PRESERVICE TEACHERS' PATTERNS OF METACOGNITIVE BEHAVIOR DURING MATHEMATICS PROBLEM SOLVING IN A DYNAMIC GEOMETRY ENVIRONMENT

by

ANA KUZLE

(Under the Direction of James W. Wilson)

ABSTRACT

As an experienced and passionate problem solver for years, I wanted to better understand the metacognition that students exhibit when solving nonroutine geometry problems in a dynamic geometry environment. In this study, dynamic tool software—namely, the Geometer's Sketchpad—was used by the participants. My intention was to focus on participants' decision making, reflection, reasoning, and problem solving as well as to understand what situations and interactions in a dynamic geometry environment promote metacognitive behavior.

Case studies were conducted of two mathematics education preservice teachers who had previously completed a semester of college geometry and had prior experience working in Geometer's Sketchpad. Artigue's (2002) instrumental approach and Schoenfeld's (1981) model of episodes and executive decisions in mathematics problem solving were used to identify patterns of metacognitive processes in a dynamic geometry environment. Data sources for this study consisted of think-aloud protocols, individual interviews after each problem-solving session, students' written solutions, researcher's observation notes, video files of problem solving sessions and a final interview. All collected data were analyzed using the constant comparative method for both the within-case and the cross-case analysis.

Problem solving of the two participants was described through identifying the metacognitive processes within each problem-solving episode, and associating them with the Geometer's Sketchpad use. During the reading, understanding, and analysis episodes, the participants engaged in monitoring behaviors such as sense making, drawing a diagram, and allocating potential resources and approaches that helped make productive decisions. During the exploring, planning, implementation, and verification episodes, the participants made decisions to access and consider knowledge and strategies, make and test conjectures, monitor the progress, and assess the productivity of activities and strategies and the correctness of an answer. Geometer's Sketchpad played an important role in supporting these metacognitive processes. Their use of metacognitive questions helped prompt a metacognitive activity. The effectiveness of solution approaches was dependent on the presence of managerial decisions. Cognitive problem-solving actions not accompanied by appropriate metacognitive monitoring actions appeared to lead to unproductive efforts. Redirection and reorganizing of thinking in productive directions occurred when metacognitive actions guided the thinking and when affective behaviors were controlled.

INDEX WORDS: Problem solving, Metacognition, Nonroutine geometry problems, Preservice teachers, Dynamic geometry software

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CHAPTER 1

BACKGROUND AND DESCRIPTION OF THE PROBLEM

We work hard to extract something helpful from our memory, yet, quite often, when an idea that could be helpful presents itself, we do not appreciate it, for it is so inconspicuous. The expert has, perhaps, no more ideas than the inexperienced, but appreciates more what he has and uses it better.

(Pólya, 1945/1973, P. 223–224)

At the beginning of the 21st century, "the rapid mathematization of work in almost all areas of business, industry, personal decision making, and the social and life sciences dictates that most students learn more and different mathematics than school mathematics programs provide" (Fey, Hollenbeck, & Wray, 2010, p. 41), creating unprecedented challenges in schooling practices. Nowadays, topics taught in mathematical classes require more than mere arithmetic or calculation skills, but rather extension and adaptability of previous knowledge, and flexibility in thinking. Hence, on the one side, educational institutions are obliged to educate citizens that are able to participate in an increasingly global and technological society, while at the same time, they need to reflect on current research findings in the teaching and learning of mathematics in their instructional practices. "Problem solving is the cornerstone of school mathematics. Without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited" (NCTM, 2000, p. 182). Despite the emphasis given to mathematical problem solving by various professional organizations (National Council of Teachers of Mathematics [NCTM]), reform curricula (Connected Mathematics Project [CMP], Core-Plus Mathematics Project [CPMP]), educators, and researchers, students' performance on challenging problems rarely meets delineated expectations. Research (Garofalo & Lester, 1985; Lester, 1980; Schoenfeld, 1985a, 1987; Silver, 1994) shows that students' low problem-solving performance is not due to the inadequacy of mathematical content knowledge and facts, but rather is associated with students' inability to analyze the problem, to fully understand it, to evaluate the adequacy of given information, to organize knowledge and facts they possess with the goal of devising a plan, to evaluate the feasibility of the devised plan before its implementation, and to evaluate the reasonableness of the results. One's thought processes of how to organize, plan and implement, and monitor what they already know is crucial when problem solving; that is, these researchers suggested metacognition is the "missing link," the "driving force." *Metacognition* refers to individual's awareness, consideration, and control of his or her own cognitive processes (Flavell, 1976). In this study, my passion for problem solving put me on a path of trying to better understand the elements of metacognition students exhibit when they are in the process of solving nonroutine geometry problems, and how are these processes related to interactions with a dynamic geometry environment.

Background

Mathematical problems have been central in the mathematics school curriculum since antiquity, but since the 1980s, mathematics educators have agreed upon the idea of developing problem-solving ability, and problem solving has become a focus of mathematics education as a means of teaching curricular material and seeking the goals of education (Stanic & Kilpatrick, 1989). Problem solving is a directed cognitive processing (Schunk, 2008); that is, *problem* *solving* refers to a cognitive process in which the student determines how to solve a problem that he or she does not readily know how to solve. Lesh and Zawojewski (2007) defined *mathematical problem solving* as

> the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing, and revising mathematical interpretation—and of sorting out, integrating, modifying, revising or refining clusters of mathematical concepts from various topics within and beyond mathematics. (p. 782)

A deep and comprehensive view of problem solving as an art form emerged from the work of Hungarian mathematician George Pólya. Following Pólya (1945/1973), for whom problem solving was a major theme of doing mathematics in his book, *How to Solve It*, the National Council of Teachers of Mathematics (NCTM, 1989, 2000) has strongly endorsed the inclusion of problem solving in school mathematics for several reasons: (1) to build new mathematical knowledge, (2) to solve problems that arise in mathematics and in other contexts, (3) to apply and adapt a variety of problem-solving strategies, and (4) to monitor and reflect on the mathematical problem-solving processes. Furthermore, problem solving as context, problem solving as skill, and problem solving as art, that involves creativity, reasoning, and discovery of mathematical truth, are three general themes that characterize the role of problem solving in the school mathematics curriculum (Stanic & Kilpatrick, 1989). "The art of problem solving is the heart of mathematics" (J. W. Wilson, Fernandez, & Hadaway, 1993, p. 66). Without a doubt problem solving plays a prominent role in the curriculum, as it is an essential part of mathematical knowledge and performance. Furthermore, inclusion of problem solving in school mathematics is also important to meet our society's needs with respect to work, school, and life

(NCTM, 1989, 2000) as well as to stimulate the interest and enthusiasm of students (Pólya, 1945/1973).

Problem-solving activities in mathematics require skills and understanding that are often not readily apparent to novice problem solvers, even though it has been a goal of mathematics educators to provide students with the skills necessary for success in problem solving. Pólya's (1945/1973) four phases of problem solving—understanding the problem, devising a plan, carrying out the plan, and looking back—are held to be essential in mathematics problem solving. Many researchers (e.g., Garofalo & Lester, 1985; Lester 1980; Schoenfeld, 1985a, 1987; Silver, 1994) have conducted studies with results suggesting that students' rich store of mathematical knowledge and facts is necessary but insufficient in solving mathematical problems. These authors have also pointed that the presence of cognitive processes without sufficient control and monitoring decisions render a problem-solving endeavor incomplete. Particularly, students' inability to (1) access and organize knowledge already possessed, (2) plan strategies for implementing what is known, and (3) monitor the effectiveness of these strategies as factors adversely affecting problem-solving performance appeared to be a major obstacle during problem solving. Lester (1980), for example, suggested that in addition to mathematical knowledge and experiences, students' successful problem solving is a function of at least five components, three of which involve knowledge about their own cognition before, during, and after the problem-solving episode and the ability to regulate and control procedures used during problem solving as well as the ability to use a variety of heuristic strategies in order to be successful when solving problems. Similarly, Mayer (1998) emphasized that successful problem solving relies not only on possession of mathematical knowledge but also on integration and execution of three components: skill (domain specific knowledge), metaskill (strategies on how and when to use and control knowledge), and will (motivation and task interest). He argued that expertise in any of the components is not sufficient for successful problem solving, but rather the problem solver must be proficient in coordinating the three components simultaneously. The problem solver needs to know what strategies to use, under what conditions to use them, and how to use them in a flexible matter. The interaction among these three variables is a challenge for many students. Furthermore, Schoenfeld (1992) pointed at the importance of "resource allocation during cognitive activity and problem solving" (p. 354), highlighting the impact of beliefs and attitudes about the nature of mathematics on one's management of resource allocation.

Metacognition

The term *metacognition* was coined by Flavell (1976) to explain thinking about thinking. In the 1970s, psychologists criticized the state of research on memory and, particularly, the lack of research on something unique to humans and human memory: the fact that people have knowledge and beliefs about their memory processes (Campione, Brown, & Connell, 1989). Flavell then started studying children's metamemory. More precisely, he was interested in what children know about memory and when they come to know it. His work demanded that children reflect on their own memory process, emphasizing knowledge about cognition. Since the 1970s, researchers in cognitive and educational psychology have investigated the development of metacognition and the role of metacognition in cognitive functioning (Schunk, 2008). However, writings on metacognition can be traced back at least as far as *De Anima* and the *Parva Naturalia* of the Greek philosopher Aristotle. For instance, Aristotle in the fourth book of *De Anima* writes about reflexivity of thought; that is, thinking about thinking is really thinking about the thinking that thinks specific things. In other words, one does not think about thinking without having already thought about something since the later is actual. For Aristotle, thinking is the mental grasp of the form of a thing, where the form of things and the thinking become essentially the same. Contemporary philosophers refer to this construct as self-consciousness.

Metacognition is a form of cognition and a higher order thinking process that has been defined as any knowledge or cognitive activity that takes as its object, or regulates, any aspect of cognitive endeavor (Flavell, 1976). That is, metacognition has a managerial and a regulatory role of cognitive processes; it overlooks and governs the cognitive system, while at the same time is a part of it (Veenman, Van Hout-Walters, & Afflerbach, 2006). Metacognition focuses one's attention on the importance of learners having management of their own thinking and consists of the strategies that students use to plan for their learning, monitor their thinking, and control their thinking (Flavell, 1976). Furthermore, according to Schunk (2008), it can facilitate deeper processing of information and improve performance as it activates feedback performance and knowledge acquisition components of cognition.

Flavell (1979) identified various components of metacognition that occur through the actions of and interactions among four classes of phenomena: (1) *metacognitive knowledge,* which refers to acquired knowledge addressing cognitive matters and incorporates person, tasks, goals, actions, and experiences, (2) *metacognitive experiences* that refer to conscious cognitive or affective experiences that accompany and pertain to any intellectual enterprise, (3) *goals* (or tasks), which refer to the objectives of a cognitive enterprise, and (4) *actions* (or strategies), which refer to behaviors employed to achieve them. Metacognition is considered to be conscious, flexible, indispensible, and expressible in respect to knowledge of one's knowledge processes, cognitive states and affective states (Flavell, 1976, 1979; Schraw, 1998). Furthermore, it is considered to be able to be increased through promotion of awareness of its importance, improvement of knowledge of cognition, improvement of regulation of cognition, and fostering of environments that promote metacognitive awareness (Schraw, 1998).

Research conducted on the role of metacognition in mathematical problem solving has considered metacognitive processes not only as driving forces influencing all stages of problem solving, but fruitful problem solving also was connected with a wide range of noncognitive factors, such as personality, empathy, persistence, and confidence (Lesh, 1982; Schoenfeld, 1982; Silver, 1994). Veenman et al. (2006) argue that research on metacognition should not be studied in isolation, but take in consideration individual differences (e.g., metacognitive experiences, beliefs, motivational processes, self-efficacy) and how they interact with various components of metacognition.

Metacognition in problem solving helps the problem solver to recognize the presence of a problem that needs to be solved, to discern what exactly the problem is, and to understand how to reach the goal (solution). For the successful solution of any complex problem-solving task, a variety of metacognitive processes is necessary. Metacognitive processes during problem solving include regulatory activities of planning, monitoring, testing, revising, and evaluating throughout problem solving, especially in making the mental representation and selecting and assessing the effectiveness of the strategies employed; that is, the knowledge and processes used to guide thinking are directed toward the successful resolution of a problem (Flavell, 1992; Schraw, 1998). The use of metacognitive processes supports problem solvers during the solution process and improves their ability to obtain the goal, and problem solvers who are effective in using control and monitoring the strategies are better able to solve a problem (e.g., Schoenfeld, 1992). Therefore, metacognition is a critical component in cognitive function and cognitive development. Although the educational community holds a general acceptance of the role metacognition plays in problem solving, it yet remains to understand how students acquire metacognitive processes, how metacognitive processes emerge in problem-solving situations, the extent to which students act metacognitively, and how metacognitive behaviors can be encouraged in mathematics instruction.

Technology

"For many people the appropriate use of technology has significant identity with mathematics problem solving. This view emphasizes the importance of technology as a tool for mathematics problem solving" (J. W. Wilson et al., 1993, p. 69). In the *Principles and Standards for School Mathematics* (NCTM, 2000), the authors emphasize the importance of technology in teaching and learning mathematics. Specifically, the standards advocate use of dynamic geometry technology to teach geometry at all levels of mathematics education, emphasizing that the dynamic interactive nature of this technology helps in engaging students in meaningful mathematical activities and promotes deeper understanding of concepts. Various dynamic geometry software packages such as Geometer's Sketchpad, Cinderella, Cabri, and GeoGebra are available on discs for installation on computers as well as online for download that can be used in classrooms. In addition, recent versions of calculators like TI 83 Plus and TI-Nspire of Texas Instruments come with some of these packages. Geometer Explorer is an APP now available for the iPad that will examine Geometer's Sketchpad files. Thus, new and emerging technologies continually transform the mathematics classroom and redefine ways mathematics can be taught (Fey et al., 2010).

The dynamic geometry environments can be useful with nonroutine problems using a variety of available functions. Nonroutine problems are the type of problems where students are faced with an unfamiliar problem situation without an apparent solution path. These problems require problem solvers to use information and strategies in unfamiliar ways; that is, they demand strategy flexibility; thinking flexibility, such as logical thinking; abstract thinking; and transfer of mathematical knowledge to unfamiliar situations as well as extension of previous knowledge and concepts (Schoenfeld et al., 1999). In some cases, important mathematical ideas are introduced and developed through working on problems (teaching via problem solving), rather than taught first and applied later (teaching for problem solving) (Schroeder & Lester, 1989). According to Kantowski (1977), true problem-solving activity takes place only when problems turn out to be nonroutine as they help elicit variety of cognitive and metacognitive processes. Researchers (Garofalo & Lester, 1985; Montague & Applegate, 1993) have found that type of the problem influences which metacognitive processes get activated and to what extent.

In this study, I discussed the well-received interactive geometry software, The Geometer's Sketchpad (GSP), and its relationship with metacognitive processes. Sketchpad was first introduced at the beginning of 1990s by its software developer, Nicholas Jackiw. The GSP is dynamic mathematics visualization software used to explore algebra, geometry, calculus, and other areas of mathematics as well as sciences (NCTM, 2005). Leading researchers on the teaching and learning of geometry have emphasized the benefits of using dynamic environments (DeVilliers, 1999; Fey et al., 2010; Hollebrands, 2007; Jones, 2000; Laborde, 2000). These studies point in an important direction focused on understanding the impact of working in dynamic geometry environments, such as the GSP, on student mathematical problem solving and learning in mathematics. Such research must be continued and extended if we are to obtain convincing evidence concerning students' mathematical achievement with dynamic technology tools. This study's holistic design complements the literature base and adds to what we know about individual mathematical problem solving in dynamic geometry environments.

Purpose Statement

Problem solving is an extremely complex human endeavor that involves a complex interplay between cognition and metacognition (Schoenfeld, 1992). Metacognition starts developing early, at the age of 5 to 7, reaching its full development at the age of 12, but continues to do so during life span (Alexander, Carr, & Schwanenflugel, 1995). However, the research shows that even when students are able to monitor their work, they often do not do so (Schunk, 2008). Hence, although psychological and educational researchers share a common agreement about the important role of metacognition in problem solving, before we as educators focus on promoting metacognitive processes with a goal of improving problem-solving outcomes and performance, we need to better understand the concept of metacognition; that is, how students acquire metacognitive processes and how metacognitive processes emerge in problem-solving situations. For instance, Brown (1978) and Lesh and Zawojewski (2007) assume that metacognition develops much like cognitive areas following a developmental course during which metacognitive skills become more powerful and effective as a result of years of accumulated experience in making thought the object of thinking. On the other hand, Alexander, et al. (1995), showed that metacognitive knowledge develops incrementally throughout the school years parallel to the development of one's intellectual ability.

From a sociocognitive perspective, I wanted to better understand the elements of metacognition that preservice teachers elicit when they are in the process of solving nonroutine problems. *Principles and Standards of School Mathematics* (NCTM, 2000) states, "When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving" (p. 24). These characteristics correlate with the knowledge base, problem solving strategies, monitoring and controlling, and practice components of Schoenfeld's (1985a) model of mathematical thinking, of which metacognitive processes are part. Therefore, this study focused on identifying and describing situations in the context of problem solving with technology that participants described. Careful analysis of participants' perceptions regarding their thinking while engaged in problem solving provided an opportunity to explore how participants explain the emergence of metacognitive processes (organizing, planning, monitoring, evaluating progress) when working in a dynamic geometry environment.

Research Questions

The primary purpose of this study was to investigate the patterns of metacognitive processes preservice teachers exhibit when solving nonroutine geometry problems in a dynamic geometry environment; that is, to investigate how preservice teachers experience working individually in a dynamic geometry environment and how these experiences affect their own mathematical activity when integrating content (nonroutine problems) and context (technology environment). Hence, the study was designed to better understand what circumstances, situations, and interactions in a dynamic geometry environment promote metacognitive behavior. This study was designed to seek an understanding about how and why observed metacognitive processes emerged when problem solving in dynamic geometry environment: Were participants investigating mathematical ideas with the use of GSP? Were their metacognitive processes influenced and shaped by its use? Did their metacognitive processes influence their use of the GSP? As a mathematics teacher, a graduate student, and a problem solver, I thought the GSP made a difference, and I wanted to better understand how GSP interacted with my participants' mathematical thinking and metacognitive processes while solving nonroutine geometry problems. With these interests in mind, I used the following questions to guide the study:

1. What are some of the metacognitive processes exhibited by preservice teachers when engaged in solving nonroutine geometry problems using Geometer's Sketchpad?
2. What metacognitive processes appear to be associated with the Geometer's Sketchpad use during problem solving?
3. How do preservice teachers perceive the importance of Geometer's Sketchpad when faced with nonroutine geometry problems?

This study provides knowledge about the mathematical problem-solving processes used by mathematics education preservice teachers when solving nonroutine geometry problems in a dynamic geometry environment. The study findings help to extend the current research on student thought processes by revealing what circumstances, situations, and interactions in a dynamic geometry environment promote metacognitive behaviors.

Overview of the Study

In Chapter 2, I present review of relevant literature and the theoretical framework. Included in the review are topics, such as grounding problem solving and metacognition in constructivist theory, problem solving in mathematics, metacognition, research on metacognitive aspects of problem solving, problem solving in dynamic geometry environments, and the study's framework. The description of the study, which includes the methods and also the theoretical perspective that guided the study, is provided in Chapter 3. In Chapters 4 and 5, I present two case studies, the case study of Wes and the case study of Aurora, respectively, whilst in Chapter 6, I present the analysis of the two case studies, conclusions, implications, and recommendations for future research.

CHAPTER 2

LITERATURE REVIEW

As I considered the literature related to my dissertation, a substantial body of empirical and theoretical areas became apparent. Therefore, in this chapter, I provide a review of literature in several domains that informed the theoretical framework for my study. The first was literature related with discussion of problem solving and metacognition in a constructivist paradigm. The second area was that of mathematical problem solving. The third area was the literature that addressed metacognition, while the fourth domain addressed the research on metacognitive aspects of problem solving. The fifth area was that of problem solving in dynamic geometry environments. Finally, drawing on the past studies discussed in previous sections, I describe in the sixth section the framework used to conduct the research and analysis in this study.

Grounding Problem Solving and Metacognition in Constructivist Theory

This study was guided by a constructivist theoretical perspective. Constructivism is a theory of learning and knowledge development that assumes that knowledge is actively built up in the mind of the learner, not passively received, and that function of cognition is adaptive and serves as organization of experiential world, not discovery of an ontological reality (von Glasersfeld, 1982, 1984). Hence, constructivism assumes that "every knower has to build it [knowledge] up for himself. The cognitive organism is first and foremost an organizer who interprets experience and, by interpretation, shapes it into a structured world" (von Glasersfeld, 1982, p. 612). Moreover, in a constructivist context, the problem-solving activity is an act of knowledge construction. On the other hand, metacognition is a form of cognition that includes both knowledge of one's own cognition and regulation of one's behaviors in response to that knowledge (Brown, 1987, Flavell, 1976). In light of these two considerations, I view metacognitive processes in the forefront in problem solving and knowledge construction.

I view learning as activity where an individual uses, modifies, and creates schemes through the cognitive processes of assimilation and accommodation, as described in constructivist theory. According to von Glasersfeld (1980), *cognitive schemes* consist of three parts: an assimilated situation, an action or operation related to the situation, and a result of the action or operation. For a scheme to be initiated, an individual must receive a stimulus that allows him or her to recognize a situation relevant to his or her prior experience when the scheme was used (Hackenberg, 2010; Piaget, 1964; Steffe, 1994; von Glasersfeld, 1980). This process is known as *assimilation*, which is a basic ground for modification of an existing scheme and construction of a new superseding scheme. However, not every structure can already have a template ready to be recognized as relevant to a particular scheme. This occurrence is known as *perturbation* and is often followed by both positive and negative affective behaviors, such as surprise or frustration (Hackenberg, 2010). For eliminating the state of perturbation and achieving the state of *equilibrium*, accommodation, modification, or reorganization of a scheme must occur; that is, an individual needs to accommodate new knowledge into his or her existing knowledge made to his or her current understanding of the situation (Hackenberg, 2010; Piaget, 1964; Steffe, 1992). However, the individual needs to monitor his or her activities for reorganization of a scheme to happen (Steffe, 1992). If these accommodations involve coordination of two or more schemes, or a scheme and an operation, or operations external to that scheme, the result of so-called *functional accommodation* is a more powerful scheme that can be used to solve a problem that the scheme could previously not solve (Hackenberg, 2010).

This view of learning can be applied to problem-solving situations. One aspect of problem solving is the knowledge base, while a second is metacognition. Schemes, as noted already, are structures that are the basis for the knowledge, whereas metacognitive processes are used to choose the knowledge appropriate in the given situation. Variety of concepts, facts, strategies, and experiences come in play when solving a problem, as well as metacognitive behaviors that can make difference between problem-solving success or failure. A problem solver needs to understand what the problem is asking, analyze what needs to be done, choose and evaluate a particular strategy to solve the problem, and verify the reasonableness of the result. Hence, when given a problem, a problem solver is faced with an unfamiliar situation where his or her prior knowledge and experience influence how he or she interprets the problem. In order to solve the problem, the problem solver needs to decide what prior knowledge and strategies are relevant and apply them to a novel situation or think of novel composition of strategies available. However, this process is dynamic and cyclic, and the problem solver needs to use false moves as a feedback system until the goal is reached. Consequently, through the process of problem solving, new knowledge, and experiences are constructed.

As a problem solver for many years, I experienced that nonroutine problems can be difficult, given I did not possess preexisting procedures for solving them. In those situations I went off track quite often, but sometimes I managed getting on the right course and eventually was able to successfully solve the problem and sometimes I remained stuck. That phenomenon was a riddle for me for years, and to some extent it still is. Constructivists believe that problem solving actively builds knowledge and skills using schemes. Students construct their knowledge by creating their own internal representations. They bring information from their prior knowledge and modify the information they remember. A novel situation is produced, often preceded by an "aha!" experience or a feeling of suddenly knowing what needs to be done (Schunk, 2008). I believe that the ability to get on the right track when solving nonroutine problems often involves insight or the sudden awareness of a likely solution, and involves change of perception. What happened? According to constructivists, in such situations monitoring operations produce a systematic perturbation, giving the mind the opportunity to organize itself (Steffe, 1992; von Glasersfeld, 1984). This type of accommodation is known as a *metamorphic accommodation* or a *metamorphosis,* where a modification of a scheme occurs independently and not in any particular application of the scheme as a result of reflective abstraction (Steffe, 1992; Steffe & Cobb, 1988). This knowledge is constructed as the learner organizes experiences around preexisting mental structures or schemata; that is, knowledge is constructed in the mind of the learner as the mind organizes itself. In other words, insight occurs when a problem solver moves from not knowing how to reach problem's goal to a deep understanding of the problem and its solution. I firmly believe that looking at a dynamic relationship between knowledge and metacognition through constructivist lenses can help explain how individuals solve nonroutine problems.

Mathematical Problem Solving

What is Mathematical Problem Solving?

Problem solving is considered to be one of the most important aspects of today's mathematics curriculum. A deep and comprehensive view of problem solving as an art form emerged from works of Pólya (1945/1973, 1962/1981) and strongly endorsed the inclusion of problem solving in school mathematics. The National Council of Teacher of Mathematics (1980) through its report *An Agenda for Action* recommended "problem solving be the focus of school mathematics" (p. 1), which was the first of eight recommendations. Moreover, NCTM (1989,

2000) considered problem solving to be a main focus of school mathematics. NCTM (2000) stated:

> Solving problems is not only a goal of learning mathematics but also a major means of doing so. By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations. (p. 52)

That approach allows student to think strategically while at the same time learning mathematical content. Hence, NCTM (2000) seeks to have teachers prepare students for challenges of a new technological world by becoming mathematical problem solvers through developing their own ability to think mathematically and acquire mathematical power (the ability to do and have insight into learning mathematics). More specifically, students are required to be able to analyze the relationship among the different problem parts, represent the problem situation, decide on the solution path, monitor their progress, check the solution, and evaluate the reasonableness of the solution. That said, the report emphasized the importance of teachers promoting reflection because through these processes mathematical skills and ideas can be developed; that is, their role is not merely to help students solve problems but to help students learn how to develop processes needed for successful problem solving. Even though problem solving is now more prominent in school mathematics than ever before and there has been an ongoing research starting already in the 1970s and carried more intensively in the 1980s, Stanic and Kilpatrick (1989) nicely noted in their review of history of problem solving that though problems and problem solving have been a part of mathematics since antiquity, they just recently gained the increased attention of mathematics education community.

NCTM (1980, 1989, 2000) and many researchers (Lesh & Zawojewski, 2007; Lester 1994; Schoenfeld, 1992, 1994; Silver, 1982) have shown interest in problem solving through emphasis on its various benefits in teaching and learning of mathematics. Students are rarely given opportunities, however, to solve authentic, puzzle, or domain knowledge problems for which the students do not have an immediate solution pathway. Research has shown that problem solving as a starting point for mathematical investigations focused on mathematical reasoning and proof (Schoenfeld, 1994) has the potential to increase students' interest in mathematics and problem solving (Mayer, 1998), and can create perturbations essential for constructing new knowledge and building new meanings of mathematical ideas (Hackenberg, 2010).

Problem solving is an extremely complex human endeavor involving much more than the simple recall of facts, recall of concepts, or the application of well-learned procedures. Rather, it requires numerous cognitive activities and many types of knowledge. Skills in planning, monitoring, and revising strategies are as important as having a large domain of knowledge (Schoenfeld, 1985a). Moreover, Dewey (1933) considered reflective thinking to be a particular form of problem solving; thinking to solve a problem involved a certain ordering of ideas linking each with its predecessors. Problem solving for many students is not an easy task. Schoenfeld (1985a) noted that students often lack problem-solving skills; that is, students often perform meaningless calculations without giving much thought to the problem, solve problems without any planning and evaluating of their problem-solving approach, and give up easily if the problem is not solved in a short time as a result of affective emotions, such as frustration. Even though it is accepted that prior knowledge plays a role in successful problem solving, Schoenfeld argued that students lack metacognition.

Following Pólya's work, not only researchers in mathematics education but also in artificial intelligence made great contributions to the development of problem solving in the school curriculum through their influence on the changing nature of research emphasize and methodologies. For instance, Newell and Simon (1972) argued that human problem solving can be viewed as "an information processing system that manipulates symbolic structures" (p. 920). Through conjunction of AI and cognitive sciences they developed the General Problem Solver (or problem-solving space theory) model of problem solving in the form of a simulation program. They postulated the model consisted of *initial state* (current state of a person), *goal state* (state a problem solver wants to achieve), and *problem state* (state of all possible actions that can be applied to the problem). Through simulations of problem solving, Newell and Simon helped better understand mathematics problem solving.

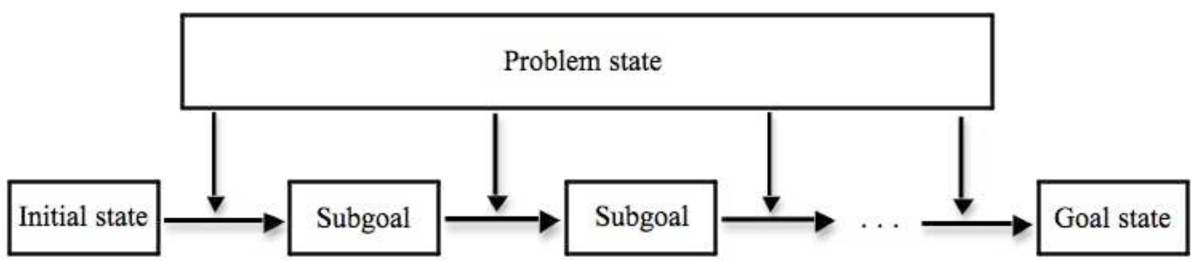


Figure 1. Problem-solving space.

Despite a plethora of research on problem solving, the field of mathematics education lacks agreement as to what constitutes a problem and problem solving, enabling valuable perspectives for future research (Schoenfeld, 1992). Most definitions of *problem solving* emphasize nonroutine problems that require problem solvers to use information and procedures in unfamiliar ways. For that reason, the following definitions for the term *problem* and *problem solving* are used for the purpose of this study. Problem by Lesh and Zawojewski (2007) is defined as, “A task, or goal-oriented activity becomes a problem (or problematic) when the 'problem solver' (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation” (p. 782), while problem solving was defined already in Chapter 1.

As noted earlier, many of the new curricula call for students to work on nonroutine problems over extended periods of time (Craine & Rubenstein, 2009). The nature of nonroutine problems allows multiple solution paths and eliciting a number of cognitive and metacognitive processes. Hence, the fundamental idea is that students will need to have opportunities to develop both the content and process understandings.

Problem-Solving Models

Various models are proposed that describe the processes that problem solvers use from the beginning until they finish their tasks (e.g., Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer, 2003; Montague & Applegate, 1993; Pólya, 1945/1973; Schoenfeld, 1981, 1985a). In the following discussion, I outline the problem-solving models of Pólya, Schoenfeld, and Garofalo and Lester that informed my study by explaining how students regulate their thinking during problem-solving situations.

Pólya’s Problem-Solving Model

One cannot talk about problem solving without starting with the work of Hungarian mathematician George Pólya, who started to investigate how students solve mathematical problems at the end of 20th century. Problem solving was a major theme of doing mathematics in his books, *How to Solve It* and *Mathematical Discovery,* in which he introduced the term *heuristic* to describe the art of problem solving. The term *heuristic*, or heuretic, was used to describe a field of study with an aim “to study the methods and rules of discovery and invention”

(Pólya, 1945/1973, p. 112). However, the educational use of the term *heuristic* has developed historically to describe a process. For instance, J. W. Wilson et al. (1993) view heuristics as "kinds of information, available to students in making decisions during problem solving, that are aids to the generation of a solution, plausible in nature rather than prescriptive, seldom providing infallible guidance, and variable in results" (p. 63), whereas interpretation by Lesh and Zawojewski (2007) is somewhat different, stating that heuristics involve strategies "intended to help problem solvers think about, reflect on, and interpret a problem solving situation more than they are intended to help them decide what to *do* when 'stuck' during a solution attempt" (p. 768). Pólya's problem-solving strategies such as working backwards, drawing a figure, looking for a similar problem, rephrasing the problem, identifying the givens and goals, and so on, are used for working on unfamiliar problems and have been advocated as important abilities for students to develop for successful problem solving (Lesh & Zawojewski, 2007).

Pólya (1945/1973) suggests a well-known model of problem solving that consists of four phases that are held to be essential in problem solving when applied to a mathematics problem:

Understanding the problem. Understand the verbal statement of the problem. Determine the unknown, data and condition. Analyze the data and condition. Draw a figure and introduce suitable notation. Separate specific parts of the condition for better understanding.

Devising a plan. Try to find a connection between the data and the problem. Consider whether the earlier methods or auxiliary problems can be used now. Develop a plan considering which calculations, computations and/or construction to perform in order to obtain the unknown.

Carrying out the plan. Examine the solution plan. Check each step of the plan carefully to make sure that each step is correct. Implement the solution plan.

Looking back. Reexamine and reconsider the solution by checking the result and checking the argument. Examine the reasonableness of the solution. Consider deriving the result differently, using the result, or the method or some other problem, or stating a new problem to solve.

Of course, this process is not linear, but rather is dynamic, cyclic, and iterative in its nature. False moves may occur, and the problem solver needs to simultaneously monitor his or her progress and go back to previous moves again and again, and change strategies, if necessary, until the goal is reached. For example, in the attempt of making a good plan the student may discover a need to understand the problem better; the student then has to go back to develop a new understanding of the problem. Figure 2 illustrates the dynamic and cyclic nature of Pólya's model, but adds problem-posing phase based on the work of Brown and Walter (cited in J. W. Wilson et al., 1993, p. 61). In this problem-solving model, any of the arrows can describe students' thinking processes during mathematics problem-solving activity.

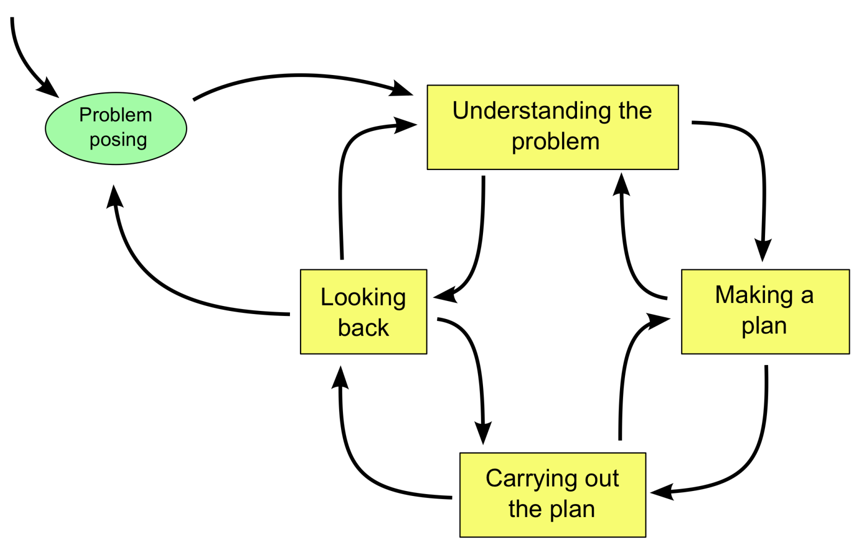


Figure 2. Dynamic and cyclic interpretation of Pólya's model.
Note. Reprinted from *Research ideas for the classroom: High school mathematics* (p. 62), by P. S. Wilson (Ed.), 1993, New York, Macmillan. Copyright 1993 by Macmillan. Reprinted with permission.

Even though Pólya in his writings never mentions the term *metacognition*, each phase of four-phased model of mathematical problem solving are metacognitive in nature. According to Silver (1982), "If we adopt a metacognitive perspective, we can view many of Pólya's heuristic suggestions as metacognitive prompts" (p. 21). Pólya expected students to think about various strategies (using representations, looking at special cases, working backwards, creating a simpler problem), ways of thinking, and patterns available to them (e. g., analysis and synthesis; variation; analogy; generalization and specialization; abstraction and concretization; distinguishing cases; superposition of particular cases; experiment; the method of undetermined coefficients; substitution and the method of recurrence relations). However, this is a simplistic interpretation of Pólya's writing (1945/1973, 1962/1981), and no theory developed so far can explain the complexity of his works (J. W. Wilson et al., 1993).

Pólya's problem-solving model resulted in a model for analyzing problem solving processes by Schoenfeld (1985a), who in his book *Mathematical Problem Solving* offers an extensive overview of his theoretical framework and methodological approach to the exploration of undergraduate mathematics problem solving as well as comprehensive descriptions of experimental and observational studies he conducted. Schoenfeld's model incorporated findings from research on problem solving by information-processing theories into Pólya's model.

Schoenfeld's Problem-Solving Model

Schoenfeld (1985a) proposed a framework for mathematical problem solving that relies on the theory that student ability to solve nonroutine problems is a function of how well they use and regulate relevant cognitive and affective characteristics. Schoenfeld (1985a) identified four categories of knowledge and performance fundamental for mathematics problem solving that are used in his model of problem solving: resources, heuristics, metacognitive control, and belief systems.

Resources refers to factual and procedural knowledge, that is, to the mathematical knowledge possessed by a student including facts and algorithmic procedures, routine nonalgorithmic procedures, facts, intuitions, and "understandings about the agreed upon rules for working in the domain" (Schoenfeld, 1985a, p. 15). These are the foundation upon which all the other categories depend. To illustrate the point, suppose a student who does not know how to find derivatives is attempting to solve a problem that requires knowing the slope of a tangent to a curve. Regardless of how clever or persistent the attempt, success is virtually impossible because the student does not possess the tools essential for success.

Heuristics refers to strategies used to explore, analyze, and probe aspects of nonroutine problems in an attempt to formulate pathways to a solution. Pólya (1945/1973) describes the tools and techniques for processes that have come to be called heuristics. These processes have generally been credited as being the fundamental framework for the development of most efforts to improve student problem-solving ability (Schoenfeld, 1985a). Examples of heuristic strategies include working backward, drawing figures, using diagrams, looking for patterns, reconstructing the problem, restating the problem, examining special cases, guessing and checking, creating an equivalent problem, and creating a simpler problem, and so forth, that are used for working on unfamiliar problems.

Metacognitive control refers to global decisions regarding the selection and implementation of resources and strategies, and *control* refers to "how one selects and deploys the resources at one's disposal" (Schoenfeld, 1985a, p. 13). That is, metacognitive control refers to the metacognitive behaviors (regulation of cognitive activities), such as decisions (decision-making) that problem solvers make regarding if (assessing), when (planning), and how (monitoring) they will use their factual knowledge, procedural knowledge (resources), and heuristics while attempting to deal with nonroutine problems.

Belief systems refer to the student's view about self, about the environment, about the topic, and about mathematics. *Beliefs about self* refers to student's feeling about himself or herself (e.g., confidence, perseverance), whereas *beliefs about mathematics* refers to what it means to think mathematically, how unfamiliar problems should be viewed, and the perceived usefulness of mathematics are issues related to this first kind of belief.

Most researchers nowadays in general accept a framework for mathematical problem solving similar to Schoenfeld's (1985a).

Garofalo and Lester's Problem-Solving Model

Garofalo and Lester (1985) devised a cognitive–metacognitive framework for studying young children's mathematical performance in order to determine appropriate mathematics problem solving instruction. Their framework is a blend of the work of Pólya, Schoenfeld, Sternberg, and Luria (cited in Garofalo & Lester, 1985, p. 170). The framework illustrates four categories of activities important in performing a mathematical task: orientation, organization, execution, and verification. Each category lists possible cognitive and metacognitive behaviors that are likely to influence cognitive actions relevant to that category.

Orientation refers to strategic behavior to assess and understand the problem. It includes comprehension strategies, analysis of information and conditions, assessment of familiarity with task, initial and subsequent representation, and assessment of level of difficulty and chances of success.

Organization refers to planning of behavior and choice of action. It includes identification of goals and subgoals, global planning, and local planning.

Execution refers to regulation of behavior to conform to plans. It includes performance of local actions, monitoring progress and consistency of local and global plans, and trade-off decisions (speed vs. accuracy, or degree of elegance).

Verification refers to evaluation of decisions made and of outcomes of executed plans entailing two components: evaluation of orientation and organization, and evaluation of execution. Evaluation of orientation and organization includes adequacy of representation, adequacy of organizational decisions, consistency of local plans with global plans, and consistency of global plans with goals, whereas evaluation of execution includes adequacy of performance of actions, consistency of actions with plans, consistency of local results with plans and problem conditions, and consistency of final results with problem conditions.

"The problem solver's ability to determine knowledge and strategies they posses in solving a task leads one to monitor one's task understanding and regulate one's strategy usage" (Garofalo & Lester, 1985, p. 165). That is, the model assumes that a student uses metacognitive knowledge about strategies and monitors those processes in order to successfully solve the problem. Furthermore, the amount of metacognitive behaviors used when solving a problem depends on the type of problem (Garofalo & Lester, 1985). For instance, working on a computation problem will require very little orientation and organization, but metacognitive activities will be prominent during the execution and verification phases. On the other hand, metacognitive activities related to the orientation and organization phases will be more evident when working on a word problem together with those related to execution and verification phases. Garofalo and Lester indicate that the transition from one phase to another occurs when solvers use their metacognitive decisions.

In summary, an operational definition of *problem* and *problem solving* by Lesh and Zawojewski (2007) is used for this study. Even though each of the presented models of problem solving are supported by many prominent mathematics educators, I decided to adapt the model by Pólya (1945/1973), Schoenfeld (1985a), and Garofalo and Lester (1985) and compile them into one. This decision was guided by a desire to obtain a model that allows capturing the variety of the student's cognitive and metacognitive practices.

Metacognition

What Is Metacognition?

In the last 40 years, mathematics educators have begun to focus on the role of metacognition in problem solving. This phenomenon has been studied in several areas, such as reading (Berkowitz & Cicchelli, 2004; Brown, 1978; Schunk, 2008), writing (Cross, 2009; Pugalee, 2001), memory development (Flavell, 1976), and mathematics problem solving (e.g., Campione, Brown, & Connell, 1989; Lester, Garofalo, & Kroll, 1989; Kramarski, Mevarech, & Arami, 2002; Schoenfeld, 1985a; Silver, 1982).

In the literature, terms such as *self-regulation*, *monitoring*, *control*, and *executive decisions* are frequently used interchangeably to describe the concept of metacognition. Often metacognition is explained as "thinking about one's thinking," "cognition about cognition," and "awareness of one's thinking." It is thought to be an elusive concept because of the difficulty distinguishing between cognitive and metacognitive processes. In this study, I use the definition by Flavell (1976) that clearly distinguishes between the two:

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning relevant properties of information or data. For example, I am engaging in metacognition … if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as a fact; if it occurs to me that I had better scrutinize each and every alternative in a multiple choice type task before deciding which is the best one. … Metacognition refers, among other things, to active monitoring and consequent regulation and orchestration of these [cognitive] processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective. (p. 232)

Flavell (1979) believed that metacognition occurs through actions and interactions among four classes of phenomena: metacognitive knowledge, metacognitive experiences, goals (or tasks), and actions (or strategies), where the last two help describe the first two components. Metacognitive knowledge refers to the interactions of the beliefs and knowledge with respect to cognitive tasks, goals, actions, and experiences. Metacognitive experiences refer to conscious cognitive or affective experiences that are related to personal intellectual enterprise before, during, or after task execution. Moreover, they have the potential to change one's knowledge base through processes of assimilation and accommodation of new information into long-term memory (Flavell, 1976). This is where metacognitive experiences and metacognitive knowledge overlap, "Some experiences have such knowledge as their content and some do not; some knowledge may become conscious and comprise such experiences and some may never do so" (Flavell, 1979, p. 908). Goals (or tasks) refer to the objectives of a cognitive enterprise (regulation and control of cognitive activities), where actions (or strategies) refer to behaviors employed to achieve them. It is through actions and interactions among four classes of phenomena that individuals know about what they know and who they are in relation to a task or event, why they engage or withdraw from a task or event, and how they attempt to accomplish the task or event.

Metacognition is related with cognitive development, and linked with the age of a person; that is, it follows the same trajectory as Piaget's theory of child's cognitive development (Schunk, 2008). The research shows, however, that even when students are able to monitor their work, they often do not do so (Schunk, 2008). Flavell (1979) explained that this failure occurs because metacognition is not an automatic process nor a spontaneous way of thinking for most students. Rather metacognition is a result of development of the cognitive system and years of accumulated experience in making thought the object of thinking; that is, students who have been taught and encouraged to behave in such a way start to routinely think about their thinking.

Metacognitive Theories

Four constructs are typically associated with the concept of metacognition: knowledge of cognition, regulation of cognition, beliefs about cognition, and awareness of cognition (Brown, 1978, 1987; Flavell, 1976, 1979; Schoenfeld, 1987, 1988; J. Wilson & Clarke, 2002, 2004). In the following discussion, I review each of the four constructs, where beliefs about cognition are mainly integrated within other constructs.

A first construct of metacognition as described by Flavell (1979) and Brown (1978) is *knowledge of cognition*. Knowledge of cognition refers to stable, expressible, and fallible information that an individual knows about his or her own cognition or about cognition in general. According to Brown, knowledge of cognition consists of three categories of metacognitive awareness: declarative, procedural, and conditional knowledge. Declarative knowledge refers to knowing about things (e.g., facts); procedural knowledge refers to knowing how to do things (e.g., rules, algorithms); and conditional knowledge refers to knowing when and why to apply the previous two forms of knowledge. Furthermore, Brown contends that metacognitive knowledge is age dependent but not context dependent but remains stable consistent within individuals.

Flavell's Theory of Metacognitive Knowledge

Metacognitive knowledge refers to one's "stored world knowledge that has to do with people as cognitive creatures and with their diverse cognitive tasks, goals, actions, and experiences" (Flavell, 1979, p. 906). That is, metacognitive knowledge refers to personal awareness of how one thinks, where this type of knowledge resides in one's long-term memory. It consists of "knowledge or beliefs about what factors or variables act and interact in what ways to affect the course and outcome of cognitive enterprises" (p. 907). These factors or variables influencing students' learning and problem solving are divided in three major categories: person, task, and strategy. Person variables or knowledge refers to beliefs about one's own and others' thinking. This category can be further subcategorized into beliefs about intraindividual differences (e.g., I learn better by listening than by reading), beliefs about interindividual differences (e.g., one of my friends is more socially sensitive than another), and beliefs about universals of cognition (e.g., there are various degrees and kinds of understanding). Task variables or knowledge consists of two categories: metacognitive knowledge about the demands and goals of a cognitive task, and knowledge about what information is available during a cognitive enterprise. Depending on what information is available (e.g., abundant or meager, familiar or unfamiliar, redundant or densely packed, and well or poorly organized), metacognitive knowledge refers to how variations in the available information may affect the outcome of the task and, therefore, how a cognitive enterprise should be managed to attain the goal given what information is available. Lastly, strategy variables or knowledge refers to metacognitive knowledge of what cognitive and metacognitive strategies are likely to be effective to achieve a specific goal in a given cognitive undertaking.

Finally, metacognitive knowledge is concerned with interactions or combinations among two or three of these types of variables. Flavell (1979) believed that this type of knowledge can have a myriad of important effects on the cognitive enterprises of children and adults: (1) evaluating, revising, and abandoning cognitive tasks, goals, and strategies based on the relationship between them and one's own abilities and interests; (b) metacognitive experiences concerning oneself, goals (or tasks), and actions (or strategies); and (c) interpreting the meaning and implications of these metacognitive experiences.

In summary, researchers believe that skilled learners possess three types of knowledge about cognition as described by Brown (1978) through which one improves one's own performance. Metacognitive knowledge starts developing at an early age, and continues to develop at least throughout the adolescent years, whereas adults possess more knowledge about their own cognition and are better able to describe it (Schunk, 2008). Even though knowledge of cognition is not necessarily easy to explicitly describe, it may facilitate thinking and self-regulation (Schraw, 1998; Schraw & Moshmann, 1995).

A second construct of metacognition as described by Flavell (1976), Brown (1978, 1987) and Schoenfeld (1987) is *regulation of cognition*. Regulation of cognition is the executive component of metacognition that involves variety of activities used by an individual in order to control his or her own cognition. Theories and research in this category are rich and offer various models (e.g., Brown, 1987; Garofalo & Lester, 1985; Schoenfeld, 1981; Zimmerman, 2002) of different cognitive and metacognitive processes students exhibit when solving problems. In the following section, I discuss models by Brown (1987), Schoenfeld (1981) and Zimmerman (2002).

Brown's Theory of Regulation of Cognition

Brown's (1987) theory examines different components of the regulation of cognition, where she defines *metacognitive regulation* as activities used to help control one's thinking or learning. Three different components of regulation of cognition are planning of cognitive activities, monitoring of cognitive activities, and evaluating the outcomes of former activities. Planning entails selection of appropriate strategies and allocation of resources that affect performance, such as making predictions, strategy sequencing, and trial-and-error. Monitoring refers to one's own awareness of comprehension and task performance, such as testing, revising, and rescheduling one's learning strategies. Checking outcomes (evaluation) refers to assessing the results and assessment of regulatory processes of one's learning, such as reevaluation of goals and conclusions. These regulatory processes can be affected by affect, such as fear, anxiety, interest, self-esteem, and self-efficacy. In contrast to Brown (1987), who argues that with the growth of problem-solving skills, executive components regulate metacognitive processes without conscious control and depend on a context, Flavell (1979) contends that a learner is consciously aware of utilizing thinking strategies. Brown (1987) argued that these regulatory processes may not be necessarily stable, conscious, or expressible in learning situations. Schraw and Moshmann (1995) and Schraw (1998) theorized that disagreement was a result of development of the processes without conscious reflection and was automated. In addition, Brown (1987) introduces the construct of *automatic state*, where the problem solver acts as an *automatic pilot*; planning, monitoring, and evaluating activities become automatic. This construct helps to explain why problem solvers sometimes are not able to describe their problem-solving strategies and metacognitive knowledge; however, they are confident they solved the problem correctly.

Zimmerman's Theory of the Self-regulated Learner

Zimmerman (2002) defined *self-regulation* as "not a mental ability or an academic performance skill; rather it is the self-directive process by which learners transform their mental abilities into academic skills" (p. 65). He views learning in a proactive way, where students learn for themselves and not as a reaction to the action of teaching. Self-regulation of learning entails selective use of the following component skills: setting specific proximal *goals* for oneself, adopting powerful *strategies* for attaining the goals, *monitoring* one's performance selectively for signs of progress, *restructuring* one's physical and social context to make it compatible with one's goals, managing one's *time use* efficiently, *self-evaluating* one' s methods, *attributing* causation to results, and *adapting* future methods.

Zimmerman's (2002) model of self-regulation learning is a three-cycled structure consisting of a forethought phase, a performance phase, and a self-reflection phase, where each phase includes two subprocesses. The forethought phase consists of two classes of processes: task analysis processes (e.g., goal setting, strategic planning) and processes related to self-motivation beliefs (e.g., self-efficacy, outcome expectation, intrinsic value, learning goal orientation). The performance phase consists of self-control processes (e.g., imagery, self-instruction, attention focusing, task strategies) and self-observation processes (e.g., self-recording, self-experimentation). The self-reflection phase consists of self-judgment processes (e.g., self-evaluation, casual attribution) and self-reaction processes (e.g., self-satisfaction/affect, adaptive/defensive). Within this cyclical model, self-regulatory processes occur at different points within the learning process; that is, the forethought phase occurs before learning efforts, the performance phase occurs during behavioral implementation of learning, and the self-reflection phase occurs after each learning effort. Zimmerman (2002) in his model of self-regulation emphasizes the role of reflection. For him, reflexive thinking is a complex practice but it is the foundation of metacognitive elements, such as metacognitive awareness, metacognitive strategies, metacognitive knowledge, feeling of knowing, and other, that have the power to enhance problem solving efforts.

Schoenfeld's Model of Episodes and Executive Decisions in Mathematics Problem Solving

According to Schoenfeld (1987) regulation of metacognition or control or self-regulation includes four components: (1) fully understanding the problem before attempting a solution, (2) planning, (3) monitoring, and (4) allocating resources (e.g., deciding what to do, for how long). In the following paragraphs I describe Schoenfeld's problem-solving model for which these four constructs are a great part.

Whereas frameworks by Brown, Lester and Garofalo, Pólya, and Zimmerman outline distinct phases of activity, Schoenfeld (1981) described-problem solving behaviors without creating clusters of categories. Schoenfeld presented a framework for analyzing problem-solving protocols (cognitive and metacognitive actions) students use to solve mathematical problems that was developed as a result of findings from research on problem solving by information-processing theories and his own experimental and observational studies conducted into Pólya's model. Problem-solving protocols (transcripts) are parsed into episodes that present periods of time during which a problem solver is engaged in a particular activity, such as exploring different possibilities or planning the best solution. The resulting model is characterized by the following episodes: reading, analysis, exploration, planning/implementation, and verification, together with junctions between episodes (transitions). Decision-making behaviors are analyzed by examining each episode and the transition between them using a set of predetermined questions attributed to each episode or transition. However, this model fails to address local indications of metacognitive behaviors (Artzt & Armour-Thomas, 1992; Schoenfeld, 1985b).

In a reading episode, student reads the problem statement silently or aloud, may note conditions of the problem, state the goals of the problem, and assess his or her current knowledge relative to the task. In the analysis episode, the student attempts to understand the problem, decompose the problem in its basic elements, examine the relationships between the given information, conditions and the goals of the problem, and choose appropriate perspectives to solve the problem using different strategies to assist him or her (e.g., draw diagram if at all possible, examine special cases, simplify problem). Whereas an analysis episode is well-structured, an exploration episode is less structured and removed from the given problem. In an exploration episode, the student searches for relevant information that can be used in an analysis, planning, and implementation sequence relying on his or her previous knowledge and experience. A variety of problem-solving heuristics may be used, such as consider equivalent problems, consider slightly modified problems, or consider broadly modifying problems. Because of the nature of the episode, new approaches may occur, and if the student decides to follow a new approach, the coding is closed, transition is denoted, and another exploration episode is opened. In a planning/implementation episode, the student creates a plan and implements it. A planning episode engages students in the selection of steps and strategies for a solution plan, drawing from previous knowledge, experiences and as a result of being engaged in any of the problem solving episodes, whereas an implementation episode involves executing a strategy or a plan made through the student's understanding, analysis, or planning phases in finding a solution through computation or proving a conjectures. During this episode it is important whether the plan is well-structured, the implementation is high quality and most importantly the current solution state is monitored or assessed both locally and globally. In a verification episode, the student reviews and tests whether his or her solution passes specific or general tests in relation to requirements of the problem. Besides assessing the correctness of a solution and perhaps considering how to apply the result to another problem, the student may also assess the aesthetic quality of the solution. A transition episode is a junction between the other episodes and occurs only when a student assesses the current solution state and makes decisions about pursuing a new direction to solve the problem. According to Schoenfeld (1981), this is the episode where managerial decisions or their absence "will make or break a solution" (p. 26).

A third construct of metacognition as described by Flavell (1976, 1979), Brown (1978), Schoenfeld (1987), and J. Wilson and Clarke (2002, 2004) is *awareness of cognition.* Awareness of cognition refers to "individuals' awareness of where they are in the learning process or in the process of solving a problem, of their content-specific knowledge, and of their knowledge about their personal learning or problem solving strategies" (J. Wilson & Clarke, 2004, p. 27). Moreover, it entails knowledge of what has been done, what needs to be done, and what might be done in order to attain a specific goal related to a learning or a problem-solving situation. That is, it involves "an individual's cumulative knowledge of acquired competencies and on-going knowledge of mental processes in progress" (p. 27). For that reason, work in this area is of great importance. If educators are interested in helping students develop these skills, "it is important to know how likely it is that students will reflect on their thinking and how accurate those reflections will be" (Schoenfeld, 1987, p. 190). These skills have proved to be important during problem-solving activity; if a problem solver is not aware of what he or she knows, being an efficient problem solver comes into question (Schoenfeld, 1987).

A fourth construct of metacognition as described by Schoenfeld (1987, 1988) is *beliefs and intuitions about cognition.* Beliefs about cognition refer to broadly agreed upon ideas and theories one has about his or her own and other people's thinking. Moreover, it refers to capacities, effectiveness, and limitations of one's own thinking in a particular problem-solving situation as a result of self-attributes, such as attribution, self-esteem, and motivation. He argues that students' beliefs about cognition shape cognition and determine the way they solve problems. For instance, if a student believes that every mathematical problem can be solved within 5 minutes or is unrelated to real life, discovery, and problem solving, he or she may stop working on a problem (Schoenfeld, 1988). Hence, beliefs and intuitions, such as self-awareness and self-regulation, are indicators that may explain students' mathematical behavior. Even though Schoenfeld (1987) distinguishes between three constructs of metacognition—knowledge about one's own thinking, control, and beliefs—he places emphasis on the importance on all of the three constructs for successful problem solving;

> "Knowing" a lot of mathematics may not do students much good if their beliefs keep them from using it. Moreover, students who lack good self-regulation skills still may go off on wild goose chases and never have the opportunity to exploit what they have learned. (p. 198)

In light of the theoretical frameworks presented here, research on metacognitive aspects of problem solving was considered to have a great promise by researchers. "There is little doubt that intervention research based on metacognitive principles has been remarkably successful in improving [childrens'] performance on a range of academic tasks" (Reeve & Brown, 1984, p.

20). Therefore, in the following section, I review research in an effort to explain what role metacognition plays in mathematical problem solving.

Research on Metacognitive Aspects of Problem Solving

Different aspects of metacognition in problem solving became the focus of research in the late 1980s (Lester, 1994). Researchers investigated metacognitive processes students engaged in during problem solving that resulted in several cognitive-metacognitive frameworks (e.g., Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Schoenfeld, 1981, 1985a); the role of metacognition in problem solving (e.g., Artzt & Armour-Thomas, 1992; Kilpatrick, 1985b; Lesh, 1982; Lester et al., 1989; Mayer, 1998; Silver, 1982; Schoenfeld, 1982, 1992); the role of writing on metacognitive processes during problem-solving activities (e.g., Cross, 2009; Pugalee, 2001); the role of noncognitive factors, such as affective states in the problem-solving processes (e.g., Carlson & Bloom, 2005; Goldin, 2000; Kilpatrick, 1969; Mayer, 1998; Montague & Applegate, 1993); and the effects of metacognitive instruction on problem solving (e.g., Cardelle-Elawar, 1995; Hoek, van den Eeden, & Terwel, 1999; Kramarski, Mevarech, & Arami, 2002; Lester et al., 1989; Mevarech, 1999; Mevarech & Kramarski, 1997, 2003; Schoenfeld, 1987, 1992).

Research on the role of metacognition in problem solving considered metacognitive processes as “driving forces” that influenced cognitive behavior at all stages of problem solving (Lester, 1994), but were also connected with a wide range of noncognitive factors, such as beliefs, attitudes, and motivation (Kilpatrick, 1969; Mayer, 1998; Schoenfeld, 1985a). The role of metacognition in problem solving was examined through two related components: metacognitive knowledge (knowledge of one’s own thought processes) and regulation and monitoring of thought processes during problem solving (Lester, 1994).

