Analyzing Free Quantum Field Theories on the $ax+b$ Space-time and Wigner Contractions to the Minkowski Plane

by

Maurice Joseph LeBlanc, III

(Under the Direction of Dr. Robert Varley)

Abstract

While much work toward constructing quantum field theories has already been done in the case of a flat space-time, the discipline is not nearly as well-developed in the case of a curved space-time. The axioms offered by Wightman provide a mathematically rigorous system for the construction of a quantum field theory on Minkowski space. I extend this axiomatic approach in order to develop a model for a free scalar quantum field theory in the case of a two-dimensional curved space-time by offering a set of axioms for a quantum state field which can then be used to create a quantum operator field by second quantization using the Segal field operator. I then demonstrate an example of a mathematically rigorous free scalar quantum field theory on a curved space-time satisfying these axioms. Finally I use Wigner's contraction method to demonstrate that these free quantum fields on the curved $ax+b$ space-time limit to certain quantum fields on the flat plane, $\mathbb{R}^2$.

Index words: homogeneous two-dimensional curved space-time, quantum state field, free scalar quantum field theory, Wightman axioms, Mackey machine

Analyzing Free Quantum Field Theories on the $ax + b$ Space-time and Wigner Contractions to the Minkowski Plane

by

Maurice Joseph LeBlanc, III

Bachelor of Sciences, University of Texas at Austin, 2000

Master of Arts, University of Georgia, 2010

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2013

Analyzing Free Quantum Field Theories on the $ax + b$ Space-time and Wigner Contractions to the Minkowski Plane

by

Maurice Joseph LeBlanc,III

Major Professor: Robert Varley

Committee: Edward Azoff
Michael Usher
Shuzhou Wang

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2013

Contents

Acknowledgments

I would like to thank my advisor, Dr. Robert Varley, for his incredible patience and endless support without which this thesis would surely not exist.

I would also like to thank the members of my committee, Dr. Edward Azoff, Dr. Brad Bassler, Dr. Michael Usher, and Dr. Shuzhou Wang for their time, guidance, and insights.

I also owe a debt of gratitude to all of the members of the mathematical-physics seminar for their feedback, ideas, and discussions.

Finally I would like to thank my friends and family for their ongoing encouragement – in particular my father for igniting my love of mathematics, my mother for always believing that I would be a doctor one day, and my wife, Sara, whose infinite support made all of this possible.

"The mainspring of scientific thought is not an external goal toward which one must strive, but the pleasure of thinking." -Albert Einstein

Chapter 1

Introduction

The reconciliation of general relativity with quantum mechanics has been one of the greatest challenges in physics for almost one hundred years. Much work has been done to integrate the observations of quantum mechanics into a field – a function of time and position – which can be used to model a changing number of particles. Concurrently much work has been accomplished in understanding general relativity, especially in the use of differential geometry to describe curved space-time. However these two physical theories remain distinct and discordant. This thesis is a mathematical attempt to continue in the physical tradition of marrying these two ideas.

Physicists traditionally approach the construction of a quantum field theory by considering operator-valued fields which can be used to study identical particles. However much of their work is not presented in a mathematically rigorous fashion. By formulating axioms for a quantum field theory and requiring certain physical properties hold, we can give quantum field theory a firm mathematical basis.

One such attempt to axiomatize quantum field theory was offered by Wightman. These axioms offer an elegant framework by which a quantum field theory can be constructed on (flat) Minkowski space. We hope to extend this axiomatic approach in order to develop a model for a free scalar quantum field theory in the case of a two-dimensional curved space-time. The quantum fields throughout this thesis are assumed to be scalar, i.e. they have no spin. Studying scalar fields allows for a thorough study of quantum field theories without the added complexity of spin. The lack of interaction between the particles makes this study simpler physically while maintaining compatibility with special relativity. Mathematically, by considering a free quantum field theory, we are able to reduce these axioms and then using these simplified rules adapt them to the curved case.

One test for any such theory is to see how it limits to the flat case as the space-time becomes less curved. We will equip the curved space-time with a non-negative parameter, ρ, and study the effects of the contraction on the space-time and its symmetry group. The contraction technique demonstrated by Wigner will be invaluable in this respect.

In this introductory chapter, we introduce the necessary background material concerning the flat and curved space-times as well as the main tools needed throughout this thesis. In the next chapter, we state a simplified version of the Wightman axioms and offer an example using these simplified axioms. In chapter three, we demonstrate the extension of these simplified axioms to the curved case in order to construct a quantum field theory on a two-dimensional space-time and show how such a field theory limits to the flat case considered with reduced symmetry. The final chapter discusses future research directions and unresolved questions.

1.1 The Minkowski Plane, Representations, and Fock Spaces

The **Minkowski space-time** is the manifold model most often used to study special relativity. As this thesis is an attempt to extend to a two-dimensional curved case, the **Minkowski plane**, which is defined as $M = \mathbb{R}^{1+1}$ with coordinates (t, x) and the metric

$$\langle u, v \rangle = \langle (u_0, u_1), (v_0, v_1) \rangle = u_0 v_0 - u_1 v_1$$

will serve as an excellent progenitor. This metric can alternatively be written in matrix form:

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This metric induces a causal structure on the Minkowski plane. Two space-time points, $u = (u_0, u_1)$ and $v = (v_0, v_1)$, are said to be **light-like separated** if $\langle u - v, u - v \rangle = 0$, **time-like separated** if $\langle u - v, u - v \rangle > 0$, and **space-like separated** if $\langle u - v, u - v \rangle < 0$. Two space-time points can be causally connected only if they are not space-like separated. Physically this corresponds to space-time points which can be reached from one another without exceeding the speed of light, which we have set to be unity.

The **Lorentz group**, $\mathcal{L} = O(1, 1)$ is the group of linear isometries of the Minkowski plane. It can be realized as $\{G \in GL_2(\mathbb{R}) | G^\top \mu G = \mu\}$. This group has four connected components:

$$\mathcal{L}_{++} = \left\{ \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ \sinh(\tau) & \cosh(\tau) \end{pmatrix} | \tau \in \mathbb{R} \right\} \qquad \mathcal{L}_{+-} = \left\{ \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ -\sinh(\tau) & -\cosh(\tau) \end{pmatrix} | \tau \in \mathbb{R} \right\}$$

$$\mathcal{L}_{-+} = \left\{ \begin{pmatrix} -\cosh(\tau) & -\sinh(\tau) \\ \sinh(\tau) & \cosh(\tau) \end{pmatrix} | \tau \in \mathbb{R} \right\} \quad \mathcal{L}_{--} = \left\{ \begin{pmatrix} -\cosh(\tau) & -\sinh(\tau) \\ -\sinh(\tau) & -\cosh(\tau) \end{pmatrix} | \tau \in \mathbb{R} \right\}$$

Viewing the first component of our space-time point as time, $\mathcal{L}_{++} \cong SO(1,1)^o$ is the subgroup of isometries that preserve the direction of time and the orientation of space, while elements of $\mathcal{L}_{\pm -}$ reverse the orientation of space and elements of $\mathcal{L}_{-\pm}$ reverse the direction of time. For physical reasons, we need only consider the first subgroup. It will be convenient to refer to elements of this group by the tau argument (Λ_τ). We will be interested in orbits of the Minkowski plane under $SO(1,1)^o$, particularly the points of the mass hyperbolae – defined to be $H_{(m,0)} = \{(E,p) \in \mathbb{R}^2 | E^2 - p^2 = m^2, E \geq 0\}$ for $m \geq 0$ – the zero subscript denotes the spin. Since the mass hyperbola is an orbit under the action of $SO(1,1)^o$ in the dual space to the Minkowski plane, for each $m > 0$ there exists a bijection between $H_{(m,0)}$ and $SO(1,1)^o$ given by $(E,p) \mapsto \lambda$ where $\lambda(m,0) = (E,p)$. Note that $E = \sqrt{p^2+m^2}$. Thus $SO(1,1)^o \cong H_{(m,0)} \cong \mathbb{R}$ since $\lambda(m,0) \mapsto (\sqrt{p^2+m^2}, p)$ is a one-to-one map. However instead of Lebesgue measure, we will employ a measure on $H_{(m,0)}$ which is invariant under Lorentz transformations: $d\Omega_{(m,0)} = \dfrac{dp}{\sqrt{p^2+m^2}}$. We require any physical measurement to be preserved by the action of this group; we also wish for invariance of measurement under translations in M. Hence we consider the identity component of the **Poincaré group** [1]:

$$\mathcal{P}_2^o = \mathbb{R}^2 \rtimes SO(1,1)^o$$

where by abuse of notation $\mathbb{R}^2$ denotes the translations on $\mathbb{R}^2$ – $((T,X), \mathbb{1})$ will be used to denote $Trans_{(T,X)}$, translation by (T,X). The group operation for the Poincaré group is given by $((t,x),\Lambda)((t',x'),\Lambda') = ((t,x)+\Lambda(t',x'), \Lambda\Lambda')$ and $((t,x),\Lambda)^{-1} = (-\Lambda^{-1}(t,x), \Lambda^{-1})$. The Poincaré group acts on the Minkowski plane by:

$$((T,X),\Lambda)\cdot(t,x) = Trans_{(T,X)} \circ \Lambda(t,x) = \Lambda(t,x) + (T,X)$$

[1] also known as the inhomogeneous Lorentz group

With our space-time in place, we now turn our attention to the space of states. The states of a quantum field can be modeled by points in a separable Hilbert space in which the inner product corresponds to measurements within the model. Thus to construct a quantum field in Minkowski space-time, we will require a unitary representation of $\mathcal{P}_2^o$ on the Hilbert space of states.

Definition 1.1.1. *A* ***group representation*** *of a group G on a vector space V is a group homomorphism from G into $GL(V)$.*

- *If $\Phi : G \to GL(V)$ is a representation, then $\Phi(g \cdot h) = \Phi(g) \cdot \Phi(h)$ for all $g, h \in G$.*
- *A representation is called (algebraically)* ***irreducible*** *if the only subspaces of V which are invariant under the action of G are the zero-dimensional subspace and V itself.*

In this thesis, we will be primarily concerned with representations on separable Hilbert spaces.

- *A representation is* ***unitary*** *if it preserves the inner product on $\mathcal{H}$*

$$\langle \Phi_g(v), \Phi_g(w) \rangle = \langle v, w \rangle \text{ for every } g \in G \text{ and } v, w \in \mathcal{H}.$$

- *A representation on a Hilbert space which has no invariant closed subspaces except the zero-dimensional subspace and $\mathcal{H}$ itself is said to be* ***irreducible****.*
- *Two representations of a group G, Φ_1 on $\mathcal{H}_1$ and Φ_2 on $\mathcal{H}_2$, are* ***equivalent*** *if there exists an isometric isomorphism $\gamma : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\Phi_2(g) = \gamma \circ \Phi_1(g) \circ \gamma^{-1}$ for all $g \in G$.*
- *Finally a* ***strongly continuous*** *representations is one in which for each fixed $v \in \mathcal{H}$, the map $g \mapsto \Phi_g(v)$ is continuous with respect to the Schwartz topology.*

When considering a free field, it is often convenient to regard the Hilbert space $\mathcal{H}$ as the one-particle state space for a boson[2] and its induced **Fock space** as the space of finite-particle states. A Fock space is an infinite-dimensional vector space and is a natural tool for quantum field theory. This mathematical construction is used to manufacture the quantum states of a multi-particle system from a single particle system. In order to model the changing number of particles in a given state, for each space-time point in the Minkowski plane M we demand operators on the Hilbert space of states that model creation and annihilation of a particle in a small neighborhood of that point. The creation and annihilation operators are used to account for the introduction and removal of particles, allowing us to describe a system with a variable number of particles.

In order to construct a Fock space from the Hilbert space $\mathcal{H}$, we must first construct the tensor powers of $\mathcal{H}$. The tensor product $\mathcal{H} \otimes \mathcal{H}$ describes a system consisting of two identical non-interacting particles.[3] We then denote the Hilbert space completion of the ordinary algebraic tensor power by $\mathcal{H} \hat{\otimes} \mathcal{H}$. Similarly we will denote the Hilbert space completion of the ordinary algebraic m-fold tensor product by $\mathcal{H}^{\hat{\otimes} m}$.

As this thesis is concerned with particles with spin zero, we need only consider the bosonic case which corresponds to symmetric tensors.[4] The symmetric part of a general tensor T of order m can be found by utilizing the Sym operator:

$$SymT = \tfrac{1}{m!} \sum_{\sigma \in S_m} \tau_\sigma T$$

where $\tau_\sigma T_{v_1, v_2 \dots v_m} = T_{v_{\sigma(1)}, v_{\sigma(2)} \cdots v_{\sigma(m)}}$

[2]for simplicity, we will consider only the neutral pi meson

[3]In quantum mechanics identical particles are indistinguishable. Thus in a Fock space, all particles must be identical. In order to consider multiple types of particles, one must take the tensor product of different Fock spaces - one for each species.

[4]Recall a symmetric tensor is one that is invariant under any permutation of its vector indices.

For notational convenience, we will denote the Hilbert-space completion of the symmetric m-fold tensor products by $Sym^m(\mathcal{H})$. We define $\mathcal{F}_0$ to be the set of elements of the form

$$\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots)$$

where only finitely many $\psi^{(i)}$ are non-zero, $\psi^{(0)} \in \mathbb{C}$, $\psi^{(1)} \in \mathcal{H}$, and $\psi^{(m)} \in Sym^m(\mathcal{H})$.

Finally we define the Fock space induced from the Hilbert space $\mathcal{H}$ to be:

$$\mathcal{F}_{\mathcal{H}} = \overline{\mathcal{F}_0} = \overline{\bigoplus_{m=0}^{\infty} Sym^m(\mathcal{H})} = \overline{\mathbb{C} \oplus \mathcal{H} \oplus Sym^2(\mathcal{H}) \oplus Sym^3(\mathcal{H}) \oplus \cdots}$$

where $\mathbb{C}$ has been used to denote the vacuum state.

This Fock space is itself an infinite-dimensional complex Hilbert space with inner product:

$$\langle \phi | \psi \rangle := \sum_{m=0}^{\infty} \langle \phi^{(m)} | \psi^{(m)} \rangle_m$$

which is in turn defined by the inner product on each of the constituent spaces. The inner products on $\mathbb{C}$ and $\mathcal{H}$ are multiplication and the Hilbert space inner product respectively. The inner product on $Sym^2(\mathcal{H})$ is given by:

$$\langle \phi_1^{(2)} \otimes \phi_2^{(2)} | \psi_1^{(2)} \otimes \psi_2^{(2)} \rangle_2 = \langle \phi_1^{(2)} | \psi_1^{(2)} \rangle \langle \phi_2^{(2)} | \psi_2^{(2)} \rangle$$

The inner product for higher tensor powers is defined analogously. We require that for all $\psi \in \mathcal{F}_{\mathcal{H}}$, $\|\psi\|^2 = \langle \psi, \psi \rangle < \infty$.

In order to pass from an m-particle state to an $(m+1)$-particle state, we use the **creation operator** $c_u^m : Sym^m(\mathcal{H}) \to Sym^{m+1}(\mathcal{H})$ given by:

$$v_1 \otimes \cdots \otimes v_m \mapsto \sqrt{m+1}\ Sym\,(u \otimes v_1 \otimes \cdots \otimes v_m)$$

To pass from an m-particle state to an $(m-1)$-particle state, we use the **annihilation operator** $a_u^m : Sym^m(\mathcal{H}) \to Sym^{m-1}(\mathcal{H})$ given by:

$$v_1 \otimes \cdots \otimes v_m \mapsto \sqrt{m} \langle u, v_1 \rangle v_2 \otimes \cdots \otimes v_m$$

The creation operator c^m is also denoted $(a^{m+1})^\dagger$ since it is the adjoint to the annihilation operator a^{m+1}. One other property of these operators that we will find useful is that $c_u a_u$ gives the number of particles in the normalized state u. These well-known relations can be found in a number of physics and mathematical physics texts including [4] and [12]. The root factors included in these operators are there for physical reasons and are necessary for the number operator of quantum field theory; they have been included in order to be consistent with the existing literature but have little bearing upon the mathematics. For each $u \in \mathcal{H}$ we create operators C_u^m and A_u^m on the Fock space which act as the identity on all the constituent spaces except the m-particle state where they act by c_u^m and a_u^m respectively. Finally we construct operators C_u and A_u which act by c_u^m and a_u^m respectively for each $m = 0, 1, 2, \ldots$

For a unitary operator U_g on $\mathcal{H}$, we can define an operator $U_g^{(m)}$ on the simple tensors:

$$\otimes_{i=1}^m f_i^{(m)} \mapsto \otimes_{i=1}^m U_g f_i^{(m)}$$

We can then induce an action on finite sums of simple tensors and subsequently extend to the Hilbert-space completion of these sums, $\mathcal{H}^{\hat{\otimes} m}$. By restricting to the symmetric subspace $Sym^m(\mathcal{H})$, we can produce an operator mapping the Hilbert-space completion of the symmetric m-fold tensor product to itself. Finally we can produce an operator $\mathcal{U}_g$ on $\mathcal{F}_\mathcal{H}$ which, by abuse of notation, we will write as:

$$\mathcal{U}_g f = \oplus_{m=0}^\infty U_g^{(m)} \left(\otimes_{i=1}^m f_i^{(m)} \right) = \oplus_{m=0}^\infty \otimes_{i=1}^m U_g f_i^{(m)}$$

1.2 The $ax + b$ group

In order to simplify the study of a quantum field on a curved space-time, we have chosen to consider such a field in the 1+1-dimensional case. While lower-dimensional analysis is not directly applicable to the physical 1+3-dimensional space-time, it captures much of the theory and nuance of the curved case without extra complexity.

The group of affine linear transformations on $\mathbb{R}$ is known as the $ax+b$ group due to its action on the real line: $T_{(a,b)} : \mathbb{R} \to \mathbb{R}$ given by $T_{(a,b)}(x) = ax+b$ for $a > 0$ and $b \in \mathbb{R}$. In this paper, we have adapted these coordinates to be $(b, \alpha) \in \mathbb{R}^2$ where $\alpha = \ln(a)$. This ordering of the coordinates follows the physical convention of listing the temporal coordinate before the spatial one. The $ax+b$ group is isomorphic to a two-dimensional subgroup of $\mathcal{P}_2^o$, the three-dimensional connected Poincaré group – the inclusion map will be shown presently.

This Lie group follows the group law: $(B, A) \cdot (b, \alpha) = (B + e^A b, A + \alpha)$. It follows that $(b, \alpha)^{-1} = (-e^{-\alpha} b, -\alpha)$. By considering the conjugation action, it follows that the subgroup $\mathbf{B} = \{(b, 0) : b \in \mathbb{R}\}$ is normal. Furthermore we can define a map

$$\phi : \mathbf{A} = \{(0, \alpha) : \alpha \in \mathbb{R}\} \to Aut(\mathbf{B})$$

$$\text{by} \qquad \phi_{(0,\alpha)}(b, 0) = (0, \alpha) \cdot (b, 0) \cdot (0, \alpha)^{-1} = (e^{\alpha} b, 0).$$

We can therefore identify this non-abelian, simply-connected, two-dimensional, solvable[5] Lie group as being isomorphic to $\mathbb{R} \rtimes \mathbb{R}$.

