RELATIONSHIPS BETWEEN PRESERVICE SECONDARY MATHEMATICS TEACHERS' BELIEFS, KNOWLEDGE, AND TECHNOLOGY USE

by

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(Under the Direction of Denise A. Spangler)

ABSTRACT

This multi-case study examines the relationships between preservice teachers' beliefs and knowledge regarding teaching mathematics with technology. Based on the theoretical framework on teacher beliefs and Technology, Pedagogy, and Content Knowledge (TPACK), I investigated four preservice secondary mathematics teachers' technological pedagogical content knowledge (TPCK) and their beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology in the mathematics classroom. Three semi-structured interviews (beliefs, task-based, and performance interviews) were used to collect data about the preservice teachers' TPACK components, beliefs, and how to use a technology tool in their imaginary mathematics teaching.

The findings of this study indicated that preservice teachers with sophisticated or student-centered beliefs about the nature of mathematics, learning mathematics, and technology use displayed higher levels of mathematical content knowledge, pedagogical content knowledge, and technological content knowledge, respectively, than preservice teachers with traditional or teacher-centered beliefs about mathematics, learning mathematics, and technology use. In addition, this study suggested that in order to effectively use technology to teach mathematics, preservice teachers should develop their beliefs and knowledge in all areas of mathematics, pedagogy, and technology.

Understanding the relationships between preservice teachers' TPACK and beliefs provides insights into how teacher education programs can support preservice teachers to develop TPACK and integrate technology into their future mathematics instruction.

INDEX WORDS: Preservice secondary mathematics teachers, Teacher beliefs, Teacher knowledge, Technology, Technological pedagogical content knowledge

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TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The emergence and advancement of technology has brought great changes to mathematics as an academic discipline and to the learning and teaching of mathematics (Guerrero, 2010; Habre & Grundmeier, 2007). Technology has let us develop, explore, and expand new and existing mathematical ideas by providing concrete modeling and applications through various advanced computer technologies (Grandgenett, 2008; Guerrero, 2010). With the variety of technologies available, techniques for learning and teaching mathematics have become dynamic, diverse, and effective. Substantial research has demonstrated technology's positive effect on the mathematical learning process. Technology can help students to acquire not only computation skills but also mathematical ideas, conceptual understanding, and connections among various representations (Abboud & Habre, 2006; Kaput, Hegedus, & Lesh, 2007; Roschelle et al., 2010).

In 2000, the National Council of Teachers of Mathematics (NCTM) offered its vision of mathematics teaching using technology and the Technology Principle in its publication *Principles and Standards for School Mathematics*: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). The NCTM emphasized not only technology's capabilities to engage students in high-level thinking and in-depth mathematics learning but also the role of a teacher in a technology-rich classroom: "The teacher plays several important roles in a technology-rich classroom, making decisions that affect students' learning in important ways. Initially, the teacher must decide if, when, and how technology will be used" (NCTM, 2000, p. 26). Therefore, the use of technology in mathematics class is encouraged, and preservice and current mathematics teachers are expected to be able to make prudent decisions when integrating technology. Staying in line with the NCTM's vision, the Association of Mathematics Teacher Educators (2006) recommended that "mathematics teacher preparation programs must ensure that all mathematics teachers and teacher candidates have opportunities to acquire the knowledge and experiences needed to incorporate technology in the context of teaching and learning mathematics" (p. 1).

With the increase in the need for training that integrates technology into teaching, many teacher education and professional development programs offer technology courses for preservice and current teachers. Teacher educators and researchers encourage teachers' student-centered technology uses that "support inquiry, collaboration, or re-configured relationships among students and teachers" (Culp, Honey, & Mandinach, 2005, p. 302) and enable students to engage in higher levels of thinking with less cognitive load by providing visualization and representation of problems (Jonassen, 2003) as the best educational technology practices (Bigatel, 2004; Coppola, 2004; Ottenbreit-Leftwich, Glazewski, Newby, & Ertmer, 2010). Teachers have tended to use technology, however, to display lesson content or support their existing practices rather than to implement inquiry-based, collaborative, or problem-solving activities and projects (Culp et al., 2005; Ottenbreit-Leftwich et al., 2010). According to Project Tomorrow (2008), in its "Speak Up 2007" survey, 51% of the responding teachers reported that when using computers they primarily assign homework or drill-and-practice work as a way to "facilitate student learning." In addition, Project Tomorrow (2011) compared the results from two of its "Speak Up" surveys (2008, 2010) to show how technology use in the classroom has changed over time.

Although some relatively sophisticated uses of technology (e.g., conducting investigations, creating graphic organizers) were significantly higher in 2010 than in 2008, the majority of teachers' technology uses were still limited to providing homework and practice. Moreover, the percentage of teachers using technology for assigning homework and practice work had increased from 2008.

Why do teachers have a difficult time using technology effectively in their teaching? Why is technology used in such different ways among teachers with the same relevant knowledge? According to Ertmer (1999, 2005) and Hew and Brush (2007), there are two sets of barriers to the integration of technology into teaching: first-order barriers, which refer to factors such as environmental readiness (e.g., lack of time, computers, or Internet access), and second-order barriers, which refer to internal factors such as teachers' beliefs. As technology integration into the classroom has been encouraged and funded, however, problems caused by external constraints (access, support, etc.) have been resolved in the majority of schools (Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012). Teachers' beliefs may constitute a more deeply ingrained barrier to student-centered technology use (Ertmer, 2005; Ottenbreit-Leftwich et al., 2010).

As an international student, I was surprised when I first noticed students use simple or graphing calculators in mathematics classrooms in the United States. The general perspective in South Korea is the use of calculators in the mathematics classroom hinders students' mathematical thinking and increases the students' dependence on the calculator, and mathematics teachers do not use calculators in their classrooms. Initially, I shared this attitude. While studying in the United States, however, I came to understand the great potential of technology integration to facilitate students' mathematics learning, and I no longer think technology obstructs students' conceptual understanding. I became interested in strategies for effectively using technology to teach mathematics and developing preservice mathematics teachers' knowledge of how to use technology effectively in their future teaching. With much research on the influence of beliefs and knowledge on technology use—and having experienced a change in my own beliefs about technology integration—I wondered whether or how preservice mathematics teachers' beliefs and their knowledge of technology use in the mathematics classroom are related to each other. I suspected preservice teachers' knowledge of how to use technology to teach mathematics, together with their beliefs, formed a strong basis for integration of technology into their future teaching. I was also interested in the use of a dynamic geometry environment (DGE) such as *Geometer's Sketchpad* (*GSP* [Jackiw, 2009]), which helps students acquire a deeper understanding of geometric concepts (Laborde, Kynigos, Hollebrands, & Strässer, 2006). *GSP* provides an environment in which students can freely investigate in nontraditional ways to learn mathematical ideas (Marrades & Gutiérrez, 2000), and it can enable students to construct deductive explanations while providing a foundation for mathematical ideas of proof (Jones, 2000). Thus, I chose *GSP* as a technology tool to see how preservice teachers use *GSP* and what their uses reveal about their beliefs and knowledge of technology integration.

Background

I found that a great deal of the research addressed mathematics teachers' beliefs, knowledge, and classroom practices regarding technology while looking for related research literature. Mishra and Koehler (2006) posited that in order to effectively integrate technology into the classroom, teachers need to have specialized and interwoven knowledge—that is, technological pedagogical content knowledge (TPACK; originally TPCK). The TPACK framework comprises three main components—content, pedagogy, and technology—and the intersections between and among them. TPACK is “an understanding that emerges from interactions among content, pedagogy, and technology knowledge” (Koehler & Mishra, 2009, p. 66). Mishra and Koehler’s TPACK framework has provided an important means to understand the complexity of technology integration and examine teachers’ knowledge of how to use technology to teach subject matter.

Much of the research I found on teachers’ pedagogical beliefs and teaching practice regarding technology showed teachers’ beliefs are strong indicators of their teaching practice with technology. Kim, Kim, Lee, Spector, and DeMeester (2013) indicated “teachers’ beliefs predict, reflect, and determine their actual teaching practice” (p. 77), as many studies have asserted (Kagan, 1992; Pajares, 1992; Wilkins, 2008). Thus, teachers with similar knowledge and skills can practice different teaching styles because of their different beliefs (Ernest, 1989b). In addition, the procurement of technology and technology-related knowledge does not always guarantee successful technology integration (Polly, Mims, Shepherd, & Inan, 2010). Teachers’ beliefs, defined as intrinsic barriers that “hinder technology integration, can interfere with teachers’ technology integration even when first-order barriers are overcome” (Kim et al., 2013, p. 77). More specifically, effective technology incorporation requires the use of technology consistent with or compatible with teachers’ existing pedagogical beliefs (Ertmer, 2005).

Many studies addressed preservice teachers’ beliefs and attitudes toward technology use. Some studies (Messina & Tabone, 2015; Turner & Chauvot, 1995) indicated preservice teachers’ beliefs about technology use tended to be teacher centered. By contrast, preservice teachers in Choy, Wong, and Gao’s study (2009) showed their confidence and intention to use technology for student-centered learning.

Although the goals of teacher education include bringing preservice teachers’ beliefs into alignment with student-centered learning and developing their TPACK for effective incorporation of technology into mathematics teaching, little research has explicitly examined both preservice teachers' pedagogical beliefs and their TPACK. Instead, most researchers focused on preservice teachers' pedagogical beliefs even though many researchers acknowledged the importance of beliefs in the nature of mathematics (e.g., Ernest, 1989a, b; Raymond, 1997). Moreover, many researchers used self-report surveys for preservice teachers in elementary or early childhood education that were not designed to accurately capture the nuances of individuals' beliefs and intentions and their mathematics-related TPACK. Thus, in this study, I used a multi-case methodology to investigate preservice secondary mathematics teachers' TPACK in a specific mathematics context, along with their beliefs about the nature of mathematics as a discipline, learning and teaching mathematics, and using technology in mathematics class.

Rationale

Mouza, Karchmer-Klein, Nandakumar, Ozden, and Hu (2014) stated "although this generation of preservice teachers is more technologically savvy and actively engaged with digital media, knowledge and skills alone are not sufficient conditions for curricular use of technology in support of rigorous standards" (p. 206). To encourage and improve effective technology use of preservice mathematics teachers, it is essential for teacher education programs to focus on both preservice mathematics teachers' beliefs and knowledge regarding mathematics, pedagogy, and technology (Crompton, 2015).

Through an investigation into preservice mathematics teachers' beliefs, their TPACK, and the relationships between the two, teacher educators can design courses or experiences that provide preservice teachers with opportunities not only to construct appropriate knowledge to teach mathematics using technology but also to reflect on their beliefs about mathematics, teaching, learning, and technology. Thus, teacher education programs can develop strategies to foster mathematics teachers who can teach mathematics through the appropriate use of technology. In addition, it is possible for professional development programs to provide resources to address teachers' beliefs to increase their teaching quality with technology. Ultimately, students can learn mathematics with the benefits of using technology. Therefore, this study on the relationships between preservice secondary mathematics teachers' beliefs (that is, beliefs about the nature of mathematics, learning and teaching mathematics, and technology use) and their TPACK in mathematics context is important and can contribute to a deeper understanding of both preservice teachers' beliefs and their TPACK and the development of their future teaching with technology.

Research Questions

With the goal of clarifying and understanding the relationships between preservice secondary mathematics teachers' beliefs and their TPACK, and using a multi-case methodology with three semi-structured interviews (beliefs, task-based, and performance interviews), I investigated the following research questions:

1. What are preservice secondary mathematics teachers' beliefs about the nature of mathematics, learning mathematics, teaching mathematics, and the use of technology in the mathematics classroom?
2. What levels of TPACK components do the preservice secondary mathematics teachers have in the context of geometry?
3. How do the preservice secondary mathematics teachers' beliefs (that is, their beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology in the mathematics classroom) relate to their TPACK components?

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Teachers' beliefs and knowledge affecting their teaching practice have been of interest to mathematics educators and researchers for several decades. Of particular interest in this study is the relationship between preservice mathematics teachers' beliefs and their knowledge about teaching mathematics with technology. Although many researchers have investigated mathematics teachers' beliefs or knowledge (e.g., Ball, Thames, & Phelps, 2008; Pajares, 1992; Philipp, 2007; Shulman, 1986), relatively few studies have directly considered the relationship between preservice teachers' beliefs about and knowledge of how to use technology in the context of teaching mathematics. In this chapter, I focus on preservice and current mathematics teachers' beliefs and knowledge about teaching mathematics with technology. I begin by defining beliefs and then discuss knowledge, belief systems, and current and preservice mathematics teachers' beliefs and their use of technology. Next, I address knowledge that teachers need in order to effectively integrate technology into their teaching. Finally, I review research related to teachers' beliefs and knowledge about teaching with technology.

Defining Beliefs

In the 1970s, interest in teachers' beliefs became heightened as the focus of research on teaching shifted from teachers' behaviors to their thinking and decision-making processes (Clark & Peterson, 1986; Munby, 1982; Thompson, 1992). It was not until 1980, however, that teachers' beliefs about the nature of mathematics, teaching mathematics, and learning mathematics were considered as one of the crucial research topics in mathematics education (Pehkonen & Pietilä, 2003; Thompson, 1992). Although a large amount of research addresses teachers' beliefs, the definitions of beliefs in the research are not the same. Pajares (1992) suggested clarifying the definition of belief in the research: "It will not be possible for researchers to come to grips with teachers' beliefs ... without first deciding what they wish belief to mean and how this meaning will differ from that of similar constructs" (p. 308). It is not easy to define or distinguish among beliefs, conceptions, and knowledge, however, because these terms are sometimes used as synonyms (Pajares, 1992).

Some researchers (e.g., Singletary, 2012; Thompson, 1992) view conception as "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007, p. 259). These researchers consider beliefs as a subset of conceptions. Pehkonen (2004), however, defines conception as one's conscious or professed beliefs, that is, a subset of beliefs. Although my perspective aligns with Pehkonen's description, I also agree with Thompson's (1992) statement about conception in terms of mathematics: "It will be more natural at times to refer to a teachers' conception of mathematics as a discipline than to simply speak of the teachers' beliefs about mathematics" (p. 130). Thus, I view conception with respect to mathematics as comprising conscious or professed beliefs concerning the discipline of mathematics.

Furinghetti and Pekhonen (2002) investigated the multiple definitions of beliefs by surveying 18 mathematics educators and identified two types of knowledge: objective and subjective. The researchers indicated objective knowledge is accepted by the mathematics community, such that individuals are able to approach this knowledge and construct "their own conceptions of mathematical concepts and procedures, i.e. they construct some pieces of their subjective knowledge" (p. 53), whereas subjective knowledge is informal, personal, and private knowledge that is not necessarily made public and evaluated by other people. Pehkonen and Pietilä (2003) considered beliefs related to subjective knowledge as personal, experience-based, and tacit knowledge, and my perspective aligns with theirs. I view knowledge as objective knowledge, that is, formal, justifiable, or verifiable, and beliefs as subjective knowledge, that is, individual understandings about the world constructed based on personal experience.

Belief Systems

One of the goals of my study is to identify preservice secondary mathematics teachers' beliefs about mathematics, teaching, learning, and technology. In this study, my view of teachers' belief systems is conceptualized based on Green's (1971) structure of belief systems.

In *The Activities of Teaching*, Green (1971) suggested three dimensions of belief systems: the relationships between beliefs, the degree of strength of beliefs, and the characteristics of clustering beliefs. The first of these dimensions concerns a "quasi-logical" structure of belief systems. According to Green, belief systems have a particular order between beliefs. This order cannot be said to be logical, however, because beliefs are arranged according to the logic in people's belief systems. Green called some beliefs *primary* and others *derivative*. Given three beliefs A, B, and C that a person holds, it can be the case that "A is seen as the reason for B, and B, in turn, as the reason for some other belief, say C" (p. 44). Thus, in this system, A is a primary belief and B and C are derivative. Green's second dimension considers the "psychological strength" of beliefs. These beliefs are viewed as either *central* or *peripheral* depending on how strongly they are held. According to Green, central beliefs are the most strongly held, and peripheral beliefs can be more easily challenged and changed. The third dimension is related to the claim that "beliefs are held in clusters, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs" (p. 48). This dimension implies that it is possible to simultaneously hold conflicting core beliefs that reside within different belief clusters. Each of these characteristics of Green's belief system "has to do not with the content of our beliefs, but with the way we hold them" (p. 48).

In another perspective on belief systems, a sensible system, Leatham (2006) argued teachers develop beliefs in ways that make sense to them. This perspective is informed by considering Thagard's (2000) coherence theory of justification that individuals' beliefs are justified when they cohere with their other beliefs mutually supportive of each other. In addition, "to justify a belief . . . we do not have to build up from an indubitable foundation; rather we merely have to adjust our whole set of beliefs . . . until we reach a coherent state" (Thagard, 2000, p. 5). Beliefs are viable within a belief system when they make sense in the context of individual's other beliefs. Thus, when contradictory beliefs in different clusters are revealed, the individual must redress or adjust the conflict because sensible belief systems do not allow for overt contradictions (Singletary, 2012). In sensible systems, beliefs influence teachers' decision making and their actions. If teachers' actions appear to contradict their beliefs, it is possible that the researcher has misunderstood the implication of the belief or is unaware of what beliefs of teachers influenced their actions in the particular situations.

Inconsistencies are often revealed, however, not only between teachers' beliefs and their practices but also within their beliefs. Teachers can hold conflicting beliefs at the same time "without becoming evident because they [their beliefs] are connected to different contexts, certainty and consciousness" (Drageset, 2010, p. 32). This perspective is in line with Green's (1971) third dimension of belief systems that contradictory beliefs can be held in different clusters. In addition, Drageset noted a belief can be simultaneously derivative and psychologically central or simultaneously primary and psychologically peripheral. For example, beliefs lacking psychological strength may not influence teachers' decisions or actions even if the beliefs are held as primary. Therefore, it is important to investigate the psychological strength of a teacher's beliefs in a given situation to better understand how the teacher makes decisions (Drageset, 2010). Green's structure of belief systems as a theoretical framework helped me to gain insight into how preservice teachers' beliefs are organized and are related to their other beliefs; it also explains away conflicting beliefs and inconsistencies between beliefs and behavior.

Teachers' Beliefs and Use of Technology

In the literature, I found diverse terms that many researchers have used to define types of teaching or learning. I address these terms in this section first. In general, there are two contrasting sets of adjectival terms: constructivist/student-centered and traditional/teacher-centered. First, "constructivist" is a derivative of "constructivism" that refers to a learning theory. In this perspective, learners are viewed as creators of their own understanding by combining what they already believe to be true based on past experiences with new experiences (Richardson, 1997). In addition, knowledge is viewed as a product of an individual's construction of the experiential world. Thus, mathematics is viewed as a human creation that is continually expanding. "Student-centered" is aligned with "constructivist" in that students are the main agents of their own learning. Student-centered approaches tend to emphasize interactive activities in which students can address unique learning interests and needs to deepen their understanding (Hannafin & Land, 1997).