One of the first studies on the role of metacognition in problem solving was a project by Lester et al. (1989). Lester et al. (1989) conducted an extensive study on seventh graders' mathematical problem solving. In their study, they investigated qualitative aspects of mathematics in mathematical problem solving of middle school students; that is, the study was designed to study the role of metacognition, both knowledge and control of cognition. The students were taught by one of the investigators 3 days per week for a period of 12 weeks. The investigator played three different roles: an external monitor, a facilitator of metacognitive development, and a model of a metacognitive problem solver. Data collection included videotapes of students working individually or in pairs, journals, and individual interviews. Results indicated that among the four categories of their framework (Garofalo & Lester, 1985), the orientation phase had the most effect on students' problem solving. Mathematical journals and the pair problem-solving method (thinking aloud, questioning) were revealed to be effective as they helped students develop conceptual understanding and metacognition. Moreover, Lester et al. found that the metacognitive instruction was most effective when it was provided in a systematically organized manner over a prolonged period of time under the direction of the teacher. This study influenced subsequent related research studies in the next two decades.

Research conducted by Schoenfeld (1981, 1985a, 1987, 1992, 1994, 2010) in the last 30 years created a vast opus of work on various aspects of problem solving, such as metacognitive processes, knowledge, attitudes and beliefs, and heuristics. Schoenfeld's research entailed teaching a course on problem solving at the undergraduate level that offered an explicit instruction on heuristics (teaching about problem solving). As a result, he noted that students can learn to use a variety of problem-solving strategies as suggested by Pólya (1945/1973) that can subsequently improve their problem-solving performance. Moreover, he observed that the way mathematics is taught shapes not only their views about mathematics, but their mathematical behaviors. For instance, instruction that gives emphasis to understanding the problem and its analysis is portrayed in students' problem-solving behaviors. However, he found that some students are not able to solve nonroutine problem successfully even when they possess sufficient knowledge for solving it as a result of lack of monitoring skills. He hypothesized that instruction giving emphasis to regulation of cognition can improve students' problem-solving performance, which became his focus later on.

Schoenfeld (1987, 1992) conducted research with undergraduate students on the role of metacognition in terms of monitoring and controlling learning and problem solving. In his problem-solving course students received explicit instruction on monitoring their progress when problem solving. Explicit instruction entailed a set of questions that aimed at students' knowing the what, why, and how aspects of problem solving; "What (exactly) are you doing? (Can you describe it precisely?), Why are you doing it? (How does it fit into the solution?), [and] How does it help you? (What will you do with the outcome when you obtain it?)" (1992, p. 206). This type of instruction influenced not only their mathematical behaviors but improved their problem-solving success.

Lester (1994) in an overview of mathematical problem solving research from 1970 to 1994 reports on generally accepted results on the role of metacognition in problem solving:

1. Effective metacognitive activity during problem solving requires knowing not only what and when to monitor, but also how to monitor;
2. Teaching students to be more aware of their cognitions and better monitors of their problem-solving actions should take place in the context of learning specific

mathematics concepts and techniques (general metacognition instruction is likely to be less effective); and

3. The development of healthy metacognitive skills is difficult and often requires "unlearning" inappropriate metacognitive behaviors developed through previous experience. (pp. 666-667)

Many researchers used Schoenfeld's theory or extended it to further explore the role of metacognition in problem solving (e.g., Artzt & Armour-Thomas, 1992; Lawson & Chinnappan, 1994, 2000). Artzt and Armour-Thomas (1992) examined the role of cognition and metacognition in a small-group problem-solving setting, where modification of Schoenfeld's (1985a) framework was used to delineate problem-solving processes. Results suggested that problem solving without attention given to choosing and planning activities, and their monitoring is unsuccessful. They suggested lack of these processes is the main difficulty in students' problem solving, where the synergy of both cognitive and metacognitive behaviors is necessary. Furthermore, the data suggested the dynamic nature of problem solving; that is, students go back and forth between episodes using different heuristics. Finally, Artzt and Armour-Thomas suggested a feasible framework for research on mathematical problem solving.

In a study of geometry problem solving, Lawson and Chinnappan (1994) studied the interplay between the quality of low- and high-achieving students' knowledge organization and their problem-solving performance. Five major categories of processing events involved in problem-solving activity were identified: identification of given information, problem control or management, generation of new information, self-assessment (metacognitive knowledge use), and error. Lawson and Chinnappan indicated that during geometry problem solving, students often fail to solve problems they might have solved because they waste a great deal of time and effort pursuing inappropriate directions. Compared to low-achieving students, high-achieving students activated greater number of identification, managements, and generation events and consequently produced more knowledge and information, which gave them better problem-solving performance. Furthermore, low-achieving students could retrieve certain relevant knowledge only with retrieval prompts, suggesting that ineffective knowledge organization hindered their problem performance. That is, success in problem-solving performance depends greatly on the problem solver's ability to retrieve more knowledge and activate links among knowledge schemas and related information (Lawson & Chinnappan, 2000).

J. Wilson and Clarke (2004) developed a multi-method technique to better understand student mathematical metacognition. Metacognition was parsed into three different constructs: metacognitive awareness, metacognitive evaluation, and metacognitive regulation. Although they were able to describe metacognitive actions (purposeful activity) through students' self-reports during problem solving showing that students use metacognitive language to describe awareness-evaluation-regulation cycles, they were unable to relate these metacognitive behaviors to improved future problem solving performance (Lesh & Zawojewski, 2007).

In the last decade, there has been ongoing research on the role of writing on metacognitive processes during problem solving (e.g., Pugalee, 2001; Cross, 2009). Pugalee (2001) conducted a qualitative research study on 20 ninth-grade students to describe and examine the role of writing in the mathematical problem-solving process. The framework by Garofalo and Lester (1985) was used to identify metacognitive behaviors in the four-stage model of problem solving processes. Results obtained revealed that students' written descriptions (their solution thinking, mathematical problem solving processes, ideas coming to their mind) showed evidence of various metacognitive behaviors during mathematical problem-solving as described by Garofalo and Lester (1985). Metacognitive behaviors were most important during and initial understanding of the problem (orientation), and for moving students towards identifying a successful solution plan (organization). The results revealed that writing allows demonstration of mathematical reasoning and metacognitive behaviors during the process of problem solving. More study is needed, however, that would reflect how writing influences metacognitive behaviors of students (Pugalee, 2001). Cross (2009) conducted a multi-method study examining the individual and combined role of writing and argumentation on mathematical achievement. Results of the study revealed that students who were engaged in either of the activities had greater achievement than students who had none at all, and students engaged in argumentation and writing had greater knowledge gains than students who were engaged in argumentation alone or in neither activity. The qualitative analysis revealed that writing activities helped the writing group to develop a deeper conceptual understanding of their current knowledge, whereas the same served as a heuristic for the argumentation-writing group. Hence, writing is a challenging cognitive process that requires a careful examination of one's thinking that one wants to articulate.

The interrelationship between metacognition, and cognition with noncognitive factors, such as affective behaviors (e.g., attitudes, beliefs, emotions, values) during problem solving is of high significance and may influence students' problem-solving performance (Carlson & Bloom, 2005; J. Wilson & Clarke, 2004). Although, the role of affective attributes, such as motivation, interest, pleasure, impatience, anxiety, and persistence in problem solving has a long history, it is not stressed often (Mayer, 1998). Mayer (1998) borrows from interest theory, self-efficacy theory, and attribution theory when arguing that "the will to learn depends partly on how the problem solver interprets the problem solving situation" (p. 56). *Interest* refers to natural curiosity or willing participation in activity. According to interest theory, students engage in deeper thinking when they are interested rather than uninterested in a particular activity. *Self-efficacy* refers to an individual's judgment of his or her own capabilities. According to self-efficacy theory, students who have high self-efficacy understand the material better and are more successful in problem solving. *Attributions* refer to the explanations one uses to make for success (e.g., effort) or failure (e.g., difficulty). One's feelings and beliefs about one's interest and ability to solve problems can help or hinder the problem-solving process (Mayer, 1998). Similarly, Goldin (2000) refers to the affective domain more broadly as a tetrahedral model including beliefs and belief structures, attitudes, emotional states, values, ethics and morals. Affective states are described as "local changing states of feeling that the solver experiences and can utilize during problem solving—to store and provide useful information, facilitate monitoring, and evoke heuristic processes" (p. 209). Affective states interact productively or counterproductively with problem solving and include curiosity, puzzlement, bewilderment, frustration, anxiety, fear and despair, encouragement and pleasure, elation and satisfaction. "Affect is not incidental but fundamental" (p. 218) during problem solving; it can foster mathematical ability and creativity, but it can also inhibit the problem-solving process and have consequences in future problem-solving situations. Moreover, Carlson and Bloom (2005) showed that effective management of negative affective behaviors, such as anxiety and frustration, was instrumental for participants' perseverance during problem solving. These studies point in an important direction focusing on understanding the influence of complex construct of affect during problem solving and extend it to characterizing these affective states and their use during problem solving.

At the same time that researchers were exploring the role of metacognition in problem solving, parallel research was conducted on the effect of metacognitive instruction on problem solving. In this regard, attention has been given to strategies that promote development of thinking, metacognitive skills and argumentation skills. Metacognitive instruction entailed teachers directly modeling the problem-solving process and metacognitive strategies, such as self-questioning (comprehension, connection, strategic and reflection questions), self-monitoring, self-evaluation, and organizing relevant knowledge for their students. Studies (e.g., Kramarski et al., 2002) have shown that students who received metacognitive instruction were better able to organize their knowledge and explain their reasoning than students who did not receive metacognitive instruction. The following are some instruction strategies that have been found over the past 25 years to be effective in metacognitive instruction: effective questioning techniques (Kramarski et al., 2002; Mevarech & Kramarski, 1997, 2003; Lester et al., 1989; Schoenfeld, 1987, 1992), mathematical journals (Lester et al., 1989), and pair and group problem solving (Kramarski et al., 2002; Lester et al., 1989). Despite positive results reported from these studies, it is yet unknown why the performance improved. Was the improvement due to learning metacognitive processes, or were concepts learned better or differently? (Lesh & Zawojewski, 2007).

In summary, research on the metacognitive aspects of problem solving reports on several different components: knowledge base and retrieval of metacognitive knowledge, metacognitive experiences, and metacognitive instruction supporting the former two. The knowledge base and retrieval of metacognitive knowledge amplifies efficient problem-solving attempts. Metacognitive experiences allow students to capitalize on their experience, where the execution of a cognitive action prompts metacognitive experience. However, each component alone does not ensure productive problem solving; that is, students need instruction in how to manage their metacognitive knowledge and metacognitive experiences. Even though there has been extensive research on metacognitive aspects of problem solving, it is still unknown how and why metacognitive behaviors emerge, to what extent students act metacognitively, and to what degree metacognition influences problem-solving activity, in both desirable and unproductive ways.

Problems Solving in Dynamic Geometry Environments

In the past two decades a new genre of educational software has become prominent as a classroom tool to support the teaching and learning of geometry. This type of software has been labeled *dynamic geometry software or system* (DGS) or *dynamic geometry environments*, a term coined by Nick Jackiw (Olive & Makar, 2010), describing in its name the main features of the software: direct manipulation of geometric figures possible via a pointing device, for instance, by dragging parts of the figure (Hoyles & Noss, 2003). DGS such as GSP uses a few primitive building objects such as points, lines, line segments, rays, circles, and polygons, and the interface of a DGS creates an opportunity to transform a mathematics classroom into an environment of investigation of interesting phenomena where students engage in observing, manipulating, predicting, conjecturing, testing, and developing explanations for observed phenomena (NCTM, 2000, 2005). For instance, once objects are constructed, the dynamic feature of the software allows for the objects to be dragged via any of their constituent parts while maintaining their constructed properties and underlying geometric relationships, in contrast to a paper-and-pen representation. This feature creates a kind of feedback to the user and an opportunity for a perturbation that was not available before the introduction of DGS (Olive & Makar, 2010). Olive and Makar (2010) provide an in-depth discussion through constructivist lenses on the potential for DGS to transform the way mathematics is learned.

As this type of software allows for direct construction, manipulation, and measurement of geometric figures, it became increasingly popular in mathematics education; particularly it found a role in the geometry curriculum and made a contribution to problem-solving strategies (Fey et al., 2010; Olive & Makar, 2010; J. W. Wilson et al., 1993). In understanding the problem, the students can use various functions of the software, such as making a representation of the problem to help understand what is meant by the problem statement or parts of the problem. In the exploration phase, where the student is searching for a solution plan, the software can be used to make possible explorations or enhance explorations of a problematic situation by helping the student gather relevant data that can be used later on in analysis, planning, or implementation, to consider various problem-solving strategies (J. W. Wilson et al., 1993), and to test their conjectures that may play an important role in devising a solution plan (Hölzl, 2001). Furthermore, educational experiences with the use of technology can enhance the education experience of individual's building of conjectures. NCTM (2000, 2005) states that teachers should create opportunities for students that foster such students' mathematical power, and consistently challenge and encourage them by raising questions and formulating conjectures. In the planning phase, the software can be used to examine variety of strategies, to assess the plan, and to revise it if needed. Although during an implementation episode, the student is involved in the execution of a strategy or plan, the software can help with more complicated questions that go beyond student's competency, such as noting where the problem solving activity is leading. In the verification episode, capabilities of the software can be used extensively. The student can not only test the obtained result and the strategy used, but also extend the problem, extend the solutions, extend the process and develop self-reflection (J. W. Wilson et al., 1993). Hence, the exploratory nature of DGS provides students with a genuine problem-solving activity, which is expected in technology use in mathematics education.

Furthermore, Goldenberg et al. (1988) argue that providing the opportunities and dynamic tools for students' explorations promotes the habits of mind that constitute true mathematical power. According to Goldenberg et al. (1988) and Cuoco, Goldenberg and Mark (1996), habits of mind are the ways of thinking that are there to allow students to develop a myriad of approaches and strategies that can be applicable in situations varying from challenges in school to those in life. Some of the habits of mind they discuss include but are not limited to looking for patterns, exploring, communicating, argumenting, conjecturing, refuting, and generalizing. The habits are there to enlighten students about the creation of mathematics, and most importantly to help them learn the way mathematicians think about mathematics. Consequently, students' engagement in these habits helps develop and increase their ability to determining on their own how to think mathematically; deciding what information is needed, choosing a particular strategy, testing their conjectures, and examining what is learned and how it can be applied to a different problem-solving situation. On the down side, users may tend to abuse the power of DGSs. Instead of appreciation for the structure of the system, the system becomes a tool in the hands of the user; the user uses the system because he or she wants to get the problem done which takes away the cognitive load of mathematical thinking (Hoyles & Noss, 2003; Olive & Makar, 2010).

At the beginning, the research on DGSs was primarily focused on their potential as a conjecturing tool to investigate students' processes of construction in geometrical contexts (Goldenberg & Cuoco, 1998). Later studies raised new issues of constraints and student difficulties of interpreting and constructing while attending to the multiple functionality of the dragging mode and considerating how the tools reconcile the nature of explanation, verification, and proof (Hoyles & Noss, 2003; Zbiek & Hollebrands, 2008). Goldenberg (2001) illustrated that DGSs can also generate new problem-solving heuristics. For instance, his work draws attention to the heuristic discussed by Pólya (1945/1973), the strategy of relaxing constraints of a problem, that allows the problem solver to construct a set of solutions to an open problem in which the solution to the original problem appears as a particular case.

Hollebrands (2007) investigated the ways in which high school students use the GSP mediating the understanding that they developed about geometric transformations by focusing on their uses of technology. Three themes related to use of technology emerged from what was reported: students' mathematical interpretations as a result of dragging, students' mathematical interpretations as a result of measuring, and students' uses of proactive and reactive strategies. The main difference between the two strategies was whether the choice of action was in response to what computer produced or already in advance of what the student anticipates the software was supposed to do. The results of the study showed that the feedback provided by the GSP was critical for the students' later actions. Therefore, that study points in an important direction with a focus on understanding how DGSs, such as the GSP influence student learning in mathematics.

Zbiek, Heid, Blume, and Dick (2007) provided an in-depth summary of existing research on technology in mathematics education presented through several different constructs focusing on five interest points: the student, the teacher, curriculum content, the mathematical activity, and the tool. For the purpose of this report, I focus on what informed the present study, that is, a critical role technology plays among student, content, and activity. Zbiek et al. differentiate two different types of mathematical activities — procedural and conceptual activity — based on the technical dimension of mathematical activity. A procedural mathematical activity relies on mechanical procedures where new knowledge or concepts are acquired through procedures and facts. In contrast, a conceptual mathematical activity relies upon understanding, communicating, and making connections between and among mathematical concepts. In such activity, processes such as pattern recognizing, conjecturing, generalizing, and abstracting are noticeable.

Artigue (2002) reports on a decade of French research projects that focused on the use of graphing programs and computer algebra systems in the teaching of high school mathematics. Two aspects of her report need to be highlighted: the instrumental approach and framing schemes. The instrumental approach to tool use incorporates ideas borrowed from both the theory of instrumentation developed in cognitive ergonomics and the anthropological theory of didactics initiated by Chevallard. The main tenet in the instrumental approach is the difference drawn between an artifact and an instrument: The artifact is the object that is used as a tool, whereas the instrument involves techniques and schemes developed by the user during its use (instrumentalisation) that then guide the way the tool is being used and the user's thinking (instrumentation). *Instrumentalisation* is a psychological process that refers to the construction of schemes directed towards the technological tool; that is, it refers to the appropriation and transformation of the tool by the student. *Instrumentation* is directed towards mathematical concepts; that is, it refers to the effects of tool use on the student activity, where the use of a certain tool becomes internalized as the user executes the task. When an artifact becomes an instrument through the process of instrumentalisation and instrumentation, then this process is called *instrumental genesis*. The notion of instrumental genesis is similar to the theory of shaping by Hoyles and Noss (2003), who describe student-tool interaction as a two-way process in which the tool shapes the thinking of the user, but is also shaped by the user's thinking.

Researchers (e.g., Artigue, 2002; Lagrange, 2003; Thompson, 2009) have recognized the instrumental approach as a promising framework within the context of using a computer algebra system (CAS) in mathematics education. Artigue's report covers several selected studies that report on students' use of calculators with CAS capabilities to investigate and solve "framing schemes" or mathematical activities being procedural or conceptual. Artigue's experience and reflection on these projects led her to conclude that often routine techniques were credited to procedural mathematical activities, whereas conjectured based techniques were credited to conceptual mathematical activities. The research overview presented here attests to the fact that DGSs provide the user a well-tuned system within which different mathematical concepts and mathematical problems may be explored (Hoyles & Noss, 2003).

Theoretical Framework

Works (Artigue, 2002; Garofalo & Lester, 1985; Pólya, 1945/1973; Schoenfeld, 1981, 1985a) cited in previous sections offered an insight onto models that describe problem solving, metacognitive processes, and technology. For this study, the overall framework is a compilation of frameworks presented in previous sections of this chapter. Adapting the model by Pólya (1945/1973), Schoenfeld (1981, 1985a), and Garofalo and Lester (1985), I developed a model capturing together student's cognitive and metacognitive practices. The instrumental approach framework (Artigue, 2002) offered a model describing the effects of tool use on the participants' activity and transformation of the tool to fit participant activity, and what circumstances, situations, and interactions in GSP promote metacognitive behaviors to occur during mathematics problem solving.

As mentioned before, because of the dual nature of cognitive processing, the problem-solving episodes (reading, understanding, analyzing, exploring, planning, implementing, and verifying) were categorized as cognitive or metacognitive but not purely cognitive or purely metacognitive. Cognition is implicit in any metacognitive activity, but metacognition might or might not be present during a cognitive act (Artzt & Armour-Thomas, 1992) or cognition is necessary to perform a task, whereas metacognition is of great importance to understand how a task was preformed (Garner, 1987, as cited in Schraw, 2001, p. 113). That is, metacognition is necessary to understand how and why is task performed whereas cognition in only necessary to merely perform the task.

The framework outlines the complex interplay between metacognitive and cognitive processes in mathematics problem solving. To improve the compilation of frameworks discussed earlier, I considered the nuances for each episode with respect to students' metacognitive awareness, metacognitive evaluation and metacognitive regulation (J. Wilson & Clarke, 2004) to better understand the nature of the cognitive and metacognitive processes.

Even though this framework can provide some codes in connection to problem solving, it does not take technology into consideration. To supplement these categories, I used the instrumental approach by Artigue (2002) to help explain student-tool interaction during geometry problem solving. Given that this framework was promising within the context of using CAS in mathematics education, in addition I used a grounded theory approach. These three models allowed me to focus this study on the problem solving and metacognitive processes used when students engage in problem solving and how is GSP integrated into students' problem-solving processes.

In this chapter, I reviewed literature related to problem solving. Moreover, I discussed studies related to metacognition, metacognitive aspects of problem solving, and problem solving in dynamic geometry environments. The review of literature provided a sound foundation for the frameworks that I used to analyze my data and for developing the research methodology.

CHAPTER 3

METHODOLOGY

The purpose of this study was to investigate the patterns of metacognitive processes preservice teachers exhibit when solving nonroutine geometry problems in a dynamic geometry environment. Hence, the study was designed to better understand what circumstances, situations, and interactions in a dynamic geometry environment promote metacognitive behavior. The study was qualitative. The research questions for the study are as follows:

1. What are some of the metacognitive processes exhibited by preservice teachers when engaged in solving nonroutine geometry problems using Geometer's Sketchpad?
2. What metacognitive processes appear to be associated with the Geometer's Sketchpad use during problem solving?
3. How do preservice teachers perceive the importance of Geometer's Sketchpad when faced with nonroutine geometry problems?

This chapter provides a detailed description of the research design, verbal reports, description of the pilot study, participant selection, data collection procedures, mathematical problems, timeline for data collection, data analysis, validity and reliability, and chapter summary.

Research Design

For this study, I used a multiple case study qualitative research design (Merriam, 1998; Patton, 2002). In qualitative research, the researcher is the primary tool to conduct the research (Patton, 2002). Qualitative research study is reflexive as data collection, data analysis, development and modification of theory, and modification of research questions occur simultaneously each influencing all of the others (Mason, 2002; Patton, 2002). Hence, the focal point of qualitative research is on the research process as a whole, allowing the researcher to understand, organize and report on the phenomena under investigation rather than focusing on a specific outcome (Patton, 2002).

I chose a multiple case study approach for this study. According to Merriam (1998) "a case study design is employed to gain an in-depth understanding of the situation and meaning for those involved. The interest is in process rather than outcomes, in context rather than a specific variable, in discovery rather than confirmation" (p. 19). That is, a case study allows one to answer questions such as how and why the specific phenomenon, such as problem solving, occurred (Merriam, 1998), pushing the study beyond description alone and explaining the phenomenon in depth, in real context, and holistically (Mason, 2002; Patton, 2002). Since this study focused on metacognitive processes of preservice teachers, how these emerged when working in a dynamic geometry environment, and problem solving being an individual activity in its nature, the multiple case study design was a logical choice, and each preservice teacher was classified as a case.

Verbal Reports

Research on problem solving uses different methods that can help elicit problem-solving processes. Based on a review of several studies (e.g., Garofalo & Lester, 1985; Schoenfeld, 1985a) and reports on verbal reports as data (Ericsson & Simon, 1980), the following list of verbal report techniques emerged: think-aloud, both individual (Ericsson & Simon, 1980) and pair protocols (Schoenfeld, 1985a, 1985b); clinical interviews (Ginsburg, Kossan, Schwartz, & Swanson, 1983); concurrent probing (Ericsson & Simon, 1980); retrospective probing (Ericsson & Simon, 1980); retrospective general report (Ericsson & Simon, 1980); and retrospective clinical interview (Schoenfeld, 1985b). Each technique has its weaknesses and strengths. It is up to the researcher to consider the constraints of each technique, and try to eliminate drawbacks that might occur when deciding which method to use.

The think-aloud protocol or verbal protocol analysis is a rigorous data collection method used in variety of research areas to gather data that "aims to elicit the inner thoughts or cognitive processes that illuminate what's going on in a person's head during the performance of a task, for example, painting or solving a problem" (Patton, 2002, p. 385). Think-aloud protocols involve participants thinking aloud, such as verbalizing what they are thinking, doing, focusing on, felling, and so on, as they do a task. Individual, pair, or group think-aloud protocols have been used in research on metacognition and mathematical thinking (e.g., Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; Kantowski, 1977), to analyze individual differences (Reeve & Brown, 1984), or to communicate self-regulation (e.g., Kroll, 1988; Reeve & Brown, 1984; Schoenfeld, 1985a) during problem solving. According to Patton (2002) in an individual think-aloud protocol, the researcher often asks questions and uses prompts, such as "keep explaining aloud what you are thinking," "keep talking," "keep telling what you are doing," "say everything in your mind" or "speak out loud what you are doing" (Ericsson & Simon, 1980; Lawson & Chinnappan, 1994; Montague & Applegate, 1993), to get the person to talk about what he or she is thinking. In both scenarios, subjects may fail to remember, may misremember, or may invent what has just occurred to them because of automatization, lack of motivation, or other factors (Ericsson & Simon, 1980).

Given that the basic strategy of think-aloud protocol requires participants to verbalize their thoughts and feelings, many researchers take time "to train" participants to get used to what usually they do internally with themselves (Patton, 2002). On the downside, however, the training in verbal report of data could influence the type of reporting that students do and cue responses expected by the researcher (Afflerbach & Johnston, 1984). Nevertheless, practice ensures the quality of self-reports to the researcher, and provides participants the experience to better understand and get used to the technique before actual data collection (Patton, 2002). Furthermore, the development of the think-aloud protocol was described in the study of Ericsson and Simon (1993) where they stated that brief training in verbal reporting of data and concurrent verbal reports do not change the course and structure of the cognitive processes, nor does the verbalization under these conditions slow down those processes.

Data from think-aloud protocols are often confirmed by data from other sources, such as concurrent probing, clinical interviews, retrospective clinical interviews or probing, and observations of problem solving (Goos & Galbraith, 1996). Concurrent probing allows the researcher to ask the participant to report on specific aspects that are of interest (Ericsson & Simon, 1980), such as “Why did you do that?” or “How does this help you?” Although concurrent verbalization methods are critiqued for their reactivity, incompleteness, inconsistency, idiosyncrasy, and subjectivity (Eriscsson & Simon, 1980), verbal methods provide rich descriptions of metacognitive processes during problem solving activity (Goss & Galbraith, 1996). On the other hand, retrospective clinical interviews or probing are conducted after the participant has solved the task. The participants are asked to explain their problem solving, clarify their think-aloud reports, and recall specific actions or events (Ericsson & Simon, 1980). However, Patton (2002) reports that the concurrent approach is more reliable than the retrospective approach because the participant’s verbal data are independent of his or her short-term memory recall of cognitive processes and strategies.

In summary, to provide a description of both cognitive and metacognitive processes, instead of utilizing only think-aloud protocol, concurrent verbalization methods with retrospective methods (probing and clinical interview) were used in this study.

Summary of the Pilot Study

In preparing for designing this study, a small pilot study was performed in the spring of the 2009–2010 academic year, and in the fall of the 2010–2011 academic year. Data and experiences gained from that pilot study were considered in preparing the methodological direction of the main study. The pilot study influenced the number of participants, criteria for selecting them, research questions, and data collection protocols.

During the spring semester there were two participants in the pilot study, one female (Summer) and one male (Josh). Both participants were college sophomores, dual majors in mathematics and mathematics education, and enrolled in Technology and Secondary School Mathematics course at the University of Georgia (UGA), that I had been a teaching assistant for at the time of piloting. I have identified them as college students who were good mathematical problem solvers, who possessed knowledge of geometry and who would willingly share their thinking in the class. During the pilot study, I conducted five problem-solving sessions with Josh, and one problem solving session with Summer.

The pilot study contained five problems, and one problem was solved in each session. The first two problems were construction problems, the third one was an applied problem, and the last two problems were exploration problems. After transcribing and analyzing the first session, where Josh and Summer were given a construction task with and without problem representation, respectively, and the rest of the problem-solving session with Josh, I recorded the following preliminary observations: (1) exclusion of problem representation aided developing better understanding of the problem, (2) the nature of student-tool interaction differed between both participants, (3) their problem-solving strategies and decisions were guided by the context and the use of GSP, (4) their problem-solving processes exhibited differed with respect to the type of problem, (5) a rich store of geometry knowledge was of great importance to solving the problems, and (6) the participants appeared to believe that their instructors had been helpful in learning how to use the software and approach the problem. These observations and results were helpful in rethinking my methodology. First, I omitted the problem representation for the construction problem (Problem 1) solved by both Josh and Summer. Second, for the next four problem-solving sessions conducted with Josh, in order to confirm data from the think-aloud protocol I used concurrent verbalization. With time, I became skilled in undirected probing, ensuring that I as a researcher did not help, make suggestions, or prompt Josh's metacognitive processes. Concurrent verbalization methods provided a description of Josh's metacognitive processes and influence of problem-solving context on Josh's problem-solving processes. For that reason, concurrent verbalization methods were included as another data source method in the main study. However, Josh was not successful in solving one of the construction problems and the exploration problems. For that reason and in order to get a better insight into the solving processes of participants, I chose three problems for the main study. In addition, passing a college geometry course was added to the participant selection criteria. Even though at first I intended to examine only the metacognitive processes exhibited by my participants when working in a dynamic geometry environment, the observations listed above—namely, (2) and (3)—helped in formulating a research question that would explore the relationship between exhibited metacognitive processes and the use of GSP. Last, conducting this pilot study was helpful in getting experience as a researcher.

Participant Selection

For the main study, I used a purposeful sampling strategy was as a way of collecting rich and deep data from the research participants. The participants were two preservice teachers, Wes and Aurora (pseudonyms), each serving as a unique case, from the mathematics education program at the UGA. Given that qualitative research is highly interactive, and that the researcher is constantly in the process of questioning and refining the research tools (Patton, 2001), the basic criteria for participation in this study were based on the four following conditions: (1) participants wanted to be a significant part of investigating and communicating the experience of technology in mathematical problem solving (TMPS), (2) participants had previously experienced investigating mathematics in a dynamic geometry environment and are comfortable working with it, (3) participants were comfortable with me (and I with them) discussing not only their mathematics but also their experiences and feelings while engaged in TMPS, and (4) participants have taken college geometry course. However, since this study used the GSP as a context for problem solving, I tried to recruit subjects from the Technology and Secondary School Mathematics course at the UGA I co-taught in the Fall 2009 semester or the Spring 2010 semester. Based on research and personal experience, and after consulting with my major professor, I determined several people that would be ideal, that worked well individually, and were reflective thinkers who articulated their thinking well. Not only had they been used to working in a dynamic geometry environment, but they had substantial mathematical background, and we had established a rapport where they felt comfortable interacting with me on a variety of levels. Participant recruitment started at the beginning of February. However, because of the work expected for senior students, I was able to recruit only one such participant (Wes), whereas Aurora approached me herself with the idea of participating in the study. Nevertheless, both participants met all the criteria outlined above and had somewhat similar mathematics backgrounds.

Data Collection Procedures

Data collection methods for this study consisted of the following: (1) interviews, (2) audio/video taping of the think-aloud sessions as the participants were engaged in solving problems, (3) document review (of participants' written solutions, .gsp files), and (4) my observations of the sessions to investigate preservice teachers' thought processes during problem solving in a dynamic geometry environment.

The first phase of formal data collection started by using the interview protocol (see Appendix A for preliminary interview protocol) in my office. This first part was intended to elicit the participant's background, such as age and year, views about problem solving, problem solving experience, nature of their problem-solving, nature of mathematics, mathematical learning, and prior technology learning experiences and perceptions. The general protocol questions were used as a guideline for the interview. During the preliminary interview, I described the nature of the study to each participant and what it entailed before giving them the consent form.

Prior to the data collection, both participants briefly practiced the think-aloud protocol with a sample problem. With Wes, practice occurred immediately after the preliminary interview, whereas with Aurora practice occurred before the first problem-solving session. At that time, I again described the nature of the study and what was expected from them during the think-aloud protocol. During the practice period, I used prompts to encourage the participants to speak their thoughts, such as: "keep explaining aloud what you are thinking," "keep talking," "say everything you are thinking and doing," "say everything in your mind," or "speak out loud what you are doing." Practice with the think-aloud method took around 10 minutes. I discussed ways to improve the participant's skill with this research technique. The procedure provided participants with practice for understanding and developing confidence prior to utilizing the technique with the research problems.

At the core of the present study, the next stage of data collection concentrated on the participants' involvement in solving of three mathematical problems in a dynamic geometry environment (see next section for the mathematical problems). Data collection occurred in a one-to-one setting between the participant and me, as in the preliminary interview and a practice think-aloud session. Next, the participants were asked to solve the given problem using the think-aloud method. After receiving the problem, the participants began by reading the problem aloud or in silence. When needed, they asked questions they had to make sure they understood the problem's wording before beginning to work on it. I videotaped the working screen using Camtasia and ScreenFlow during all the sessions to record the participants' mathematical behaviors and how they responded to the problems. They continuously thought aloud and engaged in a conversation with me while working on the problems, describing their thinking and behaviors. However, during extended periods of time I used prompts to encourage the participants to speak their thoughts. Each participant made three appointments for solving the problems, and sessions were arranged so that the participants were not under time constraints. They used as much time as they needed in solving each problem. However, I made sure we never went over 60 minutes in each session. The individual interviews took place shortly after the participants finished solving each problem. We stayed in the same room and talked about the participant's problem-solving session that had taken on average 35–60 minutes. The interview protocol (see Appendix B for the problem-solving interview protocol) consisted of two parts and took on average 20–40 minutes. I used the problem-solving interview protocol questions as a guideline for the interview. They were intended to elicit the participant's views about the problem-solving task as well as to try to understand what situations, and interactions in a dynamic geometry environment promote metacognitive behaviors and to elicit the participant's experience about using technology solving the particular task. The field notes included the descriptions of questions, reactions, and behaviors that occurred during data collection that were then used during a retrospective interview. The same procedures were used for the following two mathematical tasks. At the end of each appointment, we discussed the date and time for the next problem-solving session.

Mathematical Problems

In this study, the participants individually solved three mathematical problems using the think-aloud protocol and concurrent probing. I asked the participants to explain their solutions and the processes they used as they worked on the problem. The mathematical problem solving tasks included three nonroutine problems selected and modified from a variety of sources, including mathematical journals, textbooks and web sites (Craine & Rubenstein, 2009; Johnston-Wilder & Mason, 2005, J. W. Wilson, n.d.). The problems that met the following criteria were considered for study inclusion:

1. The problems are nonroutine; that is, the participants are faced with an unfamiliar problem situation without an apparent solution path. These problems require problem solvers to use information and strategies in unfamiliar ways; that is, they demand strategy flexibility; thinking flexibility, such as logical thinking; abstract thinking; and transfer of mathematical knowledge to unfamiliar situations, as well as extension of previous knowledge and concepts (Schoenfeld, 1992).

2. "The problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation" (Pólya, 1945/1973, p. 6).
3. The problems provide students with opportunities to engage in metacognitive activity.
4. The problems cover mathematical content in geometry, and solutions should not require mathematical concepts and skills that participants have not learned in their college geometry course in order to allow them to engage in problem-solving profitably.
5. The problems should challenge participants to experiment, conjecture, and prove, if possible, and should invite different strategies and extending existing knowledge and problems to new problems.

For the purpose of this study, I examined a vast pool of geometry problems together with my major professor for content validity. After choosing a smaller pool of problems, based on the results of the small pilot study conducted in the spring of the 2009–2010 academic year, and in the fall of 2010-11 academic year, I developed three types of problems, construction, applied, and exploration problem for this study. In the pilot study, the participants' approaches to solving a problem varied based on the different type of problems, which led me to believe that using a variety of problems would enhance my understanding of their problem-solving processes and how their problem solving processes were tied to their use of GSP.

The first task used in this study was *The Longest Segment Problem,* taken from the course material of a problem-solving class taught by Nicolas Oppong in the Spring 2009 semester at the UGA. I pilot-tested a variation of the Longest Segment Problem in a study I conducted during the Spring 2010 semester and knew the problem could be solved in several different ways. Problem 1 was a construction problem.

Given two intersecting circles. Draw a line through one of the intersection points, say, A. That line also intersects circles in exactly two points, say, B and C. What choice of the point B results in the segment BC such that the segment BC is the longest?

a. Formulate and prove your conjecture.
b. Find the construction for a point B such that the length of BC is the longest.

Justify your answers as best as you can.

Figure 3: The Longest Segment Problem.

A second problem, *The Airport Problem*, was taken from *Understanding Geometry for a Changing World* (Craine & Rubenstein, 2009), one of the NCTM Yearbooks, and adapted in this study. Problem 2 was an applied problem.

Three towns, Athens, Bogart and Columbus, are equally distant from each other and connected by straight roads. An airport will be constructed such that the sum of its distances to the roads is as small as possible.

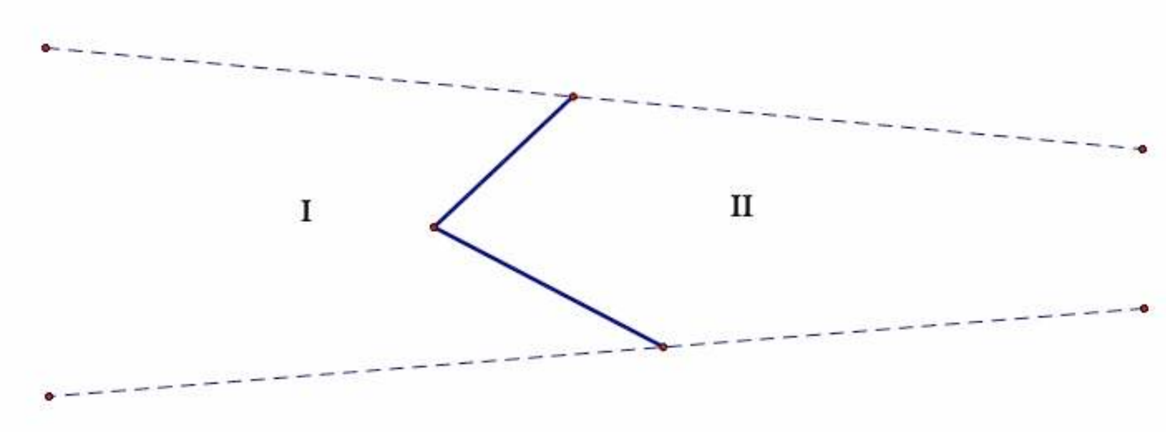
a. What are possible locations for the airport?
b. What is the best location for the airport?
c. Give a geometric interpretation for the sum of the distances of the optimal point to the sides of the triangle.

Justify your answers as best as you can.

Figure 4: The Airport Problem.

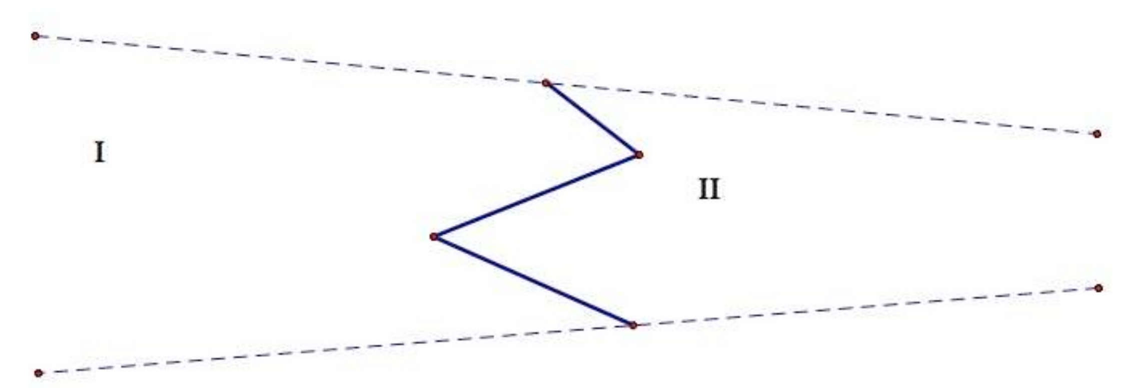
A third problem, *The Land Boundary Problem,* was taken from the book *Euclidean and Transformational Geometry: A Deductive Inquiry* (Libeskind, 2008), but it was also available from the TIMSS video study–Japan. Problem 3 was an exploration problem.

The boundary between two farmers' land is bent, and they would both like to straighten it out, but each wants to keep the same amount of land. Solve their problem for them.



Justify your answers as best as you can.

What if the common border has three segments?



Justify your answers as best as you can.

Figure 5: The Land Boundary Problem.

One applied task and one construction task utilized in this study was used in order for participants to apply their knowledge to construct interactive diagrams that model the problem situation. On the one hand, drawing a static diagram requires understanding of the structure of the problem, but constructing an interactive diagram, on the other hand, requires a deeper understanding of the relationships among the geometric objects involved in the problem solving situation as well as knowing how to construct those objects. Therefore, these two types of problems allowed participants opportunities to use and apply their knowledge, and translate verbal statements into an interactive representation. For the exploration task, they were expected to investigate a mathematical idea, to deal with a situation that may not have a single solution and to make, test and verify their conjectures. These problems were used to encourage the participant to use the GPS to explore ideas, to conjecture, to test his or her conjecture, to generalize, and to verify generalizations. An exploration problem was included in this study, because it illustrated the participants' mathematical thinking processes, and also provided an alternative method for observing the participants' problem solving behavior and different uses of the GSP. The participants had been involved in classroom activities in which they were asked to explore ideas, conjecture, generalize, and verify. Thus, solving the exploration problems was not novel for them.

Timeline for Data Collection

In this study, both participants participated in an initial interview, three or more problem-solving sessions, and a final interview during the Spring 2011 semester. Data collection with Wes started in February where he participated in three problem solving sessions. The first meeting was on February 17 and the final one was one April 20. Data collection with Aurora started at the beginning of March. Aurora participated in four problem solving sessions (we met twice in the process of solving Problem 3 because of time constraints) starting on March 21 and ending on April 27. The final interview was conducted after the last problem-solving session. It served as a reflection on the process of participating in this study (see Appendix C for final interview protocol). A detailed schedule for the interviews and the problem sessions is given in Table 1. At least one week separated the problem-solving sessions for each participant.

Table 1

Timeline for Data Collection

Participant	*Nature of Meeting*	*Date*
Wes	Preliminary Interview	02/17/2011
Wes	Problem 1-The Longest Segment Problem	02/17/2011
Wes	Problem 2-The Airport Problem	02/25/2011
Aurora	Preliminary Interview	03/02/2011
Wes	Problem 3- The Land Boundary Problem	03/16/2011
Aurora	Problem 2-The Airport Problem	03/21/2011
Aurora	Problem 1-The Longest Segment Problem	03/28/2011
Aurora	Problem 3- The Land Boundary Problem (part I)	04/05/2011
Aurora	Problem 3- The Land Boundary Problem (part II)	04/13/2011
Wes	Final Interview	04/20/2011
Aurora	Final Interview	04/27/2011

Data Analysis

In multiple case study research, data analysis is an ongoing process from the initial collecting of data throughout the entire data collection process; that is, data collection and data analysis occur simultaneously (Merriam, 1998). This ongoing analysis ensures rich but focused data during the meaning-making processes of both the participant and the researcher (Merriam, 1998).

For the purpose of this study, I conducted two stages of analysis, the within-case analysis and the cross-case analysis, as suggested by Mason (2002) and Merriam (1998) using inductive analysis. For the within-case analysis, each case (preservice teacher) was treated as a comprehensive case that helped describe the metacognitive processes each participant exhibited and perspectives on the experience of using technology during each problem-solving session.

First, all data from the think-aloud sessions, the interview sessions, hard copies and GSP sketches of participants' solutions, and researcher's field notes were transcribed after each problem solving session. I then analyzed the data for convergence or determining which pieces of data were similar using inductive analysis. Inductive analysis contends "the patterns, themes, and categories come from the data rather than being imposed on them prior to data collection and analysis" (Patton, 2002, p. 306). That is, inductive analysis allows construction of themes or theories that are "grounded" in the data. When using inductive analysis, I focused on creating codes and categories from the data, developing or enhancing theory during the act of analysis and the use of constant comparative method during analysis of the data (Charmaz, 2006). The constant comparative approach is an inductive method that consists of continuously making comparisons at every level of data analysis in order to produce codes, categories and themes (Charmaz, 2006). Using the constant comparative method, I categorized all data that consisted of comparing and generating categories, integrating categories, and delimiting the theory to help illuminate common themes across cases and within cases. During this process it was important to keep the voice of participants present to help develop a deep and sound understanding of a phenomenon under investigation (Charmaz, 2006).

Data analysis was a continuous task that started at the beginning of data collection. The first phase of analysis began with the participant transcript of the think-aloud problem-solving session. I first read each hard copy of the transcript a number of times, and made general comments, notes, and thoughts in the margins. After I coded each transcript, I opened two blank documents on the computer for each research question, and copy-and-pasted all the information from each transcript that would address each specific question. In addition, I labeled key phrases, words, and behaviors for which I kept a record in an audit trail.

The second phase of analysis involved generating categories from the codes. I coded each problem-solving episode based on behaviors exhibited by the participant during his or her problem solving activity. I read through each transcript sentence-by-sentence identifying actions or behaviors or words by the participant to generate categories. Initial coding helped organize the data and allowed for comparisons across participants.

During the third phase of analysis, I reanalyzed my data and refined the codes, noting key behaviors and characteristics that related each code to its category. In addition, codes were refined once again through identification of the level for each problem-solving behavior.

In the last phase of analysis I went back to the first problem solving transcribed data and analyzed the participant's interview. Each participant's interview was coded and categorized similarly to think-aloud session. This analysis also involved coding of my field notes, written solutions of problems, and video files. During this phase, multiple data sources allowed further data triangulation, comparison and confirmation of emerging patterns of behaviors. Hence, by integrating the codes, categories, my field notes, and memos, I was able to construct themes and develop theory from the data.

During these four phases of analysis, each case was treated separately, and each phase of analysis occurred after each problem solving session. This ongoing analysis during data collection allowed me to create and refine the theoretical model of each participant. Moreover, it allowed me to be thoughtful during the process of data analysis. These four phases of analysis were followed by writing a case study narrative for each participant after all data were collected.

After within-case analysis was completed, at the end I cross-analyzed the cases. The cross-analysis occurred after writing a case study narrative of each participant. Each problem-solving session of one participant was compared with the corresponding session of the other participant. During this phase, I again reanalyzed each case separately. That, together with the cross-case analysis, was used to create a sound theory offering general explanations of metacognitive processes and perspectives on the experience of using technology.

Validity and Reliability

Validity and reliability are criteria that are traditionally associated with quantitative research, but can also be used in qualitative research (Patton, 2002). Creswell and Miller (2000) note that validity and reliability are key components in a qualitative research study. Validity is defined "as how accurately the account represents participants' realities of the social phenomena and is credible to them" (Schwandt, 1997, in Creswell & Miller, 2000, pp. 124–125). Consequently, it becomes the responsibility of the researcher to interpret the participants' experiences and meanings in such a way that it reflects participant realities as closely as possible. In a qualitative research study reliability depends on the congruency of the findings and the data collection methods (Patton, 2002). Methods such as triangulation, or using multiple sources to establish consistency, can also be used to ensure reliability (Creswell & Miller, 2000).

To ensure the validity and reliability for the study of preservice teachers, I used several procedures that involve triangulation, thick rich description, and the audit trail (Patton, 2002). When talking of triangulation, I used triangulation of sources and analyst triangulation to contribute to verification and validation of qualitative analysis. Triangulation of qualitative data sources involved comparing observations with interviews and checking for the consistency of what participants said during the think-aloud session and during the interview session. Finally with regard to analyst triangulation, I used member checking, and my major professor served as a peer observer and provided critical feedback (Creswell & Miller, 2000) during the analysis and interpretation phase of the study. I also maintained a researcher journal and audit trail as a way to record reflections on the research process through memoing and journaling, keeping a research log of all activities, developing a data collection chronologically, and recording data analysis procedures clearly. I believe that employing the procedures mentioned above ensured trustworthiness and rigor.

Chapter Summary

This chapter was my attempt to outline in detail the methodological processes in which I engaged to conduct a qualitative study with two mathematics education preservice teachers. I provided a description of qualitative research methods, including verbal reports, along with a summary of the pilot study, participant selection, data collection procedures and analytic techniques. Finally, I closed the chapter by discussing issues of validity and reliability, which are important in qualitative research. The following chapters, Chapter 4 and Chapter 5, present each case study separately.

CHAPTER 4

THE CASE OF WES

In this chapter I present the case study of Wes. This chapter is divided into four sections. The chapter begins with a description of Wes, outlining his background, his learning of mathematics, his view of mathematics and problem solving, and his technology learning experiences and perceptions obtained from the preliminary interview. The second section is a careful examination of three problem-solving sessions in which a different mathematical problem-solving task was investigated each time. The report of each problem solving session is a synergy of events during the session and participant's descriptions and interpretations of the session providing the reader with a rich representation of the sessions. The report of each problem solving session ends with a synthesis and a parsing of the session protocol in which I used the following abbreviations: C–cognitive, M–metacognitive, T conjecture–transition episode characterized by conjecturing, and Green arrow– transition episode characterized by "taking a step back." Note that I present mathematical problem-solving tasks given in the order of sessions, and not in the order of tasks as outlined in the mathematical problems section. The third section presents Wes's reflection on the three problem-solving sessions and problem solving in a dynamic geometry environment as a result of participating in this study. The final section is a short summary of the case of Wes.

Getting to Know Wes

Wes was a senior student in the mathematics education department at UGA at the time of data collection. He first started studying in the engineering department at UGA but had switched to the mathematics education department after 2 years. When I asked him about this decision, he revealed that decision was guided by his attitude and views of mathematics,

> *When I got into it, I really liked the mathematics more than I did the science, and I liked applications, but I like the more simplistic side of the math, not that math is simple, but how simplistic it is and pure, so that's why I decided to switch.*

Furthermore, he added that he did not regret his decision and that since then he *"had a really good time, and a really good experience in [the] math education department."* His favorite courses in the mathematics education department were Connections in Secondary Mathematics, *"because we got to study some things I wouldn't have thought about, like dimension, the concept analysis. I like going into depth. That was really interesting,"* and Contemporary School Mathematics, that focused on conic sections. Wes stated,

> *Favorite courses since I have been here. Everything we did we had to prove, and I was pushed to the limit in that course more than I have been in any course I have taken, and I had to use all my mathematics which is great. It was awesome.*

At the time I was writing this report, Wes had started to work in a high school in Warner Robins, Georgia. However he was planning on getting his masters in mathematics within the next 5 years because he thought that *"[mathematics] can be more useful outside of school,"* with an aim of trying to find work afterwards in an air-force base, as his father worked there.

Problem-Solving Experience and Perceptions

Wes's problem solving experience was substantial. It involved a problem-solving course, Problem Solving in Mathematics, he took in his junior year, and problem solving in mathematics classes he took for which he said, *"Just regular math stuff. You know, problem 47 and page 251. Something like that."* He described the experience of taking Problem Solving in Mathematics course as being *"Very cool."* Moreover he explained how it influenced his view on mathematics, his view of problem solving, and how he went about problem solving;

> *I got to see a lot of different problems that I never encountered before, and I learned a lot of different ways going about solving problems. So I a approach problem differently now than I would have previously, probably. I am more straightforward when I am problem solving. Before I took the class [it] was very linear, but then in that class I learned there are so many different ways you can go about solving a problem; you can go, start from the bottom and go back to the top, instead of going top down. I don't know if that makes sense. Kinda knowing what the answer is, and then working back, trying to figure out how you could get there instead of just going from here to the end [hand gestures]. A lot of different techniques.*

The problem-solving course gave him the opportunity to experience genuine problem solving activity, allowing him to develop his problem solving abilities, problem solving strategies, performance and shape his views about problem solving. With respect to content, he said he learned arithmetic and geometric mean inequality, and developed a better understanding of ratios and proportions. The problems done in the class were more difficult than he had ever encountered and *"required more thinking, a lot more thinking on some of them."* Several times he mentioned that he used calculus to solve many problems, as that was his favorite subject. However, when asked about geometry, his answer was somewhat different, revealing a negative emotion of frustration when solving geometry problems;

> [Geometry] *is one of my favorites, but it's also one of the areas I struggle with the most. In my mathematics career I was always able to get on. I am usually able to solve things pretty quickly, and with geometry sometimes it is not always so apparent how to solve it and it takes me a while, and it's frustrating. It can be frustrating.*

During a class I taught, I got to observe Wes solving variety of geometry problem. Even then, I noticed his passion when given a mathematical challenge; he never wanted to be given the answer, but rather derive it on his own. He would then get consumed in the problem and never gave up. He explained mathematical problem solving as,

> *Well, there is a problem you don't know the answer and you have to use what you know, your tools. I like to think of it as a tool bag, like all the mathematics, all the logics you can develop, and you are trying to use these tools to get through the problem and to come up with the solution you think is correc*t.

He liked to solve problems because he viewed them as *"puzzles,"* adding *"I get consumed with them. I can spend hours on them. I dunno why. I'll keep thinking about it and sometimes I will wake up at night and will be: That's it. I like when it happens like that."* Out of different types of problem, he enjoyed solving applied problems, problems that use calculus, word problems, and proofs;

> *It [proof]'s like a puzzle and you have this information, and you want to show that this is true, or maybe you want to show this is never true. I just really like this idea that you start of with these basic axioms, very simple things and you can prove this and your tool bag becomes bigger and bigger, you expand which is really neat I think.*

He did not enjoy routine problems as he perceived them to be boring. Even though he was an excellent student with much mathematical and problem-solving experience, he described himself as not being a good problem solver attributing his success to persistence, *"I just work hard."* Moreover, he compared himself as a problem solver to his friends who though less mathematically proficient were better in problem solving then he was as they could "*see things differently than other people, and they see the path that I wouldn't have chosen."* Hence, he attributed allocation of ingenious solution paths as a quality of a good problem solver that in his opinion he lacked.

His problem-solving routine depended on how the problem was presented, as *"problems can be suggestive"* with respect what needs to be done or used (e.g., a picture). He furthermore emphasized the importance of reading and understanding the problem before making any other problem-solving endeavor,

> *I do take a while to actually read the problem and then I decide well how I am gonna solve it and I usually look for clues inside the problem about where to go. And, if I don't have clues, I don't see any clues, then I usually try to draw a picture.*

He did not solve problems aloud, but did the thinking and talking in his head. Interestingly he added the importance of verifying the result of a solution process,

> *When I am finished I usually try to assess whether it is reasonable. I think that's important. That's one thing I learned from 4600 [Problem Solving in Mathematics course]: Does this make sense given what they told me? So, I usually try, I like to see if that's reasonable and if someone challenges me to find a different way [to solve the problem] then I will use a different way but usually if I find a way that I like, I usually stick with that.*

Knowing that he had solved many problems throughout his education, I was interested in finding out what were the reasons he missed on a problem. He said, *"A lot of times it would be computational errors while the process would be correct,"* whereas a conceptual mistake was a result of misreading the problem, *"maybe what I thought to be as a suggestive path to go, maybe I misinterpreted that"* or would get in a situation where *"I really don't know what I am doing, and I approached the problem wrong in the beginning."* As noted before, when he got stuck on a problem, he usually continued working on the problem, or took a break explaining,

> *And then I get tired and I usually go to sleep and then I sometimes have a wake up in the middle of the might and I would be: That's how you do it. And sometimes I will wake up the next morning, start all over it, and usually when I am refreshed I can see things clearly. So I usually just keep working at it. I mean I keep trying, trying and trying.*

View of Mathematics

Wes viewed mathematics as *"very pure and simplistic,"* and *"fundamental as you can describe so many things with mathematics."* Through variety of mathematical courses he had taken during his undergraduate studies, his view of mathematics changed from seeing it as *"as perfect; there are always these steps involved, there is always an answer"* to not being so perfect:

> *And I see that now there are some problems that even haven't been solved yet. There is just something about it that my logics is different now. I cannot explain it very well. I just know I think differently about problems and how I approach things.*

Learning of Mathematics

As far as type of learner he was, Wes again emphasized the importance of reading and working on his own: "*I like to read. Watching people do it doesn't help, having someone tell me* *what to do doesn't help. I just usually have to read and try it myself."* He said he was a visual learner, explaining, *"I do like to have a type of visual that I can work with … seeing it, it's very helpful."* He also stated that practice with computations helped him learn mathematics but added, *"I don't like being given something and being told it works, just trust me it works. I like to be able to know why it works."* He also shared that he did not have to remember all of the formulas by heart, *"If I can understand the concept, I can just pretty much do whatever I need to."*

Technology Learning Experiences and Perceptions

Wes had previous experience of learning mathematics with technology in high school (GSP, Graphic Calculator) and mathematics education college courses (GSP, Phantom). When asked about problem solving using technology tools, he stated,

> *[It helps] exploring the problem. Because when you have a piece of paper, it's static, you draw once, then you have to draw it again and again. With the computer everything is just fluid and I can change things, change the parameters of the problem, test my conjectures, see if my solution is reasonable. I don't know. I like it. I like it a lot. There is so many things you can do with Geometer's Sketchpad or Phantom, that I didn't know if it will work and I am glad I had the opportunity to learn how to do it. You can do a bunch of simulations and so many things you can do with it. Just really neat.*

He also used technology to visualize a problem or problem situation, but he would use paper and pen for writing down the steps needed to solve the problem, and for solving any computational parts of the problem. However, he listed several drawbacks with using technology, such as *"It can be misleading at times,"* and the effect of it on student thinking,

> *I can see where things get lost with technology, like certain skills… Now you can just do it [geometric construction], and you are using a lot of logic doing some of constructions on GSP but you can do it so quickly that you are not really thinking about what is going on, you don't have to understand the process or you know the process more than the concept.*

In summary he had a positive view of using technology to solve problems as *"it's so much quicker than having to draw everything and it's much more precise, you can see it, you can have* *reasonable conjectures because when you draw it on a paper it's not perfect."* This characteristic was observable throughout problem solving sessions with Wes.

Solving the Problems

Problem-Solving Session 1: The Longest Segment Problem

The Problem-Solving Session 1 occurred after the preliminary interview. Wes was very excited before we started the session, as he wanted to be challenged with a mathematical problem. Thus, going into the session he had a positive attitude. The session took approximately 80 minutes. Wes spent approximately 35 minutes solving the Longest Segment Problem (see Figure 3 in Chapter 3, p. 64), and during the remaining time, we reflected on the session. For this session, I gave Wes a sheet of paper and a pen, and opened The Longest Segment Problem.gsp file on which the problem was printed. Formulation of the conjecture was done with an ease, taking Wes approximately 10 minutes for it, while its proof was a more challenging task that took approximately 25 minutes. After Part a of the problem was finished, construction of Point B, which was Part b of the given problem, was a trivial task.

Solving the Longest Segment Problem

Wes started the session by reading the problem aloud. He had never seen a problem similar to this one problem before. When he read the problem, he ranked it as 3 out of 5 in difficulty. As he was reading the problem, he did not experience any difficulty understanding the problem as posed; however, he focused on several key words: *longest* as *"it came at the end of the sentence and it's what the entire problem is about,"* and *intersecting "is key too, given two intersecting circles. That's important."* Hence, the understanding episode started in which goal of the problem was noted correctly and explicitly as well as the conditons of the problem. After he read the entire problem from the hard copy I gave him, he focused on the problem sentences.

