Stochastic Control and Optimization of Assets Trading

by

H. Tin Kong

(Under the direction of Qing Zhang)

Abstract

Stochastic optimization is an optimization method which involves probabilistic ingredients, such that there is random noise in the problem objectives or constrains, and/or there is randomness in the algorithm when making choice in the search direction. Due to the unpredictable nature of financial markets, stochastic optimization has been drawing gradually greater attention in the field. This dissertation presents stochastic optimization methods from different approaches on two common models in financial markets, namely mean reversion model and regime-switching model.

When the price of an asset is governed by mean reversion model, the objective of an investor is to find the threshold buy and sell prices such that the overall return (with slippage cost imposed) is maximized. This work provides those threshold prices that allows buying, selling and short selling of an asset. A dynamic programming approach is employed to ensure the optimality in the first part of the dissertation. It shows that the solution of the original optimal stopping problem can be achieved by solving four algebraic equations. In the last part of the dissertation, a stochastic approximation approach is implementated on the same problem for comparison. A recursive algorithm is designed to determine the threshold prices. In both aproaches, numerical examples such as Monte Carlo simulations and real market data are given for demonstration.

Considering trend-following trading strategies that are widely used in the investment world, the second part of this work provides a set of sufficient conditions that determine the optimality of the traditional trend-following strategies when the trends are completely observable. Again, a dynamic programming approach is used to verify the optimality under these conditions. The value functions are shown to be either linear functions or infinity depending on the parameter values. The results even reveal some counter-intuitive facts.

INDEX WORDS: Optimal stopping, Stochastic control, Stochastic Approximation, quasi-variational inequalities, trend following, regime-switching, mean-reverting asset

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Dedication

To my grandparents.

To my parents, Bun Yu Tung and Wing Yue Kong.

To my soulmate and wife, Hui.

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Chapter 1

Preliminaries

For the previous forty years, theory of finance has been emergencing to a new scientific discipline. Since the Black-Scholes [2] model was published in 1973, mathematical finance has become a branch of applied mathematics concerned with financial markets. The original model that Black and Scholes used to discribe the behavior of asset price was called the geometric Brownian motion (GBM) which can be represented by a stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{1.1}$$

where X_t is the price of an asset under consideration, μ is the expected return of the asset, σ is the volatility of the asset price and W_t is a standard Brownian motion. In the terminology of stochastic analysis, the term with dt is called the drift term and the term with dW_t is called the diffusion term.

Due to the complexity of the financial market, the GBM is not sufficient to capture the market behaviors. As a result, various models have been developed for the past few decades. This dissertation is concerned with optimal trading on two different models, namely mean reversion model and regime-switch model. The goal is to find the best strategy that maximize the overall profit. We begin by an introduction to those models.

1.1 Mean Reversion Model

A mean reversion model, also known as Ornstein−Uhlenbeck model, is often used in financial and energy markets to capture price movements that have the tendency to move towards an "equilibrium" level. It is one of the serval approaches used to model the stochastic behavior of interest rates, currency exchange rates, and commodity prices. The behavior of the model can be described by a stochastic differential equation

$$dX_t = a(L - X_t)dt + \sigma dW_t \tag{1.2}$$

where X_t is log of the price of an asset S_t; a is the reversion rate and L is the equilibrium level, σ and W_t are the volatility and standard Browian motion respectively as in GBM (1.1).

From (1.1), it is clear that the sign of the drift term is independent of the process. In contrast to the GBM, the drift term of mean reversion model can have different signs depend on the current value of the process. If the current value of the process X_t is less than the equilibrium level L, the drift is positive; if the current value of X_t is greater than L, the drift is negative. In other words, the value of the process X_t has a tendency to increase when it is below L and decrease when it is above. Furthermore, the farther the value of X_T from the equilibrium level, the higher the tendency to move back to the equilibrium level. This is one of the reason that mean reversion model is popular on modeling commodity prices and interest rates. When the demand is high, the price is high, but the high price suppress the demand and thus lower the price. As a reult, the price cannot be too far from the equilibrium level.

Nonetheless, there is a deficiency on modeling prices or interest rates. From the stochastic differential equation (1.2), it is not hard to see that the value of X_T could be negative, which is inappropriate in practice. Therefore, a modified version of mean reversion model is offen used. In our model (1.2), we let $X_T = \ln(S_T)$ on (1.2) to ensure that the price S_T of an asset at time T is non-negative. For interest rate models such as Cox-Ingersoll-Ross [9] model (CIR model), they describe the instantaneous interest rate r_t by the following stochastic differential equation,

$$dr_t = a(L - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

The diffusion term contains $\sigma\sqrt{r_t}$ that avoids the possibility of negative interest rates.

The stochastic differential equation (1.2) of the mean reversion model is solved by applying the following crucial theorem independently by Kiyoshi Ito [25] and Wolfgang Doeblin.

Theorem 1.1.1 (Ito-Doeblin Formula)

Let X_t be an Ito process given by

$$dX_t = u(t, X_t)dt + v(t, X_t)dW_t.$$

Let $f(t,x) \in C^2([0,\infty) \times R)$. Then

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the quadratic variations of t and W_t,

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \qquad dW_t \cdot dW_t = dt.$$

We proceed to solve (1.2). Let $f(x,t) = xe^{at}$. By Ito-Doeblin formula,

$$\begin{aligned} dX_t e^{at} &= e^{at}dX_t + aX_t e^{at}dt \\ &= e^{at}(a(L - X_t)dt + \sigma dW_t) + aX_t e^{at}dt \\ &= e^{at}aLdt + \sigma e^{at}dW_t. \end{aligned}$$

Integrate from t to T on both sides,

$$\begin{aligned} X_T e^{aT} - X_t e^{at} &= L(e^{aT} - e^{at}) + \sigma \int_t^T e^{as}dW_s \\ X_T &= X_t e^{a(t-T)} + L(1 - e^{a(t-T)}) + \sigma \int_t^T e^{a(s-T)}dW_s. \end{aligned}$$

If $t = 0$, then

$$X_T = X_0 e^{-aT} + L(1 - e^{-aT}) + \sigma \int_0^T e^{-a(s-T)}dW_s. \tag{1.3}$$

Thus, the expectation of X_T is given by

$$\mathsf{E}(X_T) = x_0 e^{-aT} + L(1 - e^{-aT}),$$

where $X_0 = x_0$ is a constant. Moreover, we can calculate the covariance of X_s and X_t by Ito isometry,

$$\begin{aligned}
\mathsf{Cov}(X_s, X_t) &= \mathsf{E}[(X_s - \mathsf{E}[X_s])(X_t - \mathsf{E}[X_t])] \\
&= \mathsf{E}\left[(\sigma \int_0^s e^{a(u-s)} dW_u)(\sigma \int_0^t e^{a(v-t)} dW_v)\right] \\
&= \sigma^2 e^{-a(s+t)} \mathsf{E}\left[\int_0^s e^{au} dW_u \int_0^t e^{av} dW_v\right] \\
&= \sigma^2 e^{-a(s+t)} \int_0^{\min(s,t)} \mathsf{E}[e^{2az}] dz \\
&= \frac{\sigma^2}{2a} e^{-a(s+t)} (e^{2a\min(s,t)} - 1).
\end{aligned}$$

It follows that

$$\mathsf{Var}(X_t) = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

By (1.3), it is clear that a mean reversion process X_T is normally distributed. Therefore,

$$X_T \sim N\left(x_0 e^{-aT} + L(1 - e^{-aT}), \frac{\sigma^2}{2a}(1 - e^{-2at})\right). \tag{1.4}$$

Since the mean reversion model (1.2) has a closed form solution, we can simulate the process with arbitrary time steps. From (1.4), the process can be simulated by

$$X_{t+\Delta t} = X_t e^{-a\Delta t} + L(1 - e^{-a\Delta t}) + \left(\sigma \sqrt{\frac{1 - e^{-2a\Delta t}}{2a}}\right) z,$$

where $z \sim N(0, 1)$ and Δt is the step size.

Figure 1.1 shows the sample path of a mean-reverting process with $a = 0.8$, $L = 2$ and $\sigma = 0.5$, where the number sample points is 10000 and step size is $\frac{1}{252} = 0.003968$. As stated in (1.2), $X_t = \ln(S_t)$. The actual prices S_t can be calculated by $S_t = e^{X_t}$, as shown in figure 1.2. It simulates the prices of a mean-reveting asset for about 40 years with 252 trading days per year.

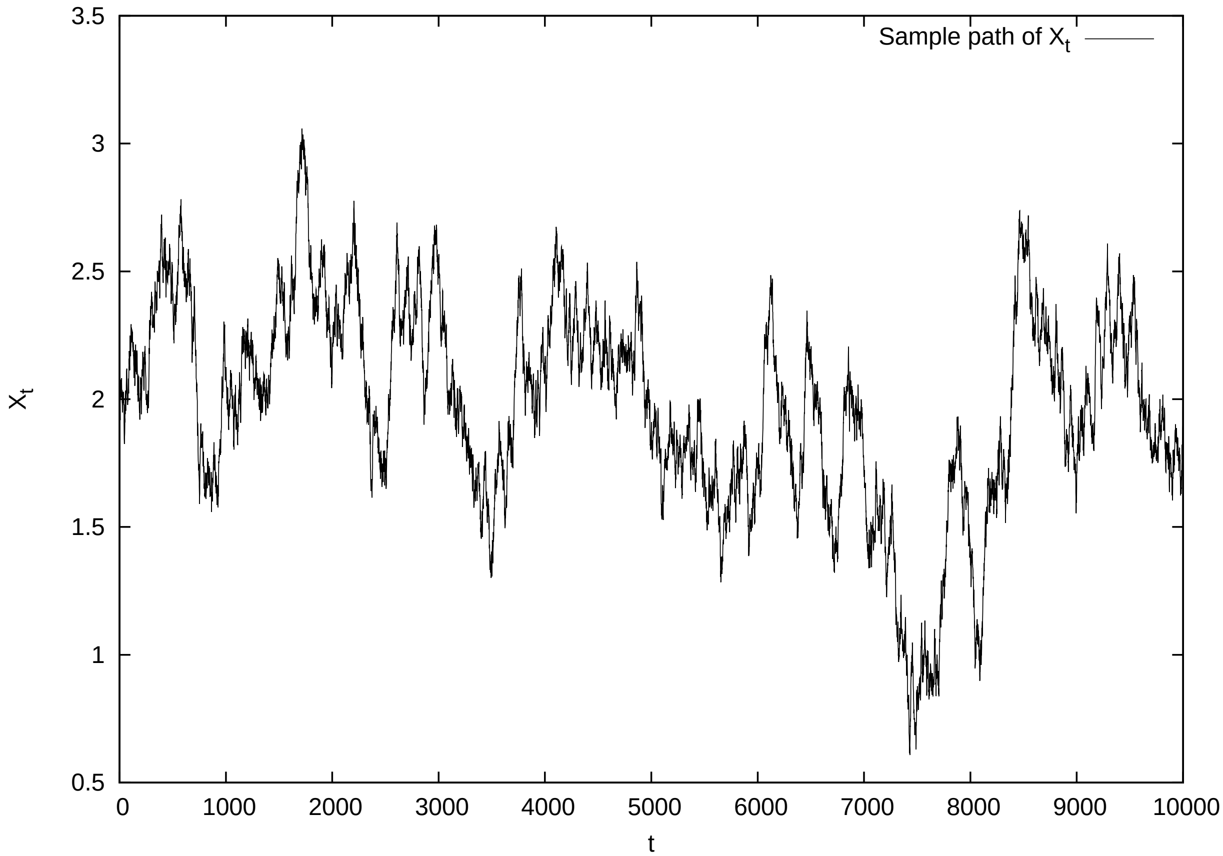

Figure 1.1: Sample path of a mean-reverting process X_t

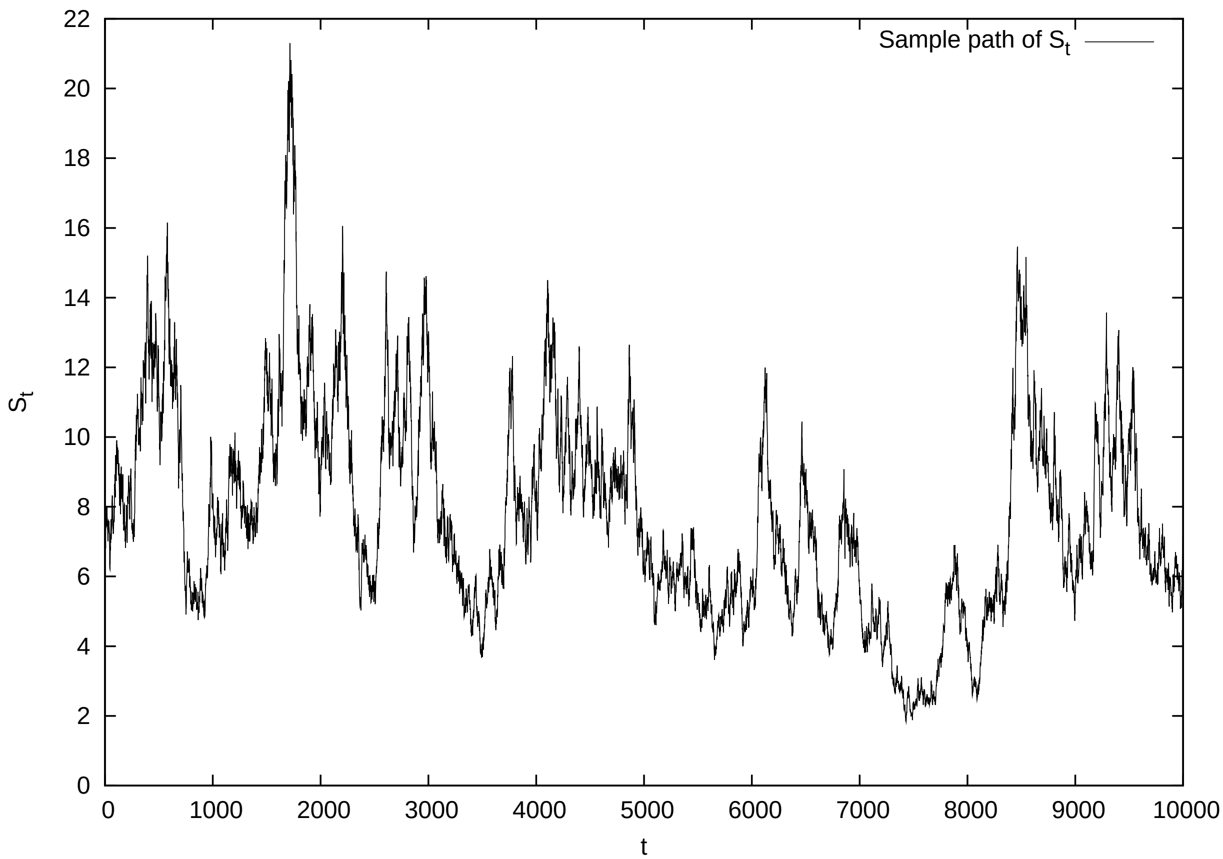

Figure 1.2: Sample path of $S_t = e^{X_t}$

For calibration of the parameters of a mean reversion model, we implement the discretized version of (1.3),

$$X_{i+1} = X_i e^{-a\Delta t} + L(1 - e^{-a\Delta t}) + \left(\sigma\sqrt{\frac{1 - e^{-2a\Delta t}}{2a}}\right) z, \tag{1.5}$$

where $\{X_i\}$ are the observation points such as the stock price on each trading day and Δt is a time step such as one trading day. It is clear that the relation between consecutive observations X_i and X_{i+1} is linear with IID normal random error z, which can be discribed by a first order autoregression,

$$X_{i+1} = \beta_0 + \beta_1 X_i + \epsilon, \tag{1.6}$$

where $\beta_0 = L(1 - e^{-a\Delta t})$, $\beta_1 = e^{-a\Delta t}$ and $\epsilon \sim N\left(0, \sigma^2 \frac{1 - e^{-2a\Delta t}}{2a}\right)$. With a collection of observations $\{X_i\}$, β_0 and β_1 can be estimated by linear regression and the model parameters can be calculated by

$$\begin{aligned} a &= \frac{-\ln \beta_1}{\Delta t}, \\ L &= \frac{\beta_0}{1 - \beta_1}, \\ \sigma &= \mathrm{sd}(\epsilon)\sqrt{\frac{-2\ln \beta_1}{\Delta t(1 - \beta_1^2)}}, \end{aligned} \tag{1.7}$$

where $\mathrm{sd}(\epsilon)$ is the standard derivation of ϵ.

Although mean reversion model is mainly employed in commodity prices and interest rates, it is not uncommon to find mean reversion in stock markets. Figure 1.3 shows the stock prices of Advanced Micro Devices, Inc. Co (NYSE: AMD) in the period 1983 - May 2010 and figure 1.4 shows the corresponding prices in log scale. Both demonstrate resemblance of mean reversion.

Figure 1.3: AMD stock prices 1983 - May 2010 (Courtesy of Yahoo Finance)

Figure 1.4: AMD stock prices in log-scale 1983 - May 2010 (Courtesy of Yahoo Finance)

1.2 Regime-Switching Model

Since the introduction of the celebrated Black-Scholes model, which assumes geometric Brownian motion (1.1) on stock prices, there is an explosive growth in derivatives trading on financial markets. Nevertheless, soon enough people realized that the geometrix Brownian motion fail to capture the changes and complexities of the markets due to the assumption of constant parameters. As a result, the Black-Scholes model has been extended in various directions.

Regime-switching model (or Markov regime-switching model) is one of the common generalization of the geometric Brownian motion. The idea was first introduced by Hamilton [23] in 1989. The model he was considering is a first order autoregression. He suggested that the parameters of the model should vary according to market events such as financial crisis or abrupt changes in government policies. To do this, the events and changes can be represented by a finite state Markov chain which is independent from the stock prices, and the parameters take values according to the Markov chain.

The idea of Markov regime-switching can be applied to different models. For an Ito process,

$$dX_t = u(t, X_t)dt + v(t, X_t)dW_t,$$

we can couple the process X_t with a finite state Markov chain so that u and v depend on the Markov chain as well. Let $\alpha(t)$ be a finite states Markovian chain with states $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and the generator $Q = \{q_{ij}\}_{n\times n}$ is given by

$$\mathsf{P}(\alpha(t+\delta) = j \mid \alpha(t) = i) = \begin{cases} q_{ij}\delta + O(\delta), \\ 1 + q_{ii}\delta + O(\delta), \end{cases}$$

where $\delta > 0$. Here q_{ij} is the transition rate from i to j if $i \neq j$ while

$$q_{ii} = -\sum_{j\neq i} q_{ij}.$$

The Ito process with Markov chain $\alpha(t)$ is given by

$$dX_t = u(t, X_t, \alpha(t))dt + v(t, X_t, \alpha(t))dW_t.$$

The Ito-Doeblin formula for an Ito process X_t shows that for a function $f \in C^2([0,\infty) \times R)$, $f(t, X_t)$ is also an Ito process. Likewise, for an Ito process X_t with Markov chain $\alpha(t)$, $f(t, X_t, \alpha(t))$ is an Ito process. The corresponding Ito-Doeblin formula with Markovian switching is the following.

Theorem 1.2.1 (Generalized Ito-Doeblin Formula)
Let $\alpha(t)$ be a Markov chain with states $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and generator $Q = \{q_{ij}\}_{n\times n}$. Let X_t be an Ito process with Markov chain $\alpha(t)$ given by

$$dX_t = u(t, X_t, \alpha(t))dt + v(t, X_t, \alpha(t))dW_t.$$

Let $f(t, x, \cdot) \in C^2([0,\infty) \times R)$. Then

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + Qf(t, X_t, \cdot)(\alpha(t)) \tag{1.8}$$

where

$$Qf(t, X_t, \cdot)(\alpha_i) = \sum_{j\neq i} q_{ii}(f(t, X_t, \alpha_j) - f(t, X_t, \alpha_i)),$$

and $(dX_t)^2 = (dX_t)\cdot(dX_t)$ is computed according to the quadratic variations of t and W_t,

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \qquad dW_t \cdot dW_t = dt.$$

In this dissertation, we focus on a geometric Brownian motion with Markovian switching. The parameters in (1.1) are no longer assumed to be constant. Instead, their values rely on a two-states continuous time Markov chain.

Let X_t be a regime-switching geometric Brownian motion governed by

$$dX_t = X_t(\mu(\alpha_t)dt + \sigma(\alpha_t)dW_t), \tag{1.9}$$

where $\alpha_t \in \{1, 2\}$ is a two-state Markov chain with generator given by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},$$

$\mu(1) = \mu_1$, $\mu(2) = \mu_2$ are the expected return rates, $\sigma(1) = \sigma_1$ and $\sigma(2) = \sigma_2$ are the volatilities, and W_t is a standard Brownian motion. We can solve (1.9) by the generalized Ito-Doeblin formula (1.2.1).

Let $f(t, x, \cdot) = \ln(x)$. By the generalized Ito-Doeblin formula,

$$\begin{aligned}
d\ln(X_t) &= \frac{1}{X_t}dX_t + \frac{1}{2}\left(\frac{-1}{X_t^2}\right)\sigma^2(\alpha_t)X_t^2 dt + Q\ln(X_t) \\
&= \frac{1}{X_t}(X_t(\mu(\alpha_t)dt + \sigma(\alpha_t)dW_t)) - \frac{1}{2}\sigma^2(\alpha_t)dt + Q\ln(X_t) \\
&= \mu(\alpha_t)dt + \sigma(\alpha_t)dW_t - \frac{1}{2}\sigma^2(\alpha_t)dt + Q\ln(X_t) \\
&= \left(\mu(\alpha_t) - \frac{1}{2}\sigma^2(\alpha_t)\right)dt + \sigma(\alpha_t)dW_t,
\end{aligned}$$

as $Q\ln(X_t) = 0$. Integrate from t to T on both sides,

$$\begin{aligned}
\ln(X_T) - \ln(X_t) &= (\mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s))(T-t) + \sigma(\alpha_s)(W_T - W_t) \\
\ln(X_T) &= \ln(X_t) + (\mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s))(T-t) + \sigma(\alpha_s)(W_T - W_t) \qquad (1.10) \\
X_T &= X_t e^{\left(\mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s)\right)(T-t) + \sigma(\alpha_s)(W_T - W_t)}.
\end{aligned}$$

If $t = 0$, then

$$X_T = X_0 e^{\left(\mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s)\right)T + \sigma(\alpha_s)W_T}, \qquad (1.11)$$

i.e.

$$X_T = \begin{cases} X_0 e^{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 W_T} & \text{if } \alpha_s = 1, \\ X_0 e^{\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)T + \sigma_2 W_T} & \text{if } \alpha_s = 2. \end{cases}$$

From (1.10), we can see that $\ln(X_t)$ is normally distributed,

$$\ln(X_t) \sim N\left(\ln(X_0) + \mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s)t, \sigma^2(\alpha_s)t\right).$$

Hence X_t is log-normally distributed. For a log-normally distributed random variable $Y = e^Z$ with $Z \sim N(\mu, \sigma^2)$, the mean $\mathsf{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ and the variance $\mathsf{Var}(Y) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$. Therefore, the mean and variance of a geometric Brownian motion X_t with Markov chain α_t are

$$\begin{aligned}
\mathsf{E}(X_t) &= X_0 e^{\mu(\alpha_t)}, \\
\mathsf{Var}(X_t) &= X_0^2 e^{2\mu(\alpha_t)t}(e^{\sigma^2(\alpha_t)t-1}).
\end{aligned}$$

By the closed form solution in (1.11), we can simulate the process by

$$X_{t+\Delta t} = X_t e^{\left(\mu(\alpha_s) - \frac{1}{2}\sigma^2(\alpha_s)\right)\Delta t + \sigma(\alpha_s)z\Delta t}$$

where $z \sim N(0,1)$ and Δt is the time step.

