar.

(Mike made a 48-bar representing a neighbor’s dog’s weight of 48 pounds and covered it. When asked to make his dog’s weight that is three-fourths of the 48 pounds, he produced a 36-bar and explained his construction.)
M: First I divided the 48 by 4 then I got 12, then multiplied 12 by the 3 and then you get that.
T: Wow, awesome. When you divided that 48 by 4 and you got 12, that’s 12 pounds, right?
M: Yeah.
T: How much was that of 48 pounds we started with?
M: One-third.
T: Is it one-third? Of 48 pounds?
M: (After a short pause) what was the question?
T: Well, I think, you said one-third. What is that one-third of?
M: One-third of 48 pounds.
T: Are you sure?
M: Oh, OK. I was supposed to divide 48 by 3.
T: Wa, wa, wait. No, no. You did it good. You did it good. I really like what you did, when you got 36 pounds, right? But what I am wondering is when you got that 12 pounds, what part is that of the original 48 pounds?
M: One-fourth.
T: How do you know that’s one-fourth?
M: Because, because if you put the 12 over the 48, then you simplify it then you can get one-fourth.
T: That’s one way, can you tell me another way how you know?
M: Divide 12 into 48, then 4.
T: Oh, yeah, right. That’s good, that’s another way. OK, so--.
O: Can I ask a question?
T: Sure.
O: What would you do with 12 pounds to make the 48 pounds? What would you have to do?
M: Multiply by 4.
T: (Nodding) All right. We know that 12 is one-fourth of that, that’s my neighbor’s dog, right?
M: Yeah.
T: So, one-fourth of how much my neighbor’s weight, now, what about how much you made? What is 12 pounds of how much you made?
M: 36.
T: What is 12 pounds, what part of 36 pounds is 12 pounds?
M: It’s … Do I have 12?

T: Yeah, when you made 12 pounds, why don't you color in 12 pounds so that is 12 in it and we can see it?
M: (He colors 12 parts in the 36-bar blue.)
T: You said that is a fourth, right, of what my neighbor's dog weighs. Now, we are saying maybe, your dog weighs this much (pointing her finger at the 36-bar), so how much is that (pointing at the blue parts) of your dog weighs?
M: This one (the blue parts)? How much is that of my dog? It's one-fourth.
T: Is it? One-fourth of this (pointing at the 36-bar)?
M: (After a short pause) one-third of that (the 36-bar).
T: How do you know that?
M: Because you already have 12 pieces, if you color another 12 (coloring the 12 parts at the most right blue) and then three pieces (moving his cursor across the 36-bar).
T: Um, I see, I see, OK. So you said, when you told me how about, how you make this, you said you divided 48 by 4 and you got 12 and then 12 times 3. Right? That makes this new amount. Wait a second. What part of 48 pounds is the amount you made?
M: 36.
T: Yeah, 36 pounds. What part is that? How much is that of the 48?
M: That is four-thirds (seemingly with no confidence). Oh, wait, that is three-fourths.
T: Oh, this (36-bar) is three-fourths? of the 48? Oh, wait, you also said four-thirds. What do you think about when you said that?
M: I was thinking about the other way. I was thinking about it is being switched. (He uncovers the 48-bar and colors alternately 12 parts in the 48-bar.)
T: Can you tell me now about the four-thirds? When you said four-thirds, what do you think? There is a bar up there that is four-thirds of another one, that's what you are thinking about?
M: No.
T: No? OK, let's try a different one unless there are some questions.

When asked to make three-fourths of the 48-bar, Mike produced a 36-bar by saying "divided the 48 by 4 then I got 12, then multiplied 12 by the 3." His dividing activity opens a possibility of his gaining an ability to construct a unit fraction of a composite unit. However, we need to carefully examine whether he related his operating, dividing and multiplying, to constructing a fraction because the 36-bar seemed to be produced by relying on calculating.

For a while, he considered the 12-pound part, resulted from dividing the 48-bar into 4 parts, as a third as he explicitly mentioned, "One-third of 48 pounds." The comment indicates that he conflated the fractional whole and used 36 rather than 48 pounds. However, I attend to the fact that he employed "dividing by 4 and multiplying by 3" to obtain three-fourths of the bar.

It seems certain that dividing and multiplying are relevant to his concept of fractions; however, I doubt he was engaging in partitioning because the “dividing the 48 by 4”activity did not imply any insight into the reconstitution of the 48 into equal parts.

Regarding his confusion about the fractional whole, I consider a possibility that the parts and the whole were not conceived of using equi-partitioning. Equi-partitioning the 48-bar into four parts implies an ability to reconstitute the bar using the produced part. So, if he was engaged in equi-partitioning when he divided the 48 by 4, the result of his dividing should mean one out of four parts because partitioning is a psychological structure that includes both producing parts and reconstituting a whole by the part produced. As a result, multiplying the 12, one part, by three would mean the collection of three of the 12s, and that would lead to three-fourths. However, the 12 pounds representing one part was not one-fourth in the sense that he knew that four of them constituted the whole. The fact that he would implement dividing and multiplying without equi-partitioning implies that the result of dividing and multiplying would just be meaningful as the 36-bar, not in terms of one part that could be iterated three times to produce 36. In addition, it means he would consider the whole 48-bar as a given rather than a reconstituted quantity by the parts. Therefore, even though he produced the 36-bar as three-fourths of the 48-bar using the 12 pounds, the 12-pound part did not relate the 36-bar with the 48-bar

However, I infer he related his multiplying activity to constructing a fraction. Mike multiplied the 12 by three to produce the 36-bar. When he conflated the fractional whole with respect to the question concerning how much the 12-pound part is of the 48-bar, his answer one-third was in reference to the 36-bar produced. He seemed to relate his multiplying by three to constructing a fraction one-third. This way of thinking shows progress in Mike’s concept of fractions because he previously neither tried to divide a composite unit to produce a fraction [cf. Protocol M2.4] nor created a reciprocal relationship with respect to repeating.

Mike finally answered “One-fourth” with respect to the question, “What part of the original 48 pounds is 12 pounds?” and elaborated on how he produced it using two procedures: “dividing 12 into 48,” and “12 over the 48 then simplifying.” The term “dividing” in his statement was just to produce a quotient 1/4, so the “1/4” that he produced did not refer to his partitioning activity. His dividing was not a conceptual activity to construct a fraction based on operating. Further, even though he was prompted to relate the result of his dividing with multiplying, he did not seem to consider how dividing and repeating could be related to each other for a fraction one-fourth because, when asked “What is 12 pounds, what part of 36 pounds is 12 pounds?” he seemed to just realize he was supposed to use the 12 for the 36-bar: “It’s … Do I have 12?” In addition, when asked to think of the 12 pounds in terms of the 36-bar, he kept saying one-fourth until he interpreted the question by looking at the 36-bar in which one 12-pound part was colored blue, which was prompted by the teacher. He did not develop a concept of a unit fraction beyond a part-whole comparison. In particular, his response “No” to the teacher’s probing “When you said four-thirds, what do you think? There is a bar up there that is four-thirds of another one, that’s what you are thinking about?” clearly shows that his mentioning four-thirds did not involve reasoning based on conceptual activities. Therefore, I conclude that his construction of three-fourths or four-thirds was not based on equi-partitioning.

Ashley’s response: Mike is not transferring “divide 48 by 4, get 12” to 12 out of 48 (03/30/06).

Ashley watched Mike engage in Protocol M4.1. The observation provided an opportunity for her to construct a concept of a unit fraction explicitly based on equi-partitioning and to conceptualize it toward its iterative aspect.

Protocol A4.1 (1): Comments on Mike's conception of one-third and one-fourth.

I: What do you think [about] his conception of one-third and one-fourth?

A: I don't think he has a picture concept or well, he has a calculation concept like he can calculate it but he is not transferring to this picture (the 48-bar alternately colored into 12 parts). Again, he knows that because she is asking about three-fourths, he has something to do with 3 and something to do with 4. But, he is not understanding those three parts of four. Or, I don't think. So, the idea getting a big picture, he understands that there is 48 pounds and I want three-fourths of it. To get one-fourth or he is using that idea, [he] divided 48 by 4, get 12, then he is not transferring that to be 12 out of 48, that is one-fourth because he knows she is asking about three parts, so he is thinking about three. Just making that connection.

I: Even though he used 12 pounds to multiply [by] three [to get three-fourths], he was not aware that this 12 pounds is as one-fourth of [the 48-bar].

A: Right, by thinking multiplied by 3, so he is thinking it must be a third. You know, I said, to get three-fourths, he said I took that 12, then multiplied by 3, so it must be one-third. Maybe that's what he is thinking.

I: Ah, with 12 pounds, he multiplied by 3,

A: To get three-fourths,

I: To get three-fourths, that's why 12,

A: That must be one-third.

I: Even though 12 of the original pounds (pointing at the 48-bar)

A: He is not thinking of it that way. There is 12 out of 48, it is 1 out of 4, it's the same ratio, he is not thinking of that.

I: He just focused on how many [times] he repeated.

A: Yes.

Ashley mentioned two concerns related to Mike's construction of fractions. The first one is that he is not relating "dividing by 4" with "1 out of 4." She pointed out he did not notice that the 12 pounds became one-fourth of the 48-bar even though he divided the 48 by 4. In addition, she interpreted his difficulty producing one-fourth for the 12 pounds as resulted from his inability to transfer "dividing 48 by 4" to "12 out of 48." In light of her conceptual progress of fractions, "out of" implies an activity for a partitive unit fractional scheme. So, "out of" in her statement requires equi-partitioning. Such a concern about his lack of the "out of" concept is very crucial in developing her insight into Mike's development of a view of fraction as operations because an ability to relate equi-partitioning with a construction of a unit fraction is critical in developing a view of fractions as operations.

The second concern is about repeating for creating a fraction. Listening to his reasoning about the answer that the 12 pounds is one-third of the 48 pounds, she realized that he was focusing on the number of repetition necessary for the 36 pounds, and the repeating activity led him to say one-third. This insight is very significant in that it would prompt her to become aware of the importance of the simultaneous aspect of the operations, dividing and repeating, performed for producing a fraction. When asked to create a question that would help the student conceptualize a unit fraction, she responded as follows (A stands for Ashley):

Protocol A4.1 (2): A question to help conceptualize a unit fraction.

A: If you take a group of 12, and you say how much is 12 out of 24, you know if you have 24 total, so he can repeat that group of 12 twice because he counts them and notice

> that 24. So there is one group, I have two sets, that's equivalent to half, one out of two. . . . . When he said if you divide 48 by 4 and you get 12, when he said that, you can say, what if you divide 36 by 12, what answer would you get, so using his calculations, 3, so how many equal pieces are there, how many groups of 12 there are, 3, so one group of 12 must one out of 3, one-third.

The above statements clarify that she attempted to associate dividing and repeating simultaneously with creating a unit fraction. Also, the sequence of the questions above shows that she had an explicit idea of an iterable unit fraction because she wanted him to think about 12 pounds in terms of the 36-bar as well as the 48-bar. So, when asked about her observation of Mike's conception of one-third or one-fourth to figure out three-fourths or four-thirds, she elaborated as follows:

> Protocol A4.1 (3): Ashley's comments on Mike's construction of four-thirds based on one-third.
>
> A: He doesn't see the connection yet. Because he is not, essentially he is guessing (inaudible) just say that, because at first he said that 12 out of 48 is one-third, and then one-fourth, he is just guessing. But [if he] had exact amounts to compare, it could help him to visualize it. Because if you had 12 three times, that would be 36 down below it. (inaudible) the 36 is divided into 3 groups of 12, one two three, and so how many of these one-thirds are up here (pointing at the 48-bar), one two three four.

While elaborating on Mike's construction of a fraction four-thirds, she explicitly used the following phrases in order to represent a fraction related to one-third: "12 three times," "divided into 3 groups of 12" and "how many of these one-thirds." Such an explicit use of the phrases involving fraction operations helped her to reify a construction of four-thirds on the basis of partitioning and iterating. According to the above comment on a way to produce four-thirds using one-third, it seems certain that she constructed one-third as an iterable unit fraction and then produced four-thirds based on the iterable unit. In particular, her saying “he didn’t get one-third of this (the 36-bar) is one-fourth of this (the 48-bar)” corroborates she established a multiplicative relationship between two bars through the iterable unit fraction.

Following the above protocol, another estimation problem was posed: Make three-fifths of a 60-pound bar without looking at the bar. Mike produced a 36-pound bar by dividing the 60 pounds by 5 and then multiplying the 12 by 3. However, he still seemed unsure that the quotient of 60 divided by 5, the 12 pounds, is a third of the 36-bar. That is, he first answered one-fifth then corrected to one-third by saying “you divide the 36 by 3 then you get 12.” It implies that he considered the given amount as a fractional whole and a fraction in terms of a part belonging to the amount, which is inconsistent with his reasoning revealed in Protocol M4.1. At this point, we need to carefully consider his statement “36 divided by 3 and get 12” compared to 36 divided by 12. Such a way of talking indicates that he did not use the 12 pounds conceptually to generate the 36 pounds; in addition, he was not aware of the 12 pounds as a result of operating.

Responding to his unconfident answer of “one-third” with respect to the question how much is the 12 pounds of the 36-bar produced, Ashley suggested following: “Just ask what is one-third of this one (the 36-bar), of yours, whatever the bottom. What is one-third of it? If he knows 12, 12 is the unit, which is measuring by.” She recognized the importance of finding a unit amount for repeating, but it is not clear that her conception of repeating went beyond implementing segmenting a presupposed amount, so that it is sufficient for understanding the iterative aspect of the unit amount.

Ashley watched Mike engage in making an estimate of his dog's weight so that three-fifths of the dog's weight is sixty pounds. He first drew a solid bar representing sixty pounds, copied the bar and divided the copied bar into three parts. He then pulled one part out and repeated it twice and joined them to the bar divided into three parts. So, he produced a new 5/5-bar that represented his dog's weight. By dividing each of the two parts added into 20 small parts, he figured out his dog weighs 100 pounds and three-fifths of the 100 is 60 pounds. However, it was very challenging to him to answer how much the 100 pounds is of the 60 pounds.

Protocol M4.2: Dominant additive reasoning in a fractional context.

(Mike was asked how to produce 100 pounds.)
M: Because here (pointing at the solid bar) is 60, then I added another 20, so that will be 80, then you added another 20, that will be 100.
T: Wow. That is a big dog, 100 pounds.
O: What's three-fifths of 100?
T: Yeah, what's three-fifths of 100?
M: What's three-fifths of 100? 60.
T: Oh, awesome, awesome.
O: How much is 100 of 60?
M: Uh? How much is 100 of 60?
T: That is what you ask? (Looking at Observer) Yeah, how much is 100 of 60?
M: I don't know.
T: Yeah, you know.
O: You've just got it.
M: Did it?
O: How did you make 100 (inaudible)?
M: Added 40.
T: Yeah, added 40. But you did something else when you were thinking about it, I think you first before you added 40. Remember what you did to make this really long one?
(There was some discussion about a technical problem)
M: First I tried to make a bar as long as this one (the covered one), then I repeat that, copy that one, pulled out one because I knew that this is just three (inaudible) and add two more.
T: It was just only three?
M: Yes. I need two more because this one is already three, because all is three-fifths. So I took out one and repeat that one like that. I divided 60 by 5, wait, hold on (pause), divide by 3, and I get 20, so I put 20 to each one, and I got 40.
T: Uh-huh, you add 40 on. Yeah, you divided that 60 by 3, you said, right? To get the 20? And what you have to do that with the 20 to make the new bar?
M: 20. (Sits quietly without answering for a long time.)

T: How many these pieces (pointing at one part in the copied bar divided into three parts) would you need?
M: How many of these?
T: Uh-huh.
M: Two.
T: Two more?
M: Yeah.
T: How many total?
M: Hund[red], forty? Like these little pieces?
T: All right, you need forty little pieces. Right.
M: I need actually two of these (pointing at the copied bar divided into three parts)
T: Ah, I see. You need two of those more, let's see, 20 pounds. How much is that of what you started with?
M: It's one-fifth.
T: Is it? Of which one?
M: Oh, it's one-third.
T: Oh, it's one-third? Of what?
M: Of the sixty.
T: Of the sixty, oh, OK. What about of the 100?
M: Of the 100? It's one-fifth.
T: Oh, how would you know that?
M: Because if you divide 20 by, by the 100, and you get 5.
T: I see. Wait a second. Let's see if we are gonna answer Mr. Hope's question. So, he asked what is the 100 of the 60.
M: What is the 100 of the 60?
T: What fraction name could you get for the 100 pounds compared to the 60 pounds?
M: Three-fifths or … or 100 out of 60 … maybe a mixed number.
T: I don't know. Would it?
M: Yeah.
T: Yeah, do you think so?
M: (inaudible) it will be one and (long pause) two-fifths.
T: One and two-fifths?
M: Yeah.
T: Oh, okay

Mike's construction of a 100-pound bar was based on an additive way of thinking in that he created the bar by adding parts to a given amount even though he produced a unit amount that can be repeated to make the 100-pound bar. However, I doubt he was engaged in adding two fractions to make a fractional whole, a desired bar, like five-fifths is three-fifths plus two-fifths. Even though he knew that the 60-pound bar is three-fifths and he needed two more parts to make five parts, there is no evidence that he related each of the two parts to the fractional whole five-fifths. It was at the end of the above episode that he first mentioned one-fifth for one part. Responding to the question concerning how much the 20 pounds is of the 100, he answered one-fifth as he said “if you divide 20 by, by the 100, and you get 5.” That is, one-fifth was based on his calculating, not his concern about the 100 as five-fifths.

Mike seemed to engage in whole number addition by transferring three-fifths to three parts. The whole number reasoning involved fractional reasoning in that his producing three parts was to represent three-fifths. In addition, it seems certain that he had a concept of fractions based on part-whole comparisons because he thought of three-fifths in terms of three parts and a whole consisting of five parts. Such a part-whole concept of fractions distracted him in a way that he did not need to consider one part one-fifth of a desired bar while elaborating on his construction of the 100 pounds from the 60 pounds. In particular, I infer his additive way of thinking allowed him to stay away from dealing with fractions in the problem context because additive reasoning does not require a referent once he set a counting unit.

According to his construction, one hundred pounds was made up with two separate amounts, two parts [forty pounds] and the given bar [sixty pounds]. Mike knew that each of the two parts consists of twenty small parts. That means he would already have divided each part of the 3-part bar [the 60 pounds] into 20 small parts in order to produce 60 pounds. Therefore, the forty pounds was related to the sixty pounds through one part [20 pounds]. However, we need to carefully examine his reasoning related to one part in order to see if one part—twenty small parts—is used as an iterable unit. If so, he would be able to establish a relationship between the 60-pound bar and the 40-pound bar as well as a relationship between the 40 pounds and the 100 pounds.

When asked “How much is that [20 pounds] of what you started with [the 60-pound bar]?” Mike first said “It’s one-fifth” and then corrected it to one-third in response to the following question “Of which one?” In addition, he knew that the 20 pounds is one-fifth of the 100 pounds by saying “if you divide 20 by, by the 100, and you get 5.” However, I argue that one part used for one-third did not involve any insight into iterating to create the 60 pounds and further the 100 pounds because the 100 was considered one and two-fifths of the 60.

The fact that one part used for the 60 pounds was not used for the 100 pounds in an iterative manner means he had yet to create an iterable unit fraction in a generative manner: for example, a part can be one-third, one-fourth, one-fifth and so on because we can think of a whole by repeating the part as many times as needed. Such a lack of an ability to construct an iterable unit fraction would affect his difficulty establishing a relationship between the 40 pounds and the 60 pounds.

The above problem context required him to engage in inverse reasoning because he needed to make a fractional whole to which the given fraction three-fifths referred. Given the 60-pound bar as three-fifths of another bar, he divided the 60 pounds into three parts. That implies he had an anticipatory fractional scheme and conceived of a fraction three-fifths through equi-partitioning. In addition, he reversed the operations performed for three-fifths: his dividing into three was justified by saying “I need two more because this one [the 60-pound bar] is already three, because all is three-fifths.” The statement indicates that he was engaged in equi-partitioning on the desired bar. However, he seemed unaware that his dividing the 60 pounds into three parts is also partitioning activity with respect to the 60-pound bar because his answer one-third in response to the question concerning how much the 20 is of the 60 seemed only based on calculating, and one-third was never used for constructing a fraction for the added 40 pounds even though the 40 pounds and the 60 pounds related to each other through the 20.

At this point, let's consider Mike's conception of equi-partitioning through both Protocol M4.1 and Protocol M4.2. In Protocol M4.1 where he was supposed to make three-fourths of 48 pounds, he deduced a fraction only from the number of repetitions. Since a whole 48 pounds was not considered as constituted by a 12-pound part, he conceived of a part only in reference to the produced 36 pounds. A 12-pound part was considered one-third of 48 pounds. As a result, his dividing the 48 pounds into four parts did not mean equi-partitioning. His dividing an amount neither involved a notion of repeating to reconstitute the amount nor did it lead to producing a unit fraction based on the operation of repeating. On the other hand, in Protocol M4.2, Mike implemented equi-partitioning on an anticipated whole even though it was not the whole but the given amount that he actually partitioned. In addition, he explicitly knew that the 60-pound bar that he divided was constituted by three parts, which shows some progress from in Protocol 4.1. So, I can say he was engaged in equi-partitioning both the 60-pound bar and the desired bar. However, he had yet to develop the equi-partitioning on the 60 pounds as the splitting operation: with respect to the question how much 100 is of 60, he answered one and two-fifths. If the splitting operation was available to him, his dividing the 60 into three equal parts would imply his ability to reconstitute the 60 in terms of three of the one part produced, which means one part should be one-third of the 60 pounds. Furthermore, the one part used for reconstituting the 60 could be used to construct a fraction for the added two parts completing the 100 pounds, so the 100 should be five one-thirds of the 60.

I argue that differentiating an operation from a result of the operation is critical in developing the splitting operation because Mike's inability to construct one-third for the question concerning how much 60 is of 100 seemed related to his confusion between dividing into three parts and producing one-fifth of a desired bar. The differentiation between operations and results of operating assumes that the individual is consciously aware of operating. So, we need to first investigate whether Mike was aware of his operations when producing a fraction. When asked "How much is that [the 20 pounds] of what you started with [the 60 pounds]?" he first answered one-fifth and then corrected it to one-third. The 20 pounds was a result of his dividing the 60 pounds into three parts. However, his answer "one-third" did not seem to involve the dividing activity. He rather seemed to arrive at one-third by finding a quotient of the 60 divided by the 20 because he did that for the 100 pounds. Therefore, I conclude that he was yet to be explicitly aware of equi-partitioning when producing a fraction even though his construction of a fraction involved equi-partitioning. That is, he deduced a fraction in terms of a quotient of two quantities, a given amount and a resultant amount.

Such a lack of awareness of fractions as operating would be closely related to an additive way of thinking. He conceived of a fractional whole additively like the 100 pounds is "two more," not two-fifths more, than the given three-fifths representing 60 pounds. Thus, his attention was limited to the parts needed for making up a whole, thereby disregarding his partitioning the given 60-pound solid bar, which was used to conceive of three-fifths. As a result, he did not establish any relation between the 60 pounds bar and one part on the basis of his operating.

Mike seemed to consider fractions as resultant amounts based on a predetermined whole. Such a view of fractions has common ground with a partitive fractional scheme in that the scheme enables one to produce a fraction by disembedding a part(s) out of a presupposed whole. However, his view of fractions as resultant amounts did not involve an explicit notion of operations of dividing and repeating different than in a partitive fractional scheme. He had difficulty identifying a whole when relating parts with a whole: he verbalized “one and two fifths” with respect to the question, “What is the 100 of the 60?” Therefore, his responses such that 20 pounds is a third of the 60-pound bar and a fifth of the 100-pound bar would be separately processed in his mind and procedurally produced. That means 20 pounds would not contribute to establishing a relationship between the two bars. As a result, he viewed fractions as resultant amounts without concerning his operating.

Ashley’s response: A big idea is finding a unit to measure by and then repeating that unit (03/30/06).

Ashley watched Mike struggle with the question “How much is 100 of 60?” The observation provided an opportunity for Ashley to solidify an insight into an iterative aspect of a unit fraction. The following protocol shows her development of an iterative aspect of a unit fraction based on her view of fractions as operations.

Protocol A4.2: Comments on Protocol M4.2.

I: He correctly answered 20 pounds is one-third of 60, one-fifth of 100.

A: Yes.

I: But do you think at the same time--?

A: No, he is not connecting it, still.

I: What do you think he would be missing?

A: Okay, 20, how much of the 60? One-third. I think I would ask, how many groups of, sets of 20 would you have to, would you need to make 100? I mean, I might say, how many sets, and then he gets 5, well, 20 is how much of 60? One-third. So, how many one-thirds?

I: Ah, how many one-thirds.

A: Five one-thirds. So 100 must be how many thirds, you know, say like that. 100 is how much of 60? I don't know, just going back to that repeated (inaudible).

I: What is the big idea of your question?

A: A big idea is finding a unit to measure by and then multiplication and repeating that unit. Those are big concepts like you take that, that whole unit, how many repeats of it would you need? And taking the unit (inaudible) relative, that group of twenty relative to 60, it's three groups, or groups of 20 relative to the 100, that is 5 groups, I don't think he is ready yet necessarily for the, like there is a wording issue, I think. It's just a bit confusing. When you just straight go into, this is, there is I can't put a finger on it, but there is a wording, kind of threw me off (inaudible) maybe he was getting confused about the same thing.

Ashley did not think that Mike's responses such as the 20 pounds as one-third of the 60-pound bar as well as one-fifth of the 100-pound bar, he produced at the end of Protocol M4.2, implied an insight into a relationship between the 20-pound part and each of the bars: "he is not connecting it, still." Mike had difficulty creating a fraction with respect to a question concerning how much the 20 pounds is of the 60 pounds on the basis of his construction. Related to the difficulty, she thought that he was missing the idea how many groups of the 20 pounds are in the 60 pounds. That is, she was concerned that he did not notice a relationship between an activity and a result of the activity: dividing into three parts produces three identical parts. Such a concern implies Mike's inability to construct a unit fraction relating it with equi-partitioning.

As long as he relied on part-whole comparisons, he would think of the parts in terms of the numerosity rather than a relation between a part and a whole consisting of the parts. She argued that once he considers the dividing activity as relating a part with the whole on which dividing was implemented, the activity will be tied with repeating that leads to generating a whole: “that group of twenty relative to 60, it’s three groups, or group of 20 relative to the 100, that is 5 groups.” Her comment was not about a part-whole comparison but about an issue of constructing a part that can be repeated. So, in light of the question she liked to pose, “how many one-thirds [to make the 100-bar]?,” it is clear that she wanted to encourage him to think of one part as an iterable unit and many parts as iterated. By observing Mike, she explicitly developed an iterative aspect of a unit fraction from the perspective of fractions as operations.

Coordinating Various Levels of Units for a Multiplicative Insight Into a Fraction

Ashley watched Mike engage in making a share for one person and producing an equation based on his construction; the posed problem was that two one-inch candy bars are shared among three people. Previously, given a similar problem “Making a sandwich so that a 2-foot long sub sandwich is three times longer than it,” he relied on a partitive fractional scheme and had difficulty implementing distributive reasoning [cf. Protocol M2.2]. Thus, three levels of units were not produced in his construction. He knew that one part is one-third of one bar and two parts representing a share for one person are one-third of the whole thing. However, the two relationships were not related at his conceptual level even though he was able to distribute a unit of three across each foot in the 2-foot bar. In addition, when producing an equation referring to his construction, he had difficulty connecting his operating with constructing a fraction. The following protocol shows that he still had difficulty producing an equation reflecting his construction even though he was able to coordinate two levels of units involved.

Protocol M4.3: Producing an equation referring to a share for one person when two bars are given to be shared among three people.

T: Make two candy bars. Make one candy bar and copy it.
M: (Makes one candy bar and copies it.)
T: If you wanna share among three people, what do you do?
M: (Divides each bar into three parts.) share among three people?
T: Make a share for one person.
M: (Pulls one part out of each bar divided.)
T: Is that the one person share? How many times bigger is the candy than the share?
M: Three times?
T: Three times. Give me, give me, give me an equation for that. You don't have to use x any more.
M: OK, it will actually be one times three equals three. Or, hold on.
T: How much of the whole candy bar do you have, how much of all the candy did you have?
M: (Vertically joins the two parts pulled out.) How much of all the candy do I have? One-third.
T: You got, how much of the whole candy?
M: One-third.
T: One-third. So can you make an equation now?
M: Hum.
T: All the candy. OK? All the candy, just (inaudible) right? So, three times.
M: Three times one third equals one third, hold on. (Seemingly calculating in his head and pretending writing something on the screen) three times one equals one (pause).
T: Three times…
M: I can't get it.
T: All the candy, the candy is a unit, OK? It's gonna be, let's gonna say that's gonna be, one, one, one gram of candy, OK?, so three times a third equals?
M: (After a short pause) one gram?
T: One gram, OK, let's do another problem.

When asked to make a share for one person using two bars, Mike constructed it by engaging in distributive reasoning: he divided each inch bar into three parts, made a share for one person using two parts, and then knew that the set of two bars is three times the share for one person and the one-person share is one-third of two bars. However, it was very challenging for him to produce an equation involving one-third on the basis of his construction. For him, knowing that a given amount is made up of "three times the share for one person" did not mean having constructed one-third through the share. Even when reminded that his construction resulted in one-third of the given amount, he was not able to produce a reasonable equation involving one-third and finally said “I can’t get it.”