However, even though he did not explicitly assess the current state of his problem solving knowledge relative to the problem, this was done implicitly. He utilized a cognitive problem solving strategy, drawing a diagram representing the problem, however selection and use of that problem solving strategy was based on thinking about his previous problem solving strategies which is a metacognitive behavior. That said, he started working aloud on the problem where his thinking shaped how the software got used; he performed various parts of the problem statement step by step on the computer as they were outlined in the problem before moving onto the next part of the problem,

> *I began by constructing the two circles, and then I read the [statement]. I constructed two intersecting circles… Draw a line through one of the intersection points, say A. So, let me make an intersection Point. A line also intersects circles in exactly two points, say B and C. So I have a line, and it's going through this Point A and, a line goes through A also intersects the circles in exactly two points, say, B and C. B and C is like right there. So, label points, A, B, and C,*

followed by explicitly uttering the goal of the problem by reengaging with problem text by reading the main question three times, *"What choice of the Point B results in the Segment BC such that the Segment BC is the longest?"* Later on during the interview, he explained to me his decision for drawing a representation of the problem through continuous engagement with the problem statements,

> *I broke down the directions and the instruction step by step so I can see, make sure that I am going on the right track. Because sometimes if you read through a problem quickly you miss a piece of information or you will misread something and then it doesn't matter what you do afterwards because it's gonna be wrong.*

It was evident that he viewed reading the problem as an important step when problem solving as well as a step-by-step procedure to make sure his interpretation of the problem was correct which shows evidence of metacognitive engagement during problem solving. He stayed mentally engaged throughout the process, monitoring and directing his knowledge and thinking. Through his actions he transformed the tool to fit his needs allowing him to visualize and develop an understanding of the problem. Thus, the tool became an instrument through stages of personalization and transformation of the tool.

After he represented the problem in the GSP, he did not read the problem again, because of his step-by-step process he was sure his interpretation of the problem was correct, which showes evidence to judging the effectiveness of his thinking processes and strategy. Immediately after reading the problem, he had the idea of using triangles to solve this problem,

> *Triangles seem to be so important in geometry because everything else depends on triangles I have noticed…See with geometry, I have taken Euclidean Geometry 5200 and the basic tool we have for doing anything in that class is a triangle. I mean that's what we worked with was triangles, that's the extent we've got to. So whenever I see geometry problem where it's proving something, I always think well I always have triangle in my tool bag to pull out, and I have a lot to work with… So, I thought I am gonna have to use a triangle somewhere.*

It was clear Wes thought about previous content-specific knowledge and experiences that might be helpful in the current problem-solving situation. Before exploring and analyzing the problem, he made the circles smaller to fit in his viewing window, and placed a point that he labeled as Point D on Line BC for dragging. Again, he repeated the problem goal by reading the main question twice, while the second one was a short utterance, *"What choice of of B?"* followed by a short pause. An exploration episode started where Wes engaged in the search for a solution plan. His first idea was to use the dynamic capabilities of the GSP by moving the Point D to get a good idea where Point B would be to satisfy the problem statement. Explicitly stating the goal of the problem, and then selecting Point D on the Line BC demonstrates his actions were focused, purposeful and goal driven. Also, his actions drove the use of the software as an exploration tool. He then moved Point D on Line BC causing Line BC to change its orientation in the plane monitoring his progress; he observed how Segment BC acted as Point D was being moved in the plane. However, movement of Point D was not random. Shortly after he said, *"There is the shortest."* He moved Point D and stopped when it seemed that BC was at its maximum length. As a consequence, he used the software capability, namely Measurement Tool to measure length of BC to verify his conjecture that this was when BC was the longest,

> *Right now I am trying to figure out where BC is the longest. So, I am measuring the length of that segment and I am trying to see where it stops going up, and where it stops going down. So, it seems like it's right here.*

The length of BC at its maximum was 19.03 cm. The analysis episode started here, although it was intertwined with exploration. Here Wes stopped for a second, assessing the relevance of the new measurement information in relation to the problem goal. In the interview, he explained that at this point he remembered the problem statement in the back of his mind, and that led him to again move Point D on the Line BC observing how the lengths increased, reached its maximum, decreased to find the location of the Point B where BC would be at its longest. He knew he needed to characterize Point B in relation to some geometric object. Hence, through reflection on the problem and the processes, his actions became driven by noting the relationship between the conditions and the goal of the problem. Again, here his thinking transformed how the software got used again as an exploration tool to fit his needs; thus, the tool became an instrument through stages of personalization and transformation of the tool.

By moving Point D he realized the following, "*Hm? It seems like it's perpendicular to this segment [segment between the two circle intersections]."* At this point that segment was not drawn. This behavior shows he directed his actions and thinking into understanding the information obtained through use of the software by imagining the solution. For that reason a choice of perspective was made; he drew that segment, labeled it AH, where Point H was the second point of circle intersection, followed by trying to make sense of the new problem-solving state, *"Now why would that be the case?"* Even though he never explicitly tested his current solution state, by his actions I can interpret that assessment of the same was done implicitly. With this, he ended the first part of Question a by formulating the conjecture, *"You have to choose Point B so that the line that joints the two intersection points is perpendicular to BC,"* and then he worked on the second part of Question a, trying to prove the above-formulated conjecture. It is important to note that before this point, he did not implement his notion of using a triangle to solve the problem that he stated in the early phase.

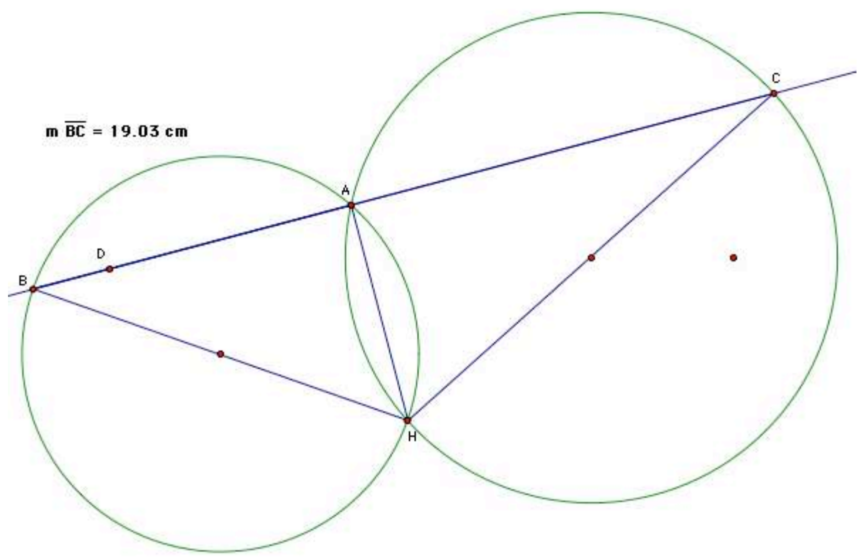


Figure 6. Wes's construction of Segment BC that triggered formulation of the conjecture.

Immediately after that, the exploration episode started. Wes started thinking of his mathematical content-specific knowledge that would explain this phenomenon which demonstrated awareness of his knoweldge in the current problem solving state. In the interview he explained that at that point he remembered that in order for that to happen, Points B and C would have to lie on the diameter as the diameter is the longest segment in a circle. Here again, he used the software to verify his thinking. He drew Segments BH and HC, and was visually convinced that they were in fact diameters of the two circles, which gave him a starting point to *"play around"* with mathematical concepts he knew. By drawing Segments BH and HC, he obtained two triangles, Triangle HAB and Triangle AHC, stating they were both right triangles. Here he finally implemented the notion of using a triangle to solve the problem, which was important to engage in productive problem solving endeavors. I asked him if he could explain why they were both right triangles. He promptly stated that a triangle inscribed in a semicircle is a right triangle, and provided a proof of his statements. He explained that in a triangle the exterior angle was always equal to the sum of its two interior angles, and since the exterior angle here was a line, and therefore equaled 180°, and the triangles drawn were isosceles triangles, the Angle BAH was equal to 90°. Therefore, he inferred that both Triangles HAB and AHC were right triangles. Wes clearly missed to properly explain why Segment BC and Segment AH were perpendicular. They were perpedicular because he drew them to be perpendicular, and not because Triangles HAB and AHC were inscribed in a semicircle.

Evaluation of the current solution state was done explicitly because of my question directed at it. Here I prompted evaluation problem solving behavior. After my interruption, he continued trying to prove his conjecture. He tried to regulate his thinking activities towards relating the two segments to the conditions of the problem,

> *So, see I drew all these auxiliary Lines [BH, CH]. I am not trying to prove BC is in relation to those, I am not trying to do that. I am trying to prove it in relation to BC, to itself, or any other point on the circle. Alright.*

His search for proving his conjecture relying on previous knowledge and experiences continued, and the actions were condition driven and purposeful. Again, his thinking directed him to again move Point D on the Line BC but this time observing how this affects the lengths of Segments BA and AC,

> *If B was somewhere else, why would it be different? Hm…m? Why is that? Hm…m? So, I got the center there, so if I keep changing D, if I keep moving this point right here, B, Segment BA is gonna start getting smaller and AC is gonna start getting bigger but still that maximum point occurs right there [when BH becomes the diameter]. This is a right Triangle [ABH], that's a right Triangle [ACH].*

It is evident he monitored his actions throughout by reflecting on the undertaken cognitive activity. Furthermore, here we can see interplay between his thinking processes and tool use. Although Wes driven by his thinking processes, started the exploration by exploring the effect of Point D movement on the Segments BA and AC, during the phase new information obtained by observing movement of Point D influenced his thinking processes. Hence, the software was no longer used as a tool in his hands but became an instrument. Moreover, he was able to relate the new information to his previous learned mathematical concepts and theorems, as can been interpreted from how he continued to explain his discovery:

> *Oh, I think I know. In a triangle the longest side is opposite the largest angle and the largest angle possible right here is gonna be 90° angle. No! Let me see. I can have a bigger angle than 90°, but then the other one is not gonna be 90°. So to get the longest segment I need both of them to be 90° because you need both of them to be 90°. So, the longest side in a right triangle is always the hypothenuse because the 90° is the maximum angle. So you need this [pointing at the diameters]. Then you have to have these two [BAH, CAH] to be a 90°, so they need to both go through the center but I think I am thinking backwards. Am I trying to assume what am I trying to prove? Let me think. Let me figure out where the Point B is. Point B has to be here, and it has to be such that BC is perpendicular to AH because you need to have two right angles, because the two right angles imply that BC is the largest, the longest. BC is the longest. Hm...m? For BC to be the longest it needs to be (pause) I have two right triangles. Okay but why is it, why is it BC the longest there? I've got this length right here, I've got this triangle right here BAH, that is, the angle is the angle BAH is greater than 90° which implies that BH is greater there than it would be if B was right here on the line passing through this center.*

We can notice that as he was trying to reason through his explanation, he was evaluating his thinking processes, and logico-mathematical statements as well as regulated those constantly. However, at this point he started questioning his last statement that BH was longer when it did not pass through the center of the circle, noticing soon his mistake, followed by an explanation of his reasoning,

> *I don't know what I was thinking. I was thinking that with the triangles, the angle that's true, if the triangle is static, if it's not changing and I was using it when I was moving it, I was thinking. Oh, I was applying that theorem incorrectly. I need the triangle not be moving.*

Assessment of the current solution state was explicit; he evaluated his thinking processes and corrected them. As a consequence of the actions described above, he was able to proceed with the proof by considering previous knowledge and new information obtained during exploring activities. He continued working on the proof but perplexed as to what were the assumptions, and what was he supposed to prove,

> *The longest segment or the BC is the longest at this point when B is located on the diameter of the circle. I know that and it's when these two lines are perpendicular. Okay. See, I am getting confused with what I am trying to prove and what I already know because for this side length to be the longest, for BA to be the longest I need.*

He never finished his thoughts here, but went ahead and again moved Point D on the Line BC and observed how that affected the length of BA, and AC, ending this with statement, *"I need to find average balance point, which would be when it's a 90° angle."* Here again student-tool interaction was evident, and moslty driven by the software influencing Wes's thinking trying to make sense out of the current problem solving state. He then tried to explain again why Segment AD needed to be perpendicular to Segment BC relying on his content-specific mathematical knowledge; that is, why the angle between the two lines needed to be a 90° angle:

> *Why is it a 90° angle? Hm? I think I got it. Okay. The longest length is the diameter so I need to have a triangle with a maximum length, or the maximum perimeter. No, I need to have BC be the longest... So, for BH and HC, when they are the longest, BC is at its maximum length. And that's always the case. Because anywhere else you are gonna have a line going through the circle that doesn't pass through the center, so they can't be the longest. So right there when these two are at their longest, length BC is at its longest length.*

In summary, he tried to make a conclusion based on the fact the two triangles were right triangles; that is, he kept using the fact that since Triangles BAH and ACH were right triangles, BH and HC were the longest chords in the circle as they were the diameters. However, his proof at that point was not complete, and was not getting him anywhere. Here he was very close to proving his conjecture entirely if his strategy involved looking at the Triangle BCH as two triangles, Triangle ABH and ACH. In that case, he would have been able to analyze each of the triangles concluding that since AH is constant, and BH and CH are the diameters of two right triangles, Segments AB and AC are of maximum length for the two inscribed triangles, and hence BC is it at its maximum length when AH is perpendicular to Segment AH. In the interview he explained that at that point he got caught up with the triangles while thinking, *"Oh, I need to use this information right now, this is important that these are right triangles,"* followed by him saying *"but not necessarily, not at that point because that comes later."* He said for this situation that GSP hindered him as it influenced his thinking processes, instead of his thinking guiding the use of GSP to aid his thinking processes. What happened afterwards was a direct consequence of taking, what he called, "a step back"; "*if I wouldn't have taken a step back just to take a minute and analyze other parts of the problem, of the sketch, I don't know where I would have gone from there."* Hence, assessing the state he was at, led him to reflect on activites undertaken during the session. The concept of stepping back was very interesting. In the interview he explained,

> *When I got stuck, I tried to step back, take a step back, think over what I've been thinking of because sometimes I get so entangled in the problem that I can become lost or focused on something that doesn't really matter, so taking a step back allows me to clear my head for a second and then I go back in.*

He added:

> *Take a step back to get off what I was thinking about. If I would have focused on those right triangles, I mean it was an important aspect of the problem, it just wasn't at that point it wasn't important. Because it wasn't what I needed to get to from Point A to Point B. It wasn't what I needed at that moment. I needed it later on, just not right then. And I*

> *could've been probably still focused on that if I hadn't take a step back. Well let me consider some other options.*

During his explanation of the proof, he kept moving Point D. He did so because he was lost and did not know where to go next. He stated, *"This seems fun, this maybe will show me something."* Though his comment seems as the action was purely cognitive, selection and use of such strategy seemed to be however a metacognitive behavior. He drew on his previous knoweldge and experience of how and why to use the particular strategy and by using his executive skill to optimize the use of his resources. In the following, the effect of the use of the software as an exploration tool on Wes's thinking and actions was more than evident. The leap of faith he took was a success. After his last statement that when BH and CH were the longest, BC was at its maximum length, he asked, "*Can I use that angle right there maybe? Hm…m?"* which was the consequence of using the software as possible new information was available to be retrieved. He noticed Angle BHC, moved again Line BC by moving Point D and conjectured that it might be constant, followed by accepting this conjecture by using the Angle Measurement Tool offered in the GSP toolbar. His thinking directed him towards the use of the software, demonstrating again bidirectional student-tool interaction. He stated that measuring the angle was important because then all Triangles BCH obtained by moving Point D would be similar triangles, even though the lengths of BH and HC changed. He used that fact in continuation of his proof:

> *So, since I know that angle [BHC] was constant, that it's not changing, I know that BC is going to be the longest when it goes through, when the two lines go through the center of their circles, so when BH goes through the center of the first circle and HC goes through the center of the second circle. But then since B is the endpoint of the diameter and C is the endpoint of the diameter, so if you make a Segment BA and the Segment BH and the Segment HA and the Segment AC you have a right triangle because any triangle in a semicircle is a right triangle so that means BC has to be perpendicular to HA.*

His approach to the proof was somewhat backwards, which he also stated, and he was not particularly satisfied with it. Furthermore, his evaluation of the thinking processes was explicit though not entirely complete. At this point, I pointed out that his line of reasoning was based on measurement. Here, I definitely interferred with this thinking processes as I prompted to verify the statement that the angle is constant in a mathematically correct way. However, I cannot be sure that he would not have come to this realization on his own because on numerous occasions he stated that measuring is not equal to proving. This led to these two questions he asked himself, *"How can I show? Why wouldn't the angle change?"* demonstrating evalution of the current problem solving state, and trying to make sense of it. It is evident that the concept of angle was prominent in proving the conjecture as he kept playing with it for a long time, which relates to knowledge of what needs to be done, and what has been done in order to regulate what might be done to achieve the goal. Later in the interview, he explained why he thought that the angle concept was a fruitful approach,

> *Because that's very unique, it's a unique situation when something is constant and it's not changing. It doesn't happen very often, especially since the side lengths are changing. Everything else is changing in the problem except for those angles which is very intriguing.*

After a short pause he said, *"I wonder if those Angles [*HBC and HCB*] change?"* He went ahead and measured Angle HBC, as he knew he only needed to measure one of them to verify his conjecture again utilizing the the Angle Measurement to accept his assumption. Here, his thinking processes influenced the use of the software as a measuring tool with the goal of verifying his conjecture. He then concluded that for any circle combination he would have Triangles BCH unique to similarity, finishing his line of thoughts by asking, *"But why, but why? That's another thing I have to prove. Why don't they change?"* He then restated in his own words what was given in the problem, commenting also that the Line AD that he drew was important as it related to the conjecture he was trying to prove. He had a sense he was close to proving it, but missing a link as his proof was based on mesauring Angle BHC. He used GSP to move Line BC by moving Point D and observed the Triangle BHC. Decision to move Line BC was a metacognitive behavior where use of the exploratory capability of the tool influenced his following thinking processes. Shortly after a moment of silence was interrupted by an idea that *"popped in"* his head,

> *O…oh., dah. I know this! This Angle [HBC] doesn't change because any angle that subtends the chord… this chord [AH] is equal to one another, they are congruent. That's one of my favorite theorems. I don't know why cuz I always thought it was so neat. So, this Angle [HBC], as long as it subtends this chord [AH], it's the same.*

When I asked him if he could provide the conjecture now, he replied affirmatively and engaged in construction of logically connected mathematical statements; that is, the implementation episode started. During his first attempt his reasoning was scattered, *"That said, let me think about it. Did I answer the questions why is BC the longest? Oh, maybe I need to state that again,"* but provided him with the opportunity to share out loud mathematical statements, to evaluate them and to organize his thoughts into logical statements and gave it another attempt,

> *BC is gonna attain its maximum length when BH and HC are at their maximum length which is when they pass through the center of the circles, which means they are the diameter of the circles. Yes, it's kinda like, I think of it like an alligator mouth, like the mouth of the alligator, if it's like here, if the end of the mouth is right here, then the end is here, the segment that connects them will be shorter than if it's a bigger alligator mouth.*

However, since he did not mention the fact that the Angle BHC remained constant, I asked him how was this important for the proof,

> *So, what I was thinking was since this Angle [BHC] which we proved was constant in here, it's always gonna be constant so since that's the case all that matters now is the length of BH and HC because I need them to be at their maximum, their maximum length for BC to be at its maximum length because that angle is constant and that only occurs when they are the diameters of the circles.*

Hence, evaluation of the current solution state was done explicitly because of my question directed at it. I prompted here his metacognitive behaviors. His reflection on the exploration episode where new information was obtained allowed him to piece together his previous knowledge and deductions he made during the exploration episode to prove his conjecture. During the process of proving the conjecture, student-tool interaction was evident, and the software became an instrument through his appropriation of the tool and through effects of the tool on his problem-solving processes and strategies. Although, proving the conjecture required retrieval and coordination of a variety of geometry concepts, and problem solving strategies, Wes did not exhibit any of the negative affect, but rather persevered in his problem-solving activity.

After completing Part a of the problem, he focused on Part b of the problem, which did not pose a big challenge. He read the statement of the Part b and made the decision of doing the construction on the GSP,

> *Given any two circle, I would construct the intersection points right there [Points A and H], and I would construct the Segment [AH] between those two points, and then I would go here [Point A] and I would make a perpendicular line, perpendicular to this Line [AH]. Construct perpendicular line and this is where Point B [at intersection of the perpendicular and the circle] would be. This is where Point B would be and Point C would be right there [at the other intersection of the perpendicular and the circle].*

Planning and implementation occurred simultaneously. Furthermore, applying his knowledge and skills, as well as monitoring and evaluation of his progress was automatized.

After solving the problem Wes was satisfied with the outcome. Interestingly, during the process of solving the problem he said the problem difficulty raised to about 3 to a 4. However, once he solved the problem, he never explicitly assessed the solution. During the interview, he said he was confident he solved the problem correctly. When asked how did he knew he solved it correctly, he said, *"I just had this feeling now when I know I completed a problem and it's hard to describe but you just know it."* I probed more asking him how could be sure, and he replied, *"Because all my logics seems consistent with my prior knowledge. Things are working how they should,"* finishing this statement by listing mathematical content he used to solve this problem to support his statement. Hence, evaluation was based on judgments he made regarding his knowledge and thinking processes employed in this problem-solving situation.

He again ranked the problem difficuly being of 3 out of 5, and stated his view of the problem,

> *[I] liked it required accumulation of mathematical knowledge in solving it. I mean it is not one class that I used to solve this problem, it's from lots of classes, little bits and pieces that not necessarily I would attribute to that class, but just little tiny things that come together eventually.*

When I asked him what were the top things that were most important to solve the problem, three categories emerged;

1. mathematical knowledge—included knowledge of perpendicular, diameter is the longest segment in a circle, and angles that subtend the same chord in a circle are congruent. Mathematical content was very important during problem solving because his reasoning depended on it without which he would not have had *"a very good argument"* and *"could have [not] convinced someone else."* He noted that the above-mentioned mathematical knowledge was important for proving the conjecture, while that was not the case with stating the conjecture and constructing Point B.

2. visualization—

   needed that [visualization aspect] cuz if I was just to read this statement and had to draw on the piece of paper there is no…o way I would have been able to have done this at all. Because that would have required me having to draw all these segment and measure each one of them, and my hand isn't perfect. I can't draw perfect circles. And that would have made me question my strategies going forward, I don't know if my strategies would have been the same if I would have had to drew it by hand,

3. dynamic software—

   If I didn't have some kind of dynamic tool to use I would have been playing around on the paper which would be incredibly difficult because they are so many options, there is infinitely many options and being human there is so many mistakes you can

> *make when drawing it and just being able to do it so quickly. I know how to construct these things so what's the point in constructing it over, and over, and over again when I know that. I need to be able to solve the problem now rather than taking forever and keep drawing it. Where does it get me? With GSP you can do it instantly.*

His use of GSP as an exploration tool was evident. He used the dynamic aspect of GSP, such as changing the parameters of the problem (size of the circle), moving Point D on the Segment BC to see how BC acts. He used the Measurement Tool several times in the process of solving the problem to verify his thinking and then as a result to decide on a strategy going forward. Moreover, it is important to comment on his use of the Measurement Tool. He stated,

> *The measurement came second, it didn't come first. I wasn't just measuring things to see when it's gonna be the same, or when it's gonna change. I only measured when I had an idea and wanted to verify that idea.*

Hence, the appropriation of the software to a measurement and verifying tool was influnced by his thinking processes. He also used GSP to draw auxiliary objects, such as lines, segments, and line segments. When I asked him if he wanted to use some other aspect of GSP to solve this problem, he said that he wanted to graph the length of the Segment BC as Point B moved along the circle. He explained,

> *Cuz for some reason I was thinking of parabola, the length is getting, the length gets to its max and then it started to fall back down. And I wanted to graph that to see if that was in fact the only maximum that occurs because I was thinking: Well could the maximum occur here, could it occur two times. I was interested in that. I am trying to think right now if I could have done that.*

As I was wrapping up the session by thanking him for the session, Wes said, "*It was fun. I liked that problem.*"

Synthesis of the Problem-Solving Session 1

Wes read and understood the problem. During the understanding episode he put effort in to make sense of the problem information in a diagram, monitored his progress and reengaged with the problem by numerous times reminding himself of the conditions and requirements of the problem. Because of the exploratory nature of the problem, he decided to use the dynamic capabilities of the software in his search for a solution plan during which he accessed knowledge that might be helpful in solving the problem. His actions were related to the problem and its goal, and regularly monitored. The nature of student-tool use led him to the conjecture. In order to prove his conjecture he tried to examine the explicit relationships between the given information, new information and the goal of the problem. Even though he mainly used the software as an exploration tool, the feedback the software provided influenced his problem-solving activity; new information was obtained and coordinated with personal knowledge which led him to prove his conjecture by constructing logically connected mathematical statements. Through his actions he transformed the tool to help him visualize the problem, develop a better understanding for it, analyze the problem, and explore different possibilities. Thus, the tool became an instrument through stages of personalization and transformation. During this process, allocation of resources, namely content-specific knowledge and the software together with regulation of these resources was of great importance to engage in productive efforts. Once he proved his conjecture, solving the rest of the problem was done with ease. Verification of the results was never done explicitly, but rather implicitly. Figure 7 demonstrates Wes's proble- solving cycle.

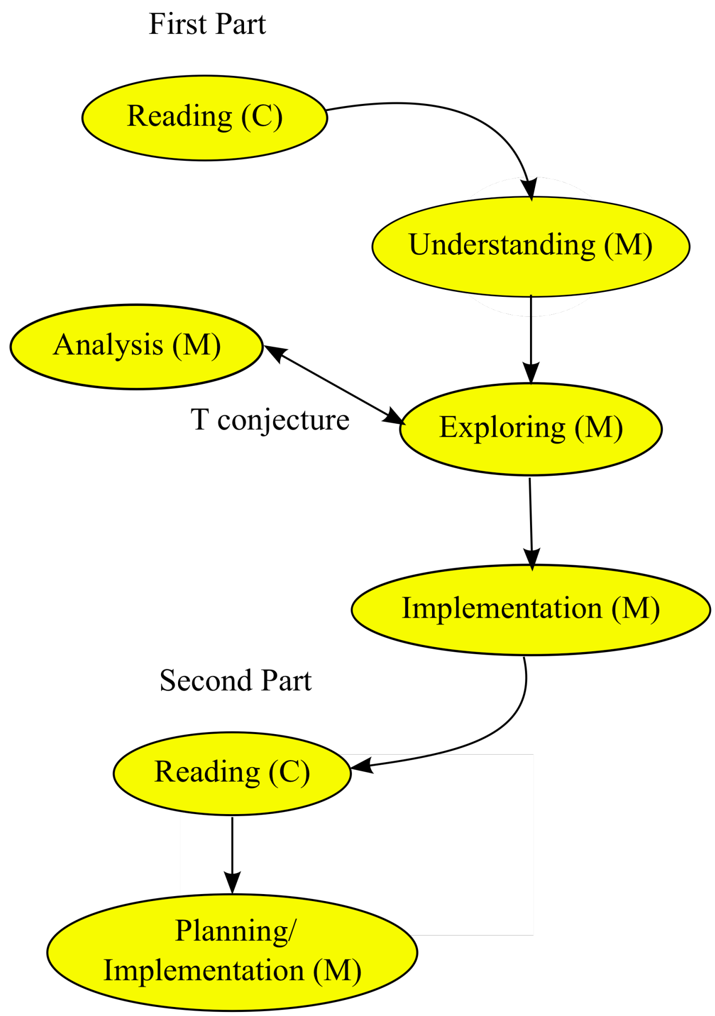


Figure 7. Parsing of Wes's Problem-Solving Session 1.

Problem-Solving Session 2: The Airport Problem

I met Wes for the Problem-Solving Session 2 on a Friday afternoon after his student-teaching. Coincidentally, he had just finished teaching points of concurrency a week before we met. The problem solving session 2 took approximately 80 minutes. Wes spent approximately 45 minutes solving the Airport Problem (see Figure 4 in Chapter 3, p. 64), and during the remaining time we reflected on the session. For this problem solving session, I gave Wes a sheet of paper and a pen, and opened The Airport Problem.gsp file on the problem was printed. Anwering the first two questions and providing a sound explanation for his answer took approximately 30 minutes. The last part of the problem, providing a geometric interpretation of the sums, was trivial. It took barely 15 minutes, but Wes provided two possible solutions, one of them being extremely creative drawing from his problem-solving experience.

Solving the Airport Problem

Wes started the session by reading the problem aloud from the hard copy I gave him which was a matter of preference, *"if it's not something I am familiar with, I like having hard copies. There is something about it."* This monitoring strategy was a metacognitive act that helped him maintain focus during the reading episode. Wes never saw this problem before; however, he had the opportunity to explore a similar problem twice, where he had to construct a point such that the sum of the distances from that point to its vertices is small as possible, known as the Fermat's point. After he read the entire problem twice, he took the pen and started representing the problem on the problem sheet. Hence, the understanding episode started. He used a cognitive problem solving strategy, drawing a diagram representing the problem, however selection and use of that problem solving strategy was based on thinking about his previous problem solving strategies, which is a metacognitive behavior. When he read the problem, he *"broke down the question to its parts"* optimizing the use of his cognitive resources; effort was made to carefully make sense of the problem in a diagram. He read the first sentence, drew an equilateral triangle and labeled the triangle vertices with A, B, and C. He explained,

> *It says that the three towns are equally distant from one other so C has to be equidistant from B and from A and A has to be equally distant from B and C. So I figured it would be an equilateral triangle which would be right here [Figure 8], I have equal angles.*

Hence, he made sense of the problem situation; he interpreted the problem statement and the conditions of the problem were explicitly and correctly noted. He read the second sentence, restated it, placed a point labeled Air, which stood for Airport, inside the triangle and drew perpendicular lines from the Airport Point to the sides. At this point it is clear that he assessed the current state of the his knowledge relative to the problem; Point Airport was chosen, placed inside of the triangle and perpendicular segments were drawn from the point onto the sides. At this point, I interrupted him as he was quiet by encouraging him to keep thinking aloud. He continued to share his reasoning, *"I just put that point in the middle, there is no exact location at this point yet,"* explaining as well why he drew perpendiculars to the sides of the triangle, *"since I am thinking that the distances need to be perpendicular distances because the distance from a point to a line should be the smallest distance possible which is a perpendicular line dropped from there [Point Air]."* At this point, both conditions and the goal of the problem have been correctly and explicitly noted. He did not have any difficulty understanding the problem, as it was *"straightforward."* The following demonstrates effort he put in understanding and making sense of a mathematical problem that was put in a real life context,

> *What are the possible locations? What's the best location for the airport? That's more of a practical question because you are you have to think about the context of the problem. You have three towns and you have an airport, you want to minimize the distance from each one and you want it to be an equal distance so that it's just an efficient location.*

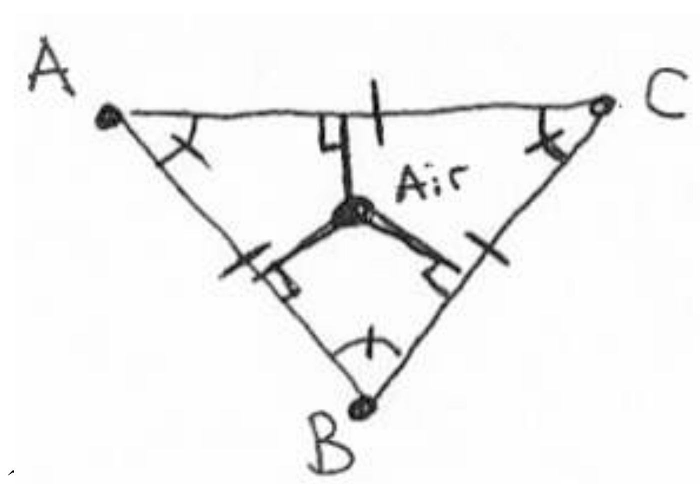


Figure 8. Wes's pen-and-paper representation of the Airport Problem.

He then reengaged with problem text by reading again two of the questions, Parts a and b of the problem, and immediately thought of his first idea *"it looks like the perpendicular bisectors"* but promptly realized and said, *"well if it's an equilateral triangle all the points of the concurrency are in the middle."* It is important to note that up to that point he did not yet use technology but drew a sketch on a sheet of paper,

> *I was getting a rough idea in my head cuz I like to when I break a question at its parts I like to have some kind of visual representation so I can as I go along add things to the representation and eventually in the end I have a visual representation of what the question is asking me.*

This behavior was aligned to what he shared during the preliminary interview; during the paper and pen experience he *"started thinking, brainstorming what [he] thought was gonna be the answer, circumcenter, centroid, basing it up on [his] previous knowledge of centers,"* before moving onto using technology as he had *"a better idea what [he] was being asked."* It was clear Wes thought about previous content-specific knowledge and experiences that might be helpful in current problem solving situation. Before testing his conjecture, he decided to switch from paper and pen to using technology with an idea on his mind, *"Now's the time to use this. Now I have an idea what's going on,"* which again shows evidence that effort was put forth to fully comprehend the problem, and allocate knowledge that might be helpful in solving the problem before trying to solve it and use technology to do so. On the GSP sketch with the problem, he repeated what he had done on the paper; he started constructing an equilateral triangle explaining this time his line of reasoning,

> *So, we know we have a triangle of some sort and in fact we have an equilateral triangle because for that condition to be satisfied they all must be equally distance parts. I need to construct an equilateral triangle. I will draw a segment, and I will label that Segment AB, and then I will take two circles, one centered at A to B, and one centered from B to A. So I can conclude this since this radius [AB] is also the same as this radius [AC], and this radius [BA] is same to this radius [BC], by the transitive property I have an equilateral triangle. So this intersection point then is Point C, which is Columbus in the problem.*

That said, his thinking shaped how the software got used; he performed variuos parts of the problem statement step by step on the computer as they were outlined in the problem before moving onto the next part of the problem. Simultaneously making a representation of the problem, and explaining his thinking and rationale behind it, did not present any difficulty; he was able to successfully direct his thinking staying mentally engaged throughout. After he constructed Triangle ABC, he placed the Airport Point outside of the triangle, although in the sketch earlier he placed the Point Airport inside the triangle. As he placed the point outside, he continued to share his thinking,

> *I am gonna put it out here just because I have an idea it might be in there but it's not necessarily true. It might be able to be outside as well. Because the way I am gonna construct this is to move that move that point wherever I need it to be.*

Wes was aware of the obstacles he might encounter in the process of solving the problem; that is, he was aware of the knowledge as to what needs to be done and what might be done in this particular context that would allow him proceed with his solution plan that was for me at that time overt, but became apparent later on. He drew upon his knowledge of problem solving using the software and used executive skill of planning to optimize the use of his own cognitive resources, which is evidence of regulation of his thinking processes. He continued to think aloud hypothesizing the solution of the first part of the problem, *"I had an idea that the airport will need to be on the inside to minimize the sum cuz I mean if the airport is up here [outside], it's really gonna be far away from C, I feel that."* Planning episode started during which he engaged in devising a solution plan for the first part of the problem relying on his previous content-specific knowledge. At this point his plan was not overt, but rather implicit. It was later explained that because of the exploratory nature of the problem, he decided to use the dynamic capabilities of the software to test his conjecture. Hence, construction of new knowledge was directed towards the software. Next he connected the Airport Point to the vertices A, B and C, measured all three lengths, explaining, *"And now what I am gonna do is, I am going to calculate the sum of those because that's what it asks. It says an airport will be constructed such that the sum of its distances to the roads."* His actions were purposeful and goal driven, though monitoring of progress was not apparent at this point. As he verbalized what he did by checking at the same time the problem statement again, he immediately realized his mistake, *"O…OH, I am doing distanced from the cities themselves."* Although, evaluation of the current state was missing, and could have been costly, by explaining what he had done and noting the conditions of the problem he became aware of the inaction described above. This part of the episode shows that lack of monitoring of progress can hinder the problem solver's path to finding a solution of a problem. He left the segments connecting the airport point to the three vertices, however, but changed the line width to dash, and constructed three segments that were perpendicular to the bases of the triangle *"because that's the distance from the airport to the roads,"* and changed the line width of the three segments into thick allowing him to focus on them more easily. Again, here his problem solving experience directed the use of the software as a visualization tool. While he was constructing the three perpendiculars, I asked him what was his expectations to where the airport should be built replying, *"My original expectation was gonna be the centroid of the triangle for an airport will be constructed such that the sum of its distances to the roads is small as possible. I think that's where it will be."* Already at an early stage of this session before being fully engaged in solving the problem he conjectured that points of concurrency might be the solution to the problem. For me as a researcher and an observer it was interesting to find out how he related the problem to the content needed to solve the problem. That said, he revealed afterwards in the interview his reasoning,

> *It [points of concurrency] was in the back of my head probably because today I was teaching, or I was testing my students over some of that content so it was fresh in my mind and the key part there was the circumcenter is equidistant from all the vertices so I thought well if it's equal distant from all of the vertices then the paths I take from the cities should be equal at the circumcenter.*

Hence, it seems that conjecture was driven by his recent experience, accessing and considering mathematical concepts and facts that might be useful, and ultimately by piecing together knowledge of what needs to be done in this particular real life problem solving context. It is, however, yet intriguing as to why that actually happened as the problem did not allude to points of concurrency.

Even though he conjectured circumcenter might be the solution of the problem, he added that there might be more than one solution implying the problem being suggestive, such as *"the word smallest,"*

> *But there might be more than one location but I am not just positive yet. It says the smallest but there might be two places where it's the smallest. I don't know why I am thinking that but I just have a feeling.*

Here the assumption of the number of solutions for the first and second part of the problem was driven by keyword that he focused on "possible locations" and "best location,"

> *That [possible locations] led me to believe there are more than one…The first after I read that one [Part a] I thought there is gonna be more than one location. And then when I read this [Part b], there is gonna be something different about the best location than the all the possible ones. There is gonna be a unique one. There is gonna be something special about the best location.*

This shows evidence he fully put effort in understanding and analyzing the goals of the problem, and at the same time made sense of it by considering at the same time mathematical content and context. Hence, during the planning episode new insight allowed for better understanding of the problem situation. At this point his problem solving path became overt; his strategy going into the problem was using GSP to *"make a simulation of the problem on the GSP, make a dynamic* *point and use that point to test the conjecture of the circumcenter."* Hence, being aware of the goal of the problem, and then wanting to test his conjecture demonstrates his actions were focused, purposeful and goal driven. Although this may be seen as a purely cognitive action, selection and use of such strategy is, however, a metacognitive behavior; he drew on his previous problem solving experience of how and why to use the particular strategy and by using his executive skill to optimize the use of his resources, namely the software as a verification tool. At this point, however, the circumcenter was not yet constructed. He decided to measure the three lengths, and to add them together before using GSP to move the Airport Point helping him to discover a very interesting occurrence,

> *At this point I have a dynamic location of the airport that can actually go anywhere where I need it to go. All right so here it's 6.84 [cm], that's the sum of the distance to the roads [Smile]. Inside the triangle it's always the same. No matter where you put it. When you are inside the equilateral triangle the sum of the distances is always 6.84 cm, so no matter where they put the airport it's always 6.84 cm.*

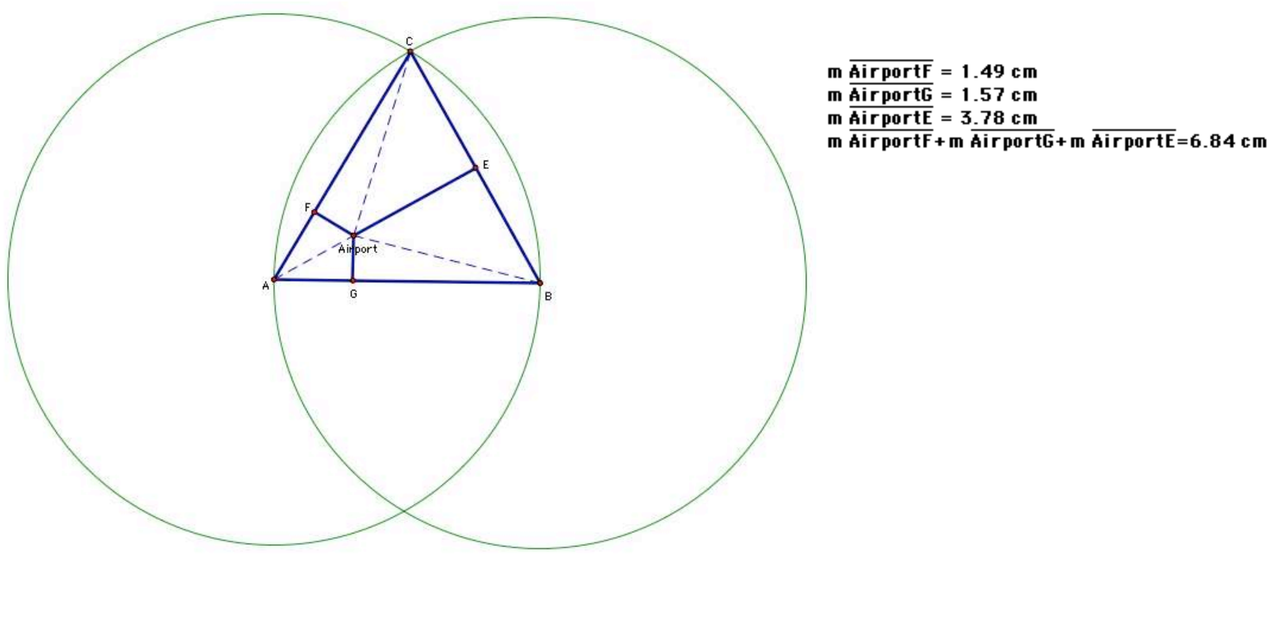


Figure 9. Solving the first part of the Airport Problem: Wes's investigation inside the triangle.

He was surprised by the discovery, revised his previous conjecture, and tested the new result by moving the Airport Point outside of the triangle. Though, at first his actions influenced the use of the software, feedback that it provides to the user then shaped the use of the software as an exploration tool that continued to influence his actions. At that point, one segment disappeared enabling him to have the sum. He began to self-question what he had done, *"Let me think about this. Why would they be disappearing?"* Directing his thinking to judging the effectiveness of his previous actions, led him to quickly realize that he drew the perpendiculars to the sides of the triangle instead to the lines containing the sides of the triangle, and decided to reconstruct the perpendiculars. Hence, he assessed the relevancy of new information provided by the software that helped him again to direct his knowledge of perpendiculars and use then executive skills of self-correction to optimize the use of his cognitive resources.

He again moved the Airport Point in the plane, both outside and inside the triangle observing how the sum of the distances changed, *"6.84 [cm] and it's constant inside of here and when we go outside it starts to change. It starts to increase and then it goes back to a constant [inside] and it starts to increase [outside]."*

After this investigation he stated his conjecture, *"Well, I think I know now what the possible locations are. I would say the possible locations are anywhere inside the equilateral triangle."* With this implementation episode ended, and the first part of the problem was solved although he was quite perplexed by the result.

During the process of solving the first part of the problem that included making and testing a conjecture, student-tool interaction was evident. Even though at the beginning, Wes's thinking influenced the appropriation and transformation of the software to a tool that allows construction, exploration, and measurement, the software became an instrument through effects of the tool on his problem solving processes and strategies.

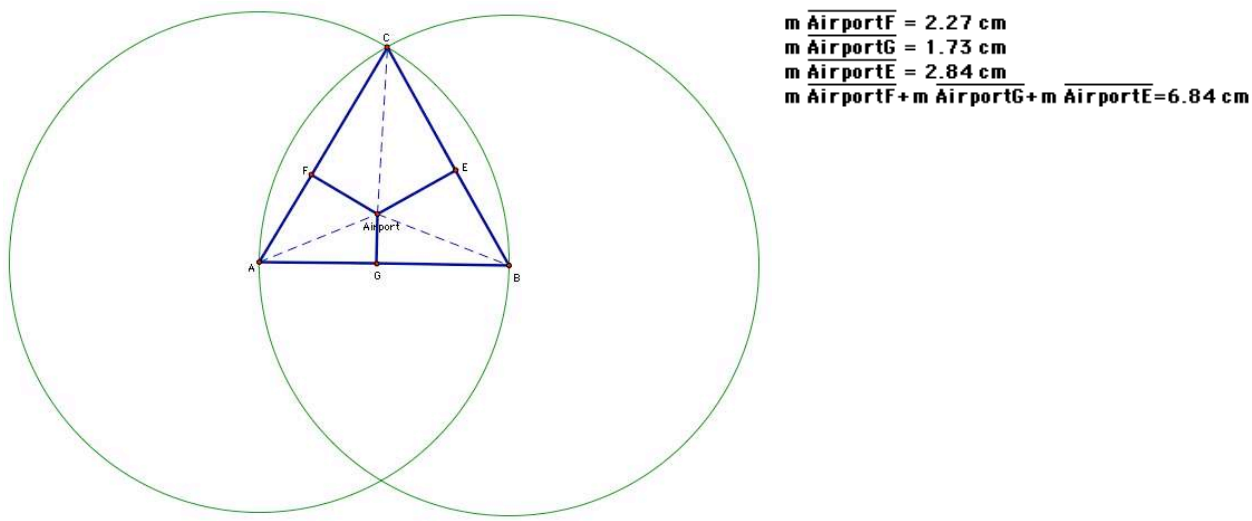


Figure 10. Wes's solution for all possible locations for the airport.

He immediately started working on the second question. He again read the question, followed by understanding the problem and analyzing what needs to be done that occurred intermittently. He tried to answer it quickly but again questioned his conclusion,

> *What is the best location for the airport? Since it's anywhere in the triangle it really doesn't matter. That's just it. It can be anywhere inside this triangle and well, hold on. What are the possible locations for the airport such that the sum of its distanced to the roads is as small as possible? Sum of the distances from the roads is as small as possible. So if I was going from this city and I wanted to go to the airport, I can't just go straight there as the birds fly, I have to go along the road here [side of a triangle] and then go there [towards the Airport Point]. Okay.*

He answered the question based on solving the first part of the problem noting at first goal of the problem. However, as soon as he verbalized his thoughts aloud, he evaluated his thinking. He then reengaged with the problem statement; he reminded himself of the requirements of the problem keeping in mind the context of the problem. That is, he examined the explicit relationship between the conditions and the goal of the problem through which he gained a better understanding of the problem. At the end, it seemed as he had at that time selected a perspective driven by the goal and the conditions of the problem to solve the problem, though it was not completely overt to me. For that reason, I encouraged him to continue thinking aloud, *"At this point I would say at the centroid. It's really no difference. I can interchange the letters of my vertices and it wouldn't really change the problem because you have symmetry in all the cases."* Since before he concluded that all of centers of a triangle are concurrent in an equilateral triangle, I was interested in his decision it was only one of them, being centroid. Therefore, I directly asked him to explain his answer, that prompted a different answer at the end of his reasoning,

> *How come the centroid? Because I see the centroid as the center of the equilateral triangle. Oh well, actually in an equilateral triangle all the points of concurrency are at the same point so it really doesn't matter which one I am saying. But if I say the circumcenter, I can say that it's an equal distance from all of the vertices, and then I might be able to say, well, if it's equally distant from all of the vertices then the path along the roads itself I think would be the shortest distance possible but it would be the same for all three cities. But, let me go ahead and do what I was thinking.*

Hence, evaluation of the current solution state was done explicitly because of my question directed at it that could have suported his following actions. I cannot be sure, however, that he would not have come to this realization on his own. Again, allocation of the knowledge directed by the problem context became apparent. After my interruption, he continued trying to prove his conjecture that opened planning and implementation episode in which he continued to use the software.

He sat in silence working on the problem while I observed him. For each of the vertices he measured two possible paths from that vertex to the Airport Point. For instance, he measured distance from A to G, G to Airport Point and added them together. Likewise, he measured the distance from A to F, from F to Airport Point, and added them together. Similarly, he did the same for the vertices B and C carefully lining up each calculation to its corresponding section on the sketch that helped him monitor his actions. As he worked on getting all the measurements, he continued to explain what he was thinking and doing,

> *I am thinking, well the best location for the airport would be somewhere were everyone can access the airport at in equally distance, like your route from C is no different than it is from A, no different than it is from D. I am gonna add those two together and then I am gonna do that for each of the other cities and I am gonna see perhaps how the distances are from the cities if you take the car. I am gonna have to start paying really close attention to what I am measuring.*

Later in the interview he explained his problem solving strategy in more detail,

> *I did the pairs because I wanted to consider all the routes possible, well not every route possible but the most direct routes possible and then I wanted to add them together to find the sum and to find the best location; the sum of the distances from each of the cities should be at minimum and if possible the same from each one which since I am in the equilateral triangle, it should be the same.*

At this point his plan became overt. The plan was well structured, and certainly relevant to the problem solution. I could argue, however, that he failed to assess the quality of the selected method strategy with respect to, for instance, speed and the degree of elegance. Nevertheless, this matter will be explained later. Measuring appropriate paths in pairs and placing measurements close to the path measured demonstrates that implementation of the plan occurred in a structured way and that the actions were monitored. Though this seems as the problem solving method was purely cognitive, selection and use of such strategy is, however, a metacognitive behavior; he drew on his previous problem solving experience of how and why to use the particular strategy and by using his executive skill to optimize the use of variety of available resources–knowledge, facts and procedures, and the software. Moreover, he monitored his progress throughout. However, he failed to effectively manage the vast repertoire of resources. During this episode his thinking, that is, his plan to test his conjecture, affected the use of the software as a measurement tool before being used as an exploration and verification tool.

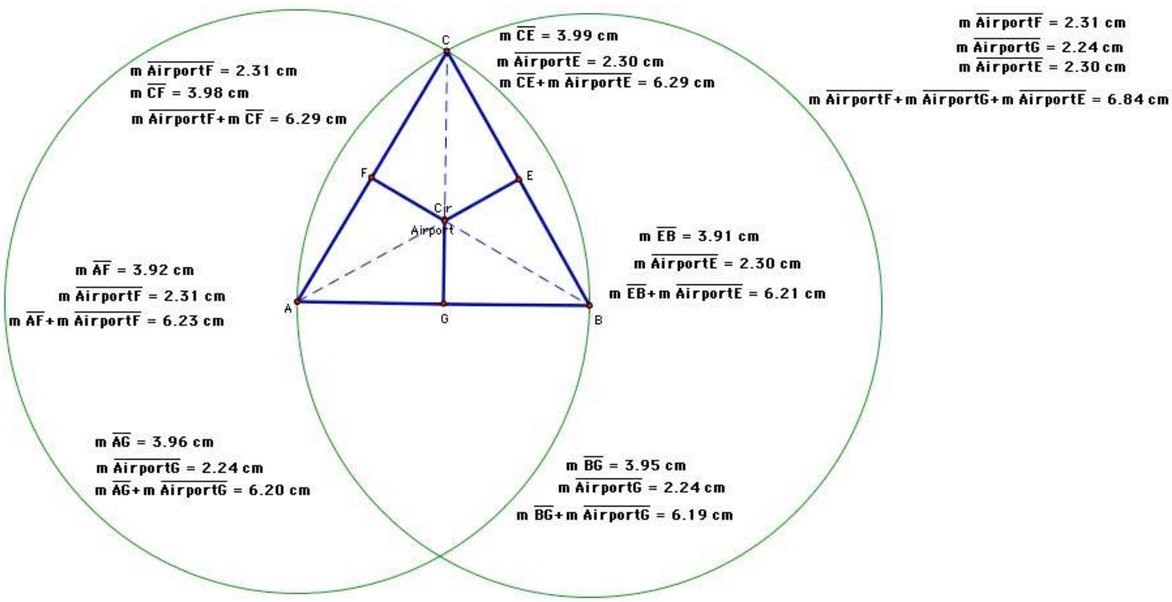


Figure 11. Wes's solution for the best possible location for the airport.

When he was done, he moved the Airport Point and observed the measurements, *"I am looking how the sums are changing relative to one another. I am trying to get these numbers as close to one another as possible,"* followed by decision to construct the circumcenter based on the observations, *"Because it looks like Airport to G is about one third the distance from C to G which would be the centroid but the centroid is also the circumcenter."* Since already before he noted that points of concurrency are concurrent in an equilateral triangle and conjectured them being the solution, his statement was perplexing, at this point I was not sure if the software provided him feedback that triggered allocation of knowledge of 1:2 ratio points of concurrency divide perpendicular bisectors or he tried to gain a deeper insight into the problem during which he wanted to pretest his conjecture as circumcenter was yet not constructed by using dynamic capabilities of the software. Later in the interview he explained the latter,

> *I constructed the circumcenter and I was moving this point around near the Airport and I was observing on the screen the measurements I had taken cuz I have taken all those measurements where approximately it [circumcenter] is to justify my conjecture that it would be the circumcenter. So then I moved the point away, constructed the circumcenter and took the Airport Point and moved it to the circumcenter to see if it all the distances were equal, all the sums were equal.*

He constructed the circumcenter, labeled it Cir for circumcenter and compared it to the Airport Point, *"Now I am gonna take my Point Airport and move it to the circumcenter to see what I observe. It is pretty much right there, that's where they are all the same."* When he used GSP, however, he was not using trial-and-error, but rather he was using GSP to test his conjecture,

> *I had visualized the airport being at the circumcenter. I had visualized that. I saw it in my head but I was looking at it…I was thinking well the airport can be anywhere in here but when I am looking for the best location it couldn't be down here because that's too far to travel from cities. So I knew it was gonna be somewhere in this area. Yeah. That's why I stayed so close to this area.*

Hence, his thinking transformed the tool to be used as a verifying tool. He concluded by stating his conjecture, which ended implementation episode, and confirming its alignment with the first question that opened verification episode,

> *So from any city, the path I choose is gonna be equal as if I am going from another city to the airport. So I think that would be the best location for the airport because the distance is equal from each of the cities but I think that would be the most efficient method and still since I am inside the triangle my first condition is still satisfied. The airport is constructed such that the sum of the distances to the roads is as small as possible, so, I think this is the best location.*

Thus, during verification episode Wes reflected on the reasonableness of the result as well as assessing the consistency of the final result with the problem conditions. Nevertheless, implementation episode took quite some time, even though earlier he stated that he thought the solution might be the circumcenter, and could have easily derived the solution using knowledge he had about the properties of concurrency points of an equilateral triangle that raised a question of lack of regulatory skills. In the interview he explained his strategy,

> *I wanted to be sure but I was kind of in a procedural mode right there: I want to measure these, I want to measure these [different paths]. I was very systematic about it. Usually when I am solving problems I want to be systematic and detailed. Cuz you never know when something you assumed to be true or take something for granted maybe that's that step you took for granted maybe wasn't supposed to be taken for granted. Maybe there was something to it.*

Thus, he used a cognitive problem solving strategy, measure and add. Though completion of that task and similar ones is basically a cognitive process of using a cognitive strategy, it is the metacognitive behavior of the problem solver that selects and uses these cognitive tasks and should not be taken for granted. However, as soon as he verbalized this he realized another path he could have gone with,

> *Now that I think about that, I shouldn't have done that. Like, if I am looking at this now I see that this triangle right here with that hypotenuse from the Airport to A, that these two [AFAir, AGAir] are congruent, these two [BGAir, BEAir] are congruent, these two [CEAir, CFAir] are congruent. I should have seen that immediately.*

Hence, the regulation at that point was not perfect, but verbalization of his undertaken activities allowed him to judge their effectiveness. Nevertheless, this episode highlights the importance of considering utility of potential alternative knowledge and actions. He did not consider first two parts of the problem challenging rating them being of 2 out of 5 difficulty.

After he finished the first two parts of the problem, he started working on the last part of the problem; that is, trying to give a geometric interpretation for the sum of the distances with respect to the optimal point to the sides of the triangle. He read the statement three times, restated it, *"So, I am trying to characterize that point in terms of these three lines, sum of those three lines. Hm?"* before asking me to help him understand it which opened the understanding episode. Based on the preliminary interview, I knew he had looked into the geometric interpretation of the arithmetic, and geometric mean with respect to an isosceles trapezoid in the Problem Solving course he took. I used that as an example to explain what the problem statement meant. After my example, he read the statement two more times, each time followed by a short pause where he again put out effort and energy to understand the problem statement before trying to characterize the airport point using the knowledge of properties about the concurrency points that opened the analysis episode,

> *What I am struggling is, I am at what I call the circumcenter but that's also the location of the centroid and the incenter and the orthocenter, but the orthocenter I think really doesn't matters. The centroid is the balance point. But I can characterize that point geometrically. It's 1/3 the distance of the median but so it's 1/3 the way of the base of each, it's 1/3 of the median in height. Does this make sense? So from the Airport to G is 1/3, from the base to the airport that's 1/3 the height of the equilateral triangle. So, I can characterize the point that way geometrically.*

Hence, conditons and the goal of the problem were explicitly noted. Furthermore, he decomposed the problem in its parts, and examined the relationship between each part with respect to the goal of the problem. Here it was clear Wes thought about previous content-specific knowledge, namely 1:2 ratio, that might be helpful in current problem solving situation which demonstrated awareness of his knoweldge in the current problem solving state. At this point he was perplexed how to continue from this point on, *"I characterized the location of the airport in terms of the sides of the triangle, and not the sum,"* repeated what he already said about the relationship, adding, *"But I need to, I am worried about the sum of these. That's what I am worried about."* Though, he was able to deduce individual relationships between the Airport point and length of the needed segments, those were yet not logically conneected into a coherent whole. Clearly, he evaluated his reasoning thus far, at the same time being aware of what needs to be done in this problem solving situation, he tried to direct his thinking processes to that goal. For that reason, he engaged himself into self-questioning that got him to realize the solution to the problem,

> *How can I characterize the sum of the distances geometrically? Give a geometric interpretation for the sum of the distances of the optimal point to the sides of the triangle.*

> *Sum of the distances? What is it? What is the sum of the distances geometrically? What is the sum of the distances geometrically? OH, hold on. It's the height of the triangle!*

Here he put effort to piece together his previous knowledge and deductions he made during the analysis episode to solve this problem. However, since he uttered the solution without any explanation, I asked him if he could explain his thinking. He explained, *"Because each one of these is 1/3 the distance of the altitude of the triangle and there is three of them."* However, he promptly brought this into relation to Fermat's point that was for me completely unexpected,

> *So if I were to kind of translate these Segments [EAir, FAir, GAir], I could actually construct a median or I can construct the altitude from these three segments. I remember about Fermat point... You can manipulate those segments in the proof of Fermat to where you end up making a straight line. Oh, I think I know what I can do.*

He devised his plan and implemented it basing it of the proof of the Fermat's point. Before copying segments on the Altitude AE, he colored the Segments GAirport, EAirport and FAirport in green, blue and red, respectively, and concluded,

> *So what I am gonna do is, I am gonna choose airport and I am going to change colors of these segments. Green [GAir], this one is gonna be red [FAir] and then I am going to construct a circle by center and radius and place it there. Red. And then from this point I am gonna construct one that has the length of the green radius and it goes through it and I built the altitude out of it.*

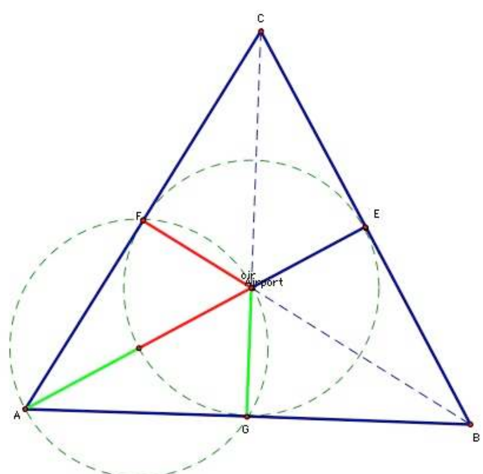


Figure 12. Wes's geometric solution of the geometric interpretation of the sums.