The $ax+b$ group can be naturally generalized by considering a family of Lie groups $\{G_\rho : \rho \geq 0\}$ with adapted group law $(B, A)_\rho \cdot (b, \alpha)_\rho = (B + e^{\rho A} b, A + \alpha)_\rho$. For $\rho > 0$, G_ρ can be seen to be isomorphic to G_1 by mapping $(b, \alpha)_\rho$ to $(b, \rho\alpha)_1$. Notice that when $\rho = 0$, the group multiplication deteriorates to addition in $\mathbb{R}^2$. In Wigner's terminology, this corresponds to the contraction of the $ax+b$ group with respect to the B subgroup. This will be discussed in further detail in Section 3.3. For notational simplicity, we will normally suppress the ρ subscript and note it only where confusion might arise. The group will play two distinct roles in our constructions:

- a two-dimensional space-time $\mathbb{R}^2$ with a Lorentz metric to be discussed shortly
- the two-dimensional group of symmetries of this space-time

[5] the subnormal series $1 \lhd B \lhd G$ has abelian factor groups

In order to distinguish when we have used the group as a space-time and when we have used it as an acting symmetry group, we will denote it as S_ρ and G_ρ respectively. This ρ parameter will allow us to control the curvature of the space-time, S_ρ.

The group G_ρ can be included into the Poincaré group by the map $\iota : G_\rho \hookrightarrow \mathcal{P}_2^o$:

$$(b,\alpha) \mapsto \begin{pmatrix} \cosh(\rho\alpha) & \sinh(\rho\alpha) & \frac{1}{\rho}(\cosh(\rho\alpha)-1)+b \\ \sinh(\rho\alpha) & \cosh(\rho\alpha) & \frac{1}{\rho}\sinh(\rho\alpha)+b \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for } \rho > 0$$

Allowing $\rho \to 0$ and applying L'Hôpital's rule where necessary, we may extend this inclusion map to the case where $\rho = 0$:

$$(b,\alpha) \mapsto \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & \alpha+b \\ 0 & 0 & 1 \end{pmatrix}$$

Since the group multiplication is preserved by this map ($\iota(B,A)\cdot\iota(b,\alpha) = \iota\left((B,A)\cdot(b,\alpha)\right)$), we conclude that the map is an injective homomorphism.

This inclusion produces an action of G_ρ on the Minkowski plane M:

For $\rho > 0$

$$\iota_{(b,\alpha)}(0,0) = \begin{pmatrix} \cosh(\rho\alpha) & \sinh(\rho\alpha) & \frac{1}{\rho}(\cosh(\rho\alpha)-1)+b \\ \sinh(\rho\alpha) & \cosh(\rho\alpha) & \frac{1}{\rho}\sinh(\rho\alpha)+b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\rho}(\cosh(\rho\alpha)-1)+b \\ \frac{1}{\rho}\sinh(\rho\alpha)+b \\ 1 \end{pmatrix} = \left(\tfrac{1}{\rho}(\cosh(\rho\alpha)-1)+b, \tfrac{1}{\rho}\sinh(\rho\alpha)+b\right)$$

For $\rho = 0$

$$\iota_{(b,\alpha)}(0,0) = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & \alpha+b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ \alpha+b \\ 1 \end{pmatrix} = (b, \alpha+b)$$

Unlike the Minkowski plane which has the same Lorentz metric at each point of the space-time, the metric for S_ρ varies from point to point. The standard Lorentz metric with respect to the basis $\{\frac{\partial}{\partial b}, \frac{\partial}{\partial \alpha}\}$ on $T_e S_\rho$, the tangent space to S_ρ at the identity, $e = (0, 0)$, is:

$$\left.\begin{pmatrix} \frac{\partial}{\partial b} \cdot \frac{\partial}{\partial b} & \frac{\partial}{\partial \alpha} \cdot \frac{\partial}{\partial b} \\ \frac{\partial}{\partial b} \cdot \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial \alpha} \cdot \frac{\partial}{\partial \alpha} \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In order to calculate the metric at a generic point $(b, \alpha) \in S_\rho$, we translate the tangent space $(T_{(b,\alpha)} S_\rho)$ at that generic point (b, α) to the tangent space at the identity e using the map induced by multiplication by the inverse from the left.

$$x \left.\frac{\partial}{\partial b}\right|_{(b,\alpha)} + y \left.\frac{\partial}{\partial \alpha}\right|_{(b,\alpha)} \xrightarrow{(L_{(b,\alpha)^{-1}})_\star} x \left.\frac{\partial}{\partial b}\right|_{(0,0)} + y \left.\frac{\partial}{\partial \alpha}\right|_{(0,0)}$$
$$(x, y) \mapsto (b, \alpha)^{-1}(x, y) = (-e^{-\rho\alpha} b + e^{-\rho\alpha} x, -\alpha + y)$$

Therefore the Lorentz metric at a generic point $(b, \alpha) \in S_\rho$ with respect to the basis $\{\frac{\partial}{\partial b}, \frac{\partial}{\partial \alpha}\}$ is:

$$\gamma = \left.\begin{pmatrix} \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} \end{pmatrix}\right|_{(b,\alpha)} = \begin{pmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}$$

This matrix is referred to as the **First Fundamental Form** (I_p) of S_ρ. This matrix corresponds to the metric $ds^2 = e^{-2\rho\alpha} db^2 - d\alpha^2$. Since the metric is independent of time coordinate b, our curved space-time, S_ρ, is a **static space-time** like its flat cousin the Minkwoski plane. These space-times are the simplest Lorentzian manifolds as the geometry of these space-times do not change over time.

Proposition 1.2.1. *The Lorentz metric is invariant under the left multiplication action of the group.*

Proof. Consider the left multiplication action $L_{(B,A)}$ on a point (b, α) of the space-time S_ρ:

$$L_{(B,A)}(b,\alpha) = (B,A)(b,\alpha) = (B + e^{\rho A}b, A + \alpha)$$

Thus the action on the tangent space can be given by the matrix:

$$(L_{(B,A)})_\star = \begin{pmatrix} \frac{\partial}{\partial b}(B + e^{\rho A}b) & \frac{\partial}{\partial \alpha}(B + e^{\rho A}b) \\ \frac{\partial}{\partial b}(A+\alpha) & \frac{\partial}{\partial \alpha}(A+\alpha) \end{pmatrix} = \begin{pmatrix} e^{\rho A} & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $(L_{(B,A)})_\star : T_e(G) \to T_{(B,A)}(G)$ is given by

$$(x,y) \mapsto \begin{pmatrix} e^{\rho A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^{\rho A} \\ y \end{pmatrix}$$

The Lorentz norm at the identity is: $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$

while the Lorentz norm at the point (B, A) is: $\begin{pmatrix} xe^{\rho A} & y \end{pmatrix} \begin{pmatrix} e^{-2\rho A} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} xe^{\rho A} \\ y \end{pmatrix} = x^2 - y^2$

Hence the Lorentzian norm is preserved by the multiplication from the left. □

We can exploit the left-invariance of the Lorentz metric to calculate the effect that right translation has upon the metric. Consider:

$$R_{(B,A)} \circ L_{(B,A)^{-1}}(b,\alpha) = (-Be^{-\rho A} + e^{-\rho A}(b + e^{\rho\alpha}B), \alpha)$$

Thus $\left(R_{(B,A)} \circ L_{(B,A)^{-1}}\right)_\star$ is given by the matrix

$$\begin{pmatrix} e^{-\rho A} & e^{-\rho A}\rho e^{\rho\alpha}B \\ 0 & 1 \end{pmatrix}$$

Since $\left(R_{(B,A)} \circ L_{(B,A)^{-1}}\right)_\star : T_e(G) \to T_e(G)$, we can calculate that

$$(x,y) \mapsto \begin{pmatrix} e^{-\rho A} & e^{-\rho A}\rho B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^{-\rho A} + y\rho Be^{-\rho A} \\ y \end{pmatrix}$$

The Lorentzian norm of this point is:

$$x^2e^{-2\rho A} + 2xy\rho Be^{-2\rho A} + y^2\rho^2B^2e^{-2\rho A} - y^2$$

which can also be described by:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} e^{-2\rho A} & \rho Be^{-2\rho A} \\ \rho Be^{-2\rho A} & \rho^2B^2e^{-2\rho A} - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since the left translation by $(B, A)^{-1}$ has no effect on the metric, we conclude that right translation by (B, A) is responsible for the new Lorentz metric with matrix representation:

$$\begin{pmatrix} e^{-2\rho A} & \rho Be^{-2\rho A} \\ \rho Be^{-2\rho A} & \rho^2B^2e^{-2\rho A} - 1 \end{pmatrix}$$

Our study will primarily employ the first fundamental form, which can be thought of as the matrix representation of the "standard" left-invariant Lorentz metric. The first fundamental form $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}$ can be used to compute the Gaussian curvature of the surface, S_ρ. Since this parametrization is orthogonal, we may use the simple formula offered in O'Neill's *Semi-Riemannian Geometry* text:

$$K = \frac{-1}{\sqrt{|EG|}} \left(\frac{\partial}{\partial b} \left(\frac{\frac{\partial}{\partial b}\sqrt{|G|}}{\sqrt{|E|}} \right) - \frac{\partial}{\partial \alpha} \left(\frac{\frac{\partial}{\partial \alpha}\sqrt{|E|}}{\sqrt{|G|}} \right) \right)$$

$$K = -e^{\rho\alpha} \left(0 - \tfrac{\partial}{\partial\alpha}(-\rho e^{-\rho\alpha}) \right) = -e^{\rho\alpha}(-\rho^2e^{-\rho\alpha}) = \rho^2$$

Notice that S_ρ has positive curvature for $\rho > 0$ and the Minkowski plane (S_0) has zero curvature, which agrees with our intuition that it is flat.

We note that this formula for Gaussian curvature is similar to the formula offered on page 60 of Shifrin's *Differential Geometry* text [14]:

$$K = \frac{-1}{2\sqrt{EG}} \left(\frac{\partial}{\partial \alpha} \left(\frac{\frac{\partial E}{\partial \alpha}}{\sqrt{EG}} \right) + \frac{\partial}{\partial b} \left(\frac{\frac{\partial G}{\partial b}}{\sqrt{EG}} \right) \right)$$

However since the product EG is negative, it must be adapted from the Riemannian case to the Lorentzian case, paying special attention to signs.

We can also use the first fundamental form to write the d'Alembert operator following the algorithm given by Jost on page 110. [6]

$$\Box_\rho = A\frac{\partial^2}{\partial b^2} + 2B\frac{\partial^2}{\partial b\partial\alpha} + C\frac{\partial^2}{\partial\alpha^2} + D\frac{\partial}{\partial b} + E\frac{\partial}{\partial\alpha}$$

where $\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \begin{pmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} e^{2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}$

In order to compute the lower order terms, let

$$d = det(I_p) = det\begin{vmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{vmatrix} = -e^{-2\rho\alpha}$$

$$D = \frac{1}{\sqrt{|d|}}\left(\frac{\partial}{\partial b}(A\sqrt{|d|}) + \frac{\partial}{\partial\alpha}(B\sqrt{|d|})\right) = e^{\rho\alpha}\left(\frac{\partial}{\partial b}(e^{2\rho\alpha}e^{-\rho\alpha}) + \frac{\partial}{\partial\alpha}(0)\right) = 0$$

and

$$E = \frac{1}{\sqrt{|d|}}\left(\frac{\partial}{\partial b}(B\sqrt{|d|}) + \frac{\partial}{\partial\alpha}(C\sqrt{|d|})\right) = e^{\rho\alpha}\left(\frac{\partial}{\partial b}(0) + \frac{\partial}{\partial\alpha}(-e^{-\rho\alpha})\right) = \rho$$

Hence the d'Alembert operator for the space-time S_ρ is given by:

$$\Box_\rho = e^{2\rho\alpha}\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial\alpha^2} + \rho\frac{\partial}{\partial\alpha}$$

Having a firm understanding of the mathematical structure of the $ax + b$ space-time G_ρ, we can now impose physical interpretations. The causal influence of a space-time point (b, α) is called the **forward light cone** of (b, α). These are the points in space-time which can be reached from (b, α) by traveling forward in time without exceeding the speed of light. Geometrically, the **light-like directions** for a point $(b, \alpha) \in S_\rho$ are given by the $(x, y) \in T_{(b,\alpha)}(S_\rho)$ with Lorentz norm equal to zero – i.e. (x, y) such that

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = e^{-2\rho\alpha}x^2 - y^2 = 0$$

Thus in the light-like directions, $x = \pm e^{\rho\alpha} y$. It follows that $\dfrac{b'(T)}{\alpha'(T)} = \dfrac{x}{y} = \pm e^{\rho\alpha}$
We now consider two cases, which will turn out to be the two light-like directions.

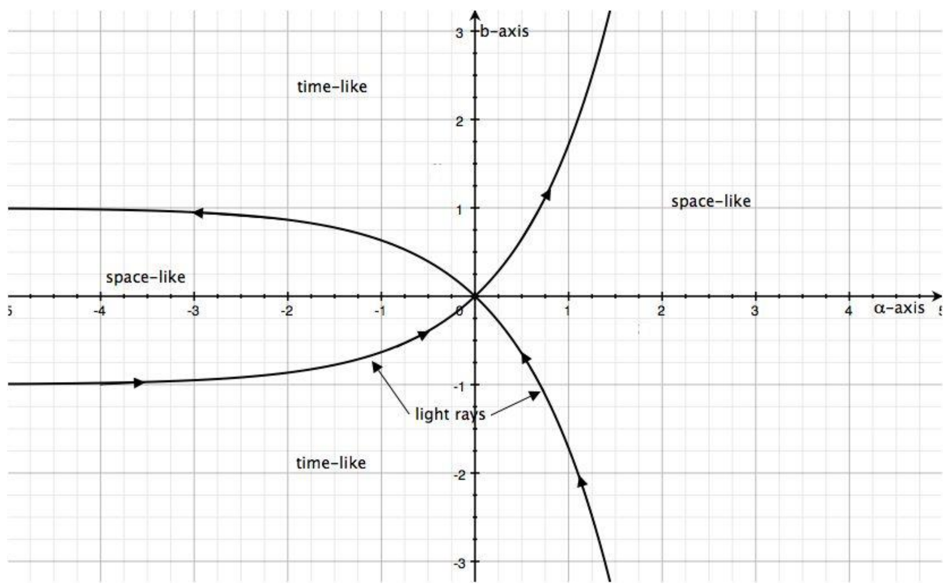
- Suppose $\dfrac{b'(T)}{\alpha'(T)} = e^{\rho\alpha}$ and $b'(T) = 1$.
  Solving the system of differential equations $b'(T) = 1$ and $e^{\rho\alpha(T)}\alpha'(T) = 1$ and using the initial condition $(b(0), \alpha(0)) = (b_0, \alpha_0)$, we conclude that $b(T) = T + b_0$ and $\alpha(T) = \frac{1}{\rho}\ln(\rho T + e^{\rho\alpha_0})$ for $T > -\frac{1}{\rho}e^{\rho\alpha_0}$
- Suppose $\dfrac{b'(T)}{\alpha'(T)} = -e^{\rho\alpha}$ and $b'(T) = 1$.
  Solving the system of differential equations $b'(T) = 1$ and $e^{\rho\alpha(T)}\alpha'(T) = -1$ and using the initial condition $(b(0), \alpha(0)) = (b_0, \alpha_0)$, we conclude that $b(T) = T + b_0$ and $\alpha(T) = \frac{1}{\rho}\ln(-\rho T + e^{\rho\alpha_0})$ for $T < \frac{1}{\rho}e^{\rho\alpha_0}$

Thus the forward light cone at a point (b_0, α_0) can be defined to be the set:

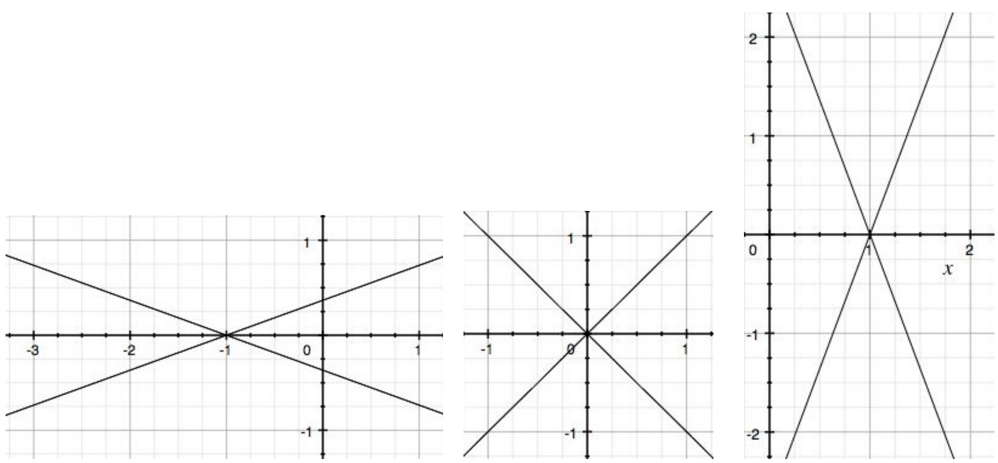
$$C_{(b_0,\alpha_0)} = \left\{(b, \alpha) \in S_\rho \mid b - b_0 \geq \tfrac{1}{\rho}|e^{\rho\alpha} - e^{\rho\alpha_0}|\right\}$$

Points on the interior are said to be **time-like separated** from (b_0, α_0), while points on the boundary (where $b - b_0 = \frac{1}{\rho}|e^{\rho\alpha} - e^{\rho\alpha_0}|$) are said to be **light-like separated** from (b_0, α_0). A backward light cone can be constructed by considering all of the points (b, α) for which $(b_0, \alpha_0) \in C_{(b,\alpha)}$. These are the space-time points which could have causal influence on (b_0, α_0). If two space-time points are such that neither lies in the forward light cone of the other($|B - b| < \frac{1}{\rho}|e^{\rho A} - e^{\rho\alpha}|$), they are said to be **space-like separated**.

Since the $ax + b$ space-time is homogeneous, understanding the light cone at any point allows us to understand the light cone at any other point. Furthermore since the space-time is simply connected, these geodesics are all complete. Note that the light cone does not look symmetric with "Euclidean eyes," but this is unsurprising as we are regarding a curved space-time with a Lorentz metric on a flat page in our Riemannian world. The light cone seems to flatten out as α decreases. This is due to the dependence of the metric on α.



The light-like directions in the tangent space to S_ρ at $(b, \alpha) = (0, 0)$ form the typical light cone of flat space-time, $|\frac{\partial}{\partial b}| = |\frac{\partial}{\partial \alpha}|$. However the light-like directions in the tangent space at $(b, \alpha) = (0, 1)$ form a steeper light cone, $|\frac{\partial}{\partial b}| = e^{\rho}|\frac{\partial}{\partial \alpha}|$, while the light-like directions in the tangent space at $(b, \alpha) = (0, -1)$ form a shallower light cone, $|\frac{\partial}{\partial b}| = e^{-\rho}|\frac{\partial}{\partial \alpha}|$. Since the metric is independent of b, the light-like directions are unaffected by changes in the b coordinate.