Figure 1.5 shows the sample path of a regime-switching geometric Brownion motion with $\mu_1 = 0.7, \mu_2 = -0.4, \sigma_1 = 0.2, \sigma_2 = 0.6, \lambda_1 = 2, \lambda_2 = 3$, where the time step is $\frac{1}{252} = 0.003968$. It simulates the prices of a regime-switching asset for over 30 years with 252 trading days per year. Clearly, model in (1.9) and mean reversion model are different. However, if we consider a case of regime-switching geometric Brownian motion with the same volatilities σ's and transition rate λ's (hence same duration of regimes), and same magnitude of μ's but opposite signs on both regimes, it represents an asset price that goes up and down alternatively which is also the behavior of mean-reversion. Figure 1.6 demonstrates the sample path of such a case and reveals significant difference from mean reversion. The reason is that a regime-switching asset has no tendency to the equilibrium level, so the prices can flustrate dramatically.

Assets with regime-switching geometric Brownian motion are ubiquitous in financial markets. Figure 1.7 shows an example, Golden Sachs in the period 1999 - May 2010.

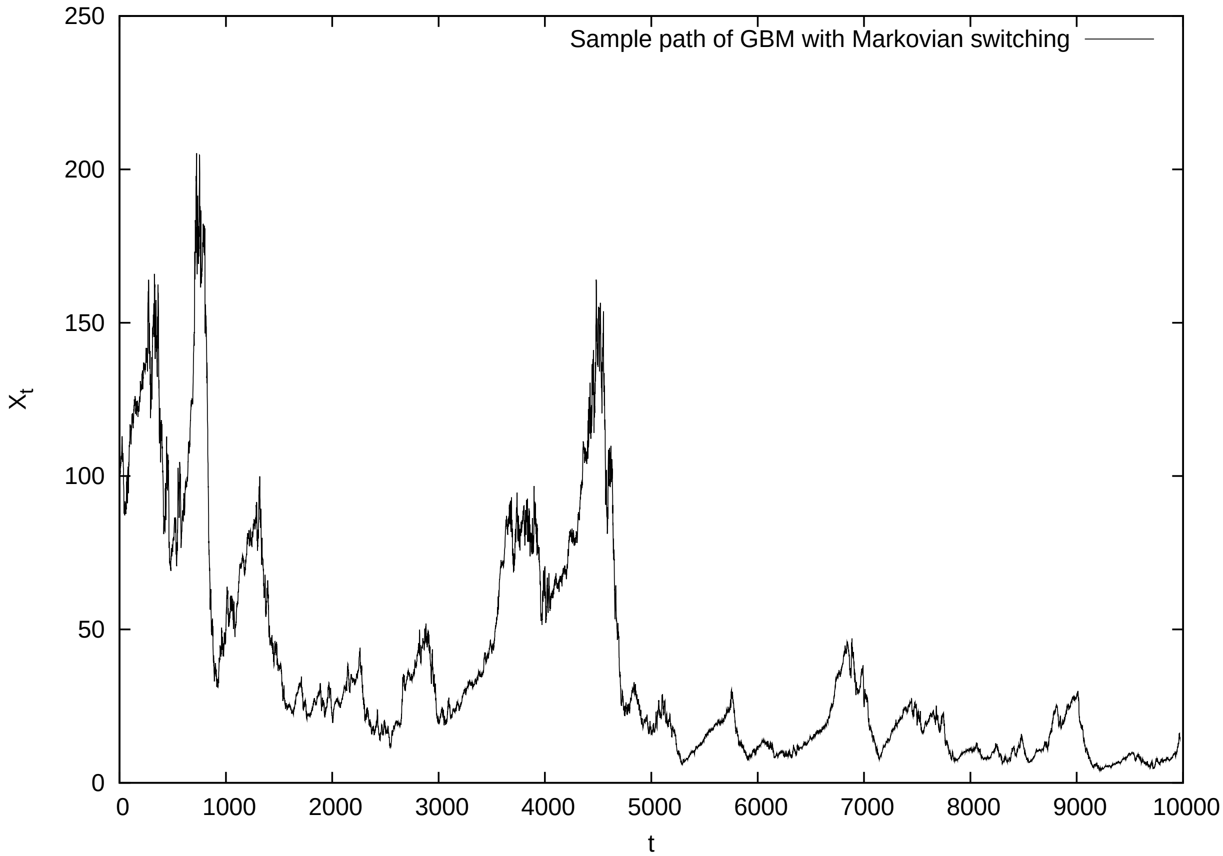

Figure 1.5: Sample path of a regime-switching process X_t with $\mu_1 = 0.7, \mu_2 = -0.4, \sigma_1 = 0.2, \sigma_2 = 0.6, \lambda_1 = 2, \lambda_2 = 3$

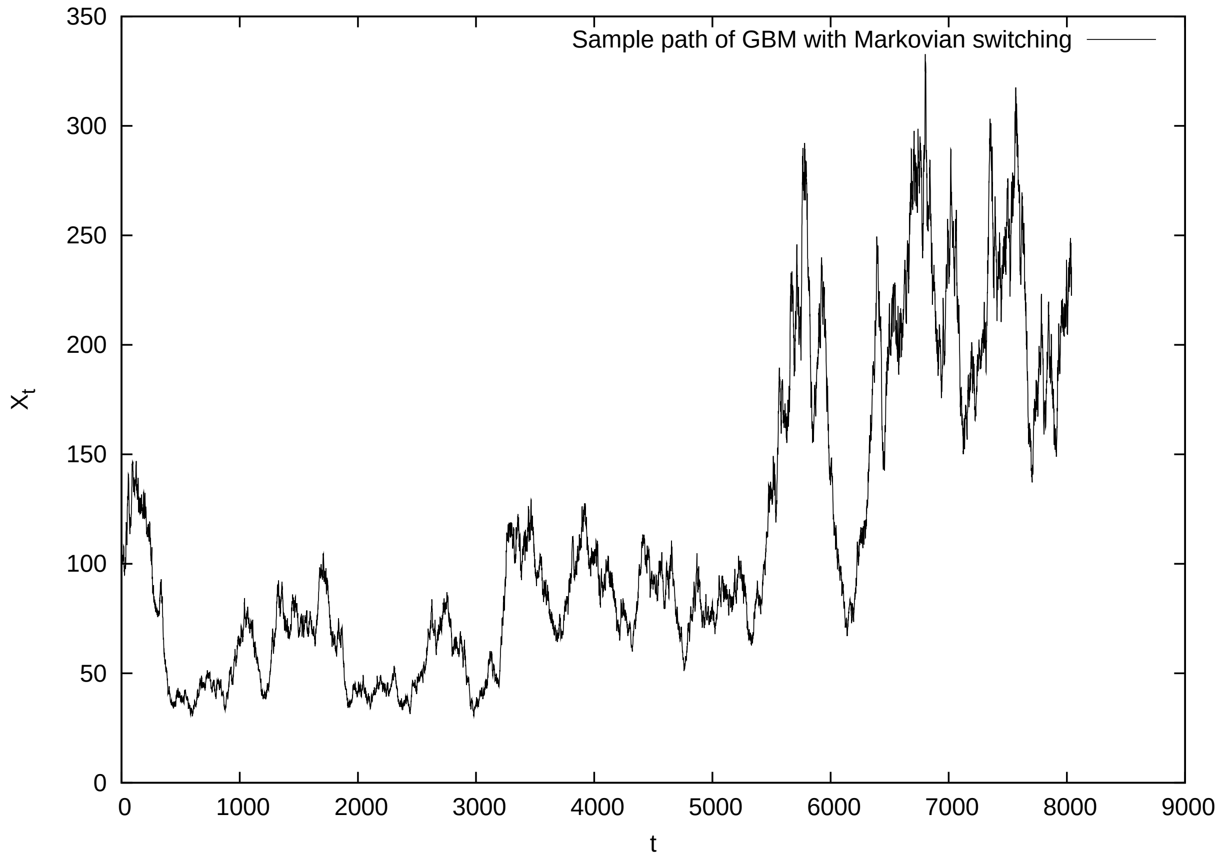

Figure 1.6: Sample path of a regime-switching process X_t with $\mu_1 = 0.5, \mu_2 = -0.5, \sigma_1 = 0.4, \sigma_2 = 0.4, \lambda_1 = 3, \lambda_2 = 3$

Figure 1.7: Goldman Sachs 1999 - May 2010 (Courtesy of Yahoo Finance)

1.3 Outline

We present the dissertation in the following order.

In chapter 2, we reveal an optimal trading rule that allows buying, selling and short selling of an asset when its price is governed by mean-reverting model. The goal is to find the buy and sell prices such that the overall return (with slippage cost imposed) is maximized. This chapter shows that the solution of the original optimal stopping problem can be achieved by solving four algebraic equations. Numerical examples are given for demonstration.

In chapter 3, we investigate a trading strategy under regime-switching model, namely trend following strategy. This chapter provides a set of sufficient conditions that determine the optimality of the traditional trend following strategies when the trends are completely observable. The value functions are shown to be either linear functions or infinity depending on the parameter values. The results reveal two counter intutive facts: (a) trend following may not lead to optimal reward in some cases even the investor knows exactly when a trend change occurs; (b) stock volatility is not relevant in trend following when trends are observable.

In chapter 4, a stochastic approximation algorithm is developed to estimate the buy and sell prices. The model under consideration and the objective are the same as those in chapter 2. However, instead of the theoretical approach, a recursive approach is presented. Numerical examples in chapter 2 are re-calculated in this chapter by the stochastic approximation algorithm for comparison. Moreover, real market data are experimented for demonstration. The algorithm indicates a weaker model requirement and faster computation compare to the method in chapter 2.

Chapter 2

An Optimal Trading Rule of a Mean-Reverting Asset

2.1 Introduction

In this chapter we study the optimal trading rule of an asset with random fluctuation in its price. In particular, we consider the trading rule which involves three aspects: buying, selling and shorting. Shorting or short selling is the practice of selling financial securities the seller does not then own, in the hope of repurchasing them later at a lower price. A typical trading strategy in financial markets is to buy low then sell high, or sell high then buy low if one is selling short. However, identifying these buy and sell prices poses a great challenge. Our objective is to determine those threshold price levels when the behavior of the asset price follows a mean-reversion model.

Mean-reversion model is frequently employed in financial and commodity markets to characterize price movements that have inclination to move towards a median line. A decent amount of studies has been done in this model (e.g. Fama and French [18], Gallagher and Taylor [20]). It is also used to model interest rate and energy price (see Blanco and Soronow [3] and Jong and Huisman [16]). Apart from that, there are some results in option pricing for mean-reverting asset (see Bos, Ware and Pavlov [5]).

A significant volume of literature was concerned with trading rules in financial markets; see for instance, Zhang [41] and Guo and Zhang [22]. Treating a mean-reverting asset, Zhang and Zhang [43] was devoted to optimum trading strategy. It established two threshold prices (buy and sell) that maximize overall discounted return if one trades at those prices. Nonetheless, the net positions of its formulation are limited to either flat or long. In other words, short selling, a common operation in stock market, was not included in their studies. In addition to the results obtained in [43] along this line of research, an investment capacity expansion/reduction problem is considered in Merhi and Zervos [33]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model is treated by Løkka and Zervos [31] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [26] studied an optimal timing of investment problem under a general diffusion market model. The objective is to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution is obtained in [26]. Other related literature in connection with taxes of capital gains can be found, for example, in Cadenillas and Pliska [7], Constantinides [10], Dammon and Spatt [15], and references therein.

In order to obtain a more realistic trading rule, short selling is taken into account in our formulation. In addition, we allow either one share long, or flat, or one share short at any given time. One is allowed to choose between shorting and buying when one has no share in holding. In this chapter, we also consider slippage costs associated with each transaction because it becomes noteworthy in frequent transactions. In our formulation, a fixed rate of slippage cost is incurred in each transaction. The objective is to buy, sell or short so as to maximizing a discounted reward function. We follow a dynamic programming approach to resolve the problem. We obtain the corresponding Hamilton-Jacobi-Bellman (HJB) equation for the value functions. Using these HJB equations, we solve the optimal stopping problem by determining four threshold levels corresponding to buying and selling points. These levels are then used to convert the HJB equations into quasi-algebraic equations via a smooth-fit technique. We also provide a verification theorem to assure the optimality of our trading rule. The behavior of these buy and sell prices is investigated by varying different parameters in numerical examples.

We present the results in the following order. In §2.2, problem setup is constructed. In §2.3, the associated HJB equations and their solutions are studied. In §2.4, a verification theorem withAn Optimal Trading Rule of a Mean- sufficient conditions is proved. Finally, numerical examples are given in §2.5.

2.2 Problem Setup

Let S_t denote the price of the asset under consideration at time t and let $X_t = \log(S_t)$ be the mean-reverting process governed by

$$dX_t = a(L - X_t)dt + \sigma dW_t, \ X_0 = x, \tag{2.1}$$

where $a > 0$ is the rate of reversion, L is the equilibrium level, $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion.

Remark 2.2.1 Traditionally, a geometric Brownian motion is used to capture equity price movements in finance literature. This hypothesis is validated by the long bull market from the early 80's to the 00's. However, if one examines the Dow Jones Industry Average from the 60's to 80's instead, it is easy to notice that the index is trapped in a so-called trading range and the market follows a mean reverting model.

In this chapter, we allow to buy or sell at most one share at a time. Three net positions are allowed in this setup. It can be either short (with one share of stock on loan), flat (no stock holding) or long (with one share of stock holding).

Let

$$0 \le \psi_1 \le \tau_1 \le \psi_2 \le \tau_2 \le \dots$$

denote a sequence of stopping times. Selection from either buying or selling a share is allowed at ψ_i. If one chooses to buy at ψ_i, then only selling is allowed at a later time τ_i. Alternatively, if selling is chosen at ψ_i, then one can only buy at τ_i. Either buying or selling at τ_i sets the net position back to flat.

Let k_t denote the net position with

$$k_t = \begin{cases} -1, & \text{short one share}, \\ 0, & \text{flat}, \\ 1, & \text{long one share}. \end{cases}$$

We define the sequence of stopping times for each initial net position k_0 as follows:

$$\begin{aligned} \Lambda^0_{-1} &= (\tau_0, \psi_1, \tau_1, \psi_2, \tau_2, \ldots) \\ \Lambda^0_0 &= (\psi_1, \tau_1, \psi_2, \tau_2, \ldots) \\ \Lambda^0_1 &= (\tau_0, \psi_1, \tau_1, \psi_2, \tau_2, \ldots) \end{aligned}$$

If initially the net position is flat ($k_0 = 0$), then one can either buy or sell a share. Recall that one is limited to no more than one share either long or short. Therefore, if the initial net position is short ($k_0 = -1$), then one can only buy a share at τ_0. Likewise, one can only sell a share at τ_0 if the initial net position is long ($k_0 = 1$). Note that Λ^0_{-1} can be regarded as a buy at τ_0 followed by Λ^0_0, and Λ^0_1 can be regarded as a sell at τ_0 followed by Λ^0_0. At time ψ_i, for $i \geq 1$, we have a choice of either going long or short. At time τ_i, we have to go long if shorted at ψ_i or go short if bought at ψ_i.

In this chapter, we focus on a threshold type problem. Intuitively, under the mean-reversion model, one should buy if the price is low and sell if it is high. In terminology of stochastic control theory, we look for a control function $u(x, k)$ of the following form:

$$u(x,k) = \begin{cases} 1, & \text{if } k = -1 \text{ and } x \leq b_1, \\ 1, & \text{if } k = 0 \text{ and } x \leq b_0, \\ -1, & \text{if } k = 0 \text{ and } x \geq s_0, \\ -1, & \text{if } k = 1 \text{ and } x \geq s_1, \end{cases} \tag{2.2}$$

where b_1 is the buy price when the net position is short one share, b_0 and s_0 are the respectively buy and sell prices when the net position is flat and s_1 is the sell price when the net position is long one share. $u(x, k) = 1$ represents buying one share and $u(x, k) = -1$ represents selling or short selling one share.

Let ρ be the discount rate and K be the percentage slippage rate. (Here K can also be used to account for transaction commission in percentage.) Recall that one only has choice of buying or selling at time ψ_i. Let $u_i = u(X_{\psi_i}, k_{\psi_i})$. If $u_i = 1$, buy one share and subtract $e^{-\rho\psi_i} S_{\psi_i}(1+K)$ from payoff. If $u_i = -1$, sell one share and add $e^{-\rho\psi_i} S_{\psi_i}(1-K)$ to payoff.

Given the initial state $X_0 = x$ and initial net position $k_0 = -1, 0, 1$, and

$$\begin{aligned}
\Lambda_{-1} &= (\tau_0, (\psi_1, u_1), \tau_1, (\psi_2, u_2), \tau_2, \ldots) \\
\Lambda_0 &= ((\psi_1, u_1), \tau_1, (\psi_2, u_2), \tau_2, \ldots) \\
\Lambda_1 &= (\tau_0, (\psi_1, u_1), \tau_1, (\psi_2, u_2), \tau_2, \ldots),
\end{aligned}$$

the reward functions of decision sequences, Λ_{k_0}, are given as follows:

$$J_{k_0}(x, \Lambda_{k_0}) = \begin{cases}
\mathsf{E}\{-e^{-\rho\tau_0} S_{\tau_0}(1+K) + \sum_{i=1}^{\infty} \{[e^{-\rho\tau_i} S_{\tau_i}(1-K) - e^{-\rho\psi_i} S_{\psi_i}(1+K)] \mathbb{1}_{\{u_i=1\}} \\
\quad + [e^{-\rho\psi_i} S_{\psi_i}(1-K) - e^{-\rho\tau_i} S_{\tau_i}(1+K)] \mathbb{1}_{\{u_i=-1\}}\}\}, & \text{if } k_0 = -1, \\[2ex]
\mathsf{E} \sum_{i=1}^{\infty} \{[e^{-\rho\tau_i} S_{\tau_i}(1-K) - e^{-\rho\psi_i} S_{\psi_i}(1+K)] \mathbb{1}_{\{u_i=1\}} \\
\quad + [e^{-\rho\psi_i} S_{\psi_i}(1-K) - e^{-\rho\tau_i} S_{\tau_i}(1+K)] \mathbb{1}_{\{u_i=-1\}}\}, & \text{if } k_0 = 0, \\[2ex]
\mathsf{E}\{e^{-\rho\tau_0} S_{\tau_0}(1-K) + \sum_{i=1}^{\infty} \{[e^{-\rho\tau_i} S_{\tau_i}(1-K) - e^{-\rho\psi_i} S_{\psi_i}(1+K)] \mathbb{1}_{\{u_i=1\}} \\
\quad + [e^{-\rho\psi_i} S_{\psi_i}(1-K) - e^{-\rho\tau_i} S_{\tau_i}(1+K)] \mathbb{1}_{\{u_i=-1\}}\}\}, & \text{if } k_0 = 1.
\end{cases}$$

The term $\mathsf{E} \sum_{n=1}^{\infty} \xi_n$ for random variables ξ_n is interpreted as

$$\limsup_{N \to \infty} \mathsf{E} \sum_{n=1}^{N} \xi_n.$$

For $k_0 = -1, 0, 1$, let $V_{k_0}(x)$ denote the value functions with the initial state $X_0 = x$ and initial net positions $k_0 = -1, 0, 1$. That is,

$$V_{k_0}(x) = \sup_{\{\Lambda_{k_0}\}} J_{k_0}(x, \Lambda_{k_0}). \tag{2.3}$$

Remark 2.2.2 In this chapter, the optimal trading rule is formulated as an impulse control problem. Such approach is also used often in studying portfolio selection problems with transaction costs. In these studies, a typical model consists of geometric Brownian motion price and investment/consumption utility function. We refer the reader to Fleming and Soner [19] and references therein for related results.

2.3 HJB Equations

In this section, the corresponding HJB equation will be presented and the smooth-fit method will be used to solve the optimal stopping problem.

Let $\mathcal{A}$ denote the generator of X_t, i.e.,

$$\mathcal{A} = a(L-x)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}.$$

Formally, the associated HJB equations should have the form:

$$\begin{cases} \min\{\rho v_{-1} - Av_{-1}, v_{-1} - v_0 + e^x(1+K)\} = 0, \\ \min\{\rho v_0 - Av_0, v_0 - v_1 + e^x(1+K), v_0 - v_{-1} - e^x(1-K)\} = 0, \\ \min\{\rho v_1 - Av_1, v_1 - v_0 - e^x(1-K)\} = 0. \end{cases} \tag{2.4}$$

Naturally, a buy decision should be made when the price is low ($b_0,\ b_1$ in (2.2)) and a sell decision is made when the price is high ($s_0,\ s_1$ in (2.2)). Moreover, one would be more willing to buy when the net position is short than it is flat, because one has to bear risk in loaning a share. Similarly, one would be more willing to sell when the net position is long than it is flat, as one has to bear risk in holding a share. As a result, the $b_0,\ b_1,\ s_0,\ s_1$ in (2.2) should satisfy the following inequalities,

$$b_0 \le b_1 \le s_1 \le s_0.$$

Moreover, in view of the bounds obtained in [43] for value functions, we solve (2.4) for functions v_{-1}, v_0, and v_1 that satisfying the following conditions:

$$\begin{aligned} &-e^x(1+K) \le v_{-1}(x) < K_0 \quad \text{and} \\ &0 \le v_1(x) \le K_1 e^x + K_2 \end{aligned} \tag{2.5}$$

for some constants K_0, K_1, and K_2.