As for the first solution he did not use the software at all, this was obviously not the case for the alternative solution where his thinking guided the way the software was being used. As his was proving me with an alternative solution his behavior changed; he exhibited joy and pleasure. He finished the think aloud part of the session by saying, *"That is a beautiful problem. I like that."*

Even though he solved the last part of the problem quickly, he still considered this part being the most challenging part of the problem and rated the problem being of 5 out of 5 difficulty. He explained, *"Because I was struggling with the words more than anything trying to see, trying to interpret what you what the question was asking."* Hence, this provides again an excellent opportunity to highlight the importance of understanding the problem that is an important preparatory aspect during problem solving as recognized by Pólya. Nevertheless, past initial difficulty of understanding the problem, he rated the proof of the last part of the problem as being 4 out of 5 difficulty. Furthermore, he was certain that he solved the problem correctly as it was consistent with his mathematical knowledge.

As I asked him what knowledge did he anticipate to use as he started to work on he gave an interesting answer where his problem solving experience was more than evident,

> *Actually, I was pulling from my class with Dr. Wilson. For some reason I thought I was gonna have to rearrange those segments like I did. I was thinking there is something about these three segments, I am gonna have to do some kind of transformation on them and move them around to see where am I going because I didn't see any clear geometric content. I mean we are working with very simple objects. We are working with an equilateral triangle, circles, line segments. That's it. There is not very much else we are working with. So I thought I am gonna have to do some kind of unique, kind of cleaver, something cleaver that I wouldn't think of, like put that three squares problem Dr. Wilson gave about reflecting them over the line. Before I came here today, I was like I am gonna have to use something interesting and I need to keep all the options open.*

Therefore, as he was solving the problem he was aware of his content-specific knowledge and problem solving strategies that could be useful in this particular problem solving situation.

When I asked him what were the top things that were important for him to solve this problem, two categories emerged,

1. mathematical content—content that was important to solve the problem included knowing that the shortest distance from a point to a line is a perpendicular, properties of concurrency points in an equilateral triangle for the first two parts of the problem, and 1:2 relationship for solving the last part of the problem, and
2. geometry software—technology assisted him for the first two parts of the problem whereas the interactions were conjecture based as he wanted to either formulate or test a conjecture.

GSP aided in formulating the conjecture that the sum of the segments is an altitude,

> *I formulated the conjecture about the altitudes, that the three segments when you add them together form the length of an altitude, the sum of the three segments. I had it in this something like this position and I was like, if I were to make a circle and it would touch the line here and then I can make a circle with this radius and it would complete the altitude. Kinda what I did in the Fermat up here. That was after GSP, that gave me that idea cuz I was moving it [Airport Point] around and when it was like at that exact position, and I was like: "O…oh, okay,*

whilst it aided in testing the second conjecture, *"The conjecture I had already formulate was that the centroid, circumcenter, that was gonna be the best location and I used GSP to confirm that."* In these situations he used the dynamic feature of the software, *"dynamically move that Point Airport and observe the measurements changing as I move the Airport Point."* He used a lot of measurements, especially for the second part of the problem. The nature of the measurements was not "a wild goose chase," however, but rather a confirmation, *"it gave me more confidence that I was on the right track."* He also added that visual helped him as well especially when reconstructing the Fermat proof,

> *The visual appearance of it, being able to color code that was really important. It's very difficult to distinguish when you get so many lines. If I was not able to change the colors,*

> *it would have been very difficult for me to proceed on with work. So, it helped me and in general, I would have said I needed it definitely.*

He did not think technology hindered him at any point.

When I asked him what if anything he learned as a result of solving this problem, he said that he did not learn any new mathematical content but rather the importance of *"taking a step back"* during problem solving, *"You might wanna take a step back every now and then. Don't get so focused on one thing."* In essence, the problem solver needs to constantly assess the usefulness of undertaken activities, consider utility of potential alternative actions and monitor these actions otherwise problem solving may turn into a fruitless endeavor.

At the end of the session he shared that he enjoyed the problem adding the most interesting part of the problem for him, *"if I add up those three segments I am gonna get the altitude again. It's like, that blows my mind."*

Synthesis of the Problem-Solving Session 2

Wes started the problem solving session by reading the problem. In order to get a better understanding of the problem, he made a static representation of the problem. During that time he was aware of his knowledge of what needs to be done, and what might be done in this particular problem solving situation. Already at this early stage he conjectured that points of concurrency are a plausible result of the problem. He then moved onto the computer where he made a dynamic representation of the problem. Often he reminded himself of the conditions and requirements of the problem. Through this process he developed a deep understanding of the problem relating together both mathematical content and context. Because of the exploratory nature of the problem, he decided to use the dynamic capabilities of the software in his search for a solution plan; make and test his guesses. His actions were related to the problem and its goal, and monitored, which allowed for false moves to be quickly noticed and altered. During this process, allocation of resources, namely content-specific knowledge and the software was of great importance. The nature of student-tool use led him to refuting his conjecture, and making a new one with respect to solving the first part of the problem turning the software into an instrument. He then focused on the second part of the problem, where soon after anaysis of the problem a perspective of the problem was chosen based on examining the relationship between the given information and the goal of the problem. Though a perspective was appropriate to solve a problem, nevertheless absence of reflecting on the effectiveness and efficiency of selected method was time and ellegance costly. In order to implement his plan he used the software which allowed him to engage in coherent and well-structured seriuos of steps, and verify his conjceture. Nevertheless, he evaluated the reasonableness of the result based on his content-specific knowledge.

Solving the last part of the problem proved to have been the most challenging part of the problem as he was not able to understand the problem statement. This highlighted the importance of understanding the problem during problem solving. In order to solve the last part of the problem, he tried to examine the explicit relationships between the given information, and the goal of the problem. During this process, allocation of resources, namely content-specific knowledge and the software (for the second solution) was of great importance. He was extremely content after the session. Figure 13 demonstrates Wes's problem-solving cycle.

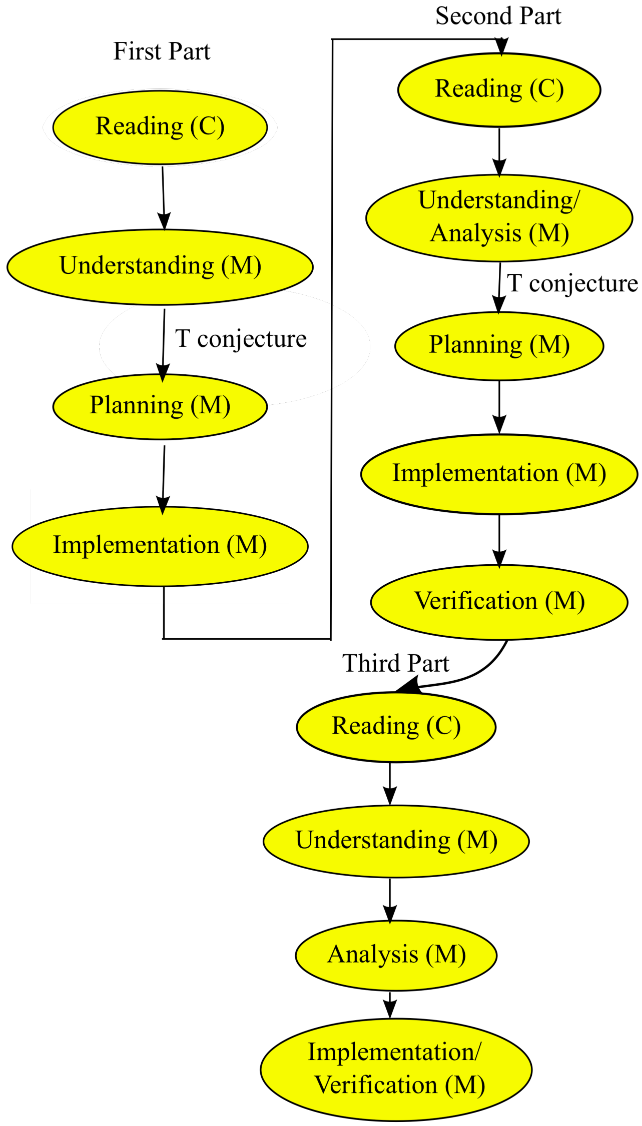


Figure 13. Parsing of Wes's Problem-Solving Session 2.

Problem-Solving Session 3: The Land Boundary Problem

This was my last problem solving session with Wes. We met on a Wednesday afternoon after he was done student teaching. He first shared experiences from student teaching as he got observed that day by a university supervisor before starting the session. For this session I prepared a problem that had two parts, however, I have decided to present the second part after completion of the first part of the problem. At the beginning of the session, I gave Wes a sheet of paper containing the first part of the Land Boundary Problem (see Figure 5 in Chapter 3, p. 65) and a pencil, and opened the first subfile of the Land Boundary Problem.gsp file on the problem was printed. Solving the first part of the problem was not a difficult challenge for Wes taking him approximately 15 minutes, whereas during the next 25 minutes we reflected on the problem solving session.

Solving the Land Boundary Problem: First Part

Wes started the session by reading the problem aloud from the hard copy I gave him. He said he never seen a problem similar to this one before. When he saw the problem, he thought it was of 3.5 to 4 difficulty on a scale from 1 to 5. He did not have any difficulty understanding the problem but he added that "bent" could have been a problematic situation if he was not given a representation of the problem. Nevertheless, immediately, he pointed on the lands of both farmers on the hard copy emhasizing the problem condition, *"So this is the first farmer's land; this is the second farmer's land. Okay. They wanna keep the area the same."* Thus, understanding episode started in which the goal of the problem was noted correcly and explicitly as well as the conditions of the problem. Later in the interview, he said that at the beginning he was not sure how he would solve the problem immediately, sharing that he was stuck. He explained this was result of generality of the problem, *"I was like: Where am I gonna go with this, because it's so general. There is just a drawing."* Thus, already at the early stage of problem solving, he tried to direct his knowledge and thinking as to what might be done in this particular context.

He then engaged in the analysis of problem by examining the explicit relationship between the given information and the goal of the problem where a choice of perspective was made,

> *I need to somehow make two congruent triangles so I can say their areas are equal and I need those congruent triangles to be in such a way that I have straight line so I can get rid of the hump.*

It was clear Wes thought about previous content-specific knowledge and experiences that might be helpful in current problem solving situation. Thus, effort was put to make sense of the problem statement; he interpreted the goal of the problem in mathematical terms, keeping the area of the land the same, which then influenced the following actions. Keeping the same amount of land was equated with rearranging the two farmers' lands by finding two congruent triangles because congruent triangles have congruent areas. Later during the interview Wes stated that "*context was helpful in proceeding with the problem"* allowing him to direct his thinking to mathematical concepts that might be helpful in this situation. He continued to analyze the problem by examining two cases that would not produce a solution based on his previous observations,

> *Well I know what you can't join these two points [Points C and D] because that would give him some of his land and that just wouldn't be fair. So if you were to continue this way [line through Points C and E]. Would that be? No, you can't do that. I know that's [boundary CD] not gonna work cuz that would be essentially cutting of his land, and giving it to the other guy. You can't do this [boundary CH] because that would be the same thing; you would be cutting of his land and giving it to him.*

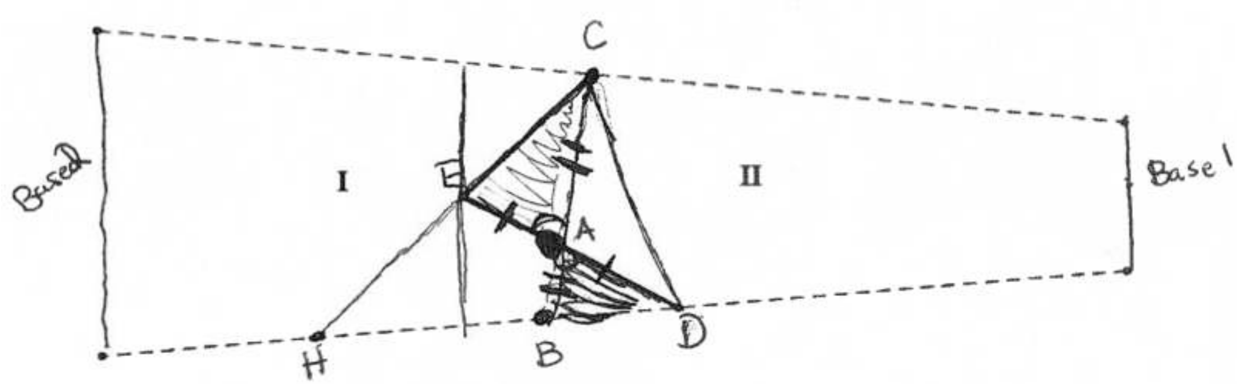


Figure 14. Wes's first attempts solving the Land Boundary Problem with paper-and-pen.

Before he engaged in devising a plan, he reengaged with the problem text restating explicitly the goal of the problem again in his own words,

> *My goal is to make this area [sections I and II] equal. I need to have a straight line going from their boundaries and it needs to keep the areas the same so I need to make this line straight cuz it's bent right now. We need to determine where this straight line is gonna be.*

His first plan reflected the choice of perspective outline in the analysis episode—obtaining two congruent triangles, and rearranging them such that the border between the two lands becomes straight. This problem solving strategy was something he had seen in his mathematics classes, and thought it might be helpful in solving the problem. Thus, he drew on his previous mathematical experience as to how and why to use this strategy in this problem solving situation and by using his executive skill (planning) to optimize the use of this helpful resource.

His first attempt entailed creating a Midpoint A of Segment ED obtaining two triangles, Triangle AEC and Triangle ADB,

> *So what I am thinking is, what if I were to find a midpoint of that section [ED]. This is a Midpoint [E], I have this [EA] would be equal to this [AD], this angle [AEC] would be equal to this Angle [ADB]. So I was thinking maybe if I can make these triangles [Triangles AEC and ADB] congruent maybe then it would work, but you are not guaranteed that this [sides AC and AB] is equal.*

He decided to draw the Midpoint A in order to evaluate if two triangles obtained would be congruent triangles as the idea of congruency was important in order to have equal areas,

> *I was thinking well maybe I could make these two triangles congruent so that they are. I was thinking that Triangle AEC would be congruent to Triangle ADB and then their areas would be the same. Would that help me at all? Because then yeah cuz then this portion that I am cutting off right here of his land would essentially be put over here and this portion that I am cutting off of his land would essentially put back over here. Then they would be equal.*

The plan as devised was relevant to the problem solution where relevant mathematical concepts and knowledge were accessed and considered. At this stage, however, he failed to reflect on the effectiveness of the plan as outlined. But visualizing the resulting plan is not as easy task and might have been out of his imagining capabilities. As he implemented his plan, however, he reflected on the undertaken activity realizing this approach would not work because the Triangles AEC and ADB were not congruent, *"But that's not gonna work. Because I have a side and an angle but what I don't have is another angle so I can't have ASA or SAS."* This episode highlights the importance of monitoring of the progress of planned actions allowing the problem solver to revise them.

Up to that point Wes was still working of the hard copy before moving onto GSP. This was driven by previous session where he immediately decided to use GSP. This time he took his time, about 7 minutes, working just with paper and pen. This was a desirable and productive metacognitive behavior that helped him develop a better understanding of the problem, and devise plausible solution paths. Here he moved from hard copy to the .gsp file as, *"it was getting confusing on the paper because I had already tried two things and I was getting confused: well which line am I looking at, is this his land, or the original one? It was just getting messy."*

As he moved from paper and pen to using the software the exploration episode started. He pointed with the mouse on the two lands. He engaged in self-questioning, and moved Point E within land I, and land II, *"Is there a geometric way to do this? Cuz obviously it's not gonna matter about that [Point E]."* He furthermore analyzed the screen as he was moving the Point E to get a better understanding of the problem, *"I am trying to think a little bit about does it matter where that Point is [E]. In reality does it? It's just a general point. I think this is a just a visual representation of the problem."* Later on in the interview he explained that he was trying to reason if the shape of the region would influence the answer. Also, he moved the upper and bottom boundary to further explore the problem, *"I was moving this and I was seeing how does* *that influence how large or how small their land was."* At this stage in the search for a solution plan he used the dynamic nature of the software in order to find relevant information that could potentially be useful in planning episode. This problem solving strategy is a cognitive activity, however, he utilized it to make sense of the problem situation, and such selection and use of the software is a regulatory skill. However, his actions did not seem focused, and I was not sure what was the perspective on future actions.

He sat in silence for a few seconds before deciding to bound the lands, *"I am gonna join these two points and just act like that's their land so there are defined boundaries now. So now I am just working with this. I need to make the areas equal,"* reminding himself again of the goal of the problem. Moreover, he added land not being bounded completely on the representation was somewhat confusing *"because when I think of land I think of it as, especially when people own it, it's closed off."* Nevertheless, he was aware that bounding the land was not necessary for solving the problem, but helped him to solve the problem, as he was able to put it in a concrete context. He drew Segment CD, and moved Line DH vertically for a few seconds. Within next few seconds he uttered, *"Oh! If I construct a line through there [E] and it's parallel to that one [base 1]."* He explained that as he was moving Point E horizontally on the sketch, the idea of drawing a parallel line came to his mind,

> *I was moving this Point [E] around and I was like well if I can have it move so that the altitude doesn't change then I can just say that the area of the triangle is remaining constant and I decided, well okay I will make it parallel.*

During the process of search for a solution plan, student-tool interaction was evident. The leap of faith he took was a success. At first he directed the software through search for relevant information, then, however, through effects of the tool on his activity the software became an instrument. The feedback software provides, helped access and consider mathematical knowledge (area between two parallel lines) and consequently unfolded a possible solution approach. He explained his solution plan before implementing it,

> *Well what I was thinking is that, as long as that line remains parallel, the area of that Triangle [CDE] doesn't change. So if Point E is right here, I can just slide it parallel to that side [base 1] of this land and the area of the triangle remains the same because the base remains the same and the altitude is remaining the same. Yeah!*

Plan was overt, and relevant to the problem solution. However, he failed to assess the quality of the plan with respect to its structure. He continued to implement his plan in a structured way, but again at this point there was yet no assessment of the current solution state. Once he implemented his plan, he tried to verify his statement by moving Point E along the line. However, he noted his conjecture was flawed, *"If they move it parallel, the altitude shouldn't change and the base isn't changing so the area of the triangle wouldn't change. Hold on! But the altitude is changing."* He sat perplexed for a few seconds trying to make sense of the current solution state before realization how and why was his approach flawed. However, he did not abandon the entire approach but rather altered a flaw in his reasoning, and refined his plan, *"O…OH! I need it to be parallel to this Line [CD] because then it [altitude] wouldn't be [changing] because that would ensure that the altitude is not changing."* Thus, this time he assessed the quality of the plan with respect to appropriateness and correctness. During the process of proving the conjecture, student-tool interaction was evident, and the software became an instrument through his appropriation of the tool and through effects of the tool on his problem solving processes and strategies. He explained this awareness in more detail later,

> *then I was just looking at that line and it occurred to me: Oh it's a triangle so I need this, the altitude is measured from, it's the height and as long as the altitude is moving and it's not changing its length then the area is remaining the same so I need the base to be parallel to that point, or to the line going through the Point E to maintain that altitude to be constant height because it's a perpendicular distance between those two lines [CD and EF].*

He immediately explained his reasoning with excitement, *"Triangles that are on the same base and in the same parallels have equal areas. That's what Euclid said. And even they don't have to be on the same base. They can just be on the congruent bases."* He then implemented his plan in a structured way monitoring his steps constantly,

> *So through Point E, I am going to make a line that's parallel to DC which is the line that joins the endpoints of their land, is that what it is? Yeah. Of the fence. Yeah. So now that altitude is not gonna change and that doesn't mater where the Point [E] is. So, there. The area of that triangle is not gonna change so there! They would have a straight line and their areas would remain constant.*

He was confident he got the problem correct as it relied on proper mathematical reasoning, *"I think that's right because that Triangle [CFD] is on the same base and they are in the same parallel so it has the same area. Yes! Yes, that looks a lot better now."*

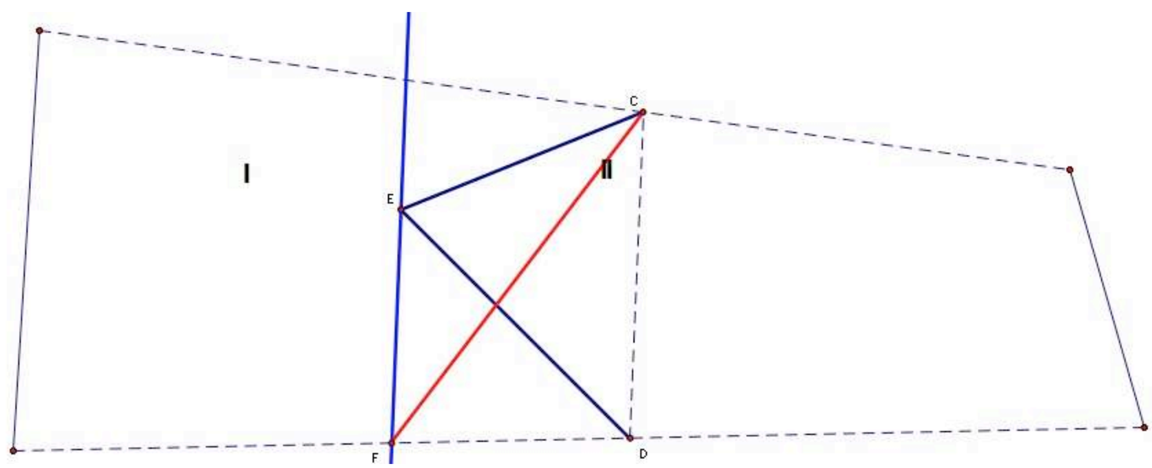


Figure 15. Wes's solution of the Land Boundary Problem – Segment CF.

As he was done, he added, *"That's interesting."* Interestingly, he added that because of the context he was able to solve the problem because he was able to relate the problem to a real life situation,

> *If it would have just said make these two areas equal, I don't know how far I would have gotten. I like the context that is given in so I have something concrete to imagine; oh so*

> *this is what I am doing. I am trying to make these lines, or I need to make their lands equal in area and land is a unique. It relates well to area like the area of the land.*

When I asked him what were the top things that were most important to solve the problem, two categories emerged;

1. mathematical knowledge related to the concept of area, and
2. geometry software.

He thought GSP helped him explore the problem, and get a deeper understanding of the problem, *"I guess it's just exploring a little bit of what you have. I think I have a better understanding of the problem when I am able to interact with it a little bit more."* Influence of technology was most evident when moving Point E. By moving Point E he got on a right solution track,

> *I was moving it vertically so then I was like: Well that triangle would have the same area as long as it moves parallel to something and I just chose that side [base 1] out of, I guess too quickly if I would have thought through it I think I might have chosen the base [CD] of the triangle before I chose the other one.*

When I asked him how important was technology in solving the problem, he shared his thoughts if he could have solved the problem just with paper and pen, "*I am trying to think if I could have done it over there [paper], if it wouldn't have got messy? Well, probably not because it's very static where this is more dynamic."* He was not sure if the would have solved the problem without technology *"cuz it was moving that Point E that gave [him] that idea."*

He did not use GSP Measurement Tool in any situation during problem solving. This was result of previous problem solving sessions,

> *That was my strategy the last couple of times, to start measuring stuff and, before I started the problem I was: I am gonna try and do without measuring. No measuring allowed cuz the last time I said I wanted to find the solution, and then I wanted to work backwards on how to get there, and so this time I tried a more direct path instead of doing that, to start from the top and go down.*

I probed more if he as a problem solver can make a decision, and assess when is technology helpful and should to be used. His reply revealed second possible solution for the problem,

> *Maybe, maybe, [it depends] how the general the problem is. This problem is so general. I mean there is infinitely many solutions based on upon where E is. I mean, if you move E you can have a solution every time. But when everything is fixed, there are two solutions.*

He then drew second possible solution.

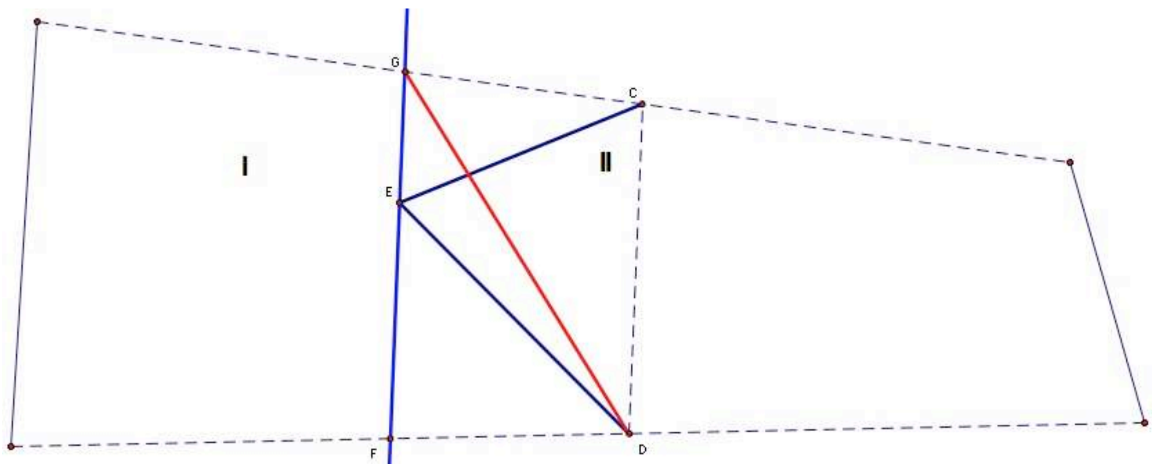


Figure 16. Wes's second solution of the Land Boundary Problem – Segment DG.

As we were about to finish the first part of the interview, I asked him if he could think of a way to make this problem challenging. He replied, *"Yeah. Instead of having just, having it bent in two ways if it was bent like in 15 ways,"* which led into the second part of the problem solving session.

Synthesis of the Problem-Solving Session 3—First Part

Wes started the session by reading and he understood the problem. At the beginning he put effort to make sense of the problem information. During the analysis episode he examined the explicit relationship between the conditions and the goal of the problem. Through interpretation of the problem statement he considered the concept of area and congruent triangles being relevant to the problem, and a choice of perspective was selected. His first solution plan, though made as a result of his understanding and analysis stage, was not successful due to lack of assessment of the plan. He decided to use the dynamic capabilities of the software in his search for a solution plan during which he accessed knowledge that might be helpful in solving the problem. Even though the exploration episode lacked structure, he assessed the relevancy of the new information, which eventually led him to devise a new solution plan. He promptly jumped into implementation of it, without carefully examining his plan.The nature of student-tool use allowed him to refine his plan, and execute it. The verification of the plan was done explicitly as a result of previous false moves. Figure 17 demonstrates Wes's problem solving cycle.

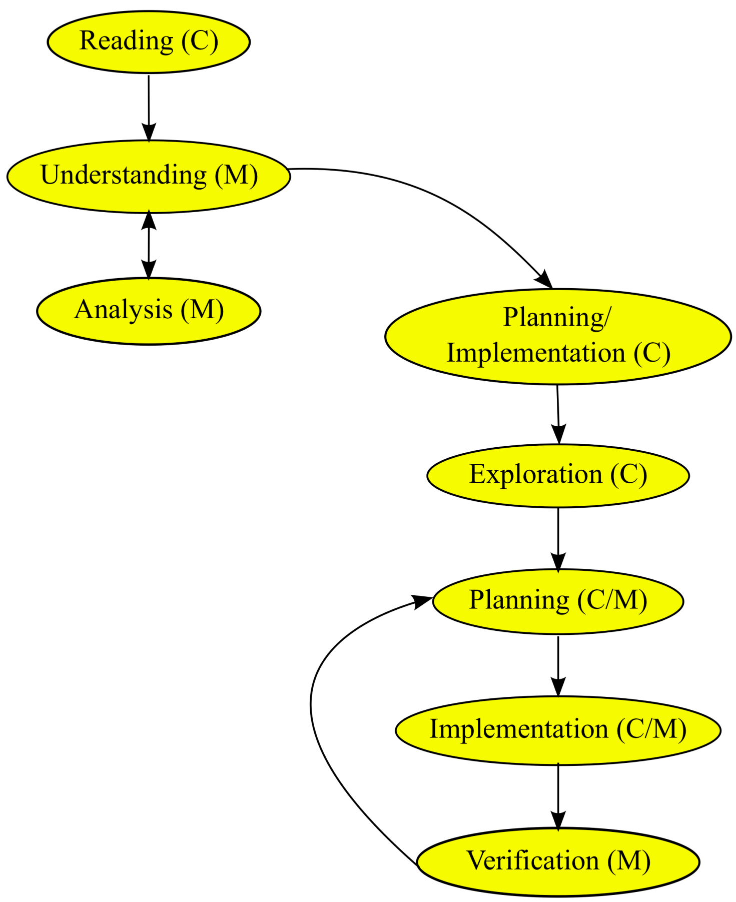


Figure 17. Parsing of Wes's Problem-Solving Session 3: First part.

Once he solved the first part of the problem, I provided a second sheet of the paper and opened second file on the Land Boundary Problem.gsp file on which the second part of the problem was printed. Solving the second part of the problem proved to be a bigger challenge taking approximately 30 minutes. He struggled figuring out how to use the first part of the problem to solve the second part of the problem. Nevertheless, his solution path allowed a deeper insight into his problem solving strategies and processes.

Solving the Land Boundary Problem: Second Part

Receiving an additional problem was a shock. He expressed a concern that his confidence in his problem solving abilities might get *"shattered."* When he read the extension of the problem, he ranked it as 4 out of 5 in difficulty as it had another bent. Pass the initial shock, Wes immediately started working on the problem by engaging himself in self-questioning which opened an understanding and an analysis episode. He questioned if he could use the same strategy, and how to use it effectively, *"Can I use the same strategy? It's bent three times. Can I use the same strategy?"* by which he meant using parallel lines. He noted the conditions of the problem explicitly, and he considered if the strategy he used in the previous task could be used in this problem solving situation. He explained he decided to use the same strategy because he *"figured that these two problems are somehow related."* Thus, a choice of perspective was made as a result of reflection on the previous problem that he perceived might be helpful in current problem solving situation. However, there was no serious assessment of a direction to come; he was not sure how to use the strategy that might be costly before such comes.

He started working on his solution plan in which he wanted to use the same approach *"at once".* His plan involved connecting Points H and J, and sliding the two Triangles, HKM, and MLK, along parallel lines through K and L, respectively to land boundary, "*What I am* *visualizing is connecting the two Points [H and J] that are on the outer most boundaries, and then making lines that are parallel to that base, and then I guess sliding this Segment [KL] right here until it meets [the boundaries].”* When he tried to assess if his plan would work or not by visualizing it in his head, he said, *“that’s not gonna work.”* However, he then again reflected on the effectiveness of the selected method not yet ready to abandon his problem solving approach. For that reason, he then used his fingers, and simultaneously sliding and visualizing an image assessed the effectiveness of the approach, *“Slide that down, slide that up. Yeah, I think it will work.”* Thus, the plan was overt, well structured and relevant. He considered its appropriateness and assessed it before its implementation by imagining it mentally; unfolding of a solution plan was imagined. The evaluation was not correct, however, suggesting that noting where his activity was leading went beyond his competency.

He went ahead and implemented his plan in a structured way, he joined Points H and J, changed color of Segment HJ to green, drew lines through Points K and L parallel to Segment HJ, obtained two Triangles (HKM, and JLM) and constructed intersection points of the lines with the boundaries. He was surprised with the outcome and shortly assessed its influence on the solution plan, *“Oh! There is four intersection points. Does it matter? See it intersects here, here, here, and here.”*

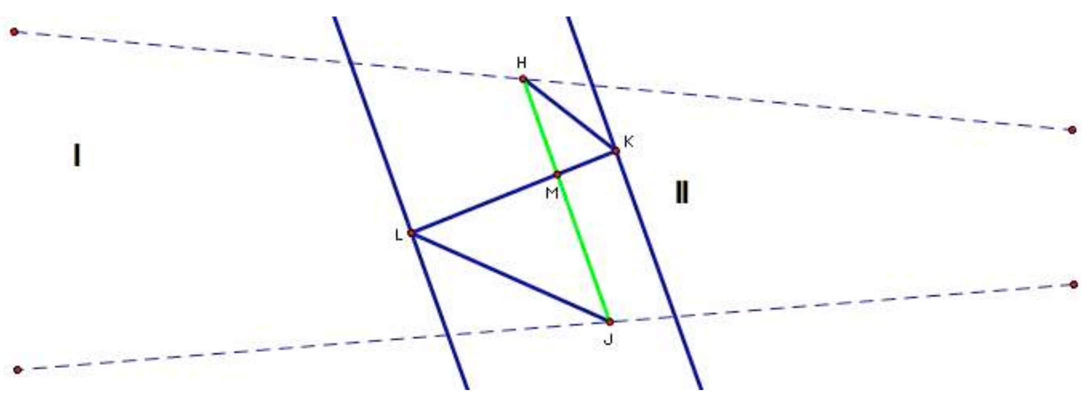


Figure 18. Wes’s implementation of the first solution plan by using the strategy at once.

In order to verify his solution plan, he decided to move Points L and K along the lines in first position, and second (see Figure 18 and 19, respectively). Now, having a concrete representation of his plan, he was visually convinced his solution plan was correct, *"So, yeah it will work."* Nevertheless, he continued to test his solution by moving Point L along its line, and analyzing what he noted by observing the screen, *"Hold on. Does the area change? I see that now the base [JM] is changing. I don't have a constant base, the base is not constant."* Furthermore, he noted, *"that the area of that Triangle [JLM] is not remaining constant"* as the base was changing, even though the height was remaining constant. He was perplexed as to why his plan was not correct. On the one side he devised and implemented a plan for which he was sure it was going to work. Once he implemented his plan, however, he used dragging options by the software to test his solution. Here we can see interplay between his thinking processes and tool use. Although, he used the software to implement his plan, the result of the implementation and using the dynamic capabilities of the software influenced his thinking processes and actions. Thus, the software was no longer used as a tool in his hands but became an instrument. As a result, this created a feedback that was not expected creating a perturbation in his mind. Nevertheless, though he engaged in verification of the result, he never assessed the quality of the process, in this case, implementation of the strategy at once.

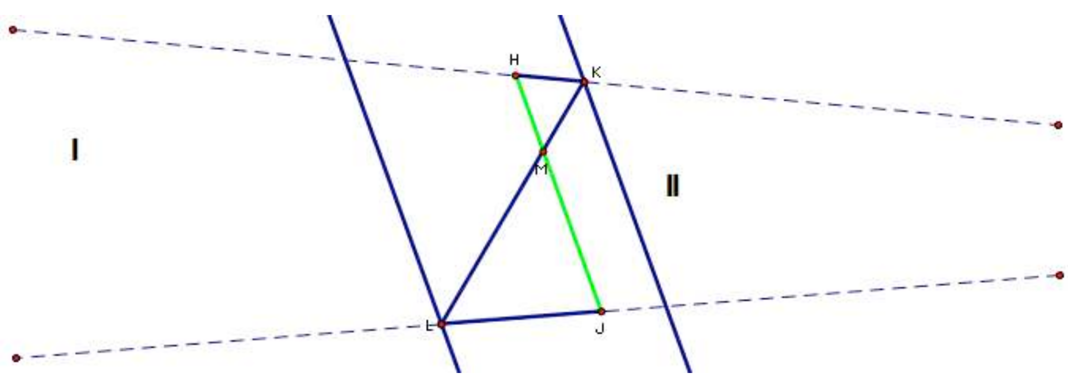


Figure 19. Wes's verification of the first solution plan using technology.

Even though, his result was not as expected, he decided to verify his solution by engaging himself in a serious of computations using algebra; that is, he decided to analyze the rate of gain/lose change in the area*, "It's good if it's [Triangle JLM] getting big at the same rate the other one [Triangle HKM] is losing area,"* followed by a metacognitive comment of monitoring the quality of reasoning, *"I've got to be careful with the area."* He was still not sure if his further analysis of the result would be fruitful or not, *"Would it be getting at the same?"* He decided to draw on the hard copy containing the problem the visual representation of the problem, and to use algebra to verify his statement. He labeled the length of Segment HJ with *l*, and drew two lines through Points K and L parallel to Segment HJ (see Figure 20). He noted that the magnitude of Segment HJ and the magnitude of the altitudes. He labeled attitudes of Triangles HKM and JLM with h_1 and h_2, respectively. He again used his fingers to help visualize the effect of straightening out the bent fence on the area of the two triangles, *"But however as I straighten it out this triangle gets smaller because the base is not becoming smaller in this triangle, and now this triangle is getting bigger because its base is becoming bigger."* Analysis was followed by implementation of his verification plan using algebra as stated previously.

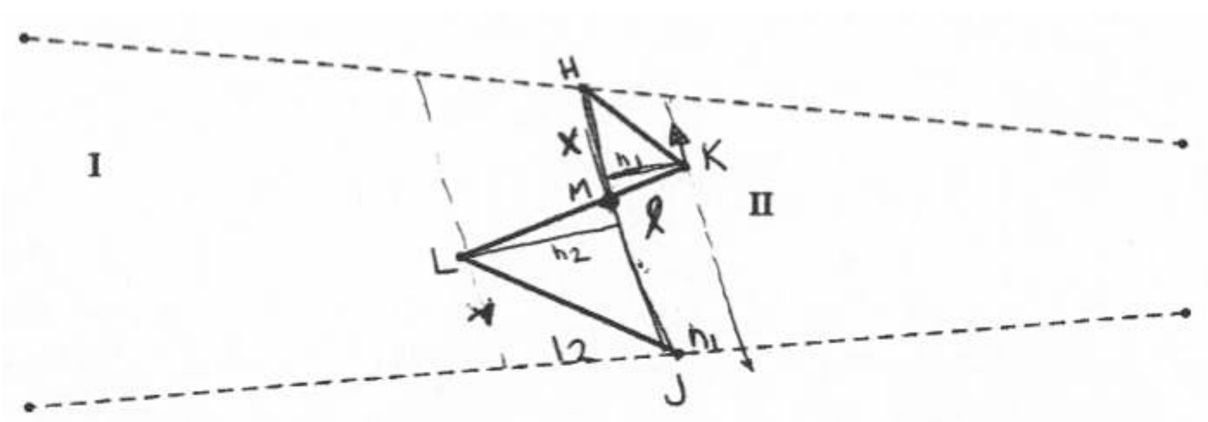


Figure 20. Wes's pen-and-paper representation of the first solution plan.

He labeled Segment HM with *x*, Segment JM with *l-x* and calculated the area of Triangles HKM, and JML (see Figure 21). He again assessed his conjecture,

> *How does the area change relative to another? I mean they are not the same area. But do they change at the constant rate? How much area is this one gonna lose as I change that? How much is this one gonna gain?*

He examined the last line in the area of Triangle JML circling $\frac{1}{2} x h_2$ as the area lost, while the first part always stayed the same. He again tried to analyze and visualize the problem situation, *"If it was there, how would it look like? So as this moves down here."* He realized he could express h_1 in terms of h_2 as the distance between two parallel was equal to h_1+h_2. He never implemented his observation, but uttered, *"I am stuck,"* explaining further, *"I don't know. I mean that seems like a legitimate strategy but then as you move it, one triangle is gaining area, and the other one is losing area."* Thus, he tried to make sense out of the current problem solving stage by evaluating his problem solving strategy, but was not sure how to do it. Later on in the interview he admitted that he did not monitor his planning activities accordingly, *"I jumped right into it. This is how is I am gonna do it."* These two episodes highlight that absence of evaluation, regulatory and monitoring activities may doom problem solving attempts to failure. With the statement *"I am stuck"* ended first possible solution plan as he did not know how to verify his conjecture nor if it was correct, and decided to apply a more geometric approach—*"a systematic approach."*

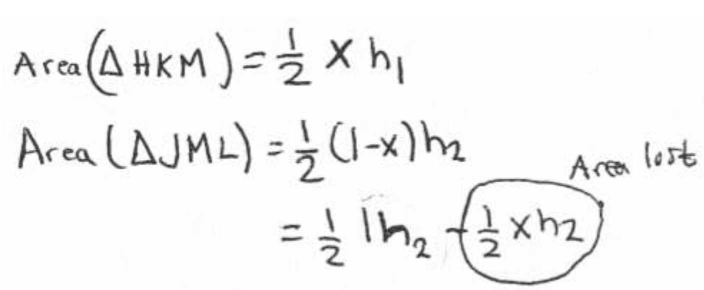


Figure 21. Wes's verification of the first solution plan using algebra.

He hid all of the lines, remaining on the screen the original problem. First he bounded the land similarly to what he had done in the first part. He then shortly assessed the current solution state. He reengaged with the problem noting the problem conditions, *"Now it is bent twice"* and tried to decide on a next perspective, *"So maybe if I dealt with each one separately. But how would I deal with this separately?"* Thus, he did made an attempt to salvage strategy used in the previous problem solving approach directing his thinking and previous experience how to do it,

> *Cuz I was trying to deal with both of them [segments] at the same time, and that wasn't working. And then I thought of the idea; well instead of trying to tackle it at the same time why don't I just try doing just one at a time and see where that gets me and that's how it developed. It was instead of dealing with both of them just trying to deal with one.*

Hence, this episode shows what power can engagement in a metacognitive activity have. In contrast to previous attempt, where a perspective was chosen without apparent structure to its efficiency and effectiveness, now the choice of perspective became structured but still not coherent. He was aware of what needs to be done in this situation, evaluated it as a promising approach but was not able to regulate his thought processes at that time. In addition, this highlights the importance of transition between episodes; he did not simply jump into the new approach but assessed short-term effects of the new approach on the solution.

Once he devised, considered, and selected his strategy, *"I will deal with each bent by itself. Now I can repeat the process I did before"* he implemented his new plan. He extended middle segment of the bent fence and changed the color of the new segment to green. He continued on implementing his plan by drawing a line parallel to Segment HK through Point I. The last step of the plan involved moving Point I along the parallel until reached the upper boundary. However, he noted that the base HK was changing which was again unexpected and not desirable occurrence. He stepped back from the problem by erasing the auxiliary lines as he *"was getting confused with what was going on"* and going back to the original problem. He then verbalized again his second solution plan as he tried to implement it once more, *"I liked my idea before but for some reason it wasn't working. I was looking at this Triangle [HIK]."* He recreated a figure similar to one in Figure 5, and analyzed it that result in noticing that the Triangle HIK was not the triangle defining the bent fence. He then explained more in detail what he mean by the systematic approach in this episode, and how the following, successful approach developed as a result of first unsuccessful approach,

> *What I was thinking was, that's how I was gonna do it systematically at first [by extending Line IL]. I was gonna make the same problem as last time but then I realized: Oh, you can't do that. That line doesn't exist in the problem. I was adding on to this line segment something that shouldn't have been there. So I was changing the problem, and I shouldn't have done that.*

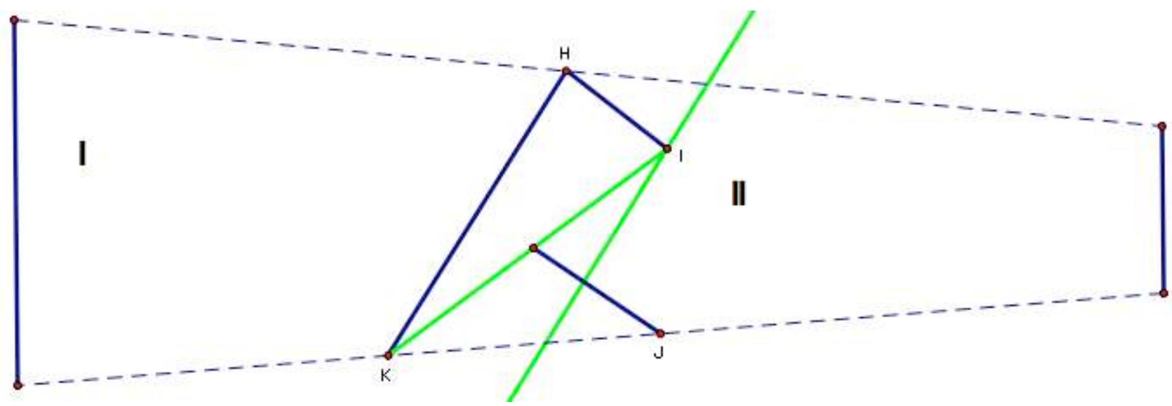


Figure 22. Wes's implementation of the second solution plan: An unsuccessful attempt to use the strategy systematically.

Thus, again we can see an interplay between his thinking processes and tool use. Although, at first used to implement his solution plan, the effect of moving Point I, influenced his thinking processes. Again importance of assessing the current state of problem solving comes evident. He was able to reflect on the process and solution, and move forward in the solution process by planning his following actions accordingly to his monitoring activities. Hence, the software was no longer used as a tool in his hands but became an instrument.

With this ended his second solution plan, but the problem solving approach was not abandoned entirely. Again he reflected on the undertaken activities, and decided to again salvage his problem solving strategy,

> *And that's when I thought, well just break it off to these two triangles with the line green lines by joining HL and JI and I worked on HL first and then after I got HL and then I was like: Oh, it's the same problem as before.*

He carefully assessed the current solution state and directed his actions towards a new solution plan. He noted two Triangles, Triangle HIJ and Triangle IJL, though at this point Segments HL and IJ were not drawn. He then drew Segments HL and IJ, and shared his solution plan,

> *I would construct a line parallel to the base of that triangle and then I would, hm. Which point should I choose first? Because I am trying to think ahead of what I am gonna [do]. If I work this systematically, which one am I gonna have to choose?*

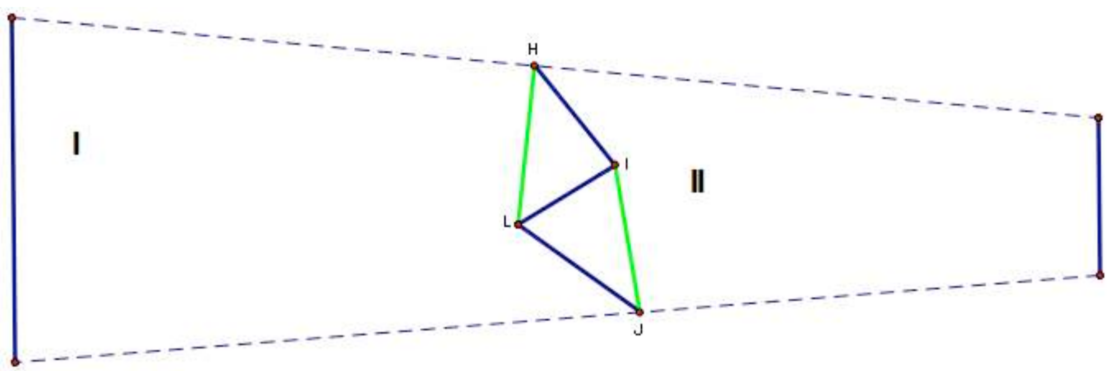


Figure 23. Wes's third attempt solving the extension by using the strategy systematically.

He continued implementing his plan by drawing a line through Point I parallel to Segment HL. Obtaining a visual representation of the problem situation aided planning behaviors, *"I will need to choose this one [Point M]."* He drew Segment LM and changed the color to red (see Figure 24). He assessed the outcome before stating, *"Our new one is right here. And we need to make it in a straight line with this one [LJ]."* He also added Triangles HML and HIL have the same area as they have the same base and lie in the same parallel.

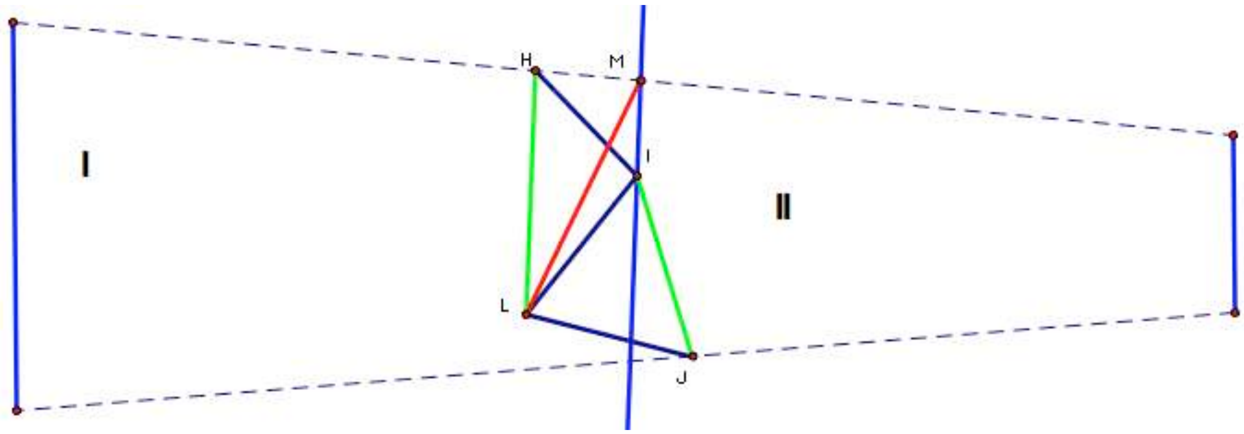


Figure 24. Wes's implementation of the third solution plan: Eliminating the first bend.

He sat in silence for a few seconds before coming to an important realization, *"O…OH! So now we are back at the original problem! Our boundary is LM and LJ so now I can connect the base of MJ."* Planning and implementation occurred simultaneously. Though his plan again was not entirely thought through, careful monitoring of implementation activities, and regulation of these optimized the use of his problem solving strategy. He drew Segment MJ, and constructed a line through Point L parallel to Segment MJ labeling the intersection of the parallel and upper boundary with Point N. He connected Points M and N, and changed the color of Segment MN to yellow (see Figure 25). He finished the implementation of his plan by saying, *"There we go. MN is our new boundary. Yes!"*

He explained he solved the problem correctly, *"Because I've reduced the problem to something I solved before by getting rid of that one line and we reduced them to two which we solved before in the first problem."* He concluded the problem solving process by saying, *"You can actually probably do that for any number of segments in the middle. Yeah."* He was confident he solved the problem correctly.

He explained that the problem was *"a little bit more challenging"* as it involved three segments. Though he said the problem was challenging, he explained that solving the first part of the problem made the problem easier as it gave him a starting point, *"idea of what [he] wanted to do,"* adding, *"so I already got a part of the problem solved."*

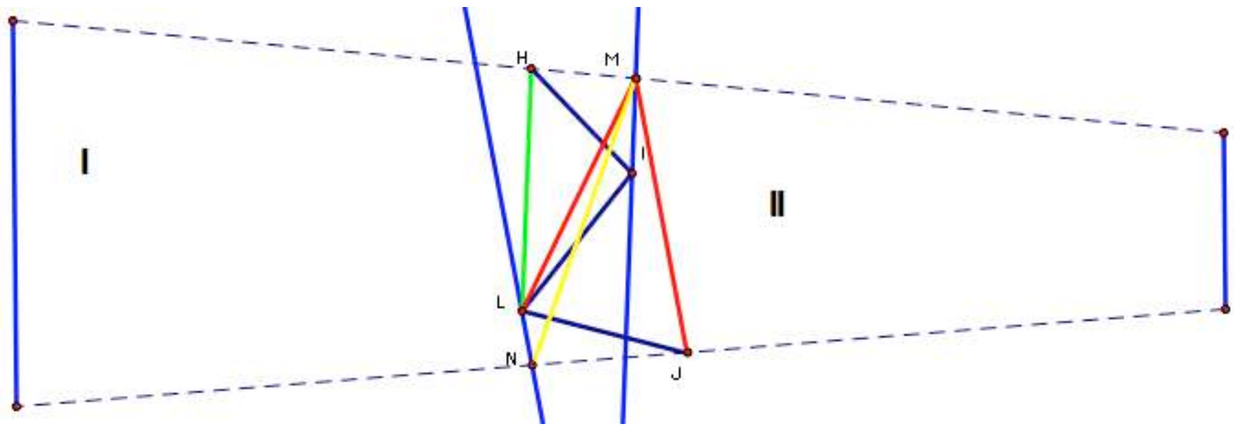


Figure 25. Wes's solution of the extension of the Land Boundary Problem – Segment MN.

One of the behaviors that was frequently exhibited during the second part of the session was engagement in internal dialogue or self-questioning (e.g., *"Does it matter?" "Will it work?" "How will I deal…?"*). Verbalization of comments during planning and implementation episode was a metacognitive behavior that directed his thinking in productive directions.

When I asked him what were the top things that were most important to solve the problem, three categories emerged; (1) mathematical knowledge, (2) problem solving strategy, and (3) geometry software. For the first two categories he said,

> *Triangles that are on the same base and are in the same parallels have the same area. And? Yeah. That's about it. That's all you really needed. And then for this problem that plus what you did before. The first one was like a lemma (mild laughter) I needed to solve this.*

Whereas in the first part technology had a minor role, here it was much more important especially when testing his fist problem solving path when its use corrected his inactions,

> *I started straight ahead trying to apply what I used before… since this is static, the base to me wasn't changing inside in my head as I moved this Point L towards here cuz I am trying to straighten out these two, pull them in a line. But then when I used the technology I was wait the triangle areas are changing so I can't do that because as one is gaining area the other one is losing area.*

Technology was helpful rather than hindering him in solving the problem, *"It helped me, the only thing it did was help me."* Even when working on the systematic approach, he viewed technology as useful during implementation,

> *being able to draw the parallel lines and everything whereas with here [paper] I wasn't able to draw parallel lines. Well with rulers and compass I could have done it but then it would have been a lot more difficult, it would have been a lot more messy.*

It was evident that during the process of solving the problem, he was monitoring his use of technology, *"I didn't go on any like leads that lead me in the wrong direction or anything. The wrong leads came from this because I was imagining that this point wasn't moving, but it does."* In addition as a result of this problem solving session, he noted the significance of relaxing the number of constrains as a valuable problem solving strategy, *"Once again reduce the constraints, get rid of as many restrictions as you can."* At the end of the sessions he shared his view about the problem, *"This one was difficult, but I liked it."*

Synthesis of the Problem-Solving Session 3—Second Part

Wes first read the problem before considering what needed to be done. He immediately decided on a choice of perspective–use the strategy from the first part of the problem. In his first attempt he tried to use the strategy at once; that is, applying the strategy on both bents at the same time. He promptly jumped into implementation of it without carefully examining his plan However, since that attempt was not successful, he revised his problem solving approach deciding to use the strategy this time is a systematic way. For both plans, although he had strategy ready at hand, selection of steps was not assessed prior to their implementation. Before devising his third and final solution plan, he took a step back and reflected on the process and solution before directing his thinking to a new plan. Though his plan as a whole was not coherent, careful evaluation of undertaken activities and regulation of his thinking and knowledge, led him to solving the problem. Hence, when a result was refuted, he assessed the approach, revised the plan prior to moving to a new problem solving cycle as shown on Figure 26. The nature of student-tool use allowed him to verify and refine his thinking that proved to be paramount for this problem. Furthermore, the second part of the session highlighted the importance of monitoring one's work—absence of managerial decisions impacts the overall problem solving process, and might otherwise end up in unproductive efforts. Figure 26 demonstrates Wes's problem-solving cycle.

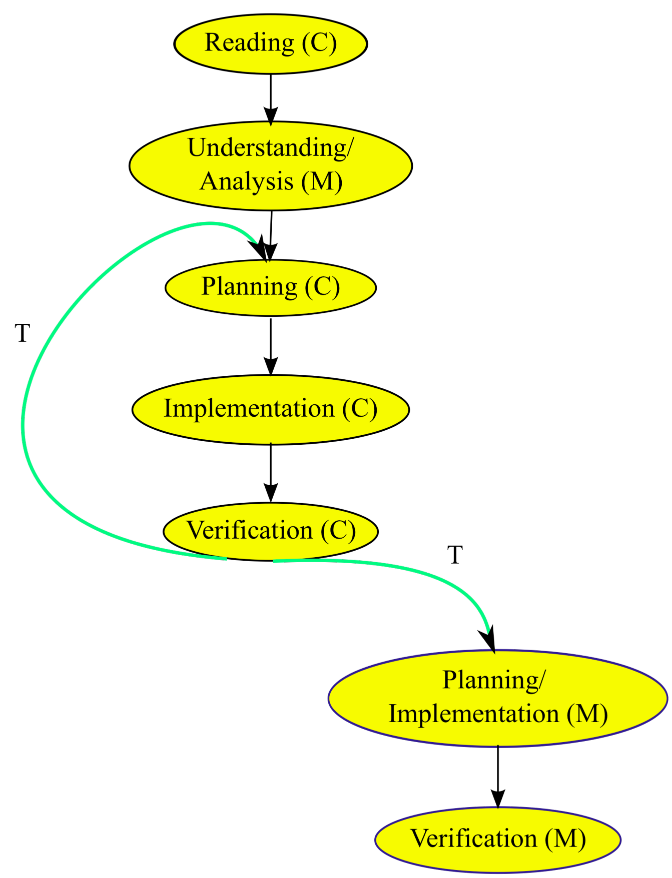


Figure 26. Parsing of Wes's Problem-Solving Session 3: Second part.

Reflection on Problem Solving in a Dynamic Geometry Environment

The third section presents Wes's reflection on the three problem solving sessions and problem solving in a dynamic geometry environment as a result of participating in this study.