(a) $T_{(0,-1)}S_\rho$ (b) $T_{(0,0)}S_\rho$ (c) $T_{(0,1)}S_\rho$

Figure 1.1: Light Cones in Different Tangent Spaces

1.3 The Wightman Axioms

Quantum field theory developed in the early twentieth century as an attempt to amalgamate quantum mechanics and special relativity using the classical models for electromagnetic fields. The founders of quantum field theory – namely Heisenberg, Pauli, and Dirac – encountered many mathematical difficulties in developing such theories and the fields themselves were left only vaguely defined. Physicists of the 1940's established systematic perturbation theory. The resulting calculations agreed strongly with experimental verification suggesting that there were sound mathematical models within quantum field theory. This prompted Gårding and Wightman to devise a definition of a "quantum field" by suggesting a set of mathematical properties which they contended every quantum field theory should have. These properties are called the **Wightman axioms**. The axioms provide a mathematical construction of a quantum field – or what we will call a **quantum operator field**[6]– on (flat) Minkowski space by providing precise mathematical requirements corresponding to physical properties desired of such a quantum field.

A **quantum field theory** in this context can be described by a quadruple $\{\Phi, U, \mathcal{H}, \mathcal{D}\}$ where $\mathcal{H}$ is a separable Hilbert space with dense linear subspace $\mathcal{D}$, U is a unitary representation of $\mathcal{P}_2^o$ on $\mathcal{H}$, and Φ is an operator-valued distribution on the Minkowski plane M. That is to say that Φ is a map from the smooth, compactly-supported functions on M – denoted $C^\infty_{cpt}(M)$ – to the endomorphisms of $\mathcal{D}$. Since $C^\infty_{cpt}(M) \subset \mathscr{S}(M)$, the Schwartz space of function on M, we can place a topology on $C^\infty_{cpt}(M)$ by restricting the Schwartz topology. Recall [15] that in the Schwartz topology, a sequence $\{f_n\} \subset \mathscr{S}(M)$ converges to $f \in \mathscr{S}(M)$ if for each $r, s \in \mathbb{Z}_{\geq 0}$,

$$\lim_{n\to\infty} \sum_{|k|\leq r} \sum_{|l|\leq s} \underset{x\in M}{sup} |x^k D^l (f_n - f)(x)| = 0$$

[6]The use of the extra qualifier "operator" will be made clear in the next chapter

For a quantum operator field theory, we require that the quadruple $\{\Phi, U, \mathcal{H}, \mathcal{D}\}$ satisfy the following axioms [18, 12]:

1: (Regularity) For $f \in C^\infty_{cpt}(M)$, $f \mapsto \langle \psi, \Phi(f)\tilde{\psi}\rangle$ is a continuous $\mathbb{C}$-linear functional for each $\psi, \tilde{\psi} \in \mathcal{D}$, using the Schwartz topology restricted to $C^\infty_{cpt}(M)$.

2: (Vacuum Cyclicity) There is a unique (up to scalar multiple) U-invariant vector $\Omega \in \mathcal{D}$ with $||\Omega|| = 1$ such that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C^\infty_{cpt}(M)$ is dense in $\mathcal{H}$.

3: (Symmetry) $\langle \psi, \Phi(f)\tilde{\psi}\rangle = \langle \Phi(\bar{f})\psi, \tilde{\psi}\rangle$ for all $f \in C^\infty_{cpt}(M)$ and $\psi, \tilde{\psi} \in \mathcal{D}$.

4: (Equivariance) $U_{((T,X),\Lambda)}\Phi(f)U^{-1}_{((T,X),\Lambda)} = \Phi(f \circ ((T,X),\Lambda)^{-1})$ for all $f \in C^\infty_{cpt}(M)$ and $((T,X),\Lambda) \in \mathcal{P}^o_2$.

5: (Causality) $[\Phi(f), \Phi(g)] = 0$ if the supports of $f, g \in C^\infty_{cpt}(M)$ are space-like separated.

6: (Spectral Condition) The infinitesimal generator of $\mathcal{U}((t,0),\mathbb{1})$, the time translation subgroup, has non-negative spectrum.

Regularity requires that Φ be a tempered distribution, which makes the mathematics far more manageable. The other axioms correspond to physical properties desired of the quantum field theory. The second axiom allows us to generate particles from the vacuum by exchanging energy with mass in order to produce any state with a finite number of particles. Furthermore, the density restriction ensures that the Fock space is not "too large" to be described by a single field. The symmetry axiom asserts control over the spectrum of the operators generated by Φ; in particular, if f is real-valued, then $\Phi(f) = \Phi(f)^\dagger$ and $\Phi(f)$ must have a real spectrum and thus can describe an observable. The equivariance condition guarantees that transformations of space-time are consistent with transformations of measurements of states. Places of space-time that cannot communicate without exceeding the speed of light should not affect measurements in one another's regions. This is axiomatized by the causality stipulation. Finally the spectral condition asserts that the energy of a configuration of states must be bounded below.

1.4 The Mackey Machine

The Mackey Machine is an algorithm that can be used to discover irreducible unitary representations of a group by using representations of a normal subgroup. This is best developed in the case of semi-direct products (which the Poincaré group and the $ax + b$ group are). For a summary of these methods, see Mackey [8]; for a detailed treatment also see chapter 17 of Barut and Raczka [1].

Let $G = N \rtimes_\phi H$ with N an abelian normal subgroup of G, a locally compact group. The action of H on N ($\phi : H \to Aut(N)$) induces an action of H on the (one-dimensional) characters of N, $\hat{\phi} : H \to Aut(\hat{N})$ where $\hat{N}$ is the group of continuous homomorphisms from N to the units in the complex plane. A typical element of $\hat{N}$ has the form $\chi_x(n) = e^{ix \cdot n}$. Since N is abelian, each irreducible representation must be one-dimensional; hence $\hat{N}$ contains all of the irreducible representations of N. In order to construct an irreducible representation of G from an irreducible representation of H, we use the following method:

(1) Consider the orbits $\{\mathscr{O}_j\}$ of $\hat{N}$ under the action of H

(2) Choose a representative element from an orbit $\chi_j \in \mathscr{O}_j$

(3) Let $L_j = Stab_H(\chi_j) = \{h \in H | \hat{\phi}_h(\chi_j) = \chi_j\}$

This is often referred to as **Wigner's Little Group**

(4) Fix an irreducible unitary representation of L_j

$$V^j : L_j \to \mathscr{U}(\mathcal{H})$$

(5) Form the semi-direct product $\mathscr{G}_j = N \rtimes L_j$

(6) Define a representation of $\mathscr{G}_j$, $M^j : \mathscr{G}_j \to \mathscr{U}(\mathcal{H})$, by $M^j_{(n,h)} = \chi_j(n)V^j(h)$

- this coincides with $V^j = 1 \cdot V^j$ on $\{e\} \rtimes L_j$
- this coincides with $\chi_j = \chi_j \cdot \mathbb{I}$ on $N \rtimes \{e\}$

(7) The representation $U^{M^j} = Ind_H^G$ of G induced from M^j is the desired irreducible unitary representation. An induced representation can take one of two related forms:

- U^{M^j} can be a representation on the induced representation space

$$\{f : G \to \mathcal{H} | f(hg) = M^j(h)f(g) \text{ and } f \in L^2(G, Haar)\}$$

on which G acts by right translation: $g \cdot f(x) \mapsto f(xg)$

- Alternatively, U^{M^j} can be a representation on the induced representation space

$$\{f : G \to \mathcal{H} | f(gh) = M^j(h)f(g) \text{ and } f \in L^2(G, Haar)\}$$

on which G acts by left translation by the inverse: $g \cdot f(x) \mapsto f(g^{-1}x)$.

The former is the more common construction and will be used in chapter three; the latter will be employed in chapter two.

Mackey showed that every irreducible representation of G can be obtained in this manner and that two representations are equivalent if and only if they come from the same orbit and use unitarily equivalent representations V^j and $\tilde{V}^j$. If H is also abelian, then the irreducible representations V^j are well-known and a complete description of the irreducible representations of G can be given.

Chapter 2

Simplification of the Wightman Axioms

In this chapter, we present a set of axioms for a quantum state field which can be used to create a free quantum field theory which satisfies the Wightman axioms. This is accomplished with the aid of the Segal field operator. We then consider an example of a quantum state field on the Minkowski plane.

2.1 Axioms for a Quantum State Field

The Wightman axioms depict a general quantum field theory. One goal of this thesis is to modify these axioms in the simplified case of a free quantum field theory for particles of mass $m \geq 0$ and spin zero. We begin by taking a step backwards. Instead of describing an operator-valued distribution (Φ), we will consider a Hilbert-space-valued distribution on M, which we will refer to as a **quantum state field**:

$$\Psi : C^{\infty}_{cpt}(M) \to \mathcal{H}$$

The proposed axioms for the quantum state field characterized by the Hilbert-space-valued distribution Ψ on the space-time (M, μ) with values in the dense subspace $\mathcal{D}$ of $\mathcal{H}$ and (strongly continuous) unitary representation[1] U of $\mathcal{P}_2^o$ in $\mathcal{H}$ are:

0: Ψ satisfies the Klein-Gordon equation – the equation of a free quantum scalar field:

$$(\Box + m^2)\Psi = 0 \; ^{2,3}$$

1: For $f \in C^\infty_{cpt}(M)$, $f \mapsto \langle \psi, \Psi(f) \rangle$ is a continuous (with respect to the Schwartz topology) $\mathbb{C}$-linear functional for each $\psi \in \mathcal{H}$.

2: The image of $C^\infty_{cpt}(M)$ under Ψ is dense in $\mathcal{H}$.

3: $\langle$ this space is intentionally left blank in order to have axiom numbers align $\rangle$

4: $U_{((T,X),\Lambda)}\Psi(f) = \Psi(f \circ ((T,X),\Lambda)^{-1})$ for all $f \in C^\infty_{cpt}(M)$ and $((T,X),\Lambda) \in \mathcal{P}_2^o$.

5: If $f, g \in C^\infty_{cpt}(M)$ are two **real-valued** functions whose supports are space-like separated, then $Im\langle \Psi(f), \Psi(g) \rangle_{\mathcal{H}} = 0$.

6: The infinitesimal generator of the time translation subgroup, $i \; dU((1,0),\mathbf{0})$, has non-negative spectrum.

As in the case of the Wightman axioms we may describe this quantum state field theory as a quadruple $\{\Psi, U, \mathcal{H}, \mathcal{D}\}$. We will call a quantum state field **irreducible** if the representation U is irreducible.

[1]To avoid trivialities, we do not allow a one-dimensional Hilbert space with a trivial group action.
[2]More precisely $(\Box + m^2)\Psi(f)$ is the zero vector in $\mathcal{H}$ for all $f \in C^\infty_{cpt}(M)$.
[3]This additional axiom determines the mass of the particle under consideration.

2.2 Connection of Quantum State Field to the Quantum Operator Field

One powerful tool for constructing a quantum operator field from a quantum state field is **second quantization** which describes the many-particle system using a basis that expresses the number of particles occupying each state. One such operator is the **Segal field operator** [12] which maps from a Hilbert space to the self-adjoint operators on its induced Fock space. Letting the separable Hilbert space $\mathcal{H}$ denote the single-particle states and $\mathcal{F}_0$ denote a dense linear subspace of the induced Fock space $\mathcal{F}_{\mathcal{H}}$, the Segal field operator

$$\Phi_s : \mathcal{H} \rightarrow End(\mathcal{F}_0)$$

is given by:

$$\Phi_s(u) = \tfrac{1}{\sqrt{2}}(A_u + C_u) \text{ for } u \in \mathcal{H}$$

where A_u and C_u are the operators defined on page 8.

We can define the quantum operator field $\Phi : C^{\infty}_{cpt}(M) \rightarrow End(\mathcal{F}_0)$ by

$$\Phi(f) = \Phi_s(\Psi(f_1)) + i\Phi_s(\Psi(f_2)) = \Phi(f_1) + i\Phi(f_2)$$

$$= \tfrac{1}{\sqrt{2}}\left(A_{\Psi(f_1)} + C_{\Psi(f_1)} + iA_{\Psi(f_2)} + iC_{\Psi(f_2)}\right)$$

where $f = f_1 + if_2$ for f_1 and f_2 are real-valued. Φ is $\mathbb{C}$-linear by construction.

Notice however

$$\Phi(f) \neq \Phi_s(\Psi(f_1 + if_2)) = \Phi_s(\Psi(f_1) + i\Psi(f_2)) = \tfrac{1}{\sqrt{2}}\left(A_{\Psi(f_1)+i\Psi(f_2)} + C_{\Psi(f_1)+i\Psi(f_2)}\right)$$

$$= \tfrac{1}{\sqrt{2}}\left(A_{\Psi(f_1)} - iA_{\Psi(f_2)} + C_{\Psi(f_1)} + iC_{\Psi(f_2)}\right)$$

This is due to the fact that while the creation operator C_u is $\mathbb{C}$-linear in u, the annihilation operator A_u is $\mathbb{C}$-anti-linear in u.[4] Hence Φ_s is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

[4] The operators C_u and A_u are $\mathbb{C}$-linear over $\mathcal{F}$ for all $u \in \mathcal{H}$

The Segal field operator has many interesting properties, many of which are shown in **Theorem X.41** of Reed and Simon [12]. One of the most useful to us is:
For each $\psi \in \mathcal{F}_0$ and $u, v \in \mathcal{H}$

$$[\Phi_s(u), \Phi_s(v)]\psi = \Phi_s(u)\Phi_s(v)\psi - \Phi_s(v)\Phi_s(u)\psi = i \ \mathrm{Im}\langle u, v\rangle_{\mathcal{H}}\psi.$$

We can now construct a free scalar quantum operator field satisfying the Wightman axioms from the quantum state field in the Minkowski plane using the Segal field operator.

2.3 An Example: A Free Scalar Quantum Field Theory on Minkowski Space

In order to verify the consistency of the axioms of section 2.1, we now construct an example of an irreducible quantum state field. The necessary "ingredients" for a free quantum field theory are:

- Space-time with a Lorentz metric (M, γ)
  - we will use $(\mathbb{R}^{1+1}, \mu)$ where the flat metric μ has signature $(+, -)$
- A *connected* Lie group of symmetries of the space-time M, $G \subset Isom(M, \gamma)$
  - we will use the connected Poincaré group, $\mathcal{P}_2^o = \mathbb{R}^{1+1} \rtimes SO(1,1)^o$
- A one-particle Hilbert space $\mathcal{H}$
- A (strongly continuous) unitary representation of G on $\mathcal{H}$

We use the Mackey Machine to construct an irreducible representation of $\mathcal{P}_2^o$.
The action of $SO(1,1)^o$ on $\mathbb{R}^{1+1}$ is given by $\phi_\Lambda(t,x) = \Lambda^{-1}(t,x)$
This induces an action $\hat{\phi}$ of $SO(1,1)^o$ on the characters $\widehat{\mathbb{R}^{1+1}}$ given by

$$\left(\widehat{\phi_\Lambda}(\chi_{(E,p)})\right)(t,x) = \chi_{\Lambda^{-1}(E,p)}(t,x) = e^{i\Lambda^{-1}(E,p)\cdot(t,x)} = e^{i(E,p)\cdot\Lambda(t,x)}$$

Since $d\Omega_{(m,0)}$ is invariant under the action of the Lorentz group, the inner product is preserved and gives the last equality.

(1) The families of orbits of $\widehat{\mathbb{R}^{1+1}}$ under $SO(1,1)^o$ are:

  (a) $E^2 - p^2 = m^2$ with $E > 0$ for each $m > 0$

  (b) $E^2 - p^2 = m^2$ with $E < 0$ for each $m > 0$

  (c) $E^2 - p^2 = 0$ with $E = 0$

  (d) $E^2 - p^2 = 0$ with $E > 0$, $p > 0$

  (e) $E^2 - p^2 = 0$ with $E > 0$, $p < 0$

  (f) $E^2 - p^2 = 0$ with $E < 0$, $p > 0$

  (g) $E^2 - p^2 = 0$ with $E < 0$, $p < 0$

  (h) $E^2 - p^2 = -m^2$ for each $m > 0$, $p > 0$

  (i) $E^2 - p^2 = -m^2$ for each $m > 0$, $p < 0$

  Since we are interested in the physical case in which there is positive mass and positive energy, the first family of orbits is the only one of interest.

(2) For fixed $m > 0$, we choose $\chi_{(E,p)} = \chi_{(m,0)}$ as the representative of the orbit $\mathscr{O}_m = H_{(m,0)}$

(3) $L_m = Stab_{SO(1,1)^o}(\chi_{(m,0)}) = \{\mathbb{1}\}$

(4) Let V^m be the trivial representation of L_m on $\mathbb{C}$

(5) $\mathscr{G}_m = \mathbb{R}^{1+1} \rtimes L_m = \mathbb{R}^{1+1} \rtimes \mathbb{1}$

(6) $M^m = \chi_{(m,0)} \rtimes V^m : \mathscr{G}_m \to \mathscr{U}(\mathbb{C})$ given by $M^m_{((t,x),\mathbb{1})}(z) = \chi_{(m,0)}(t,x)V^m(\mathbb{1})(z) = e^{imt}z$

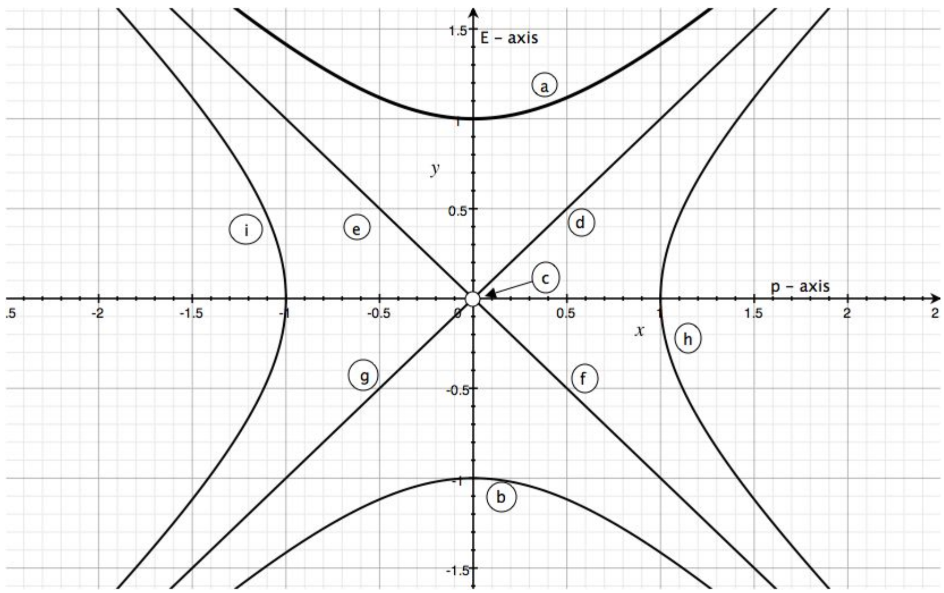


Figure 2.1: Generic Orbits of $\widehat{\mathbb{R}^{1+1}}$ under $SO(1,1)^o$

We are now ready to construct the induced representation of $\mathcal{P}_2^o$

$$U = ind_{\mathscr{G}_m}^{\mathcal{P}_2^o}(M^m)$$

on a Hilbert space

$$\mathcal{W} = \{F : \mathcal{P}_2^o \to \mathbb{C} | F\big(((t,x),\lambda)((T,X),\mathbb{1})\big) = M^m((T,X),\mathbb{1})F((t,x),\lambda),$$

$$F \text{ is measurable, and } \int_{\mathcal{P}_2^o/\mathscr{G}_m} |F(0,0,\Lambda)|^2 \, d\nu < \infty\}$$

- The modulus of functions is fixed on the left cosets of $\mathscr{G}_m$ in $\mathcal{P}_2^o$:

$$|F\big(((t,x),\lambda)((T,X),\mathbb{1})\big)| = |M^m((T,X),\mathbb{1})F((t,x),\lambda)|$$

$$= |\chi_{(m,0)}((T,X))V^m(\mathbb{1})F((t,x),\lambda)| = |e^{i(m,0)\cdot(T,X)}F((t,x),\lambda)| = |F((t,x),\lambda)|$$

- Thus the right action of $\mathscr{G}_m$ does not affect the norm of the function.