Second, "traditional" beliefs or approaches are based on the idea that teaching is mainly the transmission of knowledge and that learning is the passive reception of transmitted knowledge. In traditional classrooms, teachers have authority and can control students' learning activities. From this perspective, mathematics is viewed as a collection of facts, rules, and skills that is fixed, absolute, certain, and applicable (Raymond, 1997). “Teacher-centered” approaches are closely related to “traditional” approaches in that knowledge is primarily transmitted by the teacher through telling. Teacher-centered approaches tend to focus more on content knowledge than on student thinking or processing and place “control for learning in the hands of the teacher” (Brown, 2003, p. 50).

Drageset (2010) stated that “Beliefs influence the decisions that individuals make and also serve as the best indicators of their decisions” (p. 32). As a result of this perspective, many researchers have investigated teachers’ beliefs and the influence of their beliefs on their teaching practices (e.g., Leder, Pehkonen, & Törner, 2002; Pajares, 1992; Richardson, 1996; Thompson, 1992, 1984). Although some researchers have shown that practice is not always consistent with beliefs (Cooney, 1985; Raymond, 1997; Thompson, 1992, 1984), teachers’ beliefs are still a strong foundation for their teaching practices (Kim, Kim, Lee, Spector, & DeMeester, 2013; Pajares, 1992; Wilkins, 2008). Teachers’ particular methods of teaching mathematics or of using their knowledge are affected by their beliefs about what mathematics is and how to teach and learn it (Brown & Cooney, 1982).

Ernest (1989b) argued “It is necessary to consider beliefs to account for the differences between mathematics teachers” (p. 20). He suggested three aspects of mathematics teachers’ beliefs: (1) conception of the nature of mathematics, (2) model of teaching mathematics, and (3) model of learning mathematics.

Ernest classified three different views of the nature of mathematics in a hierarchy: instrumentalist, Platonist, and problem-solving. Mathematics teachers who hold an instrumentalist view believe mathematics is a useful but unrelated set of facts and rules. Teachers with a Platonist view believe mathematics is a static and unified body of knowledge. Teachers who have a problem-solving view believe mathematics is a human creation that is dynamic and continually expanding.

Ernest (1989b) described teachers' beliefs about teaching mathematics as "the teacher's conception of the type and range of teaching actions and classroom activities contributing to his or her personal approaches to the teaching of mathematics" (p. 22). The categories of beliefs about teaching are described through teachers' roles: instructor, explainer, and facilitator (Ernest, 1989b). Ernest indicated teachers who hold an instructor view believe mathematics teachers provide facts, procedures, and skills mastery. Teachers with an explainer view believe mathematics teachers foster students' conceptual understanding of a unified body of knowledge. Teachers who have a facilitator view believe mathematics teachers help students become autonomous problem posers and problem solvers.

Finally, Ernest categorized mathematics teachers' beliefs about learning as passive reception of knowledge and active construction of knowledge. Teachers who hold a passive view of learning believe a student is transmitted mathematical knowledge directly from a teacher, whereas teachers who hold an active view of learning believe students actively construct their own mathematical knowledge.

Ernest (1989b) explained that teachers' beliefs about the nature of mathematics are associated with their views of teachers' roles and students' learning. For example, teachers with a Platonist view—who see mathematics as a certain body of knowledge—tend to be explainers in the classroom and to see students' learning as the passive reception of knowledge. Thus, teachers may teach in different ways because of their different views concerning the nature of mathematics, teaching, or learning, even if they have similar mathematical knowledge (Ball, 1991; Ernest, 1989a).

Ottenbreit-Leftwich, Glazewski, Newby, and Ertmer (2010) stated teachers' beliefs can play a crucial role not only in their general instructional practices but also in specific technology integration practices (Dwyer, Ringstaff, & Sandholtz, 1991; Ryba & Brown, 2000; Yocum, 1996). Many researchers have investigated the influence of teachers' pedagogical beliefs on their use of technology in the classroom and have provided diverse findings (e.g., Cope &Ward, 2002; Ertmer, Gopalakrishnan, & Ross, 2001; Hermans, Tondeur, van Braak, & Valcke, 2008; Judson, 2006).

In an early study, the *Apple Classrooms of Tomorrow* research project, Dwyer and colleagues examined the impact of technology on teaching and learning in K–12 classrooms and developed a model of teachers' evolution in high-tech classrooms (Dwyer, Ringstaff, & Sandholtz, 1991). Each classroom in this project was equipped with Macintosh® computers with software (e.g., word processors, CAI software, spreadsheets, HyperCard), printers, scanners, and videotape players. The authors' model categorized five phases of teachers' development: entry, adoption, adaptation, appropriation, and invention. As teachers participated in the project over the years, their beliefs about and teaching practices with technology shifted toward "child-centered rather than curriculum-centered instruction; towards collaborative rather than individual tasks; towards active rather than passive learning" as they passed through the five phases (Dwyer et al., 1991, p. 50). Dwyer and colleagues also found teachers continuously struggled with their traditional beliefs about teaching and learning and with new technology practices even after their experiences of successful instruction with technology. This research provided evidence teachers' beliefs about teaching and learning are critical underliers of their resistance to change.

Many studies indicated teachers' pedagogical beliefs align with their use of technology in their teaching practices. Teachers who hold constructivist beliefs tended to use technology in student-centered approaches, while teachers who hold traditional beliefs tended to use technology in teacher-centered approaches (e.g., Cope & Ward, 2002; Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012; Hermans et al., 2008; Kim et al., 2013).

Hermans and colleagues (2008) investigated the relationship between primary school teachers' educational beliefs and their use of Information and Communications Technology (ICT) using multilevel modeling (n = 525). The researchers used the constructivist beliefs and traditional beliefs scale of Woolley, Benjamin, and Woolley (2004) to measure participants' educational beliefs (independent variable) and used a modified version of the *Class Use of Computers* scale of van Braak, Tondeur, and Valcke (2004) to measure participants' use of computers to support teaching and learning (dependent variable). This study provided empirical evidence teachers' beliefs about teaching and learning are significant determinants of their use of computers in the classroom. In particular, teachers' constructivist beliefs about teaching and learning are significant predictors of their computer use, whereas traditional beliefs have a negative impact on integrating ICT into the classroom. The results of the study suggest teachers with constructivist beliefs tend to adopt technology aligning with student-centered learning approaches.

In my opinion, however, it is difficult to see how teachers' constructivist beliefs are associated with their constructivist use of technology based on the results from Hermans and colleagues. In the questionnaire, Hermans and colleagues distinguished among different uses of computers in the classroom. Although the *Class Use of Computers* items included both non-constructivist (e.g., encouraging pupils to train on certain skills, teaching about the possibilities of computers) and constructivist (e.g., encouraging cooperative learning) uses of computers, the scale showed adequate internal consistency (α = 0.76). Thus, because the *Class Use of*

Computers scale does not reflect a solely constructivist orientation, it is not appropriate to conclude that teachers with constructivist beliefs appear to implement technology using constructivist approaches even though the results showed that constructivist beliefs have a significant positive impact on classroom use of computers. The results of this study indicated teachers' constructivist beliefs are an indicator of *how often* they use computers in the classroom rather than *how* they use them.

In a study conducted by Ertmer and colleagues (Ertmer et al., 2012), 12 K–12 classroom teachers who had been recognized as technology-using teachers were chosen to examine the correspondence between teachers' pedagogical beliefs and their technology practices in the classroom. The researchers collected data from teachers' personal and/or classroom websites for evidence of their classroom technology practices and conducted follow-up interviews to examine the teachers' beliefs supporting their practices. Ertmer and colleagues found 11 of the 12 teachers in the study implemented technology in ways well-aligned with their pedagogical beliefs. Moreover, teachers with student-centered beliefs tended to support student-centered curricula, and teachers with teacher-centered beliefs were more likely to implement teacher-centered curricula. For example, teacher Barnes believed teachers should be facilitators who serve in "the learning process, answering questions along the way and providing just-in-time learning" (p. 429); on Barnes' website, students could access a wide variety of ideas for their projects. By providing a project-based approach, Barnes facilitated students' use of technology and encouraged students to be the main agents of their own learning. In another case, teacher Cross revealed her beliefs about the role of technology, saying, "I think that the main goal has still got to be delivering the content," and her use of technology in the classroom aligned with those beliefs. Cross also explained how she used technology not only for instruction but also for class management: To be able to work with one group of students, she instructed the other students to use technology to reinforce specific skills. The results of this study provided evidence suggesting that teachers' pedagogical beliefs are well aligned with their technology use in the classroom but did not imply that student-centered beliefs and technology use are more appropriate than teacher-centered beliefs and technology use.

Some researchers report, however, there is not always a consistency between teachers' beliefs and their teaching practices with technology (e.g., Berg, Benz, Lasley, & Raisch, 1998; Ertmer, Gopalakrishnan, & Ross, 2001; Judson, 2006). Although teachers may possess constructivist beliefs, their technology use may not reflect those beliefs, that is, they consistently use technology in traditional ways such as in drill-and-practice exercises (Ertmer et al., 2001).

For example, Judson (2006) investigated how teachers' beliefs regarding instruction and their attitudes toward technology related to the practice of integrating technology in teaching. The *Conditions that Support Constructivist Uses of Technology* survey (Ravitz & Light, 2000) was used to measure 32 K-12 teachers' beliefs about what constitutes quality instruction and their attitudes about using technology. The *Focusing on Integrating Technology: Classroom Observation Measurement* (Judson, 2002) was used to measure teaching practices that integrated technology in the constructivist context. Judson found no significant correlation, however, between beliefs about instruction and teaching practices using technology. Although most teachers revealed strongly constructivist convictions, they failed to exhibit these beliefs in their teaching practices using technology.

Some researchers have suggested possible reasons for the inconsistency between teachers' beliefs and their classroom technology practices (Berg, Benz, Lasley, & Raisch, 1998; Ertmer et al., 2001; Ravitz, Becker, & Wong, 2000). Ertmer et al. and Ravitz et al. indicated the disparity between teachers' beliefs and practices appeared to be linked to external barriers such as predetermined curricula, assessment practices, external forces, and expectations. In Berg et al.'s (1998) study, participants reported that access to technology or the Internet and time constraints hindered their higher-level technology use.

I found that many studies used self-report surveys to measure teachers' pedagogical beliefs. However, Likert-type questionnaires, which are well documented as inadequate to accurately state participants' beliefs, both because individual items may be open to interpretation and numerical results do not provided detailed information about beliefs. To measure teachers' use of technology, most researchers observed teachers' teaching, often using quantitative counts of behaviors based on observations of teachers' teaching. Most studies showed that K-12 teachers have consistency between their pedagogical beliefs and technology use in their teaching. In particular, teachers who held constructivist or student-centered beliefs tended to use technology in student-centered approaches, while teachers who had traditional or teacher-centered beliefs tended to use technology in teacher-centered approaches. Although not all participants in these studies were secondary mathematics teachers, the findings still gave me insights that teachers' pedagogical beliefs are important indicators of their technology practices.

Preservice Teachers' Beliefs and Use of Technology

Preservice teachers' beliefs also have a strong influence on their behavior, including teaching and learning (e.g., Pajares, 1992; Calderhead & Robson, 1991; Ross, Johnson, & Smith, 1992). Preservice teachers bring with them "highly idealistic, loosely formulated, deeply seated, and traditional" entering beliefs about teaching and learning into their teacher education programs (Richardson, 2003, p. 6), and these beliefs have been developed as they observed, deduced, and evaluated teachers' roles during their thousands of hours as students (Lortie, 1975;

Richardson, 2003). In particular, preservice teachers' beliefs when they enter teacher education programs strongly affected their "interpretations of particular courses and classroom practices and played a powerful role in determining how they translated and used the knowledge they possessed and how they determined the practices they would later undertake as teachers" (Pajares, 1992, p. 310).

Kay and Knaack (2005) stated preservice teachers' beliefs, attitudes, knowledge, and skills are also crucial factors in their integrating technology into their future teaching. Studies of preservice teachers' beliefs and attitudes toward integrating technology into teaching have been widely conducted and published (e.g., Amado & Carreira, 2006; Messina & Tabone, 2015). Some researchers reported preservice teachers tended to have limited or teacher-centered beliefs about technology use (Messina & Tabone, 2015; Turner & Chauvot, 1995), whereas other researchers indicated preservice teachers showed their confidence and intention to use technology for student-centered learning (Amado & Carreira, 2006; Choy, Wong, & Gao, 2009).

For example, Messina and Tabone (2015) investigated 79 preservice teachers to identify their technology proficiency, knowledge, and beliefs regarding the value of technology in teaching and learning. To measure the teachers' beliefs regarding the role and value of technology in education, the researchers used the *Teacher Technology Integration Survey* (Vannatta & Banister, 2009), partially revising the cognitive attitude section of the Computer Attitude Measure (Kay, 1993). The results indicated the majority of preservice teachers viewed technology as devices assisting teaching such as providing teaching aids or creating materials to teach rather than as enhancing student collaboration, creativity, and active involvement.

In Turner and Chauvot's (1995) study, which focused on two preservice secondary mathematics teachers' beliefs about technology, both preservice teachers believed successfully exploring the topic of mathematics with technology requires students already have knowledge about the topic. The preservice teachers stated they would use technology with their students after they had taught the students to perform mathematics calculations by hand.

Crompton (2015) found certain experiences in student teaching or the beginning of teadhing in the field may obstruct preservice teachers use of technology in teaching. Also, preservice teachers' beliefs can develop from their previous experiences with technology. If they have few or negative experiences, they may not use technology in their teaching (Crompton, 2015). Preservice teachers may also choose not to use technology in their student teaching even if they are competent in its effective integration (Amado & Carreira, 2006; Choy, Wong, & Gao, 2009). This decision can be due to preservice teachers' lack of knowledge of how to teach mathematics or external barriers such as lack of time or supports from the teacher community or the schools.

Choy, Wong, and Gao (2009) studied 118 preservice elementary school teachers in Singapore to explore their intentions and actions regarding technology integration in their classrooms. In this study, using survey instruments they designed, Choy and colleagues examined preservice teachers' intentions to integrate technology in their future teaching (as stated before and after they completed a technology course) and their actual actions in integrating technology during their student teaching. In the post-student teaching survey, the preservice teachers were asked to evaluate their actual actions in their student teaching. In addition to the survey, the researchers selected 10 volunteer preservice teachers to collect in-depth data through one-on-one semi-structured interviews and through observing the preservice teachers' lessons during their student teaching. The results of this study showed preservice teachers had positive intentions to incorporate technology in their future teaching to facilitate student-centered learning. During their student teaching, however, the preservice teachers tended to use technology to prepare handouts, record grades and attendance, or communicate with other teachers rather than to facilitate student-centered learning. According to the analysis of interviews and lesson observations, 8 out of 10 preservice teachers used technology as an instructional tool to convey information and gain students' attention—for instance, using PowerPoint or the Internet to show images or videos. The preservice teachers showed their competence at and confidence in using technology for student-centered learning in their practice, but the results of this study indicated that they were unable to reflect their positive intentions in their technology integration.

Similar to studies of teachers' beliefs, half of the studies that I found on preservice teachers' beliefs or attitudes toward technology used self-report surveys to measure participants' beliefs and attitudes. In the literature, I found that, although there are diverse findings in studies of preservice teachers' beliefs and attitudes regarding technology use, recent studies showed that preservice teachers have positive attitudes toward technology use and their beliefs are more student-centered. In addition, little research has investigated preservice teachers' teaching with technology, such as their student-teaching or the beginning of their teaching practice. Most of the studies I found showed that preservice teachers tend to use technology in traditional or teacher-centered approaches due to external constraints or to their lack of pedagogical content knowledge even though they have student-centered beliefs and competence in technology use. Thus, in contrast with current teachers, preservice teachers tend to show inconsistency between their beliefs about technology use and the actual use of technology in their teaching.

Technological Pedagogical Content Knowledge (TPACK)[1]

Teaching is a complex and ill-structured practice that requires interlacing many kinds of specialized knowledge (Koehler & Mishra, 2009). The complexity of integrating teaching and technology makes it difficult for teachers to use technology. In addition, recognizing specific technologies' properties, affordances, and constraints that make them appropriate for certain tasks (Bromley, 1998; Bruce, 1993; Guerrero, 2010; Koehler & Mishra, 2008) and understanding how the features of technologies have an impact on what teachers do in their teaching are not straightforward (Koehler & Mishra, 2009).

An effective integration of technology into instruction requires teachers' appropriate knowledge about how to use technology in their instruction. Mishra and Koehler (2006) proposed the construct of Technological Pedagogical Content Knowledge (originally TPCK) which is now known as TPACK, or Technology, Pedagogy, and Content Knowledge Framework, building on Shulman's (1986) descriptions of Pedagogical Content Knowledge (PCK). The TPACK framework consists of three main components: Content, Pedagogical, and Technological Knowledge (CK, PK, and TK), and the intersections between and among them, represented as PCK, TCK (technological content knowledge), TPK (technological pedagogical knowledge), and TPACK (see Figure 1). The summarized descriptions of the TPACK framework components, proposed by Mishra and Koehler (2006), are as follows:

[1] Because Mishra and Koehler changed Technological Pedagogical Content Knowledge (TPCK) Framework to Technology, Pedagogy, and Content Knowledge (TPACK) Framework, many researchers have used both acronyms in their studies. In this dissertation "TPACK framework" refers to the whole Technology, Pedagogy, and Content Knowledge framework, "TPACK components" refers to the knowledge components that comprise the TPACK framework, and "TPACK" or "TPCK" refers to a specific type of knowledge that intersects with all three: content, pedagogical, and technological knowledge.

- **Content Knowledge (CK):**

  Knowledge of the actual subject matter to be learned or taught, including central concepts, theories, and organizing or connecting ideas.

- **Pedagogical Knowledge (PK):**

  Knowledge of the processes and practices or methods of teaching and learning, including classroom management, development and implementation of lesson plans, and student assessment.

- **Pedagogical Content Knowledge (PCK):**

  Knowledge of pedagogy that is applicable and appropriate to teaching specific content.

- **Technology Knowledge (TK):**

  Knowledge of the standard and advanced technologies, including the skills to install, remove, and operate particular technologies.

- **Technological Content Knowledge (TCK):**

  Knowledge of the manner in which technology and content relate to, influence, and constrain each other.

- **Technological Pedagogical Knowledge (TPK):**

  Knowledge of the capability of various technologies including affordances and constraints that influence pedagogical designs and strategies in a teaching and learning setting.

- **Technological Pedagogical Content Knowledge (TPACK):**

  Knowledge of the interaction among content, pedagogical, and technological knowledge that requires an interweaving of specialized knowledge for teaching with technology (Abbitt, 2011b; Mishra & Koehler, 2006).