Reflecting on the Problem Solving Sessions

Wes's favorite problem from this study was the Land Boundary Problem. He explained that he *"enjoyed using the first part of the problem to solve the second part. I was able to generalize a specific technique to solve a complex problem."* He said he was successful solving the problem because of his "*knowledge of Euclid's proposition."* In addition he referred to helpfulness of technology in problem solving this problem, most notably dynamic nature of it, "*The picture was static; however, the GSP file was dynamic. I could manipulate the boundaries and explore the problem further. Sometimes, changing the sketch and watching the screen can give me an idea."* Lastly, after solving that problem he was confident in his "*knowledge of Euclidean Geometry and in [his] ability to solve multistep problems.*"

As a result of participating in this study, Wes learned "*several problem solving strategies," "found that technology is not always helpful"* and changed his attitude towards his ability to problem solve and his use of technology when problem solving. Relaxing the number of constraints was "*the most important strategy [he] learned related to problem solving"* because it makes "*the problem easier*" and gives *"insight into the original problem."* While in the preliminary interview he state that he did not consider himself being a good problem solver, this view changed significantly, "*this study gave me more confidence in my ability to solve challenging problems. It may take me a while to solve a problem, but I frequently persevere. Perseverance is an important problem solving characteristic."*

Even though Wes was able to solve all of the problems, his problem solving was not every time fruitful or successful. He contended this was a consequence of *"heavily reliance upon technology to solve problem,"* not taking *"a step back"* and assessing *"the situation more frequently."* He noted that *"technology can be distracting"* hindering his "*ability to solve problems effectively"* when becoming too consumed with it. Nevertheless, he used this experience into his advantage,

> *I should use technology when it is appropriate to do so. In addition, I need to give myself more time to understand the problem. Before using technology (i.e. GSP), I should plan accordingly. For instance, I should use pencil and paper before proceeding to use GSP. Furthermore, constantly assessing my progress and my understanding of the problem is crucial. For example, I may ask myself: How will this help me solve the problem?*

Lastly, through participation in this study he became aware of his general geometry problem solving process, that will influence how he approaches a problem, goes about solving it, and how he teaches problem solving. He added that in teaching problem solving he will highlight ideas of understanding the problem, planning accordingly and using *"technology when it is absolutely necessary."*

Reflecting on Problem Solving in a Dynamic Geometry Environment

Wes relied heavily on the GSP when problem solving. He viewed technology as *"an incredible resource"* he used for different purposes using various capabilities of the software,

> *I find it difficult to draw pictures accurately. In addition, static drawings are often misleading. However, GSP provides me with a dynamic geometry environment. This gives me an opportunity to explore a variety of problems with relative ease. For example, I can confirm or dismiss my conjectures. I can easily manipulate the sketch and monitor the resulting change.*

Use of GSP to represent mathematical problems was extremely notable. He explained that because of presence of technology he can *"create accurate geometric representations with ease," "can enhance each sketch by changing its appearance (color of geometric objects, width* *of lines, etc.),"* and "*eliminate redundant steps (i.e. constructing parallel/ perpendicular lines)."* These benefits of GSP helped him *"stay organized and focused."*

Despite many benefits of using technology when problem solving, he was aware of several disadvantages of problem solving using technology. He said that technology is *"addictive," "distracting"* adding that often he lost himself in it spending a great deal of energy using it where *"this energy could have been put to better use."* His attitude towards using technology when problem solving changed,

> *In the past, I have used technology immediately. My goal is to use technology appropriately. To do so, I usually record my thoughts on paper and create a preliminary sketch. Taking the time to think the problem through before using technology can be incredibly beneficial…it should not serve as a crutch.*

In summary, the following are suggestions he would give to another person when problem solving in a technology environment,

> *I would suggest using technology as a tool. Technology is not the answer, it is simply a resource. Before using technology, you must plan accordingly. I often record my ideas on a piece of paper and make a preliminary sketch before using GSP. This helps me organize my ideas. In addition, this helps me understand the problem. Furthermore, I would recommend assessing one's progress regularly. Sometimes, technology can be overwhelming and you may lose sight of your goal. By monitoring your progress regularly, you can ensure that you don't lose focus. I often ask myself: How will using technology help me solve this problem? Finally, finding alternative solutions is a great way to advance your problem solving skills.*

Summary of the Case of Wes

The analysis of Wes indicated he had great prior experience working mathematics problems in a dynamic geometry environment. He was confident in his geometry content knowledge and use of technology during problem solving. The framework used in this study provided useful information with respect to when, where, and how Wes used both cognitive and metacognitive processes and how these affected his successfulness when problem solving. It was very easy for him to arrive at a solution in the presence of managerial skills, subject-matter knowledge, and positive self-evaluation behaviors. The situation was very much different, however, in the absence of metacognitive skills. Although his subject-matter knowledge would be seen as sufficient, lack of metacognitive processes in exploring and planning episodes seemed to have a big effect on the problem situation. Nevertheless, Wes became aware of his general problem solving processes that might influence how he will approach problem solving in future. Influence of technology on his problem solving was more than evident. Technology was mainly used to explore and better understand the problem, and aided attaining accurate visual input and "fitting" all the pieces together. However, often the software itself affected his problem solving activity. Nevertheless, instrumental genesis was paramount for effective problem solving performance. Lastly, at the beginning of the study Wes lacked confidence in his problem solving skills attributing this to lack of allocation of ingenious solution paths, whereas at the end of the study he reported that his confidence in solving challenging problems has grown.

CHAPTER 5

THE CASE OF AURORA

In this chapter I present the case study of Aurora. This chapter is divided into four sections. The chapter begins with a description of Aurora, outlining her background, her learning of mathematics, her view of mathematics and problem solving, and her technology learning experiences and perceptions obtained from the preliminary interview. The second section is a careful examination of three problem-solving sessions in which a different mathematical problem solving task was investigated each time. The report of each problem-solving session is a synergy of events during the session and participant's descriptions and interpretations of the session providing the reader with a rich representation of the session. The report of each problem-solving session ends with a synthesis and a parsing of the session protocol in which I used the following abbreviations: C–cognitive, M–metacognitive, T conjecture–transition episode characterized by conjecturing, and Green arrow-transition episode characterized by "taking a step back." Note that I present mathematical problem-solving tasks given in the order of sessions, and not in the order of tasks as outlined in the mathematical problems section. The third section presents Aurora's reflection on the three problem solving sessions and problem solving in a dynamic geometry environment as a result of participating in this study. The final section is my summary of the case of Aurora.

Getting to Know Aurora

Aurora at the time of data collection was a first year master student in the mathematics education department at UGA. Throughout her high school years she planned to be a pre-med student because of her love towards biology and science, but it was during her senior year of high school that she decided taking on a different path, *"I always enjoyed math the most, that was always my favorite subject … it always interested me as a subject."* Since she declared her major by that time, she began as a pre-med student, but switched to having a dual major in mathematics and mathematics education for grades 9 through 12 in the second semester of her freshman year at a large southeastern university. She decided to become a mathematics educator because of her passion of working with students, but not necessarily because of the subject itself,

> *It's about the passion I have for the kids and where they are going to go with their lives. I mean I love mathematics but frankly probably I could have been a teacher in any subject and still loved it simply because I like working with the students themselves.*

Her favorite undergraduate courses were advanced calculus where she got to see *"the logic behind [it]"* such as *"what the chain rule is, proving it, and seeing how it applies to everything,"* professional seminar in teaching mathematics in which she learned a lot about teaching as she was *"able to go back and listen to them [peers] discuss how their teaching experiences were going,"* and geometry. Since substantial knowledge in college geometry was one of the criteria for participant sampling in this study, I wanted to learn more about the geometry course she took, especially given it was one of her favorite classes. Since she took geometry in high school, she was hesitant taking another geometry course in college (this being the only geometry college course she took) offered in the mathematics education department. She said of the geometry class,

> *It was part content [course] but it was things that we can use in the classroom later. But the way that this course was set up was not let me write on the board and you guys copy this down and we will talk about it. It was here are the theorems, definitions. I want you to create a notebook in Sketchpad that has everything you would present in a high school geometry course to your students. It ended up being 420 pages of Sketchpad. So, it was a lot… I mean just very, very, very in depth.*

After finishing her undergraduate studies she enrolled immediately into the masters program in mathematics education at UGA because she knew she would not have done it otherwise. Moreover, she did not want to be a part time student because she knew she could not do both at once revealing again her passion towards working with students,

> *I just knew that either my students would suffer when I was teaching because I was trying to work on other things or my school work would suffer. I had a feeling that it was probably gonna be the school work because I would try to give everything to the students.*

Even though during data collection Aurora was at her second semester of her masters program, she had thus far positive experiences at UGA. She shared that she liked the courses here giving her opportunity of gaining a lot of professional knowledge, *"it's more geared towards increasing your knowledge of how to teach rather than what to teach,"* but being much more competitive compared to her undergraduate experience. Despite having few courses taken thus far, interestingly her favorite one was Contemporary School Mathematics, that focused on 3-D Geometry. She enjoyed being in that course as she had no previous background in 3-D geometry, and because she got familiar with Cabri 3-D. As far as her life goals, she plans on teaching for a little while after getting her master of mathematics education degree (M. Ed.) but eventually would like to have a supervisor position in the educational system in her home state, such as a superintendant of a school or of a school system, or a principal.

Problem Solving Experience and Perceptions

Aurora had problem solving experience throughout her education, but mainly during early years of education (elementary and middle school) where teachers encouraged problem solving as opposed to mere filling out worksheets, and in her undergraduate studies in mathematics education courses where they solved problems from perspective of a teacher and a student. She never had a course, however, that focused explicitly on problem solving.

She viewed problem solving as, *"In terms of real world experience, it's investigating phenomenon that had been within the world that are related to mathematics and discovering how mathematics can help you arrive at the solution or fix a problem,"* but also added that its *"more of an interactive approach to basic problems that we would give [to students]."* She liked routine, word, and applied problems, but disliked proofs. She viewed word problems as *"brain teasers,"* routine problems, while give her satisfaction of being right, were boring, stating, *"I don't know if [solving routine problems] really does the enjoyment level justice,"* whilst she was very passionate about applied problems stating they are *"a wonderful learning experience."* As far as proofs, she stated she viewed them as mathematical problems *"because you have to think about okay, well what steps can I go through to get to this particular solution, to get to what I am trying to show is always true,"* but added that although she disliked them she *"gained an admiration for them because in some sense they are beautiful."*

Often during the interview she replied to questions posed in addition to how they related to teaching and students. The following quote was particularly insightful offering the reader a nice opportunity to get a sense of her passion for problem solving in the educational system,

> *We've taken all the creativity out of mathematics. And so to me, in order to be able to solve problems later on in life when you are not sitting in a mathematics classroom, and someone staring you down, I think that you need those techniques that you learned from being able to go through in a mathematics classroom. I just think that you can apply them later on and that's why I think it's beneficial. An algorithm is not gonna help you when you are tying to solve something, say in a job later in life and you can't remember it specifically right of your head. I just don't feel many people can connect the idea between algorithms and formulas to real life situations because they don't think we emphasize it.*

She described herself as being a persistent problem solver, *"I do get very discouraged when I cannot arrive at an answer. I get frustrated. I am very critical of myself. I try to push myself not to give up,"* but being fairly good at it unless it is something she had never seen before. She elaborated,

> *Some [problems] do require you to think outside of the box. I don't feel I think outside the box that much. I wish I was more creative in that sense. I tend to do things like I've been told, perpetuate that, but I do see a lot of value in being able to approach something in a novel way.*

When solving a problem she does it quietly starting *"with any relevant information that I know about the subject or about the mathematics behind it,"* by making an outline in her mind of steps that would lead her to the solution of the problem. In a completely novel situation, however, she tends to write down related mathematical content and make visual representations, if the problem relies on it that might help her solve the problem by getting *"a general overview of that particular problem."* Though she sees a value in many different solutions, and enjoys reading through them, she does not tend to solve the same problem in multiple ways. She said she preferred solving problems on her own, *"I really like solving it [problems] by myself first."* She also added that she tends to take breaks when problem solving, *"I'll work on a problem, can't really arrive at anything, give it a day or two and something will pop in my head about it and will go back and work on it"* allowing her to get *"a different perspective on the problem."*

Knowing that she solved many problems throughout her education, I was interested in finding out what were the reasons she missed out on problems. She said it was usually *"the combination of the strategy and misunderstanding of the problem"* and taking "*the route that was not fruitful."* She also added that further reasons include mistakes in calculations as a result of speed, and if she *"conceptually just do not understand what the problem is asking."* Such situations are very challenging for her, and she tends to struggle finding out what went wrong but when reaching a roadblock, she mainly consults internet, technology, textbook (definitions, similar problems), and other resources before reaching out to ask someone for help.

View of Mathematics

Her view of mathematics changed throughout her education. When she was younger, she used to love *"that it had an answer. It didn't matter if you got to the right one, but it had something you can get to."* Moreover she viewed mathematics as linear, *"you had the problem, you had the process to get to it and you can always arrive at an answer and I liked that aspect of it because it was stable."* Her view of mathematics as procedure changed to an art form later years,

> *I tend to enjoy it now more for an aesthetic level. To me mathematics is kind of an art form in itself. I feel solving a problem to me is I guess my artistic expression of something I am good at.*

Learning of Mathematics

When talking of preferred ways of learning mathematics, Aurora said she is an auditory and a visual learner. She likes having information presented to her, *"I feel I retain more information if I hear what someone is saying… I tend to listen and when they've stopped talking I'll write down what they wrote down…. I think it really just concretes it for me."* For being a visual learner, she said, *"I am a visual learner in mathematics when it comes to problems that really rely upon a visual representation"* adding another strategy that helps her learn mathematics was to look at a problem on a big picture such as a board, or go to a dynamic geometry program. Furthermore, she likes procedures, and stated practice helps with having *"that technique down."* She disliked when material *"is simply presented and not explained"* hindering her from fully understanding the topic.

Technology Learning Experiences and Perceptions

Aurora had an extensive experience of using technology in learning mathematics. As an undergraduate student she took a technology course where they mainly used CPRs and TI-83 that allowed studying movements, but stated she was most comfortable with using dynamic geometry software, namely Geometer's Sketchpad and Cabri 3-D. For both software she stated,

> *I really enjoy it [GSP] in terms of things that I wouldn't be able to easily conceptualize just on a piece of paper. It's really helpful to have something that can move and be manipulated. I also with the 3-D geometry course last semester became pretty familiar with Cabri 3-D. I really liked that program. I thought it was very cool to be able to see volume and explore the third dimension.*

In her opinion using technology to solve problems has its benefits and drawbacks. As benefits she listed: *"it allows you to look at the problem in alternative ways," "the visualization thing is key," "the benefit of experimenting with the different tools to get to something," "make it [sketches] very quickly"* adding that it is useful for manipulation and modeling in terms of solving problems. As far as drawbacks for using technology to solve problems she mentioned *"an over reliance on it"* elaborating that,

> *You can't rely on it to completely give you the right answer, it might be able to give you a solution but if you can't explain why that solution works, then was it beneficial for understanding the mathematics? I don't think so.*

In summary, she viewed technology important tool in problem solving, but it was not the most important part of problem solving in her opinion. That was not exactly the case observed throughout the problem solving sessions.

Solving the Problems

Problem-Solving Session 1: The Airport Problem

The Problem-Solving Session 1 occurred two and a half weeks after the preliminary interview due to Spring Break. For that reason, at the beginning of the session I again described the nature of the study and most importantly what is expected from her during the session, namely during the think-aloud protocol. Going into the session Aurora was somewhat nervous. The session took approximately 80 minutes. Aurora spent approximately 50 minutes solving the

Airport Problem (see Figure 4 in Chapter 3, p. 64), and during the remaining time, we reflected on the session. For this session, I gave Aurora a sheet of paper and a pen, and opened The Airport Problem.gsp file on which the problem was printed. Aurora first worked on finding the best location, before solving the first part of the problem; that is, before finding all possible locations for the aiport. Once, Parts a and b were solved, she focused on interpreting geometrically the above obtained optimal solution, that proved to be the most challenging part of the problem.

Solving the Airport Problem

Aurora started the session by reading the problem in silence from the screen. When she read the problem, she thought it would be a problem of medium difficulty giving it a 2 or 3 out of 5 difficulty. At this point she read only the problem statement, skimmed over parts a and b disregarding the last part as she realized she needed *"probably get to an answer first before [she] can interpret it."* She never saw a problem similar to this explaining, *"I don't think I ever had anything that was a problem solving experience regarding centers."* Her experience regarding triangle centers, namely centroid, orthocenter, circumcenter, and incenter, was very traditional involving just the process, such as constructing each of the centers, and learning their names. It is rather interesting that here she immediately mentioned centers of a triangle as a plausible solution path though the problem itself did not mention the triangle centers in any way. Later in the interview she explained that idea of using centers *"immediately jumped in [her] head"* adding that her decision was driven by her previous knowledge of geometry where any center would create an optimal point, however, she was not sure if any of the centers would work admitting *"it was more just a hunch."* The rest of the session was spent on exploring if any of the centers was the solution of the problem.

As she was reading the problem she bolded the following part of the problem "such that the sum of its distances to the roads is as small as possible," that represented noting the second condition of the problem. Hence, the understanding episode started. She then started to represent the problem on the GSP which is a cognitive problem solving strategy. The selection of GSP representation, however, and the use of that problem solving strategy was directed towards making sense of the problem, which is a metacognitive behavior. She chose three arbitrary points in the plane A, B, and C for Athens, Bogart and Columbus, respectively, before realizing the problem required the cities to be equally distant from each other, *"Since it's three towns, I will create three arbitrary Points A, B and C or Athens, Bogart and Columbus. Oh, they are equally distant. They can't be arbitrary like I originally did. I will take this away."* She then bolded "equally distant" explaining, *"The bold basically highlights specific things that are very important to the problem to me, so I used that in order to keep it in the back of my mind ... that helps me keep on track,"* that represented noting the first condition of the problem. Hence, verbalization of the problem condition during representation of the problem and reengagement with the text contributed to evaluation of the current solution state and correcting a false action. In the absence of monitoring this might have been costly on the solution.

She then inferred since the problem sentence said the three towns had to be equally distant from each other that the triangle would have to be an equilateral triangle. She continued representing the problem by constructing an equilateral triangle; she drew a circle with radius AB, and then created another circle with radius AB with center at Point B. She obtained two points of intersection by using Construct Intersections, and decided to choose the top point *"since it's it easier for [her] to see."* She finished the construction by drawing the two remaining segments, labeling the top point as Point C, and hiding the two circles. When she was done, she said,

> *Since AB is a radius and now we know that AC is also a radius because Point C lies on the outside of the circle, we know that this [AB] is equal to this one [AC], but it's also equal to BC. So it's an equilateral triangle and they are connected by straight roads, so that's good.*

During the construction of a diagram she stayed engaged throughout; she interpreted the problem statement and conditions of the problem explicitly and correctly. In addition, when she was done she reflected on the undertaken activity and if those reflected the problem conditions before continuing any other activities.

She decided to first solve the second part of the problem explaining she is *"good with finding one answer and then working backwards to finding other answers that could be possible rather than the other way around."* Thus, her personal experience affected the process of problem solving and a choice of strategy, which is a metacognitive act. She reengaged with the problem text expressing a difficulty understanding the second problem statement as posed, *"I am confused by the plural on the distances,"* asking me, "*It only has to be one distance to each of the roads? Not the sum of all like of available points right?"* where I replied *"to the roads."* She acknowledged my answer and stated her conjecture, *"Okay. So it would seem you would want one of the centers of the triangle. I forget the specific names, so I will just start making them."* Conjecture followed immediately after she fully understood the problem conditions; though her plan was overt there was no evidence of planning at all, it was not well-structured and there was no assessment of the relevancy or appropriateness of the plan to the problem solution. Furthermore, it seems that she thought about previous content-specific knowledge and experiences that might be helpful in current problem solving situation, but lacked the ability to manage these resources in a productive way. At that point, however, it was not clear how and why she chose that perspective.

After she stated her conjecture, she decided to use dynamic capabilities of the software to implement her plan, that is, show one of the centers to be the solution of the problem. She constructed an altitude through Point A in the equilateral triangle, but erased it in order to make three more copies of the Triangle ABC *"so that it's a little bit easier to see each of the centers"* using Copy-Paste feature in the GSP. I asked her if she already anticipated which one of the four centers she knew off would be the answer for her stated conjecture, but at that point she answered *"truthfully no."* On the first triangle she drew three perpendiculars to all of the sides of the triangle, and labeled the intersection point as orthocenter. She then moved onto the second triangle where she wanted to draw angle bisectors. She drew angle bisector of the Angle CAB, stopped for a moment realizing that centers of triangle are concurrent in an equilateral triangle, *"Wait a second, that's the same thing as the other one but that makes sense because it's an equilateral triangle."* Though at first, Aurora used the software as a tool to help her implement her plan, feedback that it provided, in this case visualization of the centers, guided her thinking into realizing the centers were concurrent which she had forgot. However, she still decided to construct the incenter *"just to show they are the same"* by drawing angle bisectors for Angle CBA and Angle ACB. She then stated her conjecture, *"I am thinking that this [pointer on orthocenter] is the shortest that this is where the airport could to be constructed for it to be the shortest to all of the roads,"* providing explanation for her reasoning *"because if you were to move the point, say along any of these lines [altitudes], it would get closer to these two [AB, AC] but further away from this one [BC]. I don't know if that balances out or not."* She sat in silence for a moment observing the screen before deciding to test her conjecture. To verify her conjecture she chose a random point on one of the altitudes, and measured distances from that point, Point H, to the roads, *"I am just gonna pick a point on that [Segments GH, HI] and also do it to that one [HJ] and I realize that this is not completely mathematical but I am gonna measure each of them."* Furthermore, she drew Segments KI, KJ, and KG from the orthocenter K to the roads. She then calculated the sum for both sets of segments, as shown in Figure 27, concluding,

> *Okay, so actually where they are all equal is obviously less than when it just lies on the perpendicular bisector and if you move that you can see that it just gets larger as you go towards a vertex [A] and then once you go closer this way [towards Point K], it gets [smaller] and it starts to go up again [towards Point J].*

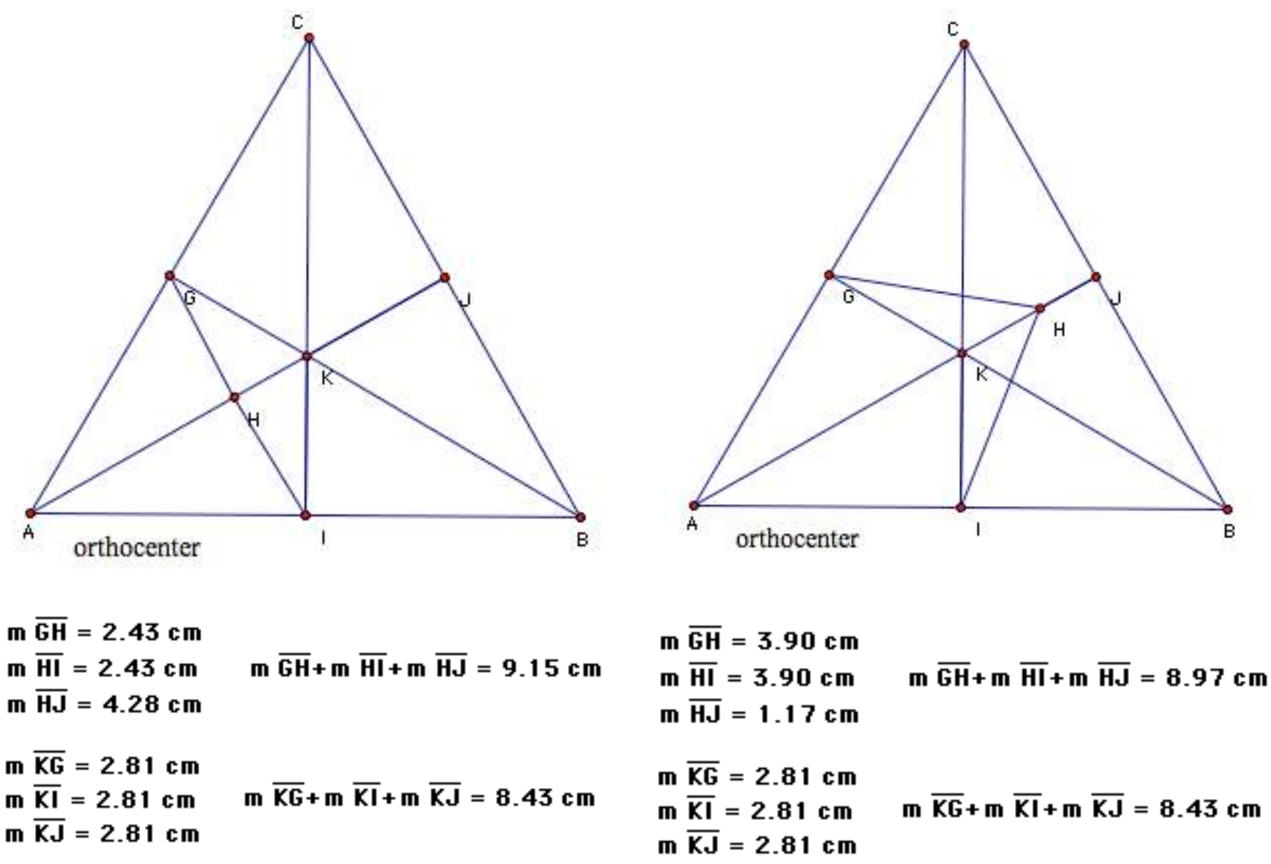


Figure 27. Aurora's verification process of the second part of the Airport Problem.

With this she provided answer to Part b of the problem, *"this convinces me that the orthocenter/circumcenter for an equilateral triangle is the only location that the sum is the smallest you can get it."* During the verification episode student-tool interaction was evident. Aurora's thinking influenced the appropriation and transformation of the software to a tool that allows making and testing conjectures through use of measurement and dynamic capabilities of the software. Thus, the tool became an instrument through stages of personalization and transformation of the tool. Nevertheless, though she reviewed the solution, she never reflected on the reasonableness of the result with the problem; that is, mathematical content and context were not considered as one entity.

As noted earlier, she quickly conjectured that one of the centers of the triangle might be the solution to the problem. For me as a researcher and an observer it was interesting to understand this phenomenon. During the interview she shared that when she read the problem, she started thinking about things that she had seen before and might help her. One of the things she immediately thought of were the centers of the triangle, trying to remember them before realizing they were the same in an equilateral triangle. She was not, however, completely sure if one of them was going to give her the answer to the problem, but she had *"a hunch,"* adding *"I had a feeling that it would be one of them just because they are so important with triangles… I really don't have any mathematical evidence to back that up but it just seems it would be the best."* Interestingly she added if she was given the same problem but with four cities, she *"wouldn't know really where to start except in the dead center of it."* Hence, it seems that conjecture was driven by her learning experience, but also a leap of faith because she did not consider this mathematical concept with respect to this particular real life problem solving context that would trigger productive thinking.

Next part of the session was spent on solving the first part of the problem. She read the statement before trying to understand the problem, and analyze what needs to be done. She noted explicitly the goal of the problem, followed by trying to make sense of the information, and considering a choice of perspective driven by the goal of the problem,

> *In terms of possible locations that would mean that would be where all of them would have to be equal to whatever this value is [pointing at 8.43 cm value]. But I am not seeing another way of doing it unless it lies at that Point [K]. Hm? I am assuming that I wouldn't be asked unless there are multiple, so it's throwing me off a bit.*

She was clearly perplexed by the problem statement and reasoned about it. She paused for almost half a minute before asking me to explain her what I meant by geometric interpretation for the last part of the problem that being another aspect of the problem for which she needed clarification. Before I answered her questions by providing her with an explanation, I asked her if she was moving now onto the last part of the problem before answering first questions of the problem. She replied affirmatively adding, *"I am moving around. I just, I haven't read c. yet."*

She sat is silence for a moment before I encouraged her to tell me what she was thinking. Her response revealed that she was thinking about Part a of the problem again. Moreover, she said she was thinking if other two centers would give her something different. She was still perplexed by the problem statement trying to make sense of it. That was why she thought looking at other centers before realizing they were concurrent. For that reason, she fast abandoned that concern. She quickly changed direction of solution path conjecturing that the solution to the problem is the same as in the second part of the problem revealing her strategy for a solution plan, *"I guess the only option I haven't considered is if all three of the measures are unequal."* Since she, as seen on Figure 27, tested only her conjecture with airport point lying on either one of the altitudes or on the point of concurrency, she decided to test her conjecture with airport point lying inside one of the quadrants bounded by sides and altitudes of the triangle *"to* *see if they could be approaching this [sum of 8.43 cm] or lower."* When I asked if she thought this was a fruitful approach she replied affirmatively, *"Yeah, it can't hurt. I might as well. So I will just pick a random point that could be moved."* Thus, as for the second part of the problem, her plan was overt, but lacked evidence of planning, and evaluation of its efficiency and effectiveness. The role of dynamic capabilities of the software was paramount for implementing the plan. Nevertheless, the choice of strategy and a resource should not be underestimated. It is the metacognitive behavior of the solver that deals with its selection and use; that is, it appears that she was aware this would be a feasible strategy with this type of problem. This claim was supported later on in the interview,

> *So essentially what I did was, I realized that if it was a center well that's one points, so if I have one point, then I can compare that one point to other points that could be within that particular figure to figure out if there are more than one possible locations.*

On the third triangle, she placed a random point inside the triangle revealing the rest of her plan and implementing it at the same time,

> *Originally I used the midpoints of the line segments, but it doesn't necessarily have to be that. It is the easiest one to tell. So just for consistency sake I will just use the midpoints of these and then maybe switch that later to something that's not the midpoint along the sides, just a random point and see what it does there.*

Surprisingly, here she constructed midpoint of triangle sides, and connected them to a random point, instead of dropping perpendicular lines from the random point onto the sides as that is the shortest distance from a point to a line. Unfortunately, she never realized her oversight, and I never followed up on that decision. It appears that she either did not fully understand the problem statement because she did ask for a clarification of it at an early stage of the session, or she was not aware that the sum of perpendiculars from the airport point to the sides would be the smallest distance possible. After she drew all the segments, she measured each one of them individually obtaining Segments LM, LN, and LO, and used Measure Calculate to sum of the three segments. She moved point inside the triangle observing how the sum changed,

> *I was just moving it randomly around the plane to see if it could ever approach 8.43 [cm] without being at the general vicinity of where K is on the original, and it doesn't look like it anywhere … I can get very close to it, but it never looks like it drops below 8.43 [cm]… This suggests that is the best possible place for it if you wanted it to be closest to the roads. So that convinces me with the midpoints.*

Using capabilities of the software she was able to accept her conjecture. However, since she was able to confirm it in a specific situation, she then decided to do it in a more general context. Similarly to the previous planning episode, her plan was overt, but she did not assess the quality of the plan as to appropriateness and structure.

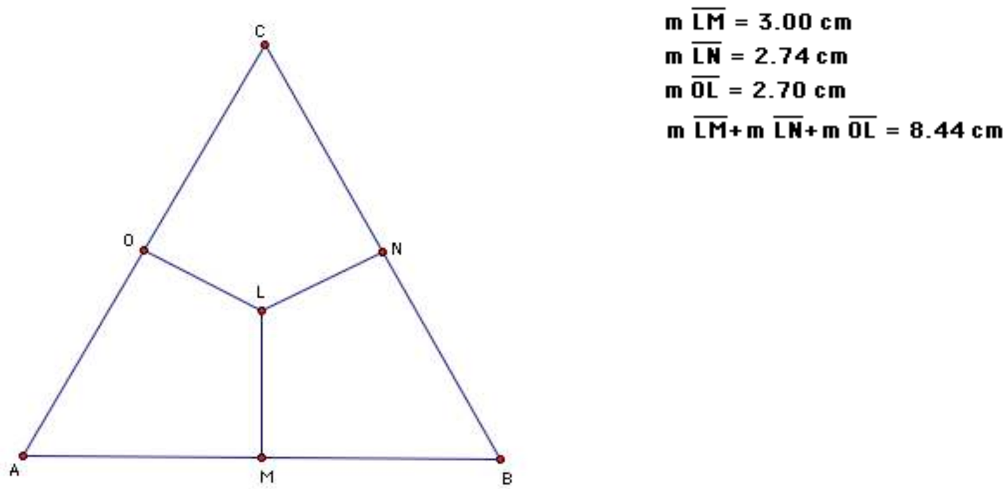


Figure 28. Aurora's implementation and testing of the first solution plan of the first part of the Airport Problem by testing Point L against Point K.

She then tested her conjecture by choosing random points along the sides of the triangle, as she felt this was something she needed to investigate as well, though she was aware of *"a lot of different variables."* She chose a random point on each of the three sides of the triangle, points T, V, W, placed a random Point U inside the triangle, drew Segments UT, UV and UW, measured them and calculated their sum. She first moved Point U, and observed how it influenced the sum. She concluded*, "that's not even going below 10, so that's not giving me anything."* She then moved Point T, followed by moving Point U to line up with Points T, U and W, concluding again, *"not even close to it anyway."* She moved Point W along ending that exploration with the same conclusion. Lastly, she moved Points U and V, after which she offered her final conclusion of her investigation,

> *I am thinking the greatest distance is always gonna be at the vertex, so the airport lying at the vertex would be the greatest distance from the other two. But in terms of when you move it to the middle of the figure, I basically convinced myself that this K point is the best location. In terms of possible location elsewhere, I don't think anything is as optimal as that, because I couldn't find anything that would come close to the 8.43 [cm] unless it was right around that center of vicinity [pointer on L, K] but still even if it was around that center of vicinity it would still be a little bit higher, even it's only by a fraction of say a mile or something.*

Hence, she verified her conjecture that the one of the triangle centers is the only possibility for the airport to be constructed. Even though she used the GSP to test her conjecture relying on its dynamic capabilities and measurement, she viewed her solution as *"reasonable"* and it *"made sense that an airport would need to be located at the center of a triangle."* Though during the think-aloud protocol she never integrated content and context, it was during the interview she elaborated that context allowed her to evaluate the reasonableness of the result,

> *Say I was driving from one side of the triangle to the airport or from another side of the triangle to the airport or from another side of the triangle to the airport. It makes sense to me that would be the smallest because even though we are talking about sums, if I had to drive say from J to and if the airport lied outside of the triangle, say to the other side of AC well that wouldn't be a reasonable fit you know to go to the airport. I don't know in my head it just makes sense.*

Later in the interview, however, she added that she was not entirely sure she solved the first part of the problem correctly because she focused *"more on this problem has one solution"* as a result how she viewed *"problem solving in the past; I need to get to one answer."*

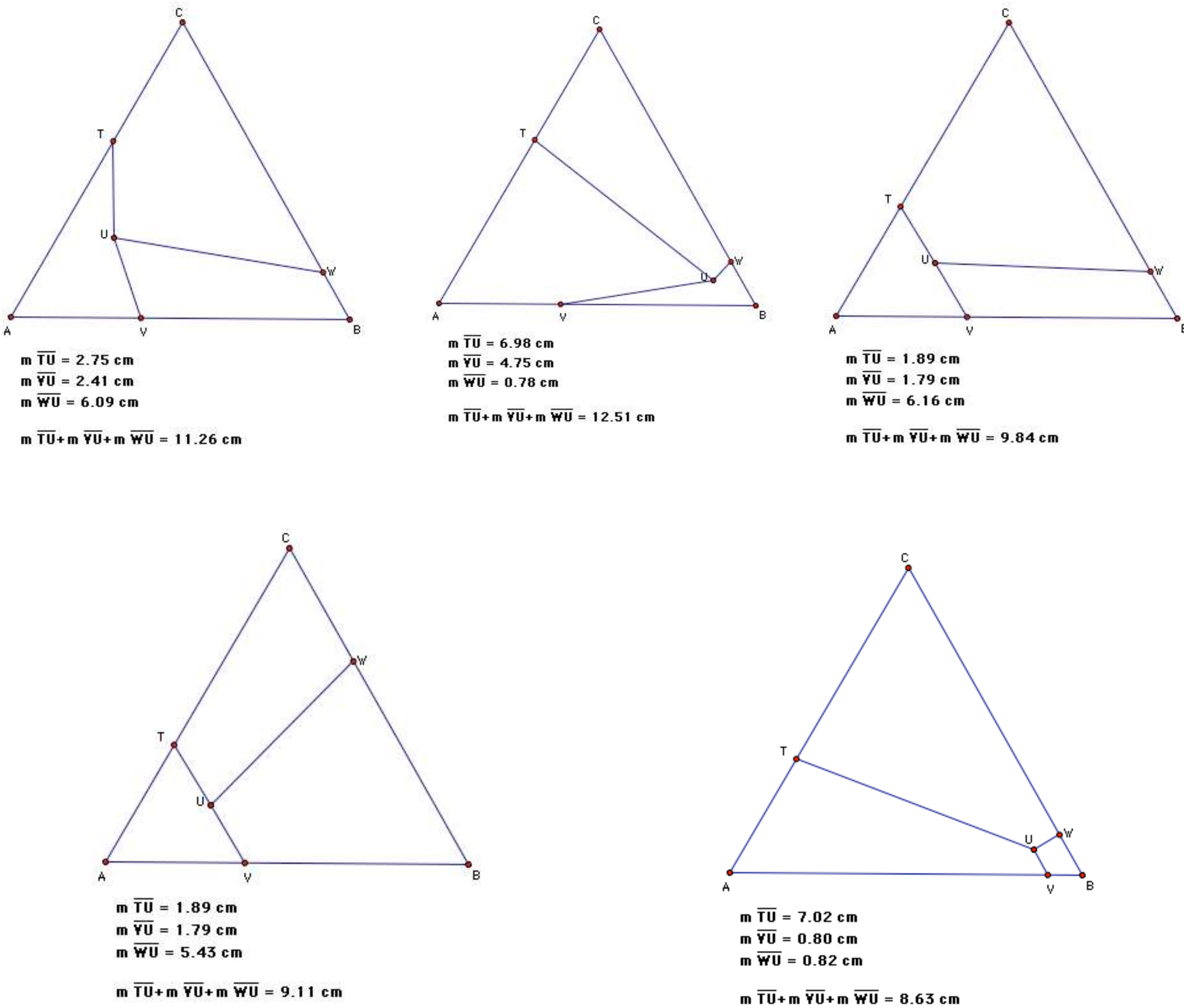


Figure 29. Aurora's implementation and testing of the second solution plan of the first part of the Airport Problem by choosing random points along the sides of the triangle.

The most important action in solving the first part of the problem, upon which the success or failure rested upon, was one that did not take place—monitoring was absent. Already when devising her plan she failed to assess the plan with respect to the problem. In addition during implementation phase monitoring was absent as inactions were not detected. She herself later on in the interview confirmed, *"I was always thinking about reasonableness… I don't think I really* *checked back solutions if they were aligned with the problem conditions."* Since progress was never monitored or (re)assessed, in consequence, Part a of the problem was solved incorrectly.

Similarly to first part of the session, use of the software was paramount. The decisions she made influenced how the tool got use; she was able to implement her plan, and by using dynamic capabilities of the software, measurement and calculation tool to compare the different centers and other points in the plane as to which one would be the solution of the problem. Hence, the software was no longer used as a tool in her hands but became an instrument. She referred to this strategy as *"trial-and-error"* adding *"that was [her] main strategy"* that she used because she *"wanted to back up [conjectures]."* The choice of the strategy and awareness of its helpfulness in the process of problem solving is a metacognitive act; she normally uses it in novel situations allowing her to better understand the problem and solve the problem. It is yet to understand, however, how she acquired this strategy (e.g., experience, vicariously).

She then moved to solving the last part of the problem. She read the problem, and asked if she could do the proof on paper, which I naturally approved. The understanding episode started. She sketched an equilateral triangle, labeled the vertices A, B and C, and sides of the triangles with *a*, *b* and *c*. She drew three altitudes and marked its parts with *x* and *y* (see Figure 30). When I asked her how she decided the label the pieces, she said, *"I am basically using the same principle as when we looked at like here, KG, KI and KJ. So the little ones are all equivalent,"* that is, she based that property on measurement. During the understanding episode, goal and conditions of the problem were established and represented by symbols in a diagram. It also seems that she decomposed the problem in its basic elements, parts of altitude, in order to examine the relationship between it. In addition, she assessed the current state of her knowledge relative to the problem solving state; her decision to move onto the paper, and use algebra was based of interpreting the problem goal,

> *When it says a geometric interpretation for the sum of the distances automatically makes me go to algebra because, okay well I know I will need to add GK, KJ, and KI. I am gonna have to do that so what's the best way to start doing that, well I probably need to write it down on paper and see if I can find a relationship between those two too.*

Therefore, goal of the problem was noted explicitly, and for that reason directed her thinking into using algebra (resource) to solve the problem. Decision to use algebra was driven by problem goal, that is, her knowledge and experiences of what might be done in this current problem solving state.

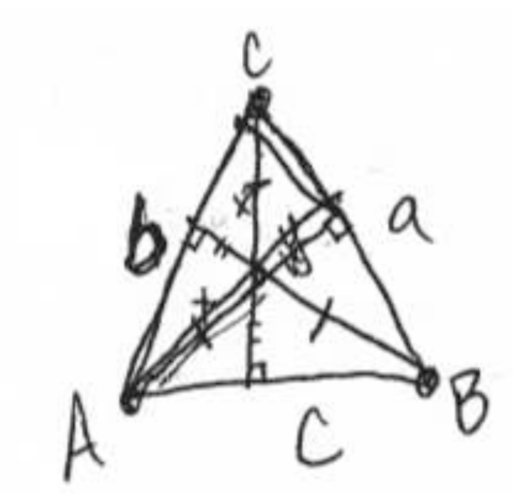


Figure 30. Aurora's pen-and-paper representation of the Airport Problem.

She then started working on the proof. However, at that point she did not think *"through exactly what the proof"* would be; there was no assessment of the quality of her actions as to relevance or appropriateness or even structure. She decided to use the Pythagorean Theorem and apply it to Triangle AJC,

> *I know that this [AJC] is a right triangle so I know that if we use the Pythagorean Theorem we have $(x+y)^2+(1/2a)^2=b^2$ just using this one here. But ½ a is also equal to ½ b because a and b are the same since it's an equilateral. So this is just a hunch by the way I*

don't know if this is actually gonna come to anything but b^2=b^2. So I want to look at, I figured out that the shortest distance is gonna be when GK, KJ and KI are… basically when you draw a perpendicular to the sides from the airport. So if I wanted to do a geometric interpretation of the sum of the distances to the optimal point, then I need to somehow get a relationship between GK, KJ, and KI which is why I was looking at those pieces but you can't know those pieces in relation to, oh yes you can, hold on. That would give me the same thing. Hold on a second.

Thus, as she was trying to solve the problem her plan was becoming structured. She identified what needed to be done in this situation and tried to direct her thinking in that direction. However, her actions did not reflect her goal. Figure 31 shows calculations in trying to determine the length of part *x* of the altitude.

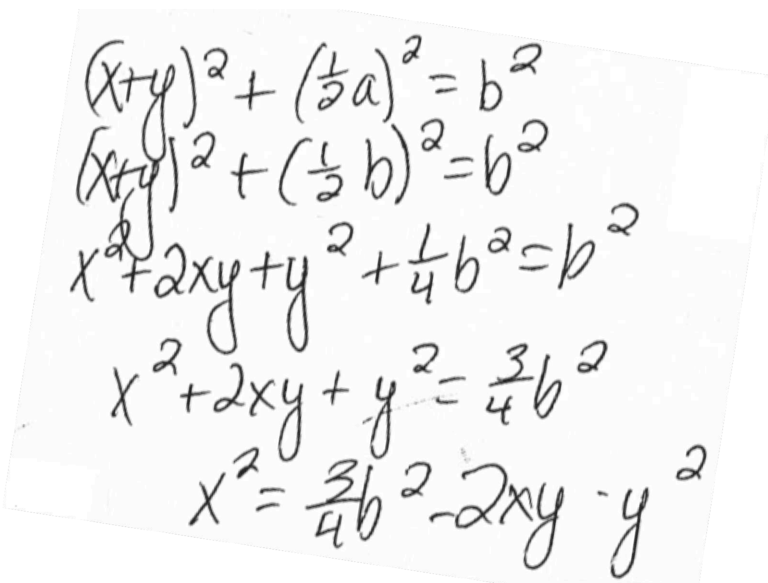


Figure 31. Aurora's work on the third part of the problem: Finding the length of Segment AK by applying the Pythagorean Theorem to Triangle AJC.

However, she was perplexed what to do next. She directed her thinking towards her goal; that is, eliminating *x*, and expressing the distances in terms of *y*. She then applied the Pythagorean Theorem to the Triangle AKG (see Figure 32), before equating the two expressions.

I want it to be in relation to the side, so I want to take x out of the picture because I am not looking at the distance between the airport and the vertices. I am looking at it to the *sides of the triangle. So I can use [Triangle AKG]. Since I have this equal to x^2 then I am gonna solve for x^2 on that side and see, it might give me something that cancels out.*

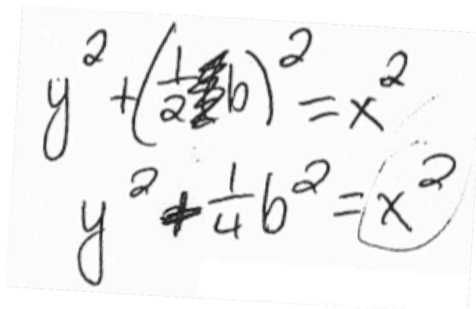


Figure 32. Aurora's work on the third part of the problem: Finding the length of Segment AK by applying the Pythagorean Theorem to Triangle AKG.

Though, she monitored her thinking, there was still no imagining to where these actions will lead her, but were rather incremental decisions she made as she progressed on the proof. She then equated the two expressions obtaining,

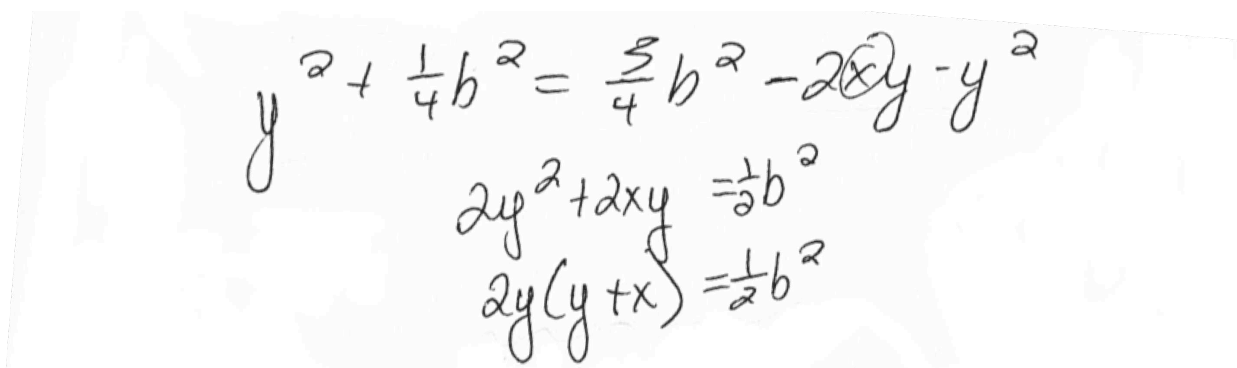


Figure 33. Aurora's work on the third part of the problem: Equating the two equations.

She concluded, *"There is a relationship but unfortunately it relates the whole, not just the sums of the airport to the road but the sums of the vertex to the opposite road or relationship to the side."* She paused for a second after observing the screen saying, *"but in an equilateral triangle the x is always 2/3 and the y is always 1/3."* She seemed satisfied,

> *I don't have x and y, I can have a relationship between x and y so if, hold on, I called x the big piece so I know that y must be equal to 2x if this is 2/3 and this is 1/3.*

She wrote on paper *x=2y*, *y=2x*, *y/2=x* and substituted *x* from the last equation*, "So now I can solve for x, substitute that into this. I don't know if this will work but."* However, here she made a careless mistake since *y=2x* and not the other way around based on her representation. She then substituted *x* for *y* as shown in Figure 34.

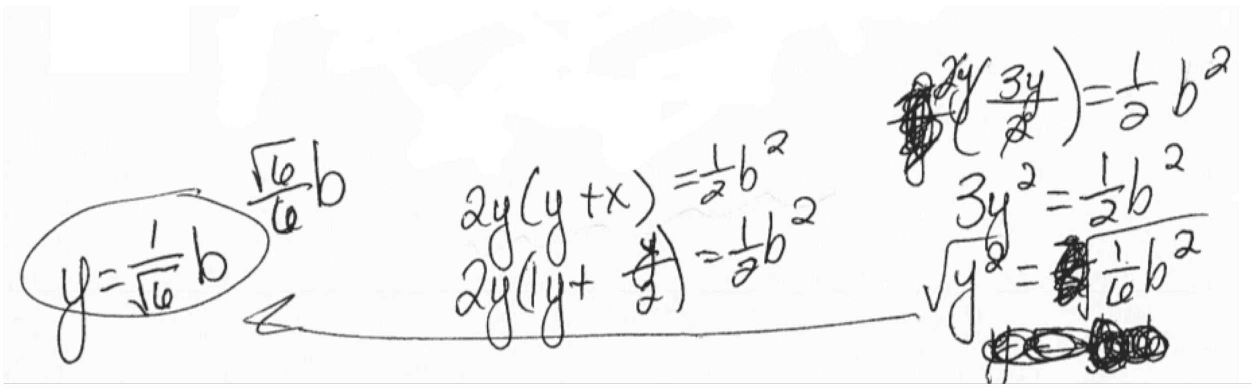


Figure 34. Aurora's work on the third part of the problem: Solving the equation by substitution.

However, once she got the result she was skeptical, *"Hm? That doesn't seem right. Oh, I can't cancel that. Hm…m. I don't think that that's right because y can't be, I mean y is short but is not that short. It can't be* $1/36^{th}$ *of the side."* She realized then mistake from the last step, *"Oh, yeah, mercy. I think I am brain dead today. Okay, so y equals* $1/\sqrt{6}b=\sqrt{6}/6b$*."* She viewed this result as reasonable after calculating the obtained result in the GSP, *"That's approximately .4 of a side [*√(/6=0.41*]. That's seems reasonable."* Then she decided to compare *y/b*, "*So what I was thinking is if this works then I would get same thing [*0.41*] which doesn't give me the same thing [*0.41*]."* Instead she got 0.29. At that point she thought she might have made another mistake somewhere, and decided to use the original equation *"because it's easier"* and substituted *x* for *y* as shown in Figure 35.

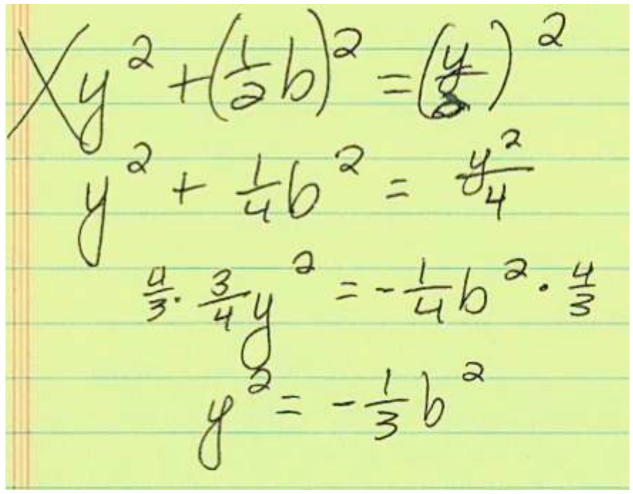


Figure 35. Aurora’s verification of the result by redoing the computational operations.

She was taken back by the outcome, *“I am getting a negative. This can’t be right. How am I getting a negative? This is definitely not right.”* She then went over her work, but again did not notice the mistake made at the beginning, *“I don’t think I have the relationship here wrong though. Let’s see. y equals 2x.”* After she checked her work and did not find her mistake, she felt stuck, *“Not sure where to go from there. I mean I got a viable answer but it’s obviously not the same relationship.”*

Thus, the implementation episode was characterized by execution of steps in a systematic way with a goal of transforming givens into the goal of the problem. There was no evaluation prior to implementation, however, or careful monitoring during it to evaluate the potential utility of her actions. Although the approach she took could have been successful, lack of careful monitoring rested the entire endeavor unsuccessful. There was potential to salvage the solution during the verification episode, but overseeing the mistake in the relationship between x and y was costly. She was certain in her approach; she tried to verify the result using both algebra and technology, but left puzzled as reflection on sensibility of solution progress was astray. In consequence, a long time of the session was spent on calculations.

She sat in silence for a moment before saying, *"We get 1/3, 1/3, 1/3. Oh, so you get just one altitude."* She explained that JK, JG, and JI made 1/3 of the altitude and when you sum them up you obtain the altitude. She double-checked her result by measuring the length of the altitude and comparing it to the sum of the distances from the optimal point to the roads. I asked her to explain me this insight she got, and she said,

> *I guess I didn't see how that would help in relationship to the sides cuz I was still thinking that it had to be in relationship to the side of a triangle…So in my mind I don't know how the altitude distance relates to the side of the triangle, I guess that's why I didn't go that route.*

Clearly, she evaluated her reasoning thus far, at the same time being aware of what needs to be done in this problem solving situation, she tried to direct her thinking processes to that goal. Taking "a step back" was helpful allowing her to realize the solution to the problem. Interestingly, she backed up as far as entire problem goes, evaluating her work thus far. She admitted that though she checked all the decisions thus far she took the ratio as *"that's gotta be right, not really looking at the values to closely."* She also added that once *"algebra didn't work"* and *"trying to relate it [sum] to a side"* she got frustrated. She then considered if there was a potential alternative route. It is then she put effort to piece together her previous knowledge and deductions she made during the planning/implementation episode to solve this problem. In this situtation she was able to access and manage useful resources at the right time which is a strong metacognitive behavior. Furtheremore, her last comment shows evidence that understanding the problem is a crucial prerequisite in solving a problem.

During the process of solving the problem, her perceived difficulty of the problem did not change, that is, she gave it 2 to 3 out of 5 difficulty. She felt *"it was a pretty consistent challenge* *across,"* because it required different steps in her thinking and even though the problem was straightforward and broken into parts she was *"challenged to come up with how it led to the next part."* She added that despite her vast geometry experience she *"didn't find it that easy, it required a lot of different aspects of geometry that you had to remember of the top of your head."*

When I asked her what were the top things that were important for her to solve this problem, three categories emerged:

1. mathematical knowledge of centers and triangles,
2. dynamic software, and
3. problem presentation.

The software was proven to be important during the session. Interactions Aurora had during the session included manipulations, constructions, testing conjectures, and calculations. She thought aspects of the software allowing her to manipulate points was *"beneficial"* in solving this problem allowing her to *"concrete things in [her] mind*" whereas this would have been impossible on paper and pen. She added that using the software to have a visual representation of the problem was paramount; it allowed her to discover some of the relationships, such as centers of triangle being concurrent in an equilateral triangle, and 2:1 ratio adding that she *"wouldn't have been able to solve this problem without knowing that relationship for sure."*

When asked if she learned anything from solving this problem, she said it gave her the opportunity to revisit some of the mathematical content she may have forgotten, such as triangle centers, and relationship regarding them (e.g., concurrent in an equilateral triangle, 2:1 ratio), and it gave her the opportunity to experience a real world application involving triangle centers.

In general, she shared that she liked the real life situation flair the problem and that she *"learned a good amount from it."* At the end of the session I thanked her for her time and effort.

Synthesis of the Problem-Solving Session 1

Aurora read and developed understanding of the problem as she was representing the conditions of the problem by a diagram. She decided to first solve the second part of the problem before solving the first and third part of the problem. She quickly conjectured that the solution was one of the triangle centers without assessing the quality of the plan, and used the capabilities of the software as a way of showing it. The nature of student-tool interaction allowed her to access knowledge of triangle centers in an equilateral triangle she had forgotten and revise her conjecture as a result of execution of her plan. Once she solved the second part of the problem, she engaged in solving the first part of the problem by testing points inside of the triangle against the solution from the second part of the problem (orthocenter/circumcenter) for which she thought was the solution of the the first part of the problem as well. Though she was able to accept her conjecture by using the capabilites of the software, monitoring of her acitivites during planning and implementation was absent, and resulted in incorrect solution. Afterwards, a great deal of time was spent on solving the last part of the problem. This was result of lack of evaluation of the plan and monitoring of quality thinking during the execution of a plan causing negative affective behaviors (frustration) to arise. Nevertheless, once she took a step back, and reconsidered the problem, the realization was instantaneous. This part of the session highlighted the importance of understanding the problem before a meaningful plan can be devised as well as the importance of regulating and monitoring processes during devising a plan, and regulation of affective behaviors. Figure 36 demonstrates Aurora's problem-solving cycle.

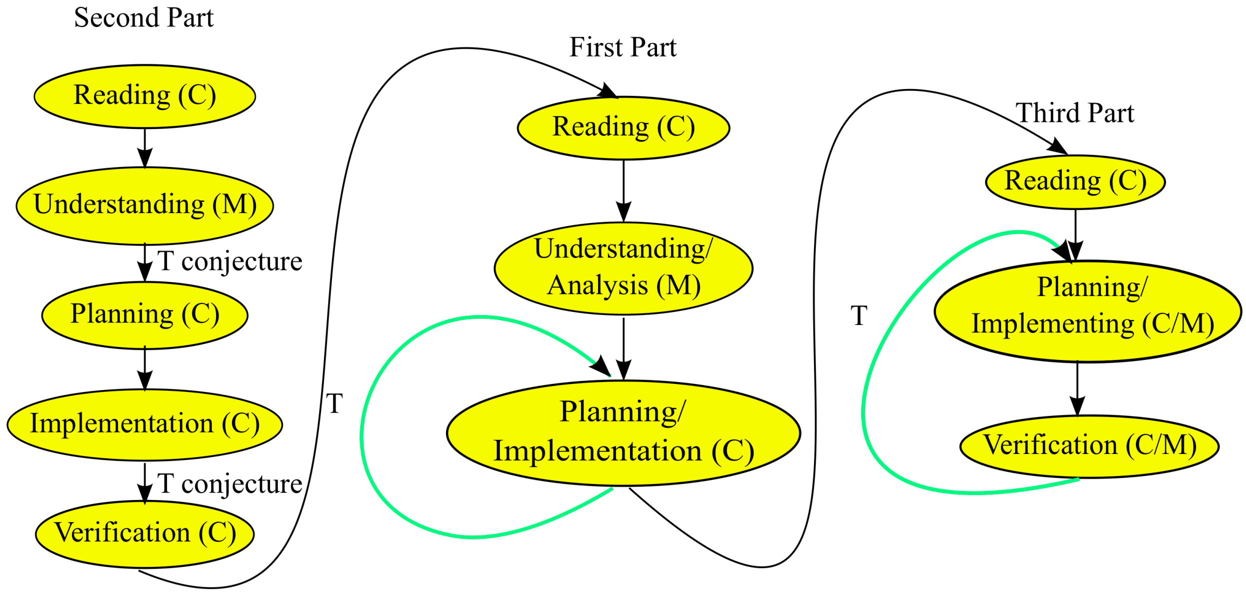


Figure 36. Parsing of Aurora's Problem-Solving Session 1.

Problem-Solving Session 2: The Longest Segment Problem

I met Aurora for the Problem Solving Session 2 a week after the first session. At the beginning of the session I again encouraged her to think aloud throughout the session. The problem solving sessions 2 took approximately 80 minutes. Aurora spent approximately 60 minutes solving the Longest Segment Problem (see Figure 3 in Chapter 3, p. 64), and during the remaining time, we reflected on the session. For this problem solving session, I gave Aurora a sheet of paper and a pen, and opened The Longest Segment Problem.gsp file on which the problem was printed. Formulation of the conjecture proved to be done with an ease, taking Aurora approximately 10 minutes for it, while its proof was a more challenging task that took approximately 50 minutes. After first part of Problem a was finished, construction of Point B, which was Part b of the given problem, was an easy task, followed by solving the second part of Problem a.

Solving the Longest Segment Problem

Aurora started the problem solving session 2 by reading the given problem from the screen and in silence. She never saw a problem similar to this one, and when she read it she did not have a sense how difficult the problem might be. She did not have any difficulties understanding the information given in the problem and felt the problem as presented was straightforward with directions easy to follow. Reading the problem was followed by highlighting and bolding the following parts of the problem, "two intersecting circles," "through one of the intersection points" and "line also intersects circles in exactly two points." Hence, understanding episode started in which conditions of the problem were noted correcly and explicitly.

After she read the entire problem from the screen, she focused on the problem sentences. Even though she did not explicitly assess the current state of his problem solving knoweldge relative to the problem, this was done implicitly. She used a cognitive problem solving strategy, drawing a diagram representing the problem. The selection and use of that problem solving strategy was based on thinking about her previous problem solving strategies, which is a metacognitive behavior. She explained that making a representation of the problem helps her visualize the problem, and helps her in understanding of the problem; that is, the tool helped develop the activity. She started working aloud on the problem where her thinking shaped how the software got used. She performed various parts of the problem statement step by step on the computer as they were outlined in the problem, monitoring her actions before moving onto the next part of the problem. When she was done with the first sentence of the problem, she continued onto the next sentence of the problem before moving further.

> *So one of the intersection points, say, A, so I just picked the top one here, and label it, label A, and then draw a line through that intersection point, that line intersects circles in*

> *exactly two points, say B and C. So, I need two points, but I guess I can just make it anywhere, so the two points that the sentence is talking about would be where they intersect the circles in two others, just label, okay, so we have B and C. What choice of Point B results in the Segment BC such that the Segment BC is the longest?*

In addition to monitoring, she put effort to make sense of the information where conditions and the goal of the problem were established and represented on the representation of the problem. Through her actions she transformed the tool to fit her needs allowing her to visualize and develop an understanding of the problem. Thus, the tool became an instrument through stages of personalization and transformation of the tool.