It seems that his conception of one-third relied heavily on a part-whole comparison while lacking an insight into an iterative aspect of one-third. In addition, he seemed to have problems with conceiving of two bars as a unit for fractional operating. He correctly answered the two questions: how much of all the candy the share for one person is as well as how many times bigger the given amount is than the share. However, he was not able to connect “three times” resulting in two bars with “a third” produced for the share for one person. Furthermore, even though he made an equation “three times a third,” he did not arrive at either one or two, which might imply two bars. That opens two possibilities related to his concept of fractions. First, a concept of one-third was not related with repeating three times to generate a whole, especially in the case of a composite unit, at his conceptual level. Second, a composite unit was not conceived as a unit for fractions. He might have noticed that the answer of “three times a third” should be “one” procedurally but have hesitated to verbalize the answer because the “one” conflicted with “two” perceived by two bars in his mind. As a result, he had difficulty constructing one-third of a composite unit and understanding one-third as a multiplicative relationship between the given two bars and a share for one person.

Distributive reasoning requires coordinating two levels of units and understanding an iterative aspect of a unit amount. In particular, an awareness of commutative reasoning seems necessary for distributive reasoning involving partitioning. For example, consider a situation that four people share a 3-foot long bar. We notice two quantities, three and four, and construct two composite units—a unit of four and a unit of three. The composite unit of four is distributed into each foot, and the other composite unit of three is restructured by “three of four.” Then, we have to think of a share for one person from the new structure—three of four. The construction of a composite unit of three based on an iterable unit prompts iterating a share for one person within one foot three times, thereby producing another structure for the 3-foot bar, "four of three." Since both structures, three of four and four of three, are results of coordinating the unit of three and the unit of four, they should be equivalent.

According to such a concept of distributive reasoning, Mike had yet to develop distribute reasoning. He knew that two parts, each of which came from each bar divided into three parts, make up one-third of the whole candy but seemed unable to reconstitute the two bars in terms of three of two parts while considering the bars as a fractional whole. Rather, he seemed to produce one-third perceptually like two parts out of six parts. However, it is very interesting that he was able to make an equation using "a third" and "three" when the teacher explicitly introduced a customary unit, a gram. That shows a high possibility that he resisted saying "one" for "three times a third" because of his inability to conceive of two bars as a unit for operating. That implies constructing a composite unit as a unit for fractional operating would be necessary for distributive reasoning involving equi-partitioning.

Ashley's response: I don't know why calling two bars as a certain unit helped him to produce a multiplication better (04/18/06).

Protocol A4.3 (1): Comments on Protocol M4.3.

A: He knew he had one-third. And it's back to that the same idea we've been working with before, he knows that he has to do with one-third. He knows he has to do with one and three. But he still isn't quite getting how to put those together.

I: One and three?

A: Right, he knows its, its one-third (pointing at two parts joined vertically), he knew that it was one-third. And he knew he had something to do with the whole set, one whole set of candy bars but he couldn't quite figure out what, what to do to get there, to get to that one whole set.

I: Um, yeah.

A: But I feel like the teacher was asking the right questions to lead him the direction.

I: How?

A: By saying, well you've got how much? One-third, Oh okay, and we are thinking of this is a unit. So that is one whole unit, so one-third times what gives you one unit. But he didn't know the answer still. He still couldn't think one-third times what will give me that one. Um, he, the teacher might say how many times do you have to have this one-third to make the one, the whole set of candy bars we started with. How many, how many times do we have, that something like that.

I: So, here, yeah, right. He had no problems making one-third for a share for him. It's one-third, these two candy bars, from these two candy bars. But when the teacher asked him, just give me a multiplication for the situation, he couldn't do that. So, my question is what kind of understanding of this problem might make him think of this situation multiplicatively, or make [him produce] a multiplication for this situation?

A: OK, so what would make him think about a multiplication in this problem? Well, OK, I still think leading him in the, OK, you've got the one-third, how many times saying the word times, but not using it as the word that means multiplication, you know because we say 3 times 2 that means the same thing as 3 multiply by 2, it means the same thing. But if you say how many times would you use this, then he'd say, oh we used this one-third three times, now can you tell to me as a math problem?

I: Yeah, he explicitly said I need three times this one, but [even though] he already knows this one is one-third, he couldn't make three times one-third,

A: He still didn't get.

I: Yeah, what would be the problem for him?

A: I am not exactly sure.

I: When the teacher just gave him some specific unit amount, let's say this candy bar is one gram.

A: When he called it a gram, uh-huh.

I: He was able to say.

A: (Simultaneously) do it. Well, that's true. I forgot about that. You're right. But I don't know the difference there, I don't know why calling it as a certain unit helps it, helped him to do it better. I don't know I don't know the reason why that would make it easier. Because I would tend to think that candy bar is a unit that students (inaudible) instead of a gram, it would be easier than thinking candy bars. But I am not sure why that was easier

I: Do you think if the, if the teacher just gives this problem with one candy bar, do you think he could solve the problem?

A: Like a…

I: Saying make a multiplication.

A: I don't think that would make that much difference because he understands the concept of one-third of it, still, I mean he did that part right so I don't think having one two and three or more of bars there in the beginning, because he knows that pull that unit out, that one-third of it.

Ashley assumed that Mike's difficulty producing an equation was due to his inability to put together three kinds of quantities—one-third, one, and three—in terms of his construction. That means she assumed that he would already have conceived of one-third, one, and three through his construction. So, her concern about his difficulty directly led to her wondering whether he could find a relationship between one-third and one: "he still couldn't think one-third times what will give me that one."

I am concerned with her assumption that he would not have a problem to consider two bars as a unit. I infer that such interpretation of the student's work would be closely related with her conceptions of fractions. In the previous section [cf. the text following Protocol 3.7], I discussed that she was finally aware of a composite unit as a unit for fractional operating, but it was not accomplished independently. The above protocol shows that she might still have been considering equi-partitioning without a notion of a composite unit as a unit for partitioning. Given such a lack of the notion, it seems clear that she set the two bars as a fractional whole without a deep understanding of two bars as "one" consisting of two units of one. This is the indicator that she did not consider any possibility that his difficulty producing an equation might be due to confusion of "one."

I was wondering about Ashley's interpretation of Mike's construction of one-third. Did she explicitly consider that dividing each bar into three parts results in dividing the whole bar into three parts? Did she look at two parts for a one-person share in terms of three groups of two parts or one group of six parts? Did she use a concept of a unit of two units of one for fractional reasoning? When asked whether the number of candy bars would matter for him to solve the problem, she responded it would not affect him because he understands a concept of one-third and knows what to pull out: “he knows that pull that unit out, that one-third of it.” Such a response makes my answers to the above inquiries in the negative because she seemed to consider a third as relying only on the whole bar (two bars) without any concern about a composite unit as a unit for partitioning.

Ashley seemed more interested in reversing an operation like multiplication as a reverse of division than the issues discussed above such as distributive reasoning, constructing a composite unit as a unit for partitioning, and two levels of units. She thought that a division problem is more concrete and straightforward than thinking of it in an inverse way, in terms of a multiplication problem: 36 divided by 9 vs. 9 times something equals 36. Reflecting her experience, she said:

> Protocol A4.3 (2): Ashley’s comments on making a multiplication problem from a division problem.
>
> A: With the idea of reversing the operation, they [students] know that, they may know that 36 divide by 9 is 4, they know that is a fact in their mind. But to say the inverse of that operation and to be able to know the connection, that 9 times x will give you 36, I am saying 9 times what, which is the same as 36 divide by 9, they don’t make that connection, from my experience.

Ashley was concerned that multiplying would need a different conceptual process than dividing. So, when Mike showed some difficulty producing an equation using a times-statement, she seemed to think that he knew that he divided one by three and got a third but he did not know that the division problem can be transferred to a multiplication, a third times three equals one. I think that could be a main reason that she focused only on the process of going to “one whole” from one-third.

Ashley watched Mike establish an equivalence relationship between one-ninth and seven sixty-thirds based on his construction. The teacher probed Mike’s units-coordinating scheme involving equi-partitioning and further distributive reasoning in a fractional context because he previously had difficulty constructing a composite unit as a unit for partitioning, which is necessary for distributive reasoning [cf. Protocol M4.3].

Protocol M4.4: Constructing commensurate fractions, 7/63 and 1/9.

T: Make seven candy bars.
M: (Makes one bar on the upper left corner of the monitor, and copies the bar three times down the below and another three times right below. So, four bars were made on the left side of the monitor and three bars on the right side.)
T: If you share that among 9 people, how much of all the candy will one person have?
M: (Divides each bar into 9 parts and produces seven 9/9-bars.) (Moves the top bar on the right side to further to the right side) that’s (the moved bar) is how much one person gets. Wait, not that.
T: Make a share for one person.
M: (By pulling one part out of a bar or moving a bar, he makes seven 1/9-parts disembedded from the seven bars.)
T: Terrific. What fraction of all the candy is that?
M: It’s one-ninth of all the candy.
T: Really, why is that?
M: Because I divide each everyone into nine, each of candy bar is divided into nine-ninths, so then I took out one from each one and then all that, if you’d line them all up, like that (placing a 9/9-bar right below another 9/9-bar), then it all be one-ninth right here (pointing his cursor at the right most parts of the bars on the left side; he seems to assume all the seven bars are lined up on the left side.)
T: So, all the candies, how many times a share for one person?
M: How many--
T: I am asking again. All the candy is how many times a share for one person.
M: (Makes all the seven 9/9-bars vertically adjacent; the result looks like a big rectangle, which has 7 rows and 9 columns.)
T: That (pointing at the seven 1/9-parts disembedded) is a share for one person, right? All the candy is how many times that?
M: All the candy is times it by 9? (He vertically is aligning all the seven 1/9-parts.)
O: Say that a little louder, Mike.
M: Um. (He completes the vertical alignment.) How much is this (the seven 1/9-parts vertically lined up) times this (the stacked seven 9/9-bars)?

T: How much is this (the seven 1/9-parts) of the whole candy?
M: How much is this of the entire candy? All this (moving his cursor over the stacked seven 9/9-bars), right here?
T: Right.
M: Let's see, it would be nine sixty-thirds. (Nine sixty-thirds is his mistake of seven sixty-thirds, and the teacher did not notice it.)
T: Okay, nine sixty-thirds. This is nine sixty-thirds of all the candy (pointing at the stacked seven 9/9-bars), right? Can you give me another fraction for this (the seven 1/9-parts)?
M: Another fraction?
T: How much is that of the whole candy? You said it before.
M: One-ninth.
T: Why is that?
M: Because if you divide, hold on, if I took out one from each of everyone in here, so now that will give me one-ninth because I divide each of everyone into nine.
T: So, that is one-ninth of all the candy, right? How much is one piece of this candy bar (pointing at the 9/9-bar at the top of the stacked bars)?
M: This is one-ninth of that candy bar (the top bar).
T: So, this is one-ninth of that (the top bar)?
M: Uh-hum.
T: So, each piece is one-ninth of each candy bar, right?
M: (Nods but seems unsure.)
T: So, all seven pieces (pointing his finger at the seven 1/9-parts lined up) is how much of the whole candy bar?
M: Seven and sixty thirds.
T: Seven sixty-thirds, you said seven sixty-thirds equals one-ninth?
M: (Stares at the monitor for a while as he moves the parts.)
T: Do you think seven sixty-thirds is equal to one-ninth?
M: Seven sixty-thirds is equal to one-ninth? (He looks like calculating in his head.) If you divide 7 by itself, I mean divide this sixty thirds by seven and you get one-ninth because 7 goes into one and.
T: Okay, can you explain why seven sixty-thirds is one-ninth just using this picture (pointing his fingers at all the bars and all the parts) without the (pointing at his head) but what you just did is fine, you divided them out, okay. Can you explain just using this, what you did here?
M: Using this (leaning toward the computer as he grasps the mouse)?
T: (Inaudible) for you. So, you said if you take one piece out of each, the candy bar, in this candy bar, nine pieces, so this (one part) is one-ninth of that candy bar (one bar), right? So, each piece is one-ninth of each candy bar (matching each part with each bar using his finger), that's what you told me. Does that help you explain why one-ninth equals seven sixty-thirds?
M: (Stares at the monitor for a long time. He matches each part with each bar using his cursor. Then he sat quietly without answering for a long time.) Yes, because each of every one of those is already one-ninth, and this entire row (pointing at the first column in the seven 9/9-bars lined up) is one-ninth, and each of every one of these where, where-- each of every bar is split into 9, so that means that--.

Mike coordinated two composite units, a unit of seven and a unit of nine, by distributing the unit of nine into each of seven 1-inch bars and producing seven 9/9-bars. To produce a share for one person, he pulled each 1/9-part out of each of the 9/9-bars and lined the 1/9-parts up and produced seven 1/9-parts aligned vertically. He then promptly called the seven 1/9-parts "one-ninth of all the candy" and showed such an answer was based on one bar as nine-ninths: "I divide each everyone into nine, each of candy bar is divided into nine-ninths." This is the clear indication that he coordinated two levels of units. However, it is not clear how he transferred "one-ninth" as referring to a 1/9-part into "one-ninth" as referring to seven 1/9-parts. When asked concerning "All the candy is how many times a share for one person?," he sat quietly for a while and answered nine times. That indicates his previous prompt response "one-ninth of all the candy" had not involved a view of seven 9/9-bars as a composite unit for partitioning. By saying "nine times," he began to deal with seven 1/9-parts as well as seven 9/9-bars as units for fractional reasoning. At this point, we need to examine how he used one of the 1/9-parts in order to establish a relationship between seven 1/9-parts and seven 9/9-bars, that is, how he constructed commensurate fractions.

Responding to the question "How much is this (the seven 1/9-parts) of the whole candy?" Mike answered nine sixty-thirds, but I guess he meant seven sixty-thirds. He considered the seven 9/9-bars as a fractional whole based on 1/9-parts, and he realized a 1/9-part should be one sixty-thirds. As a result, he developed an insight into converting one level of unit into a higher level of unit in terms of one 1/9-part: from a 1/9-part as a unit to a unit of seven 1/9-parts. However, he seemed unsure of the teacher's comment "each piece is one-ninth of each candy bar." Even though he verbalized "I divide each of everyone into nine" "This (a particular 1/9-part) is one-ninth of that candy bar (the bar including the part)," he never mentioned one-ninth for a 1/9-part while considering all the bars. Thus, the teacher's comment on associating each 1/9-part with each bar as an element of seven bars confused him because it required him to consider seven 1/9-parts in terms of one bar as well as seven bars.

When asked to relate one-ninth and seven sixty-thirds , Mike calculated to simplify seven sixty-thirds into one-ninth without any concern about the involved fractional units: "if you divide 7 by itself, I mean divide this sixty thirds by seven and you will get one-ninth because 7 goes into one." I doubt that his saying "7 goes into one" involved any sense of constructing an iterable composite unit. However, by the teacher's probing questions, he finally elaborated his understanding of the equivalent relationship between seven sixty-thirds and one-ninth on the basis of his constructs: "because each of every one of those is one-ninth, this entire row (pointing at the first column in the seven 9/9-bars vertically lined up) is one-ninth, and each of every one of these, each of every bar is split into 9, so that means that--." This statement clearly shows that he coordinated two levels of units—a unit of one bar and a unit of seven bars. Furthermore, it opens an ability to coordinate three levels of units because he became aware of a unit of a 1/9-part while considering a unit of seven units of one bar.

Ashley's response: Mike was missing the point that there were seven bars (04/18/06)

Protocol A4.4 (1): Comments on Protocol M4.4.

A: He was getting the idea. He was missing the point that there were seven bars. He can say more I have one-ninth of each one. He does. But one-ninth of how many bars, and I don't think stacking them up like this helped particularly. I mean he was kind of solve (inaudible) little bit, moving them around, but he wasn't making the connection that I have one-ninth or I divided each individual bar into nine pieces, and I have seven bars.

I: Ah, that is the missing part.

A: I think, I mean. He knew that there were seven. But he wasn't saying that, that's why he couldn't figure, I mean he knew mathematically using calculations, 7/63 is the same thing as 1/9, but--

I: [Do you mean that] even though he [knew there are] seven little pieces [in] this stack (pointing at the seven 1/9-parts vertically aligned), he wasn't aware of each little piece from one candy bar?

A: Right, I mean, he knew that from one candy bar, but he wasn't making that connection that there are sixty three little pieces. I think that's what he was missing. There were sixty three of the smallest pieces.

Ashley pointed out Mike's unawareness that seven bars were involved in the seven 1/9-parts he produced. In addition, she thought that he was not associating one-ninth of each bar with one-ninth of seven bars. This insight is very important in her development of fractional reasoning because she previously never mentioned anything about the relationship between two kinds of units in an explicit way. In Protocol A2.6 (March 2nd), where she watched Mike make a bar so that a 20-centimeter bar is 2/5 of the bar, she commented that Mike did not care about centimeter units, so he lacked an insight that the 20-centimeter bar consists of twenty 1-centimeter units. In Protocol A3.3 (March 9th), she implemented distributive reasoning by coordinating two levels of units implicitly: one-fifth of a 3-foot bar was constructed only as three-fifteenths, and an insight into one-fifth of one foot was not involved in her reasoning about one-fifth of the 3-foot bar. Therefore, she had been able to coordinate two levels of units but had yet to improve her ability to coordinate two levels of units toward three levels of units. In addition, in Protocol A4.3 (1), she tried to understand Mike's difficulty producing an equation involving a times-statement based on his equi-partitioning. Rather than focus on various levels of units, she paid attention to his conceptual difficulty converting division into multiplication. Finally, she made no comment on Mike's difficulty relating one-third of one bar to one-third of two bars.

Coordinating three levels of units involves many aspects of fractional reasoning: a units-coordinating scheme in fraction contexts, constructing a unit fraction of a composite unit, constructing an iterable unit fraction, and distributive reasoning involving partitioning. In that sense, her explicit comment on Mike's unawareness of seven bars for one-ninth of one bar would affect her fractional reasoning toward coordinating three levels of units. However, Protocol A4.4. (2), the continuation of Protocol A4.4 (1), shows that she was not still associating one-ninth of one bar with one-ninth of the entire bar in an explicit way.

Protocol A4.4 (2): (Cont.)

I: For him, [do you think] this equivalence relationship [between] seven sixty-thirds and one-ninth, this equivalence relationship, [is] only just by calculation?

A: I think so.

I: Not in terms of this kind of thinking (what Ashley pointed out in Protocol A4.4 (1))?

A: Not yet.

I: Could you give me a problem related to this problem that you think he could solve?

A: That he could solve (pause). I would say something along the lines of, how many candy bars, just questioning, how many candy bar were there total? Just start with. Then hopefully he would remember there were seven. And then you divided each one into how many pieces,

I: In total or--

A: I just ask him and he would say I divided each one into nine pieces, so how many small pieces do you think there were now? And so, he hopefully come back to that 63 because these are the easy ones, you said that this (one column in seven 9-bars vertically lined up) is one-ninth. So how many those small pieces do you have?

I: [Did you say] this (a seven 1/9-parts column) is one-ninth?

A: He said that early, I would come back to it.

I: All together.

A: Uh-huh.

I: This (one column consisting of seven 1/9-parts) is one-ninth.

A: How many those little pieces is that one-ninth? 1, 2, 3, 4, 5, 6, 7 so is that the same amount? Is seven sixty-thirds, seven of these sixty three, is that the same as this one string, one-ninth? I will try to lead him back to that.

I am interested in how Ashley related one bar with seven bars in terms of Mike's way of thinking. She first attended to that there were seven bars, and then she turned her attention to dividing each bar into nine parts and then to the number of the parts comprising seven bars. However, she did not make any comment on a relationship of one part based on nine parts with respect to sixty-three parts. That implies she distributed a unit of nine across each bar without noticing two interrelated units—a unit of one bar and a unit of seven units of one bar. Therefore, her ability to coordinate three levels of units still remains unanswered even though she established an equivalent relationship between seven sixty-thirds and one-ninth in terms of the seven 1/9-parts.

To investigate Mike's construction of commensurate fractions, the teacher posed a sharing problem to Mike with a bigger number than in Protocol M4.4: sharing eleven bars among 13 people. Ashley watched Mike engage in the sharing problem. He first divided each bar into 13 parts and pulled one part out of one of the bars divided into 13 parts, and then repeated the 1/13-part 11 times for a share for one person. Ashley differentiated his repeating activity from the previous activity—pulling one part out of each bar and joining them—as she said, "Maybe this is the first time, maybe he made that connection, that it's the same little piece." This insight would give her an opportunity to relate a 1/13-part based on one bar with 11 bars.

Protocol M4.5: Constructing a composite unit as a unit when finding thirteen times one-thirteenth.

(Mike was asked to make a share for one person using 11 bars if they were shared among 13 people. He first produced a 1/13-part by dividing one bar into 13 parts and an 11-part bar by repeating the 1/13-part 11 times.)
T: How much is that (the 11-part bar) of all the candy? How many pieces did you get?
M: I got 11.
T: Okay, 11-bar, right? How much is that of the whole candy? Give me two fractions.
M: How much is that of the entire candy? It's one-thirteenth and it's also 11 over
T: (inaudible) How many pieces did you make? Just give me a multiplication. How many pieces did you make?
M: How many,
T: How many little pieces did you make? You put thirteen in each of these (pointing at the bars). Just give me a multiplication fact for that, what 11 times--.
M: 11 times 13.
T: 11 times 13, that's enough. You don't have to figure that out. Okay, that's kind of doing mental math but did you agree how much is that of the whole bar?
M: That's 11, that's one-thirteenth of them,
T: All the candy, how many times that piece (the 11-part bar)?
M: All the candy is how many times this (the 11-part bar)?
T: Uh-hum. How many times bigger is all the candy than up there?
M: Thirteen.
T: So, what is thirteen times one-thirteenth?
M: Thirteen times one-thirteenth equals one (seemingly he got it by calculating in his head).
T: One, so one refers how much? What does one refer to in this case?
M: One refers to eleven, to one-thirteenth. Wait. One refers to (pause).
T: Thirteen times one-thirteenth is one. I ask you how many times, how many times bigger is all the candy than one little piece if one little piece is one-thirteenth of all the

candy, right? So, you took thirteen times one-thirteenth and got one. What does one refer to?
M: One refers to one of these little bars? (I am not sure what he is indicating.) One refers to (pause).
T: This (the 11-part bar) is one-thirteenth of all the candy, right?
M: Yeah.
T: So, if you picked this (the 11-part bar) thirteen times, what would you get?
M: Then you get, all this (pointing his cursor at the 13 bars).
T: You can get all the candy, right?
M: Yeah.
T: So, thirteen times one-thirteenth is one, repeat this, repeat this thirteen times, this is one-thirteenth of all the candy, so repeated one-thirteenth thirteen times.
M: Hold on.
T: Right? So, what does one refer to?
M: One refers to all the candy.
T: One refers to all, how much you have. Okay. So, one refers to all the candy. That is how much you have.

Mike constructed one-thirteenth of 11 bars by coordinating one bar with 11 bars through equi-partitioning one bar into 13 parts. For the 11-part bar consisting of eleven 1/13-parts, he produced two kinds of fractions referring to all the 11 bars: one-thirteenth and 11 over 11 times 13. He seemed to clearly know that the 11-part bar indicates one-thirteenth of the entire bar based on his construction. However, it took him some time to answer that all the 11 bars are 13 times the 11-part bar. This indicates he was in the process of developing a concept of an iterable composite unit and constructing a composite unit as a unit for partitioning. He also seemed aware that he divided each bar into 13 parts to get one-thirteenth of all the bars. This shows some progress in his fractional reasoning from that in Protocol M4.4 where he noticed dividing one bar into nine parts by distributive reasoning but did not seem to consider one-ninth of seven bars based on one-ninth of one bar.

Mike's saying 11 over 11 times 13 shows he conceptualized one-thirteenth of the entire bar through a relationship between the 11-part bar and the given 11 bars. In response to "What is thirteen times one-thirteenth?" he promptly answered "one." However, the question "What does one refer to in this case?" that followed challenged him. He seemed to arrive at the response "one" by mental calculation without referring to his construction. The teacher reminded him of his construction: "one-thirteenth" for the 11-part bar and "thirteen" he produced with respect to the question "How many times bigger is all the candy than up there (the 11-part bar)?" However, without the teacher's intensive intervention, he did not find a referent for his answer "one." That implies two important aspects of his fractional reasoning. First, he had yet to construct a composite unit as a unit for fractional reasoning. When saying "eleven" as an answer to "What is 13 times 1/13?" he might have thought of an 11-part bar as an answer because that is what he would need to iterate 13 times. Second, he had yet to develop distributive reasoning involving partitioning. That means he did not reconstitute eleven 13/13-bars in terms of thirteen 11-part bars even though he distributed equi-partitioning a bar into 13 parts across the 11 bars and produced an 11-part bar made up with eleven 1/13-parts, or an 11/143-bar.

Ashley's response: The operation of repeating calls Mike's attention to connecting the parts based on one bar with the entire bar (04/18/06).

Ashley paid attention to Mike's repeating one 1/13-part 11 times for making a share for one person. Regarding his repetition, she commented that such an activity would influence his ability to associate one bar with 11 bars. Her comment seemed to be a result of the teacher's provoking Mike to associate "repeating one 1/13-part" with "repeating the 11-part bar." She thought that such a repeating activity led him to consider 11 bars in terms of "11 times 13" by encouraging him to imagine one 11-part bar through one 1/13-part. In Protocol A4.4 (1), concerning Mike's difficulty relating one-ninth of one bar with one-ninth of all the 7 bars, she pointed out his lack of awareness that there are 7 bars: "He was missing the point that there were seven bars." Reminding herself of that concern, she anticipated that his activity of repeating one

1/13-part 11 times for a one-person share would help him associate the 11 bars with a notion of one-thirteenth of one bar because he would know each 1/13-part comes from each bar.

Regarding his difficulty finding a referent for “one,” an answer for “thirteen times one-thirteenth,” she raised language issue: “‘Refers to’ is not something, I think, he used a lot in seventh grade everyday language.” That is, she seemed to not consider his confusion concerning the referent of “one” would be related to a conceptual issue such as an issue of units. She argued that repeating activity was not involved in his construction of one-thirteenth of the 11 bars by saying “because the teacher helped him to get there to figure out the one-thirteenth 13 times equals the one whole, but he still needed the help.”

Toward Three Levels of Units

Ashley watched Mike engage in a sharing problem, share a 7-inch long bar equally among 11 people. He first divided each inch into 11 parts and pulled one part out, then produced a 7-part bar by repeating the part pulled out, an 1/11-part, seven times. Referring to the 7-part bar, he confidently answered the following questions: “How much is that (one part) of one inch?” “It’s 1/11”; “How much is it (one part) of the whole 7 inches?” “1/77”; “How many pieces would one person get?” “They will get 7”; “How much is that (the 7-part bar) of one inch?” “It’s 7/11”; and “How much is it (the 7-part bar) of the whole thing?” “It’s 7/77.” Mike seemed able to perfectly coordinate three levels of units. However, in order to make sure of his ability to coordinate three levels of units, it is necessary to carefully investigate how he was conceptualizing one-eleventh of the 7-inch bar using the 7/11-bar. In addition, we need to consider that it is not clear how he arrived at the above answers because the teacher did not ask him to elaborate how to get the answers at that time.

Protocol M4.6: Producing commensurate fractions through various levels of units, one-eleventh and seven seventy-sevenths.

(Mike was asked to make a share for one person using a 7-inch bar to share among 11 people. He produced a 7-part bar by repeating one 1/11-part 7 times.)
T: How much is it (the 7-part bar) of the whole thing?
M: It's seven (pause) seventy-sevenths.
T: Seven seventy-sevenths. Well, seven seventy-sevenths, is that the share for one person?
M: Yeah. How many people?
M: Eleven.
T: So, how much is that share of the whole candy?
M: One-eleventh.
T: Wouldn't you tell me seven seventy-sevenths?
M: Because I was thinking about the entire bar like all little lines (pointing at the 7-inch bar he produced by dividing each inch into 11 parts) so I thought that.
T: Would you think seven seventy-sevenths equals one-eleventh?
M: Yeah. Hold on, yeah.
T: Yeah, why is that?
M: Because if you divide it then you divide 7 by 7, which equals one, and divide 7 by 77, which equals 11.
T: Okay, that's computations. Why don't you think about it in terms of this up here (pointing at the bars, the 7-part bar and the 7-inch bar divided into 77 parts, on the monitor)?
M: In terms of this?
T: Can you explain that? Okay, you did the calculation to tell me that seven seventy-sevenths equals one-eleventh (inaudible). I agree with that, 100 percent. Okay, totally right. But I want you to explain up here why is, why is that one-eleventh of the bar (the 7-inch bar) is seven seventy-sevenths of the bar?
M: Why is that one-eleventh is seven seventy-sevenths of the bar?
T: Of the whole thing, of the whole 7.
M: Of the entire 7,
T: Why is that one-eleventh of the thing, by the way, you put it unit bar here. Okay, accept the unit bar (pointing at a function button to set a unit bar), just check it out, and then you can measure it. What do you think you get if you measure?
M: One-eleventh or seven seventy-sevenths.
T: Okay, either way.
M: (Measures the 7-part bar, and "1/11" appears on the 7-part bar)
T: You got one-eleventh, right. Why are those two things equal up here?
M: Because (pause), because if you multiply that (the 7-part bar) by 11 you get the entire unit bar, and that, because you need 11 pieces to make the entire bar-- (pause).
T: You know they're equal, right?
M: Yeah.
T: 11 pieces in each one of seven, right?
M: Yeah.
T: And then, so then you have pulled out, so how many little slivers do you need?

M: You need 11 of them to make one-seventh.
T: Yeah, how about to make one-eleventh?
M: To make one-eleventh, you need seven.
T: You need seven of them to make one-eleventh, right. Did you take, did you think, did you take one, what if taking one little sliver out of each one of the seven inches?
M: You get one-eleventh.
T: You get one-eleventh, right. And then also seven what?
M: Seven seventy-sevenths.
T: Seven seventy-sevenths because each one of those is one seventy-seventh. You are right on, okay.