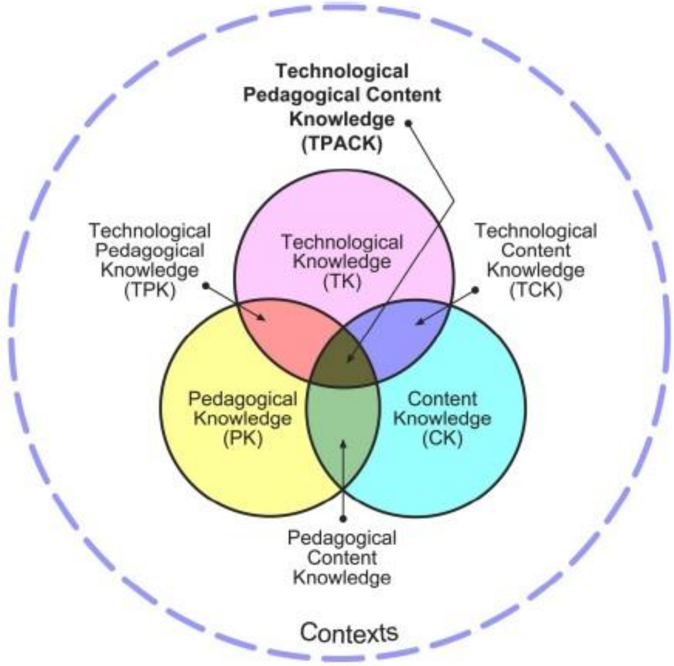


Figure 1. The TPACK framework and its knowledge components (Koehler & Mishra, 2009, p. 63)

Mishra and Koehler (2006) described TPACK this way:

> The basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (Mishra & Koehler, 2006, p. 1029)

Since the TPACK framework has been introduced, it has been used to provide insight into the complex phenomenon of the integration of technology and to promote research about teachers' use of technology in educational technology, teacher education, and teacher professional development (Koehler & Mishra, 2009). A substantial number of researchers have used the TPACK framework to describe, analyze, and evaluate preservice and current teachers' knowledge needed to incorporate of technology into the learning and teaching environment (Koehler, Shin, & Mishra, 2012). I used the TPACK framework to examine and describe preservice teachers' knowledge related to integrating technology into their mathematics teaching.

Preservice Teachers' TPACK

Recent studies provide evidence that the preservice teachers' TPACK is related to their integration of technology into mathematics instruction (e.g., Choy, Wong, & Gao, 2009; Pamuk, 2012; Özgün-Koca, Meagher, & Edwards, 2010). Preservice teachers in Choy, Wong, and Gao's (2009) study did not seem to have internal barriers to technology use in their teaching. They were aware of the benefits of the use of technology and were not reluctant to integrate technology. In addition, the schools where the preservice teachers taught during student teaching were well equipped with computers and Internet access. The researchers indicated that the results from the survey showed the reason preservice teachers had difficulties in integrating technology into their lessons was their lack of pedagogical knowledge and skills (PCK), and it might influence their use of technology in their teaching. In my opinion, it is difficult to determine whether the results of the survey provided evidence of preservice teachers' lack of PCK because the survey was designed to investigate preservice teachers' beliefs and intentions regarding technology use. Moreover, self-report surveys may not accurately reveal preservice teachers' knowledge.

Pamuk (2012) researched preservice middle or high school teachers' technology integration using the TPACK framework. The researcher collected multiple sources of data through open-ended questionnaires, teaching products, a final project report, and observations. Pamuk (2012) found that preservice teachers demonstrated a lack of TPACK and had a difficult time developing intertwined knowledge, such as TPK or PCK. In addition, preservice teachers' lack of pedagogical experience and knowledge hindered their development of appropriate technology integration approaches. The preservice teachers also displayed limited TPK even though they had well-grounded technology backgrounds. Pamuk suggested preservice teachers' deficiency of PCK may cause their limited TPK and argued developing preservice teachers' PCK is important in overall technology integration.

Similarly, in Özgün-Koca et al.'s (2010) study, preservice secondary mathematics teachers showed naive and superficial use of technology in a field setting at the beginning of the research. For example, one participant generated a task about the Pythagorean Theorem using a dynamic geometry environment (DGE). The task was no different from a traditional activity with paper and pencil, however, because it did not use the dynamic capabilities of DGE (e.g., constructing or dragging). The researchers suggested that preservice teachers struggle to design exploratory tasks due to a shortage of PCK and TK.

In the literature researchers used diverse instruments to examine preservice teachers' TPACK components. Except for one study that used self-report surveys, the other studies collected data through open-ended questionnaires or teaching materials that preservice teachers created for students. I found that preservice teachers displayed low levels of TPACK overall. In addition, what was commonly found in all three studies is preservice teachers' lack of PCK. The researchers indicated that preservice teachers' PCK may be an important factor that influences their TPACK.

Relationship Between Beliefs and Knowledge for Teaching with Technology

"TPACK cannot be considered as a body of knowledge that exists independently of teachers' beliefs" (Crompton, 2015, p. 243). According to Crompton, beliefs and TPACK are closely related to each other, and TPACK should be considered with beliefs. However, in the preceding studies about beliefs and TPACK, I found that there are varying results of studies about consistency between preservice or current teachers' beliefs and their TPACK.

According to Abbitt (2011a), there is a positive correlation between preservice teachers' beliefs and their TPACK. Abbitt (2011a) investigated TPACK and self-efficacy beliefs toward technology integration of 45 preservice teachers in early childhood education. He used the *Survey of Preservice Teachers' Knowledge of Teaching and Technology* (Schmidt et al., 2009), which reflects preservice teachers' self-assessment of their knowledge, to measure perceived knowledge in the TPACK domains and used the *Computer Technology Integration Survey* instrument (Wang, Ertmer, & Newby, 2004) to measure self-efficacy beliefs regarding the use of technology in teaching. Although Abbitt divided content knowledge into several different variables depending on subject matter, such as mathematics content knowledge (M-CK) or science content knowledge (S-CK), these distinctions were not reflected in the other content-related knowledge such as PCK, TCK, and TPCK. Thus, if Abbitt had also divided the other content-related knowledge into several types of knowledge depending on subject matter, it may have influenced the results of the study. Abbitt (2011a) found TPACK may be predictive of self-efficacy beliefs about the integration of technology. He suggested efforts to improve teachers' TPACK may result in enhanced self-efficacy beliefs. Knowledge of the intersections between technology and the other two knowledge domains (TPK, TCK, and TPCK), especially, may support higher self-efficacy beliefs about technology integration.

Mudzimiri (2010) studied preservice secondary mathematics teachers enrolled in a mathematics teaching methods course and a technology-intensive mathematical modeling course. To examine preservice teachers' development of beliefs about the use of technology in mathematics teaching and their TPACK, the researcher collected both quantitative and qualitative data through pre- and post-surveys and preservice teachers' lesson plans. The results suggested that preservice teachers' TPK, TCK, and TPACK were improved, and there was a progression in their beliefs about technology through the course. For example, one of the preservice teachers' beliefs about the use of technology changed from considering technology as a tool used only to calculate numbers or gain answers to considering technology as helpful for developing higher-order mathematical concepts.

In Smith, Kim, and McIntyre's (2015) study, which is the pilot study for this dissertation study, the researchers investigated the relationships between preservice middle-school mathematics teachers' beliefs and their TPACK. Two semi-structured interviews (belief and task-based interviews) were conducted with four preservice teachers to identify and examine preservice teachers' beliefs and TPACK. In the beliefs interview, the preservice teachers were asked about their beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology in the mathematics classroom. In the task-based interview, four separate tasks were used to assess their levels of TPACK components (CK, PCK, TCK, and TPCK). Each task was designed to have the preservice teacher examine mathematics concepts, analyze a student's mathematical understanding, and create a task or activity to develop the student's deeper understanding. Task-based interview data were analyzed and scored using rubrics, but there are some mathematical or pedagogical aspects that the rubrics did not cover. The findings indicated that preservice teachers’ TPCK levels were the lowest among TPACK components, and preservice teachers who had sophisticated and student-centered beliefs about mathematics, learning and teaching, and technology use displayed higher levels of CK, PCK, and TPCK, respectively. Thus, this study suggested that preservice middle-school mathematics teachers’ beliefs about mathematics, pedagogy, and technology use are aligned with their levels of TPACK components.

On the other hand, Chai et al. (2013) revealed a discrepancy between teachers’ beliefs and their TPACK. Chai et al. (2013) measured Singaporean Chinese language teachers’ TPACK and their pedagogical beliefs and investigated the relationship between them. This study adapted and modified Chai, Koh, & Tsai’s (2011) survey and Teo and Chai’s (2008) survey to measure 287 Chinese teachers’ TPACK and their beliefs, respectively. The researchers designated a TPACK survey called the *Technological Pedagogical Chinese Language Knowledge* (TPCLK) survey and a beliefs survey called the *Teacher Pedagogical Belief* (TPB) survey. The results of the data showed the strongest knowledge teachers perceived themselves as having was CK, and the weakest was TPACK. The technology-related knowledge (TK, TPK, TCK, and TPACK) was lower than non-technological knowledge (CK, PK, and PCK). In addition, the results from the analysis of the teachers’ pedagogical beliefs revealed that the teachers possessed highly constructivist-oriented pedagogical beliefs. They believed teaching should consider students’ individual differences, promote students’ knowledge construction through active thinking, and support inquiry and discussion. Therefore, although the teachers generally had constructivist-oriented pedagogical beliefs, the findings suggested they still needed more knowledge regarding technology integration.

So and Kim (2009) investigated the complexity of preservice elementary and secondary teachers' TPACK in the context of problem-based learning (PBL) and ICT. A collaborative lesson design, a task in which a PBL-based instruction is designed with the integration of ICT tools, was used to assess preservice teachers' TPACK. The researchers used a survey instrument with five open-ended items on perceptions of (pedagogy) and ICT (technology) to examine the preservice teachers' understanding, misconceptions, and difficulties regarding the integration (beliefs) of PBL and ICT. The results of this study indicated preservice teachers recognized a number of advantages of student-centered learning approaches (PBL) and were able to see the benefits and potential of integrating technology into teaching and learning. Preservice teachers faced difficulties, however, in applying their pedagogical beliefs or understanding about PBL in creating tasks and problems, integrating ICT tools, and identifying their role in lesson design artifacts. However, the survey and lesson-plan rubrics that the researchers used did not focus on aspects of subject matters. Thus, the questionnaires and rubric categories were too broad and general to measure participants' content-related knowledge.

Taking into account the processes and results of the four studies, the majority of studies used self-report surveys to investigate preservice teachers' beliefs. To measure preservice teachers' TPACK, two out of five studies used self-report surveys, and the others used task-based interviews or the lesson plans that preservice teachers created. The studies that investigated both preservice elementary and secondary teachers did not consider subject-matter context in their instruments for TPACK, so it may be difficult to accurately measure preservice teachers' content-related knowledge (i.e., CK, PCK, TCK, and TPCK). Given the results of the studies on beliefs and TPACK, preservice or current teachers showed varying results in terms of the consistency between their beliefs and TPACK. However, it is clear that preservice or current teachers need appropriate knowledge of integrating technology even though their beliefs about the nature of content knowledge or learning and teaching are constructivist and student-centered. This suggests that both preservice and current teachers need to improve their TPACK. Relatively few studies have directly considered both beliefs and TPACK in the context of mathematics. Therefore, more research on how preservice or inservice teachers' beliefs and their TPACK relate to each other is needed.

CHAPTER 3

METHODOLOGY

As revealed in the studies cited in the literature review, there are challenges in measuring Technological Pedagogical Content Knowledge (TPACK) in addition to the complexity of the research area. To assess mathematics teachers' TPACK accurately, we need reliable and valid assessment tools. The self-report measure, one of the frequently-used methods, has some limits. Among four studies I reviewed above, three studies used a survey. For instance, the S*urvey of Preservice Teachers' Knowledge of Teaching and Technology* (Schmidt et al., 2009) Abbott (2011a) used is for preservice teachers in elementary or early childhood education. Survey items do not provide fundamental questions or statements related to specific mathematical content knowledge. Instruments of measurement need to be customized to certain content knowledge. Moreover, we cannot say the self-reporting system assesses or measures teachers' actual TPACK because this is based on teachers' subjective, not objective, thoughts or self-judgments. That is, "as with any self-reporting measure, the ability of the instrument to accurately represent knowledge in the TPACK domains is limited by the ability of the respondents to assess their knowledge and respond appropriately to the survey items" (Abbitt, 2011b, p. 291). Thus, I selected a qualitative research methodology, a multiple-case study, to understand and identify participants' beliefs and TPACK components (CK, PCK, TCK, and TPCK) and how they are related to each other and understand the relationship between preservice teachers' beliefs and their TPACK. Yin (1984) defined a case study as "an empirical inquiry that investigates a contemporary phenomenon within its real-life context; when the boundaries between phenomenon and context are not clearly evident; and in which multiple sources of evidence are used" (p. 23). In addition, a multiple-case study methodology allows a researcher to construct contextualized experiences and systemic analysis processes (Stake, 2006). Therefore, a multiple-case study methodology was the most appropriate method to describe, identify, and examine individual participants' beliefs and TPACK, and to find possible relationships between their beliefs and TPACK from across-case analysis (Creswell, 2013).

I collected varied sources of data to infer participants' beliefs and to examine their knowledge. I investigated the participants' beliefs (the nature of mathematics; teaching and learning mathematics; and the incorporation of technology into instruction) and their TPACK components, using three semi-structured interviews with follow-up questions: (a) a beliefs interview, (b) a task-based interview, and (c) a performance interview. The data was used to examine possible relationships between beliefs and TPACK components.

Participants

Participants included four undergraduate preservice teachers enrolled in a mathematics teacher education program at a university in the southern region of the United States. They were enrolled in a secondary mathematics pedagogy course focused on learning and teaching geometry, probability, and sequences and series. One of the goals of the course was to develop preservice teachers' knowledge about technology in mathematics teaching and learning as well as how technology influences student thinking and conceptual understanding. The instructor regularly provided activities facilitating the preservice teachers' use of technology to explore mathematical concepts, including dynamic geometry environments (DGEs), for example, *Geometer's Sketchpad 5* (*GSP* [Jackiw, 2009]). The preservice teachers in the course were familiar with using technology and solving and explaining problems about geometry. I attended all classes except test days to observe how preservice teachers worked with technology, and I occasionally participated in some activities and worked with the preservice teachers. I collected my data after the course was over so that a) participants had been exposed to all relevant content and b) there was no implied connection between participating in the study and the course grade.

For a multiple-case study, Stake (2006) stated four to ten cases are enough to provide substantial information on the interaction between the cases without overwhelming amounts of differences, thereby restricting comparisons. To recruit participants, I asked all pre-service teachers in the course to participate in my study and provided consent forms at the end of the semester. Only four preservice teachers volunteered to participate in the study and signed a consent form. Each participant was paid $30 after completing the procedures for this study. The participants consisted of 1 man and 3 women between the ages of 20 and 24. Their names (pseudonyms) were Terry, Diane, Rebecca, and Jane. I next provide brief descriptions of each participant, which I obtained from the pedagogy course instructor. Thus, the opinions about their mathematical knowledge are from the course instructor and are not based on data from the study.

Terry was an energetic person who considered students' thinking and needs. He did not possess strong mathematical knowledge, and he would sometimes become frustrated using *GSP* in the pedagogy course, but he had the potential to be a good mathematician. After completing the pedagogy course, Terry took a technology course offered from the mathematics education program during the May semester. The purpose of the technology course was to appropriately select and use technology in secondary mathematics instruction, and many different technologies were used (i.e., Geogebra, *GSP*, Fathom, Tinkerplots, TI 83, Desmos, Excel, etc.).This study was conducted after Terry completed the technology course. Thus, the technology course may have influenced his view of technology use or his ability to use *GSP* to teach mathematics.

Diane was an international student. She spent most of her formative years in Vietnam where she graduated from high school and completed the English as a Second Language (ESL) program before entering the university. At the time of the study, she had been in the United States for six years. In the pedagogy course, she struggled with some of the mathematical concepts, but her struggles may have been due to her difficulties with the language.

Rebecca was one of the most responsible students among her peers. She was always the first one to begin an assignment, and she liked to get things done quickly. Rebecca loved to use technology and was very active in social media. She had earned an advanced degree or a certificate in Educational Technology. She was a good mathematician but not outstanding.

Jane was one of the top students in the mathematics education program. She had a perfect GPA when she graduated and received outstanding student awards from both the mathematics and the mathematics education departments. Jane was a dual major, earning bachelor's degrees in mathematics and mathematics education simultaneously. She was an outstanding student in every way. All participants had experience with educational technologies (e.g., Internet, PowerPoint, SMARTboard, Graphing Calculator, or Tinkerplots) in their high school days and/or college courses.

Data Collection

In this section, I explain how the interviews were designed and how the theoretical framework influenced the design.

Data Sources

A beliefs interview. I defined the areas of beliefs I would investigate in order to design an interview protocol to assess preservice teachers' beliefs of interest to my research. Many researchers have studied preservice or inservice teachers' technology-related beliefs (e.g., self-efficacy or attitudes toward technology use) or pedagogical beliefs (e.g., beliefs about learning, teaching, or students) as factors affecting their knowledge of technology integration (e.g., Abbitt, 2011a; So & Kim, 2009). Ernest (1989b) claimed "teachers' views of mathematics evidently affect the extent to which such curriculum innovations or movements take hold, through the way mathematics is taught" (p. 22), and both Ernest (1989b) and Thompson (1984) observed teachers reflect their beliefs about the nature of mathematics in their models of the teaching and learning of mathematics as well as their pedagogical beliefs. Thus, I added a beliefs about technology category and designed a semi-structured beliefs interview protocol to assess beliefs in four categories: the nature of mathematics, learning mathematics, teaching mathematics, and the use of technology in the mathematics class. I developed the interview questions based on the work of Raymond (1997) and Zakaria, and Musiran (2010) for the first three categories. Raymond (1997) studied the relationships between novice elementary school teachers' beliefs and their teaching practices, and Zakaria and Musiran (2010) investigated trainee teachers' beliefs about the nature of mathematics, mathematics teaching, and learning. I also developed the interview questions for beliefs about the use of technology in mathematics class based on the work of Landry (2010) on an instrument for measuring middle school mathematics teachers' TPACK (Appendix A). Table 1 provides example questions of each of the areas of beliefs I posed to all of the participants.

Table 1

Sample Questions from Beliefs Interview Protocol

Area	Questions
Nature of Mathematics	When you hear the term mathematics, what do you think of? Why do you think you view mathematics in this way? Could you describe what you are thinking about the difference

	between mathematics and the other subjects?
Learning Mathematics	Why do we need to learn mathematics? How do you think students learn mathematics? How do you remember feeling about your mathematics experiences in middle school? What do you think is the most important aspect of mathematics that students should learn? In other words, what part of mathematics do you want students to be really good at?
Teaching Mathematics	In order to be a good mathematics teacher, what do you think are the most important things for a teacher to do? What do you think the role of mathematics teacher should be? You can give more than one role Could you describe your thoughts on your mathematics teachers in middle school and the instructional strategies they used to teach mathematics? What do mathematics teachers need to know in order to be successful?
Learning and Teaching Mathematics with Technology	How do you think the use of technology affects students' mathematical thinking? Describe your confidence in your ability to use technologies for mathematics instruction. What technology has been available for you to use to teach mathematics? When preparing lessons that incorporate technology, what do you take into account?