Once she represented the problem, she decided to measure Segment BC. Immediately afterwards, she said, *"Well I guess first I'll just measure the distance."* However, here she selected Points B, and C as if she wanted to make Segment BC and measure it, but instead she moved parent point on the Line BC, which resulted intersection Points B and C to disappear. She evaluated her actions concluding that she made a mistake when representing the problem sentence as she was not able to observe Segment BC, *"I am gonna reconstruct the line because it says what choice of Point B results in it."* She did not at that time think of a problem as a whole; that is, she did direct her actions with respect to the goal of the problem. For the first part of the problem she started drawing the representation of the problem but put a point in the middle of the line within the second circle. That is, the parent point of the drawn line was not the same as Point B; the parent point was independent of Point B. She made Point B the parent point so that it was more in line with problem directions and moreover, to be able to see how that moved C. Thus, she was able to construct a diagram that would allow her to implement her plan taking into consideration affordances and constraints of the software, which is a metacognitive behavior.

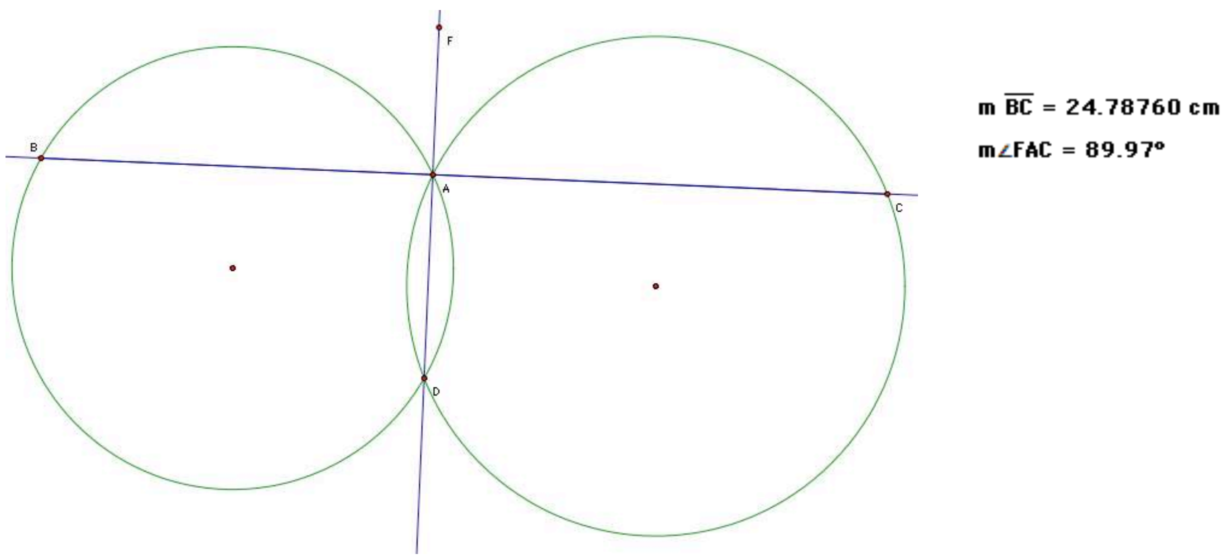
Before she continued solving the problem, she reengaged with the problem statement and reflected on the process and the problem, *"It says that the line intersects the circles in exactly* *two points say B and C. So those are good"* before continuing working on the problem. Since she did not know how she would solve problem, she decided to *"just play around with it"* explaining, *"I turned to trial-and-error just to see if like that gives me a more solid foundation for a conjecture I can make."* Therefore, the exploration episode started. She decided to measure the length of Segment BC. Before she actually got the chance to measure Segment BC, I asked her if she already had any ideas to where Point B should be. Her reply testifies she was imagining the solution as a result of her geometry knowledge,

> *I was thinking that it [BC] should go through the radius because, not B but when you move B around the circle the segment from A to C will be the longest when it goes through the center of the circle. However, that means that B is going to be smaller cuz it will be over here so I don't, I don't know if that will actually work or not. But anyway, so I just have to mess around with it a little bit more because I don't know how it will work out once you move it.*

After my interruption she drew Segment BC, measured it, moved Point B along the circle which resulted in refuting her previously stated conjecture, *"So for now I am just gonna move B and see, so 24.76 [cm], it's getting smaller, okay so that conjecture was actually wrong there. But maybe if it goes through this circle center? No same thing."* She also stated, based on observing the screen, "*it's definitely not when they are approaching the intersection points."* Already at the early stage of exploration episode she monitored her progress. Her actions were directed and purposeful, and as a consequence of monitoring of her progress she redirected her actions forward in the solution process. There was no evidence at that point of assessing the current state in the problem solving space with respect to her knowledge. After a few seconds of exploration by moving Point B along the circle driven by noting the relationship between the conditions and the goal of the problem, she stopped where Segment BC was at its maximum length that she could see because of measurement and observed the screen silently for a short period of time. The moment of silence was interrupted by her saying, *"The only thing I can see that might* *possible relate to it is, if you construct the line that goes through the two intersection points of the circle, it might be the perpendicular."* She referred to this as "*bam moment"* adding that when she was imagining the solution she *"just noticed that A that where the line through A and D would be…it just looked like it was approaching the perpendicular so it seemed like a good conjecture."* That is, this new information she obtained during exploration was considered to be relevant to the problem solution. But she did not thought of it until she got BC in the place where it was at its maximum length. At this point that segment was not drawn. This shows she directed her actions and thinking into understanding the information obtained through the use of the software. For that reason a choice of perspective was made; she drew that segment, labeled it AD, where Point D was the second point of circle intersection. Since she knew *"it could possibly be the perpendicular"* she measured the Angle FAC, where F was the parent point of the newly constructed line through Point A. As she was doing that, she said*, "I know it's not gonna be exactly 90° but if it approaches close to it, so yeah that's really close… you are getting pretty close to 90°on either side."* As she was about to postulate, she interrupted herself and decided to refine her measurements by using length precision feature of the GSP. She then moved Point B slowly on the circle observing the length of BC, moreover, tenths, hundredths, thousandths, and ten thousandths place of the decimal number representing the length of the Segment BC, and the measure of the Angle FAC,

> *Okay, so 24.7876 [cm]. So it's at exactly 90° right now. So if I go up, it should be going down and if I go down, it should also be going down. So yeah 24.7876 [cm] looks like it's the highest it can possibly get it.*

Although she verified her conjecture by using measurement tool, it did not appear she assessed the solution, or evaluated it in any way for its reasonableness. After she "double-checked" her work, she stated her conjecture with certainty, *"line Segment BC is the longest when it is* *perpendicular to the line through the two intersection points, call them, A and D."* Therefore, during the exploration episode, through monitoring of the progress, she directed her actions forward in the solution process. She was able to formulate the conjecture and verify it as well, *"being able to move B around and see what happened to C and then what happened to the length when I measured, I think really helped me figure out."* The software helped her develop the activity she engaged in, as she called it *"trial-and-error."* Nevertheless, her knowledge about her problem solving strategy and knowledge of the software capabilities guided the way she used the tool, such as measurement, and moving Point B along the circle. She transformed the tool to a meaningful instrument allowing her to test and revise her conjectures, and devising one, that is, reaching the goal, which is a metacognitive behavior.



a. Line Segment BC is the longest when it is perpendicular to the line through the intersection points of the two circles (A and D in the picture).

Figure 37. Aurora's conjecture process of the Longest Segment Problem.

After she verified her solution, she reengaged with the text saying *"so formulate and prove conjecture."* She then used Writing Tool to type in above her representation and right of the problem statement *"a. Line Segment BC is the longest when it is perpendicular to the line* *through the intersection points of the two circles (A and D in Figure 37)."* When I asked her to tell me what she was doing, she replied, *"Just so I have it in my head so I can look back at it because it would be hard for me to prove something if I didn't already have something what I thought I should be proving."* Thus, after completion of the first part of the Part a of the problem she assessed her current state, what was done thus far and deciding what to do next. She decided to try to prove the conjecture immediately afterwards confident that *"If I can get the conjecture this quick [10 minutes], then the proof shouldn't be that bad."* She reengaged with the text again noting problem statement conditions "through one of the intersection points" and "line also intersects circles in exactly two points." She tried to make a choice of perspective but she got perplexed about how to establish a relationship between condition and the goal of the problem in order to prove her conjecture,

> *What choice of the point B results in it? But I don't know how to exactly relate B. I know how to relate the Segment BC to that line through the intersection points but in terms of choice of point B. Well I guess it could work for that.*

Moreover she did not know how to proceed further. That said, she decided to solve Part b of the problem before trying to prove her conjecture as she *"really didn't think too hard about it [proof] yet".* Thus, it appears she did not really assess her knowledge relative to the problem at that point. She said that she was *"going a bit out of the way"* but decided to do so for two reasons, *"I think it will help me maybe formulate the proof a little bit too, but I just think I can work of it more better if it's more exact."* Later on in the interview she explained that she found mixing the order when solving a problem very helpful and a way of getting *"thoughts in order."* This is a good example of her awareness in the process of solving a problem; her knowledge and experience about her problem solving situations directed her to choose a perspective (construction) that might help her acquire information that could be used later on in this problem solving situation.

She read Part b of the problem and added a new page and started the construction of Point B basing it of from conjecture she stated earlier,

> *So first thing I will start now with is the line through the two intersecting points instead of the other way around because in order to have a perpendicular to that line, I first have to have that line and so then that will determine where B and C can be in order for BC be the longest. And so now I can create a perpendicular line through the intersection of A so I constructed it perpendicular to that line and so now, this is now B.*

Planning and implementation occurred simultaneously. Furthermore, applying her knowledge and skills as well as monitoring and evaluation of her progress was automatic. She *"found it pretty easy to construct it*" and finished Part b of the problem by writing above the construction "Construction of Longest Segment BC." However, she was aware that she was basing the construction *"on the premise that [her] conjecture is correct."* Moreover later in the interview she added, *"If my conjecture hadn't been correct, jumping to the construction might have not been the best thing because well I would have constructed it incorrectly. But I don't feel like my conjecture is wrong."* Through her actions she transformed the tool to fit her needs allowing her to construct Point B satisfying problem requirements. Thus, the tool became an instrument through stages of personalization and transformation of the tool.

From this point on, Aurora worked on the second part of Part a of the problem, that is, she invested all her energy in trying to prove her conjecture. Again, when she started to work on the proof she did not think it would be a hard task. She engaged in search for a solution plan relying on her previous knowledge and construction of the Point B, but also searched for relevant information to help her solve the problem. As she finished the construction, she concluded that because of the construction, Points B and C would be able to move only when the two radii of the circles were changed. She changed the radius of one of the circles and observed what was happening as a way of trying to think of properties she knew from the construction in order to help her with the proof. She opened again the first page, went through the givens in the problem, and thought of relevant knowledge that might help her solve the problem that she applied on second GSP page,

> *I know that the longest segment that can be produced for the two circles are gonna go through the two radii. So what I did was, I wanted to see if that line is parallel to line Segment BC which it is, because I constructed it based of its parallel to one of the centers points to see if it went through the other center point which it does.*

Goal driven, however, this approach raised some questions as she was unsure how to relate her conjecture to the line obtained when connecting the two circle centers, *"see the problem is that I am going off the conjecture…but using that, if it's true, then you could somehow apply that same reasoning to BC where it goes through the intersection point."* Though she considered this new information might be relevant in the solution process, she was not able to anticipate actions allowing her to use it. She was perplexed and uncertain how to *"prove something is the longest out of all the possibilities."* She went back to the first page with and observed the original drawing. She connected center of the left circle to Points A and B, and center of the right circle to Point B and C. She knew that the two triangles obtained were isosceles triangles and wanted to somehow relate the two triangles. She thought the central angles might be congruent but refuted that conjecture based of observation. Since she realized that probably was not really going to help her, she then decided to try something different.

Her new approach related to the concept of chords, or at least attempted to do so. She said, *"I am sure it has something to do with principles about chords but I don't remember what it is, I don't remember any properties about chords really."* That approach could have been fruitful, but not knowing main geometry propositions about chords hindered her. She had a pretty good *"hunch"* how these might be important, *"Because you have to show that BA which is a chord of the left circle plus AC which is the chord of the right circle is, when added together gives you the greatest length when it's perpendicular to AD,"* adding yet another evidence she did not have solid ideas how to do the proof, "*I mean just can't think of another way just to relate BC all together."* Her elaboration revealed her problem solving approach of not looking BC as one entity but rather as a sum of its parts.

> *Instead of taking BC as a whole segment and looking at it that way, I was trying to break it down, well what do I know about the left circles, and what do I know about the right circle and comparing, because I know BA plus AC is gonna give me the BC. Which is probably somewhere in the proof.*

During this exploration episode she was obviously aware where she was in the process of solving a problem; she was not only aware of the content-specific knowledge that might be helpful in this situation, she was aware of the strategy available to her to solve the problem. Thus, after two of her approaches did not work out, it was a metacognitive act that dealt with reflecting on the task in directing her thinking processes towards a solution of the task, and choosing the strategy. Even though that being a good problem solving strategy, she was somewhat perplexed by it; on the one hand, she knew that BA and AC make together the Segment BC, and are going to be the longest chords when they are the diameter of the circle, but on the other side she knew that they have to lie on the same line and therefore uncertain how to relate those two chords.

She was silent for few seconds before abandoning that approach and going in a new direction. By observing the original drawing (see Figure 37) she noticed that Point D might lie on the line connecting the center and C, and verified this conjecture by drawing the line that did indeed contain Point D. She inferred the same principle would apply for Point B. She explained this decision,

> *I was trying to think of all the ways I know how to prove things and the easiest is normally to use either similar triangles or something like that when you are working with geometric constructions. So I was trying to figure out a way I could somehow get similar triangles or congruent triangles out of this picture.*

This was Aurora's fourth exploration episode thus far. Although multiple attempts through the problem solving space, her comment above again shows the attempts are directed as they are goal driven, and not random as they might seem. Similarly to the previous episode, after one approach did not work out, she cycled back to relating the problem to her knowledge of problem solving strategy including three components: knowledge what needs to be done, what was done and what might be done in this problem solving state. In addition, none of the exploration episodes thus far lasted a long time. She monitored her work and for that reason each episode was terminated either because it was considered useless or there were no means to efficient movement towards a solution.

Before she tried to provide me with the proof, she decided to do it over on the second page for *"the sake of accuracy"*. On the Figure 37 she drew two lines, each connecting Point B and C to their respective centers of the circle. She then explained,

> *I know that BD is the longest segment you can create from one point on the circle to another point on the circle because it's the diameter. So anywhere you move B in relation to D, it's just gonna get smaller from that point, and the same principle goes for C and D. So I know that those are the two longest segments that you can create in those circles individually and then I know that they are gonna share AD because that's the intersection and I know they are gonna lie perpendiculars. I haven't proved that yet so I don't know if I can really use that reasoning but anyway, if they lie perpendiculars than it makes sense if this [BD] is the longest and this [CD] is the longest, then this [AB] must me the longest and that [AC] must be the longest segment.*

After she thought about what she said, she realized that the two Triangles ABD and BCD where neither congruent nor similar. She paused for a second and decided to add back the Line EF, concluding that now she obtained two similar triangles, Triangles DBC and DEF. Thus, she fit new information from the first exploration to a new situation. But she was still unclear about her progress, *"I don't know if that helps me or not though."* She changed the segment width of Segments BC, BD, CD, and EF to thick, and focused on trying to direct her thoughts into proving the conjecture by relating conditions of the problem, goal of the problem, and knowledge she had available to her. However, she was still fuzzy about the proof,

> *I know I want to relate BC to something and so this does relate it to what I was talking about earlier DB and DC, that those were the longest segments and so then that it makes sense that BC is the longest segment but I somehow have to relate that to EF which is the distance between the two centers of the two circles.*

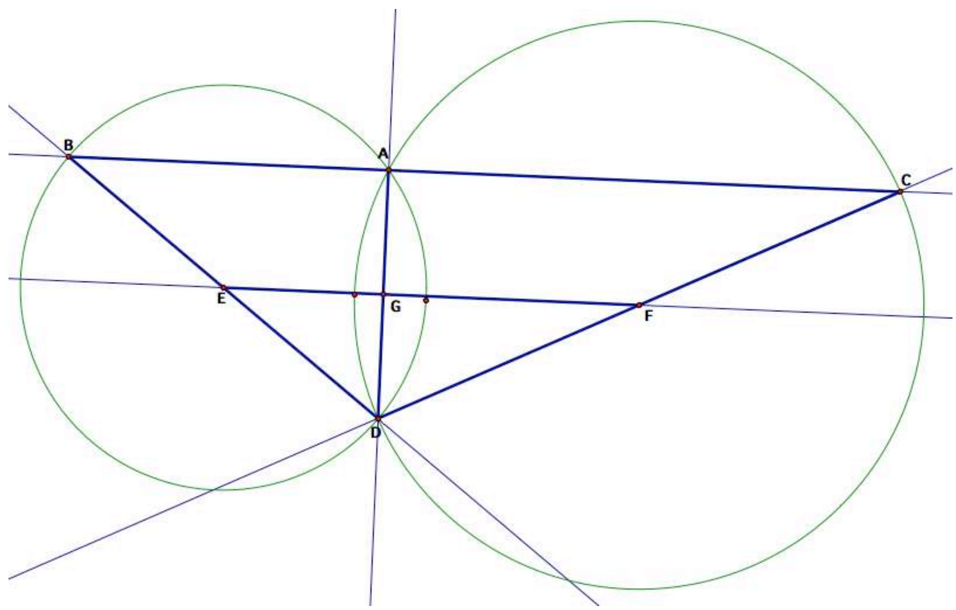


Figure 38. Aurora's work on proving the conjecture of the Longest Segment Problem.

She then realized that EF did not necessarily have to be perpendicular to AD, whose width she then changed to thick saying, *"I feel like I am missing something really obvious here."* This was her first time putting in relation Line Segments BC and EF, and she engaged in resolving the problem which results she would then eventually use in the proof. She then stated what she already knew, Triangles BAD and EGD were similar triangles with relationship ED:BD=1:2, also Triangles ACD and GFD were triangles with relationship DF:FC=1:2.

However, she realized that she could not assume AD was perpendicular to BC because that was what she was trying to prove. She paused for a few seconds, directed her thinking towards solving that issue, breaking the silence by saying, *"I think there is the property that says that intersection of the line through the two points of intersection is always perpendicular to the line through the centers of the two circles."* She opened a new page in GSP to test that property; she started by constructing two circles, drew a line through two of the centers of the circles, marked two intersection points and drew a line through those two points and used Angle Measure to measure the angle between the two lines. She obtained the measurement of 90.00°, changed the radius of both circles before concluding her conjecture was correct. Through her actions she transformed the tool to fit her needs allowing her to test the property. Thus, the software helped her develop an activity, which she then used in her advantage; the software allowed her to carry out that task necessary to continue with her proof. The tool became an instrument through stages of personalization and transformation of the tool.

She then continued working on the proof above trying to construct logically connected mathematical statements; since Triangles BAD and EGD were similar and BD was the longest segment for the first circle,

> *Since there are similar Triangles [BAD and EGD], then I know that this segment is the longest [BD] that can be for that circle and this Segment [CD] is the longest that can be for that circle and I know that this is now a right angle [angle between Line AD and EF], that this must also be a right angle [angle between BC and AD] because of the property that if you have a transversal corresponding angles are congruent. So I know that this Angle [EGD] equals this Angle [BAD] since these two are parallel [BC and EF].*

Here she became perplexed about knowing if indeed Lines BC and EF were parallel, but then realized Lines EF and AD were perpendicular (because of the property she tested), and because of the transversal property that Lines AD and BC were perpendicular, that is, she could claim that Lines BC and EF were parallel. However, she then asked herself, *"But how do I know that* *goes through BC? Oh, no, I don't, I am constructing BC."* She sat in silence for a few seconds before deciding to a write a proof, that is, what she just said, on the sheet of paper I had given her with the problem. She opened first page on GSP, to be consistent with the labeling and started writing. She wrote,

> Given circle E and circle F which intersect at Points A and D. Construct a line through Points A and D. By property ________________, we know that the line through E and F is perpendicular to AD. Construct a line parallel to EF through Point A. (Here she was constructing them to be parallel but she did not know that that would be B and C. That explains B' and C' in the following statements). Construct Lines B'D and C'D where B' is the intersection point of the line parallel to EF through A with circle E, and C' is the intersection point of the line parallel to EF through A and circle F. BD and CD are the longest segments that can be drawn for their respective circles.

When she was done she said, *"Now I have all the parts that I've used in this particular part,"* but looking perplexed she added, "*That's not actually a proof."* She again thought about what she stated before, realizing that she was not able to prove her conjecture because she was *"just basing it of the assumption that if BD is the longest, then DC is the longest then BC must be the longest."* She mentioned she originally wanted to use similar triangles to prove the conjecture and tried to combine the two approaches now. Since 2:1 ratio was the only ratio she obtained, she tried to use that in her proof,

> *Since you know that this [DF] is 1 to the whole thing [CD] of 2, then this [DG and AD] must also be 1:2 which makes these two [DG and AG] equal which also makes this ratio [GF:AC] 1:2 but see this can never be greater than a ratio of 1:2 because you can't have anything bigger than the diameter [CD]. So to me that makes sense that this [AC] would be the longest segment, then based on that reasoning just with this side length [AD]. I mean basically what I wanna say is that, I know if these [EF and AD] are perpendicular, I can use that same similar triangles reasoning on this side also [on Triangle ABD], which means that if this [AB] is the greatest segment that I can make, and this [AC] is the greatest segment I can make using those properties, then together that must be the greatest segment I can make and so then it must be BC the longest segment… That might be way off base. Yeah, so I guess that would be the end of my proof.*

Obviously she struggled proving the conjecture and tried many ideas to prove it. In that respect she said, *"I do trial-and-error with my proofs too. I'll do a proof wrong a couple of times and* *then I will walk away from it for a bit and I will come back, and maybe something else hits me."* Thus, reflection on the exploration episode where new information was obtained allowed her to piece together her previous knowledge and deductions she made during the exploration episode to prove her conjecture. Unfortunately, though her third attempt based on subidiving the problem into two parts, and then assimilating it parts into a whole could have been successful, lack of knowledge about chords hindered her. Nevertheless, she monitored her work, and her exploration episodes did not turn to an unproductive effort but became an appropriation of resources available to her, knowledge and technology, as well as experience, *"I would go over, okay well what proof techniques have I used before that could be helpful in this that I am missing that I haven't used or I haven't considered or I haven't looked up."* Furthermore her decision to solve the last part of the problem before proving the conjecture was helpful in a way of being to able to visualize it, move Point B while still preserving Segment AD being perpendicular to Line BC, to see similar triangles as opposed to working from the first representation of the problem where by moving Point B relationships would not be preserved which might would have thrown her off.

During the process of proving the conjecture, student-tool interaction was evident, and the software became an instrument through her appropriation of the tool and through effects of the tool on her problem solving processes and strategies. Although, proving the conjecture required retrieval and coordination of variety of geometry concepts, and problem solving strategies, Aurora did not exhibit any of the negative emotions, but rather persevered in her problem solving activity.

Once she was done with the problem she ranked the proof being 4 out of 5 difficulty adding that she did not think this was *"a beginner level"* proof. Even though she was confident that she got the conjecture and construction correct, she was uncertain if her proof was correct though it made sense to her, if there was an easier way to do it based on properties she did not know, and if she could make some of the assumption she made, such as Segment BA being the longest on the Line BC because BD was the longest possible segment of the circle and because of triangles similarity. She believed that *"being creative is not the best thing, if you don't get the right answer."* Nevertheless, she found it interesting to be given a problem where she had to make inferences about two circles at once, but shared that *"might have thrown [her] of a bit"* because she was used to only looking at properties of circles, and proofs of circles using only one of them, making the problem harder for her to solve.

When I asked her what were the top things that were most important to solve the problem, three categories emerged:

1. mathematical knowledge; mathematical knowledge used to solve this problem included several properties, such as *"the knowledge of similar triangles," "the knowledge of the properties of circles," "the property that the line through two intersection points of the circle is always perpendicular to the line that goes through the centers of the two circles."*
2. experience with proof writing that included *"a prior experience with proofs," "knowing where you want to go and what might help you might get there,"* and
3. dynamic software.

Dynamic software was mostly important to formulate the conjecture, and construct Segment BC. For the importance of having technology to formulate the conjecture, she said, *"I couldn't have come to my conjecture if I hadn't have been able to move B around and I don't feel I would have been able to do that with paper and pencil or anything that was not dynamic."* In formulating the conjecture, the dynamic aspect of GSP was most important, namely being able to move Point B along the circle. Manipulating Point B was important for finding where Segment BC was the longest and then relating that position of the Segment BC to Segment AD. Being able to measure the angle between Segment AD and Line BC, helped her to have a base for conjecture. She viewed that without technology it would have taken her longer to conjecture as she would have had to draw several representations in order to see Point B in several different places and infer how different representations relate one to another. She also added that noticing Segment AD was important in conjecturing and with paper and pen that would have been difficult. In constructing Segment BC she used construction tools in GSP, such as construct segment, line, and perpendicular line. Her statement that the software was important in constructing Segment BC was peculiar to me as the tools she used to construct it in GSP can be easily done also with a straightedge and a compass. She elaborated that without having technology it would have taken her a lot longer because of "*tedious constructions in order to get it perfectly right,*" and *"because it doesn't have simple construct perpendicular."* She did not think technology hindered her in any sense.

Close to the end of the interview when I asked her what did she learn from this problem, she said she needed to work on proof writing. Also, she added that solving this problem gave her the opportunity of learning about herself as a problem solver,

> *I think there is something to be said for looking at things you are not familiar with or haven't been familiar with them in a while and seeing where your difficulties are and where and where your strengths are. I can tell you that I am probably pretty good at conjecturing something and then constructing it but maybe not being able to completely explain how I can get there from what's given.*

As I was wrapping up the session talking about her experience today, Aurora said, *"It's very interesting. I enjoyed it."*

Synthesis of the Problem-Solving Session 2

Aurora started the session by reading the problem in silence and bolding conditions of the problem. She made a representation of the problem before using dynamic capabilities of the software in order to formulate her conjecture. Even though her conjecture was based on observation and verifying it by measurement, she decided to construct the Point B that satisfied the problem requirements as a means of helping her proving the conjecture. Most of the session was spent on searching for a plan where she relied on her previous knowledge and experiences with respect to which strategies to use. She stayed mentally engaged throughout exploration episodes always directing her thinking process towards the goal. By assimilating and effectively coordinating information obtained during her first exploration episode to her last exploration episode she was able to construct logically connected mathematical statements and prove her conjecture. Although availability of resources or lack of was paramount during this process, her exploration episodes did not turn to lengthy non-productive pursuits but to productive efforts allowing her to gather, consider, and use information necessary to solve the problem. Throughout the session her actions transformed the software to fit her needs allowing her to visualize the problem, test her conjectures, and refine, revise and abandon her actions. Thus, the tool became an instrument through stages of personalization and transformation of the tool. Although the session lasted almost one hour, Aurora remained enthusiastic in her problem solving activity. Figure 39 demonstrates Aurora's problem-solving cycle.

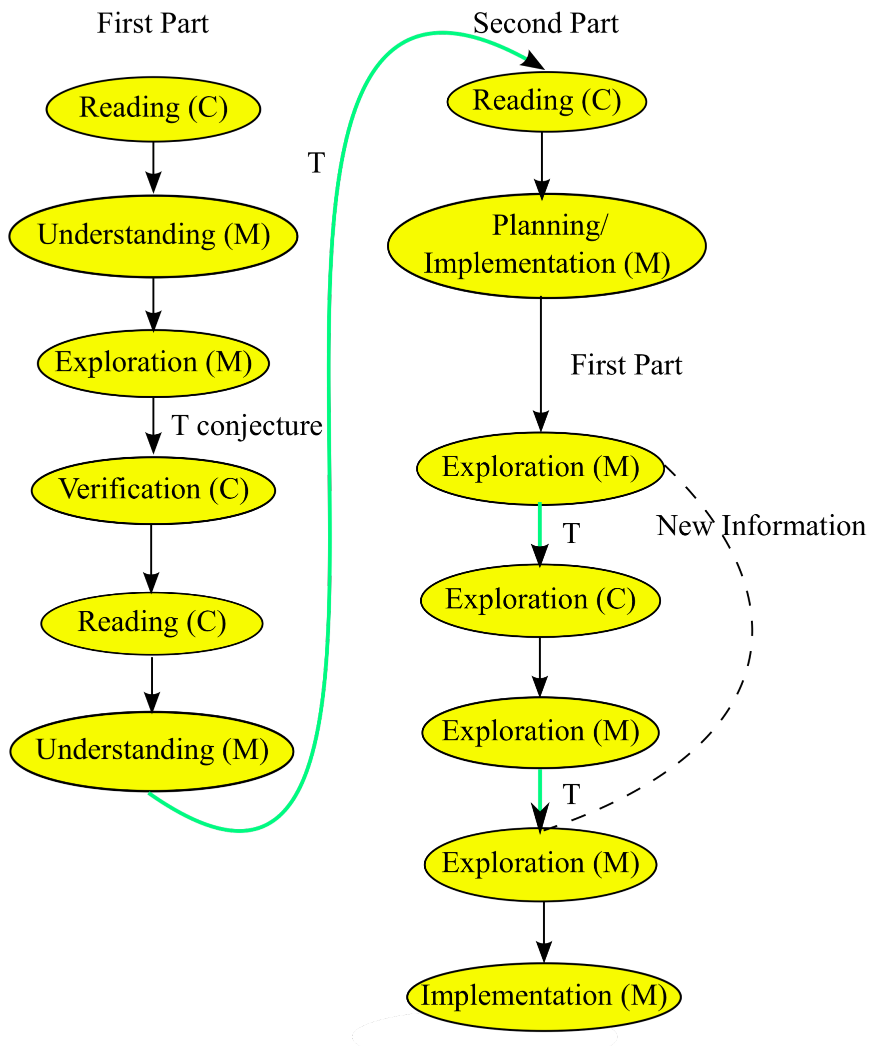


Figure 39. Parsing of Aurora's Problem-Solving Session 2.

Problem-Solving Session 3: The Land Boundary Problem

The Problem-Solving Sessions 3 comprised of two sessions that were one week apart. At the beginning of the session I had told Aurora that the problem consists of two parts, and it was up to her to decide if she wanted to solve entire problem in one session or in two sessions. For this problem solving session, I gave Aurora a sheet of paper containing the first part of the Land Boundary Problem (see Figure 5 in Chapter 3, p. 65) and a pen, and opened the first subfile of the Land Boundary Problem.gsp file on which the problem was printed. Solving the first part of the problem was somewhat a challenge for Aurora taking her approximately 50 minutes, whereas during the next 30 minutes we reflected on the problem solving session. Use of technology to solve the problem was prominent, and allowed for "bottom-up" strategy to solving the problem. Bottom-up strategy entailed taking the problem as solved, that is, starting with the problem solution (unknown) and working backwards to obtain the answer to the problem.

Solving the Land Boundary Problem: First Part

Aurora started the problem solving session by reading the problem in silence from the screen. As she was reading the problem for the second time, she highlighted and then bolded the following parts of the problem: "common border consisting of two sections," "bent, and they would both like to straighten," and "keep the same amount of land." Thus, the understanding episode started. Also, all of the problem conditions were noted explicitly as well as the goal state. She then drew the end boundaries of the land to help her visualize the problem, help in understanding of the problem as well as help with her problem solving plan,

> *it just helps me visualize it more as a plausible land…that was more related to how the problem was presented. If it's presented as land, it helps me to mark it off as plots of land and so I connected those and it just helps with the area calculations…Well mainly the biggest part was so that I can compare areas at the end.*

She had minor difficulties understanding the problem as posed because at first she interpreted it as the farmers *"needed to change the area of the land in order to keep the amount of fence the same"* but that was cleared as she verbalized it during problem solving process; verbalization of problem goal contributed to reengagement with the problem before moving forward with the problem solving process. She restated and interpreted the problem condition allowing her to make explicit relationship between the given information and the goal of the problem, *"So they want to keep the same amount of land they have right now. Well amount of land would be area."* When she started working on the problem, she did not know how to approach the problem nor which geometry concepts and propositions might help her, *"I don't think I have ever seen a problem where you were trying to keep the same amount of land by moving the border."* Thus, before starting a new episode she assessed her current knowledge relative to the problem, and partially directions to come.

Once she stated the relationship between the given information and the goal of the problem she immediately devised her plan of calculating the area of sections I and II using Construct Pentagon Interior and Measure Area Tool, and then finding an approximate solution,

> *I think first I'll find what they have already and work towards trying to straighten out the fence so they keep the same amount of land but I first need to know how much land they have at the moment.*

Once she measured the two areas, she made a duplicate of the problem representation to help her in deriving the problem solution, *"I want to keep these two areas the same so I am just going to make a duplicate of this page so I can manipulate one but still have the other one remaining the same."*

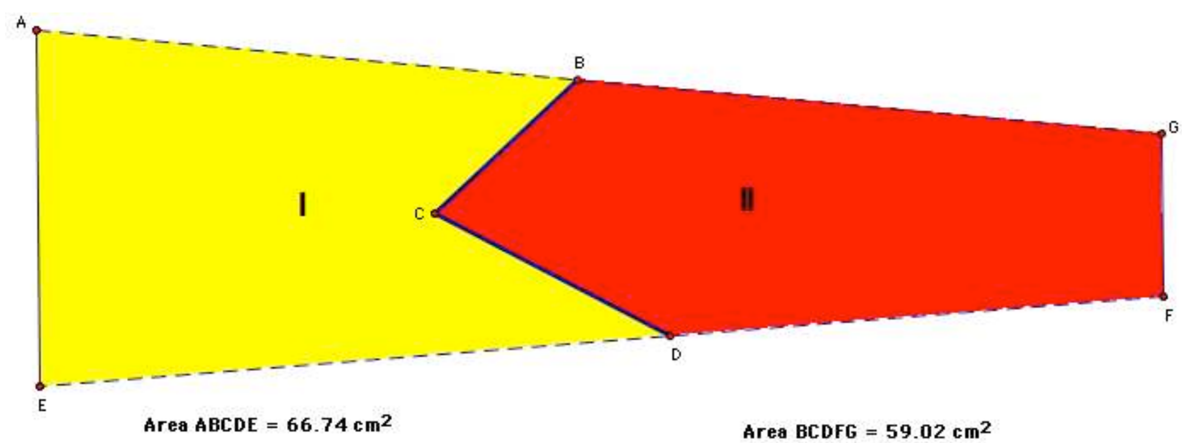


Figure 40. Aurora's solution plan of the Land Boundary Problem: Finding the area of farmers' land.

When she obtained the area of the two sections of the land, she started to analyze the outcome helping her develop a better understanding of the problem likewise. She concluded that the area I was bigger than area II, and that the amount of fence will change,

> *So it says if they each want to keep the same amount of land. That to me means that it's exactly the same amount. In that case then you can't do that unless you change the amount of fence, like the fence can't stay the same if you want to straighten it out without making the land any bigger. So you are gonna have to take out pieces of fence and reorganize them so it's just straight.*

She decided then to put a point, labeled it as Point H, on the Segment EF and drew a straight fence through Point B on the second sketch. She again constructed quadrilateral interiors of two new quadrilateral sections and measured their areas. She moved Point H on the Line EF such that the obtained areas of land sections would be the same as given in the problem,

> *Well I need straight line that will give me an area that's close to what I've already calculated. I just plotted a Point H on the Segment EF and then connected it to B because I knew that would give a me straight line for fence and I wanted to keep at least one point the same cuz I assumed that not both points would change. Then I tried to maneuver H so that it would come pretty close to what the areas were.*

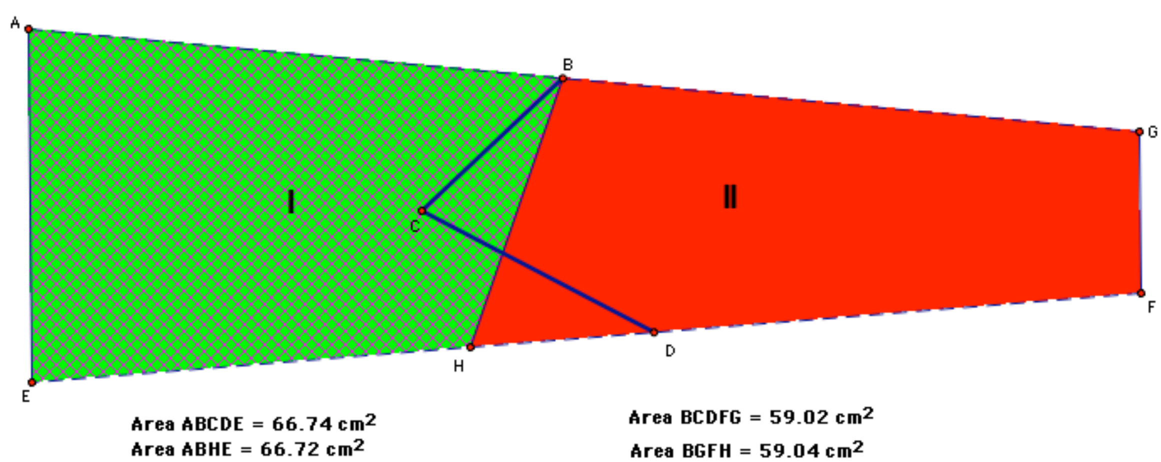


Figure 41. Aurora's solution plan of the Land Boundary Problem: Finding an approximate solution.

Her plan was overt and goal driven, that is, she wanted to keep the area of the land the same. Because of the generality of the problem, *"there is not much to work with in terms of, it's not here are some properties, go from there, here is a picture, what can you tell me about it and then how do you get to a solution"* she did not know how to approach the problem purposefully but decided to *"playing around with it."* This problem solving technique, though making her concentrate more on the answer rather than on the process, was consciously chosen as a result of previous problem solving experience and she considered it useful in problems such as this one, namely in novel problem solving situations. However, she was not sure if it would be helpful with this problem. Through her actions she transformed the tool to fit her needs allowing her to find approximate solution of the problem and help then work towards finding that solution. Thus, the tool became an instrument through stages of personalization and transformation of the tool. In addition, during and after implementation episode she did not assess her knowledge that might be helpful in the current problem solving situation nor was there evidence of sense making but rather jumped into exploration. Next paragraphs outline explorations she engaged in the search for a solution.

She analyzed the new representation (see Figure 41) ending with selecting a problem path, *"Okay so it looks to me that they are congruent Triangles [BCI and DHI] just sort of rotated or rather reflected across this Line [EG] maybe."* She thought this to be *"too much of a coincidence"* and directing her thinking trying to verify it,

> *well how do I actually construct this rather than just manipulating a point down at the bottom because I knew I needed a construction in order to have it exact, so I was trying to figure out how to get in this case BCI down to where it corresponded with D and still lie on the Line BH.*

She drew Line EG to see if Point I lie on it. However, using GSP she was able to refute her conjecture and erased the Line EG. She continued the idea of the two triangles being congruent and for that reason she wrote down on the hard copy I gave her what she needs to prove, *"BC needs to be congruent to DH in order for the area of I to be congruent to the area of II,"* and tested her conjecture by reorganizing the parts of the land as she was explaining her action,

> *I guess what you can do is simply take BC and superimpose it onto EF from the Point D. So basically what you need to do, is you need to move this piece of land [BCI] here to right there [HDI] in order to make that work.*

She turned to sketch one to implement her new idea. She intended superimposing BC on the Line EF through Point D by Mark Vector, however once she marked the vector she realized that would not work as it would *"go in the same direction."* Therefore, she made a metacognitive comment, *"What do I want to do?"* before executing her plan by drawing a circle at Point D with radius BC. Even though that circle intersected the Segment EF in two points, one within Segment ED and the other within Segment DF, she hid the circle and the second intersection point focusing on the first point that she labeled as Point H. She switched twice back and forth between the first and second sketch concluding, *"it's just a little bit so hopefully it works."* In order to test her solution idea, she constructed both quadrilateral interiors and measured the area of two new quadrilateral land sections, *"Okay, so let me just make sure that that claim is correct, so let me measure, let me construct the interior first and then measure that area. Hm? That didn't quite work."* She was surprised by the result,

> *I was working under the assumption that this Triangle [BCI] would be congruent to this one [CDH] down here if the two areas were to be congruent by simply moving that piece to down here and it doesn't look like that's the case based on doing that construction.*

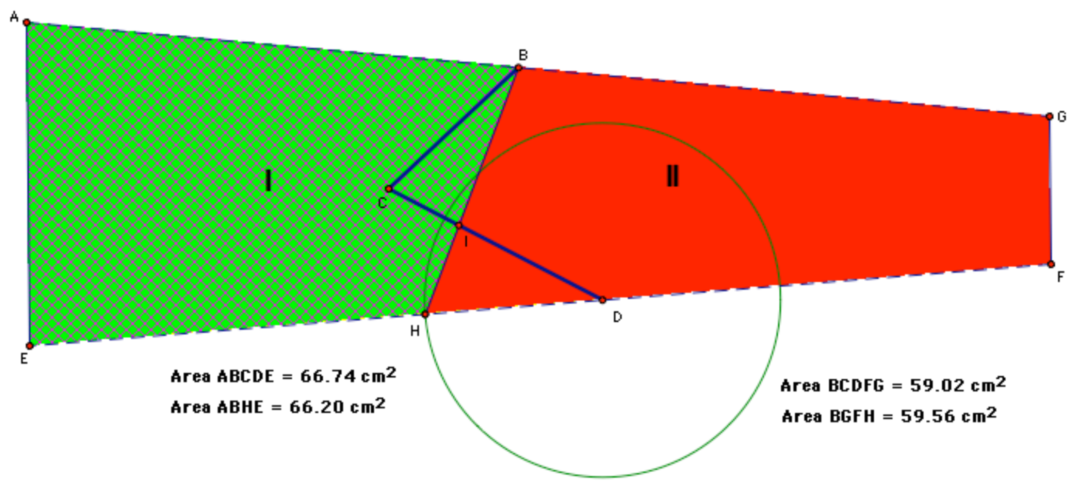


Figure 42. Aurora first exploration episode: An attempt to rearrange the land by superimposing Segment BC.

It appears, though she never explicitly stated it, keeping the same amount of land was equated with rearranging the two farmers' lands by superimposing one triangle onto another under assumption they would be congruent because congruent triangles have congruent areas. She assessed the result, helping her devise another solution path using the same idea,

> *So my problem is that even with constructing this the same distance it doesn't necessarily mean that CI and IH are congruent. So, I am just gonna move it the same rate so maybe it's not that I need BC to be the same but that instead I need IC to be congruent to IH and DI congruent to IB, and then both vertical angles you know that they are congruent.*

She continued to engage in the search for a solution plan relying on her previous experience and new relevant information. On the second sketch she constructed two circles, circle c(I, IB), and circle c(I, IC) and intersection points, two for each circle, of each circle with the Segment EF. Once she implemented her idea, she realized that it was flawed,

> *Okay, so problem is they are no longer gonna be vertical angles because they no longer lie on the same line. It should be equal because they are the same amount, well lie on the same circle with same radius so I am not exactly sure where to go with that. I know I want to put that piece of land somewhere the only problem is that it's not looking like it's gonna line up so fence is still gonna be bent.*

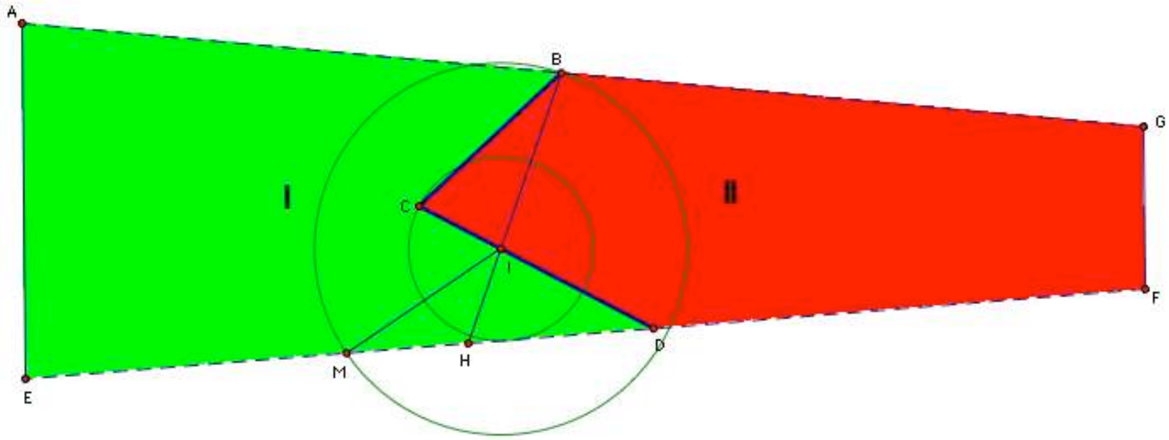


Figure 43. Aurora's first exploration episode: An attempt to rearrange the land by using the idea of congruency.

During this first exploration episode she was obviously aware where she was in the process of solving a problem; she was not only aware of the content-specific knowledge that might be helpful in this situation, she was aware of the strategy available to her to solve the problem. Though she tried to direct her actions in the same direction by trying three different solution attempts, there was little or no evaluation of the undertaken activities. In addition, there was no evidence at that point of assessing the current state in the problem solving space with respect to her knowledge. She was confident her approach was correct, *"the first two that I'd done, I was really expecting that to work so when it didn't come out to work, I was like: Ooooooh I don't know where to go,"* and was surprised when she realized after implementation of the plan the fence was still bent. Later she admitted this was her only idea that she thought might work explaining, "*I really have no idea. I feel like maybe this was probably my best hunch and since it doesn't work I just I don't really know where to go from it"* and consequently failure was costly before an assessment of both knowledge and future directions to come for the solution process.

Afterwards, she went on lengthy pursuits in order to find a correct solution path characterized by limited guidance and weak structure, *"And so from there I got a little bit frustrated and I was trying to think of all of the things I knew about it and so then I was just trying to do a few different things."* She was stuck, and tried several different problem paths (land sections created by the circle radii IM and IJ, respectively) not taking into consideration their reasonableness, *"I was just testing to see if any other radius that I have created would end up striking a point on the other boundary that would give an area that was equal but it doesn't look like it will."*

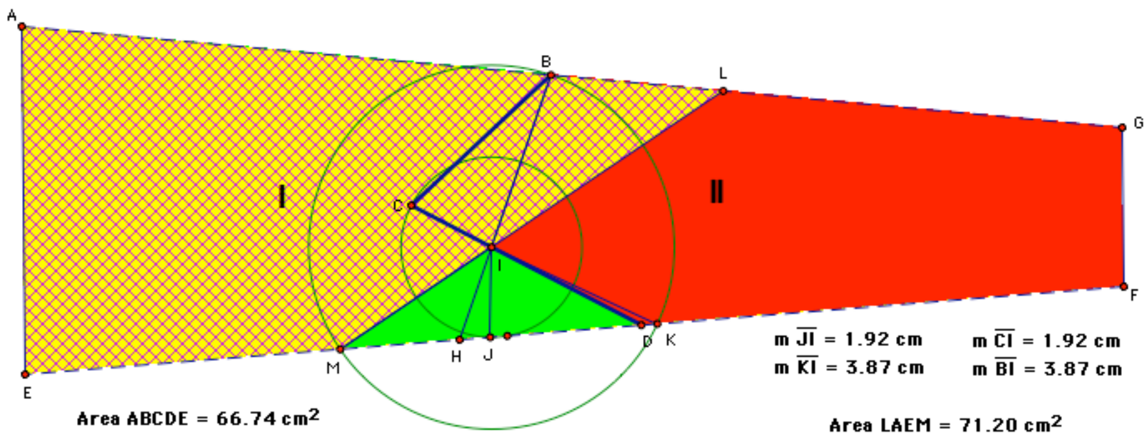


Figure 44. Aurora's second exploration episode: An attempt to rearrange the land by using Segment IM.

It was obvious her actions were purposeful; they were driven by trying to satisfy problem conditions and where the fence ought to be,

> *it was a matter of thinking about, well how can I get that triangle to stay the same but basically have the fence line at a different point. So, I would know I wanted to use side CD, I just didn't know how to get it to the place of BH, there, or BQ.*

Although she tried to direct her actions in the same direction by trying two different solution attempts, there was little or no evaluation of the undertaken activities, making it impossible to efficiently move towards a solution. Dynamic software was mostly important to carry out outlined solution ideas. This was costly — there was no evidence at that point of assessing the current state in the problem solving space with respect to her knowledge.

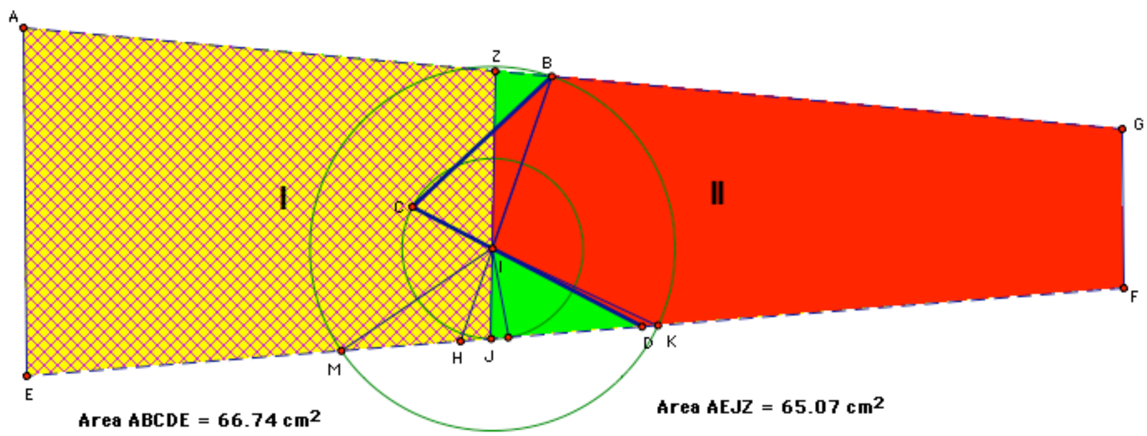


Figure 45. Aurora's second exploration episode: An attempt to rearrange the land by using Segment IJ.

After two unproductive exploration episodes, as a consequence of monitoring of her progress she redirected her actions forward in the solution process, she decided to take a step back, *"I think I am gonna start back over again sort of just by deleting all the stuff that is really not helping me at the moment,"* going back to problem representation as in Figure 41. With this Aurora began her third exploration episode. She connected Points B and D obtaining a dashed Segment BD as she directed her thinking into *"keeping the base and the height the same."* The decision to observe the Triangle BCD was driven by relating knowledge relevant to the goal. Though multiple attempts through the problem solving space, her comment above again shows they are directed as they are goal driven, but also not random as they might seem. Similarly to previous episode, after one approach did not work out, she cycled back relating the problem to her knowledge of problem solving strategy including three components: knowledge what needs to be done, what was done and what might be done in this problem solving state. Already in the first exploration episode she explored that idea by relating it only to congruent triangles. She then tried to direct her thinking into what else might be done that would have the same result– congruent areas. She observed the screen for a few seconds breaking the silence by concluding she cannot use BD as a fence, *"you can't really use that because one they don't have the same dimensions… cuz that would be a straight fence there but you obviously cut of that portion of red's area."* It appears that she was not yet able to direct her thinking far and consider relevant mathematical knowledge to be able to do so.

After a few seconds she thought of whether there could be more that one solution for the straight fence, *"See I am wondering if there are two different possibilities. What it could be? Like two different places where the fence could be,"* but then quickly changing her train of thought and explaining her previous solution attempts by trying to redistribute portions of land. She erased Segment BH and observed the representation of the problem where Segment BD was still connected.

> *I know that this whole area was there to start with, so drawing this line as a fence doesn't change anything which means… It seems like BCD would fit in BDH. I mean I need to somehow get it so I can put this amount of area down here in order to get straight line across but the problem is that I don't think that DC is going to match up to right there but I need to somehow keep the area the same*.

She was perplexed about how to manipulate a triangle for the area to stay that same, *"I mean you change any part of it and none of the areas is gonna stay the same, but I feel like I am missing something big here."* This idea resonated in her head, and played with it. She said the area of a triangle is dependent on two variables, base and height, and focused on the Triangle BCD,

> *I mean the easiest way to look at this one for me is to keep BC as the base and whatever the midpoint is to B to the height but or let me just drop it. I wanna keep BC the same and AD the same.*

However, she was perplexed how to proceed with this idea, *"I don't know, I mean if you move B or C, you change the base, if you move over D, you change the height."*

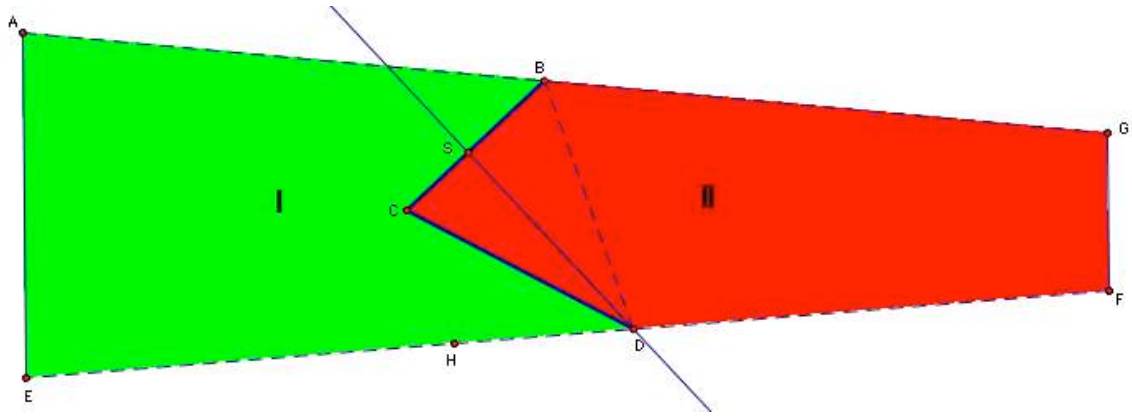


Figure 46. Aurora's third exploration episode: An attempt to rearrange the land by using the idea of parallelism.

She analyzed the problem again that aided arriving to her solution plan, *"I mean you want to get rid of a crooked part which means that BD would have to be the base you want to keep but the problem with that is that, like that changes, no it doesn't but."* She did not abandon completely her solution approach but modified it. She explained her decision,

> *I was trying to think of how I can keep the height the same and just struck me to do a parallel line to BD because I knew that the points would be equidistant no matter where I moved them along there.*

She drew a perpendicular line through Point C from BD, labeled the intersection point with P obtaining the altitude CP of the Triangle BCD. Here she got the idea of moving Triangle BCD such that Point C would lie on the Segment EF, and Segment BC would remain fixed. Nevertheless, she was worried if her solution would work as *"that wasn't originally part of a construction."* She observed the screen for a few seconds, then drew a parallel line from BD through Point C, and explained her solution plan,

> *Basically I want to keep, I wanna move BCD so the area stays the same. So, what I was thinking is, if I keep BD the same as the base of that triangle, but I can move CP now along this line that's parallel to it because I know that all points are equidistant between the two.*

She was not able to move Point C along the parallel line as it was not fixed. Therefore, she decided to translate Segment CP by Mark Vector CQ (Q is point of intersection of the Segment EF and the parallel), *"I want to move Point C to where the Point H is. It's not fixed. Okay, so I will translate it."* Using Mark Vector was interesting GSP feature she utilized to obtain Point Q one the Segment EF. However, she knew this was not the only way to obtain Point Q, but *"just had to figure out where the intersection of the line that's parallel to BD, where it hits EF."* After she implemented her plan, she concluded Segment BQ being the solution of the problem, *"so now my new triangle is going to be QDB because they have the same height because they lie in the parallel line and they have the same base because it's BD. So then my fence should be QB."* The idea to draw a parallel line did not come to her mind, but focused on creating congruent triangles, *"it's been a very long time since I've seen that particular proposition or whatever it is, theorem, but I was trying to use something based on you know creating congruent triangles."*

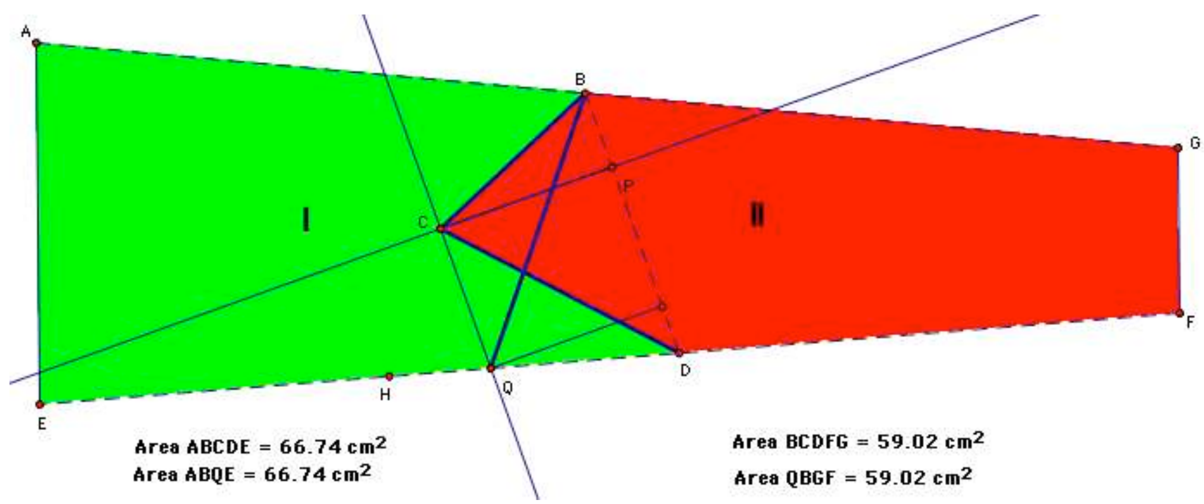


Figure 47. Aurora's solution of the Land Boundary Problem – Segment BQ.

She later added,

> *I was trying to relate it to congruent triangles so I was trying to use those properties and then I knew that some distances needed to be the same, so I was trying to use circles to create radii that might work out which that didn't go anywhere.*

Even though she gave a valid rationale for her solution and why it worked, she still measured the two areas just to *"double-check that. It seems like it's right because it was very close to where H was."* Once she measure the area of the sections divided by the Segment BQ she was certain her solution was correct, *"Okay, so yeah. QB should be the fence."* She was confident she solved the problem correctly, *"well the areas are definitely the same, so I know that that piece is right,"* and that the solution is unique, *"that's the only place where it [BQ] can go."* She further explained,

> *cuz, that's what I was debating earlier they whether there can be more than one option where BQ is my only option that I can really see is if that triangle were instead of the base being along EF, the base is along HG but then it wouldn't be adjoining two.*

Thus, search for a solution plan started very early in the problem solving session without apparent structure as to what needs to be done and what might be done. Although at first her actions were based on assessment of knowledge relevant to the situation, once her approach did not work out she felt defeated. She admitted that in terms of planning she did not do much and that she *"just wanted to look at it, and see what can [she] do in order to make the areas the same and so just first started just playing around with it."* Thus, lack of assessment of current knowledge relevant to the problem, and evaluation of undertaken activities led her in an unproductive effort. It took taking a step back, reassessing what was done, and redirecting those processes to generate efficient movements towards a solution. Thus, after two of her approaches did not work out, it was a metacognitive act that dealt with reflecting on the task in directing her thinking processes towards a solution of the task, and choosing of the strategy. Use of technology to solve the problem was prominent, and allowed use of bottom-up strategy to solve the problem. Though she used the capabilities to be able to refine, revise and abandon her approaches and arrive at a solution, she was aware that the way she used it hindered her by focusing more on the result rather than on the process.