Mike's construction of one-eleventh with respect to the 7-part bar was based on distributive reasoning. He produced the 7-part bar by repeating one 1/11-inch 7 times and justified that the 7-part bar is one-eleventh by saying "if you multiply that (the 7-part bar) by 11 you get the entire unit bar." In particular, his mentioning "the entire unit bar" for the 7-inch bar indicates he developed a composite unit as a unit for fractional reasoning even though it was each inch that he actually divided. I am interested in investigating whether Mike related such a concept of one-eleventh based on distributive reasoning with one-eleventh of one inch. I also wonder how the two kinds of one-elevenths related to each other at his conceptual level if he did.

Mike never mentioned one-eleventh for a 1/11-inch part even though he divided each inch into 11 parts and explicitly knew that eleven 1/11-inch parts comprise one inch. Instead, he used "one-seventh" to refer to one bar: "You need 11 of them (the 1/11-inch parts) to make one-seventh." In other words, he considered one inch as one-seventh because the 7-inch bar was considered as a unit for partitioning. Therefore, I conclude that his equivalent relationship between one-eleventh and seven seventy-sevenths did not involve a notion of a unit of one inch; it was only the 7-inch bar that he considered as a unit throughout the problem context. Furthermore, I argue that his distributive reasoning was based on whole number reasoning rather than fractional reasoning involving equi-partitioning. This argument sheds light on differentiating Confrey's splitting from Steffe's unit's coordination involving equi-partitioning.

Mike engaged in Confrey's splitting: he would think of one inch in terms of eleven parts and the 7-inch bar in terms of seventy-seven parts as he set a counting unit as one 1/11-inch part. That means he did not need to consider three levels of units. So, he would have difficulty engaging in distributive reasoning involving equi-partitioning such as one-eleventh of seven inches equals seven of one-eleventh of one inch.

Mike answered seven-elevenths in response to the question about the share for one person in inches. However, I doubt his answer involved an insight into three levels of units because, unlike improper fractions, seven-elevenths could be recognized by a perceptual comparison of the parts between the 7-part bar and one inch consisting of eleven 1/11-inch parts. That is, he might not have produced "seven-elevenths" for the 7-part bar in a way that it involved one-eleventh of seven as well as seven of one-eleventh of one inch. Thus, given a problem leading to an improper fraction, I infer that he would have difficulty establishing an equivalent relationship using various units involved, such that one-seventh of nine is equal to nine-sevenths of one.

Ashley's response: Mike has yet to answer the question what is 11 times 7/11 (04/20/06).

Ashley made a comment on the various units that emerged as Mike engaged in the sharing problem, a 7-inch long bar is equally shared among 11 people [cf. Protocol M4.6]. For the question, "What kinds of units do you think are involved in his mind to answer these questions?," she mentioned "the whole 7-inch bar," "one inch," "one-eleventh of one inch," and "one seventy-seventh of the whole." But interestingly she never mentioned one-eleventh of the 7-inch bar. Regarding the 7-part bar produced for a share for one person, she paid attention to the two ways that Mike responded: seven-elevenths of one inch and seven seventy-sevenths. One-eleventh of the 7-inch bar was not mentioned. This is an indication that she tended to consider units in terms of a single element, not multiple elements.

As she interpreted Mike's construction of the 7-part bar for a one-person share, she focused on his mentioning a 1/11-inch part as one seventy-seventh: "First he divided into the inches and then he said I took a sliver from each and he said seven seventy-sevenths." She pointed out that he considered a unit of one inch only when he divided a given unmarked bar into 7 parts in order to represent 7 inches, because afterwards he never used one inch as a unit. She noticed that as soon as he divided one inch into 11 parts, the produced parts were considered one seventy-seventh referring to the 7-inch bar rather than one-eleventh referring to one inch. In addition, she argued that his saying one-eleventh was based on repeating the 7-part bar 11 times, but he figured it out in terms of one seventy-seventh of the 7-inch bar: "He is counting like I have seven of them, so that makes one-eleventh because, and he said because 11 of them fit in there (pointing at the 7-inch bar), but that is not one inch. You know, he didn't call it one inch. So I don't think he is thinking in terms of inches as a unit." I wonder how Ashley related Mike's two responses, seven-elevenths of one inch and one-eleventh of the 7-inch bar, because it will give me her insight into three levels of units.

When asked "Do you think he could answer the question, what's 11 times 7/11?" she said "I just don't, he is not there yet (pause). I just don't think he would answer that" without further elaboration. Answering the question "what is 11 times 7/11" in the problem context requires an understanding of 7/11 as one-eleventh of the 7-inch bar. I doubt she had such understanding of 7/11 at that time because she never seemed to consider the 7-part bar as a unit of the 7-inch bar, which means a unit of the 7-inch bar did not involve a composite unit structure based on the 7-part bar at her conceptual level.

The teacher briefly engaged Mike in a different sharing problem, which was about sharing a 7-inch bar among 5 people. Like what he usually did, he divided each inch into five parts and collected seven of the parts by pulling one part out of each inch. He called it seven thirty-fifths and correctly stated "one-fifth of 7 inches" by using LABEL to inscribe "1/5" on the 7-part bar. After watching the above Mike's engagement, Ashley commented as follows:

> Protocol A4.6: Mike is referring to only one kind of whole.
>
> A: He has referring back to the whole bar, both of them (seven thirty-fifths and one-fifth) though. So he said one-fifth of the whole bar, seven thirty-fifths of the whole bar, and he said that each, he said something like sliver, what did he say, this sliver is one-fifth of each inch or something like that. So, I think he is still thinking of this (the 7-inch bar) as the whole.

I focus on Ashley's mentioning one-fifth in terms of two referents, the whole bar and each inch. As a matter of fact, Mike did not mention one-fifth of each inch while working on the above problem that she watched. It was not until this time that she described his dividing each inch into parts by using a unit fraction referring to a part of one inch. However, she did not maintain the notion of one part as one-fifth so that she could relate one-fifth of one inch with one-fifth of the 7-inch bar. Therefore, I infer she still had difficulty coordinating three levels of units even though she noticed one-fifth of one inch as a unit with respect to the 7-inch bar.

In Protocol M4.4, I discussed a possibility for Mike to coordinate three levels of units with the teacher's help. In Protocol M4.6, he seemed to engage in Confrey's splitting, producing multiples of multiples of an original one. The parts in his reasoning seemed recognized only in terms of multiples of the smallest part, which implies his reasoning did not involve three levels of units. The following protocol shows that coordinating various levels of units was still unavailable to him without the teacher's guide. In the protocol, he was asked the following question: if a 7-inch bar is 25 times longer than the bar you are going to make, how long is the other bar?

> Protocol M4.7: Confusion over the coordination of three levels of units: 7/25 of one inch and 1/25 of seven inches.
>
> T: Make a 7-inch bar.
> M: (Divides a given solid bar into 7 parts).
> T: I want that bar to be, just say, 25 times longer than another bar. What would you do?
> M: I would put 25 inside each one and take out one [from each] and you get seven twenty-fifths.
> T: Okay. So if you formed that bar, do you know how long the other bar is gonna be? This (the 7-inch bar) is 25 times longer than the other bar, this is a 7-inch bar.
> M: It would be.
> T: How many inches long is it?
> M: How many inches longer?
> T: Yeah,
> M: It would be.
> T: How many inches long is the other bar? This (pointing at the 7-inch bar) is 25 times longer than the other bar.
> M: Oh, (Sits quietly for a long time without answering). What the person get would be--(long pause).
> T: You know, how to make it, right?
> M: Yeah.
> T: Make it. Go ahead, make it.

Mike knew what he should do to produce a bar so that a 7-inch bar is 25 times the original bar. Even though he did not actually make it, he elaborated what he would do to produce the desired bar: "I would put 25 inside each one [inch] and take out one [from each inch, or from one inch and then repeating the pulled part seven times]." He then commented on the desired bar in his mind: "you get seven twenty-fifths." However, it is not clear what the seven twenty-fifths was referring to.

When asked "How many inches long is the other bar (the bar Mike verbally described)?" he sat quietly for a long time without answering and did not produce any kind of answer. Even though he mentioned “seven twenty-fifths” for the bar in his mind, he never revisited the fraction to respond the teacher’s question based on a specific referent. Given similar problems as above, he tended to finalize his construction by mentioning a unit fraction referring to a given whole bar, such as “you get one-eleventh (of the 7-inch bar),” “one-fifth (of the 7-inch bar),” or “you would get seven-thirty fifths (of the 7 inch bar).” However, for this time, he ended with a non-unit fraction “seven twenty-fifths” which is not referring to the given 7-inch bar.

I infer that the fraction “seven twenty-fifths” reflected the results of his units-coordinating activity—dividing each inch into 25 parts (one twenty-fifth) and repeating one part 7 times (seven twenty-fifths). However, he seemed unsure of the referent for one twenty-fifth because he would be referring to the 7-inch bar for the relationship “25 times” for the 7-part bar in his mind. He would not realize that the construction he verbally described meant using one inch as a referent, because he considered the 7-inch bar as a fractional whole throughout the problem context. In addition, he would think that the 7-part bar should be based on the 7-inch bar. As a result, he had difficulty establishing a relationship between the 7-part bar and the 7-inch bar by using one twenty-fifth of one inch. Therefore, he had yet to construct one twenty-fifth of one inch as a unit with respect to the 7-inch bar, which implies he had yet to construct three levels of units.

Ashley’s response: The bigger number threw him (04/20/06).

Ashley watched Mike engage in Protocol M4.7 and came to doubt Mike’s ability to construct a unit fraction of a composite unit. When asked to predict his response related to the 7-part bar he verbally described as the desired bar, she recalled his tendency related to similar problems to the above: “He was taking one of the fifths from each of the sevenths then he called it, he called it one-fifth.” However, she doubted that he would implement the same reasoning this time: "He took one, hold on, I try to say right, one twenty-fifth from each of them but I don't think he was gonna say it one twenty-fifth."

Protocol A4.7: Mike's struggle with a big number when coordinating three levels of units.

I: What do you think would make it difficult?

A: The bigger number threw him, I think, having the 25 (pause).

I: So the previous problem, only just 5 times longer than, he could, maybe--

A: Do the calculation and also easier to see.

I: Yeah, easier to see, right.

A: But if he did with those 25 little lines for each inch, he couldn't even see it, you know, I think it's hard to picture. But the bigger number threw him, I think.

I: So, when he said one-fifth for 7 little pieces [in a previous problem], the one-fifth maybe wasn't yet conceptualized.

A: He seemed to but maybe not, after this problem.

By watching Mike struggle with a big number, Ashley began to doubt his previous constructions like they might have been based on calculation or pictorial understanding. She seemed to feel a need to distinguish Mike's activity at a conceptual level from it based on routines because she might have imputed her understanding to Mike's activity. This kind of experience that provoked some doubt about Mike's activity was very beneficial to her later investigation of Mike's way of thinking in that she would be careful to interpret what he is doing and investigate what he can do or cannot do.

She verbalized a unit fraction using two kinds of referents: "one of the fifths from each of the sevenths" and "then he called it a fifth." This is the first time that she clearly mentioned a unit of one and a composite unit consisting of the units of one at the same time. Previously, even though she had divided a unit of one into parts, she tended to think of the parts only based on a whole amount consisting of the units of one. As a result, constructing commensurate fractions tended not to be based on distributive reasoning involving equi-partitioning. For instance, one-fifth of each inch was not involved in constructing one-fifth of seven inches. Now, she clearly showed she constructed two kinds of one-fifth, one referring to one inch and the other referring to seven bars. That means she developed a sense of three levels of units.

Ashley continued to watch Mike engage in the problem given in Protocol M4.7. Since the teacher allowed him to make the desired bar he verbally described in Protocol M4.7, he divided one inch into 25 small parts, pulled one out and repeated it seven times, so he produced a 7-part bar consisting of seven 1/25-parts.

Protocol M4.7: (Cont.)

T: Okay, each little sliver, how much is of one inch?
M: It’s one twenty-fifth.
T: Okay, how many, how much of one inch is that (the 7-part bar)?
M: It will be seven twenty-fifths.
T: Seven twenty-fifths of one inch. Right now, Mike, how many times longer is that (pointing at the 7-inch bar)?
M: That one is 25.
T: What’s 25 times 7/25?
M: 25 times 7/25 is (pause) (He seems to calculate the product in his head) one hundred and --
T: Oh, no, don’t do that. You took 25 times 7/25, right. Do you have to do that?
M: No, I’m not sure.
T: Do you have to take 25 times 7/25? We already, how would you make this? You made this (the 7-part bar), this is 7/25 of one inch, right?
M: Yeah.
T: You said that, you made that. So this (the 7-inch bar) is how many times longer than that (the 7-part bar)?
M: 25.
T: If you take 25 times, what do you have to get?
M: You get seven twenty-fifths, hold on, you get 7 inches.
T: You get 7 inches, right?
M: Yeah.

T: So, what's 25 times 7/25?
M: 7 times 7/25 is 7 inches.
T: You can get 7 inches.

With respect to the teacher's probing questions concerning 7 times 7/25, Mike produced three kinds of answers: 1) one part is one twenty-fifth of one inch; 2) the 7-part bar is seven twenty-fifths [of one inch]; and 3) the 7-inch bar is 25 [times longer than the 7-part bar]. According to his previous construction, the answer "25 times" implied that the 7-part bar is one twenty-fifth of the whole 7-inch bar. However, I infer his saying "seven twenty-fifths" for the 7-part bar would not be based on a units-coordination that his distributing activity meant. He rather seemed to produce seven twenty-fifths based on a part-whole comparison like 7 out of 25.

When asked "What is 25 times 7/25?" Mike tried to calculate it mentally without considering his construction. The teacher led him to remind himself of the construction of the 7-part bar as seven twenty-fifths of one inch. Then the teacher asked him what the relationship was between the 7-part bar and the 7-inch bar. The sequence of the questions guided Mike to see the 7-part bar as seven twenty-fifths of one bar and one twenty-fifth of the 7-inch bar. By the well-sequenced probing questions, he finally answered "7 inches" in response to the question "What is 25 times 7/25?" This answer can be considered as an indication that he coordinated three levels of units with the teacher's guidance, but corroboration is needed to consider his answer as an indication.

As I discussed in Protocol M4.7, I wonder if Mike related one twenty-fifth of one inch with one twenty-fifth of the 7-inch bar at his conceptual level, and how he did it if so. A relationship between one twenty-fifth of one inch and one twenty-fifth of the 7-inch bar is very critical in investigating three levels of units. A units-coordinating scheme generates two levels of units by coordinating two composite units, a unit of 25 and a unit of 7. The generated units are a unit of 7 ones—the 7-inch bar—and a unit of one containing 25 1/25-parts—the one inch part. At this point, we should consider the unit of 25 is commensurate with the unit of 7 because the problem was stated like the 7-inch bar is 25 times longer than another bar, so students would imagine a unit of 25 in terms of the 7-inch bar. So, the coordination shows that the unit of 25, commensurate with the unit of 7, is transferred to a unit of one containing 25 1/25-parts. That is, a unit of one becomes a referent for a 1/25-part produced by distributive reasoning. As a result, by considering a unit of 25 while referring to the generated two levels of units, we can establish a relationship between one twenty-fifth of one and one twenty-fifth of seven. The relationship requires an insight into a unit of a 1/25-part as a part comprising one twenty-fifth of 7. Due to the teacher's intervention, I cannot see his independent way of thinking related to three levels of units. However, it seems clear that he had a chance to coordinate three levels of units through the 7-part bar involved in a times-expression "25 times 7/25."

Ashley's response: He wasn't thinking in terms of inches (04/20/06).

Ashley watched Mike engage in Protocol M4.7 (Cont.) and was concerned with the units involved in Mike's activity. She pointed out that Mike did not get the answer—7 inches—until the end of the protocol and just followed the teacher's guidance. While elaborating on what Mike did, she said "he said that it's one twenty-fifth of each inch so one twenty-fifth of the whole," but Mike did not say that. Regardless of whether her comment was based on what actually happened, it makes it clear that she had a sense of one twenty-fifth of one inch as a unit with respect to the 7 inches.

Protocol A4.7: (Cont.) Comments on Protocol M4.7 (Cont.).

I: (The teacher asked him) what's the 25 times 7/25?

A: (Shaking her head) he didn't get it [though]. (After a short pause) so, he wasn't thinking in terms of inches.

I: Not in terms of inches (I did not mean to agree with her and just intended to repeat her saying), seven twenty-fifths.

A: Right, because 25 times 7/25 would give him the 7 inches. But he wasn't thinking in terms of inches for that part of it. He wasn't thinking back to the number of inches.

I: To answer the question in inches, what do you think he need to do?

A: Maybe, I mean, maybe even saying, you know, Mike, you said this is seven twenty-fifths of one inch,

I: Yeah--

A: So, just say, seven twenty-fifths of one inch, and if I multiply 7/25 of an inch times 25, how many inches will I get? Instead of saying,

I: Ah, what's 25 times, instead of questioning what's 25 times 7/25, just how much.

A: Yeah, so you said, the student said, this is 7/25 of one inch, so if I multiply that by 25, how many inches will I get? Just keep it in that unit. That would help, maybe.

Ashley thought Mike struggled with the question "What is 25 times 7/25?" because he was not thinking of the seven twenty-fifths in the question based on inches. Before the above protocol, she had admitted that he knew the 7-part bar is seven twenty-fifths of one inch. Therefore, her comment on inches means he would not have had any difficulty if he had successfully recalled that seven twenty-fifths in the given product indicates seven twenty-fifths of one inch. In addition, by emphasizing the referent for seven twenty-fifths is a unit of one inch, she seemed to focus on how many inches "25 times 7/25" will be reduced to, that is, "7 of 1/25 of one inch" 25 times will make 7 inches. Such a concern about seven twenty-fifths seemed to call her attention to one twenty-fifth of one inch rather than one twenty-fifth of the 7 inches. Therefore, I infer she had yet to develop "one twenty-fifth of the 7 inches" from her reasoning "7 of 1/25 of one inch" 25 times. That means her understanding seven twenty-fifths did not yet imply coordinating three levels of units. As a result, I conclude that she had an insight into three levels of units in a fractional context in that she deduced a fraction by repeating 7/25 of one inch to make the 7 inches. However, she had yet to coordinate three levels of units in that it was not explicit that she conceived of 7/25 of one inch based on one 1/25-part comprising one twenty-fifth of 7 inches.

Fraction Multiplication Based on Fractional Reasoning

In whole number contexts, a times-operation means repeating one of the operands as many times as indicated by the other operand; that is, "*a* times *b*" means repeating *b* as many as *a*. On the other hand, if a fraction is involved in a times-operation, we can consider two cases: regarding "*a* times *b*," 1) *a* is a whole number and *b* is a fraction 2) *a* is a fraction and *b* is a whole number. Each case requires different operations at a conceptual level. For example, 3 times 1/5 is solved using the idea of repeated addition, whereas 1/5 times 3 is solved by employing distributive reasoning. Related to conceptual understanding of a times-operation involving a fraction, I will investigate Mike's and Ashley's concept of fraction multiplication centering on the following three questions: What kinds of fractional schemes are required? How many levels of units need to be conceived of? What kind of perspectives of fractions is critical?

Ashley watched Mike engage in the question "Mr. T is 7 feet tall. He is three times as tall as the other guy. I want to know how many feet tall the other little guy is." For the other guy's height, Mike produced a 7/3-bar by dividing each foot into three parts, pulling one part out and repeating it seven times. Then he was asked "What is three times seven-thirds?"

> Protocol M4.8: Coordinating three levels of units: 7 feet, one foot, and one third of one foot.
>
> T: How many feet is that (the 7/3-bar)?
> M: That's two and one-third.
> T: Two and one-third feet?
> . . .
> T: How tall did you say, two feet and what?
> M: Two feet one-third.
> T: Two feet and one-third, can you give me a fraction for that?
> M: Seven-thirds.
> T: How do you know that?
> M: Because I took out 7 pieces (inaudible) each piece is a third.
> T: Of a foot, right?
> M: Yeah.
> T: So, a third of a foot. Wow, wow, wow. Okay, so, how many times longer is seven feet than seven-thirds feet?
> M: Three times.
> T: Three times, good. That's three times. So three times seven-thirds equals what?
> M: Seven-thirds equals--
> T: No, three times seven-thirds.
> M: Three times seven-thirds equals 7 inches [feet].

Mike had no problem with the question "three times seven-thirds." Corresponding to his construction of the 7/3-bar, he explained two and one-third consists of seven one-thirds of one foot, seven-thirds. So, when answering the question "three times seven-thirds equals what?" I infer he would think of 3 times seven 1/3-parts, which makes 21 1/3-parts, and since he knew there are 21 1/3-parts in the 7-foot bar, he could make the claim that: "three times 7/3 equals seven." However, I wonder if he viewed the 7/3-bar as one-third of the 7-foot bar and if his claim involved reciprocal reasoning.

Ashley's response: He might even reverse the operation he carried out (04/27/06).

While watching Protocol M4.8, Ashley showed some insight with respect to Mike's reasoning. She distinguished these three ways of saying: seven of one-thirds, seven-thirds, and one-third seven times as she said “he didn’t say seven one-thirds but he said seven-thirds. . . He hasn’t said that yet. . . . I don’t know that he would express it that way yet” responding to my asking about whether he meant one-third seven times. In addition, she had some ideas to investigate what he could do or could not do based on what he did.

Protocol A4.8 (1): Ashley posing questions that Mike could [or could not] answer.

I: So, could you give me some questions you think now he could answer?

A: Well, I’d like the reciprocal question that, that, the instructor asked, he said what is seven-thirds times three and he said 7 quickly, so he seems to understand that more. Um, he might even reverse the operation that could be, just to see really [inaudible], say, instead of 7 ti[mes], no, instead of 7/3 times 3, he might say what is 7/3, well, he might just reverse it, say what is 7/3 divided by 7, no times 7.

I: Seven-thirds times, seven-thirds--

A: Is that what I am trying to say? (Both laughing) I am trying to get him to do it backward. (She writes something on the table using her finger.) 7/3 divided by 7.

I: Uh-huh, 7/3 divided by 7 (pointing at the 7/3-bar)? Do you mean that? 7/3 divided by 7, what is that?

A: That will give you 3.

I: Three?

A: Right?

I: I don’t think so.

A: No? (Both laughing) Am I off today? (Inaudible) 7/3 divided by 7 (writing something on the table with her finger), yeah, I: This (the 7/3-bar) is seven-thirds [of one foot], if you divide this one (pointing at the 7/3-bar) by 7, maybe what's that?

A: (Laughs.) You are right. Okay, I am just having off today but I am trying to get it though. If the teacher was more on than I am today, being able to ask just reverse the operations to make sure that, he can do better than I can, say it reversed.

I: Reverse, you mean, reverse operation means-- the teacher's question was what is the three times this one (pointing at 7/3-bar) this little bar, so he knew that three times this little bar (the 7/3-bar).

A: It's 7.

I: 7. Seven feet. So, [do] you mean the reverse operation means from here (the 7-foot bar) to here (the 7/3-bar)?

A: Yes, thank you, yeah, that's what I wanna get. If you start with the whole thing, what could you multiply to get the 7/3?

I: Ah, could you make some question for that?

A: Well, I mean, you could say it that way. How did you get from 7 feet to seven-thirds, what would you do? What does that mean to go from, and he even showed it here (pointing at the 7-foot bar and the 7/3-bar) you told me 7 feet, and this (the 7/3-bar) is seven-thirds, so what did you do to get that? Did you divide it, did you multiply it, what did you do? I might lead him that way. So instead of starting (inaudible) him saying 7/3 times what gives you 7?

I: He could answer so, three times, he already knew that, three times. But you wanted him to answer--

A: Seven, what's gonna go from 7 feet to the 7/3? What, would you divide it? Hopefully, he would say, yes, I divided it by three and get 7 over 3, or seven-thirds.

I: What about multiplication from here (the 7-foot bar) to this one (the 7/3-bar).

A: Um.

I: Your question is focus[ing] on division or focus[ing] on multiplication? Or both?

A: Both, I would hope.

I: What do you expect as his answer?

A: If I asked him?

I: If you asked him,

A: He would probably be stuck at this, I don't think he quite got it yet.

Ashley tried to make a reciprocal question with respect to "three times 7/3 equals 7" but could not figure it out by herself. Referring to how she proceeded, I explore her concepts related to a times-operation involving a fraction. First, she intended to replace "times" with "divided by" because division would be considered the inverse of a times-operation. For that, interestingly, she maintained the structure "7/3 something equals." At this point, I am wondering what her idea would be with "three times 7/3 equals 7." Would she have constructed a relationship between the 7-foot bar and the 7/3-bar representing seven-thirds? If so, three should have meant the relationship between them. And also "three times" should have been reversed like "divided by three." However, she finalized her question with "7/3 divided by 7" and even said 3 as an answer for it. Judging from the statement, I doubt that she was thinking of seven-thirds as related to the 7-foot bar at that moment.

As soon as she sorted out her thought about reciprocal with some guidance, she tried to produce a question involving reciprocal reasoning. She first made this question “What could you multiply to get seven-thirds?” She seemed to want to emphasize an inverse relationship between “times” and “division” by asking the following questions, “What did you do to get that (the 7/3-bar)? Did you divide it, did you multiply it, what did you do?” and wished that he would say “I divided it by three.” There is no problem with understanding division as an inverse of multiplication; however, since we are now interested in fractional reasoning, we should not take for granted that “divided by three” is the same as “one-third times.” We tend to consider that a whole given amount is to be divided while we seek a specific referent when dealing with a fraction. So, “one-third times” initiates a question about a referent such as what I should refer to for one-third. That is, “one-third times” requires a concern about the units involved in the problem context. She anticipated that Mike would be able to answer a reciprocal question, which is “7 divided by 3 equals 7/3.” However, she doubted that he could produce an equation involving multiplication for the reciprocal question and commented as follows:

Protocol A4.8 (2): 7 times 1/3 as a reciprocal of three times 7/3.

A: He got the (inaudible) about 7/3 times 3 gives you what? Equal what? That was the big question he kept missing before. So, what we said before, 7 times 1/3, he might not quite be there yet. He can do the calculation of 7 times 1/3, but he might not understand yet the relationship between the bars that he has drawn (the 7-foot bar and the 7/3-bar) and that quite yet to be able to do calculation and see that.

Ashley mentioned “relationship between the bars (the 7-foot bar and the 7/3-bar)” for “7 times 1/3.” This comment is very important because it indicates that for her, “7 times 1/3” is not just repeating one-third seven times but it also implies one-third of the 7-foot bar. In other words, she seemed to think about “one-third of 7” in terms of “7 times one-third.” It is clear that she was considering two units he was dealing with—one foot and seven feet. In addition, an iterative unit fractional scheme was available to her because she constructed the 7/3-bar as an iterable unit fraction, one-third of seven feet. Based on the concepts and schemes available to him, she concluded Mike was yet to think about a relationship between 7 and 7/3 with respect to “7 times 1/3.”

Ashley watched Mike engage in a similar question to the above: “Mr. T is seven feet tall. I want to know how tall a person would be if Mr. T is 11 times as he is tall.” He produced an 11/11-bar by dividing each foot into 11 parts, pulling one part out and repeating it 11 times, and he then split the 11/11-bar into a 7/11-bar and a 4/11-bar. Regarding the 7/11-bar, he first said he got “Seven-seventy sevenths” but corrected it to “It is seven-elevenths” responding to the question “I was asking you how tall this guy was in feet. How much is this of a foot?”

Protocol M4.9: Three different operations: 1/11 of 7, 7 times 1/11, and 7 divided by 11.

T: When you take 7 times 1/11, what would you get?
M: 7 times 11?
T: No, 7 times 1/11?
M: Oh, 7 times 1/11?
T: That’s (one of the 11 1/11-parts consisting of one foot) one-eleventh of a foot, right? What’d be 7 times 1/11 of a foot?
M: 7 times 1/11 of a foot is 11/77. Hold on, is it (pause).
T: I am not saying that’s wrong. What’d be 7 times 1/11 of a foot?
M: 7 times 1/11 of a foot (pause). It’s that (pointing at the 7/11-bar), that seven-elevenths.
T: So 7 times 1/11 is how much?
M: It’s seven-elevenths.
T: Seven-elevenths. Great! What is 7 divided by 11?
M: 7 divided by 11 is seven-elevenths.
T: You got that (laughing). 7 feet divided by 11 is 7/11 of what?
M: Of the entire bar (pause), of one foot.
T: That’s right, seven-elevenths. 7 feet divided by 11 is how many feet?
M: It’s 7 feet, hold on.

T: You just did it. I've been asking a question now. You just got all these now. 7 feet divided by 11 is how much of a foot is that?
M: (Pointing at the 7/11-bar) that's seven-elevenths of a foot.

Mike had an opportunity to think about how one-eleventh of 7, 7 times 1/11, and 7 divided by 11 relate to each other. After constructing a 7/11-bar, he was asked a times-question, 7 times 1/11, and a division question, 7 divided by 11. He produced one kind of answer "seven-elevenths" for those questions with the teacher's help, but he seemed to conceive of each question separately and was yet to connect them conceptually.

Regarding "7 times 1/11," Mike did not reflect his construction of the 7/11-bar by seven 1/11-parts for a while. If he had considered 7 in "7 times 1/11" as the number of repetitions of one-seventh of one foot, he would not have missed the question. So, at the beginning of the above Protocol M4.9, I think he must have referred to the 7-foot bar for 7 in the times-question. So, he would probably consider "7 times 1/11" one-eleventh of the 7-foot bar. His first answer 11/77 reflects such an interpretation because it was based on a 1/77-part—a part in the 7-foot bar where each foot was divided into 11 parts. When the teacher revised the question by adding "of a foot" for "1/11" like "What'd be 7 times 1/11 of a foot?" he finally noticed the question indicates his construction of the 7/11-bar and 7 should be the number of the 1/11-parts.