A task-based interview. To measure participants' TPACK components, I conducted a task-based interview developed by Hollebrands and Smith (2010). The task-based interview consisted of four tasks designed to assess the participants' TPACK components within geometry topics. The four tasks are presented in Appendix B and the corresponding scoring rubrics from Hollebrands and Smith (2010) are in Appendix C. The four tasks addressed students' use of *GSP* and required participants to create activities using *GSP*. Through the task-based interview, I examined the participants' geometric concepts and their understanding of the students' understanding of the concepts or how the students think in specific technological pedagogical mathematical contexts. Then, participants were asked to design activities or tasks to help the students acquire a deeper understanding of the concepts or to remedy the students' difficulties or misconceptions using *GSP*. A sample task of the task-based interview is shown in Figure 2. During this interview, all participants were given a laptop computer with *GSP*, a copy of their textbook in the pedagogy course, a compass, a protractor, a ruler, blank paper, pencils, and markers.

TASK 1

Suppose students in your middle or high school mathematics class are studying rectangles and squares. They open a dynamic geometry sketch that contains a rectangle and a square, each of which have been constructed. Students are asked to consider properties of rectangles and squares, based on their exploration of the sketch. One pair of students has measured the diagonals and they have noticed they are always congruent. They claim, "quadrilaterals have congruent diagonals."

a. Is this claim always true, sometimes true, or never true? Explain.
b. How would you characterize their current level of geometric understanding?
c. Create a sketch using a dynamic geometry environment that you would like students to use to explore diagonals of quadrilaterals. Be sure to include directions and/or questions you would provide to students as they use this sketch.

Figure 2. Example Task for Measuring Participants' TPACK. Adapted from "Assessing prospective secondary teachers' knowledge of geometry, technology, and pedagogy," by K.F. Hollebrands and R. C. Smith, 2010, Methods and purposes for assessing high school teachers' knowledge of geometry.

Although some tasks in Hollebrands and Smith's task-based interview were at the middle school grade level, I did not change the original tasks. Because middle school mathematics content plays a basic and significant role in connecting middle and secondary mathematics, pre-service secondary mathematics teachers should know this content and how to facilitate students' learning of that content using technology. As this was also an interview, I asked additional, unstructured questions based on participants' responses to develop a thorough understanding of their knowledge in using technology to teach mathematics.

A performance interview. The third method I used to collect data was a performance interview. Abbitt (2011b) stated the following:

> Underlying the development of the performance-based measures is the idea that the products of student work are evidence of preservice teachers' instructional design and planning process. Further, by examining the design and planning process, it is possible to assess the knowledge of a preservice teacher in the TPACK domains. (p. 292)

Thus, in the performance interview, participants were asked to describe and demonstrate how they would teach a particular geometric topic using *GSP*. This interview aimed to reveal the participants' TPACK components in more detail, including their decision-making as based on their pedagogical reasoning and their ability to teach mathematics using *GSP* (Harris, Grandgenett, & Hofer, 2010). To measure the participants' knowledge not covered in the task-based interview's rubric, I conducted a performance interview. They were given the Exterior Angle Theorem (the Polygon Exterior Angle Sum Theorem): The sum of the measures of the exterior angles of a convex polygon is 360. This theorem was chosen based on the recommendation of the instructor of the pedagogy course. In the pedagogy course, the participants experienced various geometric topics. The Exterior Angle Theorem was one topic the participants did not experience in the pedagogy course. In addition, there were diverse possible strategies or ways to teach the theorem. Thus, the Exterior Angle Theorem was appropriate to differentiate the participants' levels of TPACK components. For the performance interview, the participants were allowed to prepare teaching materials (e.g., pre-constructed *GSP* materials or worksheets) in advance or make them during the interview. During this interview, all participants were given a laptop computer with *GSP*, blank paper, and pencils.

After participants completed the pedagogy course, the beliefs interview, task-based interview, and performance interview were conducted in order, with the first two interviews conducted at a one-week interval. After completing the task-based interview, I provided a handout including information about what they would do in the performance interview (see Figure 3). The performance interview was conducted one or two weeks later so that participants had time to prepare their teaching materials for the performance interview.

Performance Interview

In this Performance Interview, I would like to see how you would teach a geometry topic using Geometer's Sketchpad. For this interview, you will prepare a lesson to teach a theorem about the exterior angles of a polygon using GSP.
Think about how to teach the following theorem:

The sum of the measures of the exterior angles of a convex polygon is 360.

I will conduct an interview1 or 2 weeks later (depending on your schedule). During the interview, I will ask you to describe how you would teach the polygon exterior angle sum theorem and ask some questions about your lesson.

In your teaching, you should use GSP:

- You may prepare teaching materials in advance.
- You may make teaching materials during the interview.
- You may just use GSP during the interview.

Figure 3. A Performance Interview Instructions.

All three interviews lasted approximately 1 hour and were video and audio recorded. In the task-based and performance interviews, videos of the participants' work on their computer were recorded using a screen capture software program. I also collected any electronic files and artifacts created by the participants during the interviews.

Data Analysis

In this section, I describe the approaches I used to analyze the data. All interviews were fully transcribed, and all electronic files and artifacts created by the participants were saved or scanned for analysis. To analyze the beliefs interview data and assign codes, I used Ernest's (1989a) classification for participants' beliefs about mathematics, learning, and teaching mathematics, and I used the perspectives on technology developed by Goos, Galbraith, Renshaw, and Geiger (2003) for participants' beliefs about the use of technology in mathematics class (see Table 2). Goos and colleagues defined four types of technology use based on how teachers use technology or how they think about the role of technology in their mathematics class. In many studies, Goos et al.'s (2003) categories have been used to investigate how students interact with technology (e.g., Geiger, 2009; Nzuki, 2010) and to identify how teachers use technology in their classrooms (e.g., Goos, 2005; Morton, 2013). After coding, I found that some participants had double codes (meaning their beliefs did not fit cleanly in one category or another) for single beliefs areas (beliefs about the nature of mathematics and learning mathematics). To reinforce the reliability of my coding, I asked a reviewer who was familiar with Ernest's beliefs classification to review and assign codes for double-coded beliefs areas. The inter-rater reliability of those that were double-coded was 67%. The other reviewer assigned only one code for each category, whereas I had assigned two codes when I thought the participant showed signs of both categories. While agreeing that these participants showed evidence of both categories, the reviewer chose the more predominant category. Therefore, in that sense, there was good agreement between the raters.

I conducted additional data collection with Diane after analyzing and coding all participants' beliefs according to Ernest's and Goos et al.'s categories. Diane was an international student, and there were some moments she and I did not fully understand each other during the interview. Thus, to supplement the beliefs interview data, I emailed and asked her to answer additional written questions about her beliefs regarding mathematics; learning and teaching mathematics; and the use of technology. I chose to send an email rather than have an in-person verbal interview because I thought written questions and answers would be better for clear understanding.

I wrote each participant's narrative about his/her beliefs based on my analysis to describe and identify characteristics of each participant's beliefs in detail, using the new beliefs data from Diane's email interview for her narrative. Then, I performed a member check (Creswell, 2013) by sharing the narrative with each participant. All participants agreed that I accurately captured and described his or her beliefs.

I analyzed the task-based interview using the rubric developed by Hollebrands and Smith (2010) (Appendix C). Table 3 provides a sample rubric for Task 1. The rubric was designed to assess and interpret participants' knowledge about mathematical content, pedagogy, and technology that were required to complete the tasks. The rubric only concerned TPACK components that include mathematics content.

Table 2

Classifications of Mathematics Teachers' Beliefs

Beliefs about	Classification of beliefs	Description
Nature of Mathematics (Ernest, 1989a)	Instrumentalist	Mathematics is a set of facts and rules
	Platonist	Mathematics as a unified body of certain knowledge that does not change
	Problem Solving	Mathematics as a human creation that is continually changing
Teacher's Role (Ernest, 1989a)	Instructor	Goal of instruction is for students to master skills and perform correctly
	Explainer	Goal of instruction is for students to develop conceptual understanding of a unified body of knowledge
	Facilitator	Goal of instruction is for students to become confident problem solvers
Learning (Ernest, 1989a)	Passive Reception of Knowledge	Child exhibits compliant behavior and masters skills. Child passively receives knowledge from the teacher
	Active Construction of Knowledge	Child actively constructs understanding. Child autonomously explores self interests
Using Technology in the classroom (Goos et al., 2003)	Master	Dependence on technology, not capable of evaluating the accuracy of the output generated by technology
	Servant	Fast, reliable replacement for mental or pen and paper calculations
	Partner	Cognitive reorganization, use technology to facilitate understanding, to explore different perspectives
	Extension of Self	Incorporate technological expertise as a natural part of mathematical and/or pedagogical repertoire

Table 3

Rubric Used to Analyze TPACK Interview Task 1

Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
A. Responds that the claim is sometimes true. B. Knowledge that there exists at least one quadrilateral for which the diagonals are not always congruent. C. States that for at least the rectangle and square the diagonals are always congruent. D. Provides a correct mathematical justification for why the statement is sometimes true using proofs that involve triangles or other properties.	A. Identifies that the student is able to notice that for a square and a rectangle that the diagonals are always congruent based on their measures. B. Identifies that the student is at level 2 (descriptive) but probably not at level 3. C. Has students consider at least one counterexample of a quadrilateral that has congruent diagonals. D. Asks students to consider at least one example of a quadrilateral that has congruent diagonals.	A. Accurately constructs or draws a quad that is a counter-example using a DGE. B. Uses measures to find the lengths of the diagonals. C. Drags to create multiple examples in a DGE. D. Accurate constructions of 2 of the following quads: Square Rectangle Parallelogram Rhombus	A. Uses the DGE technology to focus students on properties of different quadrilaterals and their relationships to the diagonals in the task. B. Creates more than a single example using DGE technology to show the student that they are incorrect in the task. C. Designs an exploration for students by creating accurate constructions and utilizing the measurement and dragging features
Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	Emergent: 0-1 of A – D or no response. Beginner: 2 of A – D Intermediate: 3 of A – D Advanced: All of A – D	Emergent: 0 of A – C or no response. Beginner: 1 of A – C Intermediate: 2 of A – C Advanced: All of A – C

That is, the rubric can only be used to score the participants' levels of content, pedagogical content, technological content, and technological pedagogical content knowledge (CK, PCK, TCK, and TPCK) (Hollebrands and Smith used the abbreviation TPCK to refer to the piece of TPACK framework called technological pedagogical content knowledge). Based on the participants' work on each of the tasks, I assigned one of four levels (Beginner, Emergent, Intermediate, Advanced) for each of the four TPACK components (see Table 4). To assign an overall level of knowledge to each of the four TPACK components for each participant, I looked for the level the participant displayed most often and assigned that level (11 of the 16 categories). Next, when the participant displayed three different levels in one category, I assigned the level that was in the middle (1 of the remaining 5 categories). In another category (1 of the remaining 4 categories), the participant displayed two Advanced and two Beginner levels, so I assigned an Intermediate level for that category. For the last three categories, the participants displayed two adjacent levels of knowledge twice. In these instances, I examined the participants' work across the four tasks of the TPACK component and assigned the level that best captured their level of knowledge. In the pilot study (Smith et al., 2015), I used the same task-based interview protocol and coded similar data using the same rubric, which established the reliability of my coding.

After assigning overall levels of TPACK components, I wrote each participant's narrative about their knowledge (CK, PCK, TCK, and TPCK) based on my analysis and examples of the participants' work. The narratives provide insight into the participants' knowledge across tasks and information about what they knew and do not know in addition to their levels of knowledge.

Last, I analyzed the performance interview based on three big categories: Content, Pedagogy, and Technology with four levels (Beginner, Emergent, Intermediate, Advanced). I used the performance interview to investigate the participants' in-depth TPACK and to find evidences to support the results from the task-based interview through observing how the participants would teach the Exterior Angle Theorem using *GSP*. In the Content category, I focused on whether the participants had knowledge about mathematical concepts related to the theorem; a deductive or inductive proof; or connections between mathematical ideas or concepts (e.g., definitions of mathematical figures, the sum of interior angles of a triangle, the sum of interior angles of an *n*-sided polygon, parallel line postulates, etc.)

Table 4

Preservice Teachers' Levels of TPACK Components

Name	Task	CK	PCK	TCK	TPCK
Diane	1	Advanced	Advanced	Intermediate	Beginner
	2	Intermediate	Advanced	Intermediate	Beginner
	3	Intermediate	Beginner	Intermediate	Beginner
	4	Advanced	Beginner	Beginner	Intermediate
	Overall	Intermediate	Intermediate	Intermediate	Beginner
Terry	1	Intermediate	Advanced	Advanced	Intermediate
	2	Intermediate	Advanced	Advanced	Intermediate
	3	Beginner	Intermediate	Beginner	Intermediate
	4	Intermediate	Intermediate	Advanced	Intermediate
	Overall	Intermediate	Intermediate	Advanced	Intermediate
Rebecca	1	Advanced	Advanced	Advanced	Advanced
	2	Beginner	Intermediate	Advanced	Beginner
	3	Advanced	Intermediate	Advanced	Intermediate
	4	Advanced	Advanced	Intermediate	Intermediate
	Overall	Advanced	Intermediate	Advanced	Intermediate
Jane	1	Advanced	Advanced	Advanced	Advanced
	2	Intermediate	Advanced	Advanced	Intermediate
	3	Advanced	Advanced	Advanced	Advanced
	4	Advanced	Advanced	Advanced	Advanced
	Overall	Advanced	Advanced	Advanced	Advanced

In the Pedagogy category, I focused on what strategies they used to teach the theorem to their imaginary students and whether they could anticipate their imaginary students' thinking or potential difficulties. For example, I looked at whether they allowed their imaginary students to explore many examples to find the theorem themselves or directly provided the theorem; how they led their imaginary students to come up with proofs of the theorem; or what types of learning environment they provided, such as individual learning or collaborative learning. In the Technology category, I attended to the participants' knowledge about technology, especially *GSP*. I also observed whether they knew how to use *GSP* to implement certain tasks (e.g., constructing polygons, measuring and marking angles, using diverse tools of *GSP*). In addition, I focused on how they use technology to support their pedagogical strategies. For example, I considered whether they used the dragging feature of *GSP* to have their imaginary students explore many cases in which the theorem is true and make conjectures based on those cases, or whether they used the benefits of *GSP* to provide their imaginary students with diverse ways to explore the theorem or helped them intuitively understand the theorem (e.g., using parallel line postulates and a circle to show the sum of exterior angles is 360 degrees).

Limitations

In this section, I identify some of the limitations of this study. The first limitation was the validity and reliability of the study's interview protocols. Tests on the validity of the interview protocols were not conducted; therefore, I need expert validation of the interview protocols. In addition, it would be better if another reviewer checked the performance interview data for reliability.

Second, the task-based interview's rubric was founded upon student-centered principles. Thus, the rubric is likely biased toward participants who hold student-centered beliefs, and those participants may achieve higher levels of TPACK components than the participants who hold teacher-centered beliefs. In this study, although some participants held student-centered beliefs, they displayed low levels of some TPACK components. Therefore, the relationships between the participants' beliefs and TPACK components were not predetermined, but I recognize that some bias may exist.

Third, I examined the participants' TPACK components only within specific geometric topics and with the limited characteristics of *GSP*. The task-based and performance interviews do not cover all geometric topics, and *GSP* may lead participants to have some misconceptions.

Lastly, there was selection bias evident in choosing the four participants. Only four preservice teachers volunteered for this study; they were not randomly selected out of the pedagogy course. The participants were also enrolled in a particular university secondary mathematics pedagogy course focusing on teaching and learning geometric concepts with *GSP*; therefore, they were more familiar with, and knowledgeable about, the use of *GSP* in teaching geometric concepts.

CHAPTER 4

FINDINGS

In this chapter, I describe each participants' beliefs about the nature of mathematics, learning mathematics, teaching mathematics, and the use of technology in the mathematics classroom. In addition, I examine their levels of technological pedagogical content knowledge (TPACK) and how to use *Geometer's Sketchpad* (*GSP*) to teach a geometric theorem. Lastly, I discuss possible relationships between the participants' beliefs and TPACK.

Diane

Beliefs

The nature of mathematics

Diane explained she viewed mathematics as a set of rules and calculations with numbers when she was little. Mathematics courses she took while in college, however, changed her view of mathematics "from working with calculations to explaining the logic." Diane stated she no longer thinks mathematics is a set of numbers. She stated, "Math is more logic and proof because the more I learned math, I realized that I need to explain and have good conceptual understanding or find the connection between the concepts, rather than just calculating and follow the procedures" (Email). Diane also explained that the most important skill of mathematics students should learn is understanding "how they solve a problem and why they solve that way. Once students understand the reason of learning, they can figure out the relationship between mathematical concepts" (Email). Diane viewed mathematics as connected knowledge or concepts aligned with a ***Platonist*** view of mathematics. She believed mathematics is not a set of meaningless numbers but a unified body of mathematical concepts with relationships among them.

Diane still seemed to view mathematics, however, as a set of formulas and calculations. She said, “The most like [subject] is physics because we have to [use] diagram and calculation” (Interview 1). She also stated, “Physics is most like math because it requires critical thinking, the connection between concepts, and diagrams for visual learners. Physics also has many formulas, rules, and calculation” (Email). In her belief interview, although Diane emphasized critical thinking, a quick problem solving was still important for her. She said that we need to learn mathematics “for our logical and critical thinking. ... And mathematics helps you to understand and solve the problem quickly” (Interview 1). In addition, she said mathematics is important because “It has brought students and people, in general, enhances and improves their critical thinking and solving problems quickly and understanding the aspects of problems like, ‘How does this work this way?’” (Interview 1). When I asked how we can develop our critical thinking, Diane said, “Like use specific example ... like ... I can say a quick way to solve problems or like mental math” (Interview 1). It seemed she associated critical thinking with the way to solve problem or calculate quickly. Diane seemed to have two views of mathematics simultaneously: a set of rules and calculation and a connected body of knowledge. Thus, I categorized Diane’s view of the nature of mathematics as both ***Instrumentalist*** and ***Platonist***.

Learning mathematics

Diane described she learned mathematics in middle and high school by solving lots of similar mathematical problems repeatedly using the standard solutions from her mathematics teacher’s methodologies. Although Diane learned mathematics in a passive way in her middle and high school, she recognized that repetition of similar problems is not the way to understand mathematical concepts. Diane stated, “If I just repeat practicing more problems, I would be more understood the procedures, but not the reasons or concepts behind it” (Email). She also explained students learn the mathematical concepts by “questioning.” Diane believed students can learn best when answering their teacher’s questions or their peers’ questions and explaining what they understand to others. She explained, “By explaining to others, students can understand the reasons behind the mathematical concepts and use their own words to work through the problems” (Email). It seemed to be aligned with an active view of learning mathematics.