- We may therefore define an (right) invariant Haar measure ν on $\mathcal{P}_2^o/\mathscr{G}_m$

  $\mathbb{R}^{1+1}$ acts trivially on $\mathcal{P}_2^o/\mathscr{G}_m$, while $g' \in SO(1,1)^o$ sends $g\mathscr{G}_1 \to g'g\mathscr{G}_1$.

  Hence ν is also left-invariant.

- The measure $d\nu$ can be explicitly described using the correspondence $(E,p) \leftrightarrow \lambda$ if $\lambda(m,0) = (E,p)$ and the associated measure correspondence $d\nu(\lambda) \leftrightarrow d\Omega_{(m,0)}$

  We desire the measure to be invariant under Lorentz transformations

  – i.e. the measure should not be affected by the action $(T,X,\Lambda)\cdot(E,p) = \Lambda(E,p)$

$$(0,0,\lambda_\tau)(m,0) = (m\cosh(\tau), m\sinh(\tau))$$

  which corresponds to the measure $d\Omega_{(m,0)} = \dfrac{dp}{\sqrt{p^2+m^2}}$

  Substituting in the expressions for the transformed p,

$$\frac{d(m\sinh(\tau))}{\sqrt{m^2\sinh^2(\tau)+m^2}} = \frac{m\cosh(\tau)d\tau}{m\cosh(\tau)} = d\tau$$

  we conclude the correct measure to use is $d\tau$.

We then define $U : \mathcal{P}_2^o \to Aut(\mathcal{W})$ to be:

$$U_{((T,X),\Lambda)}F((t,x),\lambda) = F\left(((T,X),\Lambda)^{-1}((t,x),\lambda)\right)$$

We stop to verify that $U_{((T,X),\Lambda)}$ does in fact map $\mathcal{W}$ to itself.

Let $F \in \mathcal{W}$.

$$U_{((T,X),\Lambda)}F(((t,x),\lambda)) = F(((T,X),\Lambda)^{-1}((t,x),\lambda)) = F(\Lambda^{-1}(t,x) - \Lambda^{-1}(T,X), \Lambda^{-1}\lambda)$$

$$= F((\Lambda^{-1}(t,x),\Lambda^{-1}\lambda)(-\lambda^{-1}(T,X),\mathbb{1})) \overset{f\in\mathcal{W}}{=} e^{-i(m,0)\lambda^{-1}(T,X)}F(\Lambda^{-1}(t,x),\Lambda^{-1}\lambda)$$

$U_{((T,X),\Lambda)}F$ has the same norm as F over $\mathcal{P}_2^o/\mathscr{G}_m$.

Left multiplication commutes with right multiplication; therefore the transformation law is also preserved. Since F satisfies the transformation and integrability properties of $\mathcal{W}$, so does $U_{((T,X),\Lambda)}F$ and we conclude $U_{((T,X),\Lambda)}$ maps $\mathcal{W}$ to itself.

In their second volume on *Methods of Modern Mathematical Physics*, Reed and Simon offer the following unitary representation of the Poincaré group on $L^2(H_{(m,0)}, d\Omega_{(m,0)})$ – see page 4 to recall the definition of the mass hyperbola $H_{(m,0)}$ and the Lorentz invariant measure $d\Omega_{(m,0)}$.

$$V_{((T,X),\Lambda)}f(E,p) = e^{-i(ET-pX)}f(\Lambda^{-1}(E,p))$$

This representation is equivalent to the one offered above. As mentioned in the first chapter, the action of the Poincaré group on the mass hyperbola $(((T,X),\Lambda)\cdot(E,p) = \Lambda(E,p))$ gives a correspondence between $H_{(m,0)}$ and $SO(1,1)^o$:

$$(E,p) \leftrightarrow \lambda \qquad\qquad \text{for } \lambda(m,0) = (E,p)$$

Let f be a function on $L^2(H_{(m,0)}, d\Omega_{(m,0)})$.
Define a function $\tilde{f}$ on $\{(0,0)\} \rtimes SO(1,1)^o$ by $\tilde{f}(0,0,\lambda) = f(E,p)$
A simple computation shows that $\tilde{f}$ satisfies the integrability condition.

$$\int_{\mathcal{P}_2^o/\mathscr{G}_m} |\tilde{f}|^2 \, d\nu = \int_{SO(1,1)^o} |\tilde{f}(\lambda)|^2 \, d\nu(\lambda) = \int_{\mathbb{R}} |\tilde{f}(\lambda_\tau)|^2 \, d\tau = \int_{\mathbb{R}} |f(E,p)|^2 \, d\Omega_{(m,0)}(p) < \infty$$

Using the transformation law of $\mathcal{W}$, we can define $\tilde{f}$ on all of the Poincaré group:

$$\tilde{f}(t,x,\lambda) := \tilde{f}\left((0,0,\lambda)(\lambda^{-1}(t,x),\mathbb{1})\right) = e^{-i(m,0)\cdot\lambda^{-1}(t,x)}\tilde{f}(0,0,\lambda)$$

$$= e^{-i\lambda(m,0)\cdot(t,x)}\tilde{f}(0,0,\lambda) = e^{-i(E,p)\cdot(t,x)}\tilde{f}(0,0,\lambda)$$

where the second equality utilizes the preservation of the Lorentz inner product under the action of the Lorentz transformation $\lambda \in SO(1,1)^o$.

Using this correspondence we see that:

$$
\begin{aligned}
V_{(T,X,\Lambda)}(f)(E,p) &= U_{(T,X,\Lambda)}\tilde{f}(0,0,\lambda) = \tilde{f}\left((T,X,\Lambda)^{-1}(0,0,\lambda)\right) \\
&= \tilde{f}\left(-\Lambda^{-1}(T,X),\Lambda^{-1})(0,0,\lambda)\right) = \tilde{f}(-\Lambda^{-1}(T,X),\Lambda^{-1}\lambda) \\
&= \tilde{f}\left((0,0,\Lambda^{-1}\lambda)(-\lambda^{-1}(T,X),\mathbb{1})\right) = e^{-i(m,0)\cdot\lambda^{-1}(T,X)}\tilde{f}(0,0,\Lambda^{-1}\lambda) \\
&= e^{-i\lambda(m,0)\cdot(T,X)}\tilde{f}(0,0,\Lambda^{-1}\lambda) = e^{-i(E,p)\cdot(T,X)}\tilde{f}(0,0,\Lambda^{-1}\lambda) \\
&= e^{-i(E,p)\cdot(T,X)}f(\Lambda^{-1}(E,p))
\end{aligned}
$$

The last equality follows since $\Lambda^{-1}\lambda(m,0) = \Lambda^{-1}(E,p)$ and thus $\Lambda^{-1}\lambda \leftrightarrow \Lambda^{-1}(E,p)$.

Hence the representation V on $L^2(H_{(m,0)}, d\Omega_{(m,0)})$ offered by Reed and Simon can be derived from our representation U on $\mathcal{W}$ presented above using the transformation property of elements of $\mathcal{W}$.

With our representation defined and analyzed, we may now use it and the other components to construct a quantum state field.

Proposition 2.3.1. *Let $\Psi = \mathscr{F}|_{H_m}$ be the Fourier transform restricted to the mass hyperboloid and $\mathscr{S}(H_{(m,0)})$ be the Schwartz functions on the mass hyperbola. The quadruple $\{\Psi, V, L^2(H_{(m,0)}, d\Omega), \mathscr{S}(H_{(m,0)})\}$ satisfies the quantum state field axioms. When composed with the Segal field operator in the prescribed way, a quantum field theory is produced.*

Proof. Let $f \in C^\infty_{cpt}(M)$.

$$\big((\Box + m^2)\Psi\big)(f) = \Psi\Big(\big(\tfrac{\partial^2}{\partial t^2} - \tfrac{\partial^2}{\partial x^2} + m^2\big)f\Big)$$

Recall that under the Fourier transform, differentiation on the space side $(\frac{d}{dx})$ corresponds to multiplication by $i\xi$ on the frequency side.

Hence $\Box f \overset{\mathscr{F}}{\leftrightarrow} ((iE)^2 - (ip)^2)\hat{f} = (-E^2 + p^2)\hat{f} = -m^2\hat{f}$.

Therefore $\big((\Box + m^2)\Psi\big)(f) = 0$ for all $f \in C^\infty_{cpt}(M)$.

The Fourier transform is a continuous complex-linear functional [12]. Hence quantum state field axiom 1 follows immediately.

The smooth, compactly-supported functions on the Minkowski plane form a dense subset of the Schwartz functions on the Minkowski plane. Since the Fourier transform is a linear isomorphism – and therefore a continuous surjection – from $\mathscr{S}(M)$ to $\mathscr{S}(M^*)$, the image of $C^\infty_{cpt}(M)$ under the Fourier transform is dense in $\mathscr{S}(M^*)$ which is in turn dense in $L^2(M^*)$. By restricting to $H_{(m,0)} \subset M^*$, we may conclude that the image of $C^\infty_{cpt}(M)$ under Ψ is dense in $L^2(H_{(m,0)}, d\Omega)$. Hence quantum state field axiom 2 is satisfied.

In order to show equivariance, we must compare

$$V_{((T,X),\Lambda)}\Psi(f)(E,p) \text{ and } \Psi(f \circ ((T,X),\Lambda)^{-1})(E,p) \text{ for } (E,p) \in H_{(m,0)}$$

$$f(t,x) \xmapsto{\Psi} \int_{\mathbb{R}^2} f(t,x)e^{-i(E,p)\cdot(t,x)}\ dt\ dx \xmapsto{V_{((T,X),\Lambda)}} e^{-i(ET-pX)} \int_{\mathbb{R}^2} f(t,x)e^{-i\Lambda^{-1}(E,p)\cdot(t,x)}\ dt\ dx$$

$$f(t,x) \xmapsto{R_{((T,X),\Lambda)^{-1}}} f(\Lambda^{-1}(t-T,x-X)) \xmapsto{\Psi} \int_{\mathbb{R}^2} f(\Lambda^{-1}(t-T,x-X))e^{-i(E,p)\cdot(t,x)}\ dt\ dx$$

Using the change of variables $(\tau,\xi) = \Lambda^{-1}(t-T,x-X)$, this last integral becomes [5]:

$$\int_{\mathbb{R}^2} f(\tau,\xi)e^{-i(E,p)\cdot(\Lambda(\tau,\xi)+(T,X))}\ d\tau\ d\xi = e^{-i(ET-pX)} \int_{\mathbb{R}^2} f(\tau,\xi)e^{-i(E,p)\cdot\Lambda(\tau,\xi)}\ d\tau\ d\xi$$
$$= e^{-i(ET-pX)} \int_{\mathbb{R}^2} f(\tau,\xi)e^{-i\Lambda^{-1}(E,p)\cdot(\tau,\xi)}\ d\tau\ d\xi$$

We conclude $V_{((T,X),\Lambda)}\Psi(f)(E,p) = \Psi(f \circ ((T,X),\Lambda)^{-1})(E,p)$ for $(E,p) \in H_{(m,0)}$ and the equivariance condition holds.

Let $f, g \in C^\infty_{cpt}(M)$ be real-valued and have space-like separated supports. We compute $\langle \Psi(f), \Psi(g)\rangle_{L^2(H_{(m,0)},d\Omega)}$

$$= \int_{H_{(m,0)}} \left(\overline{\int_{\mathbb{R}^2} f(t,x)e^{-i(Et-px)}dtdx}\right)\left(\int_{\mathbb{R}^2} g(t',x')e^{-i(Et'-px')}dt'dx'\right) d\Omega$$
$$= \int_{H_{(m,0)}} \int_{\mathbb{R}^4} f(t,x)g(t',x')e^{i(E,p)\cdot(t-t',x-x')}\ dtdxdt'dx'\ d\Omega$$

Thus $2i\ Im\langle \Psi(f), \Psi(g)\rangle_{\mathcal{H}} = \langle \Psi(f), \Psi(g)\rangle - \langle \Psi(g), \Psi(f)\rangle$

$$= \int_{H_{(m,0)}} \int_{\mathbb{R}^4} f(t,x)g(t',x')\big(e^{i(E,p)\cdot(t-t',x-x')} - e^{i(E,p)\cdot(t'-t,x'-x)}\big)\ dtdxdt'dx'\ d\Omega$$

[5] Since Λ has determinant equal to one, $dt\ dx = d\tau\ d\xi$

Showing this integral to be zero for space-like separated supports is non-trivial and requires use of the two-point function $\Delta_+(x, m^2)$. A full discussion of this result can be seen in Reed and Simon [12], specifically **Theorem IX.48** on page 106 and **Theorem X.42** on page 214. Fortunately the corresponding axiom in Chapter 3 of this thesis requires no such outside machinery.

Since $V_{((T,X),\Lambda)} f(E,p) = e^{-i(ET-pX)} f(\Lambda^{-1}(E,p))$, it follows that $i\ dV_{((1,0),\mathbf{0})} f = E \cdot f$. Since $E > 0$, $i\ dV_{((1,0),\mathbf{0})}$ has non-negative spectrum.

The transmogrification from the quantum state axioms to the quantum field theory using the Segal field operator is treated thoroughly in **Section X.7** of Reed and Simon[12]. $\square$

Chapter 3

Extension of Axioms to the Curved Space-time S_ρ

In this chapter we present a synopsis of our free quantum theoretic results on the curved $ax+b$ space-time. We begin by using the Mackey machine to generate an irreducible representation of the symmetry group for our two-dimensional curved space-time and then show how this representation can be related to one offered by Wigner and Inönü. Next we adapt the quantum state field construction from the previous chapter to the curved case and offer an example. In fact, we construct a family of quantum state fields Ψ_η depending on a positive parameter η. Finally we utilize Wigner's contraction method to study how the proposed quantum fields in a curved space-time limit to a quantum field in a flat space-time. We will see that the flat case limit concerns a lesser-known field theory due to reduced symmetry.

3.1 Setting up an Example on Curved Space-time: Ingredients

We now repeat the process presented in chapter two for the curved space-time, S_ρ for $\rho > 0$. We first outline our "ingredients."

- Space-time with a Lorentz metric (M, γ)
  - $S_\rho = (\mathbb{R}^2, \gamma)$ where γ is the left-invariant Lorentz metric with matrix:

$$\begin{pmatrix} e^{-2\rho\alpha} & 0 \\ 0 & -1 \end{pmatrix}$$

    with respect to the basis $\{\frac{\partial}{\partial b}, \frac{\partial}{\partial \alpha}\}$

- A connected Lie group of symmetries of the space-time M
  - the group G_ρ - i.e. $\mathbb{R}^2$ acting by group multiplication from the right.

$$R_{(B,A)}(b, \alpha) = (b, \alpha)(B, A) = (b + e^{\rho\alpha}B, \alpha + A)$$

- A one-particle Hilbert space $\mathcal{H}$
- A unitary representation U of the symmetry group on $\mathcal{H}$

As before, we can use the Mackey Machine to produce an irreducible unitary representation of $G_\rho = B \rtimes_\phi A \cong \mathbb{R} \rtimes \mathbb{R}$ for $\rho > 0$, where the action of A on B is given by $\phi_\alpha(b) = e^{\rho\alpha}b$. This induces an action $\hat{\phi}$ of A on $\hat{B}$ given by $\left(\widehat{\phi_\alpha}(\chi_x)\right)(b) = \chi_{e^{\rho\alpha}x}(b) = e^{ie^{\rho\alpha}x \cdot b}$

(1) The orbits of $\hat{B}$ under A are $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$

(2) We choose $\chi_1 \in \mathscr{O}_+$ as the representative from the orbit $\mathscr{O}_+ = \mathbb{R}^+$

(3) $L_+ = Stab_A(\chi_1) = \{0\}$

(4) Let V^+ be the trivial representation.

(5) $\mathscr{G}_+ = B \rtimes L_+ = B \rtimes \{0\}$

(6) $M^+ : \mathscr{G}_+ \to \mathscr{U}(\mathbb{C})$ given by $M^+_{(b,0)}(z) \coloneqq \chi_1(b)V^+(0)(z) = \chi_1(b)z = e^{ib}z$

The representation induced by the Mackey machine will act on the Hilbert space:

$$\mathcal{H}_\rho = \{F : G_\rho \to \mathbb{C} | F\left((B,0)(b,\alpha)\right) = \chi_1(B)F(b,\alpha) = e^{iB}F(b,\alpha),$$

$$F \text{ is measurable, and } \int_{\mathbb{R}} |F(0,\alpha)|^2 d\alpha < \infty\}$$

- $\mathscr{G}_+ \backslash G_\rho = (B \rtimes \{0\}) \backslash (B \rtimes A) \cong (\mathbb{R} \rtimes \{0\}) \backslash (\mathbb{R} \rtimes \mathbb{R}) \cong \mathbb{R}$

- For any $F \in \mathcal{H}_\rho$, we may write

$$F(b,\alpha) = F((b,0)(0,\alpha)) = \chi_1(b)F(0,\alpha) = e^{ib}F(0,\alpha)$$

- The modulus of functions is fixed on the right cosets of $\mathscr{G}_+$

- Thus the left action of $\mathscr{G}_+$ does not affect the norm of the function.

- We may therefore define an invariant measure $d\alpha$ on $\mathscr{G}_+ \backslash G_\rho \cong \mathbb{R}$

Define $U \coloneqq U^{G_\rho}_{\mathscr{G}_+}(M^+) : G_\rho \to Aut(\mathcal{H}_\rho)$ by $\left(U_{(B,A)}F\right)(b,\alpha) = F\left((b,\alpha)(B,A)\right)$

- Since the action of G_ρ from the left commutes with the action of G_ρ from the right, the transformation law will be preserved.

- $\left(U_{(B,A)(B',A')}F\right)(b,\alpha) = F\left((b,\alpha)(B,A)(B',A')\right)$

$$= (U_{(B',A')}F\left((b,\alpha)(B,A)\right) = (U_{(B,A)}\left(U_{(B',A')}F\right)(b,\alpha) = \left(U_{(B,A)}U_{(B',A')}F\right)(b,\alpha)$$

  Therefore U is in fact a group action.