Aurora never saw a problem similar to this one. She said the problem was *"cool"* and thought the problem was challenging because she *"haven't seen anything like it"* and because of problem generality, *"there is not much to work with in terms of, it's not here are some properties, go from there, here is a picture, what can you tell me about it and then how do you get to a solution."* She added the problem involved only one geometric concept, the area concept, *"there is not much there besides I mean in terms of geometric terminology except for the same amount of land which gives you area."*

When I asked her what were most important things that helped her solve the problem, two categories emerged:

1. mathematical content—concept of area and theorem about area of a triangle between two parallel lines, and
2. geometry software.

The Geometer's Sketchpad mainly allowed get a better understanding of the problem as it *"allows to see it a little more dynamically"* and starting to work on the problem by calculating the areas, finding the approximate solution, and implementing and verifying different solution approaches.

She explained her use of technology when solving problems, *"any problem I get, I am like: Oh let me go and put this in Sketchpad and see what I can get out of it and sometimes that's a good thing and sometimes it's not."* Thus, the software helped develop her activity, which then transformed the tool to fit her needs. She did admit that the aftermath of that was focusing on the result rather than on the process and claimed that the situation would have been different if given this problem on paper and pen.

As a result of this problem solving session, she said she learned something about her problem solving, *"it taught me that maybe I should look for the easiest way first and then try to go from there if that doesn't work, then try something a little bit harder."* She explained, "*I was basing it on the assumption that they would be congruent and then when they weren't congruent and so I think that's a lot harder than just recognizing that BD is the most important in this construction."* Moreover, she added interesting contrast to the use of technology when solving problems, *"normally I would say that Sketchpad is best for solving most problems and I am thinking in this one it might not actually be."*

At the end of the session I asked her if she could now solve second part of the problem. However, as she did not have more time we decided to postpone it for next week.

Synthesis of the Problem-Solving Session 3: First Part

Aurora started the session by reading the problem in silence, and highlighting the conditions of the problem. After interpreting the problem, and making relationship between the problem conditions and problem goal, she decided to find the solution of the problem using capabilites of the software, and work backwards to arrive at that answer. The rest of the session was spent on searching for a solution where lack of assessment of potential utility of her planned actions rested the endeavor unproductive. However, taking a step back from the problem, evaluating what was done and what needed to be done, directed her thinking to what might be done to solve the problem. This process was sequential and monitored but rested the entire endeavor a success. Use of technology to solve the problem was prominent, and allowed for bottom-up strategy to solving the problem. It hindered her problem solving process, however, as she focused more on the result rather than on the process. This session clearly demonstrates how lack of monitoring skills, and healthy affective behaviors and the ability to manage them are the key to productive problem solving. Figure 48 demonstrates Aurora's problem-solving cycle.

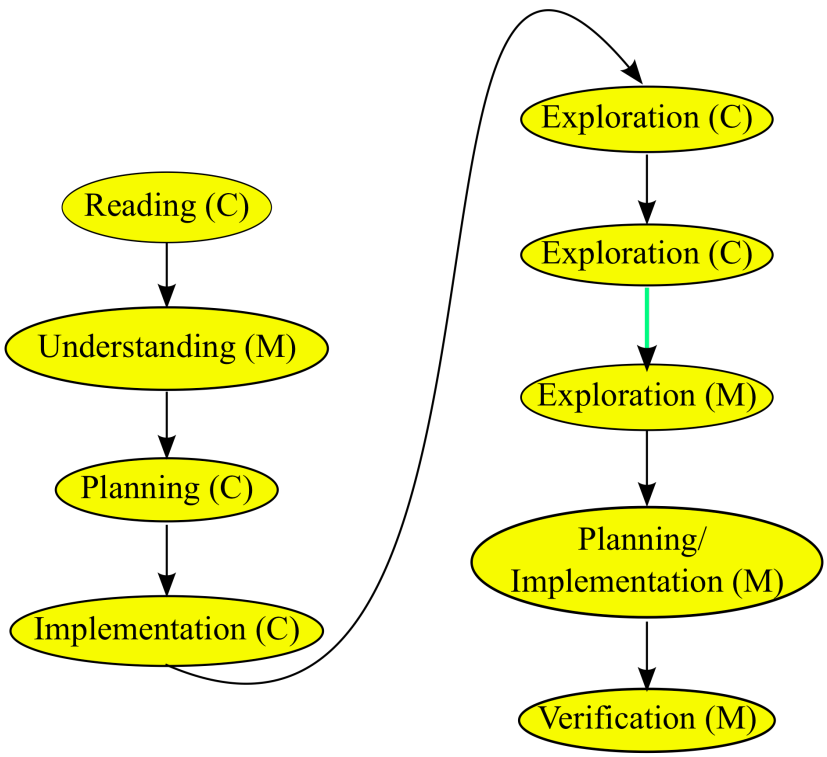


Figure 48. Parsing of Aurora's Problem-Solving Session 3: First part.

Once she solved the first part of the problem, I asked her if she had time to solve another part of the problem. As she was not able to continue the session we postponed solving the extension of the problem a week from that day. Similarly to before, in the continuation of the session, I provided Aurora with a second sheet of the paper and a pen, and opened the second subfile on the Land Boundary Problem.gsp file on which the extension of the problem was printed. Solving the second part of the problem took approximately 80 minutes. Aurora spent approximately 50 minutes solving the second part of the problem, and during the remaining time we reflected on the session. A large portion of the session was spent on figuring out how to implement the startegy from the first part of the problem to solve its extension. Again, the use of technology to solve the problem was prominent, and allowed for bottom-up strategy to solving the problem.

Solving the Land Boundary Problem: Second Part

Aurora started the session by reading the problem in silence of the screen. She did not have any difficulty understanding the problem statement adding, *"I remembered it from last time…plus I still had my bolded words up there, if I need to go back to it. But it was pretty straightforward."* She uttered, *"we want the common border to be straight again"* before turning to the first sketch observing what she had done to solve the first part of the problem. She then moved back to the second sketch with the problem deciding how she will proceed in solving the problem. She contemplated between *"jump into trying to do it in mathematically stringent way using the parallel lines"* or *"estimate it first and see if [she] can get to that line segment just by looking at it."* Understanding and analysis of the problem occurred intermittently; she noticed goals of the problem explicitly, and assessed her current knowledge relative to the problem directing her to a choice of perspective *"Can I use the same strategy?"* She decided to go with the latter, *"I will probably just start like I did last time with just finding the area"* she did not know how to use the strategy from before, *"I didn't know exactly where I wanted to go with the parallel lines. I didn't know what point I wanted it to go through."* Thus, she decided to work with what might be the ending point with hope giving her *"general idea for where the line might occur"* and *"for what to do when [she] get to the drawing parallel lines and figuring that out."*

Her plan was overt and undoubtedly relevant to the problem solution, and though she engaged in considering various solution approaches, there was no serious assessment of current knowledge. Nevertheless, the decision was consciously made to use this strategy as a useful means to solving a novel problem, *"I like estimations first and then going to the mathematics."*

She implemented her plan; she constructed hexagonal interior and used Measure Area obtaining area of land I (ARSTUD) equal to 72.88 cm^2, and constructed the second hexagonal interior and used Measure Area obtaining area of land II (RBCUTS) equal to 52.88 cm^2. Area of land I was in yellow, and area of land II was changed to orange color.

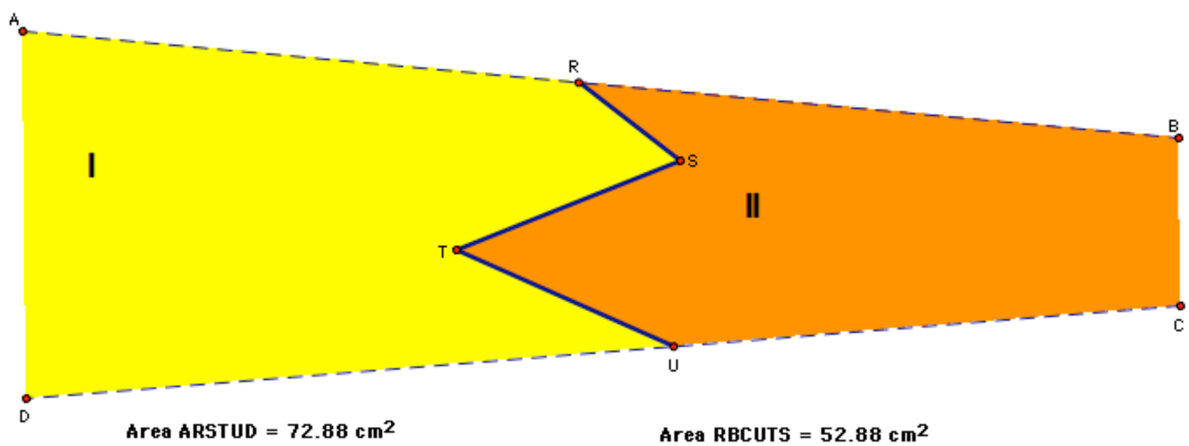


Figure 49. Aurora's solution plan of the Land Boundary Problem extension: Finding the area of farmers' land.

She continued implementing the plan of obtaining estimation by drawing the Segment RV where V was the point on the Segment CD. She constructed quadrilateral interior and used Measure Area obtaining area of quadrilateral ARYD equal to 70.87 cm^2, and constructed then the second quadrilateral interior and used Measure Area obtaining the area of the quadrilateral RBCV equal to 52.89 cm^2. She dragged Point V slowly towards and away from Point U observing at the same time how dragging Point V affected the areas of lands and areas of quadrilaterals. She stopped when she the two areas were approximately the same, *"There you go, okay. Okay, so that's about the closest I can get it using that estimation."* Point V was constructed in order to help her solve the problem noting the problem was more challenging that the first part, *"this one is a little bit more difficult cuz it's got two crooks."*

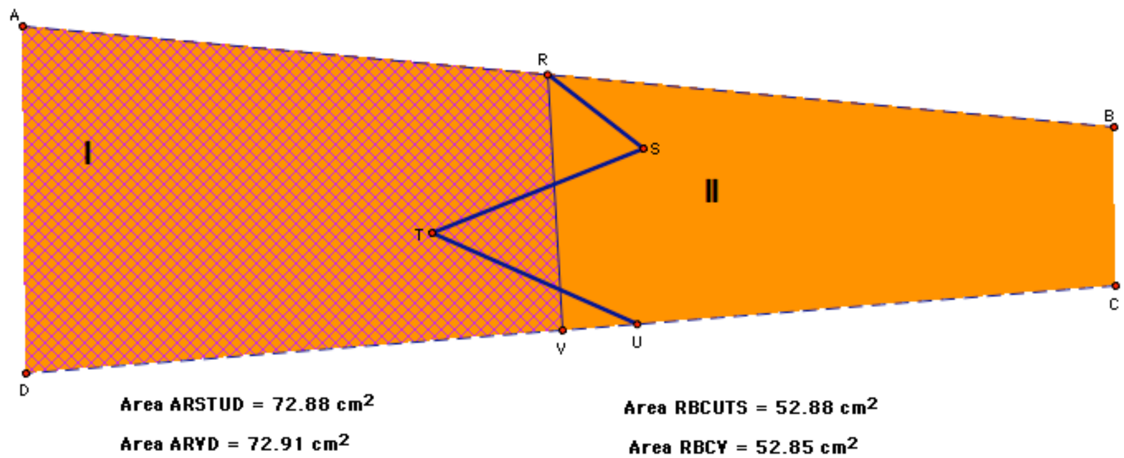


Figure 50. Aurora's solution plan of the Land Boundary Problem extension: Finding an approximate solution of the problem.

During and after implementation episode she did not assess her knowledge that might be helpful in the current problem solving situation nor was there evidence of sense making but rather jumped intro exploration. The following paragraphs outline explorations she used in the search for a solution where she often referred to first part of the problem, either stating she needed to use the strategy from the first problem or observing the sketch with the first part of the problem,

> *Let me look back at what I did last time. I created a third side of the triangle and then from the common point created a parallel line to it so it would not change the height of that triangle.*

She explained her decision, *"Because it's the same general type of problem. I figured it would have been build off of what we'd done before but not necessarily exactly the same."* She then scrolled up the sketch analyzing how she solved the first part of the problem and searched for a solution plan having two things on her mind, solution strategy from the first part of the problem and align it with what might be the solution, that is, exploration was goal driven.

She noted creating a triangle was important from the first part, and decided to create triangles concluding that there "*would be multiple ones this time."* However, she was not sure if this was a valid approach, *"I don't know if we can do the same thing that we did before where."* She drew Segment RU to see possible triangles and dashed Segment RV as that was not *"part of construction."* Once she constructed the Segment RU, she again scrolled up the screen recalling that problem solving approach of moving the triangle in a way that preserves its area, and at the same time obtaining the goal of the problem – the straight border,

> *What we did before when we did this, we have divided this up but we wanted still their areas to be the same, and so all we wanted to do was to move it so that this area of this triangle was the same as this one over here, but it had a straight border, which would make the border side of that.*

She assumed that she had to the *"a similar approach down here"* noting the two triangles, Triangle RSW and WTU, where W was point of intersection of Segments RU and ST. She decided to draw parallel lines from Segment RU through Points S and T, however she was not sure her approach was correct,

> *I think I'll maybe just try what I did last time. The problem is that since it's two it might not work out great. I am just gonna do it to each one of the segments, so parallel to this [RU], and that [RU].*

Her doubts increased with the implementation of the approach, *"My problem is though that with this Triangle [RSW], this base [RW] is simply a portion of that length [RU], so actually the triangle would need to be this whole thing [TUR]."* She then drew Segment RT and SU obtaining Triangles TUR and RSU, respectively, but erased both segments quickly after a short observation of the screen. She continued to work on the previous idea, *"Let me go back"* and stated her goal, *"I want ST to still be a straight line when I get done. The problem is that."* By

Segment ST be the straight line she meant dragging Point S to Segment AB, and Point T down to Segment CD. She implemented this idea; she constructed Points X and Y, where X and Y were intersection of parallel through Point S and Segment AB, and parallel through Points T and Segment CD, respectively. Once she obtained the Segment XY, she was not satisfied with the result, *"this is generally what we did before however, that's not even close to where V is."*

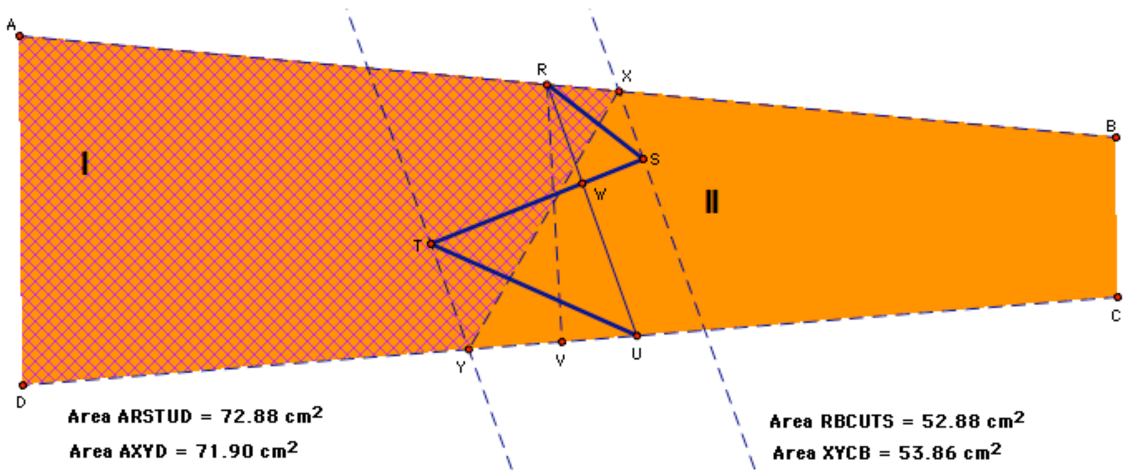


Figure 51. Aurora's first attempt to solve the problem extension by using the strategy at once.

Even though she was first taken back by the result, analysis of the solution moved her towards thinking this might be a plausible solution in the end, *"Oh, but V comes from R, instead of this Points [X and Y] over here."* She then decided to verify her problem solving approach. In order to do so, she constructed the two quadrilateral interiors, and used Measure Area obtaining the two areas, A(AXYD)=71.90 cm^2 and A(XYCB)=53.86 cm^2.

She was taken aback by the result, *"Okay, so not quite what I wanted"* and felt frustrated. She tried to make sense of the situation,

> *That reasoning doesn't seem to work because, well it's demonstrated by the areas not being the same, but it seems that simply moving the heights like we did last time but with the individual triangles while it gives you a straight border, but it doesn't give you the same areas.*

During this first exploration episode she was obviously aware where she was in the process of solving a problem; she was not only aware of the content-specific knowledge that might be helpful in this situation, she was aware of the strategy available to her to solve the problem. After that did not work out she realized she needed *"to rethink how I was approaching the problem"* and *"to head in another direction."* She erased Segment XY and at the same time tried to explain why her solution did not work, *"I think it has something to do with the fact that SW changed and WT changed when you moved them."* She sat in silence for few seconds observing the screen. During this period she *"was thinking about why it didn't work"* and *"brainstorming another way to do it."*

Her new idea involved eliminating one of the crooks, however she did not know how execute it, *"I was thinking somehow I might gonna eliminate the second crook and then get it down one, and then use that property but I don't know."* She erased parallels through Point T and S, and Points X and Y obtaining the representation as in Figure 50. She continued shortly playing with this idea in order to help her out implementing her in creation idea. As she *"didn't know exactly how to do it, and I was still hung up on need of parallel lines,"* she tried several different approaches, such as, *"tried doing parallel lines to different points just to see what would happen," "tried to make different triangles."* She thought of drawing a line through Point T, but she quickly abandoned that idea stating, *"but that won't work because they won't line up."* Here she abandoned that idea going back to trying to *"figure out how [she] can straighten this [RSTU] so that it lines up with this [RU]."* Her goal was to straighten out the Segment ST in a way it would be aligned with Segment RV.

In the following she tried out and tested several different problem solving paths in order to straighten out RST where her approaches were driven by making the straight fence that would ultimately be Segment RV, *"I am trying to think of a way that I can get the border to look like RV."* She drew a line parallel to Line ST through Point R, but quickly abandoned that idea saying, *"I just don't know where to place the parallel line."* She added, *"I feel like the parallel line should go through S."* She hid the parallel through Point R, drew Line RT, and pointed on the Point S. She again abandoned quickly the idea before it matured, *"Then you could move yeah, that's not gonna work. I was trying to keep the base the same, but this height, but the problem is that that doesn't lie on the parallel line cuz those two aren't parallel [RT and RV]."* Even though, in the process of deriving a correct solution path, she briefly considered that option, she abandoned it quickly. She was not sure how she could apply it in pieces and could not imagine it would be a straight line. Moreover, mentally her generated solution was not aligning with her estimation of where the border should be.

She then observed the screen, noting that Segment RV intersects Segment ST at a midpoint. She tested that conjecture by using Construct Midpoint concluding, *"It's a little bit further off."* She furthermore tested her conjecture by moving Point V on the Segment CD so that RV would coincide with the midpoint, and measuring the two quadrilateral areas. However, she was able to refute her conjecture. She obtained the areas equal to 71.04 cm^2 and 54.72 cm^2 concluding again her idea was not correct, adding *"Anyway it doesn't really makes sense it this context anyway,"* as this was not the part of the first part of the problem. She hid the midpoint. Midpoint problem solving approach was also fruitless.

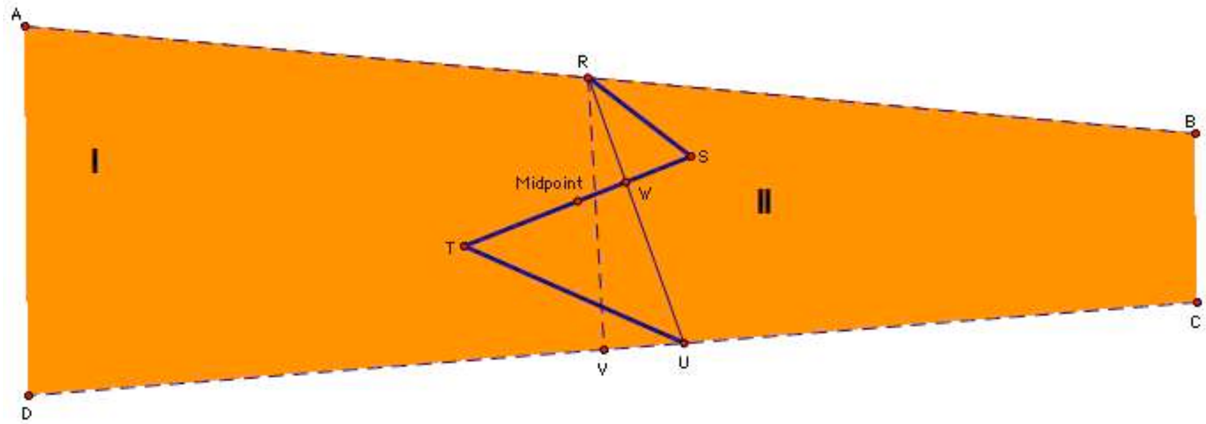


Figure 52. Aurora's attempt to solve the problem by using the midpoint idea.

Thought she tried to direct her actions in the same direction by trying two different solution attempts, there was little or no evaluation of the undertaken activities, *"I just tried a bunch of different things that I can think of which weren't necessarily on the right track"* making it impossible to efficiently move towards a solution. Furthermore, it appeared that frustration developed after her original approach did not work took control of her thinking leading her to jump from one to another exploration. Dynamic software was mostly important to carry out outlined solution ideas. This was costly — there was no evidence at that point of assessing the current state in the problem solving space with respect to her knowledge.

She was stuck; she was *"getting hung up on how to divide it up"* so that it would align with her estimate solution, Point V, and she did not how to reconcile that. She took a step back reassessing what was done, and redirecting those processes towards a solution. She then *"came to the brainstorm of trying to eliminate one crook."* She shortly observed the screen before deciding new problem solving path that was developing with time – drawing Triangle STU and working on each bent one at a time. She tried to explain how she came to that idea,

> *I remember thinking to myself that if I can somehow get it down so that I only had one crook in it that that would be the best way to do it. And so then I was thinking: Okay, well*

what can I do to make that? Can I just look one of the crooks by itself, do the same strategy that we did before in the first problem, and get it down so it's one? And then I was: Yeah.

Hence, it appears that internal dialogue she engaged herself in contributed to efficient movement toward a solution. She first drew Segment SU, before drawing a line parallel to Segment SU through Point T. She explained her decision still unaware if this approach would work or not,

because it's similar to the process before but uses different ones. The only problem is that when you get to this edge, it might not line up in a straight line to create that border. I know they won't because it's not similar to this [RV].

Then she quickly added why it might work together with revealing her problem solving strategy,

But that one could be up here and come down to CD which I am not doing here either. Okay, that could work maybe. So, okay, what if I break it down to parts, so what if I can straighten this one out [STU] so there is not a crook there anymore, and then focus on that one [RST].

She then constructed Point Z that was the intersection of the parallel through Point T and Segment CD. She quickly backed up from her thoughts; she erased Point Z and evaluated her problem solving approach,

I want to keep this Triangle [STU] the same area. So I want to keep the base the same, and the height. The problem though is that this TU might not necessarily match up here [CD] which is the border so I am still gonna have a crook in there, unfortunately.

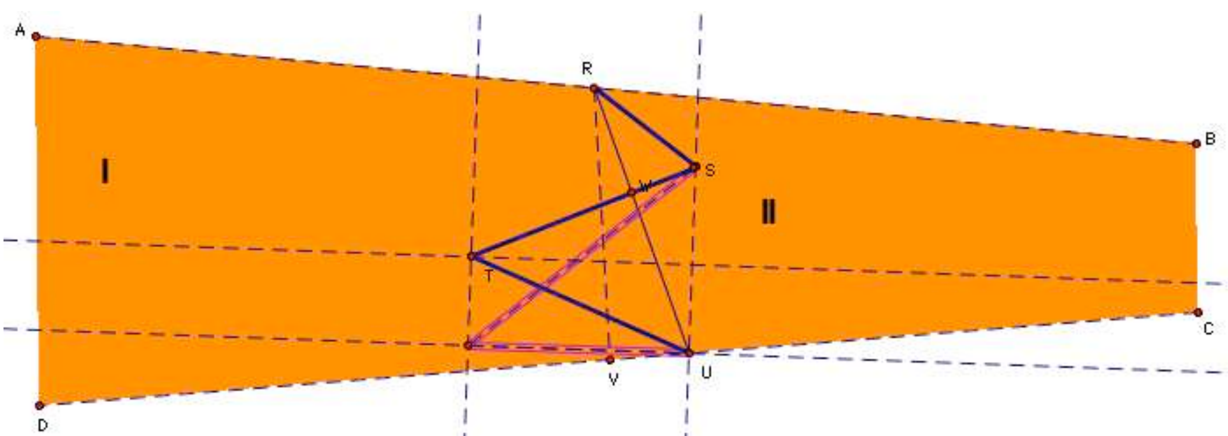


Figure 53. Aurora's implementation of the solution plan: An unsuccessful attempt to eliminate the first bend by using the strategy systematically.

She drew the altitude of the Triangle SUT through Point T and a line parallel to the altitude through Point U. She concluded her approach was not successful, *"I am still not getting rid of that crook. But that's still means that there is still one, two, three."* She observed the screen before noting a flaw in her reasoning,

> *No! That still shouldn't matter cuz I can still drop a perpendicular from there, and it's still perpendicular to the base. But, so my height is technically still the same. I was thinking because, okay it can't go outside of the boundary line, and so it was throwing me of that it couldn't be that point there which would be equal to the area of STU.*

Here she had, as she called it, *"a silly misconception"* that *"height needed to stay within the base"* but as a result of evaluating her actions she soon realized that is not the case mentioning an obtuse triangle as an example. She hid the newly constructed segments (dashed segments in Figure 53), and constructed again Point Z obtaining Triangle SZU. She verified the Triangles STU and SZU had equal areas by constructing both triangle interiors and calculating their areas. She then hid all the parallels and perpendiculars before connecting Points S and Z, and obtaining the new border containing Segments ZS and SR. She explained her solution process thus far,

> *STU is the triangle that I originally had which was that one crook there. Now I've straighten it out so that this [SU] is that side length, so that the area is still the same here…the area is still the same for this piece as now, so SUZ. So I made it basically so that these two are not there anymore. Instead of ST, and TU, I now have SZ in its place as a straightened out.*

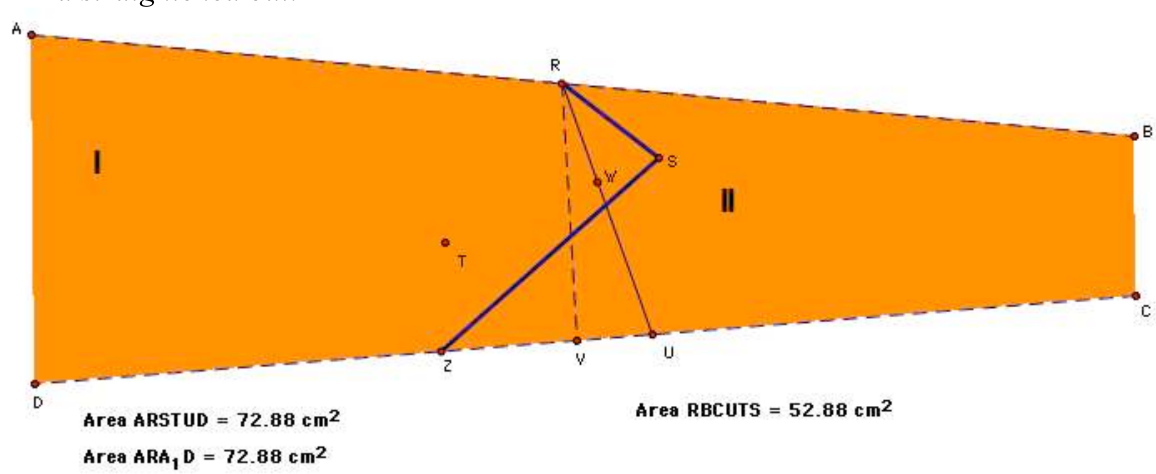


Figure 54. Aurora's implementation of the plan: Eliminating the first bend.

After she implemented first part of the solution path, she explained continuation of her plan, *"now I have this crook which means that I can just use the same principle that I did before."* However, before she implemented this she verified her solution thus far was correct, *"actually maybe I should double check this to make sure they still have the same at the moment."* As she was verifying her idea she shared that she hoped they would be the same not leaving a possibility she *"could have done that wrong."* In order to do so she constructed the pentagonal interior ARSZD and measured its area that was equal to 72.88 cm^2. When she obtained a satisfying result, she uttered, *"Excellente."* She then continued to finish implementing her solution plan. She first drew Segment RZ and drew a parallel line to it through Point S. She noted she had two possible solutions deciding to take the one resembling her estimated solution, Segment RV, *"I am gonna go with down to V cuz that's what I just originally had."* She then constructed intersection points of the Segment CD and parallel through Point S. She hid the Point V and Segment RV to be able to distinguish between Point V and the exact solution. She labeled the intersection point with A_1 and drew the solution of the problem, Segment RA_1.

Even though she was in accordance with her solution strategy, *"it makes sense to get rid of one crook, and then the next,"* she still double checked the reasoning by constructing quadrilateral interior ARA_1D and measuring its area that was equal to the area of land part I. Once she verified the solution of the problem, she said, *"I believe I have solved."* She was *"pretty confident"* she solved the problem correctly that was based on *"the original problem knowing that that is what it solved it"* and "*having a measurement there that coincides with what you think it should be."* Nevertheless, measurement was used to double-check her reasoning.

Once she stated she was done with the problem, we started to discuss number of possible solutions there are for this problem. She said she could have instead of border RA_1 also draw border ZB_1, where B_1 would be the intersection of the parallel and Segment AB. Here she explained her decision to choosing RA_1 as her solution of the plan, *"Because that's how I have done V and I wanted to see if they were at the same spot which they were."*

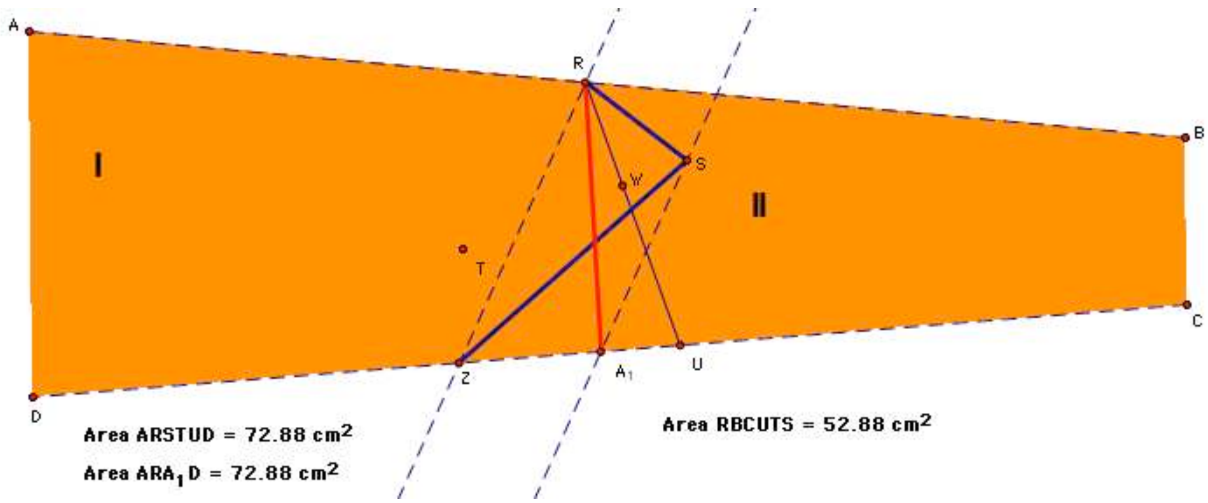


Figure 55. Aurora's solution of the extension of the Land Boundary Problem – Segment RA_1.

Her search for a solution plan through the problem solving space was driven by fitting it to the solution obtained at the beginning of the session. Her actions were goal driven but lack of assessment of current knowledge relevant to the problem, evaluation of undertaken activities and frustration after her expected approach did not work out led her to a plan that did not work. She explained that she relies on the software a lot. With Sketchpad she tends *"to play around more to get on, formulate a plan"* and that planning processes would have been different in the absence of software. Her problem solving technique, although making her concentrate more on the answer rather than on the process, was consciously chosen as a result of previous problem solving experience and she considered it useful in problems such as this one because it *"brought [her] back through all of thinking processes that [she]'ve done in the previous problem which helped outline what [she] might wanna to do with this one."* Although it helped her visualize the solution, she was aware this approach was also a hindrance as she *"caught up in how can I make that ST move so that it's there,"* that is, trying to devise a solution that would have been aligned with Segment RV. It took taking a step back, reassessing what was done, and redirecting those processes that contributed to efficient movements towards a solution. Thus, after two of her approaches did not work out, it was a metacognitive act that dealt with reflecting on the task in directing her thinking processes towards a solution of the task, and choosing of the strategy. Use of technology to solve the problem was prominent, and allowed for bottom-up strategy to solving the problem. Though she used the capabilities to be able to refine, revise and abandon her approaches and arrive at a solution, she was aware that the way she used it hindered her by focusing more on the result rather than on the process. Through her actions she transformed the tool to fit her needs allowing her to find the solution to the problem and help then work towards finding that solution. She took into consideration the affordances of the software and the tool became an instrument through stages of personalization and transformation of the tool, which was a metacognitive act.

When I asked her what were the top things that were most important to solve the problem, three categories emerged;

1. mathematicsl knowldge
2. knowledge of problem solving strategy, and
3. geometry software.

The first category was evident throughout the problem solving session involving concepts of *"parallel lines, the area, the triangles,"* whereas later she stated, *"definitely I would say that the biggest thing was probably just already having the knowledge from the last part of the problem. That was probably the most helpful."* She thought she would have not been able to solve the second part of the problem *"without having done that one first."* Interestingly, she thought the first problem was more challenging than its extension because, for the extension, she had a strategy to use although she struggled with figuring out how to use it.

Geometry software was helpful because it allowed her to *"visualize how to do stuff,"* create various geometric objects and in testing, refining, revising and abandoning her solution approaches. For example, GSP was used as a tool to check the correctness of a solution plan such as during her first solution plan (constructing parallels to RU through Points S and T) where in the absence of the software she *"would have stopped at the wrong answer."* Similarly to comment she made when solving the first part of the problem, she thought the problem did not rely on Sketchpad as it could have been solvable on paper and pen *"as long as you know the direction you want to go,"* but having it allowed her easily construct and hide different object. In this case she viewed Geometer's Sketchpad as a tool "*for the aesthetic view."*

As a result of participating in this problem solving session, she consolidated her knowledge about area of triangles, learned the need to *"take more time to think through steps"* and not make *"as prominent next time with what the ending point should be,"* and a problem solving strategy of reducing the number of constraints, which was a completely new strategy for Aurora.

> *I hadn't ever really thought about it that way but it makes sense to me; as you are going through a problem you want get rid of the constraints you can get rid of, and then work from there. So it makes sense to get rid of one, and then work from basically what you've had in the original one.*

As I was wrapping up the session by thanking her for the session, Aurora was enthusiastic and said, *"Super fun. I am going to be a farmer."*

Synthesis of the Problem-Solving Session 3: Second Part

Aurora started the session by reading the problem in silence, and immediately tried to consider knowledge and strategies relevant to the problem. Even though she contemplated between using the strategy from solving the first part of the problem or approaching the problem by measuring the area and working backwards searching for a process to arrive to that result, she decided to go with the latter as she did not know how to use the strategy from the original problem. The rest of the session was spent on searching for a solution where lack of assessment of potential utility of her planned actions and frustration made the following endeavors unproductive. She persevered throughout, however, and by taking a step back from the problem, evaluating what was done and what needed to be done, directed her thinking to how to use the strategy from the original problem, which ultimately increased her confidence. This process was sequential and monitored but made the overall endeavor a success. Thus, fruitful problem solving efforts relied upon regulation of affective behaviors as well. Use of technology to solve the problem was prominent, and allowed for bottom-up strategy to solving the problem where she again took the problem as solved and worked backwards to obtain the solution. However, it hindered her problem solving process as she focused more on the result rather than on the process. This session clearly demonstrates how lack of monitoring and evaluation skills may prompt negative affective behaviors that consequently influence the problem solving process; cognitive processes take domination over metacognitive processes until regulation of unhealthy affective behaviors and reflection on undertaken activities occurs. Figure 56 demonstrates Aurora's problem-solving cycle.

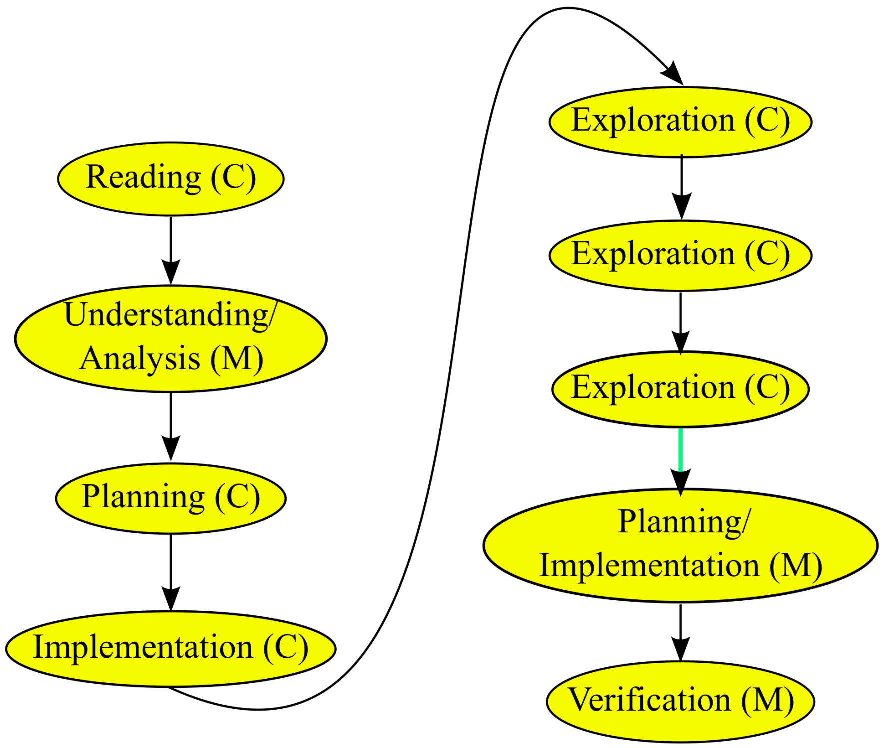


Figure 56. Parsing of Aurora's Problem-Solving Session 3: Second part.

Reflection on Problem Solving in a Dynamic Geometry Environment

The third section presents Aurora's reflection on the three problem solving sessions and problem solving in a dynamic geometry environment as a result of participating in this study.

Reflecting on the Problem Solving Sessions

Aurora's favorite problem from this study was the Airport Problem. She explained that the problem served as *"a very interesting way to approach the centers of triangles."* Moreover, she *"thought it was really interesting to see a real-world problem that incorporated a mathematical concept that is rarely applied to problem-solving situations in the classroom"* that allows applying knowledge in a novel way. She attributed successful solving of the problem to the use of GSP allowing her to focus *"on the conceptual nature of the problem," "to investigate ideas that would have taken much longer had [she] not had the use of technology,"* and to easily construct different geometric objects whereas *“by-hand construction which would have been time-consuming and ultimately detract from the purpose of the problem.”* Nevertheless, she admitted that GSP was also a hindrance *“because [she] did not think through reasons for constructing specific objects.”*

In solving mathematical tasks she was both successful and unsuccessful. She attributed success in solving mathematical tasks to her own *“mathematical knowledge and GSP to solve the problems”* where she attributed the most to the latter, and unsuccessful when she *“did not see the big picture,”* but focused instead on what the outcome should be. For example, when solving the Land Boundary Problem she attributed success to use of GSP in solving the problem,

> *It would have been a tough problem to solve without use of GSP because of the movement of triangles (i.e. ones with the same base and height, but different shapes)… I did not really know how to start the problem and had to do some experimentation with where the point might lie first. That gave me some ideas about the solution, but also made me concentrate too much on the ending point rather than the work which would allow me to arrive at that answer.*

As a result of participating in this study, Aurora learned *“not to doubt [her] problem solving ability,”* and changed the way she thought about mathematics and use of technology when problem solving. In respect to the former she said, *“I realized some of my problem-solving techniques that I typically employ and it allows me to analyze the pros and cons of my approaches,”* whereas for the latter she said,

> *I find that while I supported technology use in almost all problem-solving contexts before this study, I can now determine which problems can use technology most effectively. I am now of the opinion that certain problems are more geared towards technology, while in others, it can be a hindrance.*

With every problem she used GSP to first find an approximate answer, and then worked towards finding that solution. Consequently, she noted that she needed to change the way she used technology to problem solve to be effective as through its use it made her *“focus on the answer,* *rather than the process."* She added what changes needed to be undertaken to improve her problem solving,

> *I will focus more on the big picture and first analyze which of my approaches might be the most effective. Instead of jumping right into a problem, I will probably sit back and think about what I know which relates to the problem and the most successful way of approaching a solution.*

Lastly, through participation in this study she not only became aware of her general geometry problem solving process, but influenced how she would teach problem solving,

> *In my teaching, I think I will encourage technology in problem solving more. Additionally, I think I will be more apt to encourage different styles of approaching a problem. Before, I would have probably encouraged students to learn an algorithm for working through problems. However, after this study, I think I will allow students to work through the problem in their own way and provide helpful suggestions only if they are stuck.*

Reflecting on Problem Solving in a Dynamic Geometry Environment

Aurora relied heavily on the GSP when problem solving where she used it for different purposes using various capabilities of the software, such as *"to outline the solution and an approach," "to make mistakes, erase them, start over, go back,"* and *"[to do] quick constructions and measurements."* Participation in this study reinforced the belief of GSP being *"very helpful in working through novel problems"* as it allows fluidity of exploration,

> *I was impressed by how quickly I could arrive at an approximate answer so that I could start thinking about how to actually construct the figure…you can make a mistake and quickly get back to the last point that you knew you had correct. This is essential and technology allows students the ability to be wrong and easily correct it.*

She added that she *"will be more apt to use technology in the future."* Moreover, another benefit of technology was *"the ease of discovery provides an excellent resource."*

Despite many benefits of using technology when problem solving, she was aware of several disadvantages of problem solving using technology. She said that technology led her *"astray"* causing her to *"focus on aspects of the problem that were inconsequential to the end* *result."* She added that another drawback of using technology when problem solving is lack of reasoning to why a certain activity is being done,

> *Sketchpad would allow me to do things quickly without much thought behind why I wanted to do that. It is a blessing because I can easily erase it, but also a hindrance because I did not think through my reasons for constructing specific objects.*

In summary, the following are suggestions she would give to another person when problem solving in a technology environment,

> *Outline a process for arriving at a solution. Underline key terms and concepts. I would suggest making a list of related mathematical ideas which could be useful. Start limiting what the solution could be by playing around with the software. Although it sometimes hindered me, I think that it was also extremely helpful to know where to go. I would also suggest not getting frustrated, though I know it is hard. The hardest part of some of these problems was getting past my own thinking of "Why don't you know this? It should be so easy!" I feel that approaching a problem with no expectations is the most effective way of solving a novel problem.*

Summary of the Case of Aurora

The analysis of Aurora indicated she had very little prior experience working mathematics problems in a dynamic geometry environment. She was extremely confident in her use of technology during problem solving whereas she was not as confident in her problem-specific knowledge. Technology often served as a tool to recall specific geometry content knowledge and search for a solution plan. Again, the framework used in this study provided useful information with respect to when, where, and how Aurora used both cognitive and metacognitive processes in a dynamic geometry environment and how these affected her efforts when problem solving. A continuous interplay of cognitive and metacognitive behaviors appeared to be necessary for successful problem solving and maximum student involvement. Cognitive processes, however, were more prominent than metacognitive processes. Nevertheless, using cognitive strategies and resources were chosen as a means of solving the problem. Awareness that such strategies would contribute to solving a problem is a result of experience and for that reason a metacognitive behavior. Moreover, that was a powerful problem solving behavior for her; after her "walk" through the problem solving space, she coordinated one or more schemes and valuable new information. As a result she had a powerful "tool" to solve the problem. It yet remains to understand this exquisite and intriguing phenomenon.

During the problem solving session negative self-evaluation behaviors were sometimes evident. The combination of lack of managerial skills and frustration led to inactions or non-productive efforts. She believed false moves and multiple attempts are part of mathematics problem solving; persevering through problems using capabilities of technology software, she was able to gather necessary data, direct actions towards possible solution perspectives, and attain a correct solution in the end. Nevertheless, Aurora became aware of her general problem solving process that might influence how she will approach problem solving in future, such as outlining a solution plan, focus on the process, use technology appropriately as well as encourage different styles of approaching a problem when teaching.

CHAPTER 6

DISCUSSION OF FINDINGS AND IMPLICATIONS

In this chapter I present findings related to the research questions that guided this study, and conclusions of the study. Within conclusions of the study, I discuss additional findings of the study that went beyond the frameworks I used. Included too are study limitations, implications, and recommendations for future research that conclude the chapter.

Analysis of the Case Studies: Patterns of Metacognitive and Cognitive Behavior

The following research questions were addressed in the study:

1. What are some of the metacognitive processes exhibited by preservice teachers when engaged in solving nonroutine geometry problems using Geometer's Sketchpad?
2. What metacognitive processes appear to be associated with the Geometer's Sketchpad use during problem solving?
3. How do preservice teachers perceive the importance of Geometer's Sketchpad when faced with nonroutine geometry problems?

Case studies of two preservice mathematics teachers were used to explore these questions. This section provides an analysis of the two case studies in relation to the research questions. In the first subsection, I address the first research question consisting of descriptions of metacognitive processes preservice teachers exhibited during problem solving. The second subsection addresses the second question where the nature of student-tool interaction with respect to metacognitive processes is explained. In the final subsection of the chapter, I present preservice teachers' perceptions with respect to problem solving using the Geometer's Sketchpad.

Metacognitive Processes

The first question focused on uncovering and describing metacognitive processes preservice teachers exhibited during problem solving sessions. In order to answer that question a model adapted from Schoenfeld (1981) offered a framework describing problem solving behaviors, both cognitive and metacognitive actions, during which a problem solver engaged in a particular activity. Problem solving behaviors were described within each of seven episodes. In order to better understand the nature and interplay of the cognitive and metacognitive processes within each of the episodes, the nature of participants' answers with respect to their metacognitive awareness, metacognitive evaluation and metacognitive regulation (J. Wilson & Clarke, 2004) was taken into account. This model, however, demonstrates exhibited problem solving behaviors within each episode and should not be taken as a rigid model a problem solver goes through; episodes are not linear, but cyclic, dynamic and iterative as described earlier in Chapter 2, Chapter 4, and Chapter 5.

Reading the Problem

Both participants started each problem solving session by reading the problem statements, which was consistent with Schoenfeld's (1981) model. Although this episode is often labeled as a cognitive episode (Artzt & Armour-Thomas, 1992), both participants exhibited a variety of metacognitive behaviors during this episode. Wes always read the problem aloud and from the hard copy, aiding him not to miss any information. He typically read the entire problem before reading the main problem statement again. These monitoring strategies allowed Wes to maintain focus and identify the problem components. On the other hand, Aurora read it in silence from the screen highlighting the words with the cursor as she was reading through the problem. She typically read just the main parts of the problem if the problem had multiple goals. Similar to

Wes, these monitoring strategies helped her not to miss any information, and allowed her to maintain focus and identify the problem components. During reading episode for Aurora, the use of GSP was oriented towards the management of the tool: turning on Text Menu to highlight and bold problem statement or parts of it. During the problem-solving session both participants often reread the problem to review the problem conditions or to see if they had forgotten important parts of the problem, which appeared to be a strategy to control potential missteps. Engagement in these monitoring and control strategies, and management of the tool was a metacognitive behavior. Acting on these metacognitive processes prompted metacognitive behaviors aligned with the understanding episode that contributed to moving through the problem-solving space.

Understanding the Problem

Metacognitive behaviors that fit the understanding episode were exhibited immediately after the reading episode for both participants for all of the problems. Behaviors related to the episode usually stood alone or occurred simultaneously with behaviors related to analysis episode. Consistent with previous research, typically both participants first explicitly noted problem conditions, problem goals or key parts of the problem by either stating them aloud (Schoenfeld, 1981) or bolding them. During the understanding episode, the participants needed to consider content specific knowledge and strategies relevant to the problem, which was consistent with previous research (Carlson & Bloom, 2005; Lawson & Chinnappan, 2000; Schoenfeld, 1992). They engaged in a variety of strategies for monitoring their understanding of the problem as reported by previous research (Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985a, 1992); they were looking for the given information in the problem and what was being asked of them, restating the problem, reengaging with the problem text, asking for clarification of parts of the problem or the meaning of the problem, making sense of the problem information, representing the goals and givens of the problem by writing them down, mentally or making a representation of the problem, introducing suitable notation, and reminding him or herself of the requirements of the problem. These monitoring strategies were metacognitive behaviors that were an important attribute during problem solving that helped develop an understanding of the problem and access their knowledge, facts, and strategies.

Drawing a diagram representing the problem was a cognitive problem-solving behavior used by both participants when a diagram was not provided as a part of the problem. Although this cognitive behavior is available with paper-and-pen, both participants used the capabilities of the built-in functions of the GSP to represent the problem and quickly add secondary elements, lines, segments, rays, and points to a figure. As they were representing the problem, they stayed mentally engaged throughout the process by engaging in a metacognitive act of self-questioning showing evidence of monitoring, and at the same time they spontaneously accessed and directed their knowledge and thinking. Typically, both participants monitored their work throughout during problem representation. For instance, Wes carefully read statements, restating them or interpreting them before representing them on a paper or on GSP in increments. Furthermore, for Problem 2 and Problem 3 he engaged in problem representation and sense making of the problem with paper-and-pen first before representing it using Geometer's Sketchpad as a consequence of working on Problem 1. When working on Problem 1, he immediately started using the software. He considered that this hindered him as it did not allow him to engage deeply in understanding the problem. Once he was done he never had the need to evaluate his work because he monitored his work regularly. Aurora also read problem statements, restating them or interpreting them before representing them on Geometer's Sketchpad, but never on paper-and-pen. False moves occurred for Aurora when she was representing Problem 1 and Problem 2, however, as a result not taking into consideration the problem as a whole and directing her actions in that direction, and for not noting the main condition of the problem, respectively. These false moves were discovered, however, through reengagement with the problem statements again or verbalizing what was done. This aided reevaluating what was done and correcting false moves.

Drawing a diagram using GSP helped them visualize the problem, and attain accurate visual input. It aided accessing mathematical knowledge and facts relevant to the problem when attempting to make sense of the problem, directing their thinking processes towards working through a problem-solving space. Consequently, all these metacognitive behaviors and activities helped participants develop an understanding of what the problem meant concretely, which was consistent with the results by Goldenberg et al. (1988). An accumulation of resources, however, was not sufficient for productive paths but rather led the participants in unproductive directions in the absence of metacognitive monitoring. The ability to manage their resources and actually access useful information at the right moment was an essential metacognitive behavior in making productive and useful decisions.

During this episode they also engaged in metacognitive behaviors, such as pausing to make sense of the problem and of the current effort, and to assess productivity of their thinking (e.g., whether considered knowledge was relevant to the problem) and internal dialogue that aided to productive or desirable thinking and directions. For instance, interpreting the problem statement and sense making was most specific for Problem 3 where both participants related the problem goal of keeping the amount of the land the same with the concept of area. Neither of the participants wrote down main ideas of the problem. Nevertheless, the main ideas were verbalized by considering and organizing content specific knowledge and strategies relevant to the problem as a result of current problem solving states or previous experience. Internal dialogue consisted of posing metacognitive questions that promoted metacognitive behaviors which is consistent with other mathematics education literature (e.g., Carlson & Bloom, 2005; NCTM, 2000; Pólya, 1962/1981). These metacognitive behaviors were important and contributed to move their thinking in productive directions. Metacognitive behaviors consistent with the understanding episode were crucial in solving the problem, highlighting the importance of these preparatory behaviors also recognized by Pólya (1945/1973) and Zimmerman (2002).

Analyzing the Problem

Analysis of the problem occurred as an individual episode after the understanding episode or the exploration episode, or it occurred simultaneously with the understanding episode. Participants engaged in metacognitive behaviors to fully understand the problem and make a choice of perspective; they tried to make sense of problem information by decomposing the problem in its basic elements, reformulating the problem and seeking the relationship between the conditions and goals of the problem which was consistent with other studies (e.g., Artzt & Armour-Thomas, 1992; Schoenfeld, 1981) in order to select an appropriate perspective. That said, participants devised different perspectives, considered various mathematical concepts, facts and strategies before selecting a perspective. These behaviors were consistent with the previous research (e.g., Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; Schoenfeld, 1981, 1985a). Nevertheless, solving problems in GSP prompted or required its use. Having a diagram representing the problem, constructed using the GSP in the understanding episode, triggered accessing, considering, combining and organizing their knowledge when seeking relationships between the conditions and the goals of the problem. It also often served as a tool to recall specific geometry content knowledge to aid in problem-solving behaviors. Often having a visual input directed their actions and thinking into understanding the information obtained through the use of GSP. The decision to engage in these activities was a metacognitive act that prompted other metacognitive behaviors.

When choosing a perspective they considered knowledge of what needed to be done, and what might be done in a particular problem-solving context. In addition, both participants reengaged with the problem text and restated the problem in their own words before considering and making a choice of a perspective, which was noted in earlier research (e.g., Carlson & Bloom, 2005) as well. Participants did not always evaluate a choice of perspective with respect to effectiveness of their problem solving strategy or thinking, however, but made random associations with respect to content knowledge and problem perspective. For instance, when solving the extension of the Problem 3 both Wes and Aurora decided to use the same strategy used for the original problem without assessing how to use it, and evaluating if using it at once would be efficient. Thus, not only they did not evaluate it but also did not direct their thinking if their choice would move them towards a solution or not. Schoenfeld (1992) reported that students often do not know how, when, and whether to use their metacognitive resources to solve a particular problem and identified this as a lack of the control mechanisms. Consequently, absence of such behaviors (e.g., lack of assessment for making a choice of a perspective and strategy) or the ability to manage these resources sparked lengthy unproductive and unguided efforts in subsequent episodes.

Sometimes they considered multiple choices of perspectives such as Aurora when solving the extension of Problem 3 before deciding on a choice of perspective based on evaluation of its effectiveness. Wes also tried to construct logically connected mathematical statements, that is, organize relevant information systematically when seeking a connection between conditions and goal of the problem when working on the third part of Problem 2 before attempting to make a choice of perspective. When working on Problem 2 he also assessed and recalled familiarity of the problem with a similar problem (Fermat's point). Both participants reflected on the degree of difficulty of the problem.

The analysis episode, interestingly, was the least coded episode for Aurora, while Wes for each problem engaged in metacognitive behaviors consistent with this episode. It appeared that analysis of the problem allowed further understanding of the problem, exploration, and more analysis allowing Wes to combine, and select steps and strategies that might potentially lead to the problem solution, whereas absence of such behavior sparked lengthy pursuits for Aurora's problem solution paths.

Exploring the Problem

Behaviors consistent with the exploring the problem episode were one of the most often coded episodes, which is often the case with novice problem solvers (Schoenfeld, 1992). The problem-solving context led to exploring behaviors since the nature of the software invited participants to explore, experiment, and conjecture in the search for a solution plan, relying on their previous knowledge and experience (Hadas, Hershkowitz, & Schwarz, 2000). For this episode, the participants engaged in a variety of both cognitive and metacognitive behaviors using a variety of problem-solving strategies. Interestingly, although behaviors consistent for this episode were most coded for Aurora, it was least coded for Wes.

During this episode, when it was labeled as a cognitive exploring episode, it was characterized by weak structure, and lack of metacognitive strategies and behaviors. For instance, in the search for relevant information Aurora often relied on the use of GSP that was characterized by quick jumps into exploration lacking apparent structure to the work, did not assess of current state of her knowledge, did not assess of relevancy of actions, and lacked perspective on future steps. She engaged mainly in trial-and-error strategy consciously where she made a guess, tested it, and repeated until she assessed the feasibility of her actions. In addition to the trial-and-error strategy, the GSP allowed the bottom-up strategy where she took the problem as solved and worked backwards to obtain a solution. Aurora then focused more on the result, relying on quick guesses rather than on engaging in productive efforts to select a problem-solving path or to allocate problem-specific resources to obtain such a solution. Nevertheless, the choice of the strategy (trial-and-error and bottom-up) and awareness of its helpfulness was a metacognitive act where their knowledge of the software capabilities guided the way software was used. Often, taking a step back and regulation of negative affective behavior, namely frustration, were helpful in redirecting participants' thinking. Similarly, Wes in exploring Problem 3 moved a point and land boundaries without any apparent structure to the work, assessment of his knowledge, assessment of relevancy of actions but rather testing them, focus, and perspective on future steps. Although he used this strategy to make sense of the information, similarly, he used it to make sense of the problem situation, and such selection and use is a regulatory skill. As reported by Schoenfeld (1981, 1985a), lack of monitoring of progress often made the endeavor unsuccessful, which resulted in lenghty pursuits characterized by weak structure, absence of local and global assessment, and impetious jumps from one particular direction to another through one exploration to another before sense making occurred.

Each participant exhibited metacognitive behaviors in the search for relevant information, such as drawing away from the original problem (e.g., considering equivalent problems, looking for related problems) to ask him or herself what has been done during the exploration and how it is related to the original problem and its goal, assessing the current state of his knowledge as well as relevance of the new information with the goal of moving forward in the solution process, and assessing the efficiency and effectiveness of cognitive actions or strategies. These behaviors were consistent with previous research (Artzt & Armour-Thomas, 1992). Both participants engaged in internal dialogue, such as verbalization of self-questions, conjectures, strategies that appeared to aid efficient movement towards a solution path.

Solving problems in GSP allowed them to engage in conjecturing based on the visual representation of the problem or previous knowledge and to test their conjectures. For instance, when working on Problem 1 metacognitive behaviors Wes exhibited were: accessing and considering content specific knowledge relevant to the problem, organizing and directing that knowledge with respect to problem and solution process and testing a conjecture. Also, GSP allowed for a trial-and-error stragegy that involved also making purposeful hypotheses that allowed metacognitive behavior associated with the exploration episode to be controlled and focused. For instance, when working on Problem 3, he created a conjecture based on earlier episodes, convinced himself by testing the conjecture and then analyzed it to confirm it. In these situations the feedback provided by the GSP helped again evaluate and directed thinking processes towards devising a solution plan and consequently successfully solving the problem which is consistent with other mathematics education literature (e.g., Hollebrands, 2007; Olive & Makar, 2010). Beside conjecturing and testing of conjectures, both participants tried to imagine their actions in order to assess their efficiency or effectiveness or feasibility. It appeared that such visualization, however, was not an easy task and might have been out of their imagining capabilities.

In summary, both participants used the software's capabilities of precision, measuring, and dragging to engage in problem-solving activities that proved to be a cyclic process of generation, justification, and refinement of plausible solution paths. Hence, it appeared that metacognitive behavior of both participants considered affordances of the GSP to guide their problem-solving behaviors (Hollebrands, 2007) making the problem-solving process more fluid and allowing flexibility in the problem-solving approaches. The ability to reflect on the feasibility of their thinking processes using different resources (e.g., mathematical knowledge, facts, and technology), and manage those resources at the same time was an essential metacognitive behavior.