On the other hand, regarding a division question "7 divided by 11," he quickly answered seven-elevenths but showed a little confusion about its referent. Since he must have looked at the 7-foot bar while solving the division problem, he must have thought of the 7-foot bar as a referent for the answer seven-elevenths. However, interestingly, when the teacher specifically mentioned a unit like "7 feet divided by 11," it did not seem helpful to him. In addition, the question "7 feet divided by 11 is how many feet?" challenged him because it required him to coordinate three levels of units. First, he had to think of a unit of 7 feet according to the problem statement. Then, he would try to think of seven-elevenths in terms of one foot because he knew that 7 feet divided by 11 is seven-elevenths and the answer should be based on one foot. However, since he would have thought the answer seven-elevenths is measurable by one foot, he would finally have to consider one-eleventh of one foot for the answer. As a result, he was yet to relate one-eleventh of 7, 7 times 1/11, and 7 divided by 11 through the coordination of three levels of units.

I discussed that Mike would think of “7 times 1/11” in terms of one-eleventh of a 7-foot bar, thereby having difficulty understanding the question. Let’s closely examine what schemes and view of fractions are involved in “1/11 of 7” and how many units are considered for “1/11 of 7.” To conceive of 1/11 of a 7-foot bar, a unit of one foot should be simultaneously considered with a unit of 7 units of one foot because distributive reasoning is required. That is, a 1/11-part resulting from dividing each foot into 11 parts should be considered in terms of one foot as well as seven feet. The 1/11-part as one-eleventh is based on equi-partitioning and a view of fractions as operations, whereas the 1/11-part as one-seventy sevenths is based on recursive partitioning and a view of fractions as resultant amounts. As soon as the 1/11-part refers to one foot, “1/11 of 7” is transferred to “7 times 1/11” at a conceptual level. That means 7 comes to indicate the number of repetitions or the number of units of one for the distribution of an operation of dividing into eleven parts. Finally, seven-elevenths referring to one foot is conceived of one-eleventh of 7 feet. Such a level of understanding of the 7/11-bar indicates the following three ways of thinking:

- One-eleventh of one foot is considered with respect to the given 7-foot bar (coordinating three levels units).

- The construction of seven-elevenths is based on partitioning and repeating (an operational view of fractions).
- One-eleventh is considered in terms of two kinds of fractional wholes (an iterative fractional scheme).

Ashley's response: There is not a lot of connectivity yet (04/27/06).

While watching Mike's activity through Protocol M4.9, Ashley was pleased that she exactly anticipated what Mike would do. As she expected, Mike was not clear with the question "7 times 1/11" and she commented on that as follows:

Protocol A4.9 (1): Ashley's comments on Protocol M4.9.

A: 1/11 times 7, that took him a while, he wasn't quite there yet, but actually I think having this (one 1/11-part) helped him because he took out the 1/11 and then he said 1/11 7 times, how much of a foot is that? Well, that's 7/11, he finally started to get that. But again not all gelled yet, he was on his way. Much improved since the beginning but he kept wanting to say the piece of the whole instead of the piece of a foot.

Ashley changed the way of saying of the given question: from "7 times 1/11" to "1/11 times 7." It is not clear whether she changed it consciously or not, that is, whether she used "1/11 times 7" rather than "7 times 1/11" or "1/11 7 times." However, judging from the statement "he kept wanting to say the piece of the whole instead of the piece of a foot," she must have been concerned with two kinds of units when thinking about one-eleventh. However, her insight into three levels of unit was implicit in that she seemed to focus on seven-elevenths only based on 7 one-elevenths of one foot without being concerned with one-eleventh of 7 feet. She pointed out he tended to refer to the 7-foot bar for the 1/11-part. This is the indication that she had an insight into three levels of units and a differentiation between fractions as resultant amounts and as operations. She made a comment on his understanding of three operations—7/11, 7 times 1/11, and 7 divided by 11.

> Protocol A4.9 (2): Comments on a relationship among 7/11, 7 times 1/11, and 7 divided by 11.
>
> A: There is not a lot of connectivity yet. Again I think he probably did the calculation of the division, the computation in his head. And it took him a long time to get 7 times 1/11. But he was quick about getting 7/11 in the beginning. So, even though all three of them are the same thing, he doesn't, he doesn't think of them that way yet. So, he can get each thing separately but he is not putting them together yet, so not a lot of connection yet.
>
> I: What problem or what questions would help him make a connection of this division, multiplication and a fraction result?
>
> A: Well, I like that the teacher said what's 7 times 1/11? and let him do that, and immediately said what 7 divided by 11, that, I think that sequence of questions was good because the answers are same, very quick, very close together, that his responses were timed good or timed nicely but I would ask along those same lines. Well, it just does mean the same thing. [Pretending to ask Mike] You said that 7 times 1/11 is 7/11 and then you said 7 divided by 11, 7/11, is that the same thing? Is that the same amount? Something like that. I mean, I think I would ask him and see what he says. [Pretending to ask Mike] is it? It's called the same thing, but is it really the same thing? Maybe he can elaborate more from there.

Ashley clearly distinguished Mike's responses of 7/11, 7 times 1/11 and 7 divided by 11. Mike understood 7/11 as an answer for both questions. He struggled with 7 times 1/11 but quickly answered the division question, 7 divided by 11. Further, she was concerned that those three operations were separated at his conceptual level. As a way to help him connect them conceptually, she emphasized “7 times 1/11” and “7 divided by 11” result in the same amount. As she pointed out, the division question was very well timed to encourage him to consider the unit of 7 feet as well as the unit of one: he was engaged in the division question right after the times-question. I infer that Protocol M4.9 would help her develop an insight into three levels of units more clearly.

Improper Fractions

We tend to understand improper fractions by thinking of them as mixed numbers. For example, 7/3 is 2 and 1/3 because 1 is 3/3 and 2 is 6/3. 7/3 is also regarded as a division, 7 divided by 3, and it leads to 2 and one remaining, 2 and 1/3. Mixed numbers are produced by a measuring out strategy and part-whole comparisons for the leftover. I am concerned that procedural approaches to improper fractions by converting them to mixed numbers would restrict some important fractional reasoning, such as understanding a multiplicative relationship between two quantities, developing an operational view of fractions, constructing an iterative unit fraction, coordinating various levels of units, and so on. In this section, I will investigate Ashley’s concept of improper fractions based on her observation of Mike’s understanding of improper fractions.

Ashley watched Mike engage in the question “Mr. T, 7 feet tall, is 2/3 times as tall as an emu. Make how tall the emu is.” He divided the fourth 1-foot part in the 7-bar representing 7 feet into two parts and pulled one of the 1/2-parts and a 1-foot part out, and then he repeated the 1-foot part three times. So, he produced a 3 1/2-bar consisting of three 1-foot parts and one 1/2-part and added it to the 7-bar.

Protocol M4.10: Producing a fraction for 10 and a half.

T: How tall is that emu?
M: That's (leaning toward the monitor) around (seemingly counting) 10 and a half
T: Can you give me an improper fraction for that?
M: (As he frowns) 10 point five over three, or (pause).
T: When you said 2/3 times, what was, what was the algebraic equation for that now?
M: (Takes a deep breath.)
T: Let's go back (inaudible). Remember that Mr. T is 2/3 times as tall as the emu, so give me equation for that. Everything, you got everything right. OK, just go back to the equation. What would you say about the equation?
M: (Leaning toward the computer) 2/3 times 10 and a half equals 7 feet?

For an emu's height, Mike constructed a 10 1/2-bar consisting of 9 1-foot parts and three 1/2-parts by adding three 1-foot parts and a 1/2-part to the given 7-foot bar. He first produced a 3 1/2-bar by dividing the 7-foot bar in half and considered one-half of the 7-foot bar as the amount to add to the 7-foot bar to produce a desired bar. His way of constructing the bar that represented the emu's height indicates additive reasoning.

Mike seemed unable to think of the 10 1/2-bar in terms of the 1/2-part. In addition, he seemed yet to associate his activity with a mixed number. In other words, he did not notice "10 and a half" is "one and a half" of the 7-foot bar even though he produced the 10 1/2-bar by dividing the 7-foot bar in half and adding the half to the 7-foot bar. He said "10 point five over three" responding to the question "Can you give me an improper fraction for that (the 10 1/2-bar)?"

I infer that even though he had deduced "one and a half" of the 7-foot bar becomes the 10 1/2-bar, he would have had difficulty figuring out the 10 1/2-bar is three-halves of the 7-foot bar. Rather, in light of his response "10 point five over three," he would have been likely to say three-thirds. To conceptualize three-halves in the current context, it is necessary to understand (1) "dividing into two blocks" implies "one-half" and results in "one-third of a desired bar" (an operational view of fractions) and (2) the one-half can be repeated as many times as needed, thereby constructing three one-halves or three-halves (an iterable unit fraction). In addition, by considering that a half of the 7-foot bar is the same as seven of the halves of 1-foot parts (distributive reasoning), the conceptualization of "three-halves" is developed toward twenty-one halves by means of reasoning with three levels of units.

When asked about an improper fraction with respect to his construction, the 10 1/2-bar, he answered "ten point five over three." The response makes it possible for me to infer his construction in the following way: he produced the 10 1/2-bar to make 3/3. That means he did not engage in the splitting operation when dividing the 7-foot bar. In other words, there was no concern about "one-half"—one-half of the 7-bar or one-half of one foot. Instead, he was making one-third given that seven feet was two-thirds of the height of the emu. That indicates a half in his verbalizing "ten and a half" was an absolute amount, not related to his activity of making three and one half. However, it seemed that he clearly thought that the 7-foot bar is two-thirds of the 10 1/2-bar because he was successful in making an equation "2/3 times 10 and a half equals seven."

Ashley's response: The parts in Mike's construction are not equal sizes (04/27/06).

When asked about Mike's struggle with an improper fraction with respect to 10 and a half, Ashley first had no concern about that, but after a while, she focused on a result of his activity.

Protocol A4.10: Comments on Protocol M4.10.

I: This kind of construction (producing the 10 1/2-bar by adding the 3 1/2-bar to the 7-foot bar), do you think [it] could lead him to saying it (the 10 1/2-bar) in terms of an improper fraction?

A: I don’t know (pause). No (pause), not yet. He, again, I see where he was coming from with, with that. I understand that. But 10 and a half (taking a deep breath), no I don’t think that would be that helpful to him because when he is all set done, how many lines does he have on this one (pointing at the 10 1/2-bar), only three, right? Those are, I can only see three divisions on the (inaudible) size?

(There was some discussion to resolve her confusion about the figure of the 10 1/2-bar. The 10 1/2-bar was comprised by the parts in the following sequence; 1 foot, 1 foot, 1 foot, a half of 1 foot, a half of 1 foot, 1 foot, 1 foot, 1 foot, a half of 1 foot, 1 foot , 1 foot , 1 foot . However, she thought only three big parts are shown in the 10 1/2-bar. Through my help, she finally had an accurate figure of the bar.)

A: I don’t think, I still though, I don’t think that’s especially helpful for the improper fraction though because they are not equal pieces.

I: Um.

A: Are they?

I: No.

A: OK, again, look at it there.

I: Some of them one foot, some of them [a half of one foot].

A: Right, right.

I: A half foot

A: Right, right, they are not equal size pieces, so to say part of a whole is not gonna be helpful to have an improper fraction that way.

I: Ah, so if the result he made, he constructed, has equal amounts of pieces, then he could answer.

A: If each piece is an equal size.

I: Equal size.

A: I think he would be, it would be easier to help him get to that.

I: Uh-huh.

A: But when you've got pieces big, and then piece is this big, and you can't say, well how many sevenths is it, or how many seconds is it, you can't say that because you can't visualize it.

Ashley paid attention to the fact that the 10 1/2-bar was not equally divided. Since Mike produced the 10 1/2-bar by three of the 3 1/2-bar, which is a half of the given 7-foot bar, the 10 1/2-bar was made up of three 1/2-parts and nine 1-foot parts. She thought such non-equal parts (mixed up with 1-foot parts and 1/2-parts) would have prevented him from thinking of the 10 1/2-bar in terms of a unit part, so he had difficulty producing an improper fraction for the 10 1/2-bar. Furthermore, she inferred he could have produced an improper fraction if each part comprising the 10 1/2-bar had been in equal-sized. However, such an inference raises another issue: Would he be able to conceive of a 1/2-part as a half after realizing there are 21 parts in the 10 1/2-bar? This is very important in constructing an improper fraction because he should be able to refer to some portion of the total amount, not the total, unlike proper fractions.

Ashley watched Mike have difficulty understanding a problem statement including an improper fraction. Mike was asked "14 pounds of candy is seven-fifths times an amount of candy that I have."

Protocol M4.11: Making an equation using x so that 14 pounds is 7/5 times the amount of candy that I have.

T: Do you think I have less candy than that or more candy?

M: (After a short while) less.
T: This is 7/5 times a candy I have, how much-- less?
M: Less.
T: Less, okay. Let’s say x is how much I have. Can you give me an equation for that?
M: x times 7, I mean, 5/7 equals 14 pounds, or (taking a deep breath) let’s see.
T: x is how much I have. 14 is 7/5 of my candy, 7/5 times my candy. You said I had less, right?
M: Yeah, (pause) so 7, no, I mean, x equals 14 times 7, I mean, 5/7. (He sits quietly for a long time without answering and then takes a deep breath.)

Mike seemed to know that 5/7 is less than one, so if he multiplies something by 5/7 he will get a smaller amount. So, when asked to produce an equation, he seemed to focus on producing a smaller amount than the 14-pound bar: “x equals 14 times 5/7.” The fact that he stated the teacher’s amount correctly does not necessarily mean that he solved an equation, 7/5 times x equals 14. When he first answered “x times 5/7 equals 14 pounds,” the teacher repeated the problem statement including 7/5 several times. However, he never used 7/5 in his response.

One of the reasons students struggle with a question including an improper fraction is that they tend to think that an improper fraction does not make any sense unless a whole for the fraction being asked is given. So, before responding to the question, students tend to ask to let them know what the whole is. When asked to restate the original problem, he could not even do it using seven-fifths and kept using five-sevenths. With the teacher’s probing questions and guidance, he finally produced “7/5 times x equals 14.” Then he was asked to solve the equation.

Protocol M4.11: (Cont.)

T: Can you find out what x is now? How much candy? What would you do?
M: First, I would divide each one into five, take out five of them and then take out two more. That will be how many you have.
T: Then why are you doing that?
M: Because 5/7 is--, because 5/7 is.
T: Go ahead what you wanna do.
M: (Divides each pound in the 14-pound bar into 5 parts and pulling a 1/5-part out of them)
T: How many do you need?

M: (Repeats a 1/5-part pulled out 7 times and produces a 7/5-bar. After a short while, he moves the 7/5-bar along the 14-pound bar until he reached the end of the 14-pound bar like he measures the 14-pound bar using the 7/5-bar. Then he sits quietly for a while.)
T: What did you make?
M: I made five-sevenths, I mean, yeah, I made seven-fifths.
T: You mean seven-fifths of one pound?
M: Yeah.
T: Okay, does it help you solve x, seven-fifths of one pound. So, how many pounds did you have then?
M: I have, how many pounds right there?
T: Yeah.
M: I have one, one and.
T: Two-fifths, right?
M: Yeah, two-fifths.
T: Okay, is that the answer?
M: I am not sure yet (taking a deep breath).

To solve the equation “14 equals 7/5 times x,” Mike made seven-fifths of one pound (a 7/5-bar) and measured the given 14-pound bar using the 7/5-bar. He noticed that ten 7/5-bars fit in the 14-pound bar. However, he never tried to explain how his measuring out activity could be related to solving the equation. There are two interpretations of division: partitive and measurement. The question “14 pounds is 7/5 times the amount of candy that I have” requires him to think of the situation in terms of partitive division, which is finding a unit amount. Given an amount corresponding to seven-fifths, a unit amount is equal to five-fifths of the referent of seven-fifths. So, the unit amount should be five-sevenths of the given amount. However, Mike tried to solve the equation by employing a measurement interpretation of division, which is finding the number of divisors needed to comprise the dividend: how many seven-fifths of one pound is in the 14 pounds. Therefore, he would be confused about what the result of his activity meant—repeating seven-fifths ten times makes 14. That shows his lack of understanding a fraction as a relationship between two quantities.

We can interpret measurement division as partitive division. Mike measured out the 14-pound bar using the 7/5-bar by counting up to ten. That means he considered seven-fifths as a resultant amount. However, there is a way we can still maintain a view of seven-fifths as a relationship such as the 14-pound bar is a result of increasing one pound by seven-fifths of one pound. Thus, the number 10 that was produced by measuring out the 14-pound bar implies the number of units of one pound comprising a bar we are looking for. Mike neither considered partitive division while engaging in a measuring out activity nor did he develop a conceptual understanding of an equivalency between partitive and measurement interpretations of division.

Ashley's response: Mike was trying to figure out how many times will fit in there… That's what I am having trouble with because I am seeing in the same thing the student did (05/04/06).

Ashley had difficulty interpreting Mike's work involving an improper fraction. She had no comment on his producing a 7/5-bar consisting of 7 one-fifths of one pound and seemed to agree with his measuring out approach to the equation, "7/5 times x equals 14." She just pointed out that he mentioned the 7/5-bar as x, which he set for the teacher's amount. She thought x should be the number of repetitions of the 7/5-bar to make the 14-pound bar. Related to her idea, I raised a question: In the question "14 pounds is 7/5 times the amount of candy that I [the teacher] have," seven-fifths was not indicating 7/5 of one pound that Mike made. The question seemed to make her confused. So, I investigated her idea of seven-fifths.

Protocol A4.11 (1): A measurement interpretation for a partitive division: The 14-pound bar is seven-fifths of another bar.

I: My question is how, so just I wonder how Mike interprets this problem situation and then how to use this seven-fifths (the 7/5-bar) in the problem, how he used the fraction seven-fifths.

A: How he is going to use it? Or how he--

I: How he used or how he interpreted.

A: He was thinking of seven-fifths of a pound.

I: Yeah, he made it.

A: Yeah, but (pause) if he, if he repeats that ten times, he is gonna get the 14 pounds,

I: Yeah, 14 pounds if he repeats ten, ten times.

A: Right (pause). So he would need to, I mean, he would need to know that his was ten pounds. But his picture wouldn't show that.

I: Yeah.

A: It would show equality. But not, it's, it wouldn't be, he would have a hard time interpreting what he did as ten pounds, I think, because he would repeat ten times. But I am not quite sure where it'd go from there (pause).

I: So, just could you, could you solve, um (pause) this is my question, what does seven-fifths mean in the problem situation and then what was the Mike's understanding of seven-fifths?

A: In the problem situation, it's seven-fifths of 14, of 14 pounds, seven-fifths of the whole bar.

I: 14 is seven-fifths of the teacher's amount?

A: Right. So, 7, the amount that is missing, is seven-fifths of four((teen), am I saying right? (She says to herself.)

I: (Laughing) Opposite.

A: Yeah, oh, okay, (pause). You have 14 pounds, that one is 14 pounds. We wanna find 14 is seven-fifths of what amount.

I: Yeah, right.

A: Okay, but he found seven-fifths of one pound, not a four((teen), no (pause) because it's not seven-fifths of 14 (pause), it's about seventh of 14. I am saying it wrong (both of us were laughing) (pause). Okay, this is what I am saying. Okay, he's got the 14 and he knows he's got the equation now, so 14 is seven-fifths of x or that's the equation whether how we say it. But he got seven-fifths of something but again he, he was repeating it to see but I don't think he will, even if he gets a correct answer, even if he gets ten, I don't think he will interpret that answer correctly. Like, he won't be able to say if the teacher says ten what? Is it, he won't be able to say ten pounds. He won't understand that is, but I'm not quite sure, how to say more than that.

Ashley conceived of the equation, 14 equals 7/5 times x, based on measurement interpretation of division. She noticed repeating the 7/5-bar ten times would make the 14-pound bar, and the measuring out activity gives an answer for the equation. However, she was not able to explain how the number of repetitions is associated with 10 pounds. She was confused with the way Mike used seven-fifths: "In the problem situation, it is seven-fifths of 14, of 14 pounds, seven-fifths of the whole bar," "but he found seven-fifths of one pound, not a four[teen], no (pause) because it's not seven-fifths of 14 (pause), it's about seventh of 14."

Ashley never tried to solve the equation as she interpreted seven-fifths as a relationship between the 14-pound bar and a desired bar: (1) she never considered dividing the 14-pound bar into 7 parts, (2) she never wondered about the referent of seven-fifths in the problem statement, and (3) she never considered five-fifths compared to seven-fifths. Previously, she engaged in a problem similar to the equation "14 equals 7/5 times x": Making a bar so that a 5-inch bar is four-sevenths of it [cf. Protocol 3.5 and Protocol 3.5 (Cont.)]. At that time, she tried to make seven-sevenths compared to four-sevenths but had two kinds of difficulties. The first difficulty was reconstituting the 5-inch bar using four-sevenths as a relationship between the bar and the other bar. The second difficulty was establishing another relationship to produce the other bar using the 5-inch bar. With my guidance, she realized a need to make one-fourth of the 5-inch bar in order to think of the bar as four-sevenths of another bar. Such a realization led her to construct a desired bar by establishing a relationship such as one-fourth of the 5-inch bar is one-seventh of a desired bar. She developed an ability to establish a multiplicative relationship between two quantities by means of a fraction—four-sevenths—based on an operational view of fractions. However, she never mentioned seven-fourths for her final construct, a bar consisting of 35 one-fourth of one inch parts.

In Protocol 3.5, she developed an ability to establish a multiplicative relationship between two quantities through a proper fraction. Although she had difficulty reconstituting a 5-inch bar as four-sevenths of another bar, she considered seven-sevenths for a desired bar. On the other hand, the question in Protocol A4.11 (1) required her to establish a relationship between the 14-pound bar and a hypothetical bar by means of an improper fraction. Based on the relationship, she needed to reconstitute the 14-pound bar as seven-fifths of another bar. She never mentioned five-fifths while engaging in the protocol, and she used seven-fifths when producing seven-fifths of one pound. So, I wonder if her concept of seven-fifths was based on one-fifth and infer that she would need a given whole to think about an improper fraction. I conclude that she had difficulty generating an improper fraction without a presupposed whole.

The inference that she would need a presupposed whole led me to another issue related to improper fractions, three levels of units. Constructing proper fractions requires two levels of units—a whole and a part comprising the whole. On the other hand, conceiving of improper fractions requires three levels of units in that a notion of a fractional whole emerges as a result of partitioning a given improper fractional amount. That means partitioning the given amount requires an insight into both the operation of equi-partitioning and its result involved in the construction of fractions. In that sense, Ashley had yet to develop three levels of units. Therefore, reconstituting the 14-pound bar as seven-fifths based on one-fifth would be beyond her current fractional reasoning.

Ashley was asked to solve the equation “14 equals 7/5 times x” using JavaBars. Before starting to construct, she confessed that “(Looking at the 14-pound bar) that’s what I am having trouble with, because I am seeing in the same thing the student did. Um, I understand why he did that. That’s why I am having a hard time going from there.” Then she suggested an idea of making equivalent fractions: “7/5 is the same thing as 14 out of 10.” Since she noticed the approach is not what I expected her to do, she tried another way, dividing the 14-pound bar into 7 parts.

Protocol A4.11 (2): Solving the equation 14 equals 7/5 times x using Java Bars.

A: Right, so if I do the same bar instead of dividing it into 14, if I divided it into seven, then I took five of those. Okay.

I: Why do, why do you want to divide into seven?

A: Because I have seven pieces.

I: Seven pieces.

A: And each piece is one-fifth.

I: Ah, each piece is one-fifth. If you divided this one (the 14-pound bar) into seven, each one is 1/5.

A: 7 times 1/5.

I: And then this one-seventh, one-seventh of 14 [pound] bar is, how much is that of the bar you are gonna make?

A: Say that again.

I: If you got, when you got one-seventh of 14--

A: One-seventh?

I: You wanna, you just told me you wanna divide this 14 pounds into 7.

A: Yes.

I: And then you are gonna get one-seventh of this 14

A: Yes.

I: Then how much is the one-seventh of 14 of the bar you are gonna make?

A: One-fifth.

(She notices two of the 14 parts comprising the 14-pound bar would be one-seventh of the bar and pulls them out.)

I: So, to make your bar?

A: I have to multiply or repeat it five times.

I: Just make it.

A: (Repeats the 2-bar 5 times)

(After making the 10-pound bar, she clearly says that her bar is 10 pounds and 14 pounds is seven-fifths of 10 pounds.)

I: How much is your bar of the Mike's bar?

A: Five-sevenths.

I: How do you know that?

A: Because each two pounds is seventh. And I have one two three four five

I: So--

A: I have five-sevenths.

I: of?

A: 14

Ashley began to consider seven-fifths as seven of one-fifths. This is one of the critical developments for conceptual understanding of improper fractions. So, we need to closely investigate how she arrived at the idea of seven-fifths as seven one-fifths. At first, she had difficulty understanding what seven-fifths means in the problem context. Even though she said "14 is seven-fifths of what amount," she tended to directly associate 14 with seven-fifths like seven-fifths of 14. The idea of equivalent fractions, 14 out of 10 is equal to 7/5, was critical in considering the 14-pound bar in terms of seven parts. However, I doubt that she would be able to think that way even when she could not find any equivalent fraction including an anticipated number like 14. The 14 pounds as seven of one-fifths provoked her view of fractions as operations. So, she explicitly said "seven times one-fifth" for the 14 pounds.

When I mentioned one-seventh for the two pounds that she produced by dividing the 14 pounds into seven parts, she was so confused with my saying "one-seventh" because she produced the two pounds to represent one-fifth. That implies differentiating a fraction as an operation from a result of operating was not yet explicit in her fractional reasoning. She would have difficulty employing the splitting operation for further operations. However, my saying "one-seventh" provoked her operational view of fractions, dividing the 14 pounds into seven parts gives two pounds, which is one-fifth of what she wanted to make. The provoked operational view of fractions enabled her to engage in the splitting operation and produce an iterable unit—two pounds. Thereafter, she engaged in reciprocal reasoning and produced the 10 pounds as five-sevenths of the 14 pounds. The above progress of her fractional reasoning can be an indication that an operational view of fractions is critical in conceiving of improper fractions in a context requiring constructing a fractional whole.

Ashley then watched Mike engage in another problem involving an improper fraction differently from the previous one: “you’ve got 15 pounds. 15 pounds is three-fifths of my candy.”

Protocol M4.12: Solving an equation, 3/5 times x equals 15.

(Mike draws a bar and divides it into 15 parts)
T: Give me an equation.
M: 3/5 times x equals 15.
T: Okay, now find out what x is.
M: (Divides each pound into five parts, pulls a 1/5-part out, and repeats it 15 times, thereby producing a 15/5-bar. He then moves the 15/5-bar three times along the 15-pound bar.) So, I need three-fifths. One-fifth (aligning the 15/5-bar with the 15-pound bar along the left side).
T: How many those (inaudible)?
M: (Yawning) 15 (inaudible) pieces.
T: Just fifteen pieces? So how many, so is that the right, is that the answer?
M: (Sits quietly for a long time without answering and takes a deep breath.)
T: Not sure?
M: Yeah.

Mike tried to construct three-fifths of the 15 pounds instead of a bar so that the 15-pound bar is three-fifths of the bar. Regardless of problem contexts, he tended to consider a given amount or a given measurement unit as a referent of a fraction. That means his fractional reasoning required a predetermined whole, and he had yet to construct a reversible fractional scheme. So, he would have needed a whole in his perceptual field that he could actually take 3/5 of before 15 pounds could be considered as the result of his operating. He might still be unable to reason reversibly to produce the weight of the original, but he would know that 15 pounds is the result of operating, not the operand.

For three-fifths of the 15 pounds, he divided each pound into five parts and repeated a 1/5-part 15 times. He knew that the 15/5-bar is one-fifth of the 15 pounds, but he never noticed that the 15/5-bar is three pounds until the end of the episode. Since he was very accustomed to implementing dividing each unit into parts and repeating a part as many times as needed, he seemed to have no need to think about the situation partitively. That is, to obtain one-fifth of the 15 pounds, he did not think of five items into which the 15 is distributed. Such an insufficient attention to partitive thinking might contribute to his difficulty developing a reversible fractional scheme when solving the equation "3/5 times x equals 15." The equation requires him to develop an anticipatory, reversible fractional scheme.

Ashley's response: Mike never did a reversing operation of three-fifths, given a problem "15 pounds is three-fifths of the teacher's amount" (05/04/06).

While watching Protocol M4.12, Ashley seemed very confused about Mike's construction of a 15/5-bar because it was completely different than she expected. She anticipated he would make three-fifths of one pound and measure out the 15 pounds using that quantity as a measuring unit like he did in Protocol M4.11: "I don't really know where he got that, he has done with 15 small pieces," "I don't understand that [Mike's repeating a 1/5-part 15 times]." That indicates she understood the given problem by means of measurement interpretation of division rather than a partitive interpretation. As soon as she realized Mike solved "Make 3/5 of a 15-pound bar" rather than "15 is 3/5 of some amount," she commented on his lack of number sense: "it's the number sense thing though. I would hope and that's what you try to teach all students just to have a number sense that 3/5 of something, if that's 15, something has to be bigger." The number sense she mentioned seemed to mean an ability to consider a result of operating (the 15 pounds) as an input for reversing the scheme activated for three-fifths.

Protocol A4.12: Comments on Protocol M4.12.

I: If you were the teacher in this situation, how could you direct or help Mike figure out the problem correctly?

A: I would ask him to tell me what that means, okay, what did you make, 15 of these, 15 what, um, 15 what was it? (pause) ends up being one (pause) three-fifths of 15, right? I am trying to think. (After a while) I am getting confused, too, with all these numbers. Okay, (whispering) what did he do? He made fifteen-fifths (very weak voice), right, he made fifteen-fifths of a pound so he made three pounds.