The Mackey machine guarantees that the representation U of G_ρ is irreducible. However we must verify that it is also unitary.

$$\int_{\mathbb{R}} |U_{(B,A)}F(0,\alpha)|^2 \, d\alpha = \int_{\mathbb{R}} |F(e^{\rho\alpha}B, \alpha + A)|^2 \, d\alpha$$

$$= \int_{\mathbb{R}} |e^{ie^{\rho\alpha}B}F(0,\alpha+A)|^2 \, d\alpha = \int_{\mathbb{R}} |F(0,\alpha+A)|^2 \, d\alpha = \int_{\mathbb{R}} |F(0,\alpha')|^2 \, d\alpha'$$

This additionally shows that integrability is preserved. Since the transformation law and integrability are respected by the representation, we conclude that U maps $\mathcal{H}_\rho$ to itself.

Using the non-negative variable $\boldsymbol{x = e^{\rho\alpha}}$, we define an associated function

$$\tilde{F}(x) = F(0, \tfrac{1}{\rho}\ln(x))$$

Note that we have passed from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(\mathbb{R}^+, \times)$ and hence we pass from the invariant measure $d\alpha$ on $\mathbb{R}$ to the invariant Haar measure $\frac{dx}{x}$ on $\mathbb{R}^+$. Thus $\tilde{F} \in L^2(\mathbb{R}^+, \frac{dx}{x})$ whenever $F \in \mathcal{H}_\rho$.

We then define a related representation $\tilde{U}^i$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$:

$$\tilde{U}^i_{(B,A)}\tilde{F}(x) \coloneqq U_{(B,A)}F(0, \tfrac{1}{\rho}\ln(x)) = F\left((0, \tfrac{1}{\rho}\ln(x))(B,A)\right)$$

$$= F(Bx, A + \tfrac{1}{\rho}\ln(x)) = e^{iBx}F(0, A + \tfrac{1}{\rho}\ln(x)) = e^{iBx}\tilde{F}(e^{\rho A}x)$$

First we verify that the representation is unitary.

$$\int_{\mathbb{R}^+} |\left(\tilde{U}^i_{(B,A)}\tilde{F}\right)(x)|^2 \, \frac{dx}{x} = \int_{\mathbb{R}^+} |e^{iBx}\tilde{F}(e^{\rho A}x)|^2 \, \frac{dx}{x}$$

Letting $y = e^{\rho A}x$, $x = e^{-\rho A}y$ and $dx = e^{-\rho A}dy$. Hence $\dfrac{dx}{x} = \dfrac{e^{-\rho A}dy}{e^{-\rho A}y} = \dfrac{dy}{y}$.

Therefore

$$\int_{\mathbb{R}^+} |\left(\tilde{U}^i_{(B,A)}\tilde{F}\right)(x)|^2 \, \frac{dx}{x} = \int_{\mathbb{R}^+} |\tilde{F}(e^{\rho A}x)|^2 \, \frac{dx}{x} = \int_{\mathbb{R}^+} |\tilde{F}(y)|^2 \, \frac{dy}{y}$$

We have therefore produced a unitary representation $\tilde{U}^i$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$.

The representation $\tilde{U}^i$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$ is equivalent to the representation U on $\mathcal{H}_\rho$. Consider $\phi : \mathcal{H}_\rho \to L^2(\mathbb{R}^+, \frac{dx}{x})$ given by $F(b, \alpha) \mapsto \frac{1}{\sqrt{\rho}} F(0, \alpha) = \frac{1}{\sqrt{\rho}} F(0, \frac{1}{\rho} \ln(x)) = \frac{1}{\sqrt{\rho}} \tilde{F}(x)$
We first verify that the inner product is preserved.

$$\langle \phi(F), \phi(G) \rangle_{L^2} = \int_{\mathbb{R}^+} \overline{\frac{1}{\sqrt{\rho}} F\left(0, \frac{1}{\rho} \ln(x)\right)} \frac{1}{\sqrt{\rho}} G\left(0, \frac{1}{\rho} \ln(x)\right) \frac{dx}{x}$$

Using $\alpha = \frac{1}{\rho} \ln(x)$, it follows that $x = e^{\rho\alpha}$ and $dx = \rho e^{\rho\alpha} d\alpha$. Hence

$$\langle \phi(F), \phi(G) \rangle_{L^2} = \frac{1}{\rho} \int_{\mathbb{R}} \overline{F(0, \alpha)} G(0, \alpha) \rho \; d\alpha = \langle F, G \rangle_{\mathcal{H}_\rho}$$

Since the inner product is preserved, it follows that ϕ preserves the norm as well.

Furthermore ϕ intertwines the two representations.

$$\begin{array}{ccc}
F(b, \alpha) & \xrightarrow{\phi} & \frac{1}{\sqrt{\rho}} F(0, \frac{1}{\rho} \ln(x)) \\
{\scriptstyle U_{(B,A)}} \big\downarrow & & {\scriptstyle \tilde{U}^i_{(B,A)}} \big\downarrow \\
e^{ie^{\rho\alpha} B} F(b, \alpha + A) & & \frac{1}{\sqrt{\rho}} e^{iBx} F(0, \frac{1}{\rho} \ln(x) + A) \\
\Big\| & & \Big\| \\
g(b, \alpha) & \xrightarrow{\phi} & \frac{1}{\sqrt{\rho}} g(0, \frac{1}{\rho} \ln(x))
\end{array}$$

Alternatively we can employ standard Lebesgue measure dx on $L^2(\mathbb{R}^+)$.

We define another related representation $\tilde{U}^q$ on $L^2(\mathbb{R}^+, dx)$ by:

$$\tilde{U}^q_{(B,A)} \tilde{F}(x) = e^{\frac{\rho A}{2}} U_{(B,A)} F(0, \tfrac{1}{\rho} \ln(x)) = e^{\frac{\rho A}{2}} F(Bx, A + \tfrac{1}{\rho} \ln(x))$$

$$= e^{\frac{\rho A}{2}} e^{iBx} F(0, A + \tfrac{1}{\rho} \ln(x)) = e^{\frac{\rho A}{2}} e^{iBx} \tilde{F}(e^{\rho A} x)$$

The extra factor of $e^{\frac{\rho A}{2}}$ (sometimes called a **density factor**) ensures that the representation is unitary for Lebesgue measure.

$$\int_{\mathbb{R}^+} | \left(\tilde{U}^q_{(B,A)} \tilde{F} \right) (x)|^2 \; dx = \int_{\mathbb{R}^+} |e^{\frac{\rho A}{2}} e^{iBx} \tilde{F}(e^{\rho A} x)|^2 \; dx$$

$$= \int_{\mathbb{R}^+} e^{\rho A} |\tilde{F}(e^{\rho A} x)|^2 \; dx = \int_{\mathbb{R}^+} e^{\rho A} |\tilde{F}(e^{\rho A} x)|^2 \; dx$$

Letting $y = e^{\rho A}x$, we see that

$$\int_{\mathbb{R}^+} |\left(\tilde{U}^q_{(B,A)}\tilde{F}\right)(x)|^2 \; dx = \int_{\mathbb{R}^+} |\tilde{F}(y)|^2 \; dy$$

The aforementioned density factor accounts for the dilation of measure under the group action. The Lebesgue measure is said not to be invariant under the group action. In essence for each $g \in G_\rho$ we produce a new measure μ_g on $\mathscr{G}_+\backslash G_\rho$ by the left action of g on $\mathscr{G}_+$ from the Lebesgue measure μ by defining $\mu_g(E) = \mu(g^{-1}E)$ for each measurable set $E \subset \mathscr{G}_1$. Two measures that can be obtained from one another by:

$$\mu_g(E) = \int_E \varrho(s) \; d\mu(s) \left(= \int_{E\subset\mathscr{G}_+\backslash G_\rho} e^{-\rho A} \; d\mu(x) = e^{-\rho A}\mu(E)\right)$$

are said to belong to the same **class**. Since μ_g and Lebesgue measure are of the same class for all $g \in G$, Lebesgue measure is said to be **quasi-invariant** under the action of $\tilde{U}^q$. [8] There was no need for a density factor in the previous representation $\tilde{U}^i$ as the measure $\dfrac{dx}{x}$ is **invariant** under the action of $\tilde{U}^i$.

The representations $\tilde{U}^q$ on $L^2(\mathbb{R}^+, dx)$ and $\tilde{U}^i$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$ are also equivalent.

Consider $\phi : L^2(\mathbb{R}^+, dx) \to L^2(\mathbb{R}^+, \frac{dx}{x})$ given by $\tilde{F}(x) \mapsto \tilde{F}(x)\sqrt{x}$

We verify that the inner product is preserved by this map.

$$\langle\phi(F), \phi(G)\rangle_{\frac{dx}{x}} = \int_{\mathscr{G}_1\backslash G_\rho} \overline{e^{ib}\tilde{F}(x)\sqrt{x}}e^{ib}\tilde{G}(x)\sqrt{x} \; \tfrac{dx}{x} = \int_{\mathscr{G}_1\backslash G_\rho} \overline{\tilde{F}(x)}\tilde{G}(x)\overline{e^{ib}}e^{ib} \; dx = \langle F, G\rangle_{dx}$$

Since the inner product is preserved, it follows that ϕ preserves the norm as well.

All that remains to be shown is that ϕ intertwines the two representations $\tilde{U}^q$ and $\tilde{U}^i$. Fix $(B, A) \in G_\rho$ and $\tilde{F} \in L^2(\mathbb{R}^+, dx)$

$$\tilde{U}^i_{(B,A)}\left(\phi(\tilde{F})\right)(x) = e^{ie^{\rho\alpha}B}\phi(\tilde{F})(e^{\rho A}x) = e^{ie^{\rho\alpha}B}\tilde{F}(e^{\rho A}x)\sqrt{e^{\rho A}x}$$
$$\phi\left(\tilde{U}^q_{(B,A)}(\tilde{F})\right)(x) = \tilde{U}^q_{(B,A)}\left(\tilde{F}(x)\sqrt{x}\right) = e^{ie^{\rho\alpha}B}\tilde{F}(e^{\rho A}x)\sqrt{e^{\rho A}x}$$

Hence we conclude that ϕ is the intertwining isomorphism between $(L^2(\mathbb{R}^+, dx), \tilde{U}^q)$ and $(L^2(\mathbb{R}^+, \frac{dx}{x}), \tilde{U}^i)$. Hence we have produced three equivalent irreducible, unitary representations of the generalized $ax + b$ group G_ρ:

- U on $\mathcal{H}_\rho$ given by $\left(U_{(B,A)}F\right)(b, \alpha) = F\left((b, \alpha)(B, A)\right)$
- $\tilde{U}^i$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$ given by $\left(\tilde{U}^i_{(B,A)}\tilde{F}\right)(x) = e^{iBx}F(e^{\rho A}x)$
- $\tilde{U}^q$ on $L^2(\mathbb{R}^+, dx)$ given by $\left(\tilde{U}^q_{(B,A)}\tilde{F}\right)(x) = e^{\frac{\rho A}{2}}e^{iBx}F(e^{\rho A}x)$

Choosing the $\mathbb{R}^-$ orbit in the Mackey machine produces a family of non-equivalent representations which can derived from the above family by using the outer-automorphism $B \mapsto -B$. For each $s \in \mathbb{R}$, there is a unitary finite-dimensional representation of the G_ρ on $\mathbb{C}$ given by: $\Upsilon^s_{(B,A)}(z) = e^{isA} \cdot z$ for $(B, A) \in G_\rho$ and $z \in \mathbb{C}$. Choosing the zero-orbit in the Mackey machine produces these one-dimensional representations. Thus the collection of equivalence classes of representations of G_ρ consists of two infinite-dimensional representations (stars) and a continuum of one-dimensional representations:

$$\star$$

$$\longleftrightarrow$$

$$\star$$

In their 1953 paper, Wigner and Inönü [20] offered the following representation of the $ax + b$ group on $L^2(\mathbb{R}^+, dx)$:

$$\Phi_{(B,A)}\psi(x) = e^{\frac{A}{2}+iBx}\psi(e^A x) \qquad \text{for } (B, A) \in G \text{ and } \psi(x) \in L^2(\mathbb{R}^+, dx)$$

This representation is $\tilde{U}^q$ as defined above[1] with $\rho = 1$ (the $ax+b$ group corresponds to G_1).

$$\tilde{U}^q_{(B,A)}\tilde{F}(x) = e^{\frac{A}{2}+iBx}\tilde{F}(e^A x) = \Phi_{(B,A)}\tilde{F}(x)$$

[1] see page 36

3.2 An Example: A Free Scalar Quantum Field Theory on a Two-Dimensional Curved Space-time

With all of our components now defined, we commence with an example.

Fix $\eta > 0$ and define a Hilbert-space valued distribution on S_ρ:

$$\Psi_\eta : C^\infty_{cpt}(S_\rho) \to \mathcal{H}_\rho$$

given by:

$$\Psi_\eta(f)(b,\alpha) = e^{ib} \int_{\mathbb{R}} f(\beta,\alpha) e^{-i\beta\eta} \, d\beta = e^{ib} g_\eta(\alpha)$$

Notice that Ψ_η and $g_\eta(\alpha)$ are dependent upon η. Ψ_η is the Fourier transform of $f(\beta,\alpha)$ with respect to the β variable evaluated at η with a modulation factor e^{ib}. Since $f \in C^\infty_{cpt}(S_\rho)$, Ψ_η can be evaluated for sharply defined η; if f was simply an element of the larger space $L^2(S_\rho)$, this would not be the case.

We must first verify the integrability of $\Psi_\eta(f)$ for $f \in C^\infty_{cpt}(S_\rho)$.

$$||\Psi_\eta(f)||^2_{\mathcal{H}_\rho} = \int_{\mathbb{R}} |\Psi_\eta(f)(0,\alpha)|^2 \, d\alpha$$

$$= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(\beta,\alpha) e^{-i\beta\eta} d\beta \right|^2 \, d\alpha \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(\beta,\alpha)|^2 \, d\beta d\alpha < \infty$$

for $f \in C^\infty_{cpt}(S_\rho)$

Next we must check that $\Psi_\eta(f)$ satisfies the transformation property of $\mathcal{H}_\rho$.

$$\Psi_\eta(f)((B,0)(b,\alpha)) = \Psi_\eta(f)(B+b,\alpha)$$

$$= e^{i(B+b)} \int_{\mathbb{R}} f(\beta,\alpha) e^{-i\beta\eta} \, d\beta = e^{iB} e^{ib} \int_{\mathbb{R}} f(\beta,\alpha) e^{-i\beta\eta} \, d\beta = e^{iB} \Psi_\eta(f)(b,\alpha)$$

Since $\Psi_\eta(f)$ satisfies the transformation law and $\Psi(f) \in L^2(\mathbb{R}_\alpha)$, we conclude $\Psi_\eta(f) \in \mathcal{H}_\rho$ for $f \in C^\infty_{cpt}(S_\rho)$.

Notice however that different values of η produce different quantum state fields as the frequency term varies with η. We will offer a discussion of the interpretation of the η parameter in Section 3.3.

We hope to impose the same axioms for a quantum state field to the curved case as we did to the flat case. The proposed axioms for the quantum state field characterized by the Hilbert-space-valued distribution Ψ_η for $\eta > 0$ on the space-time (S_ρ, γ) for $\rho > 0$ with values in the dense subspace $\mathcal{D}$ of $\mathcal{H}_\rho$ and unitary representation U of G_ρ in $\mathcal{H}_\rho$ are:

1: For $f \in C^\infty_{cpt}(S_\rho)$, $f \mapsto \langle \psi, \Psi_\eta(f) \rangle$ is a continuous (with respect to the Schwartz topology) $\mathbb{C}$-linear functional for $\psi \in \mathcal{H}_\rho$.

2: The image of $C^\infty_{cpt}(S_\rho)$ under Ψ_η is dense in $\mathcal{H}_\rho$.

3: $\langle$ this space is intentionally left blank in order to have axiom numbers align $\rangle$

4: $U_{(B,A)}\Psi_\eta(f) = \Psi_\eta(f \circ R_{(B,A)})$ for all $f \in C^\infty_{cpt}(S_\rho)$ and $(B, A) \in G_\rho$
where $R_{(B,A)}$ denotes right multiplication by (B, A).

5: If $f, g \in C^\infty_{cpt}(S\rho)$ are two **real-valued** functions whose supports are space-like separated, then $Im\langle \Psi(f), \Psi(g) \rangle_{\mathcal{H}_\rho} = 0$.

6: The infinitesimal generator of the time translation subgroup[2], $\frac{1}{i} dU(1, 0)$, has non-negative spectrum.

Again we will refer to a quantum state field quadruple $\{\Psi, U, \mathcal{H}, \mathcal{D}\}$ as being irreducible if the representation U is irreducible.

[2]In the flat case, the corresponding axiom used generator $i\ dU(1, 0)$; this disparity can be reconciled by altering the transformation property or by redefining the correspondence between $H_{(m,0)}$ and $SO(1, 1)^o$ introduced on page 4 or by using the (outer) automorphism $(T, X) \mapsto (-T, X)$.

Proposition 3.2.1. *The irreducible quantum state field quadruple* $\{\Psi_\eta, U, \mathcal{H}_\rho, \Psi_\eta(C^\infty_{cpt}(S_\rho))\}$ *satisfies these quantum state field axioms.*

Proof. Let $f \in C^\infty_{cpt}$, $F \in \mathcal{H}_\rho$, $(b,\alpha) \in S_\rho$, and $(B,A) \in G_\rho$

1: Continuity of this linear functional can be shown by showing that the map Ψ_η from $C^\infty_{cpt}(S_\rho)$ to $\mathcal{H}_\rho$ is continuous - i.e. for $\{f_j\} \subset C^\infty_{cpt}(S_\rho)$ if $f_j \xrightarrow{\mathscr{S}} 0$, then $\Psi(f_j) \xrightarrow{\mathcal{H}_\rho} 0$.

Let $\{f_j\}$ be such a sequence. Notice that $f_j \to 0$ pointwise as well.

$$\|\Psi_\eta f_j\|^2_{\mathcal{H}_\rho} = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f_j(\beta,\alpha) e^{-i\beta\eta} d\beta \right|^2 d\alpha \le \int_{\mathbb{R}^2} |f_j(\beta,\alpha)|^2 \, d\beta d\alpha$$

Since $f_j \in C^\infty_{cpt}(S_\rho) \subset \mathscr{S}(S_\rho)$ and $f_j \xrightarrow{\mathscr{S}} 0$,

$$\lim_{j\to\infty} \sup_{x\in S_\rho} |\beta^4 f_j + 2\beta^2\alpha^2 f_j + \alpha^4 f_j| = \lim_{j\to\infty} \sup_{x\in S_\rho} (\beta^2+\alpha^2)^2 |f_j| = 0$$

Thus there exists an M such that for all j, $\sup_{x\in S_\rho} (\beta^2+\alpha^2)^2 |f_j| \le M$.
Similarly there exists an M_0 such that for all j, $\sup_{x\in S_\rho} |f_j| \le M_0$.
Let A be the unit disc and χ_A be the characteristic function of A. For all j,

$$|f_j| = |f_j\chi_A| + |f_j\chi_{A^c}| \le \frac{M\chi_{A^c}}{(\beta^2+\alpha^2)^2} + M_0\chi_A$$

Notice that

$$\int_{\mathbb{R}^2} \frac{M\chi_{A^c}}{(\beta^2+\alpha^2)^2} + M_0\chi_A \, d\beta d\alpha = \int_0^{2\pi} \int_1^\infty \frac{M}{r^4} \, r dr d\theta + \pi M_0 < \infty$$

Hence $\dfrac{M\chi_{A^c}}{(\beta^2+\alpha^2)^2} + M_0\chi_A$ is in $L^1(\mathbb{R}^2)$.
Applying the Lebesgue Dominated Convergence Theorem, we can conclude that

$$\|\Psi_\eta f_j\|^2_{\mathcal{H}_\rho} \le \int_{\mathbb{R}^2} |f_j(\beta,\alpha)|^2 \, d\beta d\alpha \to 0$$

Therefore the map Ψ_η from $C^\infty_{cpt}(S_\rho)$ to $\mathcal{H}_\rho$ is continuous and axiom 1 is satisfied.