Planning

Planning occurred as an individual episode after understanding, analysis, or exploration episodes, or occurred simultaneously with the implementation episode. Planning processes are normally in the literature associated with metacognitive processes a problem solver goes through during problem solving. Based on the nature of their behaviors, however, I assigned a level of cognition for this episode where lack of assessment of feasibility of the plan was considered as a leading factor. Behaviors consistent with planning were highly coded for both participants.

During this episode, when it was labeled as a cognitive episode, both participants described their intended plan or its parts but lacked any visible sequence of strategies and were without apparent structure of the plan; that is, identification of goals and subgoals, global planning, and local planning was absent or possibly not verbalized. For instance, both participants when planning for solving the second part of Problem 3 did not assess the relevancy and quality of chosen activities or strategies (trial-and-error) with respect to moving forward in the solution process, although they accessed and considered mathematical concepts and facts. However, engagement in trial-and-error strategy, and the ability to piece together different information obtained from devising several problem solving paths at the right time allowed participants, especially Aurora, to attain their goal, that is, to solve the problem.

The GSP was often used to examine variety of strategies, to examine the details of the plan, check each step carefully, monitor, assess, refine, revise, or abandon the plan according to problem goals until they arrived at a final plan, when needed. Even though the decision to engage in these behaviors was an important metacognitive behavior, however, it was not sufficient for devising an efficient and effective plan. Hence, limited metacognitive behavior, such as lack of the ability to access and coordinate useful information and strategies at the right moment, led the participants in unproductive directions. Moreover, as a result of lack of evaluating and monitoring their work, negative affective behaviors were exhibited, mainly with Aurora, influencing cognitive behaviors to take domination over metacognitive processes.

As reported by previous research (Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985a), metacognitive behaviors exhibited during this episode included: accessing, considering, and manipulating mathematical knowledge, concepts and facts relevant to the problem, assessing the plan through imagining, conjecturing and testing, and monitored and refined, revised, or abandoned the plan according to problem goals until they arrived upon the final plan.

When selecting steps and strategies for a solution plan accessing resources, such as knowledge and Geometer’s Sketchpad, and experiences relevant to the problem were paramount. For that to happen participants needed to be aware of three components: knowledge of what was done, knowledge of what needed to be done, and knowledge of what might be done in a particular problem solving situation. Manipulating their knowledge and new information obtained from previous episodes led participants to identify the following plans to find the solution to the problem. They also represented the information by adding new elements onto the sketch before using them in a solution plan. They assessed the plan with respect to the process and solution, and problem and solution. Monitoring of their plans and strategies, which was most evident with Wes, was exhibited when they verbalized thoughts and questions about their steps and strategies and by staying mentally engaged through construction of logically connected mathematical statements. There was evidence the students tested that their plans made sense, that they looked for efficient plans, and that they changed their plans during this stage. Often the participants engaged in a metacognitive act of self-questioning, such as verbalization of conjecture, questions and comments related to the plan and strategy (e.g., evalution of the current problem-solving state, and trying to make sense of it, judging the effectiveness of previous actions). Acting on these metacognitive acts prompted various metacognitive behaviors; internal dialogue contributed to move their thinking in productive directions, movement forward in solution plan, and aided assessing effectiveness and feasibility of their chosen strategies and approaches based on the key features of the problem. A balance between cognitive and metacognitive behaviors as well as noncognitive factors, such as perseverance and confidence favored productive problem solving approaches.

Implementing the Plan

Implementing the plan occurred as an individual episode most often immediately after exploration or planning episode or occurred simultaneously with planning episode. The nature of participants' behaviors differed—cognitive and metacognitive—and were consistent with implementation (e.g., execution of a plan or strategy, proving a conjecture) as exhibited in earlier research (e.g., Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985a). Behaviors consistent with the implementation episode were most coded for both participants.

Cognitive behaviors exhibited during the implementation episode included participants having executed their planned activities on paper-and-pen or using Geometer's Sketchpad in a well-structured way without assessing their activities or quality of their activities or monitoring their work. Lack of metacognitive behaviors such as control (e.g., assessing the plan with the conditions and requirements of the problem, assessing the appropriateness of actions, assessing the sensibility of the solution progress and results) made the problem-solving endeavor fruitless and led to quick jumps from one planning or implementation episode to another. These led to lenghty pursuits characterized by weak structure, and absence of assessment. Similat behaviors were exxibited in research study by Schoenfeld (1981, 1985a).

Metacognitive behaviors exhibited during this episode were consistent with previous research (Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; Schoenfeld, 1981, 1985a) and included: considering, accessing, and organizing their knowledge relevant to the problem when constructing logically connected mathematical statements, evaluating appropriateness, effectiveness and efficiency of their actions, monitoring of their actions and directing their thinking and actions towards a solution, and assessed the sensibility of the solution process and results. When implementation drew on actions that were already automatic, monitoring and evaluations of actions were automatic as well (e.g., Problem 1c). Wes also considered the degree of elegance during implementation of Problem 2. Presence of ongoing monitoring, evaluation and regulatory processes, and the ability to manage their resources and negative affective behaviors were essential metacognitive behaviors that allowed productive directions through the problem-solving space.

The problem-solving context not only allowed for easy implementation of their plans, but helped with more complex questions that extended participants' competence, such as noting where the problem-solving activity might be leading. As a result of reflecting on the activities and results through feedback provided by the GSP, they were then able to redirect their thinking processes towards a solution to the problem, choosing the strategy or the plan and assessing merits of the new strategy or plan. Hence, as noted in other problem-solving episodes, through stages of personalization and transformation of the tool they transformed it to a valuable instrument as a result of their knowledge of the software capabilities. Through engagement in these activities, they optimized the use of available resources, which was undoubtedly a metacognitve act.

Verifying the Answer

This episode, if it occurred, occurred most often individually. Interestingly, this episode was not coded very often, which will be explained in the following discussion.

The cognitive processes typical for this episode included evaluating the result by checking the computations steps, which was noted in earlier research (e.g., Artzt & Armour-Thomas, 1992; Schoenfeld, 1981) or using the Measure functions of GSP. The measuring capability of GSP made quick validations of different solution paths possible, whereas measurement capabilities to double-check the result were used only to satisfy the participant's—namely, Aurora's—validating standard. Hence, mathematical integrity was held by the GSP, and not her as a problem solver. Interestingly, Wes also on one occasion contemplated whether to accept the result he obtained using Geometer's Sketchpad followed by using Algebra to verify his answer differently (Problem 3 first part).

Besides cognitive strategies, the participants engaged in several metacognitive strategies for verifying their results. These included decisions to review their work to make sure they did not forget anything or determine if they had made a mistake, rereading the problem to make sure the solution reflected the problem conditions and answered the question, and checking the results for reasonableness of the solution of the problem. These behaviors were consistent with earlier research (Artzt & Armour-Thomas, 1992; Schoenfeld, 1981, 1985a, 1992). They also thought about a way of checking to see if their solution was correct with or without the use of GSP. The decision as to what approach to use seemed to depend on the availability and capability of resources as well as on an affective factor of value; that is, on the participant's standard for verifying that an answer is correct. When examining the solution without the use of technology, they engaged in making logically connected mathematical statements, and assessing the reasonableness of the answer (e.g., Wes, Problem 2), which was consistent with previous research (Schoenfeld, 1981, 1985a). In these situations, metacognitive awareness dealt with accessing mathematical resources needed to engage in a productive effort. Metacognitive evaluation dealt with situations where they assessed the correctness and the efficiency of the solution, and how and why were particular actions and strategies used. In these situations they relied on their content resources; that is, their conceptual knowledge informed them as to the correctness or reasonableness of the obtained solution.

The participants sometimes checked for the quality of the process and rarely assessed the aesthetic quality of the solution. Rarely, however, did they examine if the solution could have been obtained differently. For instance, Wes as a result of assessing the aesthetic quality of the solution of Problem 2b and is his actions reflected mathematical rigor, he verified the result by making logically connected mathematical statements as well.

As a result of engagement in behaviors aligned with the verifying episode with or without the use of GSP the participants spurred additional metacognitive processes. Through the process of evaluation of their actions, they made a decision whether to accept or reject the solution. The decisions to accept or reject a solution were always exhibited before moving to a new problem-solving cycle. If a discrepancy was discovered, which was often result of verbalization of their actions and thinking, the participant cycled back and engaged in correcting (e.g., refining, revising, abandoning) the incorrect cognitive or metacognitive actions by either abandoning their plan or modifying it. These behaviors were consistent with research on graduate mathematics students (Carlson & Bloom, 2005).

Metacognitive behaviors aligned with the verifying episode were important for a successful problem-solving endeavor, however, not every problem-solving session ended by the participants engaging in metacognitive behaviors of evaluation. It was done implicitly or not at all as it made sense to them but could not explain their metacognitive knowledge. Veenman et al. (2006) postulated that some metacognitive processes, such as evaluation processes, appear on a less conscious level or run in the background of one's cognitive processes as they became a regulatory habit. Future research should more closely consider how one decides to evaluate the correctness of one's solutions and how this is influenced by the problem content.

Transition

Between the episodes the participants assessed the current stage in problem solving where either decisions were made to salvage or not salvage strategies that might be valuable or assessed the value of a new direction or jump into the new approach as addressed by Schoenfeld (1981, 1985a). In these situations, metacognitive acts dealt with reflecting on the current stage in problem solving which most often occurred as a result of feedback provided by the GSP (e.g., Is this choice of perspective getting me anywhere? Do the undertaken actions reflect the problem requirements?) that then guided their thinking to current or new directions. However, lack of assessment of the current problem-solving stage between the episodes rendered the subsequent efforts fruitless and unproductive before such occurred as observed earlier by Schoenfeld (1981, 1985a, 1992). For instance, both participants several times did not assess their current knowledge or directions to come or did not assess their plan, but they immediately jumped into an implementation episode, which was costly for the solution process (e.g., Problem 3 second part).

Nevertheless, I was able to observe a new metacognitive behavior—"taking a step back." "Taking a step back" was a metacognitive behavior; it entailed reassessing what was done, putting effort to organize relevant knowledge and redirecting those processes that contributed to efficient movements towards a solution. The decision of "taking a step back" was a metacognitive act that was essential for productive problem solving, and a type of reflective behavior that promoted participants' metacognitive awareness and monitoring skills.

Patterns of Metacognitive Processes Associated with the Geometer's Sketchpad Use

The second question sought to uncover what circumstances, interactions and situations promoted metacognitive behaviors to occur. In order to answer that question an instrumental approach offered a framework describing the effects of tool use on the participant activity (instrumentation) and transformation of the tool to fit participant activity (instrumentalisation).

The Geometer's Sketchpad appeared to be integrated into the problem solving processes and strategies used by the participants. The participants' knowledge base of the Geometer's Sketchpad capabilities affected the extent to which they used the Geometer's Sketchpad while solving a mathematical problem. Both participants exhibited knowledge of several built-in functions of the Geometer's Sketchpad. The CONSTRUCT and MEASURE menus used during understanding, analysis, exploring, implementing and verifying episodes. The CONSTRUCT functions allowed the participants to make a representation of the problem and quickly add secondary elements, lines, segments, rays, and points in the figure. Drawing a diagram representing the problem was a cognitive problem solving behavior used by both participants when working on Problem 1 and Problem 2 where a diagram was not provided as a part of the problem. Although this cognitive behavior is available on paper-and-pen, Aurora for both problems immediately used the capabilities of the software, whereas Wes used it immediately when starting to solve Problem 1 but represented Problem 2 first on the paper before moving onto using the software. Most often when representing a problem or adding secondary elements to their diagrams they performed step by step procedures to make sure their interpretation of the problem was correct, which shows evidence of cognitive engagement during problem solving. They stayed mentally engaged throughout the process showing evidence of monitoring, and directing their knowledge and thinking. In addition, for both Wes and Aurora, selection and use of that problem solving strategy was based on awareness of its usefulness. Their previous problem solving experience guided them into making a representation of the problem, allowing them to visualize the problem, and helped them develop an understanding for the problem before moving onto the next problem solving activities. Hence, their actions transformed the tool to fit their needs where the tool became an instrument through stages of personalization and transformation of the tool, which is a metacognitive behavior.

The MEASURE functions allowed the participants to formulate, test, and verify a conjecture or a plan. Wes often used the Measurement Tool to verify his thinking and then as a result of that action to decide on a strategy going forward. Hence, the appropriation of the software to a measurement and verifying tool was influnced by his thinking processes. On the other side, measuring to verify a conjecture was not prominent in Aurora’s problem solving behavior. For her measuring often came first and then she tried to mathematically justify the conjecture. Furthermore, she often used the Measurement Tool during the verification episode to double-check her solution. Nevertheless, her knowledge about her problem solving strategy and knowledge of the software capabilities guided the way she used the tool as measurement tool. She transformed the tool to a meaningful instrument allowing her to test and revise her conjectures, and devising one, that is, reaching the goal while at the same time taking affordances and constraints of the software in consideration, which is a metacognitive behavior.

The dynamic character of the software allowed the participants to manipulate the figure or parts of the figure by dragging. For instance, dragging is a cognitive behavior that is not available on paper-and-pen, that is, in non-dynamic environments. Dragging as a strategy was not randomly chosen, but used to develop a better understanding of the current problem solving situation, gather information, help formulate a conjecture or a plan and assess their ideas or actions. That is, choice of dragging as a strategy for the above named purpose was a metacognitive behavior where they drew on their previous knowledge of how and why to use the particular strategy and by using their executive skill to optimize the use of their resources. Geometer's Sketchpad helped develop their acitivities whereas they transformed Geometer's Sketchpad as an exploration tool to fit their needs taking into consideration its affordances and constraints thoughout their problem solving space.

The nature of the software allowed the participants to assess their solution state and influenced their thinking and further actions. They were able to make sense of the problem solving situtation, evaluate their thinking processes, correct them and direct their thinking processes and actions as a result of previous actions towards achieving their current goals. It was a metacognitive act that dealt with reflecting on the task and undertaken activities as a result of feedback provided by the Geometer's Sketchpad in directing their thinking processes towards a solution of the task, and choosing of the strategy when working through problem solving space.

Thus, interplay between thinking processes and tool use was a prominent problem solving behavior. Furthermore, for Aurora the use of technology to solve the problem was prominent, and allowed for bottom-up strategy to solving the problem especially when solving Problem 3; that is, taking the problem as solved, and working backwards to obtain the solution. Although she used the capabilities to be able to refine, revise and abandon her approaches and arrive at a solution, she was aware that the way she used it hindered her by focusing more on the result rather than on the process. Through her actions she transformed the tool to fit her needs allowing her to work toward finding a solution to the problem. The problem solving context allowed the participants to use and apply their knowledge, translate verbal statements into an interactive representation, investigate a mathematical idea, deal with a situation that may not have a single solution, and make, test and verify their conjectures. Thus, the nature of the software allowed exhibiting different mathematical thinking processes, the use of different strategies and different uses of the Geometer's Sketchpad. Hence, a variety of cognitive behaviors became available in dynamic environments whereas non-dynamic environments do not allow for similar problem solving behaviors.

Typically, both participants used Geometer's Sketchpad to make a representation of the problem allowing them to visualize the problem and develop an understanding of the problem. Interpreting the problem and directing their knowledge relevant to the problem was important to successfully represent the problem where they stayed mentally engaged throughout the process monitoring their thinking. Visualization of the problem triggered organizing their knowledge when seeking relationships between the conditions and the goals of the problem. These metacognitive and cognitive processes often directed their actions and thinking into understanding the information obtained through the use of Geometer's Sketchpad, followed by accessing, considering and evenutally selecting a perspective as well as allowing monitoring progress of undertaken activities. In addition, technology often served as a tool to recall specific geometry content knowledge aiding following problem solving behaviors. These metacognitive behaviors were important and contributed to move their thinking in productive directions.

Furthermore, not having a static but a dynamic representation of the problem drove the use of Geometer's Sketchpad as an exploration tool. That is, availability of Geometer's Sketchpad guided the choice of strategy to solving the problem. Typically they engaged in strategy they referred to as "trial-and-error" strategy but the nature of it differed. Exploring a problem through "trial-and-error" directed Aurora to devise a successful plan. Wes used "trial-and-error" to develop a deeper understanding of the problem by examining different relationships relevant to the problem. Wes most often created a conjecture through visualization, tested it, and then tried to analyze it or reflect on the reasonableness of the result by accessing different mathematical concepts and integrating them into connected mathematical statements to confirm conjecture or a result.

Aurora on the other hand typically organized her activities driven by the novelty of a problem. That is, novelty of a problem solving situation led her to use Geometer's Sketchpad to find the solution of the problem, and then devise a plan to reach that solution. However, in doing so she considered affordances of the Geometer's Sketchpad that influenced her strategy. Typically first she would create a conjecture or a plan and then test it. If it was refuted, she most often then changed the nature of her "trial-and-error" where she would approach the problem with very little thought as to whether her approach would be fruitful or not. The capability of Geometer's Sketchpad being able to test her outlined plan quickly led her to engage in such activities. As a consequence, however, it helped elicit signs of frustration. Nevertheless, journey through problem solving space helped access problem specific knowledge relevant to the problem and decide whether to refine, revise, abandon on continue working through the chosen strategy that she then detected to solve the problem. It was Aurora's metacognitive behavior that dealt with the decision, selection and use to engage in these acts she sought to be feasible as a result of her problem solving experience.

In summary, both participants used the software's capabilities, precision, measuring, dragging, construction to engage in problem solving activities which proved to be a cyclic process of generation, justification, and refinement of a plausible solution experiencing genuine problem solving. Thus, it appears that a variety of both cognitive and metacognitive behaviors are enabled by the Geometer's Sketchpad. In addition, continous interplay of cognitive and metacognitive behaviors associated with the use of the Geometer's Sketchpad is needed for productive problem solving. It was pivotal for the participants to be able to manage these vast available activities and resources in order to meaningfully and productively engage in problem solving efforts. Thus, exhibited metacognitive processes were tied to their use of the Geometer's Sketchpad during problem solving, but their ability to decide how, when, and whether to use it determined the extent to which their efforts were productive or not.

Perceptions

The third question sought to explain participants' perceptions to the importance of Geometer's Sketchpad to their problem solving when faced with nonroutine geometry problems. Both participants expressed a belief that Geometer's Sketchpad was important and useful tool during problem solving centering around these qualities: problem solving activities and processes, visualization, speed, and accuracy.

Wes and Aurora felt strongly about Geometer's Sketchpad helping them during various stages of problem solving, most notably during exploration. Wes said that Geometer's Sketchpad gave him the opportunity to explore problems with relative ease; he was able to manipulate the figure and monitor the change that helped him gather relevant information he then used working through problem solving space. He also added that using Geometer's Sketchpad helped him assess his actions and conjectures and deciding whether to refine, revise or abandon a particular perspective. Aurora held a similar perception adding that exploring the problem aided better understanding the problem, accessing knowledge and strategies she then considered if they were relevant or not to the problem as well as aided organizing that knowledge in moving successfully towards a solution. Verifying that the answer was an appropriate solution and examining the path to obtain it using Geometer's Sketchpad's Measurement Tool was important for Aurora and was used also to double-check her actions.

During the final interview Aurora stated "The ease of discovery provides an excellent resource." She viewed Geometer's Sketchpad helpful in allowing her to outline the solution and strategy and test them cycling back making the process of problem solving fluid and less discouraging. Though both participants perceived many attributes of Geometer's Sketchpad helpful during their problem solving, they viewed it as hindrance as well. For instance, Wes stated that technology can be addictive to the extent of not planning appropriately, assessing and monitoring his actions, while Aurora said that using dynamic features of the software was detrimental to quality of her reasoning and outlined plans.

Both participants perceived the importance of being able to represent not only the problem, but an idea with Geometer's Sketchpad. Making a representation of the problem was held important in developing an understanding of the problem, examining relationships between conditions and the goals of the problem as well as considering and selecting a choice of perspective. Although imagining was exhibited for both participants, sometimes their ideas seemed to go beyond their mental visualizing abilities. The ability of being able to make a screen representation of it with Geometer's Sketchpad was paramount in the problem solving process and helped avoid false actions.

There was no session where both participants did not mention they were able to work through a problem quickly and accurately. For instance, Aurora often emphasized that she could quickly make a representation of a problem, use the measurement tool to enable her to quickly arrive at a solution. This was used to direct her thinking on how to actually construct it. Wes explained that the construction process was expedited by elimination of redundant steps helping him to stay organized and focused. Both participants became aware with time, however, that moving quickly through the problem hindered them as they relied on Geometer's Sketchpad to the extent of not taking into consideration why they were doing so. Relying on Geometer's Sketchpad with respect to accuracy was mostly specific for Wes. On numerous occasions he explained that he cannot draw accurately on paper-and-pen, and that having accurate representation of the problem helped his problem solving process and focusing on relevant objects on the representation by using appearance features of the Geometer's Sketchpad.

There were several admissions that a solution to a problem might have not been possible or successful without the use of Geometer's Sketchpad. For instance, Aurora on several occasions elaborated that it would have required a lot of hand constructions, which would be time consuming and would detract her from the purpose of the problem. In addition, she felt she would have become more easily confused about a direction to take. Wes had similar perception, but added that not having Geometer's Sketchpad in some instances might have aided to accessing greater problem specific knowledge instead of doing a construction on Geometer's Sketchpad or paper.

Both participants seemed to perceive that for all three problems, problem solving without the use of Geometer's Sketchpad would have been time consuming, detracting them from the process. Instead GSP allowed them to stay organized and focused. Aurora believed Geometer's Sketchpad was very helpful in working through novel problems, and used it as a "crutch." Wes, on the other hand, perceived it as an incredible "tool," an additional resource for working through novel problems.

Conclusions

All of the episodes as outlined in the compilation of the problem solving models were identified in this study. With respect to metacognitive processes within each of the episodes, it was evident that awareness of one's knowledge triggered selective attention, evaluation of one's thinking helped better planning for effective solution approaches, and regulation of one's thinking helped monitor progress, select appropriate problem-solving strategies, and regulate missteps. Hence, these skills proved to be important for productive problem solving activity, which was also noted earlier by Schoenfeld (1987). Similarly to previous research (Artzt & Armour-Thomas, 1992; Carlson & Bloom, 2005; J. Wilson & Clarke, 2004), it appeared that a continuous interplay between cognitive, and metacognitive behaviors and strategies was paramount for successful problem solving. Problem solvers develop cognitive actions and strategies to make cognitive progress, while at the same time these are important to monitor cognitive processes (Flavell, 1981). It yet remains, however, to understand how participants acquired or developed these strategies and their nature.

The findings furthermore showed that substantial mathematical knowledge, prior problem solving experience, reliance on the use of technology, and affective behaviors, such as perseverance and frustration were related to participants' success when problem solving. More closely, behaviors exhibited by the two participants provided a detailed characterization of the interplay between metacognitive processes and conceptual knowledge that influenced most of the episodes of the problem-solving process. In addition, ability to access the knowledge was dependent on the richness and connectedness of the participants' knowledge, as was claimed earlier by Lawson and Chinnappan (1994, 2000). Similarly to results from Pugalee (2001) and Cross (2009), writing during episodes allowed participants, namely Aurora, develop a deeper conceptual understanding of their current knowledge, analysis on current problem solving state and move towards identifying a successful solution plan. Hence, their writing demonstrated their mathematical reasoning and metacognitive behaviors, and moreover, seemed to have influenced their metacognitive behaviors. Further research is required that would examine affect of writing on problem solver's metacognitive processes.

Additional Findings of the Study

Taking a Step Back

Both participants, after unproductive engagement in particular activities in exploration, or a planning-implementation-verification episode, engaged in a behavior they referred to as "taking a step back," or "going back." This concept included a set of actions, such as thinking over and investigating their actions, chosen strategies, entire problem, that provided an important and instructional phase in their problem solving process. Data from this study confirmed that reflective activities in a problem-solving environment have the power to enhance the learning benefit of the exercises; it gave the opportunity to review previous actions and decisions before proceeding to a next stage. Consistent with the previous literature (e.g., Zimmerman, 2002) reflexive thinking is the foundation of metacognitive awareness that provides not only a better understanding of what the learner or problem solver knows, but also a way of improving metacognitive strategies, because then one can examine how he performed on a specific learning task. Hence, reflective behavior of "taking a step back" was a precondition to promote students' metacognitive awareness and monitoring skills.

Affective Domain

The findings of the study showed that affective behaviors, such as perseverance, persistence, confidence, interest, and frustration occurred frequently during the problem solving activity. These affective behaviors changed during the process of solving a problem, and were related to participants' success when problem solving. More specifically, these affective behaviors influenced variety of metacognitive processes, such as planning of cognitive activities, monitoring of cognitive activities, and evaluating the outcomes of former activities. For instance, both participants had a natural curiosity to solve the problem, and at the beginning of solving it they engaged in deep thinking. They were confident in their understanding of the problem, and an approach to solve it. The affective states of interest and confidence interacted positively with problem solving. However, once their problem solving approach did not result in an expected way, lack of evaluation of the plan and monitoring of quality thinking during the execution of a plan causing negative affective behaviors (frustration) to arise. Frustration influenced cognitive behaviors to take domination over metacognitive processes. This was mostly observed in the case of Aurora. In these problem-solving situations as a result of an incorrect path, she stopped monitoring and evaluating her progress, and engaged in lenghty pursuits characterized by weak structure, absence of local and global assessment, and impetious jumps from one particular direction to another through one exploration to another. She persevered in her problem-solving path, but never persisted too long on a chosen problem-solving path. However, she most commonly persisted on a choice of strategy—namely, trial-and-error—as a result of her previous problem solving experience. Moreover, she did not get discouraged by false attempts using this problem solving strategy, because she believed that false moves are part of doing mathematics helping her learn mathematics, and attain a correct solution. Once she engaged in a coping mechanism (taking a step back), she was able to effectively manage the affect of frustration. This consequently allowed her to assess the relevancy of new information and the current state of their knowledge making her actions focused and purposeful, and led them to a productive endeavor. Management of different affective behaviors allowed both participants to persevere in their problem solving activity. Hence, during the act of solving a problem different affective states acted both productively and counterproductively with metacognitive processes.

The observations made in this study support the arguments from other researchers (Veenman et al., 2006) that individual differences and affective behaviors one brings to a particular situation interact with and influence various components of metacognition. As Goldin (2000) pointed out, affect is fundamental during problem solving; it can both foster ability, but it can also inhibit the current and future problem-solving process. Therefore, research on metacognition should not be studied in isolation, but take into consideration complex construct of affect during problem solving and extend it to characterizing these affective states and their use during problem solving.

Negative Effects of Metacognition

There is no doubt that the participants in this study engaged in metacognitive acts. However, the data from the study demonstrated that frequent use of metacognitive acts does not equate with productive problem-solving activities. For instance, Wes spent an extensive period of time exploring if centroid was the solution to Problem 2, and took him quite some time to realize approach was unproductive. Schoenfeld (1992) argued that we are still missing an adequate theoretical model that would explain the mechanisms of metacognition. For instance, negative effects of metacognitive processes such as metacognitive awareness or regulation of one's cognitive processes can hinder participants' problem solving efforts as was observed in this study and in the study by DeFranco (1996) and Goos (2002). Teachers and educators may believe that metacognitive behaviors are important during problem solving activities and that these behaviors should be for that reason reinforced in their instruction, however, the results from the study raised important concerns with respect to what extent are metacognitive behaviors desirable or productive. Before promoting students' development of metacognitive capabilities, however, further research on metacognition, is vital to investigate unproductive aspects of metacognitive activity and its influence in obtaining a successful solution. Results of such research could contribute to an improved theory of metacognition.

Contribution of Technology

Even though the work of Schoenfeld (1981, 1985a) in which he examined problem solving of undergraduate students gave me a solid basis to look at metacognitive processes, my study extended the work of Schoenfeld by examining the patterns of cognitive and metacognitive behaviors in a different problem-solving context, that is, in a dynamic geometry environment. It was apparent that both the dynamics of problem solving processes as well as the dynamics between the participant and technology were different for the two participants. Wes perceived GSP as an incredible "tool," an additional resource for working through novel problems. His knowledge of GSP was more generative, he owned more connections, and he had well-connected knowledge that contributed to his effective use of GSP. In addition, he was often able to manage different resources, which was essential for effective problem-solving paths. On the other hand, Aurora perceived it as a "crutch" helping her in working through novel problem-solving situation. Moreover, she lacked the ability to effectively manage her own resources (knowledge, technology) and relied heavily on its use to solve the problem for her. As a consequence, such use was detrimental to quality of her reasoning and outlined plans as she did not take into consideration why she was doing so. Other researchers (e.g., Olive & Makar, 2010) had noted a similar behavior; the user impoverishes the use of the system as he or she takes away from mathematical thinking.

I have also observed that technology seems to permit participants to focus on the larger overarching concept and make connections that would otherwise be lost if technology was not available to help participants attend to details. Both participants shared the belief that GSP was an important and useful tool during problem solving centering around these qualities: problem solving activities and processes, visualization, speed, and accuracy. For instance, it helped explore, gather information, experiment, conjecture, better understand the problem, remember mathematical concepts, aid in attaining accurate visual input and fitting all the pieces together, and triggered possible solution possibilities. Moreover, the feedback provided by the GSP was critical for the participants' later decisions and actions as suggested by other researchers (Hollebrands, 2007; Olive & Makar, 2010; J. W. Wilson et al., 1993). Hence, GSP proved to be an important resource when working on nonroutine problems; it allowed participants to engage in processes, such as pattern recognition, conjecturing, abstracting, and other, and supported flexibility in thinking, transfer of mathematical knowledge to unfamiliar situations and extension of previous knowledge and concepts as reported by Zbiek et al. (2007) when working on conceptual mathematical activity. Metacognitive behaviors exhibited with GSP were oriented by some goals set by the participants to accomplish a particular task. It was hard to distinguish, however, if the effect of tool use on participants' thinking processes or schemes developed by the user during the tool use came first or second. Knowing this would allow better insight to what situations, and circumstances promoted or induced metacognitive behavior to occur. This study has contributed new insights into the problem-solving research integrating both cognitive and metacognitive behavior problem solvers exhibit within each of the seven episodes when problem solving in a dynamic geometry environment.

Limitations of the Study

This multiple case study sought to examine metacognitive processes of two preservice teachers in a dynamic geometry environment. The goal was to examine the phenomenon in depth rather than make generalizations. Because of the uniqueness in design of the study and its methodology, the participants in this study were chosen using purposeful sampling to select cases that met the selection criteria and to provide more description and a better understanding of the patterns of metacognitive behavior during a technology problem-solving experience. However, taking small sample of preservice teachers with extensive experience working in GSP and somewhat varied backgrounds and problem solving experience, not all processes were exhibited nor would they be representative of a large population. Given the nature of the study, however, the findings can still inform the mathematics education community on the issues related to the phenomenon of metacognition and preservice teachers' metacognitive practices during mathematics problem solving.

Another limitation was my potential bias of the participant's capabilities since one of the participants was in a class I co-taught, and I had worked with the other participant on several projects. For example, my experiences with the participants might have made me aware of some problem-solving approaches they have used that I would otherwise not have had inferred from only interview transcripts or participants' work. Hence, my interpretation of the data might have to some extent been affected by my previous experiences with these participants when attempting to infer metacognitive processes and problem-solving episodes so that a same study with a different researcher would elicit different findings. The inferences I have drawn from the data must be considered in terms of the former mentioned aspects.

Even though different types of nonroutine geometry problems were used in the study, the choice of the problems was another limitation of a study; although the problems were carefully chosen and fit the problem selection criteria, the results might be limited to the content area, geometry, and limited to characteristics of these particular problems, and biased by participants' prior knowledge and problem-solving experience.

Multiple methods, both online and offline, were used to provide an accurate description of thinking processes. Even though I closely examined strengths and weakness of each method, I as a researcher have to be critical of my role as an interviewer. Concurrent probing, and my intervention (prompts) during problem-solving sessions might have influenced and stimulated participants' problem solving so that a different study without researcher intervention could elicit different metacognitive processes. Moreover, my presence might have triggered metacognitive behaviors because they might have, for instance, tried harder, were careful what they verbalize, that otherwise would have not be exhibited. Furthermore, verbalizing thought processes during problem solving was hard for both participants, as both of them usually solved problems silently. For that reason participants may have not reported all their thinking, or because of subconscious thinking processes may have remained indiscernible.

Implications for Practice and Recommendations for Research

Research on metacognitive aspects of problem solving had a prominent role in mathematics education research in the 1980s and 1990s. Nowadays, however, educators cannot talk about problem solving without taking into account the role of metacognition during problem solving. Most notably, emphasis is given to inclusion of problem solving in the mathematics curriculum where the role of teachers is not merely to help students solve problems, but to help them learn how to develop processes needed for successful problem solving such as to be able to analyze the relationship among the different problem parts, represent problem situation, decide on the solution path, monitor their progress, check the solution, and evaluate the reasonableness of the solution (NCTM, 2000). Because of my own educational views and suggestions by other researchers (e.g., Kilpatrick, 1985a), explicit instruction on metacognitive practices could inhibit and restrict one's development of metacognitive skills. However, this is an issue that needs further investigation.

Ball (2003) stated, "Teachers need to be people who can work and reason with mathematics, and who possess particular mathematical qualities" (p. 7). Participants in this study reflected on their problem-solving practices in a technology environment, and how they might implement problem-solving in their future classroom; both participants became aware of their problem solving and what actions improved or hindered their own problem solving activity. Preservice teachers before becoming inservice teachers and taking those responsibilities on themselves should have experience in genuine problem solving as well as opportunities to discuss curricular, pedagogical, and learning issues with respect to that mission in variety of contexts. Moreover, taking into consideration that students often imitate their teacher's practices, it is important to improve teachers understanding, use, and promotion of metacognitive practices

(Veenman et al., 2006). Hence, mathematics teacher education programs should allow preservice teachers with opportunities not only to learn about a variety of pedagogical and learning issues, and means for implementing metacognition within their lessons, but to also experience them with respect to metacognitive aspects of problem solving (Veenman et al., 2006).

Taking into consideration the influence of an increasingly global and technological society on teaching practices, teachers need to become aware of the pedagogical and cognitive implications of technology and be able to take advantage of technology as a powerful and engaging teaching tool. The opportunity to experience genuine problem solving, reflect on their metacognitive behaviors that are consistent with the use of the GSP and identify the possible effects they have on mathematical problem solving, teaching, and learning is powerful.

In summary, the findings of this study may be applied to the development of teaching materials for methods and problem-solving courses to help consolidate preservice teachers' problem solving abilities and skills and to facilitate an understanding of their students' metacognitive activity. Characterization of preservice teachers' metacognitive processes may help educators effectively plan, develop and adjust preservice teacher programs to support their development. That is, the findings of this study can be useful in creating teaching and learning environments by revealing factors that considerably affect the metacognitive processes when problem solving in a technology context.

Some of the metacognitive processes and behaviors participants exhibited in this study while problem solving in a technology context were: connecting new information to their former knowledge, planning, monitoring, and evaluating their thinking processes, and accessing, considering and selecting thinking processes, strategies and knowledge. Thus, it is of importance to continue work in understanding what situations and interactions promote metacognitive behaviors to occur to help designing courses that can help preservice teachers learn how to guide themselves and their future students as they solve problems. Results of the study showed that participants' individual experiences affected their metacognitive processes and selection of problem solving strategy that either facilitated or hindered them.

Big research has been undertaken in the last years between neuroscientists and computer scientists to understand how people acquire knowledge, and how these processes and strategies develop with time. Advances in similar research may help answer many open questions regarding metacognition. Moreover, it is important to understand to what extent an individual acts metacognitively. As it was represented through this study, sometimes participants purposefully acted metacognitive, yet sometimes even when they could have they decided not to do so in a technology context. Furthermore, other open questions might include: (1) role of the sequence of problem solving episodes in the solution of a mathematical problem, (2) influence of different types of problems on the problem solving processes, and (3) affect of different types of dynamic environments— namely, deterministic and continuous—on problem solving processes and approaches.

NCTM (2000) calls for research that reexamines what mathematics students should learn as well as student learning because of the existence, versatility, and power of technology. This study only scratched the surface in investigating the ways technology use influences (contributes or impoverishes) to problem solvers metacognitive processes and strategies using Artigue's (2002) description of instrumental genesis. In the process of instrumental genesis both mental schemes and techniques for using an artifact and the understanding embedded into it are developed. Similar research as well as research on influence of degrees of acquisition concerning problem solvers' processes of instrumentalisation and instrumentation on metacognitive processes and strategies needs to be undertaken to enhance our knowledge in instructional technology that is present and continuously transforms mathematics classrooms. According to Goldenberg (2000) technology such as dynamic geometry systems can offer students a different perspective that helps them build a richer and deeper understanding of mathematical topics and may help to improve their problem-solving skills. The implementation of dynamic geometry environments in the classroom affects not only pedagogy, but student cognition as well (Goldenberg, 2000). Students working with this type of technology might show ways of thinking about mathematics that are otherwise difficult to observe. Results of such research could contribute to appropriation of use of emerging technologies for productive problem solving behaviors.

Though this study investigated interplay of metacognitive processes and technology other factors need to be taken into account as well, such as resources, heuristics, beliefs, attitudes, management of affective behaviors to gain better and holistic understanding of this phenomenon. The suggestions for research presented here provide insight into different research directions that may prove to be useful.

[The future mathematician] should solve problems, choose the problems which are in his line, meditate upon their solution, and invent new problems. By this means, and by all other means, he should endeavor to make his first important discovery: he should discover his likes and dislikes, his taste, his own line.

(Pólya, 1945/1973, P. 206)

REFERENCES

Afflerbach, P., & Johnston, P. (1984). Research methodology: On the use of verbal reports in reading research. *Journal of Reading Behavior, 16*, 307–322.

Alexander, J. M., Carr, M., & Schwanenflugel, P. J. (1995). Development of metacognition in gifted children: Directions for future research. *Developmental Review, 15*, 1–37.

Artigue, M. (2002). Learning mathematics in CAS environment: The genesis of a reflection about instrumentation and the dialects between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Artzt, A. F., & Armour-Thomas, E. (1992). Development of a cognitive-metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cognition and Instruction, 9*(2), 137–175.

Ball, D. L. (2003, February). *What mathematical knowledge is needed for teaching mathematics?* Paper presented at the Secretary's Summit on Mathematics, U.S. Department of Education, Washington, DC. Retrieved September 21, 2011 from http://www.ed.gov/inits/mathscience

Berkowitz, E., & Cicchelli, T. (2004). Metacognitive strategy use in reading of gifted high achieving and gifted underachieving middle school students in New York City. *Education and Urban Society, 37*(1), 37–57.

Brown, A. L. (1978). Knowing when, where and how to remember: A problem of metacognition. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 225–253). Hillsdale, NJ: Erlbaum.

Brown, A. L. (1987). Metacognition, executive control, self regulation and other more mysterious mechanisms. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation and understanding* (pp. 65-116). Hillsdale, NJ: Erlbaum.

Campione, J. C., Brown, A. L., & Connell, M. L. (1989). Metacognition: On the importance of understanding what you are doing. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 93–114). Reston, VA: National Council of Teachers of Mathematics.

Cardelle-Elawar, M. (1995). Effects of metacognitive instruction on low achievers in mathematics problems. *Teaching and Teacher Education, 11*(1), 81–95.

Carlson, M. P., & Bloom, I. (2005). The cycle nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*, 45–75.

Charmaz, K. C. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. Thousand Oaks, CA: Sage.

Craine, T. V., & Rubenstein, R. (Eds.). (2009). *Understanding geometry for a changing world* (71st Yearbook of National Council of Teachers of Mathematics). Reston, VA: NCTM.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative research. *Theory Into Practice, 39*(3), 124–130.

Cross, D. I. (2009). Creating optimal mathematics learning environments: Combining argumentation and writing to enhance achievement. *International Journal of Science and Mathematics Education, 7*(5), 905–930.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematics Behavior, 15*, 375–402.

DeFranco, T. C. (1996). A perspective on mathematical problem solving based on the performances of male Ph.D. mathematicians. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (Vol. 6, pp. 195–213). Providence, RI: American Mathematical Society.

DeVilliers, M. D. (1999). Generalizations involving multitudes. *International Journal of Mathematical Education in Science and Technology, 30*(4), 541–548.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: Heath.

Ericsson, K. A., & Simon, H. A. (1980). Verbal reports as data. *Psychological Review, 87*(3), 215–251.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal reports as data* (Rev. ed.). Cambridge, MA: MIT Press.

Fey, J. T., & Hirsch, C. R. (2007). The case of Core-Plus Mathematics. In C. R. Hirsch (Ed.), *Perspectives on the design and development of school mathematics curricula* (pp. 129–142). Reston, VA: National Council of Teachers of Mathematics.

Fey, J. T., Hollenbeck, R. M., & Wray, J. A. (2010). Technology and the mathematics curriculum. In B. J. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics curriculum: Issues, trends, and future directions* (pp. 41–49). Reston, VA: National Council of Teachers of Mathematics.

Flavell, J. H. (1976). Metacognition aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231–236). Hillsdale, NJ: Erlbaum.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive-developmental inquiry. *American Psychologist, 34*(10), 906–911.

Flavell, J. H. (1981). Monitoring social cognitive enterprises: Something else that may develop in the area of social cognition. In J. H. Flavell & L. Ross (Eds.), *Social cognitive development* (pp. 272–287). New York: Cambridge University Press.

Flavell, J. H. (1992). Metacognitive and cognitive monitoring: A new area of cognitive development inquiry. In T. O. Nelson (Ed.), *Metacognition: Core readings* (pp. 3–8). Boston: Allyn & Bacon.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*, 163–176.

Ginsburg, H. P., Kossan, N. E., Schwartz, R., & Swanson, D. (1983). Protocol methods on mathematical thinking. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 7–47). New York: Academic Press.

Goldenberg, E. P. (2000). *Think (and talking) about technology in mathematics classroom.* Retrieved May 6, 2011, from http://www2.edc.org/mcc/pdf/iss_tech.pdf

Goldenberg, E. P. (2001). Getting Euler's line to relax. *International Journal of Computers for Mathematical Learning, 6*(2), 215–228.

Goldenberg, E. P., & Cuoco, A. (1998). What is dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 351–367). Hillsdale, NJ: Erlbaum.

Goldenberg, E. P., Harvey, W., Lewis, P. G., Umiker, R. J., West, J., & Zodhiates, P. (1988). *Mathematical, technical, and pedagogical challenges in the graphical representation of functions.* (ERIC Documentation Reproduction Service No. ED294712)

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning, 2*(3), 209–219.

Goos, M. (2002). Understanding metacognitive failure. *Journal of Mathematical Behavior, 21*(3), 283–302.

Goos, M., & Galbraith, P. (1996). Do it this way! Metacognitive strategies in collborative mathematical problem solving. *Educational Studies in Mathematics, 30*(3), 229–260.

Hackenberg, A. J. (2010). Students' reasoning with reverible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432.

Hadas, N., Hershkowitz, R., & Schwarz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*(1&2), 127–150.

Hoek, D., Eeden, P. v. d., & Terwel, J. (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning and Instruction, 9*, 427–448.

Hohenwarter, M. (2010). GeoGebra (Version 3.2) [Computer software]. Linz, Austria: University of Linz.

Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*(2), 164–192.

Hölzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations — a case study. *International Journal of Computers for Mathematical Learning, 6*(1), 63–86.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 323–349). Dordrecht, the Netherlands: Kluwer Academic.

Jackiw, N. (2001). The Geometer's Sketchpad (Version 4.0) [Computer software]. Emeryville, CA: Key Curriculum Press.

Jackiw, N. (2011). Sketchpad Explorer for iPad [Computer software]. Emeryville, CA: Key Curriculum Press.

Johnston-Wilder, S., & Mason, J. (Eds.). (2005). *Deveploing thinking in geometry*. Thousand Oaks, CA: Sage.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1&2), 55–85.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for Research in Mathematics Education, 8*(3), 163–180.

Kilpatrick, J. (1969). Problem solving and creative behavior in mathematics. In J. W. Wilson & L. R. Carey (Eds.), *Review of recent research in mathematics education studies in mathematics series. Studies in Mathematics Series* (Vol. 19, pp. 153–187). Stanford, CA: School Mathematics Study Group.

Kilpatrick, J. (1985a). Reflection and recursion. *Educational Studies in Mathematics, 16*, 1–26.

Kilpatrick, J. (1985b). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–16). Hillsdale, NJ: Erlbaum.

Kramarski, B., Mevarech, Z. R., & Arami, M. (2002). The effects of metacognitive instruction on solving mathematical authentic tasks. *Educational Studies in Mathematics, 48*, 225–250.

Kroll, D. L. (1988). Cooperative mathematical problem solving and metacognition: A case study of three pairs of women (Unpublished doctoral dissertation). Indiana University, Bloomington.

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics, 44*(1&2), 151–161.

Laborde, J.-M. (2007). Cabri Geometry II Plus (Version 1.4) [Computer software]. Dallas, TX: Cabrilog.

Lagrange, J.-B. (2003). Learning techniques and concepts using CAS: A practical and theoretical reflection. In J. T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R. M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 269–283). Reston, VA: National Council of Teachers of Mathematics.

Lappan, G., Phillips, E. D., & Fey, J. T. (2007). The case of Connected Mathematics. In C. R. Hirsch (Ed.), *Perspectives on the design and development of school mathematics curricula* (pp. 67–79). Reston, VA: National Council of Teachers of Mathematics.

Lawson, M. J., & Chinnappan, M. (1994). Generating activity during geometry problem solving: Comparison of the performance of high-achieving and low-achieving high school students. *Cognition and Instruction, 12*(1), 61–93.

Lawson, M. J., & Chinnappan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education, 31*(1), 26–43.

Lesh, R. (1982). *Metacognition in mathematical problem solving*. Unpublished manuscript. Evanston, IL.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Handbook of research on mathematics teaching and learning* (2nd ed., pp. 763–804). Charlotte, NC: Information Age.

Lester, F. K. (1980). Research on mathematical problem solving. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 286–323). Reston, VA: National Council of Teachers of Mathematics.

Lester, F. K. (1994). Musing about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660–675.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes. Final report.* Indiana University, Bloomington. (ERIC Documentation Reproduction Service ED314255)

Libeskind, Shlomo (2008). *Euclidean and transformational geometry: A deductive inquiry***.** Sudbury, MA: Jones & Bartlett.

Mason, J. (2002). *Qualitative reseraching*. London, UK: Sage.

Mayer, R. E. (1998). Cognitive, metacognitive, and motivational aspects of problem solving. *Instructional Science, 26*(1–2), 49–63.

Mayer, R. E. (2003). Mathematical problem solving. In J. M. Royer (Ed.), *Mathematical cognition* (pp. 69–92). Greenwich, CT: Information Age.

Merriam, S. B. (1998). *Qualitative research and case study applications in education* (2nd ed.). San Francisco: Jossey-Bass.

Mevarech, Z. R. (1999). Effects of metacognitive training embedded in cooperative settings on mathematical problem solving. *Journal of Educational Research, 92*(4), 195–205.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A multidimensional method for teaching mathematics in heterogeneous classrooms. *American Educational Research Journal, 34*(2), 365–394.

Mevarech, Z. R., & Kramarski, B. (2003). The effects of metacognitive training versus worked-out exampls on students' mathematical reasoning. *British Journal of Educational Psychology, 73*, 449–471.

Montague, M., & Applegate, B. (1993). Middle school students' mathematical problem solving: An analysis of think-aloud protocols. *Learning Disability Quarterly, 16*(1), 19–32.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2005). *Technology-supported mathematics learning environments*. Reston, VA: Author.

Newell, A., & Simon, H. A. (1972). *Human problem solving*. Englewood Cliffs, NJ: Prentice Hall.

Olive, J., & Makar, K., (with Hoyos, V., Kor, L. K., Kosheleva, O., & Sträßer R.) (2010). Mathematical knowledge and practices resulting from access to digital technologies. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology – Rethinking the terrain. The 17th ICMI Study* (pp. 133–177). New York: Springer.

Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks, CA: Sage.

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), *Piaget rediscovered: A report of the conference on cognitive studies and curriculum development* (pp. 7-20). Ithaca, NY: School of Education, Cornell University.

Pólya, G. (1973). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press. (Original work published in 1945).

Pólya, G. (1981). *Mathematical discovery: On understanding, learning and teaching problem solving* (Combined ed.). New York: Wiley. (Original work published in 1962).

Pugalee, D. K. (2001). Writing, mathematics, and metacognition: Looking for connections through students' work in mathematical problem solving. *School Science and Mathematics, 101*(5), 236–245.

Reeve, R. A., & Brown, A. L. (1984). *Metacognition reconsidered: Implications for intervention research* (Technical Report No. 328). Urbana-Champaign: University of Illinois at Urbana-Champaign.

Richter-Gebert, J., & Kortenkamp, U. (2008). The interactive geometry software Cinderella (Version 1.4) [Computer software]. Heidelberg, Germany: Springer-Verlag.

Schoenfeld, A. H. (1981, April). *Episodes and executive decisions in mathematical problem solving.* Paper presented at the Annual meeting of the American Educational Research Association, Los Angeles, CA. (ERIC Documentation Reproduction Service No. ED201505)

Schoenfeld, A. H. (1982). Some thoughts on problem solving research and mathematics education. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 27–37). Philadelphia: Franklin Institute Press.

Schoenfeld, A. H. (1985a). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1985b). Making sense of "out loud" problem solving protocols. *Journal of Mathematics Behavior, 4*, 171–191.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well–taught" mathematics courses. *Educational Psychologist, 23*(2), 145–166.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Schoenfeld, A. H. (Ed.). (1994). *Mathematical thinking and problem solving*. Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schoenfeld, A. H., Burkhardt, H., Daro, P., Ridgway, J., Schwartz, J., & Wilcox, S. (1999). *High school assessment*. White Plains, NY: Dale Seymour.

Schraw, G. (1998). Promoting general metacognitive awareness. *Instructional Science, 26*, 113–125.

Schraw, G., & Moshman, D. (1995). Metacognitive theories. *Educational Psychology Review, 7*(4), 351–371.

Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elemntary school mathematics* (pp. 31–43). Reston, VA: National Council of Teachers of Mathematics.

Schunk, D. H. (2008). *Learning theories: An educational perspective* (5th ed.). New York: Prentice Hill.

Silver, E. A. (1982). Knowledge organization and mathematics problem solving. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 15–25). Philadelphia: Franklin Institute Press.

Silver, E. A. (1994). On mathematical problem solving. *For the Learning of Mathematics, 14*(1), 19–28.

Stanic, G., & Kilpatrick, J. (1989). Historical perspectives on problem solving in mathematics curriculum. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–31). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1992). Children's construction of meaning for arithmetical words: A curriculum problem. In D. Tirosh (Ed.), *Implicit and explicit knowledge: An educational approach* (pp. 131–168). Norwood, NJ: Ablex.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Thompson, P. W. (2009). Algebra is more than symbolic. Discussion of J. Kilpatrick's Plenary address delivered at the Annual Meeting of the International Group for the Psychology of Mathematics Education-North America, Atlanta, GA.

Veenman, M. V. J., Van Hout-Wolters, B. H. A. M., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodologial considerations. *Metacognition Learning, 1*, 3–14.

Von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In F. Bensler, P. M. Hejl, & W. K. Kock (Eds.), *Autopoesis, communication, and society* (pp. 75–85). Frankfurt, Germany: Campus Verlag.

Von Glasersfeld, E. (1982). An interpretation of Piaget's constructivism. *Revue Internationale de Philosophie, 36*(4), 612–635.

Von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17–40). New York: Norton (Originally published in P. Watzlawick (Ed.), *Die Erfundene Wirklichkeit*. Munich: Piper, 1981)

Wilson, J., & Clarke, D. (2002, April). *Monitoring mathematical metacognition.* Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, LA. (ERIC Documentation Reproduction Service No. ED465519)

Wilson, J., & Clarke, D. (2004). Towards the modelling of mathematical metacognition. *Mathematics Education Research Journal, 16*(2), 25–48.

Wilson, J. W. (n.d.). Problem solving in mathematics. Retrieved May 6, 2011, from http://jwilson.coe.uga.edu/emt725/emt725.html

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57–77). New York: Macmillan.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. (2007). Research on technology in mathematics education: A perspective of constructs. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte, NC: Information Age.

Zbiek, R. M., & Hollebrands, K. (2008). A research-informed view of the process of incorporating mathematics technology into classroom practice by inservice and prospective teachers. In M. K. Heid & G. W. Blume (Eds.), *Research syntheses: Research on technology and the teaching and learning of mathematics* (Vol. 1, pp. 287–344). Charlotte, NC: Information Age.

Zimmerman, B. J. (2002). Becoming a self-regulated learner: An overview. *Theory Into Practice, 41*(2), 64–70.

APPENDIX A

PRELIMINARY INTERVIEW PROTOCOL

Part 1: Background Information

1. Tell me something about yourself (name, year you are in).
2. What are your interests and hobbies?
3. What are your hopes and aspirations in life?
4. Tell me about your experience of studying at the University of Georgia so far?
5. What has been you favorite course you have taken in the mathematic education program and/or as an undergraduate student? Why? What has been your least favorite course you have taken in the mathematics education program and/or as an undergraduate student? Why?
6. What is mathematics problem solving from your point of view?
7. Tell me about your mathematics problem solving experience (in high school, in college).
8. Do you like to solve mathematical problems? If so, why?
9. What kind of mathematical problems do you like to solve? (e.g., routine problems, word problems, nonroutine problems, applied problems, etc.) Why?
10. What kind of problems are you good at? Why? What kind of problems are you bad at? Why?

11. Suppose you are given a problem. How do you go about solving the problem? (e.g., refer to material, ask for help, draw diagrams, etc.) What do you do when you get stuck solving a problem? When you solve a problem, do you try to solve it in a different way? When you miss on a problem, what is the reason for that?
12. Do you think you are a good problem solver? Why? Why not?

Part 2: Learning Mathematics

1. Tell me how you like to learn mathematics. (e.g., Do you like procedures, theorems? Are you a visual learner? Do you like to read textbooks? Does practice help you?)
2. What do you like about mathematics and learning mathematics?
3. What is your most liked aspect of learning mathematics? Your least liked?

Part 3: Technology Learning Experience and Perceptions

1. Tell me about your experience of using technology in learning mathematics? What types of technology have you used in learning mathematics (in high school, in college)?
2. What are the benefits of using technology in learning mathematics? What are the drawbacks of using technology in learning mathematics?
3. How does technology help you when solving a mathematical problem? How important do you view technology in solving problems?
4. How do you decide when to use technology when problem solving?
5. Describe an ideal mathematical technology learning situation. What is your role? What is happening in that situation?

APPENDIX B

PROBLEM-SOLVING INTERVIEW PROTOCOL

Part 1: Reflecting on the Problem Solving Session

A. Reading and Understanding the Problem

1. Have you ever seen a problem similar to this one before? If yes, how did it affect how you solved the problem?
2. Before you started solving the problem, how difficult did you think the problem was on a scale from 1 to 5? When you were solving the problem? At the end?
3. Did you have any difficulty understanding the problem information? If yes, explain what parts confused you.
4. Did you focus on any key words and/or aspects of the problem? If yes, explain what did you focus on.
5. Did you try to represent the problem by writing down relevant information and/or making a representation of it? If yes, explain why and how.

B. Analyzing the Problem

6. After you read the problem, did you try to simplify it? Explain.
7. After you read the problem, did you already have an idea how to solve it? Explain.

C. Exploring the Problem

8. How did you plan to solve the problem at the beginning? Explain.
9. What were the strategies you relied on when searching how to solve the problem? How did you know these strategies would be useful? Explain.
10. What mathematical content did you rely on when searching how to solve the problem? How did you know that mathematical content would be useful? Explain.
11. What mathematical procedures did you plan to use to solve the problem? Explain.

D. Planning To Solve the Problem

12. Explain your solution plan, both steps and strategies, to me.
13. Did you assess your plan? Explain.

E. Implementing the Solution Plan

14. How did you decide to implement your solution plan?
15. Did you follow your solution plan? Explain.
16. What strategies were important for you to solve the problem? Explain.
17. What mathematical content was important for you to solve the problem? Explain.
18. What mathematical procedures were important for you to solve the problem? Explain.
19. What were the most important things that helped you solve the problem? Explain.
20. What did you do when you got stuck when solving the problem?

F. Verifying the Solution of the Problem

21. How did you know you solved the problem?

22. How can you be sure that your solution is correct?

23. Did you check for the reasonableness of the result or alignment of the solution and the conditions of the problem? Explain, why or why not.

G. Miscellaneous

24. What did you learn from solving this problem? Explain.

25. What questions did this problem raise for you? Explain.

Part 2: Reflecting on Problem Solving in a Dynamic Geometry Environment

1. How would you describe interactions you had with the GSP?
2. What interactions did you have with the GSP? What were the GSP features that you used in solving the problem?
3. What role did technology play in solving the problem?
4. How important was technology in solving the problem?
5. How did the use of technology help your mathematical problem solving? How did the use of technology hinder your mathematical problem solving?
6. How did interactions with the GSP influence your problem solving?
7. Would you have solved the problem in a similar way if you had not had GSP? What would you have done differently? Explain.

APPENDIX C

FINAL INTERVIEW PROTOCOL

Directions: Please complete this on your own. The two sections below are a reflection on problem solving sessions as a result of participating in the study *"Preservice Teachers' Patterns of Metacognitive Behavior During Mathematics Problem Solving in a Dynamic Geometry Environment."* Try to be specific, referring to a specific problem solving session when possible. Please be honest.

Part 1: Reflecting on the Problem Solving Sessions

1. Here is a list of problem you solved (The Longest Segment Problem, The Airport Problem, The Land Boundary Problem). If you were to select one or two as your favorite(s), which one would it be? Why? What do you think are the (dis)advantages of using GSP to solve this/these problem?

2. Think of a time during problem solving session when you were successful in problem solving and tell me about that time. Why do you think you were successful? What did you like about your problem solving? How did you feel about that time?

3. Think of a time during problem solving session where you were not successful in problem solving and tell me about that time. Why do you think you were unsuccessful? What did you not like about your problem solving? How did you feel about that time? What did you do in that situation?

4. What and how would you change your problem solving to be effective, that is, to improve your problem solving? (You are more then welcomed to answer this question in relations to your use of GSP.)

5. What did you learn while and/or as a result of participating in this study? (This could be many things: mathematics content, how you think mathematically, ability to problem solve, new strategies, change of views on problem solving, change of views on problem solving using technology, interpretation of mathematical

problem/mathematical problem solving, change of attitude towards mathematics, change of attitude towards mathematical problem solving, change of attitude towards the use of technology, etc.).

6. As you think about the things we focused on in this study, how might this experience influence your future problem solving? Your future teaching of problem solving?

Part 2: Reflecting on Problem Solving in a Dynamic Geometry Environment

1. How would you describe your relationship between problem solving and GSP when faced with nonroutine geometry problems?

2. Did you stand on using technology when problem solving change as a result of participating in this study? If so, how? If not, why not?

3. Thinking about the limitations of just paper and pencil, do you think working in technology environment, namely GSP, helped you with your problem solving? (e.g., What made solving the problem in a technology context difficult? Easy? What were the advantages of working in a technology context on these problems? Disadvantages?, etc.).

4. What suggestions would you make to another person when problem solving in a technology environment?

THANK YOU FOR YOUR PARTICIPATION!