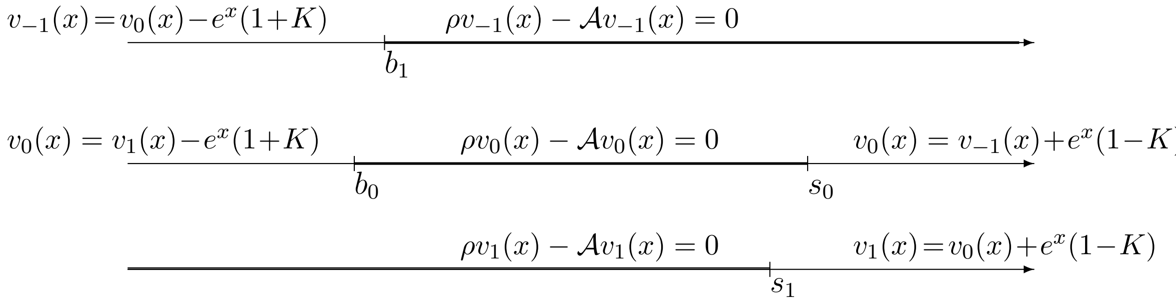

Figure 2.1: Continuation Regions (darkened intervals)

Then the continuation region when $k = -1$ should be (b_1, ∞) on which $\rho v_{-1}(x) - \mathcal{A}v_{-1}(x) = 0$. For $x < b_1$, one should have $v_{-1}(x) = v_0(x) - e^x(1 + K)$. Similarly, the continuation region when $k = 1$ should be $(-\infty, s_1)$ on which $\rho v_1(x) - \mathcal{A}v_1(x) = 0$, and for $x > s_1$, $v_1(x) = v_0(x) + e^x(1 - K)$. When $k = 0$, the continuation region is (b_0, s_0) on which $\rho v_0(x) - \mathcal{A}v_0(x) = 0$. For $x < b_0$, $v_0(x) = v_{-1}(x) - e^x(1 + K)$ and for $x > s_0$, $v_0(x) = v_{-1}(x) + e^x(1 - K)$. These continuation regions are marked by darkened lines in Figure 1.An Optimal Trading Rule of a Mean-

Next we solve the equations $\rho v_k(x) - \mathcal{A}v_k(x) = 0$ with $k = -1, 0, 1$. Let $\kappa = \sqrt{2a}/\sigma$, $\lambda = \rho/a$, and $\eta(t) = t^{\lambda-1}\exp(-t^2/2)$. Then the general solution of $\rho v_i(x) - \mathcal{A}v_i(x) = 0$ is given by (details can be found in Eloe et al. [17])

$$F_1 \int_0^\infty \eta(t)e^{\kappa(L-x)t}dt + F_2 \int_0^\infty \eta(t)e^{-\kappa(L-x)t}dt,$$

for some constants F_1 and F_2.

By the condition (2.5), $v_{-1}(\infty)$ should be bounded which implies $F_2 = 0$. Thus,

$$v_{-1}(x) = F_1 \int_0^\infty \eta(t)e^{\kappa(L-x)t}dt.$$

Similarly, the condition on boundedness of $v_1(-\infty)$ implies $F_1 = 0$. Therefore,

$$v_1(x) = F_2 \int_0^\infty \eta(t)e^{-\kappa(L-x)t}dt.$$

It is clear that both v_{-1} and v_1 are C^2 functions on (b_1, ∞) and $(-\infty, s_1)$, respectively.

The smooth-fit conditions (continuous differentiability) at b_1, b_0, s_0 and s_1 require

$$\begin{cases} v_{-1}(b_1) = v_0(b_1) - e^{b_1}(1+K), \\ \\ \dfrac{dv_{-1}(b_1)}{dx} = \dfrac{dv_0(b_1)}{dx} - e^{b_1}(1+K), \\ \\ v_0(b_0) = v_1(b_0) - e^{b_0}(1+K), \\ \\ \dfrac{dv_0(b_0)}{dx} = \dfrac{dv_1(b_0)}{dx} - e^{b_0}(1+K), \\ \\ v_0(s_0) = v_{-1}(s_0) + e^{s_0}(1-K), \\ \\ \dfrac{dv_0(s_0)}{dx} = \dfrac{dv_{-1}(s_0)}{dx} + e^{s_0}(1+K), \\ \\ v_1(s_1) = v_0(s_1) + e^{s_1}(1-K), \\ \\ \dfrac{dv_1(s_1)}{dx} = \dfrac{dv_0(s_1)}{dx} + e^{s_1}(1-K). \end{cases}$$

For simplicity in notation, define

$$y_1(x) = \int_0^\infty \eta(t) e^{\kappa(L-x)t} dt,$$

$$y_2(x) = \int_0^\infty \eta(t) e^{-\kappa(L-x)t} dt.$$

Write v_{-1}, v_0, and v_1 in terms of these two functions

$$\begin{aligned} v_{-1}(x) &= C_1 y_1(x), \\ v_0(x) &= C_2 y_2(x) + C_3 y_1(x), \\ v_1(x) &= C_4 y_2(x), \end{aligned}$$

for constants C_i, $i = 1, 2, 3, 4$. The smooth-fit conditions have the following form:

$$\begin{pmatrix} -C_1 y_1(b_1) + C_2 y_2(b_1) + C_3 y_1(b_1) \\ -C_1 y_1'(b_1) + C_2 y_2'(b_1) + C_3 y_1'(b_1) \end{pmatrix} = \begin{pmatrix} e^{b_1}(1+K) \\ e^{b_1}(1+K) \end{pmatrix},$$

$$\begin{pmatrix} -C_2 y_2(b_0) - C_3 y_1(b_0) + C_4 y_2(b_0) \\ -C_2 y_2'(b_0) - C_3 y_1'(b_0) + C_4 y_2'(b_0) \end{pmatrix} = \begin{pmatrix} e^{b_0}(1+K) \\ e^{b_0}(1+K) \end{pmatrix},$$

$$\begin{pmatrix} -C_1 y_1(s_0) + C_2 y_2(s_0) + C_3 y_1(s_0) \\ -C_1 y_1'(s_0) + C_2 y_2'(s_0) + C_3 y_1'(s_0) \end{pmatrix} = \begin{pmatrix} e^{s_0}(1-K) \\ e^{s_0}(1-K) \end{pmatrix},$$

$$\begin{pmatrix} -C_2 y_2(s_1) - C_3 y_1(s_1) + C_4 y_2(s_1) \\ -C_2 y_2'(s_1) - C_3 y_1'(s_1) + C_4 y_2'(s_1) \end{pmatrix} = \begin{pmatrix} e^{s_1}(1-K) \\ e^{s_1}(1-K) \end{pmatrix}. \tag{2.6}$$

Combine the first four equations of (2.6) in matrix form to obtain

$$\begin{pmatrix} -y_1(b_1) & y_2(b_1) & y_1(b_1) & 0 \\ -y_1'(b_1) & y_2'(b_1) & y_1'(b_1) & 0 \\ 0 & -y_2(b_0) & -y_1(b_0) & y_2(b_0) \\ 0 & -y_2'(b_0) & -y_1'(b_0) & y_2'(b_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} e^{b_1}(1+K) \\ e^{b_1}(1+K) \\ e^{b_0}(1+K) \\ e^{b_0}(1+K) \end{pmatrix}. \tag{2.7}$$

Similarly, combining the last four equations of (2.6) in matrix form, we have

$$\begin{pmatrix} -y_1(s_0) & y_2(s_0) & y_1(s_0) & 0 \\ -y_1'(s_0) & y_2'(s_0) & y_1'(s_0) & 0 \\ 0 & -y_2(s_1) & -y_1(s_1) & y_2(s_1) \\ 0 & -y_2'(s_1) & -y_1'(s_1) & y_2'(s_1) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} e^{s_0}(1-K) \\ e^{s_0}(1-K) \\ e^{s_1}(1-K) \\ e^{s_1}(1-K) \end{pmatrix}. \tag{2.8}$$

Eliminating (C_1, C_2, C_3, C_4) by (2.7) and (2.8), we have

$$\begin{pmatrix} -y_1(b_1) & y_2(b_1) & y_1(b_1) & 0 \\ -y_1'(b_1) & y_2'(b_1) & y_1'(b_1) & 0 \\ 0 & -y_2(b_0) & -y_1(b_0) & y_2(b_0) \\ 0 & -y_2'(b_0) & -y_1'(b_0) & y_2'(b_0) \end{pmatrix}^{-1} \begin{pmatrix} e^{b_1}(1+K) \\ e^{b_1}(1+K) \\ e^{b_0}(1+K) \\ e^{b_0}(1+K) \end{pmatrix}$$

$$= \begin{pmatrix} -y_1(s_0) & y_2(s_0) & y_1(s_0) & 0 \\ -y_1'(s_0) & y_2'(s_0) & y_1'(s_0) & 0 \\ 0 & -y_2(s_1) & -y_1(s_1) & y_2(s_1) \\ 0 & -y_2'(s_1) & -y_1'(s_1) & y_2'(s_1) \end{pmatrix}^{-1} \begin{pmatrix} e^{s_0}(1-K) \\ e^{s_0}(1-K) \\ e^{s_1}(1-K) \\ e^{s_1}(1-K) \end{pmatrix}. \tag{2.9}$$

Note that this is a set of four quasi-algebraic equations for (b_1, b_0, s_0, s_1).

An additional requirement for b_0, b_1, s_1 and s_0 is that the difference between buy and sell prices should offset the slippage rates K in order to make profit. Since $b_0 \le b_1 \le s_1 \le s_0$, only b_1 and s_1 need to be considered. The following inequality would be expected

$$e^{s_1}(1-K) > e^{b_1}(1+K)$$

which is equivalent to

$$s_1 - b_1 > \log\left(\frac{1+K}{1-K}\right). \tag{2.10}$$

Furthermore, $v_k(x)$ has to satisfy the following conditions to qualify for being solutions to the HJB equations (2.4):

$$\begin{cases} v_{-1}(x) \geq v_0(x) - e^x(1+K) & \text{on } (b_1, \infty), \\ v_0(x) \geq v_1(x) - e^x(1+K) & \text{on } (b_0, \infty), \\ v_0(x) \geq v_{-1}(x) + e^x(1-K) & \text{on } (-\infty, s_0), \\ v_1(x) \geq v_0(x) + e^x(1-K) & \text{on } (-\infty, s_1), \\ (\rho - \mathcal{A})(v_0(x) - e^x(1+K)) \geq 0 & \text{on } (-\infty, b_1), \\ (\rho - \mathcal{A})(v_1(x) - e^x(1+K)) \geq 0 & \text{on } (-\infty, b_0). \\ (\rho - \mathcal{A})(v_{-1}(x) + e^x(1-K)) \geq 0 & \text{on } (s_0, \infty). \\ (\rho - \mathcal{A})(v_0(x) + e^x(1-K)) \geq 0 & \text{on } (s_1, \infty). \end{cases} \tag{2.11}$$

Note that on (s_0, ∞), the inequality $v_{-1}(x) \geq v_0(x) - e^x(1+K)$ is automatically satisfied because $v_0(x) = v_{-1}(x) + e^x(1-K)$. Therefore, the first inequality in (2.11) only needs to hold on (b_1, s_0). Similarly, the forth inequality in (2.11) only needs to hold on (b_0, s_1).

Note also that on (s_1, ∞), the inequality $v_0(x) \geq v_1(x) - e^x(1+K)$ is automatically satisfied because $v_1(x) = v_0(x) + e^x(1-K)$. Therefore, the second inequality in (2.11) only needs to hold on (b_0, s_1). For similar reason, the third inequality in (2.11) only needs to hold on (b_1, s_0).

Moreover, from (2.11) we require $(\rho - \mathcal{A})(v_0(x) - e^x(1+K)) \geq 0$ on $(-\infty, b_1)$. We next show that this inequality is equivalent to

$$(\rho - \mathcal{A})e^x \leq 0 \text{ on } (-\infty, b_1). \tag{2.12}$$

Write $(-\infty, b_1) = (-\infty, b_0] \bigcup [b_0, b_1)$. Note that

$$(\rho - \mathcal{A})v_0(x) = 0 \text{ on } [b_0, b_1). \tag{2.13}$$

Thus $(\rho - \mathcal{A})(v_0(x) - e^x(1+K)) \geq 0$ on $[b_0, b_1)$ implies

$$(\rho - \mathcal{A})e^x \leq 0 \text{ on } [b_0, b_1). \tag{2.14}$$

Note also that (see Figure 2.1)), on the interval $(-\infty, b_0]$,

$$(\rho - \mathcal{A})v_1(x) = 0 \text{ on } (-\infty, b_0] \text{ and } v_0(x) = v_1(x) - e^x(1+K).$$

It follows that

$$\begin{aligned}(\rho - \mathcal{A})(v_0(x) - e^x(1+K)) &= (\rho - \mathcal{A})(v_1(x) - 2e^x(1+K)) \\ &= -2(\rho - \mathcal{A})e^x(1+K).\end{aligned}$$

Therefore, $(\rho - \mathcal{A})(v_0(x) - e^x(1+K)) \geq 0$ on $[-\infty, b_0)$ leads to

$$(\rho - \mathcal{A})e^x \leq 0 \text{ on } (-\infty, b_0].$$

Together with (2.14), we obtain (2.12).

Similarly, for $(\rho - \mathcal{A})(v_0(x) + e^x(1-K)) \geq 0$ on (s_1, ∞), write $(s_1, \infty) = (s_1, s_0] \bigcup [s_0, \infty)$ and note that

$$(\rho - \mathcal{A})v_0(x) = 0 \text{ on } (s_1, s_0] \text{ and } (\rho - \mathcal{A})v_{-1}(x) = 0 \text{ on } [s_0, \infty).$$

Consequently,

$$(\rho - \mathcal{A})e^x \geq 0 \text{ on } (s_1, s_0]$$

and

$$(\rho - \mathcal{A})(v_0(x) + e^x(1+K)) = (\rho - \mathcal{A})(v_{-1}(x) + 2e^x(1-K)) = (\rho - \mathcal{A})2e^x(1-K) \geq 0 \text{ on } [s_0, \infty),$$

which implies that

$$(\rho - \mathcal{A})e^x \geq 0 \text{ on } [s_0, \infty).$$

As a result, we have

$$(\rho - \mathcal{A})e^x \geq 0 \text{ on } (s_1, \infty).$$

These imply that

$$b_0 \leq b_1 \leq \frac{1}{a}\left[\frac{\sigma^2}{2} + aL - \rho\right] \text{ and } s_0 \geq s_1 \geq \frac{1}{a}\left[\frac{\sigma^2}{2} + aL - \rho\right].$$

Indeed, these two inequalities are equivalent to the last four inequalities on (2.11). In the next section, we show that the quadruple (b_0, b_1, s_1, s_0) satisfying these conditions leads to the optimal stopping times.

2.4 A Verification Theorem

We give a verification theorem to show that the solution $v_{k_0}(x)$, $k_0 = -1, 0, 1$, of equation (2.4) are equal to the value functions $V_{k_0}(x)$, $k_0 = -1, 0, 1$, respectively, and sequences of optimal stopping times can be constructed by using (b_0, b_1, s_1, s_0).

Theorem 2.4.1. *Let* (b_0, b_1, s_1, s_0) *be a solution to* (2.9) *satisfying*

$$b_0 \le b_1 \le \frac{1}{a}\left(\frac{\sigma^2}{2} + aL - \rho\right) \le s_1 \le s_0$$

and

$$s_1 - b_1 > \log\left(\frac{1+K}{1-K}\right).$$

Let

$$v_{-1}(x) = \begin{cases} C_1 y_1(x), & \text{if } x \ge b_1, \\ C_2 y_2(x) + C_3 y_1(x) - e^x(1+K), & \text{if } b_0 \le x \le b_1, \\ C_4 y_2(x) - 2e^x(1+K), & \text{if } x \le b_0, \end{cases}$$

$$v_0(x) = \begin{cases} C_4 y_2(x) - e^x(1+K), & \text{if } x \le b_0, \\ C_2 y_2(x) + C_3 y_1(x), & \text{if } b_0 \le x \le b_1, \\ C_1 y_1(x) + e^x(1-K), & \text{if } x \ge s_0, \end{cases}$$

$$v_1(x) = \begin{cases} C_4 y_2(x), & \text{if } x \le s_1, \\ C_2 y_2(x) + C_3 y_1(x) + e^x(1-K), & \text{if } s_1 \le x \le s_0, \\ C_1 y_1(x) + 2e^x(1-K), & \text{if } x \ge s_1, \end{cases}$$

with

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} -y_1(b_1) & y_2(b_1) & y_1(b_1) & 0 \\ -y_1'(b_1) & y_2'(b_1) & y_1'(b_1) & 0 \\ 0 & -y_2(b_0) & -y_1(b_0) & y_2(b_0) \\ 0 & -y_2'(b_0) & -y_1'(b_0) & y_2'(b_0) \end{pmatrix}^{-1} \begin{pmatrix} e^{b_1}(1+K) \\ e^{b_1}(1+K) \\ e^{b_0}(1+K) \\ e^{b_0}(1+K) \end{pmatrix}$$

and

$$y_1(x) = \int_0^\infty \eta(t) e^{\kappa(L-x)t} dt,$$

$$y_2(x) = \int_0^\infty \eta(t) e^{-\kappa(L-x)t} dt.$$

If on the interval (b_1, s_0) *the following inequalities hold*

$$\begin{cases} v_{-1}(x) \geq v_0(x) - e^x(1+K), \\ v_0(x) \geq v_{-1}(x) + e^x(1-K), \end{cases}$$

and on the interval (b_0, s_1) *the following inequalities hold*

$$\begin{cases} v_0(x) \geq v_1(x) - e^x(1+K), \\ v_1(x) \geq v_0(x) + e^x(1-K), \end{cases}$$

then,

$$v_{k_0}(x) = V_{k_0}(x), \quad k_0 = -1, 0, 1.$$

In addition, when $k_0 = 0$, *let* $\tau_0^* = 0$ *and*

$$\Lambda_0^* = ((\psi_1^*, u_1^*), \tau_1^*, (\psi_2^*, u_2^*), \tau_2^*, \ldots)$$

where, for $i = 1, 2, \ldots$, $\psi_i^* = \inf\{t \geq \tau_{i-1}^* : \ X_t \notin (b_0, s_0)\}$*; if* $X_{\psi_i^*} \leq b_0$, $u_i^* = 1$, *then* $\tau_i^* = \inf\{t \geq \psi_i^* : X_t \geq s_1\}$*; if* $X_{\psi_i^*} \geq s_0$, $u_i^* = -1$, *then* $\tau_i^* = \inf\{t \geq \psi_i^* : X_t \leq b_1\}$.

When $k_0 = -1$, *let* $\tau_0^* = \inf\{t \geq 0 : X_t \leq b_1\}$ *and* $\Lambda_{-1}^* = (\tau_0^*, \Lambda_0^*)$. *When* $k_0 = 1$, *let* $\tau_0^* = \inf\{t \geq 0 : X_t \geq s_1\}$ *and* $\Lambda_1^* = (\tau_0^*, \Lambda_0^*)$.

Then Λ_{-1}^*, Λ_0^* *and* Λ_1^* *are optimal.*

To prove the theorem, we need the following lemma which was proved in Zhang and Zhang [43]

Lemma 2.4.1. *Given* z_1 *and* z_2, *let* $\theta_1 = \inf\{t : \ X_t \geq z_1\}$ *and* $\theta_2 = \inf\{t : \ X_t \leq z_2\}$. *Then*

$$P(\theta_1 < \infty) = P(\theta_2 < \infty) = 1.$$

Proof of Theorem 2.4.1. The proof is divided into two steps. In the first step, we show that $v_{k_0}(x) \geq J_{k_0}(x, \Lambda_{k_0})$ for all Λ_{k_0}. Then in the second step, we show that $v_{k_0}(x) = J_{k_0}(x, \Lambda^*_{k_0})$. Therefore, $v_{k_0}(x) = V_{k_0}(x)$ and $\Lambda^*_{k_0}$ is optimal.

Using $\rho v_{k_0}(x) - \mathcal{A} v_{k_0}(x) \geq 0$, Dynkin's formula and Fatou's lemma as in Øksendal [35, p.226], and the choice of u_i, we have, for any stopping times $0 \leq \theta_1 \leq \theta_2$, a.s.,

$$\mathsf{E}e^{-\rho\theta_1} v_{k_0}(X_{\theta_1}) \geq \mathsf{E}e^{-\rho\theta_2} v_{k_0}(X_{\theta_2}).$$

and

$$\mathsf{E}e^{-\rho\psi_i} v_{k_0}(X_{\psi_i}) \mathbb{1}_{\{u_i=\pm 1\}} \geq \mathsf{E}e^{-\rho\tau_i} v_{k_0}(X_{\tau_i}) \mathbb{1}_{\{u_i=\pm 1\}},$$

for $k_0 = -1, 0, 1$. We have

$$\begin{aligned}
v_0(x) &\geq \mathsf{E}e^{-\rho\psi_1} v_0(X_{\psi_1}) \\
&\geq \mathsf{E}e^{-\rho\psi_1}[\mathbb{1}_{\{u_1=1\}}(v_1(X_{\psi_1}) - S_{\psi_1}(1+K)) + \mathbb{1}_{\{u_1=-1\}}(v_{-1}(X_{\psi_1}) + S_{\psi_1}(1-K))] \\
&= \mathsf{E}e^{-\rho\psi_1} v_1(X_{\psi_1})\mathbb{1}_{\{u_1=1\}} + \mathsf{E}e^{-\rho\psi_1} v_{-1}(X_{\psi_1})\mathbb{1}_{\{u_1=-1\}} \\
&\quad + \mathsf{E}e^{-\rho\psi_1} S_{\psi_1}(1-K)\mathbb{1}_{\{u_1=-1\}} - \mathsf{E}e^{-\rho\psi_1} S_{\psi_1}(1+K)\mathbb{1}_{\{u_1=1\}} \\
&\geq \mathsf{E}e^{-\rho\tau_1} v_1(X_{\tau_1})\mathbb{1}_{\{u_1=1\}} + \mathsf{E}e^{-\rho\tau_1} v_{-1}(X_{\tau_1})\mathbb{1}_{\{u_1=-1\}} \\
&\quad + \mathsf{E}[e^{-\rho\psi_1} S_{\psi_1}(1-K)\mathbb{1}_{\{u_1=-1\}} - e^{-\rho\psi_1} S_{\psi_1}(1+K)\mathbb{1}_{\{u_1=1\}}] \\
&\geq \mathsf{E}e^{-\rho\tau_1}[v_0(X_{\tau_1}) + S_{\tau_1}(1-K)]\mathbb{1}_{\{u_1=1\}} + \mathsf{E}e^{-\rho\tau_1}[v_0(X_{\tau_1}) - S_{\tau_1}(1+K)]\mathbb{1}_{\{u_1=-1\}} \\
&\quad + \mathsf{E}[e^{-\rho\psi_1} S_{\psi_1}(1-K)\mathbb{1}_{\{u_1=-1\}} - e^{-\rho\psi_1} S_{\psi_1}(1+K)\mathbb{1}_{\{u_1=1\}}] \\
&= \mathsf{E}e^{-\rho\tau_1} v_0(X_{\tau_1}) + \mathsf{E}\{[e^{-\rho\tau_1} S_{\tau_1}(1-K) - e^{-\rho\psi_1} S_{\psi_1}(1+K)]\mathbb{1}_{\{u_1=1\}} \\
&\quad + [e^{-\rho\psi_1} S_{\psi_1}(1-K) - e^{-\rho\tau_1} S_{\tau_1}(1+K)]\mathbb{1}_{\{u_1=-1\}}\} \\
&\geq \mathsf{E}e^{-\rho\psi_2} v_0(X_{\psi_2}) + \mathsf{E}\{[e^{-\rho\tau_1} S_{\tau_1}(1-K) - e^{-\rho\psi_1} S_{\psi_1}(1+K)]\mathbb{1}_{\{u_1=1\}} \\
&\quad + [e^{-\rho\psi_1} S_{\psi_1}(1-K) - e^{-\rho\tau_1} S_{\tau_1}(1+K)]\mathbb{1}_{\{u_1=-1\}}\}
\end{aligned}$$

Continuing this way, we get

$$\begin{aligned} v_0(x) \geq \;& \mathsf{E}\textstyle\sum_{i=1}^{N}\{[e^{-\rho\tau_i}S_{\tau_i}(1-K) - e^{-\rho\psi_i}S_{\psi_i}(1+K)]1\!\!1_{\{u_i=1\}} \\ & +[e^{-\rho\psi_i}S_{\psi_i}(1-K) - e^{-\rho\tau_i}S_{\tau_i}(1+K)]1\!\!1_{\{u_i=-1\}}\}. \end{aligned}$$

Sending $N \to \infty$, we have $v_0(x) \geq J_0(x, \Lambda_0)$ for all Λ_0. This implies that $v_0(x) \geq V_0(x)$. Similarly, we can show that $v_{-1}(x) \geq V_1(x)$ and $v_1(x) \geq V_1(x)$.