Diane still viewed, in part, learning mathematics as mastering skills or procedures to solve mathematical problems. When she described how she learned about Mod in a mathematics content course in college, she said, “Today when I just learned about mod, at first I didn’t know. I had to write down. ... So more I practice in doing mathematics, I can think quickly in my head to calculate and solve problems” (Interview 1). In addition, when I asked what the process of learning mathematics is, Diane just described the strategies or processes of problem solving by stating:

> You see the problem and then you question ‘Why is it this way?’... And then you look at what is given and what is asked. And from what is given, you say ‘Why is it given this way?’ And then you find a connection between the hypothesis, what do you think and then you work until [you find] the solution. (Interview 1)

Diane knew that acquisition of skills or procedures by repetition is not the best way for students to learn mathematics, but she still focused on that for her own learning mathematics. Diane seemed to have both views of learning simultaneously: learning mathematics is a ***Passive Reception of Knowledge*** for her and an ***Active Construction of Knowledge*** for others. Therefore, I classified Diane’s view of learning mathematics as both ***Passive*** and ***Active.***

Teaching mathematics

Diane's view of teaching mathematics was, in part, aligned with her view that students can learn mathematics by questioning and answering. She said:

> I think a teacher should just like [ask] question us and let us think about a problem ourselves. ... Actually teacher ask a question like a guide question, we can figure out. ... I'll work it out. Not teacher works for me. (Interview 1)

Diane believed teachers should ask guiding questions or give a hint to help student think and find out the solution of mathematical problems. In addition to asking questions, Diane believed explaining mathematical concepts, ideas, or the way to solve a problem is one of most important things for mathematics teachers to do. She stated:

> The teacher needs to be the one explaining, of course, teacher should give the student the new idea of math. And, of course, tell them how to work with math and solve the problem. In addition, new formula, the new lesson for them. (Interview 1)

During the interview, Diane continuously emphasized that mathematics teachers have to explain well to help students understand mathematical concepts. She expressed:

> I think the teacher has to work at explaining. I'm trying to work on that, how to explain things. Like some teachers, they're very good at solving and know the problem well, but when it comes to explaining to the other, they more give them answers. Teacher has to know how to explain. (Interview 1)

Diane seemed to believe teachers should provide good explanations about mathematical concepts, strategies to solve a problem, and the reasons something works rather than allow students to explore for themselves. Thus, I categorized Diane's view of teaching mathematics as an ***Explainer***.

The use of technology in learning and teaching mathematics

Diane had little experience of using technology in her mathematics classes in middle and high school. She only used a basic calculator. After entering the college, Diane had some opportunities to use different technologies (e.g., Graphing Calculator, Fathom, Geometer's Sketchpad, and Smart Board) in mathematics content and methods courses. From those experiences, Diane developed her view of the use of technology in learning and teaching mathematics. She believed technology tools were useful and efficient for learning and teaching mathematics. She explained how *GSP* helps students learn properties of shape. Diane said:

> When they [students] learn the property [of shapes] in the lecture, if we use a Geometer's Sketchpad and moving the shape around, students can know, 'Oh, so why is this stable like in the variance?' and the variance of the shapes. And they know the shape has that property. They say, 'Oh, it's cool' because they can work with it and you can see their motivation to learn. (Interview 1)

Diane also explained visualization is one of the good aspects of technology tools because students can visualize mathematics concepts, problem solving process, or graphs. She stated:

> The role of technology in mathematics is help to facilitate the math lessons and make math be more interesting. Using technologies such as Smart Board or videos, students can easily visualize some concepts, such as solving systems of equations by substitution, or listen and see mathematics video so they can be more motivated. It will help them to see the different transformations of shapes in math. Thus, students will see that math does not just contain number and calculation. (Email)

Diane's view of the use of technology was aligned with a ***Partner*** view. When she discussed a calculator or Smart Board, she displayed a ***Servant*** view of technology as quick and accurate tools to replace mental calculations or pen and paper works. Diane believed the use of calculator hinders students' mental math, and she did not want students to be dependent on technology. She said:

> I will try to avoid students to use calculators every time. I want the student to be able to do some mental math. ... I think calculator ... is just like a tool for the student to learn math faster. ... For all of math class. I just want students to see technology as a tool to learn, not to like rely heavy on it. Not too dependent. (Interview 1)

In terms of assessment using technology, Diane only focused on whether technology gives accurate and fair feedback soon. She said, "Like we have a clicker and we enter the multiple choice question, and then after that click the result, it gets back to students quickly. So, it's accurate and fair and it saves time for the student" (Interview 1). In addition, her description of how she would use Smart Board was more teacher-centered. Diane stated:

> Sometimes students can use the Smart Board, but with the teacher observing them. I don't think students are allowed to jump in Smart Board to write on themselves. ... I use Smart Board for the saving tool, for saving my lecture and lesson. So, if the student uses it, I'll save student work, too. So, if one student uses Smart Board only when they come up and solve the problem, explain the problem. But other than... the graphing work ... I'd rather use the white board. (Interview 1)

Although Diane's view of the use of technology seemed to depend on particular technology tools, overall, she seemed to view technology as an add-on that provides visual aspects and quick and accurate answers rather than as an exploring tool that facilitates conceptual understanding. Therefore, I coded Diane's view of the use of technology in learning and teaching mathematics as a ***Servant***.

TPACK Knowledge

Content knowledge (CK)

During the task-based interview, Diane displayed an ***Intermediate*** level of CK. Diane identified the properties of different quadrilaterals and provided a correct mathematical justification for why two diagonals of rectangle and square are congruent. She was able to determine whether students' conjectures were true. Although Diane was able to explain the properties of reflection and rotation and the process of how to perform reflections and rotations, she seemed to have a misconception about the center of rotation that only vertices of figure can be the center of rotation. Diane knew that the perpendicular bisectors of a triangle are concurrent and was able to recall the intersection of perpendicular bisectors is equidistant from the vertices of the triangle, but she was not able to connect what she found with the circumcenter of a triangle. Although Diane knew the properties of rectangles and squares and recognized they have an intersection with certain conditions, she seemed to view the rectangle and square as separate figures. Thus, she was unable to exactly relate the properties of rectangle and square to the reason why all rectangles are not squares or why all squares are rectangles. When I asked whether a square can be a rectangle, she initially said, "I don't think so." But right after, she explained that a square is a rectangle "with two adjacent sides that are congruent" rather than a square is always a rectangle (Interview 2). She did not seem to fully understand or articulate the inclusion relation between rectangles and squares.

Pedagogical content knowledge (PCK)

Overall, Diane displayed an ***Intermediate*** level of PCK. She was able to accurately analyze what students' mathematical thinking and understanding are and explain why the students might think in that way. She was unable, however, to identify students' levels of geometric thinking in van Hiele levels. Diane could have students consider examples and counterexamples to support mathematical claims or correct students' misconceptions. Although Diane could provide some examples to show students' claims are incorrect, she was not able to develop questions or activities to help students fully understand why their claims are true or false and lead students to discover or investigate further mathematical ideas themselves. Overall, Diane did not seem to have knowledge about how to facilitate students' mathematical understanding and learning. It is possible that Diane's limited CK was connected to her PCK. For example, in task 3, Diane did not have the students consider different types of triangles and a circumcircle of a triangle because she did not consider diverse locations of circumcenter depending on the types of triangles and was unable to derive the circumcircle concept from the equidistant property. In addition, in task 4, the reason Diane could not lead students to make a connection between rectangle and square seemed to be that she also did not recognize the inclusion relation between them.

Technological content knowledge (TCK)

In the TCK category, Diane displayed an ***Intermediate*** level. She knew how to use basic skills of *GSP* such as how to drag points and figures, label points, measure lengths and angles using *GSP* tools and was able to construct different quadrilaterals (e.g., parallelogram, rectangle, and square) and perpendicular bisectors. Although Diane finally performed rotation and reflection using *GSP*, she did not initially know that a reflection line is needed to reflect the figure. After being prompted by the interviewer, she constructed a reflection line, but she seemed to think the reflection line should pass through the center of rotation. Moreover, she believed there exists the same number of reflection lines as centers of rotation. Diane correctly drew a reflection line which did not pass through the center of rotation (that is, a vertex of a triangle) when she drew a triangle and performed reflection on the paper (see Figure 4). Diane still, however, showed her misconception about the center of rotation. She marked the vertex B of the triangle as the center of a 180 degree rotation. During the task-based interview, Diane seldom used the dragging feature, which is one of the most beneficial features of *GSP*. She just dragged a point or figure to move somewhere or make it bigger or smaller rather than dragging it with instructional purposes. Diane seemed to be more comfortable with a pen and paper than with *GSP*.

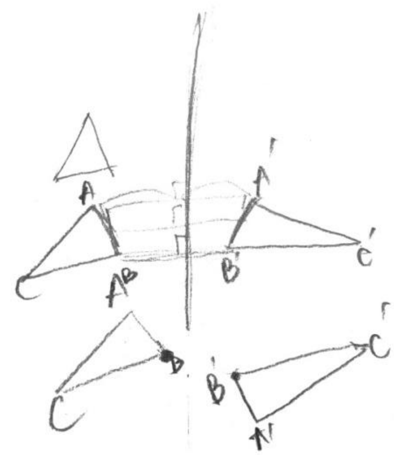


Figure 4. Diane's performance of reflection and a 180 degree rotation of a triangle on the paper

Technological pedagogical content knowledge (TPCK)

From my analysis of her task-based interview, Diane displayed a ***Beginner*** level of TPCK. She was able to create more than one example and tasks using *GSP* to show the students their conjectures are incorrect. Some of her tasks, however, did not help students develop their understanding of why their conjectures are incorrect nor deepen their understanding of the mathematical content. As I stated above, Diane did not use the dragging feature to create multiple examples, explore properties of figures, or examine conjectures. She did not fully use diverse features of *GSP*. In task 1, Diane said, “I’m going to let students do and move around the sketch. So see if the diagonals of the shape are congruent or not” (Interview 2), but she did not actually drag to show multiple examples nor create exploratory tasks using *GSP*. In addition, Diane was unable to develop tasks for students’ further mathematical learning using *GSP*. She labeled vertices of pre-image and image but did not lead students to consider the difference in orientation of rotation and reflection. And Diane could not design activities in which students would discover what a circumcenter is and its features using a circle and measuring tool of *GSP*.

Performance

Diane was the only participant who did not prepare the lesson plan for the performance interview. Diane did not open the attachment where I explained the performance interview that included what theorem participants would teach and what materials they could prepare one week before the interview. She only knew that she would teach about polygons. After reading the performance interview guide sheet during the interview, she extemporarily described how she would teach the exterior angle theorem.

Content - Intermediate

Diane began the interview by drawing a triangle on paper and tried to figure out why the theorem is true. Diane knew the extended line of sides of polygon was needed to make the exterior angle, but she was not able to state the exact definition of exterior angle. With algebraic equations, Diane explained why the sum of exterior angles of triangle is 360 degrees using her knowledge that the sum of interior angles of triangle is 180 degrees and the sum of exterior and interior angle at the same vertex is 180 degrees. Diane seemed to be more familiar with pen and paper work rather than the use of *GSP*. She found the sum of exterior angles of triangle is 180 ×

(The number of vertices of triangle) $-$ (The sum of interior angles of triangle). Diane had a hard time, however, extending this to finding the sum of interior angles for any polygon. She tried to find out a pattern between the sum of interior angles of an *n*-sided polygon and *n*, but she was not able to find it. After I gave her a hint by asking how many triangles are in the quadrilaterals, Diane was able to derive the pattern for the sum of interior angles of an *n*-sided polygon as $180 \times (n - 2)$. Finally, she could prove the theorem for any convex polygon.

Pedagogy - Beginner

To teach the exterior angle theorem, Diane wanted to start out with a triangle. First, she would draw a triangle on the board and show why the theorem is true for a triangle much like what she did on the paper in the beginning of the interview. Diane said that she would have students construct any 4-sided polygon using *GSP*. Next, the students would find, measure, and add up the interior and exterior angles of the polygon, respectively. After checking that the theorem works for the 4-sided polygon, Diane would ask the students if it works for a triangle and quadrilateral, is it true that the sum of the measurements of the exterior angles is 360 degrees for a 5-sided polygon? Because Diane wanted the students to prove why the exterior angle theorem is true for a 5-sided polygon, she would directly have the students find the sum of interior angles first rather than let the students explore the exterior angles or give them the opportunity to find the proof of the theorem themselves. Diane would let the students create, measure, and add up the interior angles of a 5-sided polygon by hand or using *GSP*. Then, Diane would have the students come up with a formula about how to calculate the sum of interior angles of *n*-sided polygon. If the students do not know what to do, she would have them consider how many triangles are in the *n*-sided polygon and write down the process of finding the formula for the sum of interior angles of *n*-sided polygon she went through. After finding the formula,

$180 \times (n-2)$, Diane would ask what the sum of the exterior and interior angle is and how to calculate the sum of the exterior angles. She hoped the students could answer her questions. As advanced tasks or questions, Diane said that she would ask the students what they learned from the activity and explore for a formula when you have a concave polygon.

When I asked about the students' thinking or misconceptions they may have, Diane stated that the students may not know what the exterior angle or the convex polygon is or how to construct the exterior angle using *GSP*. The students might think that the exterior and interior angle are the same, so they might say that the sum of exterior angle has to be 180 degrees because the sum of interior angles of triangle is 180 degrees. If the students do not know mathematical conceptions or have misconceptions, Diane said that she would explain what it is and correct their misconceptions. The definitions she provided during the interview, however, were not the precise definitions.

During the performance interview, Diane claimed she let the students find out whether the theorem is true for a quadrilateral and pentagon using *GSP*, but she provided directions step by step. She asked students to create polygons, measure interior and exterior angles, and calculate the sum of interior and exterior angles, respectively. Rather than having the students explore many cases to figure out whether the theorem is true or not using *GSP*, she was focusing on showing that the theorem is true. In addition, when Diane led the students to come up with the proof of the theorem, she seemed to give them lots of hints or ask direct questions. For example, she already showed the students the proof for the triangle case at the beginning of her teaching and asked a leading question such as what the sum of the exterior and interior angles at the one vertex is. Diane had the students consider a few examples (triangle, quadrilateral, and pentagon) and provided only the algebraic proof. Diane seemed to value explanation or transmission of her knowledge rather than students' opportunity to explore and find out why the theorem is true for any convex polygon themselves. This is consistent with her Passive view of learning mathematics and Explainer view of teaching mathematics.

Technology - Beginner

Using *GSP*, Diane was able to construct triangle, parallelogram, and pentagon, label vertices, mark and measure angles, calculate the sum of angles, and type letters. When Diane was constructing the extended sides of parallelogram, she used parallel lines to the sides. She was unable to correctly construct the extended line of sides of pentagon to find the exterior angles, however, because she actually did not know how to construct the extended line of the sides.

Diane did not think the use of *GSP* makes it easier to deal with a large number sided polygon rather than pen and paper. She said that using *GSP* is appropriate for a small number sided polygon, but it is difficult for a large number sided polygon (a hundred sided or *n*-sided polygon) because she struggled to construct the extended line of sides of pentagon and her calculation of the sum of exterior angles was not correct. She was concerned that it would be a time consuming work if the students do not know how to create a large number sided polygons using *GSP*. Although the use of *GSP* helps students see that the sum of exterior angles always stays 360 degrees even if the polygon is changing, she said, she would not use *GSP* to teach this theorem for a large number sided polygons. Overall, Diane did not fully use the dragging feature of *GSP* to explore many examples or to examine her or students' conjectures until the interviewer prompted her dragging. Diane used *GSP* for convenience to construct polygons and measure angles, not for students' better understanding or pedagogical reason. She seemed to believe that dealing with the proof or explaining processes of proof by hand is a familiar and easier way to teach the theorem than constructing conjectures and exploring many polygons using *GSP*.

Terry

Beliefs

The nature of mathematics

During the interview, Terry expressed diverse views about the nature of mathematics. He said, “Math is about numbers and facts and truths. But I also believe math is always changing” (Interview 1). Although he expressed that mathematics is continually changing, he did not seem to mean that humans create their own mathematical ideas and expand it. It was more like that humans discover or prove theories or phenomena that already exist out there. Terry stated, “We might prove something that might not have been proven now. ... Some guy might find that, ‘Oh wow’, this is actually not true and ... or even I think there’s some proofs out there that are unprovable” (Interview 1). In addition, Terry viewed mathematics as “a manipulation of numbers.” He said:

> I guess I define mathematics as a manipulation of numbers. ... There are formulas, but it’s not just about plugging into formulas. ... Because you need to understand what formulas mean. So, I guess when I hear mathematics, it’s manipulation of numbers, but also an understanding of how to use these numbers. (Interview 1)

Terry also explained that English is least like mathematics because “there’s no numbers in English”, and physics is most like mathematics because “you do a lot of mathematics, and you have to understand what you’re doing. And you’re just manipulating numbers. I guess science would be most like math. But definitely physics” (Interview 1). Based on his statements, Terry seemed to view mathematics as a set of numbers, rules, and facts.

However, he described mastery of mathematics as building a house. Terry explained:

> I would say mathematics is like building a house. Because with math, you need a really good foundation, and if you don't have that foundation, it's just going to crumble. And it won't be built right. So with math you do the same thing. You need a good foundation, then you can build up to having ... then I guess mastery of math would be the complete house. (Interview 1)

Terry believed that to be good at mathematics, having a good foundation of mathematics is important, and mathematical knowledge "all build off of each other." He also seemed to hold a ***Platonist*** view that mathematics is a unified body of certain knowledge. Thus, I categorized Terry's view of the nature of mathematics as both ***Instrumentalist*** and ***Platonist***.

Learning mathematics

Terry learned mathematics in middle and high school by repetition. He stated students just had textbooks and a calculator and then repeated the same thing. Terry practiced similar mathematical problems teachers gave him until he finally understood. Being aligned with his experiences in middle and high school, Terry held a passive view of learning mathematics. When I asked how students learn, Terry said, "Doing again and again. Repetition." And, "I believe some things should be repetition, like solving for *x*. That should just be repetition. Solving equations and repetition" (Interview 1). In addition, Terry believed that "practice" is one of the most important aspects of mathematics in terms of skills. When he mentioned "repetition" and "practice", it seemed to be related to mastery of skills or procedures to solve problems.