2: It has been shown above that Ψ_η maps $C^\infty_{cpt}(S_\rho)$ into $\mathcal{H}_\rho$. Since U is an irreducible representation and Ψ_η is not the zero map, $\overline{\Psi_\eta(C^\infty_{cpt}(S_\rho))} = \mathcal{H}_\rho$

Hence the density requirement of axiom 2 is fulfilled.

4: Recall the action of G_ρ on $\mathcal{H}_\rho$ is given by:

$$U_{(B,A)}(F)(b,\alpha) = F((b,\alpha)(B,A)) = F(b + e^{\rho\alpha}B, \alpha + A)$$

$$= F\left((e^{\rho\alpha}B, 0)(b, \alpha + A)\right) = e^{ie^{\rho\alpha}B}F(b, \alpha + A)$$

We verify that the output for $f \in C^\infty_{cpt}(S_\rho)$ of $U^2_{(B,A)} \circ \Psi_\eta$ and $\Psi_\eta \circ R_{(B,A)}$ are the same.

$$\begin{array}{ccc} f(b,\alpha) & \xrightarrow{\Psi_\eta} & e^{ib}\displaystyle\int_{\mathbb{R}} f(\beta,\alpha)e^{-i\beta\eta}\, d\beta \\ {\scriptstyle R_{(B,A)}}\big\downarrow & & {\scriptstyle U_{(B,A)}}\big\downarrow \\ f(b + e^{\rho\alpha}B, \alpha + A) & \xrightarrow{\Psi_\eta} & e^{ie^{\rho\alpha}B}e^{ib}\displaystyle\int_{\mathbb{R}} f(\beta,\alpha + A)e^{-i\beta\eta}\, d\beta \end{array}$$

5: Let $supp(f)$ & $supp(g)$ be space-like separated for $f, g \in C^\infty_{cpt}(S_\rho)_{\mathbb{R}}$.

In Minkowski space-time two points (t, x) and (T, X) are space-like separated if:

$$||(t,x) - (T,X)||^2 = \langle (t,x) - (T,X), (t,x) - (T,X)\rangle = (t - T)^2 - (x - X)^2 < 0$$

In two-dimensional curved space-time two points (b, α) and (B, A) are space-like separated if:

$$|b - B| < \tfrac{1}{\rho}(e^{\rho|\alpha - A|} - 1)$$

This follows directly from the light-cone calculation. [3]

[3]see page 16

$\langle \Psi_\eta(f), \Psi_\eta(g)\rangle_{L^2(\mathbb{R},d\alpha)} =$

$$\int_{\mathbb{R}} \left(\overline{e^{ib}\int_{\mathbb{R}} f(\beta,\alpha)e^{-i\beta\eta}d\beta}\right)\left(e^{ib}\int_{\mathbb{R}} g(\beta',\alpha)e^{-i\beta'\eta}d\beta'\right)d\alpha$$

$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(\beta,\alpha)}e^{i\beta\eta}g(\beta',\alpha)e^{-i\beta'\eta}d\beta d\beta'\right)d\alpha$$

$\overline{f(\beta,\alpha)}g(\beta',\alpha) \neq 0 \Leftrightarrow (\beta,\alpha) \in supp(f)$ and $(\beta',\alpha) \in supp(g)$

Since the supports of f and g are space-like separated, we conclude that

if $\overline{f(\beta,\alpha)}g(\beta',\alpha) \neq 0$, then $|\beta - \beta'| < \frac{1}{\rho}(e^{\rho(\alpha-\alpha)} - 1) = 0$, which is impossible.

Therefore we conclude that $\overline{f(b,\alpha)}g(\beta,\alpha) \equiv 0$.

Hence $\langle \Psi_\eta(f), \Psi_\eta(g)\rangle = 0$ if the supports of the real-valued functions f and g are space-like separated.

6: $U_{(B,A)}F(b,\alpha) = F(b + e^{\rho\alpha}B, \alpha + A) = e^{ie^{\rho\alpha}B}F(b,\alpha + A)$

Thus $\frac{\partial}{\partial t}\left[U_{(t,0)}F(b,\alpha)\right]_{t=0} = \frac{\partial}{\partial t}\left[e^{ie^{\rho\alpha}t}F(b,\alpha)\right]_{t=0} = ie^{\rho\alpha}F(b,\alpha)$

It follows that the operator $\frac{1}{i}dU(1,0)$ has positive spectrum.

□

As before we can use the Segal field operator (Φ_s) from Section 2.2 to extend from the one-particle Hilbert space to the Fock space.

Note that given an irreducible representation, it seems superficially trivial that there could not exist any invariant vector in the higher symmetric tensor product spaces if there is no invariant vector in the "basic" Hilbert space. However such invariant vectors could exist in higher symmetric spaces.

Consider the alternating representation of $S_2 = \{e, \tau\}$ on $\mathbb{C}$

$$e \cdot z = z \qquad\qquad \tau \cdot z = -z$$

The alternating representation is irreducible and the only complex number fixed by the action of τ is zero. We then build $Sym^2(\mathbb{C}) = \mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C} = \mathbb{C}$. However

$$w \otimes z \overset{\tau^{(2)}}{\longmapsto} -w \otimes -z = w \otimes z$$

for any $w, z \in \mathbb{C}$. Hence the symmetric 2-tensor space has invariant vectors. A similar issue occurs in the second symmetric tensor powers of the spin-1 representation of $SU(2)$. Therefore we conclude that the invariance condition in the second Wightman axiom is a strong requirement. Fortunately we have chosen a representation that does indeed satisfy this stringent imperative that the vacuum be unique up to scalar multiples.

Proposition 3.2.2. *The quantum operator field quadruple* $\{\Phi := \Phi_s \circ \Psi_\eta, \mathcal{U}, \mathcal{F}_{\mathcal{H}_\rho}, \mathcal{F}_0\}$ *satisfies the Wightman axioms* [4] *for a free quantum operator field.*

Proof. Let $f = f_1 + i f_2 \in C^\infty_{cpt}(S_\rho)$. Fix $\psi, \tilde{\psi} \in \mathcal{F}_0$ and write $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)}, 0, 0, \dots)$

Wightman Axiom 1:

$$f \mapsto \langle \psi, \Phi(f)\tilde{\psi}\rangle = \tfrac{1}{\sqrt{2}}\left(\langle \psi, A_{\Psi_\eta(f_1)}\tilde{\psi}\rangle + \langle \psi, C_{\Psi_\eta(f_1)}\tilde{\psi}\rangle\right) + \tfrac{i}{\sqrt{2}}\left(\langle \psi, A_{\Psi_\eta(f_2)}\tilde{\psi}\rangle + \langle \psi, C_{\Psi_\eta(f_2)}\tilde{\psi}\rangle\right)$$

Since A_u and C_u are continuous and $\mathbb{R}$-linear in u , $f \mapsto \langle \psi, \Phi(f)\tilde{\psi}\rangle$ is a continuous $\mathbb{C}$-linear functional for each $\psi, \tilde{\psi} \in \mathcal{F}_0$

Wightman Axiom 2:

$\Omega = (1, 0, 0, \dots)$ plays the role of the vacuum and is invariant under the action of $\mathcal{U}$ since it acts trivially on $\mathbb{C}$ by definition.

[4] appropriately modified from those on page 18 for the curved spacetime S_ρ for $\rho > 0$ with symmetry group G_ρ in place of the Minkowski plane M with symmetry group $\mathcal{P}_2^o$

Next we must show the uniqueness of the vacuum.

Suppose there exists $\phi \in \mathcal{F}_{\mathcal{H}_\rho}$ which is invariant under $\mathcal{U}$. Each component $\phi^{(m)}$ must therefore be invariant under $U^{(m)}$. Recall how $U_g^{(m)}$ acts on the simple tensors:

$$\otimes_{i=1}^m f_i^{(m)} \mapsto \otimes_{i=1}^m U_g f_i^{(m)}$$

Thus

$$\otimes_{i=1}^m f_i^{(m)} \overset{U_{(B,0)}^{(m)}}{\longmapsto} \otimes_{i=1}^m e^{iB} f_i^{(m)} = e^{iBm} \otimes_{i=1}^m f_i^{(m)}$$

We can then induce an action on finite sums of simple tensors – which will preserve the correspondence between $U_{(B,0)}^{(m)}$ and multiplication by e^{iBm}. Subsequently extending to the Hilbert-space completion of these sums $\mathcal{H}^{\hat{\otimes} m}$ and restricting to the symmetric subspace $Sym^m(\mathcal{H})$ does not affect this correspondence. For any $m \geq 0$ and $F^{(m)} \in Sym^m(\mathcal{H}_\rho)$, there exists a B so that $e^{iBm}F^{(m)} \neq F^{(m)}$. We conclude that there can not exist any vector $F^{(m)}$ for $m \geq 1$ such that $U_{(B,0)}^{(m)} F^{(m)} = F^{(m)}$ for all B. Therefore the vacuum is unique.

Next we show that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C_{cpt}^\infty(M)$ is dense in $\mathcal{F}_{\mathcal{H}_\rho}$.

The span of Ω is all of $\mathbb{C}$.

Since the image of $C_{cpt}^\infty(S_\rho)$ under Ψ_η is dense in $\mathcal{H}_\rho$, the image of $C_{cpt}^\infty(S_\rho)$ under $c^0_{\Psi_\eta(f)}$ is dense in $\mathcal{H}_\rho$.

Thus we conclude that the linear span of the image of $C_{cpt}^\infty(S_\rho)$ under Φ is dense in $\mathcal{H}_\rho$.

We now proceed by induction.

Suppose that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C_{cpt}^\infty(S_\rho)$ is dense in $\mathbb{C}\oplus\mathcal{H}_\rho\oplus\cdots\oplus\mathcal{H}_\rho^{\hat{\otimes} m}$; in particular the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C_{cpt}^\infty(S_\rho)$ is dense in $\mathcal{H}_\rho$ and $\mathcal{H}_\rho^{\hat{\otimes} m}$. We can tensor these two constituent spaces to create $\mathcal{H}_\rho^{\hat{\otimes} m+1}$. Taking linear combinations of the images of the dense sets, we can conclude that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C_{cpt}^\infty(S_\rho)$ is dense in $\mathcal{H}_\rho^{\hat{\otimes} m+1}$. By projecting onto the symmetric tensors, we conclude that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C^\infty_{cpt}(S_\rho)$ is dense in $Sym^{m+1}(\mathcal{H}_\rho)$. Since this argument holds for all $m > 1$, we conclude that the linear span of vectors of the form $\Phi(f_1)\cdots\Phi(f_n)\Omega$ for $f_j \in C^\infty_{cpt}(S_\rho)$ is dense in $\mathcal{F}_0$ and hence dense in $\mathcal{F}_{\mathcal{H}_\rho}$.

Wightman Axiom 3:

We compute $\langle\psi, \Phi(f)\tilde{\psi}\rangle = \langle\psi, \Phi_s(\Psi_\eta(f_1))\tilde{\psi}\rangle + i\langle\psi, \Phi_s(\Psi_\eta(f_2))\tilde{\psi}\rangle$

$$= \langle\Phi_s(\Psi_\eta(f_1))^\dagger\psi, \tilde{\psi}\rangle + i\langle\Phi_s(\Psi_\eta(f_2))^\dagger\psi, \tilde{\psi}\rangle$$

$$= \tfrac{1}{\sqrt{2}}\left(\langle(A_{\Psi_\eta(f_1)} + C_{\Psi_\eta(f_1)})^\dagger\psi, \tilde{\psi}\rangle + i\langle(A_{\Psi_\eta(f_2)} + C_{\Psi_\eta(f_2)})^\dagger\psi, \tilde{\psi}\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}\left(\langle(C_{\Psi_\eta(f_1)} + A_{\Psi_\eta(f_1)})\psi, \tilde{\psi}\rangle + i\langle(C_{\Psi_\eta(f_2)} + A_{\Psi_\eta(f_2)})\psi, \tilde{\psi}\rangle\right)$$

$$= \langle\Phi_s(\Psi_\eta(f_1))\psi, \tilde{\psi}\rangle + i\langle\Phi_s(\Psi_\eta(f_2))\psi, \tilde{\psi}\rangle = \langle\Phi_s(\Psi_\eta(f_1))\psi, \tilde{\psi}\rangle - \langle i\Phi_s(\Psi_\eta(f_2))\psi, \tilde{\psi}\rangle$$

$$= \langle(\Phi_s(\Psi_\eta(f_1)) - i\Phi_s(\Psi_\eta(f_2)))\psi, \tilde{\psi}\rangle = \langle(\Phi(\bar{f})\psi, \tilde{\psi}\rangle$$

Wightman Axiom 4:

We first consider the action of $\mathcal{U}$ on the annihilation operator.

$$\left(\mathcal{U}_{(B,A)}A_{\Psi_\eta(f)}\mathcal{U}^{-1}_{(B,A)}\right)(\psi) = \left(\mathcal{U}_{(B,A)}A_{\Psi_\eta(f)}\mathcal{U}_{(B,A)^{-1}}\right)\left(\oplus_{n=0}^\infty \otimes_{i=1}^n \psi_i^{(n)}\right)$$

$$= \oplus_{n=0}^\infty U^{(n)}_{(B,A)} a^n_{\Psi_\eta(f)} U^{(n)}_{(B,A)^{-1}} \otimes_{i=1}^n \psi_i^{(n)} = \oplus_{n=0}^\infty U^{(n)}_{(B,A)} a^n_{\Psi_\eta(f)} \otimes_{i=1}^n U_{(B,A)^{-1}}\psi_i^{(n)}$$

$$= \oplus_{n=0}^\infty U^{(n)}_{(B,A)}\langle\Psi_\eta(f), U_{(B,A)^{-1}}\psi_1^{(n)}\rangle \otimes_{i=2}^n U_{(B,A)^{-1}}\psi_i^{(n)}$$

In order to compute the inner product coefficient, we consider a generic $\phi \in \mathcal{H}$

$$\langle\Psi_\eta(f), U_{(B,A)^{-1}}\phi\rangle = \langle\Psi_\eta(f), U^{-1}_{(B,A)}\phi\rangle = \langle\Psi_\eta(f), U^\dagger_{(B,A)}\phi\rangle = \langle U_{(B,A)}\Psi_\eta(f), \phi\rangle$$

where the second equality follows from U being a unitary operator.

Thus $\left(\mathcal{U}_{(B,A)}A_{\Psi_\eta(f)}\mathcal{U}^{-1}_{(B,A)}\right)(\psi) = \oplus_{n=0}^\infty U^{(n)}_{(B,A)}\langle U_{(B,A)}\Psi_\eta(f), \psi_1^{(n)}\rangle \otimes_{i=2}^n U_{(B,A)^{-1}}\psi_i^{(n)}$

$$= \oplus_{n=0}^\infty \langle U_{(B,A)}\Psi_\eta(f), \psi_1^{(n)}\rangle U^{(n)}_{(B,A)} \otimes_{i=2}^n U_{(B,A)^{-1}}\psi_i^{(n)}$$

$$= \oplus_{n=0}^\infty \langle U_{(B,A)}\Psi_\eta(f), \psi_1^{(n)}\rangle \otimes_{i=2}^n \psi_i^{(n)} = \oplus_{n=0}^\infty a^n_{U(\Psi_\eta(f))} \otimes_{i=1}^n \psi_i^{(n)} = A_{U(\Psi_\eta(f))}\psi$$

Similarly,

$$\mathcal{U}_{(B,A)} C_{\Psi_\eta(f)} \mathcal{U}^{-1}_{(B,A)} \psi = \oplus_{n=0}^\infty U^n_{(B,A)} Sym(f \otimes U_{(B,A)^{-1}} \psi) = \oplus_{n=0}^\infty Sym(Uf \otimes \psi) = C_{U(\Psi_\eta(f))} \psi$$

Hence for real-valued f,

$$\mathcal{U}\Phi(f)\mathcal{U}^{-1} = \mathcal{U}\Phi_s(\Psi_\eta(f))\mathcal{U}^{-1} = \tfrac{1}{\sqrt{2}}\mathcal{U}(A_{\Psi_\eta(f)} + C_{\Psi_\eta(f)})\mathcal{U}^{-1}$$

$$= \tfrac{1}{\sqrt{2}}(A_{U(\Psi_\eta(f))} + C_{U(\Psi_\eta(f))}) = \Phi_s(U \circ \Psi_\eta(f))$$

Applying the quantum state equivariance axiom,

$$\mathcal{U}_{(B,A)}\Phi(f)\mathcal{U}^{-1}_{(B,A)} = \Phi_s(U_{(B,A)} \circ \Psi_\eta(f)) = \Phi_s(\Psi_\eta(f \circ R_{(B,A)}) = \Phi(f \circ R_{(B,A)})$$

For general $f = f_1 + if_2$,

$$\mathcal{U}_{(B,A)}\Phi(f_1 + if_2)\mathcal{U}^{-1}_{(B,A)} = \mathcal{U}_{(B,A)}\left(\Phi_s(\Psi_\eta(f_1)) + i\Phi_s(\Psi_\eta(f_2))\right)\mathcal{U}^{-1}_{(B,A)}$$

$$= \mathcal{U}_{(B,A)}\Phi(f_1)\mathcal{U}^{-1}_{(B,A)} + i\mathcal{U}_{(B,A)}\Phi(f_2)\mathcal{U}^{-1}_{(B,A)} = \Phi(f_1 \circ R_{(B,A)}) + i\Phi(f_2 \circ R_{(B,A)}) =$$

$$\Phi((f_1 + if_2) \circ R_{(B,A)})$$

Wightman Axiom 5:

Let the supports of $f = f_1 + if_2$ and $g = g_1 + ig_2$ be space-like separated.

$$[\Phi(f), \Phi(g)] = [\Phi_s(\Psi(f_1)) + i\Phi_s(\Psi(f_2)), \Phi_s(\Psi(g_1)) + i\Phi_s(\Psi(g_2))]$$

$$= [\Phi(f_1), \Phi(g_1)] + i[\Phi(f_1), \Phi(g_2)] + i[\Phi(f_2), \Phi(g_1)] - [\Phi(f_2), \Phi(g_2)]$$

Recall[5] that for $\psi \in \mathcal{F}_0$ and $u, v \in \mathcal{H}$, $[\Phi_s(u), \Phi_s(v)]\psi = i\ \mathrm{Im}\langle u, v\rangle_{\mathcal{H}}\psi$. Hence

$$[\Phi(f), \Phi(g)] = i\mathrm{Im}\langle\Psi(f_1), \Psi(g_1)\rangle - \mathrm{Im}\langle\Psi(f_1), \Psi(g_2)\rangle - \mathrm{Im}\langle\Psi(f_2), \Psi(g_1)\rangle - i\mathrm{Im}\langle\Psi(f_2), \Psi(g_2)\rangle$$

Since $supp(f)$ and $supp(g)$ are space-like separated, so are $supp(f_i)$ and $supp(g_j)$ for the real-valued functions f_1, f_2, g_1, and g_2. Applying axiom 5 for quantum state fields for real-valued functions with space-like separated support, we conclude $[\Phi(f), \Phi(g)] = 0$.