Now we establish the equalities. In view of Lemma 2.4.1, $\tau_i^* < \infty$ and $\psi_i^* < \infty$, a.s. Therefore, we have

$$\begin{aligned} v_0(x) &= \mathsf{E}e^{-\rho\psi_1^*}v_0(X_{\psi_1^*}) \\ \\ &= \mathsf{E}e^{-\rho\psi_1^*}[1\!\!1_{\{u_1^*=1\}}(v_1(X_{\psi_1^*}) - S_{\psi_1^*}(1+K)) + 1\!\!1_{\{u_1^*=-1\}}(v_{-1}(X_{\psi_1^*}) + S_{\psi_1^*}(1-K))] \\ \\ &= \mathsf{E}e^{-\rho\tau_1^*}v_1(X_{\tau_1^*})1\!\!1_{\{u_1^*=1\}} + \mathsf{E}e^{-\rho\tau_1^*}v_{-1}(X_{\tau_1^*})I_{\{u_1^*=-1\}} \\ &\quad +\mathsf{E}[e^{-\rho\psi_1^*}S_{\psi_1^*}(1-K)1\!\!1_{\{u_1^*=-1\}} - e^{-\rho\psi_1^*}S_{\psi_1^*}(1+K)1\!\!1_{\{u_1^*=1\}}] \\ \\ &= \mathsf{E}e^{-\rho\tau_1^*}[v_0(X_{\tau_1^*}) + S_{\tau_1^*}(1-K)]1\!\!1_{\{u_1^*=1\}} + \mathsf{E}e^{-\rho\tau_1^*}[v_0(X_{\tau_1^*}) - S_{\tau_1^*}(1+K)]1\!\!1_{\{u_1^*=-1\}} \\ &\quad +\mathsf{E}[e^{-\rho\psi_1^*}S_{\psi_1^*}(1-K)1\!\!1_{\{u_1^*=-1\}} - e^{-\rho\psi_1^*}S_{\psi_1^*}(1+K)I_{\{u_1^*=1\}}] \\ \\ &= \mathsf{E}e^{-\rho\tau_1^*}v_0(X_{\tau_1^*}) + \mathsf{E}\{[e^{-\rho\tau_1^*}S_{\tau_1^*}(1-K) - e^{-\rho\psi_1^*}S_{\psi_1^*}(1+K)]1\!\!1_{\{u_1^*=1\}} \\ &\quad +[e^{-\rho\psi_1^*}S_{\psi_1^*}(1-K) - e^{-\rho\tau_1^*}S_{\tau_1^*}(1+K)]1\!\!1_{\{u_1^*=-1\}}\}. \end{aligned}$$

Continuing this way, we obtain

$$\begin{aligned} v_0(x) &= \mathsf{E}e^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) + \mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}S_{\tau_i^*}(1-K) - e^{-\rho\psi_i^*}S_{\psi_i^*}(1+K)]1\!\!1_{\{u_i^*=1\}} \\ &\quad +[e^{-\rho\psi_i^*}S_{\psi_i^*}(1-K) - e^{-\rho\tau_i^*}S_{\tau_i^*}(1+K)]1\!\!1_{\{u_i^*=-1\}}\}. \end{aligned} \tag{2.15}$$

Similarly, we have

$$\begin{aligned} v_{-1}(x) &= \mathsf{E}e^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) + \mathsf{E}[-e^{-\rho\tau_0}S_{\tau_0}(1+K)] \\ &\quad +\mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}S_{\tau_i^*}(1-K) - e^{-\rho\psi_i^*}S_{\psi_i^*}(1+K)]1\!\!1_{\{u_i^*=1\}} \\ &\quad +[e^{-\rho\psi_i^*}S_{\psi_i^*}(1-K) - e^{-\rho\tau_i^*}S_{\tau_i^*}(1+K)]1\!\!1_{\{u_i^*=-1\}}\}, \end{aligned}$$

and

$$\begin{aligned} v_1(x) &= \mathsf{E}e^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) + \mathsf{E}[e^{-\rho\tau_0}S_{\tau_0}(1-K)] \\ &\quad + \mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}S_{\tau_i^*}(1-K) - e^{-\rho\psi_i^*}S_{\psi_i^*}(1+K)]\mathbb{1}_{\{u_i^*=1\}} \\ &\quad + [e^{-\rho\psi_i^*}S_{\psi_i^*}(1-K) - e^{-\rho\tau_i^*}S_{\tau_i^*}(1+K)]\mathbb{1}_{\{u_i^*=-1\}}\}. \end{aligned}$$

Finally, it remains to show that $\mathsf{E}e^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) \to 0$ as $N \to \infty$. Note that $X_{\tau_N^*}$ is either b_1 or s_1, so $v_0(X_{\tau_N^*})$ is either $v_0(b_1)$ or $v_0(s_1)$. It suffices to show that $\mathsf{E}e^{-\rho\tau_N^*} \to 0$.

Note that τ_n^* is monotone increasing and $\psi_n^* \le \tau_n^* \le \psi_{n+1}^*$, a.s. Let $A = \lim_{n\to\infty} e^{-\rho\tau_n^*}$. Then, $\lim_{n\to\infty} e^{-\rho\psi_n^*} = A$ and $A \ge 0$, a.s.

We next show that $\mathsf{E}A = 0$, which implies $\mathsf{E}e^{-\rho\tau_n^*} \to 0$. Note that $u_i^* = 1$ leads to $S_{\psi_i^*} = e^{b_0}$ and subsequently $S_{\tau_i^*} = e^{s_1}$; similarly, $u_i^* = -1$ leads to $S_{\psi_i^*} = e^{s_0}$ and $S_{\tau_i^*} = e^{b_1}$. Using (2.15), we have

$$\begin{aligned} v_0(x) &= \mathsf{E}e^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) + \mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}e^{s_1}(1-K) - e^{-\rho\psi_i^*}e^{b_0}(1+K)]\mathbb{1}_{\{u_i^*=1\}} \\ &\quad + [e^{-\rho\psi_i^*}e^{s_0}(1-K) - e^{-\rho\tau_i^*}e^{b_1}(1+K)]\mathbb{1}_{\{u_i^*=-1\}}\}. \end{aligned}$$

Recall that $b_0 < b_1 < s_1 < s_0$. It follows that

$$\begin{aligned} v_0(x) &\ge Ee^{-\rho\tau_N^*}v_0(X_{\tau_N^*}) + \mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}e^{s_1}(1-K) - e^{-\rho\psi_i^*}e^{b_1}(1+K)]\mathbb{1}_{\{u_i^*=1\}} \\ &\quad + [e^{-\rho\psi_i^*}e^{s_1}(1-K) - e^{-\rho\tau_i^*}e^{b_1}(1+K)]\mathbb{1}_{\{u_i^*=-1\}}\}. \end{aligned}$$

Dividing both sides by N and sending $N \to \infty$, we have

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N}\mathsf{E}\sum_{i=1}^{N}\{[e^{-\rho\tau_i^*}e^{s_1}(1-K) - e^{-\rho\psi_i^*}e^{b_1}(1+K)]\mathbb{1}_{\{u_i^*=1\}} \\ &\quad + [e^{-\rho\psi_i^*}e^{s_1}(1-K) - e^{-\rho\tau_i^*}e^{b_1}(1+K)]\mathbb{1}_{\{u_i^*=-1\}}\} \\ &\le 0. \end{aligned}$$

Note also that

$$\begin{aligned} & \Big| \tfrac{1}{N} \mathsf{E} \textstyle\sum_{i=1}^N \{[(e^{-\rho\tau_i^*} - A)e^{s_1}(1-K) - (e^{-\rho\psi_i^*} - A)e^{b_1}(1+K)]1\!\!1_{\{u_i^*=1\}} \\ & +[(e^{-\rho\psi_i^*} - A)e^{s_1}(1-K) - (e^{-\rho\tau_i^*} - A)e^{b_1}(1+K)]1\!\!1_{\{u_i^*=-1\}}\} \Big| \\ \leq\ & \frac{1}{N} \mathsf{E} \sum_{i=1}^N \{|(e^{-\rho\tau_i^*} - A)e^{s_1}(1-K)| + |(e^{-\rho\psi_i^*} - A)e^{b_1}(1+K)| \\ & +|(e^{-\rho\psi_i^*} - A)e^{s_1}(1-K)| + |(e^{-\rho\tau_i^*} - A)e^{b_1}(1+K)|\} \\ \to\ & 0. \end{aligned}$$

It follows that

$$\begin{aligned} & \lim_{N\to\infty} \frac{1}{N} \mathsf{E} \sum_{i=1}^N \{[e^{-\rho\tau_i^*} e^{s_1}(1-K) - e^{-\rho\psi_i^*} e^{b_1}(1+K)]1\!\!1_{\{u_i^*=1\}} \\ & \qquad +[e^{-\rho\psi_i^*} e^{s_1}(1-K) - e^{-\rho\tau_i^*} e^{b_1}(1+K)]1\!\!1_{\{u_i^*=-1\}}\} \\ =\ & (e^{s_1}(1-K) - e^{b_1}(1+K))\mathsf{E}A \\ \leq\ & 0. \end{aligned}$$

The condition $s_1 - b_1 > \log((1+K)/(1-K))$ implies the positivity of $(e^s(1-K) - e^b(1+K))$. Therefore, $\mathsf{E}A = 0$. Necessarily, $\mathsf{E}e^{-\rho\tau_N^*} \to 0$. This completes the proof.

2.5 A Numerical Example

In this section, we consider a numerical example with the following specifications:

$$a = 0.8,\ L = 2,\ \sigma = 0.5,\ \rho = 0.5,\ K = 0.01.$$

We solve the quasi-algebraic equations (2.9) by a simulated annealing algorithm [1] which gives

$$(b_0, b_1, s_1, s_0) = (1.4053, 1.4065, 1.6052, 1.6079).$$

[1] See Appendix A.

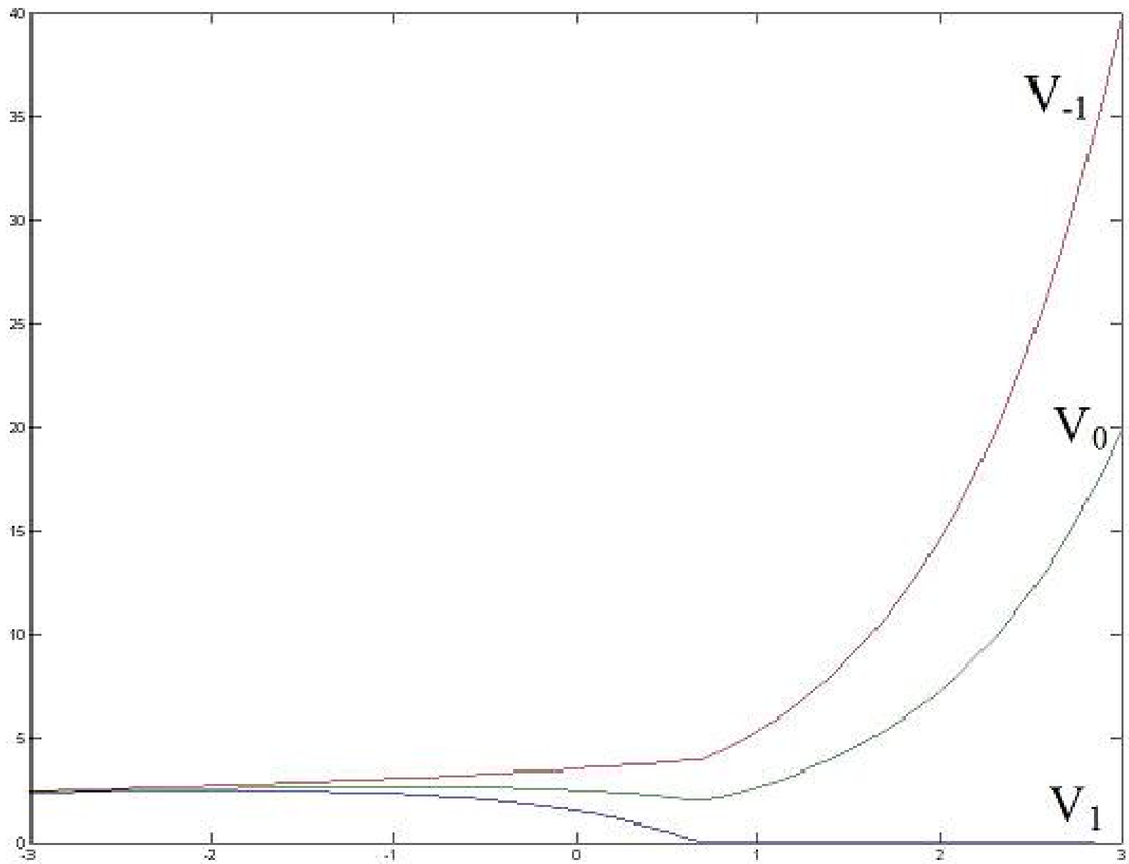

Figure 2.2: Value Functions $V_{-1}(x)$, $V_0(x)$ and $V_1(x)$.

In this example, all threshold levels in the quadruple (b_0, b_1, s_1, s_0) are below the equilibrium $L = 2$. This equilibrium serves as a pulling force that lifts the trajectory X_t from anywhere below $L = 2$. Two main factors affect the overall return: (i) the probability for X_t to go from between these threshold levels; (ii) the frequency for X_t to travel between them.

The corresponding value functions $V_{-1}(x)$,$V_0(x)$ and $V_1(x)$ are plotted in Figure 2.2.

Table 2.1: (b_0, b_1, s_1, s_0) with varying L.

L	1	1.5	2	2.5	3
b_0	0.3441	0.7934	1.4053	1.4123	2.2470
b_1	0.4047	0.7975	1.4065	1.9381	2.2471
s_1	0.6416	1.1864	1.6052	2.2977	2.6838
s_0	0.6419	1.2155)	1.6079	2.2977	2.7791
$V_{-1}(1)$	0.0450	0.1558	1.3348	2.7619	6.5644
$V_0(1)$	2.7361	2.8469	4.0803	5.5073	9.3099
$V_1(1)$	5.4272	5.5645	6.8258	8.2528	12.0553

We next vary one of the parameters at a time and examine the dependence of the threshold levels.

First we compute the threshold levels associated with varying L. Intuitively, larger L would result larger rewards and higher threshold levels (b_0, b_1, s_1, s_0). These are confirmed by the results given in Table 2.1. It can be seen that the quadruple (b_0, b_1, s_1, s_0) is monotonically increasing in L. In addition, we listed the values (V_{-1}, V_0, V_1) at $x = 1$ in Table 2.1. These values also show clear monotonicity in L.

Next, we vary a. A larger a implies larger convergence rate for X_t to reach the equilibrium level L which would result larger reward in short time. It shows in Table 2.2 that both the quadruple (b_0, b_1, s_1, s_0) and the values at $x = 1$ are monotonically increasing in a.

In Table 2.3, we vary the volatility σ. Larger σ implies greater range for the stock price $S_t = \exp(X_t)$ which is associated with larger reward functions. Table 2.3 shows again that both the quadruple (b_0, b_1, s_1, s_0) and the values at $x = 1$ are increasing in σ.

Table 2.2: (b_0, b_1, s_1, s_0) with varying a.

a	0.6	0.7	0.8	0.9	1
b_0	1.2014	1.3833	1.4053	1.4399	1.4843
b_1,	1.2074	1.3845	1.4065	1.4399	1.4934
s_1	1.4793	1.4832	1.6052	1.6789	1.7128
s_0	1.4794	1.4858	1.6079	1.6790	1.7303
$V_{-1}(1)$	0.6607	1.0586	1.3348	1.6121	1.9141
$V_0(1)$	3.4062	3.8040	4.0803	4.3576	4.6595
$V_1(1)$	6.1516	6.5495	6.8258	7.1031	7.4050

Table 2.3: (b_0, b_1, s_1, s_0) with varying σ.

σ	0.3	0.4	0.5	0.6	0.7
b_0	1.2189	1.3115	1.4053	1.4111	1.4289
b_1,	1.2189	1.3119	1.4065	1.4112	1.4293
s_1	1.5726	1.5745	1.6052	1.7413	1.8673
s_0	1.5726	1.5746	1.6079	1.7413	1.8953
$V_{-1}(1)$	0.5724	0.9088	1.3348	1.7234	2.1877
$V_0(1)$	3.3179	3.6543	4.0803	4.4688	4.9332
$V_1(1)$	6.0634	6.3998	6.8258	7.2143	7.6787

Table 2.4: (b_0, b_1, s_1, s_0) with varying ρ.

ρ	0.3	0.4	0.5	0.6	0.7
b_0	1.5540	1.4755	1.4053	1.2387	1.0159
b_1	1.5540	1.4772	1.4065	1.2387	1.0160
s_1	1.9386	1.7908	1.6052	1.5082	1.4737
s_0	1.9704	1.7927	1.6079	1.5082	1.4743
$V_{-1}(1)$	3.4650	2.1385	1.3348	0.7169	0.2446
$V_0(1)$	6.2105	4.8840	4.0803	3.4624	2.9901
$V_1(1)$	8.9559	7.6294	6.8258	6.2079	5.7356

Finally, we vary the discount rate ρ. Larger ρ means smaller reward functions and smaller (b_0, b_1, s_1, s_0). These are confirmed in Table 2.4.

Remark 2.5.1 *We select different slippage rates K. The resulting values for the quadruple (b_0, b_1, s_1, s_0) in Table 2.5 suggest that x_1 is decreasing slightly in K and x_2 stays flat. This is because larger K discourages stock transactions and has to be compensated by smaller x_1. The corresponding values at $x = 1$ seem getting smaller when K goes from* 0.001 *to* 0.02 *but shows no clear cut monotonicity in between.*

2.6 Concluding Remarks

This chapter presents an optimal trading rule which allows one to buy, sell, or sell short the underlying asset. The optimal trading rule can be determined by four threshold levels that correspond to buy and sell points. These threshold levels can be found by solving quasi-algebraic equations. This result can be used as a guide to trading in a mean-reverting

Table 2.5: (b_0, b_1, s_1, s_0) with varying K.

K	0.001	0.005	0.01	0.015	0.02
b_0	1.3979	1.4170	1.4053	1.3185	1.3619
b_1	1.4179	1.4170	1.4065	1.3259	1.3620
s_1	1.6088	1.6150	1.6052	1.6675	1.6124
s_0	1.6557	1.6153	1.6079	1.6743	1.6124
$V_{-1}(1)$	1.3249	1.3413	1.3348	1.2120	1.2860
$V_0(1)$	4.0460	4.0731	4.0803	3.9710	4.0587
$V_1(1)$	6.7670	6.8050	6.8258	6.7301	6.8313

asset. To implement the result in practice, one needs to calibrate the model. Traditionally, the least squares method is used to estimate the system parameters. An alternative is to use the stochastic approximation method to compute the threshold levels directly from the underlying prices, which is discussed in chapter 4. Related literature can be found in Yin, Liu and Zhang [40].

Chapter 3

A Trend Following Strategy: Conditions for Optimality

3.1 Introduction

Active market participants can be classified into two groups according to their trading strategies: Those who trade contra-trend and those who follow the trend. In this chapter, we focus on the trend following (TF) trading strategies. The basic premise underlying the trend following rules is that the market can be regarded either as a bull market or a bear market at a given time. Trend following strategies are concerned with trading rules that trade with the market, i.e., go long if in a bull market or go short if in a bear market. One way to capture the market trends is to use the geometric Brownian motions with regime switching.

A standard geometric Brownian motion (GBM) model involves two parameters, the expected rate of return and the volatility, both assumed to be deterministic constants. In a model with regime switching, these key parameters are allowed to be market trend (or regime) dependent. The regime-switching model was first introduced by Hamilton [23] to describe a regime-switching time series. It is extensively studied in connection with option pricing; see Di Masi et al. [32], Bollen [4], Buffington and Elliott [6], Yao et al. [39], references there in.

Stock trading rules have been studied under various diffusion models for many years. For example, Øksendal [35, Examples 10.2.2-4] considered optimal exit strategy for stocks whose price dynamics were modeled by a geometric Brownian motion. Stock selling rules under regime switching models have gained increasing attention. For example, Zhang [41] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. Under the regime switching model, optimal threshold levels were obtained by solving a set of two-point boundary value problems. In Guo and Zhang [22], the results of Øksendal [35] were extended to incorporate a model with regime switching. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. For example, a stochastic approximation technique was used in Yin et al. [40]; a linear programming approach was developed in Helmes [24]; and the fast Fourier transform was used in Liu et al. [30]. Furthermore, consideration of capital gain taxes and transaction costs in connection with selling can be found in Cadenillas and Pliska [7], Constantinides [10], and Dammon and Spatt [15] among others.

Recently, there has been an increasing volume of literature concerning with trading rules that involved both buying and selling. For instance, Zhang and Zhang [43] studied the optimal trading strategy in a mean reverting market, which validated a well known contra-trend trading method. In particular, they established two threshold prices (buy and sell) that maximized overall discounted return if one traded at those prices. These results are extended to allow short sales in chapter 2. In addition to the results obtained in [43] along this line of research, an investment capacity expansion/reduction problem was considered in Merhi and Zervos [33]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model was treated by Løkka and Zervos [31] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [26] studied an optimal timing of investment problem under a general diffusion market model. The objective was to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution was obtained in [26].

In this chapter, we consider a regime switching model for the stock price dynamics. In this model the price of the stock follows a geometric Brownian motion whose drift switches between two different regimes representing the up trend (bull market) and down trend (bear market), respectively. We model the switching as a Markov chain. In addition, we assume trading one share with a fixed percentage slippage cost. As in Zhang and Zhang [43] we introduce optimal value functions that correspond to starting net position being either flat or long.

In this chapter, we focus on a fundamental issue in TF trading. Under the framework of a regime switching market, we pose the following question: If the investor has the full knowledge of market trends, i.e., s/he knows exactly when the market turns from bull to bear (or bear to bull), will s/he always be profitable? We address this best case scenario.

In particular, we aim at classifing of parameter regions so that the optimal trading strategy varies on each of these regions. We use a dynamic programming approach, and derive a system of two variational inequalities, which can be casted into the form of HJB equations. We find solutions to these equations and construct the corresponding trading rules. We also provide verification theorems to justify the optimality of these trading rules.

The results reveal two counter intutive facts: (a) trend following may not lead to optimal reward in some cases even the investor knows exactly when a trend change occurs; (b) stock volatility is not relvant in trend following when trends are observable.