I: Yeah, but 3 pounds is only, how.[I should have asked “how would he relate fifteen-fifths to making a bar so the 15 pounds is three-fifths of the original bar?”]

A: One-fifth.

I: One-fifth of fifteen.

A: Right (pause). Well, I would probably ask how many times does this, how many times does this (pointing at the 15/5-bar) fit into the 15 pounds since he tends to go back to that. And he would measure that out then say five, what is that three, three-fifths of my amount (pretending to talk to Mike)? Or (inaudible). Is that (pause), I would just ask, but I am trying to say how, I wanna word it right, three-fifths (very low voice) (pause) I don’t know, going back to the original question, which was I wish to find three-fifths of what gives you 15.

I: Yeah.

A: So, did you show me that (pretending to talk to Mike)? Then he would probably say yes though. I’m (taking a deep breath).

I: To solve the, to solve this problem, 15 is three-fifths of another amount, to solve this problem, what's the essential element do you think of this problem?

A: 15 is three-fifths of something else, (pause) again, I was kind of thinking along the lines of what he had done last time. And it was thinking how many times will that three-fifths of one pound make, but that's not big picture of what he wanted, it will get you the answer, but I know that's not conceptually what you want. I am having a hard time with it today, sorry.

At the beginning, Ashley did not notice Mike's intention to make one-fifth of the 15-pound bar. She finally figured out the 15/5-bar is one-fifth of the 15 pounds but seemed to know that by calculation, not distributive reasoning. By saying "What is that three, three-fifths of my amount (pretending to say to Mike)?" she wanted to let him realize that there is something wrong in his understanding the original problem. However, she did not make any comment on what he should do to resolve such confusion. Since she did not engage in distributive reasoning to find one-fifth of the 15-pound bar, the 15/5-bar would not be related with each unit of one pound in her mind.

Ashley's idea of making a 3/5-bar reveals a lack of understanding the given problem in a similar way to Mike's idea of making a 15/5-bar. Neither of them considered reversing operations nor distinguishing partitive division from measurement division. She attempted to solve a problem involving partitive division by using measurement division. She considered three-fifths of one pound and the number of times it was contained in the 15-pound bar. However, she did not complete changing measurement division into partitive division.

Considering fractions as operations is necessary for reversing a fractional scheme. A fraction is constructed by activating a fractional scheme and a scheme involves a result of an operation as well as an operation. Therefore, reversing a fractional scheme implies an ability to reverse the operation(s) used to produce a result of the scheme. Partitive division, such as a 14-pound bar is 7/5 of another bar [cf. Protocol M4.11] or a 15-pound bar is 3/5 of the other bar [cf. Protocol M4.12], requires reversibility. So, for those problems, children first need to reflect on what they did to make the 14 pounds or the 15 pounds, and then they need to reverse the operations they carried out. Therefore, we should consider two aspects as necessary for conceiving of or solving a partitive division problem: an operational view of fractions and a reversible fractional scheme. In light of Ashley's and Mike's activities with respect to the partitive division problems, those two aspects seemed not involved in their construction.

When engaging in Protocol M4.11 and M4.11 (Cont.) where 14 pounds is 7/5 times the amount of candy the teacher has, Ashley understood a partitive division problem by finding a measuring unit and the number of the units needed to produce a given amount like Mike did. Mike used an improper fraction as a unit for measuring when solving a partitive division problem that involved an improper fraction. Ashley also considered the fraction 7/5 in terms of seven-fifths of one pound and tried to find the number of times it was contained in 14 pounds. That is, she solve the equation $(7/5)x = 14$ that involved an improper fraction by asking how many times is 7/5 pounds contained in 14 pounds, which is a measurement interpretation of division.

The following protocol shows Mike's difficulty related to an improper fraction more explicitly than before.

Protocol M4.13: Confusion about an improper fractional amount.

(Mike draws a bar.)
T: Let's say that bar is seven-fifths of another bar.
M: Do you want me to make the other bar?
T: Make the other bar. The other bar is gonna be more or less?
M: This one (the given bar) is seven-fifths of the other bar so this one is smaller (looking at the teacher). Hold on.
T: The other bar is smaller or this bar is smaller?
M: This one is smaller, hold on. The other one is--.
T: Tell me which one is smaller.
M: This one is seven-fifths of the other bar (pointing his cursor at the given bar). The bar is 5/7 or 7/5?
T: What if it's 5/7, the other bar is bigger or smaller?
M: Bigger.

According to his previous understanding of this kind of problem (Protocol M4.11 (Cont.)), Mike must have meant to make a 7-part bar by dividing the given bar into five parts and repeating one part seven times, which is seven-fifths of the given bar. Then he would have intended to measure the given bar using the 7-part bar. He repeatedly said "this one (the given bar) is smaller" even right after restating the problem "the given bar is seven-fifths of the other bar." However, this time he did not seem to know why he wanted to make seven-fifths of the given bar as well as what he should do with the 7-part bar. So, his answer "this one (the given bar) is smaller" can be considered as incomplete in that he did not yet measure the given bar using the 7-part bar. If he had developed partitive interpretation of division for measurement division, he would have measured the given bar using the 7-part bar and reasoned as follows: (1) the given bar is 5/7 of the 7-part bar, (2) 5/7 indicates 5/7 of the given bar, therefore (3) he has to make 5/7 of the given bar to produce the other bar. However, he did not seem to even notice he was using a measurement approach for a partitive division problem.

Mike seemed to have no idea about partitive division when the problem includes an improper fraction. However, when the teacher changed the problem by replacing seven-fifths by five-sevenths, he had no problem answering the partitive division problem, “The given bar is five-sevenths of another bar. Is the other bar is bigger or smaller?” As a reason, I consider proper fractional contexts would allow him to conceive of the problem by using part-whole operations. So, when the teacher changed the problem so that it involve five-sevenths instead of seven-fifths, he quickly answered “(the other bar is) bigger.” He might think of the given bar as five out of seven parts whereby the desired bar should be bigger than the given bar. Thus, his answer “(the other bar) is bigger” does not necessarily mean that he conceived of the situation as partitive division.

Ashley’s response: “Would it be bigger or smaller?” will become the first focus, and how many pieces he needs to divide into would be the next (05/11/06).

Ashley observed that Mike had difficulty producing a bar that a given amount is 7/5 of the original bar, but he had no problem figuring out a bar that a given bar is 5/7 of the original bar. She pointed out the inconsistency of his idea between 7/5 and 5/7 by saying that he did not apply a basic idea for 5/7 to 7/5: “you’ve got a given amount, you are finding that amount compared to another bar. Basically you are comparing two things, and it’s the same idea.” According to her idea of comparing, improper fractions can be conceived in the same way as proper fractions. So, when asked to make a bar so that a given bar is 7/5 of the bar, he should have divided the given bar into 7 parts for 7/5 like he would do with respect to 5/7. Responding to my question “What would he focus on when he solved the problem [the question including 5/7]?” she pointed out two elements: (1) number sense, 5/7 is smaller than a whole and (2) dividing a given bar into numerator. To support his approach to the fractional task, she mentioned a traditional way of learning fractions in schools: “from my experience, many teachers and students, teachers teach and students learn fractions are always parts of a whole.”

Beyond traditional perspective of fractions, her out-of concept of fractions involved her progression toward an iterative fractional scheme as follows:

Protocol A4.13: A way to help Mike's difficulty with an improper fraction.

A: (Pretends she is working with Mike.) How many pieces would you have here (a given bar)? You would have 7 because that's what you started with. Okay, so (dividing a given bar into 7 and producing a 7-bar) this is 7 pieces, and you have 7 out of 5 pieces, 7 fifths of my bar, how big is my bar? Well, (pulling one part out at the most right end) then you can ask other questions we've asked before. How much is this (the pulled part) of my bar? That's one fifth. One fifth of the bar you are making. So what would make one whole bar? (As she repeated it 5 times) 5 of them, okay, so this one (the 7-bar) is seven-fifths of this one (the 5-bar).

I: Is there another way without using a part-whole concept.

A: I am sure there is. This is what came to my mind though (pause). I mean we've talked before about, this is seven-fifths then one piece is one-fifth of the whole or it's one-seventh of this (the 7-bar).

I: Yeah, that's--, I think that's another way to understand improper fractions. Yeah, just this is seven-fifths of another bar, so if you think [of it in terms of] this one little piece (pointing at the most right end part) seven-fifths means 7 times,

A: One-fifth.

I: One-fifth of another bar. So this is one-fifth of another bar, so I can make using the fifth, the other bar, I can make the other bar.

A: Right, which is basically what we did. We just called it something different.

Ashley assumed that a fraction 7/5 as seven out of five implies there are seven parts: “you would have 7 because that’s what you started with.” In addition, her out-of concept involved an aspect of an iterative fractional scheme: “you have 7 out of 5 pieces, 7 fifths of my bar.” Such an insight into fractions is relevant to establishing a multiplicative relationship between two quantities. Her argument “Basically you are comparing two things [by means of a fraction]” indicates she had a sense of fractions as a multiplicative relation. However, she did not explicitly demonstrate such a view of fractions in her progress when constructing of an iterative fractional scheme, because a part-whole comparison based on an out-of concept dominated her fractional concept. Thus, even though she said “one piece is one-fifth of the whole (the 5-bar) or it’s one-seventh of this (the 7-bar),” she did not seem to construct one part as an iterable unit fraction to relate two quantities. Rather, she seemed to consider one part in each bar separately.

The following protocol is the continuation of Protocol M4.13. Since Mike was successful when answering a question “A given bar is 5/7 of another bar. Is the other bar bigger or smaller?” involving 5/7, the teacher continued on the original problem, a given bar is 7/5 of another bar, by replacing 5/7 with 7/5. The teacher engaged him in producing a mixed number from an improper fraction. The activity provided him with an opportunity to explicitly consider a given amount by using a unit fraction referring to a bar he was supposed to make.

Protocol M4.13: (Cont.)

T: That bar (a given bar) is seven-fifths of another bar.
M: Seven-fifths?
T: The other bar is smaller or bigger?
M: The other one is seven-fifths.
T: This is seven-fifths of another bar. You took seven-fifths of the other bar to get this one.
M: The other bar is gonna be bigger (dividing the given bar into five parts and repeating it. So he produces a bar consisting of 10 parts and then cut it to get a 7-part bar.)
(In response to the teacher’s request, he erases the remaining 3-part bar.)
T: Can you color the part you started with blue?

M: (Colors five parts in the 7-bar.)
T: Okay, the blue part is how much of the whole bar?
M: The blue part is--
T: Five-sevenths, isn't it? But I said I want you to make the blue part so that is seven-fifths of the other bar.
(Mike draws a new bar and divides it into seven parts.)
T: How much is this one of little—[the teacher could not complete the question because Mike intervened.]
M: (Interrupts the teacher's question) oh, the other bar would be smaller [than the given bar] (pulling one out of the seven parts and repeating it five times. He produces a 5-part bar).
T: Okay, so, is that (the 5-part bar) the whole bar?
M: Yeah.
T: Okay, color this seven-fifths blue.
M: (Colors the given bar divided into seven parts blue.)
T: Pull out one blue piece, how much is that of the red bar (the 5-part bar)?
M: It's one-fifth.
T: One-fifth, right, and so the blue, each part of the blue bar is a fifth of the red bar (inaudible). So, how many red bars are in blue bar?
(There was some miscommunication between the teacher and Mike. To help him understand the question, the teacher asked him to make the three more copies of the 5-part bar.)
T: How many of these whole pieces (pointing at the copied 5-part bars) are up here (pointing at the blue bar)?
M: Uhhhh (meaning to just realize the teacher's intention). One.
T: Yeah (laughing), just one of them up there, right? So, there is one, how many more, okay, what fraction, okay, color the part that's left over (inaudible) green after you do the whole piece, right?
(Since JavaBars does not provide color of green, he colored the left-over two pieces yellow.)
T: That's one whole bar, right? How many, how many more fifths?
M: Two.
T: Two, so one and two-fifths equals what?
M: One and two-fifths equal to--
T: How many fifths?
M: Seven.

This protocol shows a way to introduce a mixed number so that it can help conceptual understanding of an improper fraction. Even young children might easily convert 7/5 to 1 and 2/5 procedurally. However, as we see above, the converting involves important fractional concepts such as constructing an iterable unit fraction and coordinating various levels of units. I will investigate in what ways mixed numbers were helpful for him to understand improper fractions.

Mike seemed to have no picture of the other amount so that a given bar is 7/5 of another amount. He needed something concrete to which 7/5 refers. So, he tried to make 7/5 of the given amount. The teacher guided him to see that his construction of a 7-part bar by dividing the given bar into 5 parts did not make the other bar; rather, the given bar became 5/7 of his bar. As soon as he noticed the given bar is not 7/5 of his construct, the 7-part bar, he changed his construction into dividing the given bar into 7 parts and producing a 5-part bar consisting of 5 parts. Then, following the teacher's request, he pulled one part out of the given 7-part bar and said it is one-fifth of the 5-part bar. This is a very crucial moment because (1) he realized his dividing into 7 parts led to one-fifth of the bar he is looking for, (2) his awareness of one-fifth was prompted by or prompted an iterative aspect of a unit fraction, and (3) by conceiving of one-fifth without a given whole, he had a chance to generate a whole as a result of an activity. Based on the constructed whole, he produced a mixed number—1 and 2/5—whereby he explicitly could see a whole embedded in the given bar. Then, he was encouraged to focus on the number of fifths constituting the given bar. I expect that the above sequence of the teacher's guidance would help him conceptually establish an equivalent relationship between seven-fifths and one and two-fifths.

I was concerned with a procedural approach to improper fractions: Seven-fifths is one and two remaining because 7/5 means seven divided by five. Converting an improper fraction to a mixed number using measurement division would prevent one from establishing a multiplicative relationship between two quantities. Protocol M4.13 (Cont.) shows a promising way to help conceptual understanding of improper fractions by means of mixed numbers. The teacher provided an opportunity to construct an iterable unit fraction, generate a whole as embedded in the given amount, and reason about a fraction based on various levels of units. The following protocol shows evidence that mixed numbers corroborated Mike's conceptual understanding of improper fractions.

Protocol M4.14: Understanding an improper fraction by means of a mixed number.

T: Let's imagine that this bar (a given bar) is thirteen-sevenths of another bar. The other bar is gonna be littler or bigger?
M: Little.
T: Littler, right.
M: So, it's thirteen-sevenths.
T: So, how many, how many sevenths is gonna be in this piece (pointing at the given bar)?
M: Thirteen.
T: Thirteen of one seventh, right?
M: (Divides the given bar into 13 parts so that it becomes a 13-bar)
T: Pull out one.
M: (Pulls one out of them, so there is a 1/7-part.)
T: How much is it out of the whole bar, the other bar?
M: One-seventh.
T: One-seventh of the other bar. Make the other bar there (laughing).
M: (Repeats the 1/7-bar seven times and producing a 7-bar)
T: All right. How many whole bars is up here (pointing at the 7-bar and the 13-bar)?
M: One (clicking the 7-bar and moving along the 13-bar).
T: One whole bar, how many sevenths are leftover?
M: Six.
T: So, one and six-sevenths is equal to what?
M: To thirteen-sevenths.

Mike was never confused about an anticipated whole from an improper fractional amount. In Protocol M4.13 (Cont.), he constructed an improper fraction by considering two kinds of units—a given bar and a desired bar—and specifying a unit fraction with respect to an anticipated whole embedded in the given bar. So, given a fraction 13/7, he would be able to picture what the other bar would be conceptually. The teacher guided him along his development of a concept of improper fractions by: (1) anticipating a whole embedded in a given amount, (2) realizing that each part in the given bar was supposed to refer to the anticipated bar, (3) constructing a whole using a part in the given partitioned bar, and (4) conceiving of the given bar in terms of the constructed whole. Therefore, given an improper fraction, he would no longer worry about not having a predetermined whole because he came to know a whole should be constructed. In addition, I infer that he would be able to establish a multiplicative relationship between two quantities because conceptual understanding of an improper fraction is based on the construction of an iterable unit fraction and an awareness of various levels of units, which are critical in multiplicative reasoning in fractional contexts.

Ashley's response: Mike was not remembering one unit repeated a certain number of times for an improper fraction (05/11/06).

Ashley said that Protocol M4.13 (Cont.) surprised her because the teacher encouraged Mike to produce a mixed number. The teacher usually has had him convert a mixed number to a fraction whenever he gave an answer in a form of mixed numbers. However, she did not seem to wonder why the teacher encouraged him to think about an improper fraction by means of a mixed number. In addition, she took for granted that seven-fifths consists of seven parts each of which represents one-fifth. So, she overlooked the process to produce a mixed number by anticipating and constructing a whole with a notion of an iterable fractional unit and further coordinating various levels of units.

It took her a long time to solve the problem, "How would you color a given bar so we can see the wholes and leftover if the given bar is 3 and 1/5?" She drew a bar as a whole and produced a 3 and 1/5-bar by repeating the bar three times and dividing it into 5 parts and joining one of them to the 3-bar. She then stated there are 16 parts: "So, to change that [3 and 1/5] to improper fraction, you could say, how did you figure out the (pause) to get the 1/5, you've got three wholes, how did you figure out 1/5, well, I divided one up into five pieces, and so you've got that five pieces three times, 5, 10, 15, and you've gotten one leftover, 16." She produced 16 parts constituting the given bar without using procedural knowledge based on equivalent fractions. Therefore, for her, 3 times 5 and one more part was not just a calculation but a conceptual activity. However, I pay attention to the fact that she did not solve the problem starting with a bar representing 3 and 1/5. In other words, she related parts to a whole by presupposing a whole rather than anticipating a whole.

After watching Protocol M4.14 where Mike was successful with producing a mixed number from 13/7, I investigated her thought about his development of a concept of improper fractions.

Protocol A4.14: Comments on Protocol M4.14.

A: What is different? That's what I am trying to figure out what's different from when we started today [Protocol M4.13, M4.13 (Cont.)], when he started today. The idea of an improper fraction didn't come to little bit later, but now he is getting the ones like 13/7 or 13/5 like that, very quickly, and I don't know the difference. I am trying to pick out the difference.

. . . .

I: Maybe this question would be helpful, I think. When he faced the problem starting with an improper fraction, first time he didn't--

A: He didn't even try basically, right.

I: At that time, what he would miss, what essential point, what is the part he missed at that point?

A: (inaudible) was one unit repeated a certain number of times, I don't think he was remembering that.

I: Ah, for example, 13/5 is (simultaneously with Ashley) 13 times one-fifth.

A: 13 one-fifths. He was just getting (inaudible) down in the whole idea, 13/5, what does 13/5 look like?

I: Just as a whole. This is 13/5.

A: I think so. And the idea of making that is a portion of something else or something else is a portion of that was maybe overwhelming , I am not, then when he started thinking of it that way, even the teacher reinforced that, it made him quicker so he was understanding better.

Ashley's comments capture two important aspects of fractional reasoning related to improper fractions: inverse operation and various levels of units. By observing Protocol M4.14, Ashley saw Mike's progress and came to think about a difference from the previous episode, Protocol M4.13 (Cont.). She paid attention to a unit constituting a given bar and the number of repetitions for producing the bar. This insight is very important because it was the first time for her to doubt Mike considered an improper fraction in terms of a unit fraction like 7/5 as seven of one-fifth. In addition to the comment, she said "I don't think he was remembering that," which means Mike would not know that an improper fraction consists of a certain number of unit parts. She began to realize that reversing the operations for constructing an improper fraction is essential to considering the parts comprising the fraction and relating one part with a whole he was supposed to produce. Thus, thinking about an improper fraction in terms of parts, furthermore in terms of a unit fraction referring to a whole, was no longer considered as a preconceived concept.

Regarding his difficulty with 13/5, she mentioned he tended to see 13/5 as one thing to deal with, not in terms of parts. This comment implies his lack of an insight into various levels of units. There is a discussion about three levels of units as necessary for a conceptual understanding of improper fractions: two levels of units are sufficient for proper fractions but not for improper fractions. For example, when constructing 3/5 of one foot, we first refer to one foot and divide one foot into five parts. Then we focus on a unit of a part and produce three of the parts. This thought process shows that fractional reasoning about proper fractions is based on two levels of units—a unit of one foot and a unit of a part whose multiples comprise one foot. On the other hand, regarding 6/5 of one foot, we cannot think of six parts out of a whole foot consisting of 5 parts. It is very important to recognize that one foot can be considered five one-fifths or six one-sixths and so on. So, by partitioning one foot into 5 parts, one foot is considered as five-fifths. Then with a notion of a unit of one-fifth, one part is repeated six times. When repeating one-fifth of one foot six times, it is crucial to reason about the repetition along with a notion of various levels of units. That is, the fifth repetition of one-fifth makes a unit of a whole and the sixth repetition of one-fifth produces another level of unit. Therefore, three levels of units are necessarily involved when constructing 6/5: a unit of one foot, a unit of one-fifth of one foot, and a unit of 6 parts—6/5 of one foot.

Summary

In this section, I investigated how Ashley developed a multiplicative fractional scheme as she observed Mike engage in solving fractional tasks. The four topics related to the scheme are an iterative unit fractional scheme, a units-coordinating scheme, fraction multiplication, and improper fractions.

First, I will summarize Ashley's development related to an iterative unit fractional scheme. While watching Mike's difficulty engaging in equi-partitioning a composite unit and considering dividing and repeating simultaneously when constructing a fraction, Ashley expressed two kinds of concerns: (1) he did not relate dividing activity to producing a unit fraction; (2) he only focused on the number of repetitions when trying to produce a fraction. She paid attention to the importance of simultaneousness of the operations of dividing and repeating and constructed an iterable fractional unit based on it. She also established a multiplicative relationship between two bars by constructing an iterable unit fraction: “he didn’t get one-third of this (the 36-bar) is one-fourth of this (the 48-bar).” [cf. the text immediately following Protocol A4.1 (3)].

Ashley was also concerned that Mike did not notice a relationship between an activity and a result of the activity such as dividing into three parts produces three identical parts [cf. Protocol A4.2]. When asked to make a bar so that a 60-bar is three-fifths of the original bar, Mike produced a 100-bar by adding two parts from the given bar divided into three parts to the given 60-bar. He had difficulty establishing a relationship between the 60-bar and the added 40-bar through 20 pounds—one part. In addition, he considered the produced 100-bar as one and two-fifths of the 60-bar. Related to his construction and difficulty, she raised an issue of constructing a part that can be repeated. Based on the issue, she wanted to encourage him to think of one part as a unit that can be repeated and many parts as iterated. In conclusion, by observing Mike, she explicitly developed an iterative aspect of a unit fraction from the perspective of fractions as operations

Related to a units-coordinating scheme, Ashley did not anticipate Mike’s difficulty with conceiving of various units involved in a fractional context. She observed Mike struggle with producing an equation based on his activity—dividing each of two bars into three parts and considering two parts as a share for one person—given two bars to be shared among three people. He was yet to connect “three times,” which he verbalized when measuring two bars by using the two parts, with “a third” referring to the share for one person with respect to the whole bar [cf. Protocol M4.3]. Even though he made an equation “three times a third,” he did not arrive at either one or two, either of which might imply two bars. However, he was able to make an equation using “a third” and “three” when the teacher explicitly introduced a customary unit, a gram.

In response to Mike’s activity, Ashley said that his difficulty producing an equation was due to his inability to put together three kinds of quantities—one-third, one, and three—based on his construction: “he still couldn’t think one-third times what will give me that one” [cf. the text immediately following Protocol A4.3 (1)]. She assumed that a third, one, and three would be available to him. She did not consider any possibility that his difficulty producing an equation might be due to his confusion of “one.” Thus, when asked whether the number of candy bars would matter for him to solve the problem, she responded it would not affect him.

Ashley had a chance to think about the importance of conceiving of a composite unit as a unit for fractional reasoning. She observed Mike produce a share for one person by dividing each of seven bars into nine parts and pulling one part out of each 9/9-bar, but it took him a long time to answer the question “All the candy is how many times a share for one person?” [cf. Protocol M4.4]. Based on his reasoning about his constructs, she commented that he did not notice there were seven bars and he was not associating one-ninth of each bar with one-ninth of seven bars [cf. Protocol A4.4 (1)]. However, she was yet to associate one-ninth of one bar with one-ninth of the entire bar in an explicit way [cf. Protocol A4.4 (2)]. When Mike produced a share for one person by repeating one part as many times as needed [cf. Protocol M4.5], Ashley distinguished the repeating activity from his pulling one part out of each partitioned bar. She commented that such an activity would influence his ability to associate one bar with the entire bar.

Next, related to three levels of units, Ashley developed an insight into two levels of units along with distributive reasoning involving partitioning. When Mike successfully constructed a share for one person when sharing a 7-inch bar among 11 people and correctly responded to fractional questions concerning his constructs [cf. Protocol M4.6], I asked Ashley about her anticipation for Mike's response to the question "what is 11 times 7/11?" She doubted his ability to answer the question based on his construction, but she did not elaborate on her doubt. She never mentioned a 7-part bar consisting of seven 1/11 of one inch parts as a unit [cf. the paragraphs preceding Protocol A4.6]. As the units involved in the context, she mentioned "the whole 7-inch bar," "one inch," "one-eleventh of one inch," and "one seventy-seventh of the whole." She tended to consider units in terms of a single element, not multiple elements.

Ashley began to consider operating on a composite unit in terms of operating on each unit comprising the composite unit. Referring to Mike's dividing each inch into five parts to make an equal-share of a 7-inch bar among five people, she described his activity using two referents for one-fifth: "he said one-fifth of the whole bar, seven thirty-fifths of the whole bar, and he said that . . . this sliver is one-fifth of each inch" [cf. Protocol A4.6]. In fact, Mike never stated one-fifth of one inch. Her elaboration indicates she was considering one-fifth of 7 inches by means of one part with respect to each inch. However, she did not maintain the notion of one part as one-fifth so that she could relate one-fifth of one inch with one-fifth of 7 inches.

By observing Mike having difficulty answering the product "25 times 7/25" based on his construction [cf. Protocol M4.7 (Cont.)], she noticed he was missing a unit toward three levels of units. Mike was encouraged to consider a 7-part bar representing 7/25 of one inch as one twenty-fifth of 7 inches. Ashley focused on how many inches "25 times 7/25" will be reduced to, that is, "7 of 1/25 of one inch" 25 times will make 7 inches. As a result, she had an insight into three levels of units in that she deduced a fraction by repeating 7/25 of one inch to make the 7 inches. However, she had yet to coordinate three levels of units in that she did not explicitly conceive of 7/25 of one inch based on one 1/25-part comprising one twenty-fifth of 7 inches.

Concerning fraction multiplication, Ashley did not seem to consider Mike's equation "seven-thirds times three equals seven" while relating a 7-foot bar and a 7/3-bar multiplicatively. As a question to pose to Mike, she tried to make a reciprocal question with respect to the equation but could not figure it out by herself [cf. Protocol A4.8 (1)]. She did not think of seven-thirds as related to the 7-foot bar. With my guidance, she finally considered 7 times 1/3 as a reciprocal of three times 7/3 and mentioned "relationship between the bars (the 7-foot bar and the 7/3-bar)" for "7 times 1/3" [cf. Protocol A4.8 (2)]. She constructed the 7/3-bar as an iterable unit fraction, one-third of seven feet. Furthermore, while observing Mike's difficulty with 7 times 1/11, she demonstrated an insight into three levels of units and a differentiation between fractions as resultant amounts and as operations as she commented on his understanding of three operations—7/11, 7 times 1/11, and 7 divided by 11 [cf. Protocol A4.9 (2)].

Last, related to her development of a concept of improper fractions, Ashley thought making a quantity equally partitioned would be fundamental for producing an improper fraction [cf. Protocol A4.10]. She did not expect any difficulty related to referring to a portion of a quantity to create a fraction. Like what Mike did, Ashley conceived of an equation, 14 equals 7/5 times x, based on measurement interpretation of division [cf. Protocol A4.11 (1)]. The equation was produced by Mike from a context where 14 pounds is 7/5 times the amount of candy that the teacher has. She thought repeating a 7/5-bar ten times would make the 14-pound bar, and she noticed that the measuring out activity gives an answer for the equation. She never tried to solve the equation as she interpreted seven-fifths as a relationship between the 14-pound bar and a desired bar. She never considered dividing the 14-pound bar into 7 parts; she never wondered about the referent of seven-fifths in the problem statement; she never considered five-fifths compared to seven-fifths. She seemed to have difficulty generating an improper fraction without a presupposed whole.

Further, she had yet to reconstitute the 14-pound bar as seven-fifths based on one-seventh of one pound. That implies she was yet to reason with three levels of units. When asked to solve the equation “14 equals 7/5 times x” using JavaBars, she suggested an idea of making equivalent fractions: “7/5 is the same thing as 14 out of 10” [cf. Protocol A4.11 (2)]. The 14 pounds as seven of one-fifths provoked her view of fractions as operations. So, she explicitly said “seven times one-fifth” for the 14 pounds. In addition, my saying “one-seventh” for the two pounds provoked her operational view of fractions, dividing the 14 pounds into seven parts gives two pounds, which is one-fifth of what she wanted to make. Thereafter, she engaged in reciprocal reasoning and produced the 10 pounds as five-sevenths of the 14 pounds.

Ashley understood a partitive division problem by finding a measuring unit and the number of the units needed to produce a given amount. That means she was lacking two schemes, an operational view of fractions and a reversible fractional scheme, in a context involving improper fractions in that solving a partitive division problem requires those scheme.

Concerning a fraction 7/5, she argued that 7/5 as seven out of five implies there are seven parts and there are two quantities to compare: “Basically you are comparing two things [by means of a fraction].” In addition, she took for granted that seven-fifths consists of seven parts each of which represents one-fifth. Thus, she overlooked the process to produce a mixed number by anticipating and constructing a whole with a notion of an iterable unit fraction and further coordinating various levels of units [cf. Ashley’s response followed by Protocol M4.14].