[5]see page 24 or Reed and Simon's Theorem X.41

Wightman Axiom 6:

By quantum state field axiom 6, $\frac{1}{i}\ dU(1,0)$ has non-negative spectrum.

In other words, for all F in the domain of $\frac{1}{i}\ dU(1,0)$,

$$\langle \tfrac{1}{i}\ dU(1,0)F, F\rangle \geq 0$$

Hence on simple m-tensors ($F^{(m)}$) in the domain of $\frac{1}{i}\ dU(1,0)$,

$\langle \frac{1}{i}\ dU(1,0)F^{(m)}, F^{(m)}\rangle$

$$= \langle \tfrac{1}{i}\ dU(1,0)F_1^{(m)}, F_1^{(m)}\rangle\langle \tfrac{1}{i}\ dU(1,0)F_2^{(m)}, F_2^{(m)}\rangle \cdots \langle \tfrac{1}{i}\ dU(1,0)F_m^{(m)}, F_m^{(m)}\rangle \geq 0$$

Taking linear combinations and closure [6] preserves this non-negative spectral condition, as does restricting to the symmetric subspace. Since the spectral condition holds on each component in the direct sum, it holds on $\mathcal{F}_0$ and thus on $\mathcal{F}_{\mathcal{H}_\rho}$.

We conclude $\frac{1}{i}\ d\mathcal{U}(1,0)$ has non-negative spectrum.

□

3.3 Wigner Contraction to the Two-Dimensional Minkowski Space-time

> Classical mechanics is the limiting case of relativistic mechanics. Hence the group of the former, the Galilei group, must be in some sense a limiting case of the relativistic mechanics' group [the Poincaré group], the representations of the former must be limiting cases of the latter's representations [by taking $c^{-1} \to 0$]. ...[T]he inhomogeneous Lorentz group must be, in the same sense, a limiting case of the deSitter groups [by taking the limit of zero curvature in the space-time]. ...[T]he representation up to a factor of the Galilei group, embodied in the

[6] which is all of $\mathcal{H}^{\hat{\otimes} m}$ since the domain of $\frac{1}{i}\ dU(1,0)$ is dense in $\mathcal{H}_\rho$

Schrödinger equation, appears as a limit of a representation of the inhomogeneous Lorentz group. [20]

The symmetry group for the two-dimensional (flat) Minkowski plane is the three-dimensional group $\mathcal{P}_2^o = \mathbb{R}^{1+1} \rtimes SO(1,1)^o$; while the symmetry group for the two-dimensional curved space-time S_ρ is the two-dimensional subgroup $G_\rho = \mathbb{R} \rtimes \mathbb{R} \subset \mathcal{P}_2^o$. This is due to the flat space-time having additional symmetry. One goal of this thesis is to study the difference of these two cases and to model the metamorphosis of one into the other. Most of the current literature attempts to perturb or deform the special flat case into the general curved case. In this thesis, we use a kind of inverse to the mathematical notion of a deformation of Lie groups and algebras by contracting from the general curved case to the special flat case.

A **contraction** is a deformation of a group into another group using a series of non-singular coordinate transformations whose limit is a singular transformation. Intuitively, a contraction is performed by neglecting symmetries. Contraction is defined for a Lie group by using its Lie algebra. The structure constants of the Lie algebra are altered in a manner to be discussed below. The contracted Lie algebra is uniquely determined in terms of the original algebra. This change in structure constants results in a change in the Lie group after utilizing the diffeomorphism between the algebra and the group. Essentially all of the calculations are performed in the algebra[13]. This transformation can be used to model the limiting behavior of one group by affecting the structure of the group. A contraction is carried out with respect to a subgroup; this subgroup remains unchanged by the contraction.

We will begin our study by considering the Lie algebra of G_ρ. A Lie algebra L is a vector space (for our purposes, over $\mathbb{C}$) with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ satisfying bilinearity, anti-commutativity, and the Jacobi identity $([x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0)$ for all $x, y, z \in L$. [3] A Lie algebra can be described by giving a basis and a set of relations.

Definition 3.3.1. *Given a basis* $\{x_i\}_{i=1}^n$ *for a Lie algebra* $\mathfrak{g}$*, the relations are given by*

$$[x_i, x_j] = \sum_{k=1}^{n} C_{ij}^k x_k$$

These C_{ij}^k *are known as the* ***structure constants*** *of* $\mathfrak{g}$.

Since the Lie algebra bracket satisfies anti-commutativity, it follows that $C_{ij}^k = -C_{ji}^k$ for all i, j, k.

Example 3.3.1. *In order to discover the Lie algebra* $\mathfrak{g}_\rho$ *of* G_ρ*, we consider a matrix representation of* G_ρ

$$\Upsilon : G_\rho \to GL_2(\mathbb{R}) \text{ given by } (b, \alpha) \mapsto \begin{pmatrix} e^{\rho\alpha} & b \\ 0 & 1 \end{pmatrix}$$

- *We define* $A := \frac{\partial \Upsilon}{\partial \alpha}(0,0) = \begin{pmatrix} \rho e^{\rho\alpha} & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$
- *and* $B := \frac{\partial \Upsilon}{\partial b}(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- $\{A, B\}$ *serves as a basis for the tangent space to* G_ρ *at the identity and* $\mathfrak{g}_\rho$ *has the relation*

$$[A, B] = AB - BA = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} = \rho B$$

- *Thus the only non-zero structure constant for* $\mathfrak{g}_\rho$ *is* $C_{A,B}^B = \rho$
- *This set* $\{A, B\}$ *together with the relation* $[A, B] = \rho B$ *satisfies bilinearity, anti-commutativity, and the Jacobi identity. Thus* $\mathfrak{g}_\rho$ *is a Lie algebra.*
- *The derived series for* $\mathfrak{g}$ *is:* $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = B$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = [B, B] = 0$. *Hence* $\mathfrak{g}$ *is a* solvable *Lie Algebra.*

- *However* $\mathfrak{g}$ *is not* nilpotent *since its descending central series is:*

$$\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = B,\ \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = [\mathfrak{g}, B] = B.$$

We may contract a Lie algebra $\mathfrak{g}$ with respect to any of its subalgebras by altering the structure constants. For a fixed subalgebra $\mathfrak{s}$ construct a basis $\{x_k\}$. Now extend this basis to a basis for $\mathfrak{g}$: $\{x_k\} \cup \{x_\mu\}$. Take $\mathfrak{c}$ to be the span of $\{x_\mu\}$. (Notice that $\mathfrak{c}$ is not usually a subalgebra of $\mathfrak{g}$.) We can now write $\mathfrak{g}$ as a vector space direct sum: $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$

In their 1953 paper, Wigner and Inönü presented a contraction process which we will follow here. Let roman indices refer to basis elements which span $\mathfrak{s}$ and greek indices refer to basis elements which span $\mathfrak{c}$. After contraction of the Lie algebra with respect to the subalgebra $\mathfrak{s}$, the new "contracted" structure constants, c_{ij}^k are given by:

$c_{ij}^k = C_{ij}^k$	The structure constants of the subalgebra are not affected.
$c_{ij}^\mu = C_{ij}^\mu = 0$	Since $\mathfrak{s}$ is a subalgebra, these were zero in the uncontracted algebra.
$\boldsymbol{c_{i\mu}^k = 0}$	This makes $\mathfrak{c}$ into an ideal.
$c_{i\mu}^\nu = C_{i\mu}^\nu$	
$\boldsymbol{c_{\mu\nu}^k = c_{\mu\nu}^\vartheta = 0}$	This has the effect of abelianizing $\mathfrak{c}$.

Alternatively, we can demonstrate how the bracket between two arbitrary elements of $\mathfrak{g}$ is affected by contraction with respect to $\mathfrak{s}$. Let $[\cdot,\cdot]^o$ denote the contracted Lie algebra relations and the unadorned bracket denote the original Lie algebra relations. Fix arbitrary $s, t \in \mathfrak{s}$ and $y, z \in \mathfrak{c}$.

$[s,t]^o = [s,t]$	The structure constants of the subalgebra are not affected.
$[s,y]^o = proj_c[s,y]$	These structure constants contract to the projection of the bracket onto its component in $\mathfrak{c}$
$[y,z]^o = 0$	After contraction $\mathfrak{c}$ is abelian.

Proposition 3.3.1. *Let $\mathfrak{g}$ be a Lie algebra with subalgebra $\mathfrak{s}$.*

The Lie algebra produced by contracting $\mathfrak{g}$ with respect to $\mathfrak{s}$ can be realized as a semi-direct product of Lie algebras.

We offer two proofs – the first is by brute force.

Proof. Bi-linearity and anti-commutativity are immediately inherited from the original Lie algebra.

The Jacobi Identity $[x[yz]^o]^o + [y[zx]^o]^o + [z[xy]^o]^o = 0$ must be verified.

We can separate each bracket into its $\mathfrak{s}$ and $\mathfrak{c}$ components & write:

$$[x[yz]_s]_s + [x[yz]_c]_s + [x[yz]_s]_c + [x[yz]_c]_c + [y[zx]_s]_s + [y[zx]_c]_s$$
$$+[y[zx]_s]_c + [y[zx]_c]_c + [z[xy]_s]_s + [z[xy]_c]_s + [z[xy]_s]_c + [z[xy]_c]_c = 0$$

Thus it must be shown that

$$[x[yz]^o_s]^o_s + [x[yz]^o_c]^o_s + [y[zx]^o_s]^o_s + [y[zx]^o_c]^o_s + [z[xy]^o_s]^o_s + [z[xy]^o_c]^o_s = 0$$

and that

$$[x[yz]^o_s]^o_c + [x[yz]^o_c]^o_c + [y[zx]^o_s]^o_c + [y[zx]^o_c]^o_c + [z[xy]^o_s]^o_c + [z[xy]^o_c]^o_c = 0$$

Case 1: If $x \in \mathfrak{s}$ and $y, z \in \mathfrak{c}$, then $[yz]^o = 0, [xy]^o_s = 0, [zx]^o_s = 0$

Hence the first expression simplifies to:

$$\cancel{[x[yz]^o_s]^o_s} + \cancel{[x[yz]^o_c]^o_s} + \cancel{[y[zx]^o_s]^o_s} + [y[zx]^o_c]^o_s + \cancel{[z[xy]^o_s]^o_s} + [z[xy]^o_c]^o_s = 0$$

and the second simplifies to:

$$\cancel{[x[yz]^o_s]^o_c} + \cancel{[x[yz]^o_c]^o_c} + \cancel{[y[zx]^o_s]^o_c} + [y[zx]^o_c]^o_c + \cancel{[z[xy]^o_s]^o_c} + [z[xy]^o_c]^o_c = 0$$

Each of the remaining brackets is the bracket of elements in $\mathfrak{c}$ which is abelian after contraction and thus these brackets equal zero. Hence the Jacobi Identity is satisfied.

Case 2: If $x, y \in \mathfrak{s}$ and $z \in \mathfrak{c}$, then $[xy]^o_c = 0, [zx]^o_s = 0, [yz]^o_s = 0$

Hence the first expression simplifies to:

$$\cancel{[x[yz]^o_s]^o_s} + [x[yz]^o_c]^o_s + \cancel{[y[zx]^o_s]^o_s} + [y[zx]^o_c]^o_s + [z[xy]^o_s]^o_s + \cancel{[z[xy]^o_c]^o_s} = 0$$

and the second simplifies to:

$$\cancel{[x[yz]^o_s]^o_c} + [x[yz]^o_c]^o_c + \cancel{[y[zx]^o_s]^o_c} + [y[zx]^o_c]^o_c + [z[xy]^o_s]^o_c + \cancel{[z[xy]^o_c]^o_c} = 0$$

Since $[yz]^o_c$,$[zx]^o_c \in \mathfrak{c}$, $[x[yz]^o_c]^o_s = 0$ and $[y[zx]^o_c]^o_s = 0$. $[z[xy]^o_s] \in \mathfrak{c}$ implies that $[z[xy]^o_s]^o_s = 0$.

Hence the first expression equals zero.

In order to dispense with the second expression,

$$[x[yz]^o_c]^o_c + [y[zx]^o_c]^o_c + [z[xy]^o_s]^o_c$$

we use the fact that the Jacobi Identity was satisfied in the original Lie algebra.

Separating the expression as before ...

$$[x[yz]_s]_c + [x[yz]_c]_c + [y[zx]_s]_c + [y[zx]_c]_c + [z[xy]_s]_c + [z[xy]_c]_c = 0$$

Since $x, y, [yz]_s, [zx]_s \in \mathfrak{s}$, the first, third, and last term are zero.

Thus $[x[yz]_c]_c + [y[zx]_c]_c + [z[xy]_s]_c = 0$ and the proof is complete. □

We now consider a more elegant abstract approach.

Proof. Consider the vector space direct sum decomposition:

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c} \text{ where } \mathfrak{c} \text{ is defined as it was previously.}$$

Define the map $\phi : \mathfrak{s} \to End_{vs}(\mathfrak{g})$ where $\phi_s(x) = [s, x]$ for $s \in \mathfrak{s}$ and $x \in \mathfrak{g}$.

This map gives an action of $\mathfrak{s}$ on the vector space $\mathfrak{g}$ which maps the subalgebra $\mathfrak{s}$ to itself. Hence there is a vector space action of $\mathfrak{s}$ on the quotient vector space $\mathfrak{a} \coloneqq \mathfrak{g} \,/\, \mathfrak{s}$. We can consider this vector space $\mathfrak{a}$ as an abelian Lie algebra. Notice that $\mathfrak{c} \cong \mathfrak{a}$ as vector spaces. We now have a Lie algebra $\mathfrak{a}$ and a Lie algebra action of $\mathfrak{s}$ on this Lie algebra $\mathfrak{a}$.

Since a vector space endomorphism on an abelian Lie algebra is automatically a Lie algebra endomorphism, we may form the Lie algebra semi-direct product $\mathfrak{g}_o \coloneqq \mathfrak{s} \rtimes \mathfrak{a}$. This new object is isomorphic to $\mathfrak{g}$ as a vector space and has the desired structure constant changes. □

Example 3.3.2. *The Lie algebra for the generalized $ax + b$ group is two-dimensional with basis $\{A, B\}$ and relation $[AB] = \rho B$*

- *The contraction with respect to the subalgebra spanned by $\{A\}$ has no effect upon the Lie algebra structure since $[A, B] = \rho B$ which lies in the abelian ideal $\mathfrak{a}$ (spanned by B) as required.*

- *However the contraction with respect to the subalgebra spanned by $\{B\}$ abelianizes the algebra since $[A, B]^o$ must lie in the abelian ideal spanned by $\{A\}$.*

  Thus after contraction with respect to the subalgebra spanned by $\{B\}$, the Lie algebra for the generalized $ax + b$ group becomes abelian.

We now turn our attention to the effect this contraction has first on the representation and then on the quantum fields. With the variable parameter ρ in hand, we have a simple, intuitive model for the concept of a group contraction. By allowing $\rho \to 0$, the $ax + b$ group can be contracted to $\mathbb{R}^2$ with its usual abelian group structure. Since it has been shown that $G_\rho \cong G_1$ for all $\rho > 0$, we expect a dramatic change at $\rho = 0$.

In their paper, Wigner and Inönü presented a representation for the $ax + b$ group and then discussed the contraction process. In order to carry out the contraction process they introduced an ϵ-factor – which will play the same role as our ρ – and allowed this factor to decrease to zero.

Following Wigner and Inönü [20] we work with the Lie algebra first. In order to discover the representation of the Lie algebra $\mathfrak{g}_\rho$ for $\rho > 0$, we differentiate the representation of the Lie group. We introduce the ρ-factor by considering $\Phi_{(B,\rho A)}$ where $\rho > 0$ and Φ is the Wigner-Inönü representation[7] on $L^2(\mathbb{R}^+, dx)$. Unfortunately simply letting ρ go to zero makes the action of A trivial.

$$\tilde{U}^i_{(B,A)}\tilde{F}(x) = e^{iBx}\tilde{F}(e^{\rho A}x) \to e^{iBx}\tilde{F}(x)$$
$$\tilde{U}^q_{(B,A)}\tilde{F}(x) = e^{\frac{\rho A}{2}}e^{iBx}\tilde{F}(e^{\rho A}x) \to e^{iBx}\tilde{F}(x)$$

Thus the representation of $\mathfrak{g}_\rho$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$ becomes

- $\pi^A_\rho := \frac{\partial}{\partial A}|_{(0,0)}\left(\tilde{U}^i_{(B,A)}\psi\right)(x) = \frac{\partial}{\partial A}|_{(0,0)}e^{iBx}\psi(e^{\rho A}x) = \rho x\frac{d}{dx}\psi(x) \to 0$
- $\pi^B_\rho := \frac{\partial}{\partial B}|_{(0,0)}\left(\tilde{U}^i_{(B,A)}\psi\right)(x) = \frac{\partial}{\partial B}|_{(0,0)}e^{iBx}\psi(e^{\rho A}x) = ix\psi(x)$

and the representation of $\mathfrak{g}_\rho$ on $L^2(\mathbb{R}^+, dx)$ which is Wigner and Inönü's representation becomes

- $\pi^A_\rho := \frac{\partial}{\partial A}|_{(0,0)}\left(\tilde{U}^q_{(B,A)}\psi\right)(x) = \frac{\partial}{\partial A}|_{(0,0)}e^{\frac{\rho\alpha}{2}}e^{iBx}\psi(e^{\rho A}x) =$ ‘$\frac{\rho}{2}\psi(x) + \rho x\frac{d}{dx}\psi(x) \to 0$
- $\pi^B_\rho := \frac{\partial}{\partial B}|_{(0,0)}\left(\tilde{U}^q_{(B,A)}\psi\right)(x) = \frac{\partial}{\partial B}|_{(0,0)}e^{\frac{\rho\alpha}{2}}e^{iBx}\psi(e^{\rho A}x) = ix\psi(x)$

We overcome this obstacle as Wigner and Inönü did, by considering a representation equivalent to $\Phi_{(B,\rho A)}$:

$$\Pi_{\rho,(B,A)} := M^{-1}_{\frac{if}{\rho}} \circ \Phi_{(B,\rho A)} \circ M_{\frac{if}{\rho}}$$

where $M_{\frac{if}{\rho}}$ is multiplication by $e^{\frac{if(x)}{\rho}}$ for a fixed non-constant, real-valued, differentiable function $f(x)$. $M_{\frac{if}{\rho}}$ is an operator which maps $L^2(\mathbb{R}^+, dx)$ to itself isomorphically. Thus $\Pi_{\rho,(B,A)}$ is an irreducible representation of G_ρ on $L^2(\mathbb{R}^+, dx)$ equivalent to $\Phi_{(B,\rho A)}$.