Terry simultaneously held a different view of learning. He said, "I do believe that some should be repetition. But in a geometry class? No, it shouldn't be ... because that's such a hard class to teach procedurally anyway. So I believe that class should just be taught conceptually"

(Interview 1). And he explained that mathematics can be taught conceptually through diverse approaches including the use of technology. Terry stated:

> I think through games, activities, hands on ... if the math class is boring, you're going to lose the kids. So I'm going to try to make it fun. Fun within reason. Have them work around and use conceptual ideas, use technology, use videos, and hopefully the kids will get it. That's how I would do it conceptually. (Interview 1)

He also viewed that students should learn mathematics through struggling or thinking themselves. Terry explained:

> Let the kids struggle a little bit. That's always when I learn the most is when Dr. Moore or Dr. Smith or Dr. Izsak would just leave me be and I would just struggle and I'm like, I have no clue what to do, and then all of a sudden I would get it and I'd be just like ... I would learn it. (Interview 1)

His learning experience from his teacher education program seemed to influence his view of learning mathematics. Therefore, I coded Terry's view of learning mathematics as both ***Passive*** and ***Active***.

Teaching mathematics

Terry believed that mathematics teachers should not be "hand holders." When students are grappling with mathematical problems, teachers should not give them the answer. Terry stated:

> You can hold their hands a little bit ... like just give them crumbs. Just give them hints. But I've also learned that if you let them solve it for themselves ... they're going to feel so much more accomplished. Because I've seen it. And so they shouldn't be hand holders in that fact that they shouldn't give them answers. (Interview 1)

Terry wanted his students to do group work, solve problems together, and discuss their ideas because he believed that is where they really learn mathematics. Terry viewed that the teachers' role is to help and guide students in the right direction. He said:

> They [teachers] shouldn't be holding the students hand the whole time. Let them do it on their own. That's not my job. I already learned it. It's your turn, you know. My job is just to lead you in the right direction. Not to just hold your hand and be like ... just, 'You have to do this, then you have to do this.' Maybe ... if they get a problem wrong, and then we discuss it, we'll understand. Then I'll probably be like, 'Alright, so what do we do with this?' And then they'll say, 'Oh, you add it.' 'Alright okay.' 'Well why do you add it?' And they'll be like, blah, blah, blah, blah. 'Alright, well, what do we do next?' Like that's where the handling should be. And the answer should be coming from the students. (Interview 1)

In addition, he emphasized that teachers should teach mathematics conceptually. Terry said:

> Try to be conceptual as possible. I know like in ... like today with all the standardized tests and all that and the time, you might not be able to, but some things just need to be taught conceptually. For those, try to be as conceptual as possible. (Interview 1)

Since Terry seemed to believe that teachers should facilitate students' learning by asking guiding questions, providing an active learning environment, and letting them do mathematics their own, I classified Terry's view of teaching mathematics as a ***Facilitator***.

The use of technology in learning and teaching mathematics

Terry had a lot of experience of using technology in high school. He used a Graphing Calculator every day in Algebra II class and depended on the calculator. One of his mathematics teachers had a Smart Board, but she never used it. She often used a projector instead. In college, Terry used diverse technologies as well (e.g., Fathom, Geometer's Sketchpad, Excel, StatCrunch, JMP, and Smart Board) and liked to use technology in learning mathematics. When I asked about the effects of the use of technology on students' mathematical thinking and learning, Terry said, "I think it grows…it [technology] makes it [students' mathematical thinking] grow exponentially" (Interview 1). He believed that students can find relationships between concepts and derive a theorem or formula by having and moving figures that students can manipulate themselves when using technology, especially *GSP*. He said, "that's where I feel the conceptual understanding comes in is just by having it and just play around with it. So technology has improved their thinking way beyond what we did" (Interview 1). Terry was willing to use technology for his mathematics instruction because he viewed that technology is a huge part of our society and "it just helps the students understand more and makes it a little more flashier" (Interview 1). Since Terry believed that a huge advantage of using technology is for investigation, he said he would use technology for students' investigation or a big project. He also wanted his students to use technology freely and find out mathematical concepts themselves. For example, Terry stated:

> To find the relationship between a square and a rectangle, I think I would let them do it themselves. Because that's just clicking, moving, and all that stuff. Like constructions, I would let them do that by themselves ... I would definitely utilize them using it by themselves. (Interview 1)

In addition, Terry thought technology makes teaching easier than before not because technology is fast or replaces pen and paper work, but technology helps students understand conceptually. He explained:

> Students will love it [technology] and it will make our job ten times easier. Because then we might not have to go back ... we might not have to spend as much time teaching the subject as students might now start to conceptually understand it more. (Interview 1)

Therefore, I characterized Terry's view of using technology in learning and teaching mathematics as a ***Partner***.

TPACK Knowledge

Content knowledge (CK)

During the task-based interview, Terry displayed an ***Intermediate*** level of CK. Terry demonstrated that he was able to examine whether students' claims were correct and list properties of different quadrilaterals. Although Terry stated the diagonals of rectangle and square are always congruent, he could not correctly justify why the statement is true. Terry knew that rotation and reflection are not the same because of the difference in orientation and explained the properties of rotation and reflection. Terry was unable, however, to indentify accurate definitions of acute and obtuse triangles and the perpendicular bisectors. Although Terry found that the location of the intersection of perpendicular bisectors (circumcenter) is changing, he mentioned, "I think it [the location of the circumcenter] does depend on the length of the sides rather than the angles" (Interview 2). Terry recognized that a rectangle is not a square and explained why it is not using the properties of rectangle and square. However, when I asked, "Do you think a square is a rectangle?", Terry said, "No. I think it can be. I don't think 'is'. ... When someone says 'is' something, it means that it's always that, it can always be this" (Interview 2). He seemed to view that a square is not always a rectangle because "a square has its own identity."

Pedagogical content knowledge (PCK)

In the PCK category, Terry displayed an ***Intermediate*** level of PCK. He was able to identify what students understand and what factors or aspects influence their mathematical thinking. However, Terry did not seem to fully understand van Hiele levels. He, at times, was unable to identify students' geometric thinking levels. Terry provided appropriate examples, questions, and tasks to correct students' misconceptions, but some of his tasks did not lead students to discover properties or mathematical concepts. In task 3, Terry did not have students consider the distance from the circumcenter to each of the vertices of the triangle and a circumscribed circle because he could not develop the circumcenter concept from the intersection of perpendicular bisectors of a triangle. Terry only had students consider the location of the intersection of perpendicular bisectors of the triangle. As I mentioned above, Terry believed that a square "can be" a rectangle rather than "is." Thus, he stated that he would try to correct students if they say that a square is a rectangle. Terry used his knowledge about what a sufficient or necessary condition is for a figure to be a square or rectangle when he helped students understand why a rectangle is not a square, but he could not see that its being a square is sufficient for being a rectangle.

Technological content knowledge (TCK)

From my analysis of his task-based interview, Terry displayed an ***Advanced*** level of TCK. He was able to use the measuring, labeling, and dragging features of *GSP*. Terry used, especially, the dragging feature for diverse purposes. He dragged a point or figure to make lots of examples and explore the properties of quadrilaterals, rotation, and reflection. Terry also used the dragging feature of *GSP* to find a pattern of the location of the intersection of perpendicular bisectors. Terry was able to construct different quadrilaterals and perform a rotation. However, Terry, like

Diane, did not know that he needs a reflection line when performing a reflection. He marked the center of rotation as the reflection point because he thought that he can use a point to reflect a triangle. After being prompted by the interviewer, he recalled that the reflection line is needed. Terry already knew that the distances from corresponding vertices of image and pre-image to the reflection line are the same, but he thought that image and pre-image could be prior to the reflection line. Thus, Terry constructed a reflection line using an image and pre-image which resulted from performing a rotation and applying the distance property of the reflection line. Terry was finally able to construct the reflection line which is parallel to a side of the pre-image. In task 3, Terry struggled to construct a perpendicular line to a side of triangle because he thought that a perpendicular bisector of the side is perpendicular to the side and it bisects the opposite angle, not the side. With the triangle he had, Terry could not find a line which satisfies both conditions. Thus, reconciling his thoughts with the triangle he had, Terry initially constructed segments connecting each vertex and a midpoint of each opposite side of the triangle as perpendicular bisectors of the sides (see Figure 5).

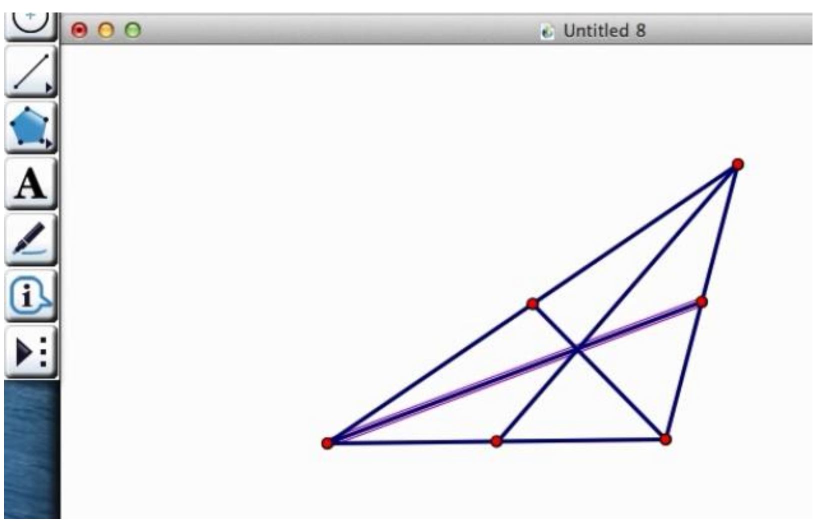


Figure 5. Terry's initial construction of perpendicular bisectors of the sides of a triangle

After being prompted by the interviewer asking about the definition or meaning of the perpendicular bisector of a side, Terry was finally able to construct correct perpendicular bisectors of the sides of the triangle.

Technological pedagogical content knowledge (TPCK)

Overall, Terry displayed an ***Intermediate*** level of TPCK. Terry demonstrated that he could provide many examples and design exploratory tasks that help students recognize their understanding about mathematical concepts is not correct using various features of *GSP* such as the dragging, measuring, and labeling features. His tasks, however, were usually for showing students that their claims are incorrect rather than developing students' deeper understanding or justifying why their claims are true or false. In task 2, Terry was asking students to change the shape of triangle using the dragging feature and see what happens to have students explore different triangles, but he did not drag to consider different locations of rotation center nor different reflection lines that are not parallel to the side of the pre-image. Since Terry did not know that the intersection of perpendicular bisectors of triangle is a circumcenter, he could not measure the distances from the circumcenter to each vertices of triangle and construct a circumscribed circle using *GSP* to help students discover the properties of circumcenter. Although Terry used his knowledge of properties of square and rectangle and posed appropriate questions to help students understand that a rectangle is not a square, his activity using *GSP* was not appropriate to develop students' understanding.

Performance

Content - Beginner

Unlike Diane, Terry knew what to do in the performance interview and prepared the pre-constructed *GSP* file consisting of several pages for his teaching. Terry did not, however, prepare the proof part of the theorem. He only focused on exploration of many polygons without the formal proof. When I asked him what if students ask how we can know that the theorem is true for every convex polygon, Terry tried to prove the theorem. Although Terry had a clue that the sum of exterior and interior angles at each vertex is 180 degrees, he struggled to develop the proof using the clue. Finally, Terry proved why the theorem is true for a triangle using parallel line postulates, but he needed some help from the interviewer. He thought if it worked for a triangle, it would work for all polygons because every polygon is based on a triangle. Terry was unable, however, to connect what he already knew to prove the theorem for all convex polygons. Using *GSP* Terry constructed polygons using rays instead of constructing the extended lines of the sides of the polygon separately. He had the misconception that the rays should be in one direction. Terry said that he would address this early to let the students recognize it when they constructed polygons using *GSP*. Terry did not, however, state the meaning of the rays and the exact definition of exterior angle.

Pedagogy - Intermediate

Terry described that he would give students the pre-constructed *GSP* file at the beginning of teaching so that the students could work themselves using *GSP*. On the first page of the *GSP* file were a triangle, exterior angles, measurements of the exterior angles, and a button "Show Angle Sum" for showing the sum of the exterior angles. He would make the students click and drag any point and see what is happening. By asking questions about the size of the exterior angles, Terry tried to have the students consider and find the pattern among the angles. After exploring, without hints or leading questions about the sum of exterior angles, Terry would let the students click the angle sum button and would ask whether the sum of exterior angles of this triangle was 360 degrees even though he had not addressed the theorem previously. Then, he would ask whether this was true for all triangles and would let the students make their own triangle by dragging vertices of the triangle. After checking it, Terry would move on to a quadrilateral and would let the students investigate the pattern of the exterior angles of quadrilateral using dragging feature. Terry anticipated the students would say that the quadrilateral's exterior angles were smaller than a triangle's because the quadrilateral had four sides. Terry said that if the students said the sum of exterior angles of quadrilateral is not 360 degrees when they explored the quadrilateral, then he would say, "Yes, because the angles are smaller" (Interview 3). After that, Terry would let the students click the angle sum button to check whether it was true. Terry was unclear about what it means to be smaller. Were all exterior angles of quadrilateral smaller than all the angles of a triangle or some of them? If Terry and his students explored only regular polygons, they would easily notice that all exterior angles of quadrilateral smaller than triangles'. That would not always happen when exploring irregular polygons. Since Terry already knew that the sum of exterior angles of polygons is 360 degrees and 360 degrees should be separated into parts depending on the number of the sides of the polygon, he might have that sense. After doing the same activity with a pentagon, Terry would lead an investigation about a concave polygon because a concave polygon could be constructed by dragging. Terry would show that the sum of exterior angles of concave polygon is not 360 degrees and would let the students investigate a definition of convex. He did not state the definitions of convex and concave polygons, however, during the interview. Then, Terry thought the students could see that the theorem only works for convex polygons, not concave. Lastly, Terry wanted the students to construct their own convex polygon (a large number sided polygon) on the last blank page of *GSP* and explore the theorem using dragging, measuring, and calculating tools. He hoped the students would see that the theorem was true for any convex polygon through his activity. For a further investigation or homework, Terry wanted the students to think about how you can prove the theorem and why the sum of exterior angles of concave polygon is greater than 360 degrees.

Terry stated that the students may not know what convex and concave mean, what a polygon is, what an exterior angle is, or the need of a certain direction of rays. If most of the class did not know mathematical conceptions or had the same problem, he would stop the class to address or discuss it.

During the performance interview, Terry would give the students many opportunities to explore the theorem themselves even though he would have them directly consider the sum of exterior angles at the beginning of teaching. Overall, he tried to ask many questions to facilitate the students' exploration rather than give the answers. However, Terry could not provide a proof of the theorem using deductive reasoning. The activity he provided was empirical.

Technology - Intermediate

During the interview, Terry was able to construct triangles, quadrilaterals, pentagons, and a 9-sided polygon using the rays, label vertices, mark and measure angles, calculate the sum of angles using the *GSP*. He also knew how to add a page on the *GSP* file and how to make a button that shows or hides a figure, measurement, or caption. In addition, Terry would let the students use diverse tools of *GSP* such as constructing, dragging, measuring, and calculating tools. He believed that the use of *GSP* would add more fun because the students could work with their own figure and actually see that the theorem works for any convex polygon. Terry stated that he prefers to use *GSP* to teach the exterior angle theorem. He said:

> That [*GSP*] helps a lot. Because hand and paper you can just mess up, and I've done proofs where I've drawn figures and it's like ... you know, that residue ... that black/grey

> residue that it leaves on your paper. I guess it just makes it neater. And then you can check [a conjecture] yourself because if you point and drag it ... and 'Well, is it true? It should be' ... I just think it looks nicer and cleaner. (Interview 3)

And, he thought that the students could learn how to use *GSP* themselves through his activity.

Rebecca

Beliefs

The nature of mathematics

Rebecca viewed mathematics as a logic puzzle. She said, "Actually I see math more as like a puzzle if that makes sense. Because every, because you're always solving a problem" (Interview 1). Rebecca thought that doing mathematics is solving problems or looking for a missing puzzle piece. She instantiated what a puzzle means:

> I think geometry for one ... when you're doing similar triangles ... you're putting the pieces together, kind of literally. Trying to figure stuff out on how things are congruent or stuff like that. Or even when you're finding missing angles ... like finding a missing puzzle piece. That's ... that's one aspect of geometry. But then algebra, I guess, you probably don't see it as much, but you're solving for X. X is your missing puzzle piece you're always trying to solve for. (Interview 1)

It seemed that mathematics is merely a set of mathematical problems for Rebecca. She did not consider the process of problem solving. Rebecca also did not focus on thinking about how students find or construct their own solutions but instead focused on solving a problem or finding the answer. When I asked what subject is most like mathematics, Rebecca stated, "I mean, it's [mathematics] got some aspects of science, because science and math always go hand in hand solving problems. But ... English and social studies, those are typically word focused and meanings of human and social things" (Interview 1). Rebecca also said:

> Another thing I really like is math and music. I'm usually, I'm actually kind of big in music. I was in high school at least. Or I played in an orchestra. And math is a big thing in music where you've got eighth notes and half notes and stuff like that. It plays a big role. And you can even graph music on a graph and it's actually really cool. So math can connect to that stuff. (Interview 1)

Due to the use of numbers for time or notes in a score, she seemed to view mathematics is similar to music. It seemed to be a superficial aspect. She mentioned neither mathematics is a unified body of certain knowledge or a human creation that is continually changing. Thus, I classified Rebecca's view of the nature of mathematics as an ***Instrumentalist***.

Learning mathematics

Rebecca learned mathematics in middle and high school through memorization of definitions and formulas and never understood what those actually are. She believed that even though teachers write all the stuff down, students do not learn it unless they want to learn it. Rebecca also viewed that students "learn math through their own ways." She said:

> It's like, it [learning] is in your own thing. It's not something that someone can make you do. Or, they can help you, but they can't make it come to you. It will just click with them like ... it just does. (Interview 1)

Rebecca recognized that learning is an individual process and not a passive reception of knowledge from teachers.

Rebecca also believed, however, in part, that she or a student learns mathematics through repetition. She described the process of learning mathematics as "a trial and error process."

Based on her definition of mathematics, she seemed to believe that students can learn mathematics through finding a correct answer to mathematical problems. Rebecca stated:

> It's like you try and you fail and you try and you fail until you get it. ... I think the kids ... the process for them is just ... trying and trying and trying and until they get it and then after that they still, I mean, this sounds very procedural and then after that they keep trying, trying, trying. But it's like, once they make sense of it with the concepts and stuff and the understanding, then they keep trying, trying and then it works. (Interview 1)

In addition, although Rebecca acknowledged that learning mathematics passively, for example, just memorizing formulas and procedures, was not the best way, she thought that this method did enable her to pass the examination. She explained:

> I was taught so procedurally. It was amazing. Literally for the AP Calculus exam our teacher gave us a front and back worksheet full of stuff we had to memorize. She literally told us, "You need to have this memorized for the calc exam." So we had flash cards of integrals and formulas and procedures. It was just regular ... I think the way I was taught may have not been the best, but I still got through it. (Interview 1)

Rebecca held an active view of learning mathematics, but she seemed to have, in part, a passive view simultaneously. Hence, I coded Rebecca's view of learning mathematics as both ***Passive*** and ***Active***.