We present the results in the following order. In §3.2, problem setup is constructed. In §3.3, classification of parameter regions are provided so that the optimal trading rules have the same structure on each of these regions. In §3.4, the associated HJB equations and their solutions are studied. Closed-form solutions are obtained. In §3.5, verification theorems with sufficient conditions are given. Finally, §3.6 concludes the chapter with further remarkd.

3.2 Problem Setup

Let X_t denote the price of the asset under consideration at time t. We consider the case when X_t is a regime swtching geometric Brownian motion governed by

$$dX_t = X_t(\mu(\alpha_t)dt + \sigma(\alpha_t)dW_t), \tag{3.1}$$

where $\alpha_t \in \{1, 2\}$ is a two-state Markov chain, $\mu(1) = \mu_1$, $\mu(2) = \mu_2$ are the expected return rates, $\sigma(1) = \sigma_1$ and $\sigma(2) = \sigma_2$ are the volatilities, and W_t is a standard Brownian motion.

In this chapter, $\alpha_t = 1$ indicates a bull market and $\alpha_t = 2$ a bear market, i.e., $\mu_1 > 0$ and $\mu_2 < 0$.

Assume α_t is observable and its generator is given by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},$$

for some $\lambda_1 > 0$ and $\lambda_2 > 0$. We also assume that $\{\alpha_t\}$ and $\{W_t\}$ are independent.

Remark 3.2.1 Assuming the observability of α_t is not based on realistic considerations. It is imposed mainly to simplify the matter to the extend that we can extract useful information without undue technical difficulties. It allows us to see through the issue in more depth and is helpful to figure out optimality conditions that are hard to see otherwise. In addition, under the best case scenario, we can identify market condtions to avoid potentially trades that deemed to be unprofitable even under the best market information. Finally, the corresponding value functions will provide an upper bound for trading performance which can be used as a general guide to rule out unrealistic expectations.

In this chapter, we allow to buy or sell at most one share at a time. Moreover, We consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding).

Let

$$0 \le b_1 \le s_1 \le b_2 \le s_2 \le \dots$$

denote a sequence of stopping times. A buying decision is made at b_n and a selling decision is made at s_n, for $n = 1, 2, \dots$.

Let k_t denote the net position with

$$k_t = \begin{cases} 0, & \text{flat}, \\ 1, & \text{long one share}. \end{cases}$$

If the initial net position is long ($k = 1$), then one should sell the stock before acquiring any share. Similary, if the initial net position is flat ($k = 0$), then one should first buy a share before a subsequent selling. We define the sequence of stopping times for each initial position k as follows:

$$\Lambda_0 = (b_1, s_1, b_2, s_2, \ldots) \qquad \text{if } k = 0,$$
$$\Lambda_1 = (s_0, b_1, s_1, b_2, s_2, \ldots) \quad \text{if } k = 1.$$

Let $\rho > 0$ be the discount rate and K be the percentage slippage rate. Given the initial states $X_0 = x$, $\alpha_0 = \alpha$, and initial net position $k = 0, 1$, the reward functions of decision sequences, Λ_k, are given as follows:

$$J_k(x, \alpha, \Lambda_k) = \begin{cases} \mathsf{E}\displaystyle\sum_{i=1}^{\infty}[e^{-\rho s_i} X_{s_i}(1-K) - e^{-\rho b_i} X_{b_i}(1+K)], \quad \text{if } k = 0, \\ \\ \mathsf{E}\{e^{-\rho s_0} X_{s_0}(1-K) \\ +\displaystyle\sum_{i=1}^{\infty}(e^{-\rho s_i} X_{s_i}(1-K) - e^{-\rho b_i} X_{b_i}(1+K))\}, \text{if } k = 1. \end{cases}$$

The term $\mathsf{E}\sum_{i=1}^{\infty} \xi_i$ for random variables ξ_i is interpreted as

$$\limsup_{N\to\infty} \mathsf{E}\sum_{i=1}^{N} \xi_i.$$

Given initial position k, let $V_k(x, \alpha)$ denote the value functions with the initial states $X_0 = x$ and $\alpha_0 = \alpha$. That is

$$V_k(x, \alpha) = \sup_{\Lambda_k} J_k(x, \alpha, \Lambda_k). \tag{3.2}$$

Let $0 \le b_1^* \le s_1^* \le b_2^* \le s_2^* \le \ldots$ denote the corresponding jump times of α_t, i.e., $b_1^* = \inf\{t \ge 0 : \alpha_t = 1\}$, $s_i^* = \inf\{t \ge b_i^* : \alpha_t = 2\}$, and $b_{i+1}^* = \inf\{t \ge s_i^* : \alpha_t = 1\}$ for $i = 1, 2, \ldots$. In the rest of this chapter, we focus on the trend following rule: Buy at b_n^* and sell at s_n^*. In the next section, we find regions for (λ_1, λ_2) so that the trend following strategy is optimal.

3.3 Classification of (λ_1, λ_2)-Regions and Assumptions

Frist we note that if $\rho \geq \mu_1$ then "no trading is optimal." Actually, it is easy to see that, for any given Λ_0,

$$Ee^{-\rho s_i} X_{s_i} - Ee^{-\rho b_i} X_{b_i} = E \int_{b_i}^{s_i} e^{-\rho t} X_t(-\rho + \mu(\alpha_t))dt.$$

Note that $(-\rho + \mu(\alpha_t)) \leq 0$ under $\rho \geq \mu_1 > 0$ and $\mu_2 < 0$. This implies that

$$E[e^{-\rho s_i} X_{s_i}(1-K) - e^{-\rho b_i} X_{b_i}(1+K)] \leq 0.$$

It follows that

$$J_0(x, \alpha, \Lambda_0) \leq 0.$$

Therefore, $V_0(x, \alpha) = 0$. Similarly, $V_1(x, \alpha) = (1-K)x$, i.e., one has to sell the share right away at $t = 0$.

Assumptions. In this chapter, we assume $\mu_1 > \rho > 0$ and $\mu_2 < 0$.

Next we determine necessary conditions that guarantee the optimality of trend following trading rules.

Let $\beta_1 > \beta_2$ denote the roots of $(\rho + \lambda_1 - \mu_1)(\rho + \lambda_2 - \mu_2) - \lambda_1\lambda_2 = 0$. In view of [22, Lemma 1], we have

$$\lim_{s_1 \to \infty} \mathsf{E} e^{-\rho s_1} X_{s_1} = \infty,$$

when $\beta_2 < \rho < \beta_1$ which is equivalent to

$$(\rho + \lambda_1 - \mu_1)(\rho + \lambda_2 - \mu_2) - \lambda_1\lambda_2 < 0. \tag{3.3}$$

In this case, the buy ($b_1 = 0$) and hold ($s_1 = \infty$) strategy is optimal and the corresponding payoff $J = \infty$.

Under our trading rule, i.e., buy at b_n^* and sell at s_n^*, in order to generate nonnegative returns, we expect

$$E[e^{-\rho s_i^*} X_{s_i^*}(1-K) - e^{-\rho b_i^*} X_{b_i^*}(1+K)] \geq 0, \ i = 1, 2, \ldots.$$

In particular, if $i = 1$ and $b_1 = 0$, we can show, by writing $X_{s_1^*}$ in terms of α_t and W_t (detailed development is developed later in this chapter in Lemma 3.5.1), that

$$Ee^{-\rho s_1^*} X_{s_1^*} = \frac{\lambda_1 x}{\rho + \lambda_1 - \mu_1}.$$

Note that $\rho + \lambda_1 - \mu_1 > 0$ when $(\rho + \lambda_1 - \mu_1)(\rho + \lambda_2 - \mu_2) - \lambda_1 \lambda_2 > 0$. It suffices to require that

$$\frac{\lambda_1}{\rho + \lambda_1 - \mu_1} > \frac{1+K}{1-K}.$$

For notational simplicity, define

$$H_1 = \frac{\lambda_1}{\rho + \lambda_1 - \mu_1} \text{ and } H_2 = \frac{\lambda_2}{\rho + \lambda_2 - \mu_2}.$$

Using this notation, we construct the following parameter regions.

$$\begin{aligned}
\text{I} \quad &= \{(\lambda_1, \lambda_2) > 0 : \ H_1 H_2 < 1, \ H_1 > (1+K)/(1-K)\}, \\
\text{II} \quad &= \{(\lambda_1, \lambda_2) > 0 : \ H_1 H_2 \leq 1, \ H_1 \leq (1+K)/(1-K)\}, \\
\text{III} \quad &= \{(\lambda_1, \lambda_2) > 0 : \ H_1 H_2 \geq 1, \ H_1 > (1+K)/(1-K)\}, \\
\text{IV} \quad &= \{(\lambda_1, \lambda_2) > 0 : \ H_1 H_2 > 1, \ H_1 \leq (1+K)/(1-K)\}.
\end{aligned}$$

It is easy to see that these four regions consist of a partition of $\{(\lambda_1, \lambda_2) : \ \lambda_1 > 0, \lambda_2 > 0\}$, as shown on figure 3.1.

We will show in the subsequent sections the following

- On Region I: Trend following gives the optimal strategies with finite optimal payoff;
- On Region II: No trade is optimal if there is no initial position; otherwise, hold the position till the first time entering a bear market;
- On Region III: Trend following is optimal with infinite optimal payoff. In this case, the buy and hold strategy is also optimal.
- Finally, on Region IV: The buy and hold strategy is optimal. Trend following on the other hand is not optimal.

In the next few sections, we focus on the optimality of trend following strategies on Region I. Then we discuss the results on other regions.

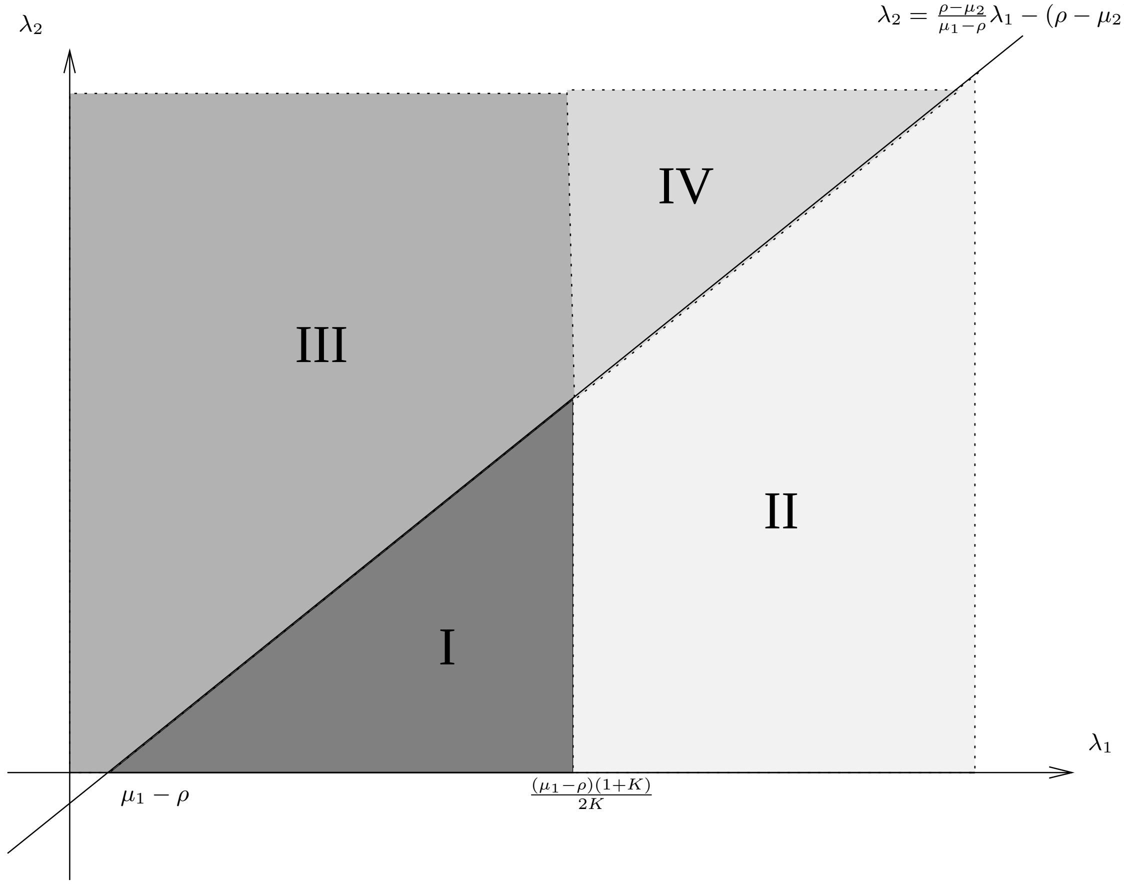

Figure 3.1: The graph of λ_2 against λ_1 with the feasible region (I) of λ_1 and λ_2 for optimality.

3.4 HJB Equations

In this section, we study the corresponding HJB equations. Let $\mathcal{A}$ denote the generator of (X_t, α) given by

$$\mathcal{A}f(x,\alpha) = \frac{x^2\sigma^2(\alpha)}{2}\frac{\partial^2}{\partial x^2}f(x,\alpha) + x\mu(\alpha)\frac{\partial}{\partial x}f(x,\alpha) + Qf(x,\cdot)(\alpha)$$

where

$$Qf(x,\cdot)(\alpha) = \begin{cases} \lambda_1(f(x,2) - f(x,1)), & \text{if } \alpha = 1, \\ \lambda_2(f(x,1) - f(x,2)), & \text{if } \alpha = 2. \end{cases} \tag{3.4}$$

Formally, the associated HJB equations should have the form:

$$\begin{cases} \min\{\rho v_0(x,\alpha) - \mathcal{A}v_0(x,\alpha), v_0(x,\alpha) - v_1(x,\alpha) + x(1+K)\} = 0, \\ \min\{\rho v_1(x,\alpha) - \mathcal{A}v_1(x,\alpha), v_1(x,\alpha) - v_0(x,\alpha) - x(1-K)\} = 0. \end{cases} \tag{3.5}$$

Our trend following rule says that one should buy at the switching times of bear-to-bull and sell at the switching times of bull-to-bear. In terms of the market trend α_t and net position k_t, we have the following strategies.

When $\alpha_t = 1$,

$$\text{if } \ k_t = 0, \quad \text{buy one share,}$$
$$\text{if } \ k_t = 1, \quad \text{hold the share.}$$

When $\alpha_t = 2$,

$$\text{if } \ k_t = 0, \quad \text{stay flat,}$$
$$\text{if } \ k_t = 1, \quad \text{sell one share.}$$

Therefore, $v_k(x,\alpha)$ has to satisfy the following conditions to qualify for being solutions to the HJB equations (3.5):

$$\begin{cases} \rho v_1(x,1) - \mathcal{A}v_1(x,1) = 0, \\ \rho v_0(x,2) - \mathcal{A}v_0(x,2) = 0, \\ v_0(x,1) - v_1(x,1) + x(1+K) = 0, \\ v_1(x,2) - v_0(x,2) - x(1-K) = 0, \\ \rho v_0(x,1) - \mathcal{A}v_0(x,1) > 0, \\ \rho v_1(x,2) - \mathcal{A}v_1(x,2) > 0, \\ v_1(x,1) - v_0(x,1) - x(1-K) > 0, \\ v_0(x,2) - v_1(x,2) + x(1+K) > 0. \end{cases} \tag{3.6}$$

From the first two equations of (3.6), we have

$$\begin{cases} \rho v_1(x,1) &= \dfrac{x^2\sigma_1^2}{2}\dfrac{\partial^2}{\partial x^2}v_1(x,1) + x\mu_1\dfrac{\partial}{\partial x}v_1(x,1) + \lambda_1(v_1(x,2) - v_1(x,1)), \\ \\ \rho v_0(x,2) &= \dfrac{x^2\sigma_2^2}{2}\dfrac{\partial^2}{\partial x^2}v_0(x,2) + x\mu_2\dfrac{\partial}{\partial x}v_0(x,2) + \lambda_2(v_0(x,1) - v_0(x,2)). \end{cases} \tag{3.7}$$

Consider the case when the value functions are linear functions of the initial state x, i.e.

$$\begin{cases} v_1(x,1) &= A_1 x, \\ v_0(x,2) &= A_2 x, \end{cases} \tag{3.8}$$

for some $A_1, A_2 \geq 0$. Then by the third and forth equations of (3.6), we obtain

$$\begin{cases} v_0(x,1) &= [A_1 - (1+K)]x, \\ v_1(x,2) &= [A_2 + (1-K)]x. \end{cases} \tag{3.9}$$

Substitute (3.8) and (3.9) into (3.7) to get

$$\begin{cases} A_1 &= \dfrac{\lambda_1[(\rho+\lambda_2-\mu_2)(1-K) - \lambda_2(1+K)]}{(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) - \lambda_1\lambda_2}, \\ \\ A_2 &= \dfrac{\lambda_2[\lambda_1(1-K) - (\rho+\lambda_1-\mu_1)(1+K)]}{(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) - \lambda_1\lambda_2}. \end{cases}$$

It is not difficult to see that these two constants are positive on Region I. Actually, for $\lambda_1, \lambda_2) \in \mathrm{I}$, we have

$$(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) - \lambda_1\lambda_2 > 0, \tag{3.10}$$

$$\lambda_1(1-K) - (\rho + \lambda_1 - \mu_1)(1+K) > 0. \tag{3.11}$$

Note that $\rho + \lambda_2 - \mu_2 > 0$ as $\mu_2 < 0$, so (3.10) implies that $\rho + \lambda_1 - \mu_1 > 0$. Consequantly,

$$\begin{aligned}
& (\rho + \lambda_1 - \mu_1)[(\rho + \lambda_2 - \mu_2)(1-K) - \lambda_2(1+K)] \\
= \; & (\rho + \lambda_1 - \mu_1)(\rho + \lambda_2 - \mu_2)(1-K) - (\rho + \lambda_1 - \mu_1)\lambda_2(1+K) \\
> \; & \lambda_1\lambda_2(1-K) - (\rho + \lambda_1 - \mu_1)\lambda_2(1+K) \qquad \text{(By (3.10))} \\
= \; & \lambda_2[\lambda_1(1-K) - (\rho + \lambda_1 - \mu_2)(1+K)] \\
> \; & 0 \qquad \text{(By (3.11))}.
\end{aligned}$$

In view of this, we have the following inequality

$$(\rho + \lambda_2 - \mu_2)(1-K) - \lambda_2(1+K) > 0 \tag{3.12}$$

Therefore, $v_k(x, \alpha) \geq 0$ on Region I.

Remark 3.4.1 The conditions (3.11) and (3.12) can be proved without assuming the linearity of the value functions, indeed it does not require any specific form on the value functions. We discuss the proof in Appendix B.

Next we show that the value functions $v_k(x, \alpha)$ defined in (3.8) and (3.9) satisfy the HJB eequations. It suffices to check the last four inequalities of (3.6). Indeed,

$$\begin{cases}
\rho v_0(x,1) - \mathcal{A} v_0(x,1) = [\lambda_1(1-K) - (\rho + \lambda_1 - \mu_1)(1+K)]x > 0, \\
\rho v_1(x,2) - \mathcal{A} v_1(x,2) = [(\rho + \lambda_2 - \mu_2)(1-K) - \lambda_2(1+K)]x > 0, \\
v_1(x,1) - v_0(x,1) - x(1-K) = 2Kx > 0, \\
v_0(x,2) - v_1(x,2) + x(1+K) = 2Kx > 0,
\end{cases}$$

by (3.10) and (3.11).

3.5 A Verification Theorem

We give a verification theorem to show that the solution $v_k(x, \alpha)$ of the HJB equations (3.5) are equal to the value functions $V_k(x, \alpha)$, and sequences of optimal stopping times can be constructed accordingly.

We need the following lemma in the proof of the verification theorem. Recall the jump times of α_t defined as $b_1^* = \inf\{t \geq 0 : \alpha_t = 1\}$, $s_i^* = \inf\{t \geq b_i^* : \alpha_t = 2\}$, and $b_{i+1}^* = \inf\{t \geq s_i^* : \alpha_t = 1\}$ for $i = 1, 2, \ldots$. Recall that

$$H_1 = \frac{\lambda_1}{\rho + \lambda_1 - \mu_1} \text{ and } H_2 = \frac{\lambda_2}{\rho + \lambda_2 - \mu_2}.$$

Lemma 3.5.1. *For each $n = 1, 2, \ldots$, we have*

$$\begin{cases} \mathsf{E} e^{-\rho s_n} X_{s_n} = \begin{cases} (H_1 H_2)^{n-1} H_1 x & \text{if } \alpha_0 = 1, \\ (H_1 H_2)^n x & \text{if } \alpha_0 = 2, \end{cases} \\ \\ \mathsf{E} e^{-\rho b_n} X_{b_n} = \begin{cases} (H_1 H_2)^{n-1} x & \text{if } \alpha_0 = 1, \\ (H_1 H_2)^{n-1} H_2 x & \text{if } \alpha_0 = 2. \end{cases} \end{cases}$$

Proof. Note that

$$X_{s_n} = X_{b_n} \exp\left(\int_{b_n}^{s_n} \left(\mu_1 - \frac{\sigma_1^2}{2} \right) dt + \int_{b_n}^{s_n} \sigma_1 dW_t \right),$$

$$X_{b_{n+1}} = X_{s_n} \exp\left(\int_{s_n}^{b_{n+1}} \left(\mu_2 - \frac{\sigma_2^2}{2} \right) dt + \int_{s_n}^{b_{n+1}} \sigma_2 dW_t \right).$$

We first consider the case when $\alpha_0 = 1$. In this case, $b_1 = 0$. Recall that s_1 is an exponential random variable with parameter λ_1. Note also that, for each u,

$$\exp\left(\int_0^u -\frac{\sigma_1^2}{2}\, dt + \int_0^u \sigma_1 dw_t \right)$$

is a martingale and is independent of s_1. It follows that, by conditioning on s_1,

$$\begin{aligned}
\mathsf{E}e^{-\rho s_1^*}X_{s_1^*} &= x\mathsf{E}\exp\left(\int_0^{s_1^*}\left(-\rho+\mu_1-\frac{\sigma_1^2}{2}\right)dt+\int_0^{s_1^*}\sigma_1\ dw_t\right)\\
&= x\int_0^\infty \mathsf{E}\left[\exp\left(\int_0^u\left(-\rho+\mu_1-\frac{\sigma_1^2}{2}\right)dt+\int_0^u\sigma_1 dw_t\right)\Big|s_1=u\right]\lambda_1 e^{-\lambda_1 u}du\\
&= x\int_0^\infty e^{(-\rho+\mu_1)u}\lambda_1 e^{-\lambda_1 u}du\\
&= \frac{\lambda_1 x}{\rho+\lambda_1-\mu_1}\\
&= H_1 x.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathsf{E}e^{-\rho b_2^*}X_{b_2^*} &= \mathsf{E}\left[e^{-\rho s_1^*}X_{s_1^*}\exp\left(\int_{s_1^*}^{b_2^*}\left(-\rho+\mu_2-\frac{\sigma_2^2}{2}\right)dt+\int_{s_1^*}^{b_2^*}\sigma_2\ dw_t\right)\right]\\
&= \mathsf{E}\left\{e^{-\rho s_1^*}X_{s_1^*}\mathsf{E}\left[\exp\left(\int_{s_1^*}^{b_2^*}\left(-\rho+\mu_2-\frac{\sigma_2^2}{2}\right)dt+\int_{s_1^*}^{b_2^*}\sigma_2\ dw_t\right)\right]\Big|s_1\right\}\\
&= \mathsf{E}\left[e^{-\rho s_1^*}X_{s_1^*}\right]\left(\frac{\lambda_2}{\rho+\lambda_2-\mu_2}\right)\\
&= \frac{\lambda_1\lambda_2 x}{(\rho+\lambda_2-\mu_2)(\rho+\lambda_1-\mu_1)}\\
&= H_1H_2x.
\end{aligned}$$

Continuing this way, we have

$$\begin{cases}
\mathsf{E}e^{-\rho s_n}X_{s_n}=(H_1H_2)^{n-1}H_1x,\\
\mathsf{E}e^{-\rho b_n}X_{b_n}=(H_1H_2)^{n-1}x.
\end{cases}$$

Similarly, if $\alpha_0 = 2$, we can show

$$\begin{cases} \mathsf{E}e^{-\rho s_n} X_{s_n} = (H_1 H_2)^n x \\ \mathsf{E}e^{-\rho b_n} X_{b_n} = (H_1 H_2)^{n-1} H_2 x. \end{cases}$$

The proof is complete.