---

[7] see page 38

We now wish to see the effect of the contraction $\rho \to 0$ on the representation.

$$\begin{aligned}\Pi_{\rho,(B,A)} &\coloneqq M^{-1}_{\frac{if}{\rho}} \circ \Phi_{(B,\rho A)} \circ M_{\frac{if}{\rho}} \\ &= M^{-1}_{\frac{if}{\rho}} \circ \Phi_{(B,\rho A)} \circ M_{\frac{if}{\rho}}\end{aligned}$$

Let $\psi \in L^2(\mathbb{R}^+, dx)$

$$\begin{aligned}\left(\Pi_{\rho,(B,A)}\psi\right)(x) &= \left(M^{-1}_{\frac{if}{\rho}} \circ \Phi_{(B,\rho A)} \circ M_{\frac{if}{\rho}}\psi\right)(x) \\ &= e^{\frac{-if(x)}{\rho}}\left(\Phi_{(B,\rho A)} \circ M_{\frac{if}{\rho}}\psi\right)(x) = e^{\frac{-if(x)}{\rho}} e^{\frac{\rho A}{2}+iBx}\left(M_{\frac{if}{\rho}}\psi\right)(e^{\rho A}x) \\ &= e^{\frac{-if(x)}{\rho}} e^{\frac{\rho A}{2}+iBx} e^{\frac{if(e^{\rho A}x)}{\rho}}\psi(e^{\rho A}x) = e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x)\end{aligned}$$

We can extend this group representation to $\mathbb{R}^2$ by considering $\lim_{\rho\to 0}\left(\Pi_{\rho,(B,A)}\psi\right)(x)$

$$\begin{aligned}\lim_{\rho\to 0}\left(\Pi_{\rho,(B,A)}\psi\right)(x) &= \lim_{\rho\to 0} e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x) \\ &= \lim_{\rho\to 0} e^{if'(e^{\rho A}x)e^{\rho A}Ax} e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x) = e^{ixf'(x)A}e^{iBx}\psi(x)\end{aligned}$$

We can therefore define a representation Π_0 of $\mathbb{R}^2$ on $L^2(\mathbb{R}^+)$ by:

$$\left(\Pi_{0,(B,A)_0}\psi\right)(x) \coloneqq e^{ixf'(x)A}e^{iBx}\psi(x).$$

Note however that this representation is no longer irreducible – for example, $R = \{\psi \in L^2(\mathbb{R}^+, dx) | supp(\psi) \subset [0,1]\}$ is invariant under the action of Π_0.

We compute the representation of $\mathfrak{g}_\rho$ on $L^2(\mathbb{R}^+, dx)$

- $\pi_\rho^A := \frac{\partial}{\partial A}|_{(0,0)} \left(\Pi_{\rho,(B,A)_\rho}\psi\right)(x) = \frac{\partial}{\partial A}|_{(0,0)} e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x)$

$$= [e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} \frac{i}{\rho} f'(e^{\rho A}x)e^{\rho A}x\rho \cdot e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x) + e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} \cdot e^{\frac{\rho A}{2}} \frac{\rho}{2} \cdot e^{iBx}\psi(e^{\rho A}x)$$

$$+ e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} e^{\frac{\rho A}{2}} e^{iBx} \cdot \psi'(e^{\rho A}x)e^{\rho A}x\rho]|_{(0,0)}$$

$$= \tfrac{i}{\rho} f'(x)x\rho\psi(x) + \tfrac{\rho}{2}\psi(x) + x\rho\psi'(x) = \left(ixf'(x) + \rho(\tfrac{1}{2} + x\tfrac{d}{dx})\right)\psi(x)$$

- $\pi_\rho^B := \frac{\partial}{\partial B}|_{(0,0)} \left(\Pi_{\rho,(B,A)}\psi\right)(x) = \frac{\partial}{\partial B}|_{(0,0)} e^{i\frac{f(e^{\rho A}x)-f(x)}{\rho}} e^{\frac{\rho A}{2}} e^{iBx}\psi(e^{\rho A}x) = ix\psi(x)$
  Notice that this operator is independent of ρ

Thus the basis for the operators in the representation of $\mathfrak{g}_\rho$ on $L^2(\mathbb{R}^+, dx)$ for $\rho > 0$ is:

- $\pi_\rho^A = \left(ixf'(x) + \rho(\frac{1}{2} + x\frac{d}{dx})\right)\cdot$
- $\pi_\rho^B = ix\cdot$

Notice that for $\rho > 0$, the operators π_ρ^A and π_ρ^B do not commute and that these operators are not bounded on $L^2(\mathbb{R}^+, dx)$. In order to discover the effect after contraction, we send $\rho \to 0$ and now use the basis:

- $\pi_0^A = ixf'(x)\cdot$
- $\pi_0^B = ix\cdot$

Notice that after contracting, the operators π_0^A and π_0^B now commute, but remain unbounded on $L^2(\mathbb{R}^+, dx)$.

The alternate approach offered in this thesis uses the representation U on $\mathcal{H}_\rho$ and considers the effect of $\rho \to 0$. This task will be simplier as the ρ parameter is already part of our construction. The representation

$$U_{(B,A)}F(b,\alpha) = F\left((b,\alpha)(B,A)\right) = F(b + e^{\rho\alpha}B, \alpha + A)$$

$$= e^{i(b+e^{\rho\alpha}B)}F(0,\alpha + A) = e^{ib}e^{ie^{\rho\alpha}B}F(0,\alpha + A)$$

becomes for $\rho = 0$ becomes the representation,

$$U_{(B,A)}F(b,\alpha) = F\left((b,\alpha)(B,A)\right) = F(b + B, \alpha + A) = e^{i(b+B)}F(0,\alpha + A)$$

on the Hilbert space $\mathcal{H}_0$. After contraction, notice that the representation remains faithful – but becomes reducible, as we will soon discuss.

As before, we study the effect the contraction $\rho \to 0$ has on the representation of $\mathfrak{g}_\rho$ on $\mathcal{H}_\rho$

- $\pi_\rho^A := \frac{\partial}{\partial A}|_{(0,0)} \left(U_{(B,A)}F\right)(b,\alpha) = \frac{\partial}{\partial A}|_{(0,0)} e^{ie^{\rho\alpha}B}F(b,\alpha + A) = \frac{\partial}{\partial \alpha}F(b,\alpha) \to \frac{\partial}{\partial \alpha}F(b,\alpha)$
- $\pi_\rho^B := \frac{\partial}{\partial B}|_{(0,0)} \left(U_{(B,A)}F\right)(b,\alpha) = \frac{\partial}{\partial B}|_{(0,0)} e^{ie^{\rho\alpha}B}F(b,\alpha + A) = ie^{\rho\alpha}F(b,\alpha) \to i\ F(b,\alpha)$

These unbounded operators do not commute for $\rho > 0$. However after contraction ($\rho \to 0$), the unbounded operators corresponding to the infinitesimal translation in the α variable (π_0^A) and to the Fourier variable $E = 1$ (π_0^B) commute.

This simpler calculation highlights another advantage of our alternate representation over the one offered by Wigner and Inönü. The drastic difference between these representations after contraction does not contradict our earlier equivalences between U and the other representations $\tilde{U}^i$ and $\tilde{U}^q$ as those equivalences only hold for $\rho > 0$.

The above limiting procedure corresponds to contracting the Lie group with respect to the $B = \{(b,0) : b \in \mathbb{R}\}$ subgroup; the contraction of the Lie group with respect to the $A = \{(0,\alpha) : \alpha \in \mathbb{R}\}$ subgroup has no effect on the group [20] and thus no effect on the representation. After contraction, the symmetry group G_ρ becomes a two-dimensional abelian subgroup $G_0 = \mathbb{R}^{1+1} \subset \mathcal{P}_2^o$ and the limiting space-time $\mathbb{R}^2$ has more symmetry – namely the Poincaré group $\mathcal{P}_2^o$ now acts isometrically on $\mathbb{R}^2 = S_0$. We have presented the contracting family of groups G_ρ as directly and abstractly as possible and have observed that the contraction can be constructed within the Poincaré group using the inclusion map offered in Section 1.2.

Consider the map $\phi : \mathcal{H}_0 \to L^2(\mathbb{R}, d\alpha)$ given by $F(b, \alpha) = e^{ib}F(0, \alpha) \mapsto F(0, \alpha) =: \psi(\alpha)$ This map is bijective and allows us to conclude that $\mathcal{H}_0 = e^{ib}L^2(\mathbb{R}, d\alpha) \cong L^2(\mathbb{R}, d\alpha)$. That is, for $\rho = 0$, the Hilbert space decomposes as:

$$\mathcal{H}_0 \cong L^2(\mathbb{R}, d\alpha) = \int_{\mathbb{R}}^{\oplus} \mathbb{C}_{(1,p)}\ dp$$

We can then define a representation u of G_0 on $L^2(\mathbb{R}, d\alpha)$:

$$u_{(B,A)}\psi(\alpha) = e^{iB}\psi(\alpha + A) = \chi_{(1,0)}(B, A)R_A\psi(\alpha) \qquad \text{for } \psi \in L^2(\mathbb{R}, d\alpha)$$

This representation is compatible with the representation U on $\mathcal{H}_0$

$$\begin{array}{ccccc} F(b,\alpha) & = & e^{ib}F(0,\alpha) & \xrightarrow{\phi} & \psi(\alpha) \\ {\scriptstyle U_{(B,A)}}\big\downarrow & & & & {\scriptstyle u_{(B,A)}}\big\downarrow \\ F(b+B,\alpha+A) & = & e^{iB}e^{ib}F(0,\alpha+A) & \xrightarrow{\phi} & e^{iB}\psi(\alpha+A) \end{array}$$

We observe that u is equal to the character $\chi_{(1,0)}$ composed with the (right) translation operator R.

The regular representation of $\mathbb{R}$ on $L^2(\mathbb{R}, d\alpha)$ decomposes as:

$$R_A \leftrightarrow \int_{\mathbb{R}}^{\oplus} \chi_p(A)\ dp = \int_{\mathbb{R}}^{\oplus} e^{iAp}\ dp$$

Thus

$$u_{(B,A)} \leftrightarrow \int_{\mathbb{R}}^{\oplus} \chi_{(1,p)}(B, A)\ dp$$

The unitary representation U is the direct integral of the characters $\chi_{(\eta,p)}$ of $G_0 = \mathbb{R}^2$

$$U_{(B,A)} = \int_{\mathbb{R}}^{\oplus} \chi_{(\eta,p)}(B,A)\ dp$$

where the energy parameter η is fixed and $\chi_{(\eta,p)}(b,\alpha) = e^{i(\eta,p)(b,\alpha)}$.

Notice that Ψ_η is independent of ρ while $\mathcal{H}_\rho$ is independent of η. The "formula" for U is independent of both ρ and η, although we must be careful to remember that U is a representation of $\mathcal{H}_\rho$ and therefore depends on ρ.

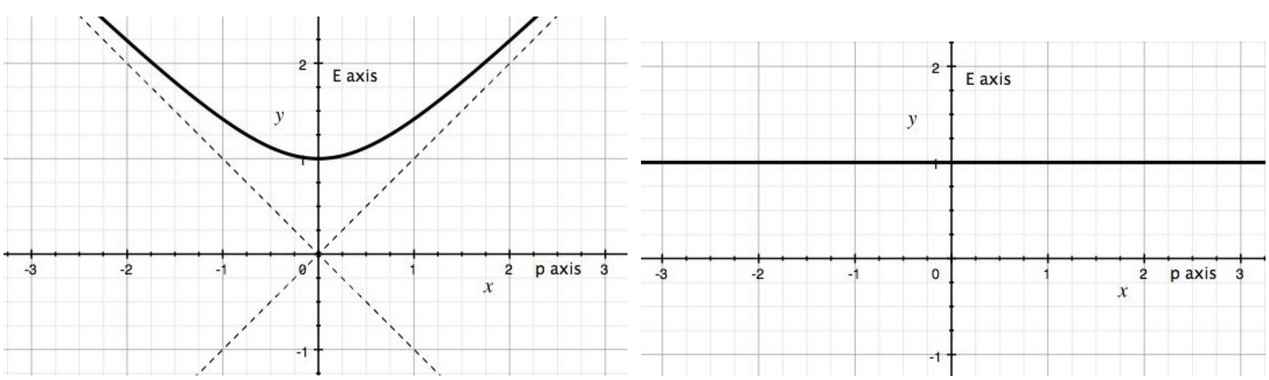
We see that when $\rho = 0$, $\Psi_\eta : C^\infty_{cpt}(S_0) \to \mathcal{H}_0 = e^{ib} \int_{\mathbb{R}}^{\oplus} \mathbb{C}_{(1,p)}\ dp$.
Hence $\Psi_\eta(f)(b,\alpha)$ can be expressed as a direct integral:

$$\Psi_\eta(f)(b,\alpha) = e^{ib} \int_{\mathbb{R}}^{\oplus} \int_{\mathbb{R}} e^{-i\alpha p} \int_{\mathbb{R}} f(\beta,\alpha) e^{-i\beta\eta} d\beta d\alpha\ dp = e^{ib} \int_{\mathbb{R}}^{\oplus} \hat{f}(\eta,p)\ dp$$

Since the Fourier transform is an isomorphism, we may simplify this process by considering this process on the transform side, in which case:

$$\hat{f}(E,p) \overset{\widehat{\Psi_\eta}}{\longmapsto} e^{ib} \hat{f}(\eta,p)$$

Thus we have constructed two different fields – the first the "standard" field with full Poincaré symmetry and the second with reduced G_0 symmetry. The former uses the Fourier transform restricted to the mass hyperbola ($H_{(m,0)}$); the latter uses the Fourier transform restricted to the line $E = \eta$ with an additional "twisting" factor e^{ib} which is essential for equivariance.



(a) Fourier transform on $H_{(m,0)}$ (b) Fourier transform on $E = \eta$ twisted by e^{ib}

Chapter 4

Remarks and Questions

The topic of quantum field theory is nearly one hundred years old and there exists a wealth of literature on the subject. The present work is certainly not the first to address constructing a quantum field theory in curved space-time – for example see Wald [17] or Zee [21]. What we have offered here is an axiomatic approach similar to Wightman in the hopes that this will place us one step closer to the physical goal of harmonizing general relativity with quantum field theory. Much work remains to be done in this endeavor.

As is often the case in scientific inquiry, new answers present new questions. Much of the work presented here might be extended and may serve as a root for future research. We now discuss several of the questions raised by the results of this thesis.

(1) In the quantum state field axioms for flat space-time presented in chapter two, we introduced a zeroth axiom using the d'Alembert operator which was used to model the free quantum scalar field on the Minkowski plane. Unfortunately, we were unable to formulate such an axiom in the curved case. It is our hope that such an axiom could be introduced and that this axiom could similarly help to define an analogue to the mass of the particle under consideration.

(2) Another aspect left unexplored in this thesis is an axiomatic consideration for the action of the reduced symmetry group $G_0 = \mathbb{R}^2$ on the Minkowski plane and how the quantum state fields for that group should be related to the quantum state fields for the Poincaré group.

(3) We may also ask how the free quantum field theories presented here are related to general quantization methods for static and stationary space-times.

(4) Several of our simplifying assumptions about the space-time might be discarded in future work. Our space-times (both the Minkowski plane and G_ρ) are simply connected and all time-like geodesics are complete. This does not hold for a general space-time. Note that many familiar spaces in Riemannian geometry do not seem to be suitable space-times. For example, S^2 does not admit a Lorentz metric and $S^1 \times S^1$ might contain closed time-like geodesics. Many of the arguments presented here might be also generalized to space-times of higher dimension in order to more accurately represent physical models.

(5) A comprehensive study of Wigner's contraction process should be conducted for other Lie groups – particularly tracing the effects of a contraction on the quantum state fields and thus on the quantum field theories.

(6) Following in the tradition of Källen and Lehmann, we can consider a "smearing" together of the quantum state fields by considering an interval of positive η's in order to create a generalized free quantum state field.

(7) Our study has also been confined to particles of spin zero. However the work done here might be extended to include positive-integer-spin particles and considerations of Yang-Mills gauge fields. Half-integer spin particles are fermions and require a study of the anti-symmetric Fock space versus the symmetric Fock space discussed here.

(8) In this thesis we have restricted our focus to the free case which does not account for the interactions between particles; consideration of these interactions is a necessary part of any final theory. This might be accomplished by considering perturbations of the free quantum field theories presented here or might require an entirely different approach.

Chapter 5

Bibliography

[1] A.O. Barut and R. Raczka. *Theory of Group Representations and Applications.* World Scientific, 1986.

[2] Manfredo P. do Carmo (translated by Francis Flaherty). *Riemannian Geometry.* Birkhäuser, 1992.

[3] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer, 1972.

[4] Claude Itzykson and Jean-Bernard Zuber. *Quantum Field Theory.* Dover, 1980.

[5] Palle E.T. Jorgensen and Gestur Ólafsson. "Unitary Representations and Osterwalder-Schrader Duality" *Proceedings of Symposia in Pure Mathematics*, Vol.68: 333-401, 2000.

[6] Jürgen Jost, *Riemannian Geometry and Geometric Analysis.* Springer, 2002.

[7] David Kazhdan "Introduction to QFT" *Quantum Fields and Strings: A Course for Mathematicians.* AMS, 1991. pg. 377-418.

[8] George W. Mackey. *Induced Representations of Groups and Quantum Mechanics.* W. A. Benjamin, 1968.

[9] Justin Manning "A Local Spectral Condition for Axiomatic Quantum Fields on the deSitter Surface" University of Georgia PhD. Thesis, Athens, Georgia, 2011.

[10] Calvin C. Moore. "Representations of Solvable and Nilpotent Groups and Harmonic Analysis on Nil and Solvmanifolds" *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, pg. 3-44, 1973.

[11] Barrett O'Neill. *Semi-Riemannian Geometry*. Academic Press, 1983.

[12] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis & Self-Adjointness*. Academic Press, 1975.

[13] Eugene J. Saletan. "Contraction of Lie Groups" *Journal of Mathematical Physics*. Vol. 2, Issue 1, 1961.

[14] Theodore Shifrin. *Differential Geometry: a First Course in Curves and Surfaces*. University of Georgia, Spring 2013.

[15] R. F. Streater and A.G. Wightman. *PCT, Spin & Statistics, and All That*. W.A. Benjamin, 1964.

[16] V.S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representations*. Springer, 1974.

[17] Robert M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. University of Chicago Press, 1994.

[18] A.S. Wightman and L. Gårding. "Fields as operator-valued distributions in relativistic quantum theory" *Arkiv For Fyskik*. 28, 1964. pg. 129-184.

[19] E.P. Wigner. "On Unitary Representations of the Inhomogeneous Lorentz Group" *Annals of Mathematics*. Vol. 40, No. 1, 1939. pg. 149-204.

[20] E.P. Wigner and E. Inönü. “On the Contraction of Groups and Their Representations” *Proc. Nat. Acad. Sci.* 39, 1953. pg. 510-524.

[21] A. Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, 2003.