Teaching mathematics

Because Rebecca believed that learning is an inner process of the individual, she viewed mathematics teachers as guides or helpers who facilitate students' learning. She said:

> I have a view of education overall as to where teachers don't actually teach. They just help the kids teach themselves, if that makes sense. Like you end up learning everything

> yourself. It’s just the teacher is the one that’s up there spitting it get out to you ... I always think like, I don’t think teachers teach. I think like teachers are just a guide. Kids teach; they teach themselves. (Interview 1)

During the interview, Rebecca expressed strongly, for example, teachers do not actually teach. It seemed she wanted to emphasize the role of teachers as a facilitator who provides an active and social learning environment. She stated, “In terms of the math part ... the role of the teacher is to help them [students] make sense of everything” (Interview 1). When I asked how teachers make students make sense of it, she answered:

> Through different tasks. Through different teaching methods and styles. If you’re using the smart board and letting the kids go up and play with it. Or using little tiles and letting them play with it. Or having them work together and use each other. (Interview 1)

She believed that each student may have different learning styles and teachers need to consider different ways to teach based on the student’s learning style. In addition, Rebecca emphasized what mathematics teachers should know. She stated:

> I think you [teacher] need to know how to reason through it [mathematics], like, how to determine what’s important. ... What do you want your kids to know and how to ... how to help their kids learn it. So it’s not really content. Content’s kind of like the last thing on my mind. How are you going to help the kids figure out what you want them to figure out? And how can you help them make sense of it? How can you determine what they need to know and how they need to know it? Those are the two biggest things I think teachers should know. (Interview 1)

Rebecca explained that it is important for mathematics teachers to know how to facilitate students’ learning and how to lead students to figure things out themselves rather than just know about mathematical content. Therefore, I categorized Rebecca's view of teaching mathematics as a ***Facilitator***.

The use of technology in learning and teaching mathematics

Rebecca described herself as a "big fan" of technology. She said:

> I love technology. I'm getting a certificate in instructional technology. So I'm a big fan. ... To me, seeing the graphs or there's this thing where there's a 3D graph you can see and twist it around and move it and like, that's where I think it's really cool. (Interview 1)

Rebecca believed that technology motivates students to learn mathematics and helps them understand mathematical concepts. She stated, "I think technology benefits because it makes kids more interested and it makes them explain more and it makes them have to understand what's happening whenever they use it" (interview 1). In addition, Rebecca's view of the use of technology was aligned with her view of teaching. She explained:

> I think the classroom shouldn't be dependent on what the teacher says. I think a teacher is just there to guide the students. Like I was saying earlier, they [students] teach themselves. With technology, they get to explore and do what they want and talk to each other about it. So I think that's really good. So it changes the role of what I grew up and it changes it to where the classroom focus is on the students and not on the teacher. (Interview 1)

Rebecca thought that using technology in the classroom influences how students learn so that they can explore mathematical ideas themselves and share their ideas with each other. Thus, she believed that teachers' role is changing; she said, "I think the teacher switches from being a drill sergeant to being a facilitator" (Interview 1). When I asked how you would use technology in your classroom, Rebecca stated:

> A lot of technology I would use for exploration. But then some of it, like I plan to have a web site and I plan my kids to interact with the web site to where I'm going to pose some kind of question and they're going to write some kind of answer on the blog or something like that. So I plan for that to happen, and that's more a use for explanation and assessment. So I can see what they're understanding. (Interview 1)

Rebecca emphasized the use of technology for students' exploration and interaction. Therefore, I characterized Rebecca's view of the use of technology in learning and teaching mathematics as a ***Partner***.

TPACK Knowledge

Content knowledge (CK)

From my analysis of her task-based interview, Rebecca displayed an ***Advanced*** level of CK. She was able to state the properties of different quadrilaterals, reflection, and circumcenter. Rebecca could determine whether students' claims were true except for the claim in task 2. In task 2, Rebecca believed that a 180 degree rotation is the same as a reflection until she designed a task for students using *GSP*. Rebecca had a misunderstanding of one of the properties of rotation. She thought the distances from the rotation center to every vertex of the triangle were the same, rather than corresponding, vertices. Rebecca realized, however, she was wrong and recalled what she learned in Dr. Smith's class. Finally, she was able to use reasoning about orientation to explain the difference between rotation and reflection. Rebecca developed the circumcenter concept from the activity in task 3 and knew that the location of the circumcenter is changing depending on types of triangles. She did not, however, explicitly state where the location of the circumcenter would be in. In the last task, Rebecca did not exactly mention that a square is a rectangle, but she said, “Everything about the square is over the rectangle but everything about the rectangle is not in the square” (Interview 2). This statement seemed to imply that a square is a rectangle, but a rectangle is not a square.

Pedagogical content knowledge (PCK)

Rebecca displayed an ***Intermediate*** level of PCK. She demonstrated that she can provide many examples to help students find out their misunderstandings. Overall, she was able to explain what students understand and how students’ mathematical thinking was influenced by using *GSP*. Rebecca also correctly indentified students’ van Hiele levels. In task 2, however, Rebecca did not correctly interpret students’ thinking because of her misconceptions about a rotation. Rebecca initially proposed “the stick guy activity” to have students find out that a 180 degree rotation is the same as a reflection, but she realized that it is not true while she was doing the activity herself (see Figure 6). Although her initial purpose of the activity was not correct, after realizing rotation and reflection are not the same, she posed appropriate questions to help students discover the distance property of reflection and rotation using that activity. In task 3, Rebecca found out that the circumcenter is equidistant from the vertices of the triangle, and it is a center of the circumscribed circle of the triangle. She did not ask questions or provide tasks, however, to help students consider the distance property of circumcenter and the circumscribed circle. Rather, she just focused on the location of the circumcenter depending on types of triangles.

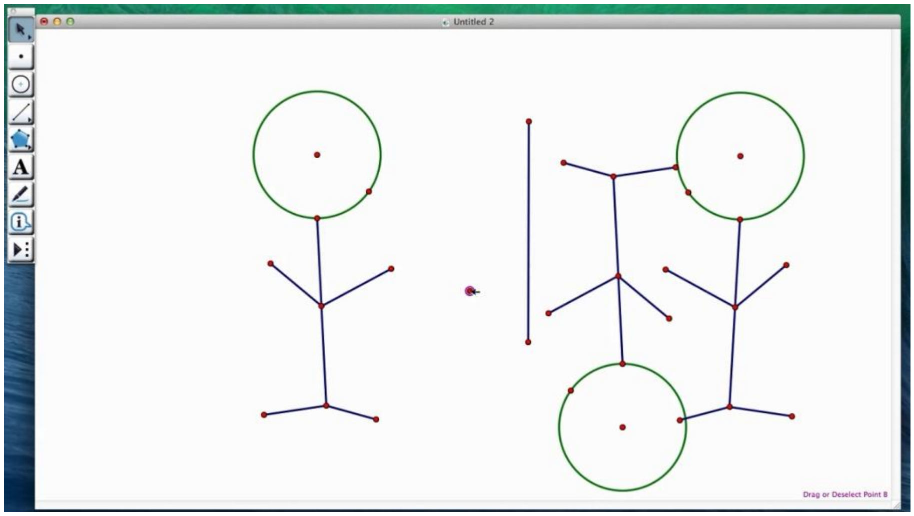


Figure 6. Rebecca's stick guy activity on *GSP*

Technological content knowledge (TCK)

During the task-based interview, Rebecca displayed an ***Advanced*** level of TCK. Using basic tools of *GSP*, Rebecca was able to measure lengths and angles, label points, and drag to create multiple examples and explore properties of figures. She correctly constructed rectangles and squares and performed rotations and reflections. Rebecca also constructed the perpendicular bisectors and used the circle tool to create a circumcircle. In task 2, Rebecca constructed a circle to demonstrate that the distances from the vertices of the image and pre-image resulted from a rotation to the rotation center are the same. In task 4, Rebecca used the marking tool to show the properties of rectangle and square. She was able to mark right angles and corresponding sides that are congruent to each other. However, Rebecca did not drag to show how a square or rectangle maintains its properties. She just showed that a rectangle can be a square by dragging a vertex of a rectangle to make it look like a square.

Technological pedagogical content knowledge (TPCK)

In the TPCK category, Rebecca displayed an ***Intermediate*** level. Rebecca was able to provide more than a single example and use appropriate figures to show the students that they are incorrect in the task. She could create activities to discover the properties of different quadrilaterals, rotation, reflection, and the circumcenter using dragging, measuring, and labeling features of *GSP*. In some tasks, however, those activities seemed to be for her investigation not for the students. In task 2, Rebecca dragged the reflection line and the center of rotation to see how images are changing when she moved the reflection line and rotation center. The reason why she moved them, however, was because the results from performing rotation and reflection were not what she expected. Moving the reflection line and rotation center, Rebecca was trying to match up an image from reflection with an image from rotation to make those look like the same. Rebecca did not consider other reflection lines or rotation centers in the task for the students. In task 3, Rebecca found the equidistant property of circumcenter, but she did not consider that property when suggesting tasks or activities for students. She just focused on the location of circumcenter depending on the types of triangles. Using marking and measuring tools, Rebecca demonstrated the properties of rectangle and square. However, she was not able to properly use the dragging feature to help students understand why a rectangle is not a square and a square is always a rectangle. In terms of dragging, there was no difference between what Rebecca did and what the student who had a misconception did in the task.

Performance

Content - Beginner

Rebecca started with a triangle in the beginning of her teaching and constructed it using rays on *GSP*. She explained the reason she would use rays was that it was easier to visualize exterior angles of the triangle. Rebecca did not state the exact definition of the exterior angle during the interview, however, and struggled to find correct exterior angles when she constructed and explored a quadrilateral as one of the examples. Rebecca correctly constructed the exterior angles when she tried again, but she still did not seem to know the exact definition of the exterior angle and the meaning of rays. She just said, “You make [the exterior angle] like where these [the rays] intersect ... That’s your angle” (Interview 3). Like Terry, Rebecca did not prepare the proof part of the theorem and only focused on the exploration of many polygons without formal proof. When I asked how to prove the theorem, she said that she needs to split 360 degrees into *n* parts for an *n*-sided polygon. Actually, Rebecca needed to prove that the sum of exterior angles of any convex polygon is 360 degrees, but she already used that fact to prove it. And, Rebecca thought that she could prove the theorem for regular polygons using mathematical induction. She explained:

> If you have the first case and then you prove the one after and so then if the first case works and then the one after works. So we know our triangle and we know the square works because the square is the next step up. So, then from there you’ll know the square works and then the pentagon has got to work ... by induction. I mean, I didn't even prove like all the way mathematically, but you know what I am saying like it will be induction. (Interview 3)

When I asked what if the students have the misconception that a polygon with a greater number of sides has a larger sum of exterior angles, Rebecca said that she would let the students know that each exterior angle measure is going to be smaller if the number of sides is going to be bigger. Like Terry, Rebecca already knew that the sum of exterior angles is fixed at 360 degrees and did not seem to consider irregular polygons.

Pedagogy - Intermediate

For the performance interview, Rebecca brought a worksheet she found on the Internet to use for her lesson (Appendix D). The worksheet was about activities with *GSP* to discover the sum of the measures of the exterior angles in convex polygons. Rather than give the exterior angle theorem to the students at the beginning of lesson, Rebecca said that she would have the students measure the length of every side and angle of a polygon and would ask what the measures of the exterior angles of the different shapes are. Rebecca wanted the students to find the pattern of exterior angles by exploring various measurements of the polygon, but she did not ask appropriate questions to lead the students to consider the sum of the exterior angles to find the exterior angle theorem. Unlike the worksheet that dealt with a pentagon, she wanted to start with a simple shape such as a triangle or square and then push the students to work with a hexagon or octagon. Rebecca thought that the worksheet is "blatant and explicit [because it] tells them [the students] exactly what to do" (Interview 3) and this would be okay for beginners because it tells them where they would go. Rebecca followed the steps of the worksheet. She would let the students construct diverse polygons, measure the exterior angles of the polygons, calculate the sum of exterior angles, drag the vertices of the polygons, and observe the sum of exterior angle measures. The students would see that the sum of exterior angles of the polygons was still adding up to 360 degrees and could make their conclusion about the exterior angle measures. Rebecca said that she would let the students use a chart to organize what they found from their work with polygons as well. The last activity on the worksheet was about another way to visually demonstrate the theorem using the dilate arrow tool of *GSP* (see Figure 7).

Double-click a point to mark it as a center.

In the Edit menu, choose **Select All**. Then click on each measurement to deselect it.

Hold down the mouse button on the **Arrow** tool, then drag right to choose the **Dilate Arrow** tool.

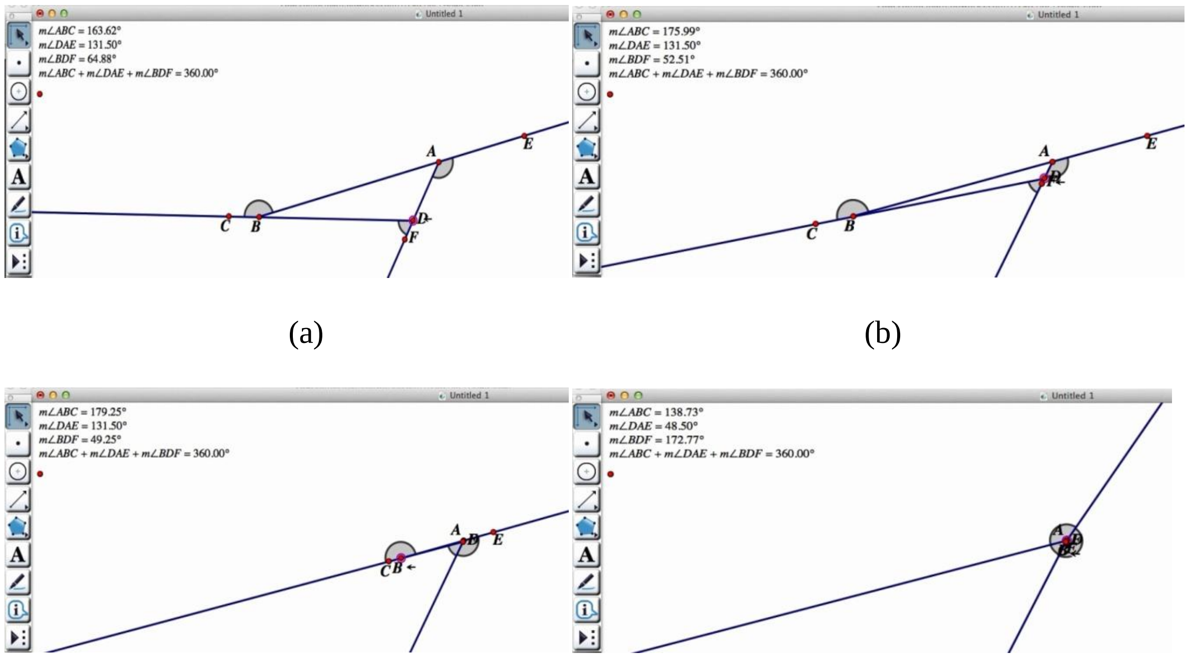
Follow the steps below for another way to demonstrate this conjecture.

7. Mark any point in the sketch as a center for dilation.
8. Select everything in the sketch except for the measurements.
9. Change your **Arrow** tool to the **Dilate Arrow** tool and use it to drag any part of the construction toward the marked center. Keep dragging until the polygon is nearly reduced to a single point.

Q2 Write a paragraph explaining how this demonstrates the conjecture you made in Q1.

Figure 7. The last activity on the worksheet that Rebecca brought

The dilate arrow tool could make the size of the polygon shrink without changing the shape of the polygon and the size of the exterior angles. If you kept dragging until the polygon was nearly reduced to a single point, you could see that only the marked exterior angles remain and the gathered exterior angles would be a circle at the end. Rebecca tried to follow the steps in the worksheet, but she could not implement them correctly. Because Rebecca only focused whether the circle appeared, she did not care whether she correctly went through the processes to demonstrate the theorem (see Figure 8). Rebecca kept trying to use the last way to demonstrate the exterior angle theorem for a triangle, quadrilateral, and seven-sided polygon, but she never succeeded. It may be due to her misunderstanding of the dilate arrow tool's role or careless reading. Rebecca said the homework she would assign would be "Prove why a regular polygon has exterior angles adding up to 360?" Then, the next day in class, she would work with the students on the proof for irregular polygons.



(a) (b)

(c) (d)

Figure 8. Rebecca's performance on the last activity on the worksheet

Rebecca kept trying to use the last way to demonstrate the exterior angle theorem for a triangle, quadrilateral, and seven-sided polygon, but she never succeeded. It may be due to her misunderstanding of the dilate arrow tool's role or careless reading. Rebecca said the homework she would assign would be "Prove why a regular polygon has exterior angles adding up to 360?" Then, the next day in class, she would work with the students on the proof for irregular polygons.

Rebecca thought that the students could make the same mistakes as she did in constructing exterior angles. If the students were confused or made mistakes when they made exterior angles, Rebecca said she would point out how to make the exterior angles asking, "How are you making your angles? What makes an angle?" (Interview 3)

Overall, Rebecca had the students examine whether the theorem is true for diverse polygons and tried to develop the worksheet by modifying it. Although she did not ask meaningful questions to help the students consider the sum of exterior angles, she tried not to directly give them the theorem or the answers to the questions. However, Rebecca was unable to develop the deductive proof and the visual demonstration of the theorem.

Technology - Intermediate

During the performance interview, Rebecca knew how to construct an equilateral triangle and quadrilateral using rays, mark and measure the exterior angles, and calculate the sum of the angles. Rebecca said she would let the students explore many polygons using dragging, measuring, and calculating tools of *GSP*. She believed that it is valuable to drag a point and see that the sum of the exterior angles of the polygon is not changing even though the side lengths and angles of the polygon are changing so that the students can find a pattern and make their conjectures. After doing the last activity of the worksheet, Rebecca said that we could do that with only *GSP*. The students could better understand what they were learning because they could see the exterior angles are gathered and make a circle.

Jane

Beliefs

The nature of mathematics

Unlike the other participants who stated that English is least like mathematics, Jane pointed out a language as a metaphor to describe mathematics. She said:

> I'd say mathematics is like a language is how I would say it because it's kind of ... its own way of describing the universe. ... I think it can be described as a language because of the way it can explain things. (Interview 1)

Jane believed that mathematics is about logical reasoning and the way to make sense of the world. When I asked what other subjects are most like mathematics, she explained:

> I'd say ... this is going to sound weird probably, but I'd say it's almost like an English class because ... and I wouldn't have said this in high school because in high school I thought that math was so much just numbers and that's basically it. But now I think writing proofs and all of that stuff, it's more like a logical flow and so I'd say it's most like in English class. (Interview 1)

Jane seemed to view mathematics as English writing including a logical flow rather than just a set of numbers. She thought that doing mathematics is similar to creating something that explains the world. In addition, she stated that mathematics is changeable. Jane said:

> I think there are a lot of things that are pretty much fact that people have discovered in the past, but I think it's always changeable and we're always able to build on that and create new things from that. (Interview 1)

As Jane stated above, she seemed to have a ***Problem Solving*** view that mathematics is a human creation that is changeable and evolving. Therefore, I coded Jane's view of the nature of mathematics as ***Problem Solving***.