Theorem 3.5.1. *Let* $(\lambda_1, \lambda_2) \in I$ *and*

$$\begin{cases} v_1(x,1) &= A_1 x, \\ v_0(x,2) &= A_2 x, \\ v_0(x,1) &= [A_1 - (1+K)]x, \\ v_1(x,2) &= [A_2 + (1-K)]x, \end{cases} \tag{3.13}$$

with

$$\begin{cases} A_1 &= \dfrac{\lambda_1[(\rho+\lambda_2-\mu_2)(1-K) - \lambda_2(1+K)]}{(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) - \lambda_1\lambda_2}, \\ \\ A_2 &= \dfrac{\lambda_2[\lambda_1(1-K) - (\rho+\lambda_1-\mu_1)(1+K)]}{(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) - \lambda_1\lambda_2}. \end{cases}$$

Then,

$$v_k(x,\alpha) = V_k(x,\alpha),$$

for $k = 0, 1$ *and* $\alpha = 1, 2$.

In addition, when $k = 0$*, let* $\Lambda_0^* = (b_1^*, s_1^*, b_2^*, s_2^*, \ldots)$*, where the stopping times* $b_1^* = \inf\{t \geq 0 : \alpha_t = 1\}$*,* $s_i^* = \inf\{t \geq b_i^* : \alpha_t = 2\}$*, and* $b_{i+1}^* = \inf\{t \geq s_i^* : \alpha_t = 1\}$ *for* $i = 1, 2, \ldots$*. When* $k = 1$*, let* $\Lambda_1^* = (s_0^*, \Lambda_0^*)$ *with* $s_0^* = \inf\{t \geq 0 : \alpha_t = 2\}$*. Then* Λ_0^* *and* Λ_1^* *are optimal.*

Proof. The proof is divided into two steps. In the first step, we show that $v_k(x,\alpha) \geq J_k(x, \Lambda_{k,\alpha})$ for all $\Lambda_{k,\alpha}$. Then in the second step, we show that $v_k(x,\alpha) = J_k(x, \Lambda_{k,\alpha}^*)$. Therefore, $v_k(x,\alpha) = V_k(x,\alpha)$ and $\Lambda_{k,\alpha}^*$ is optimal.

It is clear that $v_k(x,\alpha)$ satisfy the HJB equations (3.5). Using $\rho v_k(x,\alpha) - \mathcal{A} v_k(x,\alpha) \geq 0$ and Dynkin's formula, we have, for any stopping times $0 \leq \theta_1 \leq \theta_2$, a.s.,

$$\mathsf{E}e^{-\rho\theta_1} v_k(X_{\theta_1}, \alpha_{\theta_1}) \geq \mathsf{E}e^{-\rho\theta_2} v_k(X_{\theta_2}, \alpha_{\theta_2}),$$

for $k = 0, 1$.

Recall that $v_0 \geq v_1 - x(1+K)$ and $v_1 \geq v_0 + x(1-K)$. Given $\Lambda_{k,\alpha} = (b_1, s_1, b_2, s_2, \ldots)$, we have

$$\begin{aligned} v_0(x, \alpha_0) &\geq \mathsf{E} e^{-\rho b_1} v_0(X_{b_1}, \alpha_{b_1}) \\ &\geq \mathsf{E} e^{-\rho b_1}(v_1(X_{b_1}, \alpha_{b_1}) - X_{b_1}(1+K)) \\ &= \mathsf{E} e^{-\rho b_1} v_1(X_{b_1}, \alpha_{b_1}) - \mathsf{E} e^{-\rho b_1} X_{b_1}(1+K) \\ &\geq \mathsf{E} e^{-\rho s_1} v_1(X_{s_1}, \alpha_{s_1}) - \mathsf{E} e^{-\rho b_1} X_{b_1}(1+K) \\ &\geq \mathsf{E} e^{-\rho s_1}(v_0(X_{s_1}, \alpha_{s_1}) + X_{s_1}(1-K)) - \mathsf{E} e^{-\rho b_1} X_{b_1}(1+K) \\ &= \mathsf{E} e^{-\rho s_1} v_0(X_{s_1}, \alpha_{s_1}) + \mathsf{E} e^{-\rho s_1} X_{s_1}(1-K) - \mathsf{E} e^{-\rho b_1} X_{b_1}(1+K) \\ &\geq \mathsf{E} e^{-\rho b_2} v_0(X_{b_2}, \alpha_{b_2}) + \mathsf{E} e^{-\rho s_1} X_{s_1}(1-K) - \mathsf{E} e^{-\rho b_1} X_{b_1}(1+K). \end{aligned}$$

Continuing this way and using the fact that $v_k \geq 0$, we get

$$v_0(x, \alpha) \geq \mathsf{E} \sum_{i=1}^{N} \{e^{-\rho s_i} X_{s_i}(1-K) - e^{-\rho b_i} X_{b_i}(1+K)\}.$$

Sending $N \to \infty$, we have $v_0(x, \alpha) \geq J_0(x, \Lambda_0)$ for all Λ_0. This implies that $v_0(x, \alpha) \geq V_0(x, \alpha)$. Similarly, we can show that $v_1(x, \alpha) \geq V_1(x, \alpha)$.

Now we establish the equalities. It is easy to see that $s_i^* < \infty$ and $b_i^* < \infty$, a.s. Recall that $v_0(x, 1) = v_1(x, 1) - x(1+K)$ and $v_1(x, 2) = v_0(x, 2) + x(1-K)$. We have

$$\begin{aligned} v_0(x, \alpha) &= \mathsf{E} e^{-\rho b_1^*} v_0(X_{b_1^*}, \alpha_{b_1^*}) \\ &= \mathsf{E} e^{-\rho b_1^*}(v_1(X_{b_1^*}, \alpha_{b_1^*}) - X_{b_1^*}(1+K)) \\ &= \mathsf{E} e^{-\rho b_1^*} v_1(X_{b_1^*}, \alpha_{b_1^*}) - \mathsf{E} e^{-\rho b_1^*} X_{b_1^*}(1+K) \\ &= \mathsf{E} e^{-\rho s_1^*} v_1(X_{s_1^*}, \alpha_{s_1^*}) - \mathsf{E} e^{-\rho b_1^*} X_{b_1^*}(1+K) \\ &= \mathsf{E} e^{-\rho s_1^*}[v_0(X_{s_1^*}, \alpha_{s_1^*}) + X_{s_1^*}(1-K)] - \mathsf{E} e^{-\rho b_1^*} X_{b_1^*}(1+K) \\ &= \mathsf{E} e^{-\rho s_1^*} v_0(X_{s_1^*}, \alpha_{s_1^*}) + \mathsf{E}[e^{-\rho s_1^*} X_{s_1^*}(1-K) - e^{-\rho b_1^*} X_{b_1^*}(1+K)]. \end{aligned}$$

Continuing this way, we obtain

$$v_0(x, \alpha) = \mathsf{E} e^{-\rho s_N^*} v_0(X_{s_N^*}, \alpha_{s_N^*}) + \mathsf{E} \sum_{i=1}^{N} [e^{-\rho s_i^*} X_{s_i^*}(1-K) - e^{-\rho b_i^*} X_{b_i^*}(1+K)]. \tag{3.14}$$

Similarly, we have

$$
\begin{aligned}
v_1(x,\alpha) &= \mathsf{E}e^{-\rho s_N^*}v_0(X_{s_N^*},\alpha_{s_N^*}) + \mathsf{E}[e^{-\rho s_0}X_{s_0}(1-K)] \\
&\quad + \mathsf{E}\sum_{i=1}^{N}[e^{-\rho s_i^*}X_{s_i^*}(1-K) - e^{-\rho b_i^*}X_{b_i^*}(1+K)].
\end{aligned}
$$

Finally, it remains to show that $\mathsf{E}e^{-\rho s_N^*}v_0(X_{s_N^*},\alpha_{s_N^*}) \to 0$ as $N \to \infty$. This follows from $v_0(X_{s_N^*},\alpha_{s_N^*}) = A_1 X_{s_N^*}$ and Lemma 3.5.1 under the condition $(\rho+\lambda_1-\mu_1)(\rho+\lambda_2-\mu_2) > \lambda_1\lambda_2$. This completes the proof.

Region II:

We show in this subsection that the trend following is *not* optimal on Region II. This is mainly because λ_1 is too large which lead to the short lived subsequent bull markets. Let

$$
\begin{cases}
v_0(x,1) = 0; \\
v_0(x,2) = 0; \\
v_1(x,1) = \dfrac{\lambda_1(1-K)}{\rho+\lambda_1-\mu_1}x; \\
v_1(x,2) = (1-K)x.
\end{cases}
$$

It is direct to show that these functions solve the HJB equations (3.5). Moreover, it can be shown similarly as in Theorem 3.5.1 that

$$
v_k(x,\alpha) \geq V_k(x,\alpha).
$$

Furthermore, we consider the following strategies: If there is no existing position, do not trade; If the initial holding is one share, then sell it right away in a bear market and, if in a bull market, hold it till the end of the bull market and then sell it. It is easy to see that the corresponding payoff is given by $v_k(x,\alpha)$. Therefore, they are indeed the value functions and the above strategy is optimal.

Region III:

Note that on this region, $H_1H_2 \geq 1$ and $H_1(1-K)-(1+K) > 0$. It follows that, in view of Lemma 3.5.1,

$$\begin{aligned} J_0(x,1,\Lambda_0^*) &= \sum_{i=1}^{\infty}\left[\mathsf{E}e^{-\rho s_i^*}X_{s_i^*}(1-K) - \mathsf{E}e^{-\rho b_i^*}X_{b_i^*}(1+K)\right] \\ &= \sum_{i=1}^{\infty}\left[(H_1H_2)^{i-1}H_1(1-K)x - (H_1H_2)^{i-1}(1+K)x\right] \\ &= \sum_{i=1}^{\infty}(H_1H_2)^{i-1}\left[H_1(1-K)-(1+K)\right]x \\ &= \infty. \end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned} J_0(x,2,\Lambda_0^*) &= \sum_{i=1}^{\infty}\left[(H_1H_2)^{i}(1-K)x - (H_1H_2)^{i-1}H_2(1+K)x\right] \\ &= \sum_{i=1}^{\infty}(H_1H_2)^{i-1}H_2\left[H_1(1-K)-(1+K)\right]x \\ &= \infty. \end{aligned}$$

Note also that $\Lambda_1^* = (s_0^*, \Lambda_0^*)$. Therefore, we have

$$J_1(x,\alpha,\Lambda_1^*) \geq J_0(x,\alpha,\Lambda_0^*) = \infty.$$

It follows that the trend following strategies are optimal and

$$V_k(x,\alpha) = \infty.$$

Also, the buy and hold strategy is optimal as noted in §3.3 when $H_1H_2 < 1$. In this case the value function $V = \infty$.

Region IV:

It is clear that the buy and hold is optimal on this region and the corresponding payoff $J = \infty$, which in turn implies $V = \infty$.

We next show that the trending following is not optimal. Note that on this region, $H_1 H_2 > 1$ and $H_1(1-K) - (1+K) \leq 0$. Using (3.15), we have

$$J_0(x, 1, \Lambda_0^*) = \sum_{i=1}^{\infty} (H_1 H_2)^{i-1} \left[H_1(1-K) - (1+K) \right] x \leq 0.$$

Similarly,

$$J_0(x, 2, \Lambda_0^*) \leq 0.$$

Moreover, the trend following strategy gives

$$\begin{aligned} J_1(x, 1, \Lambda_1^*) &= \mathsf{E} e^{-\rho s_0^*} X_{s_0^*}(1-K) = \frac{\lambda_1 x}{\rho + \lambda_1 - \mu_1} < \infty, \\ J_1(x, 2, \Lambda_1^*) &= x(1-K). \end{aligned}$$

3.6 Concluding Remarks

This chapter presents sufficient conditions for optimal trend following strategies when the switching times between bear and bull markets are observable. The works in §3.3 shows that the optimality can be achieved by satisfying two inequalities in (3.10) and (3.11). Evidently, the possibility of optimality is a compromise between switching rate (λ_1, λ_2) and slippage cost (K). If we rewrite the conditions, we have

$$\begin{cases} \lambda_2 < \dfrac{\rho - \mu_2}{\mu_1 - \rho} \lambda_1 - (\rho - \mu_2), \\ \\ \lambda_1 \leq \dfrac{(\mu_1 - \rho)(1+K)}{2K}. \end{cases}$$

These two inequalities define the feasible region of λ_1 and λ_2 for optimality, as shown in Figure 3.1.

Large λ_1 implies high switching frequency which accumulates significant cost of transactions. On the other hand, small λ_1 means prolonged period of bull market that causes the investment produces infinite return. Note that if the slippage cost K approaches 0, then $\dfrac{(\mu_1 - \rho)(1+K)}{2K}$ tends to infinity which gives no boundaries for the switching rates λ_1 and λ_2. In other words, one can trade as frequent as possible since there is no cost on transactions.

Chapter 4

Stochastic Approximation on Trading Mean-Reverting Assets

4.1 Introduction

Stochastic approximation method is a category of recursive stochastic optimization algorithms that approximate quantities based on functions which are observed through noise. The idea was first introduced in an original work of Robbins and Monro [36] in 1951. It was motivated by the problem of finding a root of a continuous function which is not known but the experimenter is able to obtain noise corrupted measurements of the function at any desired values. One year later, Kiefer and Wolfowitz [27] provided an algorithm that estimates the extrema of such functions when pathwise differetiation is not possible. They proposed a finite difference form of the gradient estimate as an alternative.

The basic paradigm is a stochastic difference equation of the form $\theta_{n+1} = \theta_n + \epsilon Z_n$, where θ_n belongs to some Euclidean space, Z_n is a random variable, and the step size $\epsilon_n > 0$ goes to zero as $n \to \infty$. In this form, θ is a parameter of a system that we attempt to approximate, and Z_n could be a noised function when the parameter is set to θ_n. The parameter θ_n is recursively adjusted to meet the objective asymptotically.

Finding roots of a non-linear function is an exceedingly common problem in science and engineering. If the function is known and continuously differentiabile, it becomes a classical problem in numerical analysis and the Newton-Raphson method can be used to solve. The challenge in Robbins and Monro [36] is that the objective function $f(\theta)$ is not known or interfered by noise, then the Newton-Raphson method fails. They proposed the following recursive algorithm to approximate the root,

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

where Y_n is the noisy estimate of $f(\theta_n)$. Robbins and Monro claim that if the step sizes ϵ_n go to zero in a befitting way, there is an implicit averaging that annihilates the effect of noise in the long run. The convergence of the algorithm can be obtained with some conditions on the function f.

In addition to finding roots of a function, optimization is another essential problem in various areas. It is indeed a kind of roots finding problem since the goal is to estimate roots of the derivative or gradient of a function. If the gradient of a function is observable with or without noise, Newton-Raphson method or the Robbins and Monro algorithm can solve the problem respectively. Nonetheless, what if the gradient of a function is impossible to obtain? Kiefer and Wolfowitz [27] introduced an iterative algorithm in 1952 to overcome this challenge. Let $\tilde{f}(\theta)$ be the noisy estimate of $f(\theta)$ and

$$\tilde{\nabla}\tilde{f}(\theta_n, \xi_n) = \frac{\tilde{f}(\theta_n + \delta_n) - \tilde{f}(\theta_n - \delta_n)}{2\delta_n}$$

be the gradient estimate of $\tilde{f}$ where δ_n is a finite difference sequence such that $\delta_n \to 0$ as $n \to 0$ and ξ_n is a sequence of collective noise. $\tilde{f}(\theta_n)$ is the corrupted value of $f(\theta_n)$ with noise ξ_n. The algorithm suggested by Kiefer and Wolfowitz is

$$\theta_{n+1} = \theta_n + \epsilon_n \tilde{\nabla}\tilde{f}(\theta_n, \xi_n)$$

where ϵ_n is a step sizes sequence satisfying $\epsilon_n \geq 0$, $\epsilon_n \to 0$ as $n \to \infty$. Parallel to the Robbins and Monro algorithm, convergence of the Kiefer and Wolfowitz algorithm can be obtain with some requirements on the function f.

Since the initial work, there has been a steady increase in the investigations of applications in many diverse areas, such as queueing networks, wireless communications, manufacturing systems, problems learning, repeated games and neural nets. Yin, Liu and Zhang [40] even suggested a pioneer application in finance. They developed a class of iterative algorithms to determine the optimal timing of stock liquidation. In this chapter, we present an application of such recursive algorithm in assets trading.

In chapter 2, we give a theoretical analysis of an optimal trading rule on mean-reverting assets. The threshold buy and sell prices are obtained by using dynamic programming approach and the associated HJB equations for the value functions. The analysis assumes that the parameters of the model are known and accurate. Nevertheless, if we want to apply the method in chapter 2 on real financial market, it is not practial simply because accurate parameters are difficult to determine. Therefore, we propose a stochastic approximation algorithm to solve the same problem without assuming a specific model. In order words, model calibration is not needed. We just require a sample path with mean-reverting property. In fact, applying stochastic approximation on mean-reverting asset is not a new approach. Song, Yin and Zhang [38] suggested a stochastic approximation algorithm to estimate the optimal buy and sell prices that maximizes the profit of one trade. In this chapter, we design an algorithm to determine the optimal threshold prices that maximizes the profit of multiple trades with short selling being allowed. This is an improvement of the solution in chapter 2 for real market data.

4.2 Problem Formulation and Algorithm Design

Recall the mean reversion model in (2.1) of chapter 2,

$$dX_t = a(L - X_t)dt + \sigma dW_t,\ \ X_0 = x, \tag{4.1}$$

where $a > 0$ is the rate of reversion, L is the equilibrium level, $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion. Then the observable asset price $S(t)$ is given by $S(t) = e^{X(t)}$.

In our setup, we do not assume the asset prices follow the form (4.1) above or any specific form. We only require the asset prices have a tendancy to go back to the equilibrium. Let $0 < K < 1$ be the slippage rate for each transaction and $\rho > 0$ be the discount rate. The goal is to maximize the reward function

$$\begin{aligned} J(\theta) &= J(b_0^*, b_1^*, s_1^*, s_0^*) \\ &= \mathsf{E}\textstyle\sum_{i=1}^{\infty}\{[e^{-\rho\tau_i}S_{\tau_i}(1-K) - e^{-\rho\psi_i}S_{\psi_i}(1+K)]\mathbb{1}_{\{u_i=1\}} \\ &\quad +[e^{-\rho\psi_i}S_{\psi_i}(1-K) - e^{-\rho\tau_i}S_{\tau_i}(1+K)]\mathbb{1}_{\{u_i=-1\}}\} \end{aligned} \tag{4.2}$$

where $\tau_0 = 0$ and

$$\psi_i = \inf\{t \geq \tau_{i-1} : \ X_t \notin (b_0^*, s_0^*)\};$$

$$\text{if } X_{\psi_i} \leq b_0^*,\ u_i = 1, \text{ then } \tau_i = \inf\{t \geq \psi_i : X_t \geq s_1^*\};$$

$$\text{if } X_{\psi_i} \geq s_0^*,\ u_i = -1, \text{ then } \tau_i = \inf\{t \geq \psi_i : X_t \leq b_1^*\}.$$

Similar to the notations in chapter 2, ψ_i and τ_i are the stopping time for trading, and b_0^*, b_1^*, s_1^* and s_0^* are the threshold prices.

The idea of stochastic approximation method can be explained as follows. Clearly, $J(\theta)$ is not observable as it involves expectation. Nonetheless, we can certainly observe $J(\theta)$ with noise. As a result, we can consider a general form of noised observation of $J(\theta)$ to be $\tilde{J}(\theta, \xi)$, where ξ is the noise. In other words, complex nonlinear function form is allowed in our setup.

We proceed to elaborate the stochastic approximation procedure.

1. Initialization: Choose initial threshold estimate $\theta_0 = (b_0^0, b_1^0, s_1^0, s_0^0)$.

2. Iteration: For $n > 0$, use stochastic approximation to find θ_{n+1} from θ_n. Let e_i be the standard unit vector for $i = 1, 2, 3, 4$, $k_n = \dfrac{1}{n^{\frac{1}{6}}}$ and $\xi_{n,i}^{\pm}$ be noise sequences.

   (a) Calculate $\tilde{J}(\theta_n + k_n e_i, \xi_{n,i}^{+})$ for $i = 1, 2, 3, 4$.

   (b) Calculate $\tilde{J}(\theta_n - k_n e_i, \xi_{n,i}^{-})$ for $i = 1, 2, 3, 4$.

   (c) Compute the gradient estimate $\tilde{\nabla}\tilde{J}(\theta_n, \xi_n) = (\tilde{\nabla}_i\tilde{J}(\theta_n, \xi_n))$ by

   $$\tilde{\nabla}_i\tilde{J}(\theta_n, \xi_n) = \frac{1}{2k_n}[\tilde{J}(\theta_n + k_n e_i, \xi_{n,i}^{+}) - \tilde{J}(\theta_n - k_n e_i, \xi_{n,i}^{-})]$$

   for $i = 1, 2, 3, 4$.

(d) Update θ_{n+1} from θ_n by the stochastic approximation algorithm:

$$\theta_{n+1} = \theta_n + \epsilon_n \tilde{\nabla}_i \tilde{J}(\theta_n, \xi_n)$$

where ϵ_n is the step size sequence satisfying $\epsilon_n \geq 0$, $\epsilon_n \to 0$ as $n \to \infty$, and $\sum_n \epsilon = \infty$. In our design, we choose $\epsilon_n = \frac{1}{n}$, i.e.

$$\theta_{n+1} = \theta_n + \frac{1}{n} \tilde{\nabla}_i \tilde{J}(\theta_n, \xi_n) \tag{4.3}$$

where $\theta_n = (b_0^n, b_1^n, s_1^n, s_0^n)$.

3. Repeat the iterations until the change is less than a certain tolerence level Tol, i.e. $|\theta_{n+1} - \theta_n| < Tol$, or the iterations n reaches a number N where N is large enough.

We present an analysis of the algorithm in the next section.