However, it took her a long time to solve the problem, “How would you color a given bar so we can see the wholes and leftover if the given bar is 3 and 1/5?” She produced a bar consisting of 16 parts by first drawing a bar, repeating the bar three times, dividing it into 5 parts, and joining one of them to the 3-bar. She related parts to a whole by presupposing a whole rather than anticipating a whole.

While observing Mike’s progress in conceiving of improper fractions, she paid attention to a unit constituting a given bar and the number of repetitions for producing the bar [cf. Protocol A4.14]. Then she doubted Mike has considered an improper fraction in terms of a unit fraction like 7/5 as seven of one-fifth: “I don’t think he was remembering that,” which means Mike would not know that an improper fraction consists of a certain number of unit parts. She realized that reversing the operations used to conceive of an improper fraction is essential to considering the parts comprising the fraction and relating one part with a whole. Thus, thinking about an improper fraction in terms of parts, furthermore in terms of a unit fraction referring to a whole, was no longer considered as a preconceived concept. In addition to her concern about inverse operation related to conceptualizing improper fractions, she commented on various levels of units. Regarding Mike’s difficulty with 13/5, she mentioned he tended to see 13/5 as one thing to deal with, not in terms of parts. This comment shows her insight into his lack of various levels of units.

Ashley’s Development of Fractional Reasoning (Day 15)

Ashley was provided nine fractional tasks in a written form and solved seven problems given fifty minutes. The tasks were designed to examine her development of fractional reasoning throughout this study. I tried to minimize my intervention while she solved the problems.

She constructed an iterable unit fraction of a composite unit in a generative manner. Thus, she had no problem with finding reciprocals based on her construction. When solving the problem, “Make three-sevenths of a 21-inch bar,” she created a 9-bar by first producing a 3-part bar as one-seventh of the 21-bar and then repeating it three times. When asked “How much of 9 is 21?” she answered “seven-thirds” and elaborated as follows: “(pulls one part out of the 9-bar and repeats it three times, so she produces a 3-part bar) This amount (the 3-part bar) is one-third. . . . Okay, that fits, one two three four five six seven (as she counts up every 3-part along the 21-bar). Each group of one-third-- so, one-third seven times.”

Further, Ashley produced a fraction to establish a relationship between two fractional amounts and used it for reciprocal reasoning: “Although this one (a 9/13-bar) is 9/13 of mine, it’s nine-thirteenths of mine, but each piece is one-ninth of itself. So this (a 13/13-bar produced by repeating one part in the 9/13-bar 13 times) is one-ninth 13 times. So, 13 times 1/9. 13/9.” Responding to the question “what is the 13/9 times 9/13?” she answered “one” and clarified that the answer “one” referred to the 13/13-bar.

However, it seemed challenging for her to produce a fraction by associating an operational view of fractions with a result of reasoning with three levels of units. When asked to create seven-ninths of a 5-foot bar, she produced a 35/9-bar by dividing a 1-foot part into nine small parts, repeating one small part 7 times (a 7/9-bar), and repeating the produced 7/9-bar 5 times. In response to the question about the size of the 35/9-bar in feet in a fraction form, she measured out the bar using 9 small parts as she stated, “9 three whole times, just 27 then 27 plus 1,2,3,4,5,6,7,8, 35, it’s 35 fif[ths], No, ninths.” She created a unit of units of units: a 35/9-bar is a unit of five units of seven units of 1/9 of one foot. Her creation of the new unit based on three levels of units is grounded on an insight into two levels of units, one foot partitioned into nine parts and five feet containing five one-feet. In addition, the creation of the 35/9-bar involves an insight that five of seven-ninths of one foot will make seven-ninths of five feet. However, when trying to find a size of the 35/9-bar, she never used her construction based on three levels of units: she never mentioned “five times seven” one-ninths. Rather, she measured the 35-part bar by using one foot—nine parts, produced the number of wholes, and then thought about the total number of parts.

Ashley would know one-ninth of the 5-foot bar consists of five 1/9-parts because she already developed distributive reasoning in fractional contexts and reasoning with three levels of units. However, to investigate if her construction of the 35/9-bar involves multiplicative reasoning based on an iterable unit fraction, we need to closely examine how she related creating the 35/9-bar based on three levels of units to constructing an iterable unit fraction through distributive reasoning. By an ability to consider a 1/9-part as one-ninth of one foot as well as one forty-fifth of 5 feet, I mean that Ashley would construct an iterable unit fraction, a unit that she produced by means of distributive reasoning. To know that 1/9 of one foot is also 1/45 of five feet involves more than reasoning with three levels of units. It also involves taking the result of distributing partitioning into nine parts across the five feet as a given and uniting them together into a unit of units of units. That is, distributive partitioning and making a unit of units of units must be coordinated in the service of a goal to find what fractional part of five feet is 1/9 of one foot.

The fact that she used one-ninth of one foot to produce seven-ninths of five feet—a 35/9-bar—means she considered the 5-foot bar as a unit of units of units because she distributed partitioning each foot into nine parts across the five feet. However, she did not seem to take the results of distributive operating as an occasion for making a unit of units of units [a unit of five feet where each foot is partitioned into nine parts], which implies she lacked an operational view of fractions. She did not consider the 35/9-bar as seven one-ninths of a 5-foot bar or five seven-ninths of one foot. Although she produced seven-ninths of one foot five times, which is multiplicative, her reasoning about the 35/9-bar was not multiplicative. She did not explain how she made the 35/9-bar in terms of iterating one ninth of one foot 35 times. So, a 1/9-part would be considered as a part embedded in one foot but not as a unit to generate a newly created composite unit, the 35/9-bar. As a result, a unit of one-ninth of one foot was yet to be an iterable unit that can be used to produce the 35/9-bar, which means the one-ninth would be considered only within one foot.

Ashley's difficulty implementing an operational view of fractions for three levels of units affected her reciprocal reasoning in a fractional context. She had no problem with implementing reciprocal reasoning in a situation that requires only two levels of units—whole number context [cf. the first paragraph of this section]. However, when partitioning was involved, she had difficulty with reciprocal reasoning. She was asked to solve a fraction problem, "An 11-inch long bar is 5/7 of another bar. Find the length of the other bar," by constructing bars using JavaBars. Then she was requested to solve the same problem using mathematical notation while relating it to her construction. She first produced a 7/5-bar bar by dividing one inch in the 11-inch bar into five parts and repeating one part seven times. She then repeated the 7/5-bar 11 times and produced a 77/5-bar. The following protocol shows how she arrived at the length of the 77/5-bar (05/18/06).

Protocol 5.1: Maintaining a unit of one-fifth of one inch to measure a 77/5-bar.

I: What's the length of the bar (the 77/5-bar)?

A: Improper?

I: Yeah.

A: Okay, that's 55 [she is referring to the 11-inch bar in which each inch was divided into 5 parts.] Okay, each inch in this candy bar is five of the small pieces, so, if I take that, this is 11 inches, so this is 55 fifths. No,

I: 55/5 is 11, right?

A: Yes. Yes. That's 55 and then to count up, 56, 57, 58, I messed up.

I: Can you tell your strategy to count?

A: Somewhat?

I: To know the number of the little pieces--

A: Like, that's what I wanna, can I pull that out?

I: Yes.

A: (Pulls a 1-inch part out of the 11-inch bar and moves it over the 77/5-bar) Okay. So, that was 55 (She measures the small parts in the 77/5-bar using the pulled 1-inch part) [60,] 65, 70, 75, 76, 77. 77 fifths.

I: Do you have any other idea to directly count the number of pieces in the bottom bar (the 77/5-bar)?

A: Yes (pause).

I: Just reflect your construction.

A: Because I made each group to get the whole bar (the 77/5-bar), I made it 7 pieces of the small one-fifth. So, seven-fifths, then I repeated it 11 times, so seven-fifths 11 times would be seventy-seven fifths.

Ashley did not consider one-fifth of one inch when asked to find the length of the 77/5-bar in inches. She tried to deduce its length by figuring out how much she added to the 11-inch bar to produce the 77/5-bar. That is, even though she constructed the 77/5-bar by repeating a 7/5-bar 11 times, she considered the 77/5-bar in an additive manner without a notion of an iterative aspect of one-fifth of one inch. As a result of my probing questions, she finally reminded herself of the procedure for constructing the 77/5-bar and produced seventy-seven fifths in terms of 11 times seven-fifths.

However, we need to carefully investigate whether Ashley considered the 77/5-bar based on reciprocal reasoning, seven-fifth of 11 inches or 11 of seven-fifths of one inch, because that will show her ability to construct a fraction as a multiplicative relationship between two quantities in a situation involving partitioning. In addition, by the investigation of her reciprocal reasoning, we can see if Ashley developed an ability to use a fraction as indicating a relationship when figuring out another situation such as fraction multiplication. The following protocol shows how she solved the problem, "An 11-inch long bar is 5/7 of another bar. Find the length of the other bar," using mathematical notation while referring to her constructs produced in Protocol 5.1. She first wrote down an equation "11=(5/7)x" on a given sheet of paper, and then explained her solving the equation.

Protocol 5.1: (Cont.) Ashley's solving the equation 11=(5/7)x.

A: [An] 11-inch long bar is five-sevenths of x, and I need to relate that to this (pointing at the computer monitor). Okay, (pause) well, since I divided it, each one, each of 11 into 5 inches [5 parts], then I multiplied it by seven-fifths, I took seven of those small pieces because (pause) I took 7 of them (the 1/5-parts) because this (the 11-inch bar) bar was five-sevenths of the other, so I had multiplied this amount and I repeated it 11 times, so (she writes “7/5” in both sides of the equation 11=(5/7)x a that she already set up, so the original equation becomes “7/5·11=(5/7)x·7/5”) 11 times that 7/5 (She first points at the “11” in the equation, and then the “7/5”) I got 77/5 (as she writes 77/5=x).

In Protocol 5.1, Ashley created a 7/5-bar by dividing one inch into five parts and repeating one part seven times. Referring to the 7/5-bar, she verbalized “I multiplied it (a 1-inch part divided into five small parts) by seven-fifths.” She interpreted her multiplying one inch by seven-fifths when relating it to an activity: “I took seven of those small pieces (the 1/5-parts in the divided 1-inch part).” She nicely associated a view of a fraction as an operation with performing multiplying. Then, she considered her operating carried out for seven-fifths as a result of the operations when repeating.

For her solution of the equation 11=(5/7)x using notation, she focused on explaining how she arrived at the 77/5-bar because she knew that the bar was a solution: “I had multiplied this amount (the 7/5-bar) and I repeated it 11 times.” However, she never addressed how she produced x by multiplying (5/7)x by 7/5. She might notice she needs 7/5 to solve the equation when she wrote down the equation on a sheet of paper. Thus, I wonder if she maintained a notion of her construction of the 77/5-bar as seven-fifths of eleven inches when writing down 7/5 on the left side of 11 to solve the equation. Her repeating the 7/5-bar 11 times indicates she was aware of 11 inches as a unit of units of units. However, I doubt the repetition meant constructing a composite unit based on one-fifth of 11 inches. If her repetition implies seven one-fifths of 11 inches, she would be able to explain how the right side of the equation—(5/7)x—becomes x by multiplying it by 7/5 when asked how she arrived at x on the right side. As a result, even though she solved the equation by using a reciprocal of five-sevenths, her reciprocal reasoning did not seem to involve a reciprocal of five-sevenths with respect to 11 inches: a notion of the equivalency that one-fifth of one inch 11 times is one-fifth of 11 inches. In conclusion, 77/5 did not seem to involve a unit fraction of a composite unit, one-fifth of 11 inches.

Ashley developed an operational view of fractions as she differentiated an operation for a construction of a fraction from a result of her operating. The following protocol shows how she differentiated an operation from a result of the operation when engaging in a written problem "Using a seven-thirteenths of a 1-foot candy bar, please show me your construction to solve 7/13 times 16/7 or 16/7 times 7/13" (05/18/06).

Protocol 5.2: Solving a product 7/13 times 16/7 or 16/7 times 7/13 using a 7/13-bar.

A: (Draws a bar and produces a 7-part bar by dividing the bar into 7 parts.) Show me 7/13 times 16/7 (reading the given problem statement to herself). 7/13 times 16/7 (pause). Well, each one of these (the parts consisting of the 7-part bar) is 1/13 of the candy bar, if I want 7/13 times 16/7, but it's one-seventh of this whole bar, it's one-seventh of the whole, so I need seven-thirteenths times sixteen-sevenths, I think (pulling one part out of the 7-part bar) pull out that (the pulled part) and I repeat it, again one-seventh of this bar (the 7-part bar) but it's one-thirteenth of the whole candy bar, so I need to repeat it, one-thirteenth, I know how many times, 16 times, but I don't know [why], I'm trying to tell you why-- because I know what's your next question. Let's see 1,2,3,4,5,6,7,8,9,10,11,12,13 (repeating the one part she pulled out 13 times). Again I should make a smaller one. That's the whole candy bar. Then I need three more (repeating the pulled part three more times. So, she produces a 16-part bar.). Again I should make a smaller one to start with. This is it, but I don't know why. I am on the right track. Why did I repeat it 16 times (pause)? I am not making that connection.

I: Could you, do you have any idea of 16/7? What do you mean by 16/7 in the multiplication?

A: Okay, that helps. These 16, each one of these (the parts in the 16-part bar) is one-seventh, so I have to repeat that one-seventh of this bar 16 times, because, well because I am multiplying it by sixteen-sevenths so 16 repeats (pause) one-seventh repeated 16 times, but that one-seventh is also one-thirteenth of the whole candy bar, so is that amount, that one-seventh repeated 16 times, that one-seventh is represented by one-thirteenth of the whole, that's why 16/13 of the whole.

Ashley first produced a 7-part bar and noticed one part in the bar is "one-seventh of the bar but one-thirteenth of the whole candy bar." This is the indication that she considered one part in the 7-part bar in terms of an operation of dividing into seven part (one-seventh) as well as a result of the operation (one-thirteenth of one inch). According to the progress of her fractional reasoning, she seemed to arrive at 16/13 procedurally ahead of her construction because she knew that she had to repeat something 16 times but did not know why she had to do that. However, such an activity shows she conceived of a fraction based on operating.

When reminded that she was supposed to deal with sixteen-sevenths, she had no problem with associating her activity with the fraction 16/7: "I have to repeat that one-seventh of this bar (the 7-part bar) 16 times, because, well because I am multiplying it by sixteen-sevenths so 16 repeats (pause) one-seventh repeated 16 times." In addition, since she had an iterative fractional scheme and developed an ability to differentiate an operation from a result of it, she established a relationship between the 7-part bar and an anticipated whole bar (a 13-part bar) through one part and used it for conceptualizing a fraction multiplication: "one-seventh [of the 7-part bar] is represented by one-thirteenth of the whole, that's why 16/13 of the whole." As a result, she conceptualized a fractional product by means of an operational view of fractions.

Summary

In this section, Ashley solved seven fractional tasks in a written form. She was asked to use JavaBars, and for two tasks, she was additionally requested to demonstrate her solving using mathematical notation while referring to the construction she carried out using JavaBars.

Ashley constructed an iterable unit fraction of a composite unit in a generative manner. Thus, she had no problem with finding reciprocals by means of an iterable unit fraction. Further, Ashley produced a fraction to establish a relationship between two fractional amounts and used it for reciprocal reasoning. However, it seemed challenging for her to produce a fraction by associating an operational view of fractions with a result of reasoning with three levels of units. Responding to a question about making seven-ninths of a 5-foot bar, she produced a 35/9-bar by creating five units of seven units of 1/9 of one foot. That implies she reasoned with three levels of units. However, when trying to find a size of the 35/9-bar, she never used her construction based on three levels of units: She never mentioned "five times seven" one-ninths. As a result, although she produced the 35/9-bar by repeating seven-ninths of one foot five times, which is multiplicative, her reasoning about the 35/9-bar was not multiplicative.

Ashley did not seem to take the results of distributive operating as an occasion for making a unit of units of units, which implies she lacked an operational view of fractions. Such a difficulty implementing an operational view of fractions for three levels of units affected her reciprocal reasoning in a fractional context.

Given a problem, "An 11-inch long bar is 5/7 of another bar. Find the length of the other bar," she produced a 77/5-bar by repeating a 7/5-bar, created by dividing one inch in an 11-inch bar into five parts and repeating one part seven times, 11 times [cf. Protocol 5.1]. Even though her construction was based on a repeating activity, she conceived of the length of the 77/5-bar in inches in an additive manner without a notion of an iterative aspect of one-fifth of one inch. So, she considered how much she added to the 11-inch bar to produce the 77/5-bar. Then she was requested to solve the same problem using mathematical notation while relating it to her construction [cf. Protocol 5.1 (Cont.)]. She first wrote down an equation $11=(5/7)x$ on a given sheet of paper. She then changed the equation by $7/5\cdot 11=(5/7)x\cdot 7/5$ while elaborating as follows: "I multiplied it (a 1-inch part divided into five small parts) by seven-fifths, I took seven of those small pieces (the 1/5-parts in the divided 1-inch part)." She used reciprocal reasoning by considering 11 of seven-fifths of one inch. However, she never addressed how she produced x by multiplying (5/7)x by 7/5. As a result, even though she solved the equation by using a reciprocal of five-sevenths, her reciprocal reasoning did not seem to involve a reciprocal of five-sevenths with respect to 11 inches: a notion of the equivalency that one-fifth of one inch 11 times is one-fifth of 11 inches.

Ashley developed an operational view of fractions as she differentiated an operation for a construction of a fraction from a result of her operating. When engaging in a written problem "Using a seven-thirteenths of a one-foot candy bar, please show me your construction to solve 7/13 times 16/7 or 16/7 times 7/13" [cf. Protocol 5.2], she first produced a 7-part bar and considered one part in the 7-part bar in terms of an operation of dividing into seven part (one-seventh) as well as a result of the operation (one-thirteenth of one inch). By probing questions, she associated her activity with the fraction 16/7: "I have to repeat that one-seventh of this bar (the 7-part bar) 16 times, because, well because I am multiplying it by sixteen-sevenths so 16 repeats (pause) one-seventh repeated 16 times. . . . One-seventh [of the 7-part bar] is represented by one-thirteenth of the whole, that's why 16/13 of the whole." As a result, she conceptualized a fractional product by means of an operational view of fractions.

CHAPTER 6

IMPLICATIONS, CONCLUSIONS, AND DISCUSSION

Implications

What does it mean to engage in mathematics? How do we know one is engaging in mathematics? What is meant by developing a mathematical way of thinking? Why do we attempt to know one person's knowledge especially in mathematics?

Through this study I learned and experienced that one's conceptual activities should be placed at a core of the answers to those questions. Every mathematical concept requires conceptual activities in some way, and we engage in an activity when facing a situation involving a mathematical concept. Even when we attempt to solve a situation, however, we are sometimes unaware of the activities taking place at a conceptual level. Teachers should be able to make their students' as well as their activities explicit, and that should be a starting point for their learning to be a mathematics educator. In what follows, I will explain the ways this study affects mathematics education.

Engaging in Students' Way of Thinking Can Be a Promising Way for Teachers to Develop Mathematical Knowledge for Teaching

One of the most important responsibilities of mathematics teachers should be dedicated to finding a way to help students learn mathematics with conceptual understanding. This study suggests that engaging in a student's way of thinking can be a way for teachers to engender students' conceptual understanding. Without having a sense of what students are doing at a conceptual level, there would be no way for teachers to know what students need for a conceptual understanding of a concept. In addition, without having a chance to contrast their understanding of a mathematical concept with their understanding of a student's mathematical concept, teachers would be likely to misunderstand what constructing a concept means.

Ashley engaged in Mike's way of thinking and developed an insight into what he was doing when solving fractional tasks. Through the progress in her ways of engaging in Mike's activity, we can see in what sense the mathematical concepts she developed are worthwhile. At first, she expressed some doubts about Mike's way of solving and attempted to interpret it based on her knowledge that was separate from his. She then began to interpret his activity based on her understanding of his knowledge, and that made it possible for her to make comments that might help his learning. Finally, she analyzed his activity and extended her knowledge to reach his way of thinking.

The knowledge she developed is worthwhile for teaching in the sense that it involves an insight into schemes and operations for a mathematical concept. An operation cannot constitute a concept by itself because it cannot be singled out. Schemes cannot be developed without engaging in an intensive observation of others' way of thinking because they are the observer's conceptions. In addition, it is challenging to notice where and how a scheme or an operation is activated. Ashley might not know anything about the schemes and operations she developed. However, I do believe that she would know or have an ability to investigate what a student is doing when trying to produce, say, three-fourths, and what makes it difficult if he struggles to produce three-fourths. She also would have resources on which to draw in an attempt to help him with his difficulty or extend his understanding. Her knowledge of a fraction would involve knowledge about (1) meanings of a fraction; (2) producing a fraction in various situations; (3) using a fraction to solve a variety of problem situations; and (4) relating a fraction with other fractions. The most important aspect of her knowledge of fractions is that she developed such knowledge by interiorizing schemes and operations relevant to a meaning of fractions based on the operations of dividing and repeating. Both solving fractional tasks and interpreting Mike's activity were critical in such a development. For example, Ashley first had difficulty producing a reciprocal conceptually. By engaging in Mike's way of operating, she developed a concept of iterable unit fractions and an insight into relating two quantities multiplicatively. Based on such a progress, she constructed reciprocals and further extended her fractional reasoning.

Further research Ashley did not have a chance to interact with the student Mike but developed a repertoire of ways to interact with students conceptually. So, I'd like to do further research on how a teacher who has such a repertoire uses it in her teaching of individual students. The research would contribute to specifying interactions between the teacher and student(s) considering a dynamic nature between the teacher's zones of potential construction and student(s)' actual construction. I expect that it would shed light on teachers' knowledge that is closely related to students' learning.

Multiplicative Reasoning and Fractional Reasoning

This study showed that developing fractional reasoning can be a good way to develop fractional multiplicative reasoning, and it can be accomplished by an operational view of fractions. I consider fractional reasoning from a perspective that fractions are constructed in terms of the operations carried out, and dividing and repeating are essential operations for construction of a fraction. Based on the perspective, an operational view of fractions is defined as an interiorized concept with which one can differentiate fractions as operations from anticipated results of operating (cf. Chapter 2).

Dividing activities involve coordinating various units, and repeating activities require one to interiorize an iterable unit item. Based on those ideas, we can claim that constructing a fraction based on equi-partitioning involves an insight into multiplicative structure because creating an iterable composite unit item is fundamental to constructing a multiplicative structure. In addition, an operational view of fractions makes it possible to relate an interiorized meaning of fractions based on operations with other operations and schemes. At the interiorized level, fractions as operations involve reasoning with three levels of units, reciprocal reasoning, the splitting operation, and an ability to establish a multiplicative relationship between two quantities by means of a fraction. In that sense, developing fractional reasoning becomes a good way to develop fractional multiplicative reasoning.

Further research An operational view of fractions is based on an interiorized concept of fractions, and this study found that the interiorization is crucial in developing fractional multiplicative reasoning. However, it still need further research to clarify relationships between an operational view of fractions and other concepts related to multiplicative reasoning in fractional contexts. As an example, I'd like to investigate how some research-based fractional schemes are modified with a notion of an operational view of fractions and how the modified schemes affect developing advanced fractional concepts.

We as Teacher Educators Need to Know Learning Trajectories of Teachers That Involves Learning Trajectories of Children

This study started with my curiosity about teachers' learning of students' way of thinking from a constructivist perspective. Since I sincerely experienced how challenging it is to coordinate my knowledge with [or even understanding] a student's reasoning, I was eager to see how others develop their understanding of students' way of thinking. I did and do believe that investigating how a teacher learns mathematics while trying to understand students' ways of thinking will provide teacher educators invaluable insights into school mathematics and teachers' knowledge for teaching mathematics.

Through this study, I constructed a teacher's learning trajectory. Steffe (2004) called attention to the construction of learning trajectories:

> The construction of learning trajectories of children is one of the most daunting but urgent problems facing mathematics education today. It is also one of the most exciting problems because it is here that we can construct an understanding of children's mathematics and how we as teachers can profitably affect that mathematics. (p. 130)

I apply his comment to the field of teacher education. Mathematics teacher educators discuss knowledge that teachers should know to help students learn mathematics. Without understanding their students' mathematical concepts and operation, teachers can at best proceed as if they help their students learn mathematics. Likewise, without having a picture of teachers' learning of students' way of thinking, we as teacher educators can at best proceed as if we help teachers understand students' mathematics.

Further research In this study, I analyzed my data focusing content, so I constructed Ashley's learning trajectory of multiplicative reasoning based on fractional reasoning. In the future, I'd like to analyze the data from another perspective, focusing more on her understanding of Mike's concepts. That would provide teacher educators another kind of learning trajectory such as a teacher's learning trajectory of a student's units-coordinating scheme. Further, I'd like to propose a study the purpose of which is to investigate what kinds of and in what ways students' conceptual activities are critical in teachers' knowledge development by (1) observing where teachers pay attention while watching students' activities and (2) investigating what aspects of problems or activities are problematic for teachers. The proposed study would provide teacher educators with an insight into teachers' learning on the basis of the concept of schemes. In addition, such a perspective of teacher learning in terms of schemes would benefit teacher educators as they attempt to educate teacher education students to work with pre-college students.

Conclusions

This study investigated a middle school teacher's knowledge development by engaging her in observing a seventh grader's fractional reasoning in a constructivist learning environment. Based on the investigation, I constructed three conclusions that emerged centering around multiplicative reasoning in fractional contexts. The conclusions involve the following concepts: an operational view of fractions, various levels of units, reversibility in a context involving equi-partitioning, a composite unit as a unit for partitioning, and an iterable unit fraction.

Constructing Fractions Based on Operations is Fundamental to Multiplicative Reasoning in Fractional Contexts

This study is grounded on the idea of fractional reasoning based on an equi-partitioning scheme, whose goal is to estimate one of several equal parts of a quantity and, in a test of whether the estimated part is a fair share, iterate the part to find if iterating produces a result that is equal to the original (Steffe, 2004). Throughout the study, dividing into equal parts and repeating a part as many times as needed are considered two major activities for fractional reasoning.

I found that constructing a fraction in terms of activities—dividing and repeating (or partitioning and iterating)—is fundamental to developing multiplicative reasoning in fractional contexts. The two activities constituting equi-partitioning reinforces multiplicative ways of thinking in the sense that (1) dividing or partitioning requires one to consider various units simultaneously and to create various levels of units by implementing the activity and (2) repeating or iterating enables one to generate a quantity in terms of multiples of a unit. For instance, equi-partitioning a unitary bar into five parts creates a unit containing five units; sharing a 2-inch bar among three people requires one to consider a composite unit of 2 in terms of another composite unit of 3 and also to create each inch consisting of three parts. In addition, constructing five-sevenths of a quantity by means of iterating allows one to recognize it by five one-sevenths of the quantity.

The study is also grounded in the ideas that a units-coordinating scheme is basic to multiplicative reasoning in whole number contexts (Steffe, 1992), and construction of an iterable composite unit is fundamental to producing anticipatory multiplicative operations such as anticipating iterating nine seven times (Steffe, 1994). Based on these two ideas and the conceptual aspects of dividing and repeating that I discussed above, I conjectured that a fraction constructed based on dividing and repeating would serve in the construction of multiplicative ways of thinking in fractional contexts. Ashley's development of fractional reasoning supports my conjecture.

Constructing fractions based on operations. I will first describe how Ashley came to construct fractions based on dividing and repeating. When mentioning a fraction, she began to refer to the equi-partitioning activities she carried out as essential to producing a fractional amount. Her initial concept of fractions, based on part-whole comparisons, permitted her to focus on resultant amounts without considering equi-partitioning. For example, when asked "How much is this (a 5-bar) of your string (an 11-bar)?" she first answered two and one-eleventh and then corrected it by changing the problem statement: "this (the 11-bar) is two and one-eleventh of this one (the 5-bar)." She had difficulty thinking of the 5-bar in terms as a part constituting the

11-bar. In addition, both ways of answering indicate that she did not constitute the 5-bar as a unit to measure the 11-bar.

She watched Mike have difficulty constructing a fraction to relate two quantities [cf. Protocol M2.1] and began doubting whether producing a whole by repeating a part three times implies constructing one-third with respect to the produced whole. By engaging in his activity, she realized the importance of both dividing and repeating in relation to one another when producing a fraction and had a chance to corroborate a concept of equi-partitioning as a fundamental activity in constructing a fraction.

She eventually focused on a part that can be iterated and iteration of the part rather than part-whole comparisons. In particular, improper fractional contexts provided her with an opportunity to develop a sense of an iterable unit and further to construct a fraction based on an iterating activity. After stating a 7-part bar as seven-thirds of a 3-part bar, she elaborated her reasoning as follows: “Each one of one-thirds (pointing at one part of the 3-part bar), each one of the same lengths, size and shape would be another third. So, seven of them together make seven-thirds” [cf. Protocol 1.6]. By focusing on a part that represents so many others and thinking about multiples of the part, Ashley associated the fraction seven-thirds with iterating the part seven times: seven-thirds now meant seven one-thirds. She might eventually produce seven-thirds by comparing the parts of the 7-bar and the 3-bar. However, the constraint of not using an out-of statement for an improper fractional situation forced her to think of a quantity in terms of a part that can be repeated.

Ashley constructed a meaning of fractions based on the operations of dividing and repeating, but there were still limitations in her use of the operations. For instance, with my guidance she stated that a 5/9-yard bar is five-elevenths of an 11/9-yard bar while referring to her activity, pulling one part out of eleven parts comprising 11/9-yard bar and repeating the part five times. However, she failed to produce a product using her concepts of five-ninths, eleven-ninths, and five-elevenths: She said “I am trying to get five-elevenths” [cf. Protocol 1.8 (Second cont.)]. So, I conjectured that constructing a fraction based on operations is necessary for relating two quantities multiplicatively but not sufficient for establishing a multiplicative relationship between them.