4.3 Analysis of Convergence

We proceed to examine the asymptotic properties of the proposed algorithm in this section. The technique of the analysis was developed in Kushner and Yin [29]. We show that the recursive equation (4.3) is closely related to an ordinary differential equation, while the stationary points are the optimal threshold prices of our strategy.

Let

$$\begin{aligned} & t_n = \sum_{j=1}^{n} \frac{1}{j}, \\ & m(t) = \max\{n : t_n \leq t\}, \\ & \forall n \ \ \theta^0(t) = \theta_n \text{ for } t \in [t_n, t_{n+1}) \text{ and} \\ & \theta^n(t) = \theta^0(t + t_n). \end{aligned} \tag{4.4}$$

From the definition above, $\theta^0(t)$ is the piecewise constant interpolation of θ_n on the interval $[t_n, t_{n+1})$. To proceed, we define for $i = 1, 2, 3, 4$,

$$\begin{aligned}
\beta_n^i &= \frac{J(\theta_n + k_n e_i) - J(\theta_n - k_n e_i)}{2k_n} - \frac{\partial J(\theta_n)}{\partial \theta^i}, \\
\chi_n^i &= [\tilde{J}(\theta_n + k_n e_i, \xi_{n,i}^+) - \tilde{J}(\theta_n - k_n e_i, \xi_{n,i}^-)] \\
&\quad - \mathsf{E}_n[\tilde{J}(\theta_n + k_n e_i, \xi_{n,i}^+) - \tilde{J}(\theta_n - k_n e_i, \xi_{n,i}^-)], \\
\omega_n^i &= [\mathsf{E}_n \tilde{J}(\theta_n + k_n e_i, \xi_{n,i}^+) - J(\theta_n + k_n e_i)] - [\mathsf{E}_n \tilde{J}(\theta_n - k_n e_i, \xi_{n,i}^-) - J(\theta_n - k_n e_i)],
\end{aligned} \tag{4.5}$$

where E_n is the conditional expectation with respect to the σ-algebra of $\{\xi_j^\pm : j < n\}$, and define

$$\begin{aligned}
\beta_n &= (\beta_n^1, \beta_n^2, \beta_n^3, \beta_n^4), \\
\chi_n &= (\chi_n^1, \chi_n^2, \chi_n^3, \chi_n^4), \\
\omega_n &= (\omega_n^1, \omega_n^2, \omega_n^3, \omega_n^4).
\end{aligned}$$

Then the recursive algorithm (4.3) can be written as

$$\theta_{n+1} = \theta_n + \frac{1}{n}\nabla J(\theta_n) + \frac{1}{n}\frac{\omega_n}{2k_n} + \frac{1}{n}\frac{\chi_n}{2k_n} + \frac{1}{n}\beta_n.$$

Note that ω_n is the noise term and β_n is the bias term. Before we proceed to the analysis of the algorithm, we need to define some terminology, namely weak convergence, tightness and truncation.

Definition 4.3.1 *Let $\{Y_n\}$ and Y be $\mathbb{R}^r$ valued random variables. Y_n is defined to be weakly convergent to Y if*

$$\mathsf{E}f(Y_n) \to \mathsf{E}f(Y) \text{ as } n \to \infty,$$

for any bounded and continuous function $f(\cdot)$.

Definition 4.3.2 *A sequence of random variables $\{Y_n\}$ is defined to be tight if for each $\zeta > 0$, there is a compact set C_ζ such that*

$$\mathsf{P}(Y_n \in C_\zeta) \geq 1 - \zeta \text{ for all } n.$$

Definition 4.3.3 *Let S_v denote a v-sphere, i.e. $S_v = \{x \in \mathbb{R}^r : |x| < v\}$. Given a sequence of processes $Y^n(\cdot)$, $Y^{n,v}(\cdot)$ are defined to be the v-truncation of $Y^n(\cdot)$ if $Y^{n,v}(\cdot) = Y^n(\cdot)$ up until the first exit from S_v and satisfy*

$$\lim_{B\to\infty} \lim_n \sup \mathsf{P}\left\{\sup_{t\le T} |Y^{n,v}(t)| \ge B\right\} = 0 \text{ for each } T < \infty.$$

We can now proceed to the theorem that confirm the convergence and optimality of the algorithm.

Theorem 4.3.1 *Suppose the following conditions* **(A1)** *to* **(A5)** *are satified,*

(A1) *The second derivative of $J(\theta)$ is continuous.*

(A2) *The sequences $\{\xi_n^{\pm}\}$ are bounded.*

(A3) *$\tilde{J}(\cdot,\xi)$ is continuous for each ξ.*

(A4) *For each $0 < M < \infty$ and each $0 < T < \infty$, the set $\left\{\sup_{|\theta|\le M} |\tilde{J}(\theta,\xi_n)| : n \le m(T)\right\}$ is uniformly integrable.*

(A5) *For each θ in a bounded set and for each $T < \infty$,*

$$\sup_n \sum_{j=n}^{m(T+t_n)-1} \frac{1}{j}\sqrt{\mathsf{E}\left|\mathsf{E}_j \frac{\omega_j(\theta,\xi_j)}{2k_j}\right|} < \infty, \text{ and}$$
$$\lim_n \sup_{0\le p\le m(T+t_n)} \mathsf{E}|\nu_p^n| = 0,$$

where

$$\nu_p^n = (n+p) \sum_{j=n+p}^{m(T+t_n)+p-1} \frac{1}{2jk_j} \mathsf{E}_{n+p}[\omega_j(\theta_{n+p+1},\xi_j) - \omega_j(\theta_{n+p},\xi_j)], \text{ for } 0 \le p \le m(T+t_n).$$

Then $\theta^n(\cdot)$ converges weakly to $\theta(\cdot)$ and is a solution of the ordinary differential equation

$$\dot{\theta} = \nabla J(\theta),$$

provided that the equation has a unique solution for each initial condition.

To prove the theorem, we need the following lemma which was proved in Song, Yin and Zhang [38].

Lemma 4.3.1. *Let $q^v(\cdot)$ be a smooth function such that*

$$q^v(\theta) = \begin{cases} 1 & \text{if } \theta \in S_v, \\ 0 & \text{if } \theta \in \mathbb{R}^r - S_{v+1}. \end{cases}$$

Then under the conditions **(A1)** *to* **(A5)**,

$$\mathsf{E}\left| \sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j} \frac{\omega_j}{2k_j} q^v(\theta_j^v) \right|^2 \to 0 \text{ as } n \to \infty, \tag{4.6}$$

$$\mathsf{E}\left| \sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j} \frac{\chi_j}{2k_j} q^v(\theta_j^v) \right|^2 \to 0 \text{ as } n \to \infty, \tag{4.7}$$

and

$$\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j} \left[\frac{\omega_j}{2k_j} + \frac{\chi_j}{2k_j} \right] q^v(\theta_j^v) \to 0 \tag{4.8}$$

with probability one as $n \to \infty$ and the convergence is uniform in t.

Proof of Theorem 4.3.1. The idea of the proof is to first show that the theorem holds for a v-truncation of $\theta^n(\cdot)$, and then let $v \to \infty$ to prove that $\theta^n(\cdot)$ also hold.

Consider the truncated sequence of θ_n,

$$\theta_{n+1}^v = \theta_n^v + \left[\frac{1}{n} \nabla J(\theta_n) + \frac{1}{n} \frac{\omega_n}{2k_n} + \frac{1}{n} \frac{\chi_n}{2k_n} + \frac{1}{n} \beta_n \right] q^v(\theta_n^v). \tag{4.9}$$

Then, similar to (4.4), the interpolation of θ_n^v are

$$\theta^{0,v}(t) = \theta_n^v \text{ for } t \in [t_n, t_{n+1}),$$
$$\theta^{n,v}(t) = \theta^{0,v}(t + t_n).$$

The proof of the theorem for $\theta^{n,v}(t)$ is divided into two parts. First, we show the tightness of $\theta^{n,v}(t)$. Second, we show that the limit of $\theta^{n,v}(t)$ is a solution of the martingale problem with a certain operator.

Consider the separation of noise and bias of $\theta^{n,v}(t)$,

$$\theta^{n,v}(t) = \tilde{\theta}^{n,v}(t) + \sum_{j=1}^{m(t+t_n)-1} \frac{1}{j}\left[\frac{\omega_j}{2k_j} + \frac{\chi_j}{2k_j}\right] q^v(\theta_j^v),$$

where

$$\tilde{\theta}^{n,v}(t) = \sum_{j=1}^{m(t+t_n)-1} \frac{1}{j}[\nabla J(\theta_j^v) + \beta_j] q^v(\theta_j^v).$$

Let $t, s \geq 0$ with $0 \leq s \leq \zeta$, for any $\zeta > 0$. We have

$$\mathsf{E}|\tilde{\theta}^{n,v}(t+s) - \tilde{\theta}^{n,v}|^2 \leq K\mathsf{E}\left|\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}[\nabla J(\theta_j^v) + \beta_j] q^v(\theta_j^v)\right|^2. \tag{4.10}$$

By the boundedness of $\{\theta_j^v\}$, we obtain

$$\begin{aligned} & \mathsf{E}\left|\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}\nabla J(\theta_j^v) q^v(\theta_j^v)\right|^2 \\ \leq\ & K[(t+s+t_n) - (t+t_n)]^2 \qquad (4.11) \\ \leq\ & Ks^2 \\ \leq\ & K\zeta^2. \end{aligned}$$

Similarly,

$$\mathsf{E}\left|\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}\beta_j q^v(\theta_j^v)\right|^2 \leq Ks^2 \leq K\zeta^2. \tag{4.12}$$

By (4.6) and (4.7) in Lemma 4.3.1, it is clear that

$$\limsup_{n\to\infty} \mathsf{E}|\theta^{n,v}(t+s) - \theta^{n,v}(t)|^2 = \mathsf{E}|\tilde{\theta}^{n,v}(t+s) - \tilde{\theta}^{n,v}(t)|^2.$$

Using (4.10), (4.11) and (4.12) followed by letting $\zeta \to 0$, we get

$$\lim_{\zeta\to 0}\limsup_{n\to\infty} \mathsf{E}|\theta^{n,v}(t+s) - \theta^{n,v}(t)|^2 = 0. \tag{4.13}$$

Along with (4.8) in Lemma 4.3.1, $\{\theta^{n,v}(\cdot)\}$ is tight by the criterion of tightness in p. 47 of Kushner [28].

We now proceed to the second part of the proof for $\theta^{n,v}(\cdot)$.

On a complete separable metric space, tightness and sequential compactness are equivalent thanks to Prohorov's theorem. Since the sequence $\{\theta^{n,v}(\cdot)\}$ is tight, there exist a weakly convergent subsequence. Without loss of generality, we re-index the weakly convergent subsequence as $\{\theta^{n,v}(\cdot)\}$ with limit $\theta^{v}(\cdot)$.

By Lemma 4.3.1, we have for each $t, s \geq 0$,

$$\theta^{n,v}(t+s) - \theta^{n,v}(t) = \sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}[\nabla J(\theta_j^v) + \beta_j] q^v(\theta_j^v) + o(1), \tag{4.14}$$

where $o(1) \to 0$ in probability uniformly in t. In addition, by the condition **(A1)** and a truncated Taylor of β_n, it can be shown that

$$\beta_n q^v(\theta_n^v) = O(\frac{k_n^2}{k_n}) = O(k_n).$$

Therefore,

$$\mathsf{E}\left|\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}\beta_j q^v(\theta_j^v)\right| \to 0$$

as $n \to \infty$ and the convergence is uniform in t. Hence,

$$\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j}\beta_j q^v(\theta_j^v) \to 0 \tag{4.15}$$

in probability and uniformly in t as $n \to \infty$.

From the definition of t_n and $m(t)$ in (4.4), there exist an increasing sequence of positive integer $\{b_p\}$ and a decreasing sequence of positive real numbers $\{\alpha_p\}$ depends on n such that $m(t+t_n) \leq b_p \leq b_{p+1} \leq m(t+t_n+s) - 1$ for any $t, s > 0$ and that

$$\frac{1}{\alpha_p}\sum_{j=b_p}^{b_{p+1}-1} \frac{1}{j} \to 1 \text{ as } n \to \infty. \tag{4.16}$$

Let Γ denotes the set $\{p : m(t+t_n) \le b_p \le b_{p+1} \le m(t+s+t_n)\}$. Then

$$\begin{aligned}
&\sum_{j=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{j} \nabla J(\theta_j^v) q^v(\theta_j^v) \\
&= \sum_{\Gamma} \sum_{j=b_p}^{b_{p+1}-1} \frac{1}{j} \nabla J(\theta_j^v) q^v(\theta_j^v) \\
&= \sum_{\Gamma} \sum_{j=b_p}^{b_{p+1}-1} \frac{1}{j} \nabla J(\theta_{b_p}^v) q^v(\theta_{b_p}^v) + o(1) \qquad (4.17)\\
&= \sum_{\Gamma} \nabla J(\theta_{b_p}^v) q^v(\theta_{b_p}^v) \alpha_p \frac{1}{\alpha_p} \sum_{j=b_p}^{b_{p+1}-1} \frac{1}{j} + o(1) \\
&= \sum_{\Gamma} \nabla J(\theta_{b_p}^v) q^v(\theta_{b_p}^v) \alpha_p + o(1)
\end{aligned}$$

by (4.16).

Combining (4.14), (4.15), (4.17) and Skorohod representation, for any bounded and continuous function $g(\cdot)$, continuously differentiable function $h(\cdot)$, any positive integer ι, any $t_i \le t$ with $i \le \iota$, it can be show that there is a sequence $\tilde{\epsilon}_n$ of real numbers such that $\tilde{\epsilon}_n \to 0$ as $n \to \infty$ and that

$$\begin{aligned}
&\mathsf{E} g(\theta^{n,v}(t_i) : i \le \iota)[h(\theta^{n,v}(t+s)) - h(\theta^{n,v}(t))] \\
&= \mathsf{E} g(\theta^{n,v}(t_i) : i \le \iota) \left[\sum_{\Gamma} \alpha_p \nabla h'(\theta_{b_p}^v) q^v(\theta_{b_p}^v) \right] + \tilde{\epsilon}_n \qquad (4.18)\\
&\to \mathsf{E} g(\theta^{n,v}(t_i) : i \le \iota) \int_t^{t+s} \nabla h'(\theta^v(z)) \nabla J(\theta^v(z)) q^v(\theta^v(z)) dz
\end{aligned}$$

as $n \to \infty$. Moreover, the weak convergence and the Skorohod implies that

$$\begin{aligned}
&\mathsf{E} g(\theta^{n,v}(t_i) : i \le \iota)[h(\theta^{n,v}(t+s)) - h(\theta^{n,v}(t))] \\
&\to \mathsf{E} g(\theta^v(t_i) : i \le \iota)[h(\theta^v(t+s)) - h(\theta^v(t))]
\end{aligned} \qquad (4.19)$$

as $n \to \infty$.

As a result, (4.18) and (4.19) lead to that $\theta^v(\cdot)$ is a solution of the martingale problem with operator given by

$$L^v h(\theta^v) = \nabla h'(\theta^v)\nabla J(\theta^v)q^v(\theta^v).$$

In other words, $\theta^v(\cdot)$ is the solution of the truncated ordinary differential equation

$$\dot{\theta^v} = \nabla J(\theta^v)q^v(\theta^v).$$

This finishes the proof of the theorem for $\{\theta^{n,v}(\cdot)\}$ on arbitrary v. We can then let $v \to \infty$, and employ the technique developed on p. 284 of Kushner and Yin [29] to conclude the proof.

4.4 Numerical Examples

In this section, we investiage the numerical performance of our algorithm. In §4.4.1, we simulate the sample paths with the same paramaters as those in the numerical examples on chapter 2, then we apply the proposed algorithm on the sample paths to estimate the threshold prices. This way, we can compare the results from our algorithm with those from the theorethical approach in chapter 2. Subsequently, we generate sample paths with various parameters, follow by applying the algorithm to see how the threshold prices change. In §4.4.2, we demontrate the performance of our algorithm using real market data. A couple of stocks that resemble mean reversion are considered.

4.4.1 Comparison to the Theoretical Results

We consider the same numerical example as in chapter 2. The example has the following specifications,

$$a = 0.8,\ L = 2,\ \sigma = 0.5,\ \rho = 0.5,\ K = 0.01. \tag{4.20}$$

Hence, the corrsponding mean reversion stochastic differential equation is represented by

$$dX(t) = 0.8(2 - X(t))dt + 0.5dW_t. \tag{4.21}$$

We set $X(0) = 2$ and simulate 5000 sample trajectories for the solution of (4.21) by finite different method of 10080 steps with step size $\frac{1}{252}$. This setup simulates 40 years of stock prices with 252 trading days per year. Then we apply the stochastic approximation procedure with 300 iterations to obtain the threshold prices $(b_0^*, b_1^*, s_1^*, s_0^*)$ for each trajectory. We take the sample mean of the 5000 sets of threshold prices and compare with the theoretical results in chapter 2.

The theoretical results in chapter 2 are

$$(b_0, b_1, s_1, s_0) = (1.4053, 1.4065, 1.6052, 1.6079).$$

Since the asset price is given by $S(t) = e^{X(t)}$, the actual threshold prices are

$$(b_0, b_1, s_1, s_0) = (4.0767, 4.0816, 4.9789, 4.9923).$$

Compare with the results of the stochastic approximation proceduce on 5000 Monte Carlo simulations,

$$(b_0^*, b_1^*, s_1^*, s_0^*) = (4.0017, 4.0758, 4.9957, 5.0949),$$

we can see that the results from those two approaches are quite close.

In what follows, we vary one of the parameters in (4.20) at a time and examine the dependence of the threshold prices.

First we compute the threshold levels associated with varying L. Intuitively, larger L would result higher threshold levels $(b_0^*, b_1^*, s_1^*, s_0^*)$. These are confirmed by the results given in Table 4.1. It can be seen that the quadruple $(b_0^*, b_1^*, s_1^*, s_0^*)$ is monotonically increasing in L. In addition, we listed the theoretical results (b_0, b_1, s_1, s_0) from chapter 2 for comparison.

Next, we vary a. A larger a implies larger convergence rate for X_t to reach the equilibrium level L. It shows in Table 4.2 that the quadruple $(b_0^*, b_1^*, s_1^*, s_0^*)$ is monotonically increasing in a. Theoretical results (b_0, b_1, s_1, s_0) are also listed.

Table 4.1: $(b_0^*, b_1^*, s_1^*, s_0^*)$ with varying L.

L	1	1.5	2	2.5	3
b_0	1.4107	2.2109	4.0767	4.1054	9.4593
b_1	1.4988	2.22,	4.0816	6.9455,	9.4603
s_1	1.8995	3.2752	4.9789	9.9513	14.6406
s_0	1.9	3.372	4.9923	9.9513	16.1045
b_0^*	1.1722	2.0244	4.0017	4.6783	8.9144
b_1^*	1.2511	2.5577	4.0758	5.9718	11.5879
s_1^*	1.6349	3.5478	4.9957	8.213	16.6667
s_0^*	1.9096	4.0392	5.0949	10.1178	19.4266

Table 4.2: $(b_0^*, b_1^*, s_1^*, s_0^*)$ with varying a.

a	0.6	0.7	0.8	0.9	1
b_0	3.3248	3.988	4.0767	4.2203	4.4119
b_1	3.3448	3.9928	4.0816	4.2203	4.4522
s_1	4.3898	4.407	4.9789	5.3597	5.5445
s_0	4.3903	4.4185	4.9923	5.3602	5.6423
b_0^*	3.0288	3.3316	4.0017	4.3014	4.4994
b_1^*,	3.2454	3.5244	4.0758	4.3492	4.5047
s_1^*	3.7653	4.1168	4.9957	5.3913	5.6335
s_0^*	3.9967	3275	5.0949	5.4374	5.6437

Table 4.3: $(b_0^*, b_1^*, s_1^*, s_0^*)$ with varying σ.

σ	0.3	0.4	0.5	0.6	0.7
b_0	3.3835	3.7117	4.0767	4.1005	4.1741
b_1	3.3835	3.7132	4.0816	4.1009	4.1758
s_1	4.8192	4.8283	4.9789	5.7048	6.4708
$s_0)$	4.8192	4.8288	4.9923	5.7048	6.6545
b_0^*	3.3049	3.6562	4.0017	3.9806	4.1942
b_1^*	3.6237	3.937	4.0758	4.2633	4.4473
s_1^*	4.5033	4.6862	4.9957	5.367	6.0556
s_0^*	4.8032	4.9592	5.0949	5.7218	6.3758

In Table 4.3, we vary the volatility σ. Larger σ implies greater range for the stock price. Table 4.3 shows again that the quadruple $(b_0^*, b_1^*, s_1^*, s_0^*)$ is increasing in σ. (b_0, b_1, s_1, s_0) are the theorethical results from chapter 2.

Finally, we vary the discount rate ρ. The results in Table 2.4 show that $(b_0^*, b_1^*, s_1^*, s_0^*)$ decrease in ρ. (b_0, b_1, s_1, s_0) are listed for comparison.

After the comparisons of the results from the stochastic approximation proceduce with those in chapter 2, it is clear that they have a great deal of resemblance. First, the dependence of the threshold prices on parameters from both approaches match. They both change in the same direction as the paramaters vary. Second, the value of the thredhold prices from both approaches are very near.

Table 4.4: $(b_0^*, b_1^*, s_1^*, s_0^*)$ with varying ρ.

ρ	0.3	0.4	0.5	0.6	0.7
b_0	4.7304	4.3732	4.0767	3.4511	2.7618
b_1	4.7304	4.3807	4.0816	3.4511	2.7621
s_1	6.949	5.9942	4.9789	4.5186	4.3654
s_0	7.1735	6.0056	4.9923	4.5186	4.368
b_0^*	4.9703	4.4956	4.0017	3.7101	3.0145
b_1^*	4.8818	4.6552	4.0758	3.8498	3.2617
s_1^*	6.6615	6.0917	4.9957	4.522	3.6625
s_0^*	6.492	6.2817	5.0949	4.7104	3.9385

4.4.2 Results from Real Market Data

The major advantage of stochastic approximation algorithm over the theoretical approach in chapter 2 is the unnecessity of model calibration and the computation time. Given the prices of a stock in a period, if we want to find the threshold prices, we first need to calibrate the model by using (1.7). Followed by the simulated annealing algorithm in appendix A to solve a system of integral equations in (2.9). A result from this approach takes about two hours to compute which is extremely unrealistic in practice. On the other hand, for stochastic approximation algorithm, all we need is a sequence of historic prices, and the results can be computed within 30 seconds. Moreover, the algorithm is simple and easy to implement, so it is a much better choice on trading desks.

To demonstrate the performance of the algorithm, we search stock and time period that the prices remember mean reversion. Then we collect the daily closing prices of a qualify stock within an eligible time period. We treat the first half of the data as a historic data, which are used to run the algorithm, and the last half of the data as a future data, which are used to run the trading strategy base on the threshold prices from the algorithm. Since the strategy assume an investor start with flat (holding no share), we have to make sure the investor holds no share as well on the last trading day when we calculate the payoff. In other words, if the net position is long one share on the last, one has to sell the share; if the net position is short one share on the last day, one has to buy one share.

4.4.2.1 Example: Wal-Mart Stores, Inc.

In this example, we apply the proposed stochastic approximation algorithm on the stock prices of Wal-Mart Stores, Inc. (WMT). Figure 4.1 shows the closing prices of each trading day from Jan 3,2005 to Feb 29, 2008.

The experiment is carried out as follow. We first compute the optimal trading prices by using stock prices of first 398 trading days (historic data). After the threshold prices are obtained, we simulate the trading strategy on the stock prices of the remaining 397 trading days (future data).

1. Based on the historic data in the first 398 trading days Jan 3, 2005 - Aug 1, 2006, we run the algorithm with 600 iterations. The results for the optimal threshold prices are

$$b_0^* = 42.8648, b_1^* = 45.2162, s_1^* = 50.2103, s_0^* = 48.1956.$$

2. Using the trading prices above, we practice the trading strategy on the future 398 trading days Aug 2, 2006 - Feb 29, 2008. The total reward in the period is 12.0069. Table 4.5 records the trading dates and prices.

4.4.2.2 Example: Aloca, Inc.

This example considers the stock prices of Alcoa, Inc. (AA) from Jan 4, 1999 to Oct 15, 2002. We apply the same algorithm to determine the threshold prices. Figure 4.2 shows the closing prices of each trading day from Jan 4,1999 to Oct 15, 2002.

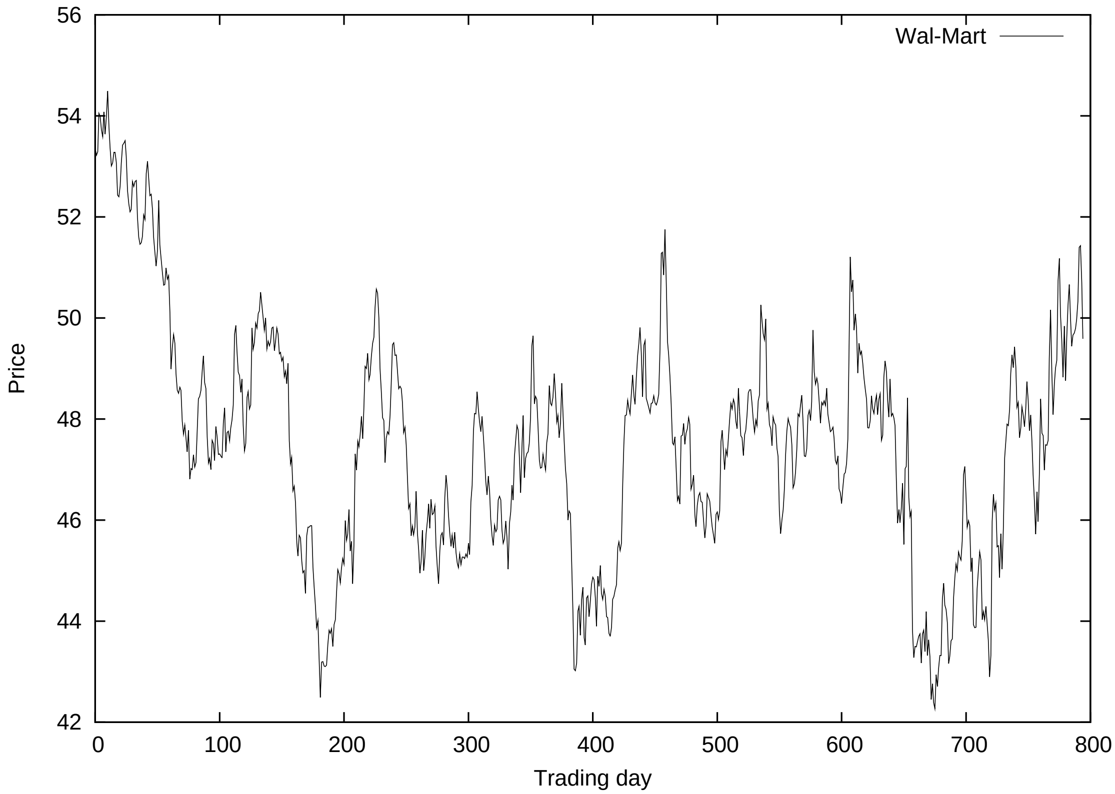

Figure 4.1: Wal-Mart Jan 3, 2005 to Feb 29, 2008 (Courtesy of Yahoo Finance)

Table 4.5: Trading record of WMT stock from Aug 2, 2006 to Feb 29, 2008

Action	Date	Price
Sell	2006-09-14	48.37
Buy	2007-08-14	43.82
Buy	2007-09-05	42.45
Sell	2008-01-31	50.74
Sell	2008-02-01	51.18
Buy	2008-02-29	49.59

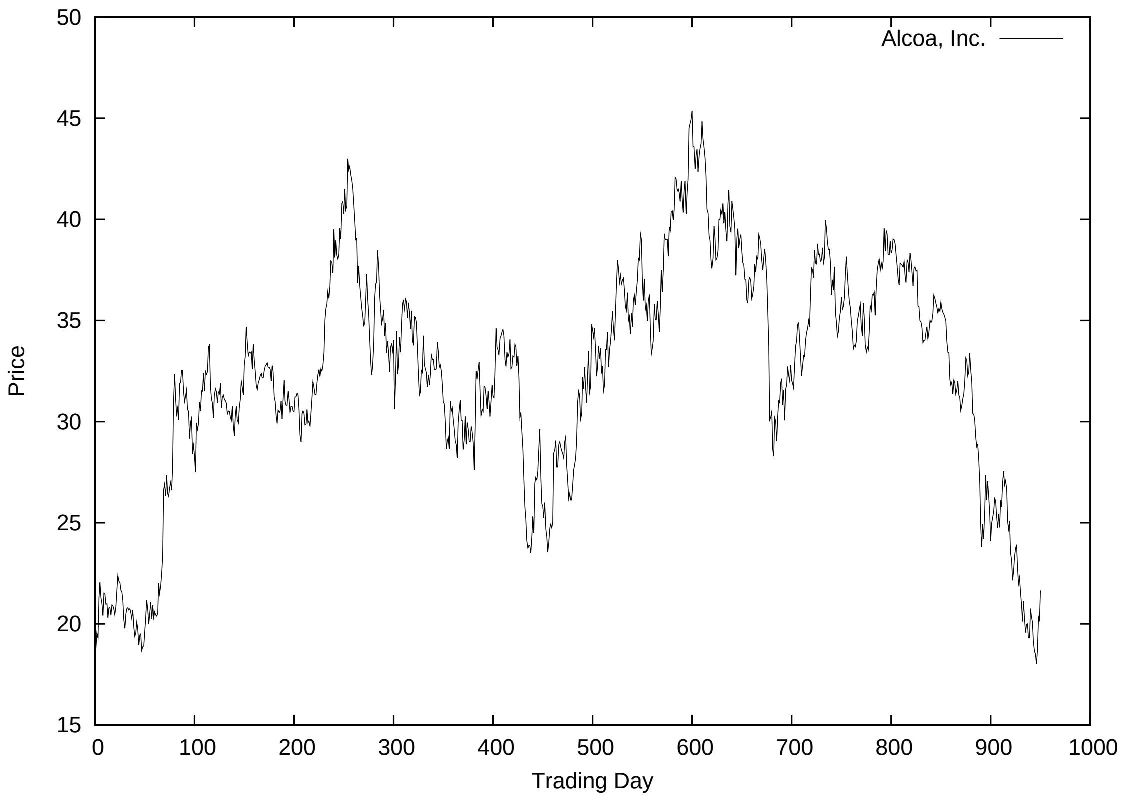

Figure 4.2: Alcoa, Inc. Jan 4, 1999 to Oct 15, 2002 (Courtesy of Google Finance)

Similar to Example 4.4.2.1, we compute the optimal trading prices by using stock prices of first 475 trading days (historic data). After the threshold prices are obtained, we simulate the trading strategy on the stock prices of the remaining 476 trading days (future data).

1. Based on the historic data in the first 475 trading days Jan 4, 1999 - Nov 16, 2000, we run the algorithm with 600 iterations. The results for the optimal threshold prices are

$$b_0^* = 29.6517, b_1^* = 29.4765, s_1^* = 33.4387, s_0^* = 38.743.$$

2. Using the trading prices above, we practice the trading strategy on the future 476 trading days Nov 17, 2000 - Oct 15, 2002. The total reward in the period is 22.5896. Table 4.6 records the trading dates and prices.

4.5 Concluding Remarks

The proposed stochastic approximation algorithm is meant to be an improvement of chapter 2. Compare to the solution of the theoretical method, the proximity of the results in the numerical examples of this chapter indicates that stochastic approximation algorithm does produce reliable estimation on finding optimal trading prices. In addition, it requires a lot less effort (such as model calibration) and computation time. All it needs is historic prices. Most importantly, the implementation is very simple. It is a great choice for automatic trading. From the performance on the real market data, it shows that our algorithm is able to provide reasonable guideline in a short period of time.

The algorithm is developed for mean reversion model. Although it does not required a specific model, it does demand the price of the underlying stock to revert to an equilibrium level. Otherwise, the results come from this algorithm would not be useful. Another caution of using this algorithm is the initial guess. Once a sequence of historic prices is obtained, one needs to observe the trajectory and make a reasonable initial guess. If the initial guess is too much out of line, the algorithm does not give reasonable threshold prices. Since if the

Table 4.6: Trading record of AA stock from Jan 4, 1999 to Oct 15, 2002

Action	Date	Price
Buy	17-Nov-00	27
Sell	19-Dec-00	33.5
Sell	7-Mar-01	39.29
Buy	20-Sep-01	28.61
Buy	21-Sep-01	28.3
Sell	23-Oct-01	33.75
Sell	23-Nov-01	38.8
Buy	15-Jul-02	29.2
Buy	16-Jul-02	28.76
Sell	15-Oct-02	21.65

trajectory does not pass the initial guess prices, the algorithm cannot compute the pay off, and thus the iteration cannot update the result.

Appendix A

Simulated Annealing Algorithm

Simulated annealing is a probabilistic optimization method used for locating global optimum of a given objective function in an enormous search space. The concept comes from annealing in material science, a process involving heating and controled cooling of a material in order to harvest crystals with larger size and fewer defects.

The heat stimulates the atoms in the material and detaches them from their initial position with local minimal internal energy. Then the atoms have a chance to ramble randomly to higher energy states. The controled cooling allows the atoms to recrystalize. If the cooling is slow enough, the nature can find a configuration of atoms with lower internal energy than the initial one. The idea was first introduced in Metropolis et al. [34] and gradually become popular on solving optimization problems such as the traveling saleman problem [8].

By analogy with this natural process, each iteration of the simulated annealing algorithm generates a new point randomly. The distance of the new point from the current point is based on a probability distribution with a scale proportional to the temperature. The algorithm accept all new points that decrease the objective function, however it also accept the new points that increase, with a determined probability. This is the reason that the algorithm is suitable for global optimization. The algorithm avoids being trapped in a local minima by accepting new points that raise the objective function. As the algorithm proceed, an annealing schedule is defined to control the temperature drop until the algorithm is terminated by some stopping conditions. In the next section, we describe the algorithm that was used to solve (2.9) in chapter 2.

A.1 Outline of the Algorithm

To solve the system of integral equations in (2.9) numerically, we find the point that minimizes the ℓ_2 norm of the difference of the left and the right hand sides.

Let

$$L = \begin{pmatrix} -y_1(b_1) & y_2(b_1) & y_1(b_1) & 0 \\ -y_1'(b_1) & y_2'(b_1) & y_1'(b_1) & 0 \\ 0 & -y_2(b_0) & -y_1(b_0) & y_2(b_0) \\ 0 & -y_2'(b_0) & -y_1'(b_0) & y_2'(b_0) \end{pmatrix}^{-1} \begin{pmatrix} e^{b_1}(1+K) \\ e^{b_1}(1+K) \\ e^{b_0}(1+K) \\ e^{b_0}(1+K) \end{pmatrix},$$

$$R = \begin{pmatrix} -y_1(s_0) & y_2(s_0) & y_1(s_0) & 0 \\ -y_1'(s_0) & y_2'(s_0) & y_1'(s_0) & 0 \\ 0 & -y_2(s_1) & -y_1(s_1) & y_2(s_1) \\ 0 & -y_2'(s_1) & -y_1'(s_1) & y_2'(s_1) \end{pmatrix}^{-1} \begin{pmatrix} e^{s_0}(1-K) \\ e^{s_0}(1-K) \\ e^{s_1}(1-K) \\ e^{s_1}(1-K) \end{pmatrix}$$

and

$$\theta = (b_0, b_1, s_1, s_0).$$

Then the objective function is

$$f(\theta) = \|L - R\|_2 \tag{A.1}$$

The following are the steps of the algorithm:

1. Choose an initial guess of the estimates $\theta_{\mathbf{0}}$ and compute $f(\theta_{\mathbf{0}})$. Initial temperature $\mathbf{t_0} = (100, 100, 100, 100)$. Initial temperature parameter $\mathbf{k_0} = (1, 1, 1, 1)$. Reanneal interval $R = 100$. Upper bound of the search space $\mathbf{UB} = (UB_1, UB_2, UB_3, UB_4)$. Lower bound of the search space $\mathbf{LB} = (LB_1, LB_2, LB_3, LB_4)$.

2. Generates a point using Student's t distribution generates a point based on the current point and the current temperature using Student's t distribution.

   Let $\mathbf{t_n} = (t_1^n, t_2^n, t_3^n, t_4^n)$ be the current (n^{th} iteration) temperature, $\theta_{\mathbf{n}} = (\theta_1^n, \theta_2^n, \theta_3^n, \theta_4^n)$ and $\mathbf{Y} = (y_1, y_2, y_3, y_4)$ be a random vecter with $y_i \sim N(0,1)$. Then $\hat{\mathbf{Y}} = \frac{\mathbf{Y}}{\|\mathbf{Y}\|_2}$ is the unit vector of $\mathbf{Y}$ and

$$\theta_{\mathbf{n+1}} = \theta_{\mathbf{n}} + \mathbf{t_n} * \hat{\mathbf{Y}},$$

where $*$ is an entry-wise multiplication. If $f(\theta_{\mathbf{n+1}}) < f(\theta_{\mathbf{n}})$, $\theta_{\mathbf{n+1}}$ is accepted as the next point. Otherwise, $\theta_{\mathbf{n+1}}$ can still be accepted by the following acceptance probability - Boltzmann probability density.

Let $u \sim \text{Uniform}(0,1)$ and $\Delta f = f(\theta_{\mathbf{n+1}}) - f(\theta_{\mathbf{n}})$, if

$$B = \frac{1}{1 + e^{\dfrac{\Delta f}{\max\{\mathbf{t_n}\}}}} > u,$$

then $\theta_{\mathbf{n+1}}$ is accepted, otherwise it is rejected.

3. After a new point is accepted, the algorithm lower the temperature by first increase the temperature parameter. Let $\mathbf{k_n} = (k_1^n, k_2^n, k_3^n, k_4^n)$, hence $\mathbf{k_{n+1}} = \mathbf{k_n} + \mathbf{1} = (k_1^n + 1, k_2^n + 1, k_3^n + 1, k_4^n + 1)$. Then we decrease the temperature by the following formula,

$$\mathbf{t_{n+1}} = \mathbf{t_0} * 0.95^{\mathbf{k_{n+1}}}, \tag{A.2}$$

again $*$ and the power operation are both entry-wise. The algorithm also stores the best solution at this step.

4. Reannealing is performed after the number of points accepted by the algorithm reaches R which is defined in step 1. This is another way to guarantee that the algorithm is not trapped in a local minima. It raise temperature vector on each entry by first re-define $\mathbf{k_{n+1}}$. Let

$$s_i = (UB_i - LB_i) \left. \frac{\partial f(\theta)}{\partial \theta_i} \right|_{\theta_i = \theta_i^n}$$

$$k_i^{n+1} = \left| \ln | \frac{t_i^0}{t_i^n} \frac{\max\{\mathbf{s}\}}{s_i} | \right|$$

where t_i^n and θ_i^n are the i^{th} entry of the current temperature parameter and point respectively, for $i = 1, 2, 3, 4$. If $\dfrac{t_i^0}{t_i^n} \dfrac{\max\{\mathbf{s}\}}{s_i}$ is so small that it is less than the tolerance Tol (which is discussed in the stopping conditions part) of the algorithm, then $k_i^{n+1} = |\ln(Tol)|$. Hence, the temperature vector can be re-defined by (A.2).

5. The algorithm stops when one of the following stopping conditions is met:

   - The average change in value of the objective function in $500 \times$ number of variables $= 2000$ iterations is less than $Tol = 10^{-6}$.
   - The number of function evaluations exceeds $3000 \times$ number of variables $= 12000$.

Appendix B

Alternative Proof of conditions (3.11) and (3.12)

Let $X_t = e^{Y_t}$. Then $v_0(X_t, 1) = v_0(Y_t, 1)$ and $Y_t = \ln(X_t)$. By the generalized Ito-Doeblin formula (1.8),

$$\begin{aligned} dY_t &= (\mu(\alpha_t) - \tfrac{1}{2}\sigma^2(\alpha_t))dt + \sigma(\alpha_t)dW_t + Q\ln(X_t) \\ &= (\mu(\alpha_t) - \tfrac{1}{2}\sigma^2(\alpha_t))dt + \sigma(\alpha_t)dW_t, \end{aligned}$$

since $Q\ln(X_t) = 0$. Then the generator $\mathcal{G}$ of Y_t is given by

$$\mathcal{G}f(y,\alpha) = \frac{1}{2}\sigma^2(\alpha)\frac{\partial^2 f}{\partial y^2}dt + (\mu(\alpha) - \frac{1}{2}\sigma^2(\alpha))\frac{\partial f}{\partial y} + Qf(x,\cdot)(\alpha),$$

where $Qf(x,\cdot)(\alpha)$ is the same as that in (3.4). The HJB conditions in (3.6) becomes

$$\begin{cases} \rho v_1(y,1) - \mathcal{G}v_1(y,1) = 0, \\ \rho v_0(y,2) - \mathcal{G}v_0(y,2) = 0, \\ v_0(y,1) - v_1(y,1) + e^y(1+K) = 0, \\ v_1(y,2) - v_0(y,2) - e^y(1-K) = 0, \\ \rho v_0(y,1) - \mathcal{G}v_0(y,1) > 0, \\ \rho v_1(y,2) - \mathcal{G}v_1(y,2) > 0, \\ v_1(y,1) - v_0(y,1) - e^y(1-K) > 0, \\ v_0(y,2) - v_1(y,2) + e^y(1+K) > 0. \end{cases} \tag{B.1}$$

From the first two equations of (B.1), we get

$$\rho v_1(y,1) = \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_1(y,1) + (\mu_1 - \frac{\sigma_1^2}{2})\frac{\partial}{\partial y}v_1(y,1) + \lambda_1(v_1(y,2) - v_1(y,1)) \tag{B.2}$$

and

$$\rho v_0(y,2) = \frac{\sigma_2^2}{2}\frac{\partial^2}{\partial y^2}v_0(y,2) + (\mu_2 - \frac{\sigma_2^2}{2})\frac{\partial}{\partial y}v_0(y,2) + \lambda_2(v_0(y,1) - v_0(y,2)). \tag{B.3}$$

Substitute the forth equation of (B.1) into (B.2) and the thirth equation of (B.1) into (B.3), we obtain

$$(\rho+\lambda_1)v_1(y,1) = \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_1(y,1) + (\mu_1 - \frac{\sigma_1^2}{2})\frac{\partial}{\partial y}v_1(y,1) + \lambda_1(v_0(y,2) + e^y(1-K))$$

and

$$(\rho+\lambda_2)v_0(y,2) = \frac{\sigma_2^2}{2}\frac{\partial^2}{\partial y^2}v_0(y,2) + (\mu_2 - \frac{\sigma_2^2}{2})\frac{\partial}{\partial y}v_0(y,2) + \lambda_2(v_1(y,1) - e^y(1+K)).$$

As a result,

$$\begin{aligned} & v_0(y,2) \\ = \; & \frac{1}{\lambda_1}\left((\rho+\lambda_1)v_1(y,1) - \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_1(y,1) - (\mu_1 - \frac{\sigma_1^2}{2})\frac{\partial}{\partial y}v_1(y,1) - \lambda_1 e^y(1-K)\right) \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} & v_1(y,1) \\ = \; & \frac{1}{\lambda_2}\left((\rho+\lambda_1)v_0(y,2) - \frac{\sigma_2^2}{2}\frac{\partial^2}{\partial y^2}v_0(y,2) - (\mu_2 - \frac{\sigma_2^2}{2})\frac{\partial}{\partial y}v_0(y,2) - \lambda_2 e^y(1+K)\right). \end{aligned} \tag{B.5}$$

Next, we want to find the conditions for $\rho v_0(y,1)-\mathcal{G}v_0(y,1) > 0$ and $\rho v_1(y,2)-\mathcal{G}v_1(y,2) > 0$ which lead to (3.11) and (3.12). By (B.4), we have

$$
\begin{aligned}
& \rho v_0(y,1) - \mathcal{G}v_0(y,1) \\
= \; & -\frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_0(y,1) + (\frac{\sigma_1^2}{2} - \mu_1)\frac{\partial}{\partial y}v_0(y,1) + (\lambda_1+\rho)v_0(y,1) - \lambda_1 v_0(y,2) \\
= \; & -\frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_0(y,1) + (\frac{\sigma_1^2}{2} - \mu_1)\frac{\partial}{\partial y}v_0(y,1) + (\lambda_1+\rho)v_0(y,1) \\
& -\left((\rho+\lambda_1)v_1(y,1) - \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial y^2}v_1(y,1) + (\mu_1 - \frac{\sigma_1^2}{2})\frac{\partial}{\partial y}v_1(y,1) + \lambda_1 e^y(1-K)\right) \\
= \; & \frac{\sigma_1^2}{2}(\frac{\partial^2}{\partial y^2}v_1(y,1) - \frac{\partial^2}{\partial y^2}v_0(y,1)) + (\frac{\sigma_1^2}{2} - \mu_1)(\frac{\partial}{\partial y}v_0(y,1) - \frac{\partial^2}{\partial y^2}v_1(y,1)) \\
& +(\lambda_1+\rho)(v_0(y,1) - v_1(y,1)) + \lambda_1 e^y(1-K) \\
= \; & \frac{\sigma_1^2}{2}e^y(1+K) + (\frac{\sigma_1^2}{2} - \mu_1)(-e^y(1+K)) + (\lambda_1+\rho)(-e^y(1+K)) + \lambda_1 e^y(1-K) \\
= \; & e^y((\mu_1-\rho)(1+K) - 2\lambda_1 K).
\end{aligned}
$$

Therefore, $\rho v_0(y,1) - \mathcal{G}v_0(y,1) > 0$ implies that

$$(\mu_1 - \rho)(1+K) - 2\lambda_1 K > 0$$

which is condition (3.11).

Similarly, by (B.3), we can show that

$$\rho v_1(y,2) - \mathcal{G}v_1(y,2) = e^y((\rho - \mu_2)(1-K) - 2\lambda_2 K).$$

In addition, $\rho v_1(y,2) - \mathcal{G}v_1(y,2) > 0$ implies that $(\rho - \mu_2)(1-K) - 2\lambda_2 K > 0$ which is condition (3.12). This completes the proof.

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