Developing an insight into various levels of units. Next, I will describe how constructing fractions based on operations is related to developing an insight into various levels of units and further establishing a multiplicative relationship between two quantities on the basis of the insight.

Given a context where a 2-foot bar is three times longer than another bar, Ashley first did not consider coordinating a composite unit of two (feet) and the relationship “three” [cf. the text preceding Protocol A2.2 (1)]. She arrived at two-thirds procedurally by dividing two by three. She disregarded Mike dividing each of two feet into three parts and taking two of the parts to produce the other bar. She wanted him to produce two-thirds of the given amount, that is, two-thirds of two feet. She did not conceive of the fraction two-thirds based on dividing and repeating. If she had done so, she would have produced a fraction one-third according to her idea of dividing two (feet) by three or would have wondered about the number of repetitions, two, she originally wanted to make. As a result, she seemed never concerned about a coordination of the composite unit 2, the relation “three times,” and the hypothetical bar. So, Mike’s producing six parts by dividing each inch into three parts was meaningful to her only in that she could take 2/3 of the six parts using the relation that 4 to 6 is proportional to 2 to 3. Therefore, I infer constructing a fraction based on operations will facilitate an awareness that a composite unit can be a unit for partitioning and an awareness that a multiplicative relation between a known composite unit and an unknown composite unit implies a partitioning of the known composite unit. The following activity supports the inference.

After a while, Ashley noticed dividing each foot into three subparts led to producing three equal groups comprising the 2-foot bar. She clearly knew that two subparts make the other bar, but never referred to the two subparts as being a fractional part of a foot. By dividing each foot into three subparts, she created three levels of units—a unit containing two feet, a unit containing three subparts, and a subpart as a unit. However, her production of three levels of units did not seem to be a result of an awareness that a multiplicative relation between a known composite unit and an unknown composite unit implies a partitioning of the known composite unit. She seemed to divide each inch into three subparts without considering that her goal was to divide the 2-foot bar into three parts. Rather, her goal seemed to be to produce 6 parts so she could take two parts as a desired bar. She was aware of the relation, “three times,” but it was not coordinated with partitioning. The 2-foot bar was not considered as a unit for partitioning and she did not produce three levels of units. She never considered the 2-foot bar in terms of one-third of one foot, thereby relating two parts with the 2-foot bar only by focusing on a part-whole relation—two out of six, which is one-third. However, when Mike created a product “two-thirds times a half” according to his activity, she interpreted the product as follows: “two-thirds of a half of the entire thing [two feet]”, which is “two-thirds of one foot” [cf. Protocol A2.2 (2)]. Based on Mike’s activity, she had a sense of constructing a fraction based on an operation along with a notion of two levels of units.

By engaging in various sharing contexts requiring equi-partitioning, Ashley corroborated her ability to deal with various levels of units. For instance, she was asked to find a share for one person given a 3-foot bar to be shared among five people [cf. Protocol 3.3]. She partitioned each foot into five parts and completed her construction by taking three of the 1/5-parts. She produced a share for one person by operating on three feet and then one foot and created three levels of units. However, her difficulties appeared in her concept of the result in that she said that it was three-fifteenths but never seemed to know that it is one-fifth of three feet based on one-fifth of one foot. That means the three parts comprising one-fifth of three feet did not involve using the three levels of units she produced in reasoning that three parts out of fifteen also constitute one composite part out of five such parts. So, I infer that she would have trouble with establishing a relationship between three parts as one-fifth of three feet and three-fifths of one foot. In addition, the fact that one-fifth was only considered as three-fifteenths implies that her construction of an iterable unit fraction of a composite unit did not involve distributive reasoning. Thus, she had difficulty figuring out the sharing situation by means of 3 divided by 5. In summary, her interiorized concept of fractions as operations helped her to conceive of a situation as multiplicative by enabling her to create three levels of units. However, the three levels of units were a constraint for her in developing recursive reasoning in that she had difficulty using a result of a fraction constructed by the activities carried out as a situation to construct another equal fraction. As another example related to lack of using three levels of units as input for reasoning further, I discuss her reversibility.

When asked to find a reciprocal of a fraction produced by her activity, Ashley had difficulty reversing a fractional scheme used to produce a fractional amount [cf. Protocol 3.1]. She produced three-eighths on the basis of her an activity—dividing a 24-bar into eighths and repeating the one-eighth part three times. However, she never considered dividing the resultant amount, three-eighths, into parts and repeating a produced part in order to produce the original amount. As a result, she had trouble establishing a multiplicative relationship between two quantities such as, "One-eighth of a 24-bar is one-third of a 9-bar."

In conclusion, Ashley constructed fractions based on dividing and repeating by using equi-partitioning as a fundamental operation. The concept of fractions as an ensemble of operations served in her creating a situation as multiplicative by engaging in reasoning with various levels of units. However, her fractional operations were not sufficient to establish a multiplicative relation between two quantities. In addition, even though she interiorized her fractional operations and related two quantities using a fraction, she was yet to use the constructed fractions as a situation for another fractional scheme.

Differentiating a Result of Operating from the Implemented Operation is Critical in Establishing a Multiplicative Relationship between Two Quantities and Developing Reversibility.

In the previous section, I discussed how the fractional operations of dividing and repeating were not sufficient to establish a fractional multiplicative relation between two quantities. Related to the issue, I found that an ability to differentiate a fraction as an operation from an anticipated result of operating is critical in developing multiplicative reasoning in that, as the distinction becomes explicit, it is more likely to take the fractional operations carried out as a situation for other fractional operations. In particular, such a differentiation is necessary for reversibility requiring inverse operations.

I will first introduce a scheme, an operational view of fractions. In Chapter 2, I defined the scheme by describing its operation in two ways. First, according to the idea of reflection (Piaget, 1950), an operational view of fractions involves a reflective level that permits the results of fractional operations being differentiated from the operations carried out. Second, according to the idea of an image (Thompson, 1996), an operational view of fractions enables one to consider an implemented fractional operation as being coordinated with other possible schemes of which the operation is a part. Therefore, one who has an operational view of fractions should be aware of his/her operating and able to use a result of operating when establishing possible schemes.

When engaging in producing an equation involving a product, Ashley did not differentiate her operating and her results of operating. However, she had no problem with the differentiation between operating and a result of operating when relating two quantities multiplicatively. For instance, when asked to make one-seventh of one pound using one-third of one pound [cf. Protocol 3.4], Ashley distinguished an operational meaning of fractions from a meaning of fractions as results of operating. The distinction enabled her to activate a recursive scheme and engage in distributive reasoning. She repeated a 1/7-part, produced by dividing 1/3 of one pound into seven parts, three times to complete her construction. This implies she created one-seventh of one pound at the level of an operational view of fractions in that she recognized the given situation—1/3 of one pound—as a situation that required dividing and iterating and used fractional language as she referred to the operations she carried out. In addition, her operational view of fractions enabled her to implement recursive reasoning. The fact that she conceived of the given bar as indicating the missing thirds indicates she conceptualized one-third in terms of two views of fractions: one-third as an operation as well as a result of operating. So, one-seventh of the given 1/3 of one pound bar could be distributed across the other thirds by the operation “a seventh of a third.” That is, considering fractions as operations interrelated with distributive reasoning.

However, Ashley’s construction of a fraction based on operations did not involve an ability to establish a fractional multiplicative relationship between two quantities [cf. Protocol 3.4 (Second cont.)]. I found that her distinction between fractions as operations and results of operating meant separating two views, not relating them in a productive way. In an operational view of fractions, she should be able to differentiate fractions as operations from results of operating simultaneously and distinctively. Responding to “how much of this one (a 1/3 of one pound bar divided into seven parts, a 1/3-bar) is this one (a 1/7 of one pound bar divided into three parts, a 1/7-bar)?,” she quickly answered three-sevenths [Protocol 3.4 (Cont.)]. However, she failed to establish a product to obtain the 1/7-bar from the 1/3-bar. That means she did not consider three-sevenths as multiplicatively relating one-third and one-seventh. She seemed confused one-seventh, referring to the weight of the 1/7-bar, with three-sevenths, referring to her operating when she produced the1/7-bar using the 1/3-bar.

With my intervention, she noticed a relational meaning of three-sevenths [cf. Protocol 3.4 (Second cont.)]. She began to consider the fraction three-sevenths as a relation between two quantities, the 1/3-bar and the 1/7-bar. However, her construction of a fraction as a multiplicative relationship between two quantities was not yet interiorized because when asked to produce a multiplication in the inverse way, she made it quickly but confessed, “I am not picturing it yet.”

I argue that a mature fractional multiplication scheme entails a relational insight between two fractional quantities prior to activity. Ashley was yet to establish fraction multiplication in that sense [cf. Protocol 3.5]. She was asked to make a bar so that a 5-inch bar is four-sevenths of the original bar. The posed question required her to consider a fraction four-sevenths as a relation between a 5-inch bar and an anticipated bar. It also implicitly asked her to establish a new relationship, the inverse relationship, to make the new bar using the given relationship, four-sevenths. She did guess that the anticipated bar would double the 5-inch bar. However, she engaged in reversible reasoning with one half, not with four-sevenths.

After a while, she seemed to develop a relational insight into the fraction four-sevenths, but the insight relied only on a quantitative comparison between four-sevenths and seven-sevenths without implementing partitioning: She just mentioned “thirty-five” compared to “twenty” [parts comprising the 5-inch bar]. She failed to construct an iterable composite unit that could be repeated four times to produce the 5-inch bar. Even though she divided each inch into four parts, her partitioning into four parts did not imply a repetition: “If I divide each piece into 4 (pause) but I don’t know about that” [cf. Protocol 3.5]. This comment indicates that she did not explicitly assimilate the problem context involving four-sevenths as a situation for partitioning into four parts and iterating one part seven times. As a result, she had yet to develop an operational view of fractions and she was yet to have a sense of reversibility.

Reciprocal reasoning in fractional contexts requires an anticipatory scheme, a reversible fractional scheme, the splitting operation, and an operational view of fractions. In particular, an operational view of fractions provides a fundamental structure for the other components in that they are based on equi-partitioning, which is a basic activity to consider fractions as operations.

Ashley’s difficulty with reciprocal reasoning seemed related to her inability to use a composite unit as a unit for partitioning. She did not know how to use a 5-inch bar in a situation where the bar is four-sevenths of another bar. Such an inability seemed closely related to an inability to differentiate operating and a result of operating. The 5-inch bar is a result of four-sevenths of another bar, and to obtain the other bar she has to be aware of the operations implemented for four-sevenths, dividing into seven parts and repeating one part four times. Furthermore, she should be able to differentiate dividing the 5-inch bar into four parts—an operation leading to one-fourth—from producing one-seventh of the other bar—a result of dividing into four parts. The 5-inch bar is then reconstituted by recursive operations and distributive reasoning: Five of four parts become four of five parts because one-fourth of the 5-inch bar consists of five parts. With my guidance, she finally reconstituted the 5-inch bar and constructed an iterable composite unit fraction: “That’s [five 1/4-parts] one-fourth of this one (the 5-inch bar), no one-seventh of this one (the 5-inch bar),” “one-seventh of the whole [a desired bar]” [Protocol 3.5 (Cont.)]. As a result, she constructed an iterable composite unit fraction based on an operational view of fractions, which means she developed multiplicative reasoning based on fractional reasoning.

In conclusion, Ashley conceived of a fraction from two perspectives: fractions as operations as well as results of operating. Instead of separating them, she distinctively related the two views of fractions: She considered fractions as results when reconstituting a given quantity and as operations when generating other operations such as recursive operations, distributive reasoning, or the splitting operation. Such an operational view of fractions permitted her to develop an anticipatory multiplicative fractional relationship between two quantities. Therefore, an operational view of fraction was critical in her development of reciprocal fractional reasoning.

Multiplicative Reasoning Can Be Developed Based on Fractional Reasoning

Throughout this study, I regarded fractional reasoning as based on equi-partitioning and multiplicative reasoning as based on units-coordinating. Thus, the statement “multiplicative reasoning can be developed based on fractional reasoning” implies that an insight into various levels of units can be developed using interiorized fractional operations. In this section, I will describe how Ashley developed an insight into various levels of units and coordinated them on the basis of her fractional operations. First, Ashley’s awareness of equi-partitioning when constructing a fraction affected constructing an iterable unit fraction in a generative manner.

Equi-partitioning, an iterable unit fraction, and reciprocal reasoning. An activity is iterable, whereas a result of an activity is likely to be constrained by perception. In that sense, considering a repeating activity as an element of making a fractional amount is critical because it opens a possibility for a part to be iterated beyond a perceptually presupposed whole. By observing Mike's inability to implement equi-partitioning to make a fractional amount, she noticed the importance of the simultaneity of the operations of dividing and repeating in constructing a fraction [cf. Protocol A4.1 (2)] and established a multiplicative relationship between two quantities using an iterable unit fraction [cf. the text following Protocol A4.1 (3)]. Furthermore, she explicitly developed a multiplicative aspect of a unit fraction based on the simultaneity of dividing and repeating [cf. Protocol A4.2].

Ashley showed that a construction of an iterable unit fraction closely relates to differentiating fractions as operations from results of operating. When asked to make a string so that one-fifth of one decameter is four-sevenths of the string [cf. Protocol 3.6], Ashley drew a bar representing one-fifth of one decameter and produced a 7-part bar by dividing the bar into four parts and repeating one part seven times. The construction of the 7-part bar implies Ashley had no problem with reversing the operations used for four-sevenths. However, she had difficulty engaging in reciprocal reasoning because dividing into four parts was considered only as a way to produce one-seventh. Even though she had no problem with anticipating a hypothetical bar and reversing the operations used for four-sevenths, she was yet to engage in the splitting operation, which requires "the operations of partitioning and iterating be implemented simultaneously rather sequentially. That is, he [a student] would need to . . . see the results of iterating as constituting the teacher's stick [a known quantity]" (Steffe, 2002, p. 288). Given the situation requiring inverse operating, she seemed to transform the operations used to make four-sevenths into a result of the operations. Thus, recognizing dividing into four parts as an operation to construct a fraction was suppressed by her inability to differentiate the result of four-sevenths—the one-fifth of one decameter bar—from the operations she carried out when considering the given bar as four-sevenths of the other bar.

When asked "How much is this (the 7-part bar) of this one (the given bar divided into four parts)?," she quickly answered seven-fourths. Although she had difficulty constructing a fraction for the 7-part bar based on her operating—dividing into four parts and repeating one part seven times, she had no problem with producing a fraction regarding the 7-part bar and the given bar. That is, producing a fraction given the produced 7-part bar and the given bar divided into four parts did not require her to be aware of equi-partitioning operating. So, seven-fourths could be constructed only by thinking of one part as an iterable unit. As a result, her construction of seven-fourths did not imply she implemented the splitting operation by differentiating her operation from an anticipated result of the operation. Her ability to construct an iterable unit and a fraction based on operations enabled her to engage in the splitting operation: "Each small piece is a fourth of this fifth [of one decameter], of this piece (the given bar divided into four parts, a 4/7-bar). So, I have seven of them." She then conceptualized a fractional product based on her construction of seven-fourths. Concerning 7/4 times 1/5, she stated as follows: "7/4 times 1/5, this whole thing (the given 1/5 of one decameter bar) is one-fifth, it's four-twentieths, so each piece is one-twentieth, and I have seven, 1,2,3,4,5,6,7" [cf. Protocol 3.6 (Cont.)]. She eventually considered the produced 7-part bar as a result of seven-fourths with respect to the given one-fifth of one decameter bar. She also interpreted "7/4 times 1/5" based on seven-fourths as an operation. In conclusion, she established a multiplicative relationship between two fractional quantities—one-fifth and seven-twentieths—by using seven-fourths. However, her reversibility remained somewhat incomplete in a sense that her reasoning about the multiplicative relationship did not involve reciprocal reasoning [cf. Protocol 3.6 (cont.)]. Now, I will describe in what ways multiplicative reasoning relates to fractional reasoning.

Multiplicative reasoning based on fractional reasoning. An ability to construct an iterable composite unit is crucial in developing multiplicative reasoning because a construction of an iterable composite unit is accomplished by reasoning with three levels of units. Ashley had yet to take a composite unit as a unit to be partitioned. I found that such a lack of taking a composite unit as a unit to be partitioned is closely related to her difficulty reasoning with three levels of units and to establishing a multiplicative relationship.

She was asked to make four-sevenths of a 5-inch bar [cf. Protocol 3.7]. She first made a 4-part bar, produced by dividing each inch into seven subparts and repeating one subpart four times, to produce a unit amount to repeat [this was 4/7 of one inch]. Then, she attempted to repeat the 4-part bar seven times. As soon as she realized four-sevenths means repeating something four times, she repeated the 4-part bar four times. Those activities indicate she did not engage in recursive as well as distributive reasoning. First of all, she disregarded a unit of five—the 5-inch bar—as a unit for partitioning in that she neither repeated the 4-part bar five times nor constructed a 5-part bar to make a unit to repeat it four times. As a result, she did not construct three levels of units for four-sevenths of five inches. Thus, even though she implicitly considered the 5-inch bar as seven-sevenths compared to four-sevenths, she never tried to make an amount that can be the 5-inch bar by repeating it 7 times. She did not consider four-sevenths as a multiplicative relationship to relate the 5-inch bar with a hypothetical bar.

Once she was aware of a composite unit as a unit for operating, she implemented her operational view of fractions. When she realized the given 5-inch bar was a unit for fractional operating, she quickly noticed what she should do for the original question—make four-sevenths of a 5-inch bar: “I need to repeat this blue part (five 1/7-parts) four times because this (the blue part) is one-seventh” [cf. the second paragraph preceding Protocol 3.8]. As a result, a construction of a unit fraction of a composite unit based on an operational view of fractions permitted her to construct an iterable unit fraction as an ensemble of operations and establish a multiplicative relationship based on the iterable unit fraction.

Reasoning with three levels of units. An ability to reason with three levels of units is essential for developing advanced multiplicative reasoning in fractional contexts in that coordinating two levels of units and constructing three levels of units are necessarily involved in equi-partitioning with respect to a composite unit. Ashley watched Mike engage in various situations requiring three levels of units: constructing commensurate fractions and multiplying fractions using reciprocal reasoning [cf. Protocol M4.4 through M4.7]. By engaging in those situations, she explicitly became aware of three levels of units.

Related to three levels of units, Ashley first noticed that Mike had difficulty conceiving of a composite unit as a unit to be partitioned even though he used distributive reasoning while coordinating two levels of units [cf. Protocol A4.4 (1)]. When asked to make a share for one person in order to share seven candy bars among nine people [cf. Protocol M4.4], he produced seven 1/9-parts for a one-person share and said the seven of 1/9-parts is one-ninth of the whole bar. However, it took him a long time to answer “All the candy is how many times a share for one person?” At that time, Ashley pointed out Mike’s unawareness of the seven bars as a whole: “he wasn’t making the connection that I have one-ninth or I divided each individual bar into nine pieces, and I have seven bars.” In addition, she doubted that he associated one-ninth of each bar with one-ninth of seven bars. Such an insight led her to develop a sense of three levels of units [cf. Protocol A4.7].

However, her insight into three levels of units was not explicitly used for conceiving of a fractional product. She pointed out Mike’s lack of insight into various levels of units given a sharing problem with a 7-inch bar to be shared among 11 people: She commented that he disregarded a notion of one inch containing 11 parts [cf. the texts preceding Protocol A4.6]. However, with respect to “11 times 7/11,” she focused only on the fact that seventy-seven one-elevenths of one inch make seven inches. In other words, she never seemed to consider a 7-part bar, produced by dividing one inch into 11 parts and repeating one part 7 times, as a unit to conceive of the 7-inch bar. That means the 7-inch bar did not involve a composite unit structure based on the 7-part bar at her conceptual level.

Producing a reciprocal question with respect to a situation involving a fractional product was challenging to Ashley. However, such a challenging experience provided her with an opportunity to reconsider a fraction as relating two quantities along with a notion of three levels of units [cf. Protocol A4.8 (2)]. As a result, she exactly anticipated Mike’s lack of an ability to associate “1/11 of 7,” “7 times 1/11,” and “7 divided by 11” [cf. Protocol A4. 9 (1)]. She pointed out he tended to consider a 7-foot bar as an only referent for a part, produced by dividing one foot into 11 parts. That is, she had an insight into three levels of units and a differentiation between fractions as resultant amounts and as operations.

It seemed challenging for her to produce a fraction by associating an operational view of fractions with a result of reasoning with three levels of units. When asked to produce seven-ninths of a 5-foot bar, she created a 35/9-bar, which is a unit of units of units in a sense that the 35/9-bar was constructed by repeating seven 1/9s of one foot five times [cf. the fourth paragraph in Final session]. However, when asked to find a size of the 35/9-bar, she never used her construction based on reasoning with three levels of units: She never mentioned “five times seven” one-ninths. Rather, she focused on counting by nine, which is the number of parts comprising one foot, and then counting by one until she reached the last part of the 35/9-bar. She lacked coordinating distributive reasoning involving partitioning with making a unit of units of units whereby she could not maintain a notion of 1/9 of one foot for the produced unit of units of units.

Such a lack of a multiplicative way of thinking with respect to a result of reasoning with three levels of units affected her reciprocal reasoning. Given a problem, “An 11-inch long bar is 5/7 of another bar. Find the length of the other bar,” she constructed a 77-part bar based on reasoning with three levels of units: first creating a 7-part bar (7 one-fifths of one inch) and then repeating it 11 times. When asked to solve the problem by mathematical notation while referring to her construction, she set up an equation 11=(5/7)x and solved it by using a reciprocal of five-sevenths of one inch. However, her reciprocal reasoning did not involve a reciprocal of five-sevenths with respect to 11 inches [cf. Protocol 5.1].

In conclusion, by conceptualizing equi-partitioning as a fundamental fractional activity, Ashley constructed an iterable unit fraction and further developed an insight into three levels of units. In particular, her development of an ability to consider a composite unit as a unit for partitioning helped associate an operational view of fractions with constructing three levels of units. In addition, as she differentiated her operating from an anticipated result of operating, she developed reciprocal reasoning involving inverse operation and the splitting operation. However, it was challenging for her to conceive of a fractional product by means of an insight into three levels of units and to produce a fraction by associating an operational view of fractions with a result of reasoning with three levels of units.

Discussion

In this section, I will first discuss four stages of fractional reasoning and then three relationships among some concepts related to multiplicative reasoning in fractional contexts: relating the splitting operation with an operational view of fractions; relating three levels of units with an operational view of fractions; relating multiplicative reasoning with fractional reasoning.

Four Stages of Fractional Reasoning

Throughout this study, four stages of fractional reasoning were considered: a fraction based on a part-whole concept, a fraction based on a resultant amount, a fraction as a result not differentiated from an activity, and a fraction as an operation differentiated from an anticipated result of operating.

The four stages are identified according to the levels of development of an operational view of fractions. At the first stage, a fraction based on a part-whole concept, one is aware of an equi-partitioning scheme when producing fractions. That means the operations of dividing and repeating are considered in construction of fractions. However, equi-partitioning is implemented only to produce parts comprising a whole or a required amount. So, the number of parts produced is the focus of attention, and partitioning as an operation is disregarded. Ashley's initial concept of fractions falls into this stage.

At the second stage, a fraction based on a resultant amount, one conceives of a part as a result of equi-partitioning but lacks an insight into a composite unit as a fractional whole. In this stage, one can be engaged in the splitting operation but the operation is constrained by a concept of a whole as an absolute amount. In other words, one can construct a part based on a multiplicative relationship between the part and a whole, but the multiplicative relationship does not involve a units-coordinating scheme. Ashley was at this stage until she developed reasoning with three levels of units.

The third stage, a fraction as a result not differentiated from an activity, implies one's awareness of operations to be involved in constructing fractions. However, at this stage, a result of operations is not differentiated from operating. One constructs a fraction in terms of an operation carried out as well as a result of it, but lack of the differentiation between them leads one to have difficulty establishing a multiplicative relationship in fractional contexts. It was at this stage that Ashley constructed a fraction while referring to equi-partitioning or splitting operation but had difficulty with reversibility.

Last, the fourth stage, a fraction as an operation differentiated from an anticipated result of operating involves reasoning with three levels of units. It permits one to engage in a situation requiring a reversible fractional scheme with a notion of two levels of the splitting operation. This stage emphasizes an operation in a reflective level and reversibility through a multiplicative fractional scheme. Ashley was in the process of developing this stage.

Next, I will discuss a relationship between the splitting operation and an operational view of fraction by focusing on a conceptual difference between equi-partitioning and the splitting operation.

Equi-Partitioning, the Splitting Operation, and an Operational View of Fractions

An equi-partitioning scheme is considered as explaining how children use their whole number reasoning in partitioning. Given an ability to construct iterable units of one and composite units, the purpose of an equi-partitioning scheme is "to estimate one of several equal parts of some quantity and to iterate the part in a test to find whether a sufficient number of iterations produce a quantity equal to the original" (Steffe, 2004, p. 132). Partitioning and iterating are necessarily involved in an equi-partitioning scheme, but within the scheme the operations are yet to be composed in the sense that the operations are anticipatory. What this means is that the operating child can posit a hypothetical part of the given whole and anticipate iterating the part to produce the whole without actually carrying out the iterations. On the other hand, the splitting operation is defined as a composition of partitioning and iterating and the child can anticipate the results of the operations as just explained.

Steffe (2004) argued that the distinction between an equi-partitioning scheme and the splitting operation "defines two distinct learning levels for fractions" (p. 156) while referring to constructing an improper fractional scheme, a fractional composition scheme, and recursive partitioning. That is, the splitting operation requires a multiplicative structure whereas an equi-partitioning scheme is additive. However, an equi-partitioning scheme is considered as necessary to develop or implement the splitting operation in that splitting a quantity into parts is based on an ability to create an hypothetical iterable unit item that could be iterated to produce the partitioned whole. That is, partitioning and iterating are realized as simultaneous operations in that a child has an immediate apprehension that the quantity can be split into equal parts and any of the parts can be used to produce the quantity.

One of the foci of this study was to investigate how an equi-partitioning scheme and the splitting operation are involved in producing fractions. The equi-partitioning scheme was considered as an assimilating scheme whose results are used to create a proper fraction, and the splitting operation was regarded as an operation to produce an improper fraction. In some situations only dealing with proper fractions, what is regarded behaviorally as equi-partitioning is more appropriately interpreted as splitting. For example, when solving a problem "How much of the unit bar is 3/4 of a 1/4-bar," equi-partitioning the 1/4-bar into four parts and realizing one of the parts as 1/16 of the whole bar involves splitting. On the other hand, when constructing 4/3 of a 1-inch bar, equi-partitioning is sufficient to partition the 1-inch bar into three equal parts. But if a child independently partitions the bar into three equal parts and iterates one of them four times to produce 4/3, this is an indicator that the child split the bar into three parts. That is, the splitting operation permits the child to posit a hypothetical part that is freed from the unit bar to iterate four times. However, Hackenberg (2007) argued the splitting operation is not sufficient to construct an improper fraction, say 4/3 of the 1-inch bar, unless the child can reason with three levels of units. This study proposed a way to use operations, such as the splitting operation or coordinating three levels of units, in the production of fractions. For instance, with respect to the above situation of producing 4/3 of a 1-inch bar, I would elicit the partitioning of the 1-inch bar into three parts or the repeating of one of the parts four times to produce that fraction. Such an approach might provoke the interiorization of the operations carried out. As a result, the study investigated a constructive path toward the splitting operation from its fundamental source, an equi-partitioning scheme.

I found that when constructing fractions based on the splitting operation involves an operational view of fractions, the splitting operation enables one to construct improper fractions. Imagine a student who has constructed the splitting operation and who is asked to produce 4/3 of a 1-inch bar. If the student does divide the 1-inch bar into three parts, she might be still constrained to the 1-inch bar in the sense that a fraction still needs to be a part of the whole for the student. An operational view of fractions permits her to be aware of the operation of iterating one-third of the bar four times, which might induce a perturbation in the meaning of the possible results of operating. If she focuses on four-thirds as one-third more than three-thirds, she would produce a new bar as a result of repeating and that bar can be considered as a unit. Based on such an insight, together with her ability to engage in the splitting operation, she might notice that one part is one-fourth of what she produced as four-thirds of the 1-inch bar. This is a major advancement because the student for the first time produces an inversion in the relation between the fractional whole and the fraction in that the fractional whole is contained in the fraction produced rather than the fraction being contained in the fractional whole. This insight is based on coordinating three levels of units.

Hackenberg (2007) reported that coordinating three levels of units is required to construct improper fractions, and the splitting operation is not sufficient for it: "construction of the splitting operation is not necessarily accompanied by the *a priori* coordination of three levels of units" (p. 30). Dividing the 1-inch bar into three parts creates two levels of units: the given 1-inch bar and one part, three of which comprises the 1-inch bar. That is, the splitting operation does not necessarily produce creating and coordinating three levels of units.

I agree with Hackenberg's argument that the splitting operation is not sufficient to construct improper fractions. An example is contained in Protocol M4.10, where Mike solved the problem, 7 feet is 2/3 of emus' height. He produced 10 and a half as an answer but failed to convert it to an improper fraction. Mike divided the fourth part of a 7-bar into two subparts and took one subpart and three whole parts. Then he joined this 3 1/2-bar to the 7-bar. He answered 10 and a half, but did not transform that into twenty-one halves. It seemed clear for him that 10 1-foot parts makes 10 feet, and 2 subparts (two halves) makes one foot. However, he did not make 20 halves, which indicates he did not create three levels of units. Instead of focusing on coordinating three levels of units as required to construct improper fractions, I introduced the concept of an operational view of fractions to account for the child's awareness of possible results of operating that produce novel results, such as iterating a 1/3-inch bar four times after splitting the 1-inch bar into three parts. If the novel results induce a perturbation as I explained above, this provides an account of the conditions that lead to the construction of improper fractions.

In fact, to produce 4/3 of a 1-inch bar, an operational view of fractions [along with the splitting operation] implies that dividing the 1-inch bar into three parts and taking one part entails an awareness of a relationship that the one part is one-third of the 1-inch bar as well as one-fourth of an anticipated bar. The awareness is based on differentiating operating from an anticipated result of operating. That is, the pulled part taken has two meanings: one is based on operating and the other is based on an anticipated result of operating. Splitting into three parts produces one-third. In addition, a result of the splitting, the one part, is supposed to be repeated four times, which means the pulled one part is one-fourth of an anticipated bar.

Therefore, if the splitting operation is implemented along with an operational view of fractions, the student would produce four-thirds by repeating one-third, referring to the partitioning, four times. This construction of four-thirds is more than two levels of units in that the differentiation is done prior to the repeating activity. In other words, the splitting of the 1-inch bar into three parts accompanies an anticipation of a unit comprised by four parts, which creates three levels of units. In conclusion, the splitting operation along with an operational view of fractions can provoke constructing improper fractions, and a result of the operational view of fractions is three levels of units. Next, I will discuss a relationship between three levels of units and an operational view of fractions based on the idea of a units-coordinating scheme in fractional contexts.

A Units-Coordinating Scheme, Three Levels of Units, and an Operational View of Fractions

A units-coordinating scheme is considered as necessary to conceive of a situation as multiplicative in whole number contexts (Steffe, 1992). Coordinating units means “distributing a composite unit across the elements of another composite unit” (Steffe, 2002a, p. 279). That is, a units-coordinating scheme creates a new composite unit by coordinating two composite units.

The idea of equi- partitioning as a fundamental operation to construct fractions permits one to use a units-coordinating scheme in fractional contexts. By equi-partitioning a unit amount, a composite unit is created. Equi-partitioning a composite unit provokes one to engage in a units-coordinating scheme. In whole number contexts, coordinating two composite units creates two levels of units. On the other hand, coordinating two composite units within a fractional context creates three levels of units rather than two levels units. Summarizing the idea, construction of a unit fraction by equi-partitioning a unit is considered as a result of creating two levels of units; construction of a unit fraction of a composite unit by units-coordinating is considered as a result of three levels of units.

However, units-coordinating in fractional contexts does not necessarily imply being aware of a three-levels-of-units structure even though such a structure might be produced. For example, given a relationship that 7 feet is 11 times a person’s height [the text preceding Protocol M4.9], Mike divided each foot into 11 subparts and produced a 7-part bar. With respect to the 7-part bar, he mentioned seven-elevenths and seven-seventy sevenths. However, he was yet to consider the 7-part bar as one-eleventh of 7 feet. Saying seven-elevenths indicates his awareness of one foot consisting of eleven subparts—two levels of units. On the other hand, his inability to construct one-eleventh of 7 feet for the 7-part bar implies that his saying seven-seventy sevenths would not involve an awareness of three levels of units. If he had thought seventy-seven subparts in terms of seven of eleven subparts comprising the 7-foot bar, one subpart would have been considered one of eleven subparts comprising one foot as well as one of seven subparts comprising one-eleventh of 7 feet. Therefore, he should have been able to establish an equivalent relationship between seven-elevenths of one foot and one-eleventh of 7 feet if he had reasoned with three levels of units.

Constructing fractions by reasoning with three levels of units at the level of awareness is related to an operational view of fractions. Let's go back to the above example, 7 feet is 11 times a person's height. The 7-foot bar is considered as a composite unit consisting of seven 1-foot bars. The given relationship "11 times" implies another composite unit—an11-part bar—in the sense that, based on the relationship, Mike would posit an imaginary bar so that repeating it 11 times makes the 7-foot bar. Coordinating the composite unit of 7 and the other composite unit of 11 entails partitioning each foot into eleven subparts and thereby three levels of units is created: 7 feet consisting of seven 1-foot bars, one foot comprised by 11 one-eleventh parts, and one-eleventh of one foot. The partitioning produces one-eleventh on the basis of an operational view of fractions. In addition, the coordination enables one to engage in recursive operations and to consider seven subparts as the person's height. An operational view of fractions permits one to produce one-seventy seventh for one subpart by using the recursive operation as well as seven-elevenths for seven subparts. However, considering seven subparts as one-eleventh is problematic in that unitizing seven subparts is required before implementing a repeating operation. The unitizing operation involves reasoning with three levels of units, which means an insight into a relationship that partitioning one foot into 11 subparts results in partitioning the 7-foot bar into 11 parts. Partitioning one foot into 11 subparts with a notion of three levels of units means partitioning 7 feet into 11 parts. That is, an operational view of fractions permits one to notice that one-eleventh of one foot produces one-eleventh of 7 feet in terms of simultaneous operation on each foot, which leads to the notion of seven one-elevenths of one foot is equivalent to one-eleventh of 7 feet.

I conjecture that coordinating three levels of units in fractional contexts requires an operational view of fractions. Within the above context, I focused on the relationship between partitioning one foot into 11 subparts and partitioning 7 feet into 11 parts in order to see if the 7-part bar involves reasoning with three levels of units. Partitioning one foot into 11 subparts produces the fraction one-eleventh as an operation as well as making it possible to produce seven-elevenths by repeating a result of the operation seven times. It can also lead to creating one-eleventh with respect to 7 feet along with a notion of an anticipated result of recursively operating on each foot. Therefore, seven-elevenths of one foot as one-eleventh of 7 feet requires an operational view of fractions. One-eleventh used for seven-elevenths involves a view of fractions as operations, whereas one-eleventh of 7 feet involves a view of fractions as anticipated results. Last, I will discuss multiplicative reasoning in fractional contexts.

Multiplicative Reasoning as Emerging from Fractional Reasoning

There can be two possible ways to relate multiplicative reasoning and fractional reasoning: fractional reasoning as based on multiplicative reasoning vs. multiplicative reasoning as based on fractional reasoning. Vergnaud (1988, 1994) considered fractions as one of the concepts constituting the conceptual field of multiplicative structures, and argued (1988) that the multiplicative conceptual field “consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide” (p. 141). That is, Vergnaud grounded the concept of fractions in multiplicative reasoning—fractional reasoning as based on multiplicative reasoning.

On the other hand, Steffe opened a possibility to consider multiplicative reasoning as based on fractional reasoning. A units-coordinating scheme is considered as necessary to construct a multiplicative structure in whole number contexts (Steffe, 1992), and an equi-partitioning scheme is considered as an assimilating scheme of whole number reasoning in fractional contexts (Steffe, 2002a). Fractional reasoning based on equi-partitioning permits one to engage in partitioning and iterating when constructing fractions. Such basic ideas encouraged me to investigate what aspect of fractional reasoning makes it possible for one to engage in multiplicative reasoning while engaging in fractional reasoning.

From a constructivist perspective, mathematical concepts are constructed by using conceptual activities, operations. So, along that line, fractions are constructed in terms of one's operating, partitioning and iterating. However, constructing fractions based on performed operations is not sufficient to reason with fractions multiplicatively because, although it permits one to produce a relation between two quantities, it does not guarantee an ability to establish a priori a fractional multiplicative relationship between two quantities. For example, when asked to produce one-seventh of one pound using one-third of one pound, Ashley divided the 1/3-bar into seven parts and created a 1/7-bar by repeating one part three times while she said she would consider three of one-third for completing her construction [cf. Protocol 3.4]. Furthermore, she had no problem with elaborating her reasoning of the 1/7-bar as three-sevenths of the 1/3-bar on the basis of her operating: "each single small piece is one-seventh of one-third because I broke the one-third into seven pieces, so that's one-seventh and I repeated it three times" [cf. Protocol 3.4 (Cont.)]. However, she had yet to establish a relationship between the 1/3-bar and the 1/7-bar using the fraction three-sevenths [cf. Protocol 3.4 (Second cont.)].

By conceptualizing an operational view of fractions, I differentiated fractions based on operating from fractions based on anticipated results of operating, and found that the differentiation is critical in establishing a multiplicative relation between two quantities by using a fraction. For instance, even though she was able to produce a fraction five-elevenths to relate a 5/9-bar and an 11/9-bar while referring to her operations, she had difficulty establishing a product using the bars. She was confused between five-elevenths, which are the operations she implemented to produce the 5/9-bar, and five-ninths, which refers to the 5/9-bar she produced using the operations: “I am trying to get five-elevenths” [cf. Protocol 1.8 (Second cont.)]. In Protocol 3.4 (Second cont.), Ashley had difficulty establishing a relationship between the 1/3-bar and the 1/7-bar by using the fraction three-sevenths she produced to relate the two bars. However, as soon as she figured out three-sevenths in the way that “[the 1/7-bar] equals three-sevenths of this one (pointing her cursor at the 1/3-bar),” she was successful in producing the multiplicative relationship “one-seventh equals three-seventh times one-third.” That indicates it was critical to notice that one-seventh is an anticipated result of her operating and three-sevenths refers to her operating.

Further, in the previous topics of this section, an operational view of fractions was investigated related with two important concepts for multiplicative reasoning in fractional contexts, the splitting operation and three levels of units. The splitting operation is considered as a fundamental multiplicative operation (Steffe, 2002a), and coordinating three levels of units is required to generate fractional numbers (Hackenberg, 2007). Through the discussion, I argued that the splitting operation based on an operational view of fractions enables one to construct improper fractions, which is a critical pathway to generate fractional numbers, and an operational view of fractions is required to coordinate three levels of units. These insights into an operational view of fractions allow us to consider multiplicative reasoning as emerging from fractional reasoning.

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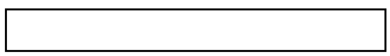
APPENDICES

A: Problems for Each Problem Solving Session

<u>The First Problem Solving Session (Day 1)</u>

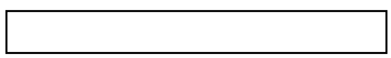
1. Given a relationship between a given bar and another bar, make the other bar.



1) The given bar is one-fifth of another bar. Make the other bar. Could you explain what one-fifth times five is using the bars?
2) The given bar is three-fifths of another bar. Make the other bar.
   - Before you make it, what would you have to do? How much is each of the pieces of the given bar and of the bar you are supposed to make?
   - What fraction is the other bar of the given bar?
   - Could you explain what three-fifths times five-thirds is using the bars, the given bar and one you just made?
3) The given bar is seven-fifths of another bar. Make the other bar.
   - What fraction is the other bar of the given bar? Could you explain what seven-fifths times five-sevenths is using the bars, the given bar and one that you just made?
4) The given bar is thirteen-ninths of another bar.
   - Make the bar that is twenty-five ninths of the other bar.

- What fraction do you need to make the twenty-five ninths bar from the thirteen-ninths bar? (Or what would you have to do to make the twenty-five ninths bar from the thirteen-ninths bar?)
- Could you give me an equation to make twenty-five ninths from thirteen-ninths?
- What about the other way?

2. The following candy bar is seven inches long. Given relationship between the bar and another bar, make the other bar.



1) You want to equally share the bar among three people.
   - How much is one share in inches?
   - What fraction is one share of the entire bar? What about of one inch?
   - Can you give me a multiplication using those facts? (seven equals …, seven-thirds equals, or one-third of seven equals seven-thirds of one inch)
2) The bar is eight times longer than another bar.
   - How much is the other bar in inches? How much is the other bar of the seven-inch bar?
   - Can you make an arithmetic equation using those facts? (one-eighth of seven inches equals seven-eighths of one inch, that is, one-eighth times seven equals seven-eighths; seven-eighths times eight equals seven)
   - Could you explain what eight times seven eighths is using the bars, the seven-inch bar and one that you just made?
3) The bar is nine-thirteenths of another bar.

- How much is the other bar in inches?
- What fraction is the other bar of the given seven-inch long bar?
- Can you give me a multiplication using those facts? (one-ninth of seven equals seven-ninths, one-ninth times seven equals seven-ninths; one-ninth of seven equals one-thirteenth of (seven times thirteen) ninths; Thirteen times seven-ninths equals seven times thirteen-ninths)

The Second Problem Solving Session (Day 2)

1. You have a five-foot ribbon. I have an eight-foot ribbon. Draw how much you have. Using the ribbon, could you make mine? I wonder how much mine is of yours. Could you give me a multiplication for that? I wonder what five times eight-fifths is. Could you give me a multiplication starting with eight such as eight times something equals…?
2. This time, you have a string, which is 13/9 of one foot. I have 25/9 of a one-foot string. Using the string, could you make mine? What would you have to multiply yours by to get mine? Using those facts, could you give me a multiplication?
3. You have a thirteen-inch candy bar. If you want to equally share the candy bar among 5 people, could you make a share for one person? What would you have to do to get your original bar from a share? What would you have to do to get a share from the original bar? (How many times the original bar makes the share for one person?) What is 13 divided by 5? Is that the same as 13 times one-fifth? How could you justify the equivalent relationship between the division and multiplication?
4. You have a 13-inch candy bar. That is 5/7 of my candy bar. Could you make it? How much is it? How much is the bar you just made of the 13-inch bar? Could you give me a multiplication starting with 13 such as 13 times something equals…?

5. You have two candy bars, red and pink. The red candy bar is one-third of one foot, and the pink bar is one-fifth of one foot.
   - Could you make the pink bar using the red bar? How would you know how to do that?
   - What do you multiply the red bar by to get the pink bar? Could you explain how to get the fraction 3/5 relying on what you did such as dividing and repeating? Could you give me a multiplication using the facts you brought? (three-fifths) What is 3/5 times one-third?
6. This time, the red candy bar is still one-third of one foot, but the pink bar you have is two-sevenths of one foot.
   - Could you make the pink bar using the red bar?
   - How much is the pink bar of the red one? Could you give me a multiplication for that?

<u>The Third Problem Solving Session (Day 6)</u>

1. You have a 12-inch candy bar. My candy bar is 3/4 of yours. Could you show me how long mine is? (3 parts of 12 is 1/4 of his as well as 1/3 of mine.) Why is the 3 parts 1/4 of 12? (repeating it 4 times makes the whole – so the 3 parts becomes a composite unit fraction: Pay attention to his saying whether he uses the 3 parts as a countable unit or an iterable unit) Does that mean 3/12 is equal to 1/4? Is there another equivalent fraction in this situation? (9/12 equals to 3/4) Could you give me a multiplication for that? (12 times 3/4 equals 9, 12 times 1/4 equals to 3)

(Variation: For this time, yours is 3/4 of mine. Could you make mine?)

2. A 36-centimeter sub, sharing it among four people – how does she approach to this problem? Partitively or quotitively? Does she consider each person as an iterative unit? (Partitive way) or does she use the number of people to segment the thirty six? (quotitive way)
3. A 3-foot sub sandwich. If you want to share it among 5 people, could you make a share for one person?

<u>The Fourth Problem Solving Session (Day 7)</u>

1. Let's suppose that a given candy bar weighs one-third of one pound. Using the given bar, could you make another candy bar that weighs one-seventh of one pound? (she would use 3/7 of 1/3 to make 1/7) How much is one-seventh pound bar of one-third pound bar? (3/7) Could you give me an arithmetic equation for that? (1/3 * 3/7 = 1/7) or what is 3/7 * 1/3? In terms of little pieces, 1/3 means 7/21 and 1/7 means 3/21. Could you explain for the equivalence? Let's do the same questions for the other way. How much is one-third of one-seventh? (7/3) What is 7/3 * 1/7? (1/3) Could you make a four-sevenths pound bar using the one-third bar? Please give me a multiplication starting with one-third and ending up with four-sevenths. (Is she aware of her doing regarding the fraction she got?)
2. A given string is one-fifth of one decameter. The length of the string is four-sevenths of yours. Using an unknown, could you give me an equation for that? (1/5 = 4/7 *x) Make your string. How long is your string? How much is yours of the given string? (7/4) How would you know that? So, give me a number sentence to represent x. (7/4 * 1/5) -------- paper & pencil

(Variation: The length of the string is 8/7 of yours)

3. You have a 9-inch candy bar. That is 5/7 of my candy bar. Could you make how long mine is? How long is it in inches? (63/5) How much is the bar you just made of the 9-inch bar? (7/5) Could you give me a multiplication starting with 9 such as 9 times something equals…? (7/5 * 9 = 63/5)

(Is she aware of two kinds of fractions for the unit amount? For example, one-fifth of 9 equals to one-seventh of the length of mine. That means mine is seven-fifths of your candy bar. Does she explicitly consider the relationship between the number of pieces in each inch and nine parts representing 9 inches?)

(Variation: For this time, I have 5/7 of your 9-inch bar. Could you show me how long mine is?)

<u>The Fifth Problem Solving Session (Day 8)</u>

1. You have a 5-inch candy bar. Could you make 4/7 of the bar? How long is it in inches? (20/7 --- I expect that she would get the answer in terms of 4/7 five times) How much is the 5-bar of the bar you just made? (7/4) How would you know 5 is 7/4 of the bottom bar? (If she explains using a reciprocal focusing on 1) Could you make 1/7 of the 5-inch bar? How much is it in inches? (Make sure 1/7 of 5 equals 5/7 of one inch) How much is it of the bottom bar? Is it helpful for you to explain why the 5-inch bar is 7/4 of the bottom bar?

(Is she aware of two kinds of fractions for the unit amount? For example, 1/7 of 5 is equal to 1/4 of the bottom bar. That means the 5-bar is 7/4 of the bottom bar. Does she explicitly consider the relationship between the number of pieces in each inch and five parts representing 5 inches?)

(Variation: For this time, your bar, the 5-inch bar, is 4/7 of mine. Could you show me how long mine is?)

2. You have 5/7 of a one-foot sub sandwich, which is three times as much as I have. Could you show me how long mine is? How do you know that's one-third of yours? (How can you make yours using mine?) How long is it in feet? Could you give me a multiplication for that? What if my sub sandwich is 2/3 of yours? Make it. How would you know the bar you just made 2/3 of yours? (Half of it makes 1/3 of mine) How long is it in feet? How much is yours of mine? (3/2) How come?
3. There are two strings, one of them is one-seventh of one foot and the other one is one-ninth of one foot. Could you find a unit you can measure both of them? (Does he know one-ninth of one-seventh, one-sixty thirds?) Could you show me how you are going to use the unit to make the seventh and the ninth? (I expect her to explain that using the word "repeating.") Now, how could you make one-ninth of one foot using one-seventh of one foot? Please give me a multiplication for that. --- If the problem doesn't work for her, I will pose this problem first: there is a 47-centimeter string. Using one centimeter, how can you make the string? (repeats it 47 times) --- Do you think your construction supports the reason that seven-sixty thirds is equal to one-ninth?

<u>The Problems for Fifteenth Session (Day 15)</u>

1. Given a 21-inch candy bar, make 3/7 of the bar.
2. Given a string, which is 9/13 of yours, make your string.
3. Given a 5-foot ribbon,
   A. I was wondering how long 7/9 of it is. Find the length of 7/9 of it using JavaBars.
   B. I was wondering how long 9/7 of it is. Find the length of 9/7/of it using JavaBars.

4. Using a 7/13 of a one-foot candy bar, please show me your construction to solve 7/13 * 16/7 or 16/7 * 7/13.
5. An 11-inch bar is 5/7 of another bar. Find the length of the other bar by using mathematical expression and by constructing using JavaBars. You can start either one, but please make sure that your second way should be based on your first way.
6. Here is 5/7 of a one-foot candy bar, which is 3/5 of your candy bar. Find the length of yours by using mathematical expression and by constructing using JavaBars. You can start either one, but please make sure that your second way should be based on your first way.
7. How much of 3/5 of a one-foot sub is 4/7 of the sub? Please give me a multiplication for your construction.
8. Referring to a given 4-inch candy bar, I will make an estimate to be 3/5 of the bar. How could you check if the estimate is good?

B: A List of Protocols From Problems Solving Sessions

<u>Ashley's Initial Concepts of Fractions (Day 1-2)</u>

A concept of fractions

- Protocol 1.1: Making a 7/7-bar from a 9/7-bar.
- Protocol 1.2: Explaining nine-sevenths using a 9-bar.
- Protocol 1.3: Employing a measuring out strategy, then using a part-whole comparison.
- Protocol 1.4: Changing a question to its inverse to use a measuring out strategy.
- Protocol 1.5: Employing a measuring out strategy for a question leading to a proper fraction.

A concept of a unit fraction

- Protocol 1.6: Reasoning that a 7-bar is seven-thirds of a 3-bar.

A concept of a whole

- Protocol 1.7: Finding a reciprocal relationship between a bar and nine-sevenths of the bar based on her construction of the 9/7-bar.

A concept of multiplication

- Protocol 1.8: Producing a multiplication using a relationship between a 5/9-bar and an 11/9-bar.
- Protocol 1.8: (Cont.)

Current understanding of fractions as operations and unit fractions as iterable units

- Protocol 1.8: (Second cont.)
- Protocol 1.9: Finding a product 17/9 times 4/17.

<u>Some Changes in Ashley's Initial Concepts of Fractions through Engaging in Constructivist Learning Environment and Mike's Mathematics (Day 6-8)</u>

Developing a concept of a unit fraction

- Protocol 3.1: Using the construction of a 9-bar as three-eighths of a 24-bar, finding a fraction for the 24-bar in terms of the 9-bar.

Developing distributive reasoning through a grouping idea

- Protocol 3.2: Creating one-fourth of a 36-bar.
- Protocol 3.3: Sharing a 3-foot sub sandwich among five people.
- Protocol 3.3: (Cont.)

An operational view of fractions as a part of a fractional multiplication scheme

- Protocol 3.4: Making one-seventh of one pound using one-third of one pound.

- Protocol 3.4: (Cont.)
- Protocol 3.4: (Second cont.)
- Protocol 3.5: Making a bar so that a 5-inch bar is four-sevenths of the original bar.
- Protocol 3.5: (Cont.)

An impact of establishing a multiplicative fractional relationship between two quantities on conceptualizing fractional multiplication

- Protocol 3.6: Making a string so that a fifth of one decameter is four-sevenths of the string.
- Protocol 3.6: (Cont.) Finding a product 7/4 times 1/5.

An impact of conceptualizing a composite unit as a unit for fractional operating on establishing a multiplicative relationship

- Protocol 3.7: Making four-sevenths of a 5-inch bar.
- Protocol 3.8: A reciprocal of 4/7 of a 5-inch bar.

<u>Ashley's Development of Fractional Reasoning (Day 15)</u>

- Protocol 5.1: Maintaining a unit of one-fifth of one inch to measure a 77-part bar.
- Protocol 5.1: (Cont.) Ashley's solving the equation 11=(5/7)x.
- Protocol 5.2: Solving a product 7/13 times 16/7 or 16/7 times 7/13 using a 7/13-bar.

C: A List of Protocols From Video Watching Sessions

<u>Ashley's Initial Concept of Mike's Mathematics (Day 3-5)</u>

A relationship between mixed numbers and improper fractions

- Protocol M2.1: Making an amount so that seven 1-inch bars are three times the amount.

  Ashley's response: Two and one-third might not be equivalent to seven-thirds (02/16/06).

- Protocol A2.1: Comments on Protocol M2.1.

A meaning of fractions through conceptualizing a whole in fractional contexts

- Protocol M2.2: Making a sandwich so that a 2-foot long sub sandwich is three times longer than it.

  Ashley's response 1: Two-thirds of a foot is right but that's not two-thirds of the whole thing (02/16/06).

- Protocol A2.2 (1): Ashley's comments on Mike's construction of a bar so that a 2-foot long bar is three times longer than the bar.

  Ashley's response 2: Saying "of" instead of "times" helped make sense of a multiplication statement (02/16/06).

- Protocol A2.2 (2): Comments on "two-thirds times one-half" Mike produced in Protocol M2.2.

Developing initial concepts of fractions through Mike's mathematics

<u>A concept of a fraction based on a unit fraction</u>

- Protocol M2.3: Making ¾ of a given amount and distinguishing it from another amount, three-fourths of which is the given amount.

  Ashley's response: Asking how many thirds are in a 4/3-bar would help figure out one part in the bar is one-fourth of itself (02/23/06).

- Protocol A2.3: A question to help Mike figure out a relationship between the original bar and four-thirds of it.

<u>A concept of a fraction based on activity</u>

- Protocol M2.4: Making a bar that is three-fourths of a 12-inch bar.

  Ashley's response: Mike is mixing up how many parts with how many times (02/23/06).

- Protocol A2.4: Comments on Mike producing four 12-bars for three-fourths of the 12-bar.
- Protocol M2.5: Making a bar so that a 12-inch bar is three-fourths of the bar.

  Ashley's response: Having the idea of *one group* before he figures out all the others could be helpful (02/23/06).
- Protocol A2.5: Comments on Protocol M2.5.

<u>A concept of a fraction based on awareness of the units involved</u>

- Protocol M2.6: Making a bar so that 20-centimeter bar is two-fifths of the bar.

  Ashley's response: Mike was thinking of a 20-centimeter bar in terms of two-fifths, but not 20 one-centimeters (03/02/06).
- Protocol A2.6: Ashley's comments on Mike's dealing with a 20-bar as two-fifths of some amount.
- Protocol A2.6: (Cont.)

<u>Ashley's Fractional Knowledge Development through Observing Mike (Day 9-14)</u>

Developing an insight into an iterative aspect of a unit fraction by considering fractions as operations

- Protocol M4.1: Establishing a multiplicative relationship between a 36-pound bar and a 48-pound bar

  Ashley's response: Mike is not transferring "divide 48 by 4, get 12" to 12 out of 48 (03/30/06).
- Protocol A4.1 (1): Comments on Mike's conception of one-third and one-fourth.
- Protocol A4.1 (2): A question to help conceptualize a unit fraction.
- Protocol A4.1 (3): Ashley's comments on Mike's construction of four-thirds based on one-third.

- Protocol M4.2: Dominant additive reasoning in a fractional context.

  Ashley's response: A big idea is finding a unit to measure by and then repeating that unit 03/30/06).

- Protocol A4.2: Comments on Protocol M4.2.

Coordinating various levels of units for a multiplicative insight into a fraction

- Protocol M4.3: Producing an equation referring to a share for one person when two bars are given to be shared among three people.

  Ashley's response: I don't know why calling two bars as a certain unit helped him to produce a multiplication better (04/18/06).

- Protocol A4.3 (1): Comments on Protocol M4.3.
- Protocol A4.3 (2): Ashley's comments on making a multiplication problem from a division problem.
- Protocol M4.4: Constructing commensurate fractions, 7/63 and 1/9.

  Ashley's response: Mike was missing the point that there were seven bars (04/18/06).

- Protocol A4.4 (1): Comments on Protocol M4.4.
- Protocol A4.4 (2): (Cont.)
- Protocol M4.5: Constructing a composite unit as a unit when finding thirteen times one-thirteenth.

  Ashley's response: The operation of repeating calls Mike's attention to connecting the parts based on one bar with the entire bar (04/18/06).

Toward three levels of units

- Protocol M4.6: Producing commensurate fractions through various levels of units, one-eleventh and seven seventy-sevenths

Ashley's response: Mike has yet to answer the question what is 11 times 7/11 (04/20/06).

- Protocol A4.6: Mike is referring to only one kind of whole.
- Protocol M4.7: Confusion over the coordination of three levels of units: 7/25 of one inch and 1/25 of seven inches.

Ashley's response: The bigger number threw him (04/20/06).

- Protocol A4.7: Mike's struggle with a big number when coordinating three levels of units.
- Protocol M4.7: (Cont.)

Ashley's response: He wasn't thinking in terms of inches (04/20/06).

- Protocol A4.7: (Cont.) Comments on Protocol M4.7 (Cont.).

Fraction multiplication based on fractional reasoning

- Protocol M4.8: Coordinating three levels of units: 7 feet, one foot, and one third of one foot.

Ashley's response: He might even reverse the operation he carried out (04/27/06).

- Protocol A4.8 (1): Ashley posing questions that Mike could [or could not] answer.
- Protocol A4.8 (2): 7 times 1/3 as a reciprocal of three times 7/3.
- Protocol M4.9: Three different operations: 1/11 of 7, 7 times 1/11, and 7 divided by 11.

Ashley's response: There is not a lot of connectivity yet (04/27/06).

- Protocol A4.9 (1): Ashley's comments on Protocol M4.9.
- Protocol A4.9 (2): Comments on a relationship among 7/11, 7 times 1/11, and 7 divided by 11.

Improper fractions

- Protocol M4.10: Producing a fraction for 10 and a half.

Ashley's response: The parts in Mike's construction are not equal sizes (04/27/06).

- Protocol A4.10: Comments on Protocol M4.10.
- Protocol M4.11: Make an equation using x so that 14 pounds is 7/5 times the amount of candy that I have.
- Protocol M4.11: (Cont.)

Ashley's response: Mike was trying to figure out how many times will fit in there… That's what I am having trouble with because I am seeing in the same thing the student did (05/04/06).

- Protocol A4.11 (1): A measurement interpretation for a partitive division: The 14-pound bar is seven-fifths of another bar.
- Protocol A4.11 (2): Solving the equation 14 equals 7/5 times x using Java Bars.
- Protocol M4.12: Solving an equation, 3/5 times x equals 15.

Ashley's response: Mike never did a reversing operation of three-fifths, given a problem "15 pounds is three-fifths of the teacher's amount" (05/04/06).

- Protocol A4.12: Comments on Protocol M4.12.
- Protocol M4.13: Confusion about an improper fractional amount.

Ashley's response: "Would it be bigger or smaller?" will become the first focus, and how many pieces he needs to divide into would be the next (05/11/06).

- Protocol A4.13: A way to help Mike's difficulty with an improper fraction.
- Protocol M4.13: (Cont.)
- Protocol M4.14: Understanding an improper fraction by means of a mixed number.

Ashley's response: Mike was not remembering one unit repeated a certain number of times for an improper fraction (05/11/06).

- Protocol A4.14: Comments on Protocol M4.14.