Learning mathematics

During the interview, Jane emphasized students' active learning. She believed that students should "learn mathematics by actually doing the mathematics." She stated, "When I say 'doing', I mean being given a problem and they have to sit down and figure it out themselves and so problem solving type things" (Interview 1). Like the other participants, Jane learned mathematics in a traditional way. For example, her mathematics teachers lectured and she took notes and practiced similar problems as homework in middle and high school. She had new experiences of how to learn and teach mathematics, however, after entering her teacher education program. She said:

> In college, when we were learning how to teach it, we were seeing that it's helpful to have the student do more than just sit there and listen the whole time, be more active in the process. And so for my middle and high school math classes, it was always just the sitting and listening and trying to absorb it. ... I think it would have been so much better and I would have been more excited about it had I been given the chance to kind of explore it on my own a little bit more. (Interview 1)

Jane believed that it is important for students to find solutions themselves and communicate with peers and their teacher in learning mathematics. She explained the learning process as follows:

> How I think it should be, at least, is that the students should get a sort of problem, they should try to think about it and figure out the best that they can and if possible, that should be a social thing, maybe talk to each other and try to figure it out. But then after that, I think you need the teacher to step in and sort of solidify things and point in the right direction ... like they [students] could come together and talk about what they found and say, 'Okay, this is what we found, this is why it's true' and all of that. So, I think it's important for them to do the figuring out and then come together to define something or say why something works. And I think that's the whole process put together, how to learn it. (Interview 1)

Jane thought that students can learn mathematics through solving mathematical problems themselves and discussing their ideas or solutions. Thus, I categorized Jane's view of learning mathematics as ***Active Construction of Knowledge***.

Teaching mathematics

As I stated above, although Jane was taught mathematics by lecture and repetition in middle and high school, from her experience in her teacher education program, she learned that teachers should facilitate students' learning through problem solving and communicating rather than lecturing and showing students what to do. Jane said, "I'm totally convinced that more problem solving and social approach to math is the way to go. And so I think it would have been better to teach it that way than the way that I was taught" (Interview 1). Her view of teaching mathematics was aligned with her Active view of learning. Because Jane believed that discussion and group work are essential for students to learn mathematics, she emphasized the role of teachers as a facilitator in the classroom discussion. Jane stated:

> You [teacher] have to be good at getting the discussion going and then kind of wrapping it up to the point you want to make and then once the students have explored everything on their own, I think that's the time where you solidify things or maybe give the definition or something like that it hasn't come up yet. But I think it's definitely just to facilitate, not really to like be, 'I'm the be all and end all source of knowledge' ... like I don't think that's the teacher should be. (Interview 1)

When I asked how teachers facilitate students' mathematical learning, Jane explained:

> I think that while they're working on things if they have a question about something, I would facilitate that by not just giving them the answer but like asking them to talk about it with other people in their group or other people in the class and I think that facilitates learning because when you're forced to explain something to someone else, it really helps you to learn that thing better and so I think it's really good for students to go to each other for help and then you can go to the teacher as a last resort. (Interview 1)

Jane believed that teachers should provide an interactive learning environment through discussion and group work and help students think through their own solutions of mathematical problems. Thus, I classified Jane's view of teaching mathematics as a ***Facilitator***.

The use of technology in learning and teaching mathematics

In middle and high school, Jane used different types of technology in her mathematics classes such as an overhead projector, graphing calculator, and Smart Board. Her mathematics teachers mostly used technology in limited ways, however, such as using it as another whiteboard to write mathematical concepts or problem solving procedures or to display teaching materials. In her teacher education program, Jane had opportunities to learn how to use technology for learning and teaching mathematics. Based on her experiences in the teacher education program, Jane thought that the visual aspect of technology was an advantage of its using in learning and teaching mathematics. She stated:

> Students can really see what's happening and so faster and better understanding are really big advantages for that and also I think it's just more exciting to the students when they can use technology, if they can do this they'll be more excited about learning it. (Interview 1)

In addition, because Jane believed that using different teaching methods for different students' learning styles is important, she thought technology should be incorporated into mathematics instruction. Jane explained:

> I think that's [technology] a great tool to use for teaching mathematics because when you're explaining something you can have a visual to show using technology and so the students are not only hearing what you're saying but they're also seeing it which like different students learn in different ways and so sometimes it's really important for students to actually see what's going on. (Interview 1)

When I asked about the influence of the use of technology on students' mathematical thinking, she answered as follows:

> Not only is it about like seeing like visually what's happening, it also helps them to figure out what's going on so they can look at it. 'I can change this one parameter. What happens to the rest of it?' And that's important because it kind of gives them the background for why something is true or what something does and that kind of thinking is what goes into writing proofs. And so you can use technology as a background for how to write proofs, too, because of this one thing happening and they can go ahead and do that on whatever technology they're using and they can see what that thing will cause to happen. ... It sounds really vague when I try to explain it. But ... they'll see what's fundamental in it, like what things never change and what things do change and that's crucially important to thinking about writing a proof because you have to think about what things are never going to change to see why something is true. (Interview 1)

Jane thought that technology could help students explore what is happening when they change mathematical conditions or test their conjectures using technology including dragging or putting specific numbers as variables, and she thought that those processes could provide a base of a proof. Hence, I coded Jane's view of using technology in learning and teaching mathematics as a ***Partner***.

TPACK Knowledge

Content knowledge (CK)

In the CK category, Jane displayed an ***Advanced*** level of CK. Jane was able to determine whether students' mathematical claims or the statements were true or false. She also correctly stated the properties of different quadrilaterals, rotation, reflection, and circumcenter. Jane proved that rectangles and squares have congruent diagonals using Pythagorean theorem. In task

2, Jane thought the statement that a 180 degree rotation is the same as a reflection is true for a specific triangle, such as an equilateral triangle, even though she knew that a reflection reverses orientation and a rotation preserves orientation. When providing examples of images from rotation and reflection do not look the same, she considered different reflection lines and centers of rotation. In task 3, Jane found that the circumcenter is equidistant from each of the vertices of the triangle and the circumcenter is the center of a circle that circumscribes the triangle. She did not, however, demonstrate knowledge about the location of the circumcenter depending on the type of triangle. In task 4, Jane recognized that a square is a rectangle and a rectangle is not a square. She also stated that a square is also a rectangle and identified common and different properties of rectangles and squares.

Pedagogical content knowledge (PCK)

Based on my analysis of her task-based interview, Jane displayed an ***Advanced*** level of PCK. She was able to analyze students' mathematical understanding and thinking and demonstrate why they might think in that way. Jane also correctly identified the students' van Hiele levels. In task 2, Jane was aware that students might think a rotation and reflection are the same if they rotate and reflect an equilateral triangle. She designed tasks to help students find differences between rotations and reflections. As another task, Jane had students provide examples and counterexamples where the images of rotation and reflection look like the same, and she would ask students why it is the case that it works at the end (see Figure 9). Although Jane suggested this task with a misconception, this activity can help students discover that the images of symmetric polygons under a reflection and rotation of 180 degree may appear to look the same. In task 3, Jane modified the task to help students explore different types of triangles and discover the equidistant property of circumcenter based on what students have already done in the previous activity. In addition, Jane created an appropriate activity that leads students to understand that square is always rectangle, but a rectangle is not always a square in task 4.

Technological content knowledge (TCK)

Overall, Jane displayed an ***Advanced*** level of TCK. Using *GSP*, Jane was able to create lots of examples and counterexamples to correct students' misconceptions. She also measured lengths, and labeled figures using *GSP*. Jane knew how to construct different quadrilaterals (e.g., rectangle and square) and a perpendicular bisector. She correctly performed rotations and reflections and created a circle using the circle tool. In task 3, although Jane used dragging to modify the original triangle to examine whether the perpendicular bisectors meet at a point in different triangles rather than examining different locations of the circumcenter, she still used dragging to explore the property of the circumcenter. In task 4, Jane understood that dragging maintains the properties of the original construct if it was correctly constructed, so she was able to show that a square can never be made into a non-square rectangle by dragging.

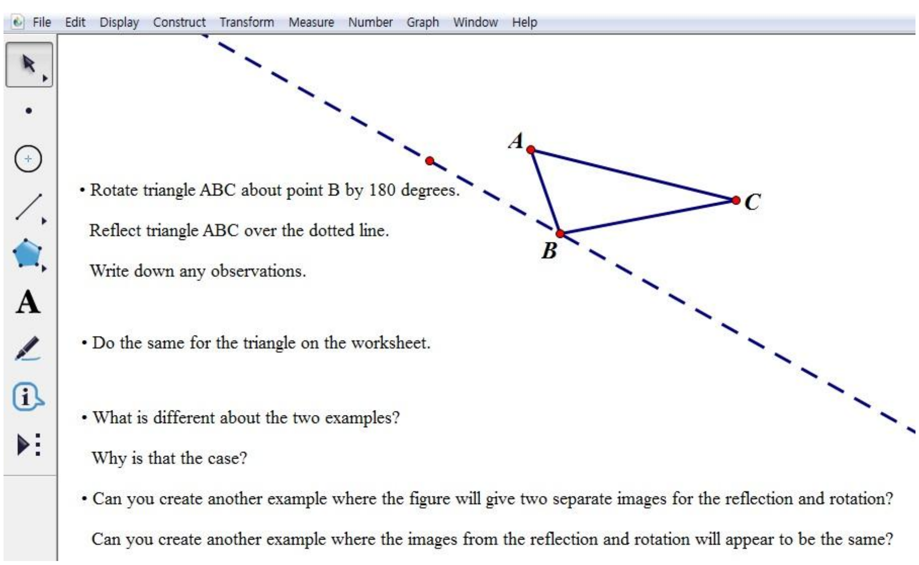


Figure 9. Jane's task to find the images of rotation and reflection that look the same

Technological pedagogical content knowledge (TPCK)

During the task-based interview, Jane displayed an ***Advanced*** level of TPCK. Jane was able to create and use appropriate figures as examples or counterexamples and use diverse features of *GSP* (e.g., dragging, measuring, and labeling features). For students, she created well-organized tasks using *GSP* that would have students investigate and deepen their mathematical understanding. In task 1, Jane provided three different quadrilaterals and let students explore different quadrilaterals and their diagonals using dragging and measuring tools. Although Jane did not exactly mention "properties," she wanted students to find out the properties of quadrilaterals and their relationships to the diagonals through that activity. In task 2, Jane knew that the location of the reflection line and the center of rotation may influence the images of reflection and rotation, but she did not consider that in the activities for students. Jane was able to design an exploratory task by developing the original paper-based task using the dragging feature of *GSP* in task 3. Through her task, students could examine whether the perpendicular bisectors are concurrent for all triangles and discover that the circumcenter is equidistant from the vertices of the triangle. In the last task, Jane provided several examples in the rectangle and square categories, let the students explore those quadrilaterals using diverse *GSP* features, and asked students why those quadrilaterals are classified in that way. While the students performed the activity, Jane wanted the students to find the relationship between rectangles and squares as well as their properties.

Performance

Content - Advanced

Jane was the only participant who correctly demonstrated two different proofs of the exterior angle theorem (e.g., algebraic and geometric proofs). At first, Jane proved the theorem for a pentagon using algebraic equations, and she was also able to develop the proof for an *n*-sided polygon. Jane knew that the interior and exterior angles add up to 180 degrees and how to calculate the sum of the interior angles of a polygon. Using her knowledge, Jane was able to show that $180 \times n - \{(180 \times (n - 2)\} = 360$, where $n =$ the number of the sides of a polygon. In addition, Jane provided another explanation using parallel line postulates to prove the theorem for the pentagon. Constructing parallel lines to each side going through one of vertices of the pentagon, she found and marked the same angles as each exterior angle of the pentagon at the vertex. Then, the same angles as each exterior angle around the vertex made a circle, and the sum of them was 360 degrees (see Figure 10). Although this was not the proof for all polygon, the students could actually see that the sum of exterior angles of the pentagon is 360 degrees and better understand the theorem through this method.

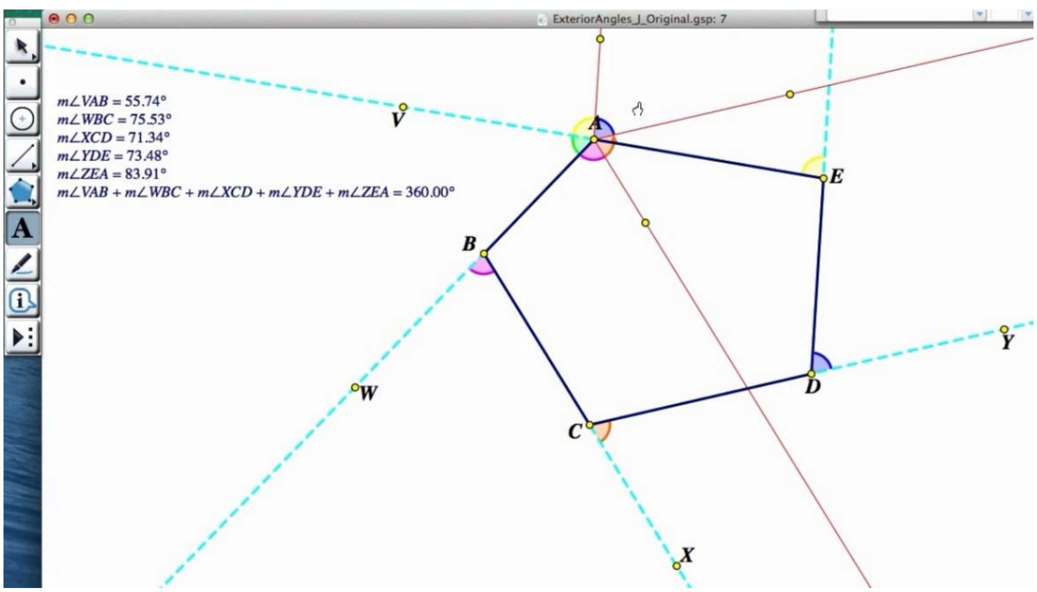


Figure 10. Jane’s proof of the exterior angle theorem using parallel line postulates

Pedagogy - Advanced

Like Terry, Jane brought a pre-constructed *GSP* file with several pages she made to the interview. First, Jane explained that she would give students the pre-constructed *GSP* file so that they could work individually. On the first page of the *GSP* file were questions about the sum of the exterior angles of a rectangle. To address the theorem, Jane said that she would ask, "What is the sum of exterior angles of the rectangle ABCD?" She wanted to start out with a rectangle because the students could see that the exterior angles are obviously 90 degrees and could be easily added together to arrive at the sum of 360 degrees. Then, Jane would ask the students to create the claim that this is true for any rectangle. She would let the students move around the rectangle and find that the exterior angles are always 90 degrees and the sum of them is always 360 degrees. Jane would not deal with a formal proof at this stage, but she would let the students discuss why that is true or not. She would have the students do the same activity with a triangle on the next page of the *GSP* file. The students would measure the exterior angles, calculate the sum, and drag vertices to explore diverse triangles to examine whether the claim they constructed is true or not. Jane expected that the students could come up with the proof showing that the interior angle and the exterior angle make a straight line and the sum of the interior angles of the triangle is 180 degrees and express this algebraically. Then, Jane would provide an octagon and let the students examine whether the claim is true for the octagon and then find a pattern. Because Jane wanted the students to explore and find the theorem, she said she would provide students with the claim: The sum of the measures of the exterior angles of any polygon is 360 degrees. Then, she would let the students construct a polygon in a blank space on the next page of *GSP* and test Jane's claim by dragging one vertex to create different shapes. Through this exploration of polygons, Jane expected the students could find out that the claim is not true for a concave polygon and could then revise the claim using the precise wording of the theorem: The sum of the measures of the exterior angles of any convex polygon is 360 degrees. When dealing with a proof of the theorem, Jane said she would use an irregular pentagon. She expected that the students could use the same reasoning as they used in the triangle case. Jane would have the students discover the proof themselves first and then would discuss it as a class. In addition, Jane would discuss another way to prove the theorem for the pentagon geometrically that the students could potentially come up with as a whole class. She explained how the students could make a circle consisting of the exterior angles of the pentagon. After completing her teaching demonstration, I asked her, “What if the students cannot find out the proof?” Jane said that for the algebraic proof, she would go back to the triangle case to recall the reasoning they had used in the triangle case because they would be more familiar with a triangle and know the sum of the interior angles of the triangle. She would have the students focus on one of vertices to see that the interior angle and exterior angle make a straight line. Then, she would ask the students to use the same reasoning for other polygons. For the geometric proof, she would ask the students to recreate the angles at one vertex. If the students still did not understand, she would give a hint such as using parallel lines. Jane thought that if the students had that starting point, this would help them to do the proof. For advanced tasks or homework, Jane would ask about an example where the theorem is not true. She thought that the students could come up with a concave polygon. And, Jane would ask them to provide both algebraic and geometric proofs as they had done in the class for the different polygons.

Jane anticipated the students might have some misconceptions. One of them is where the exterior angle is. Jane said that it would be good to have the students find where the exterior angles are first to avoid their misunderstanding of it. She did not, however, state the exact definition of the exterior angle. The other misconception the students might have would be that the sum of exterior angles is going to be different depending on how many sides the polygon has.

During the performance interview, Jane, like Diane, said she would give the students concrete directions what to do next. However, the difference between Jane and Diane is that Jane would provide the students with opportunities to explore many different polygons themselves. Jane encouraged the students to create and examine their own claims. In addition, Jane also tried to facilitate the students' understanding of the theorem and its proof through organized task questions rather than just giving the students the answer or showing the processes of proofs.

Technology - Advanced

Jane was able to use diverse *GSP* tools to construct triangles, rectangles, pentagons, and octagons using rays, label vertices, mark and measure angles, calculate the sum of angles, and type text. She also knew how to add a page to the *GSP* file and how to create parallel lines to the sides of the polygon. In addition, Jane wanted the students to explore diverse polygons themselves using diverse features of *GSP*. She thought that *GSP* is a nice tool to investigate the claim the students made because they can construct their own polygons, move them around, and see that the sum of exterior angles of the polygon is always going to be 360 degrees. When dealing with the octagon, Jane was concerned that the octagon could be made concave by dragging. Although the large number sided polygon may create issues for students, Jane said that she would use *GSP* to explore it because "It would be cool for them [the students] to see even this weird looking polygon still works for that one" (Interview 3). Jane stated that *GSP* is beneficial to dealing with, especially, the second proof because the use of *GSP* makes it more clear to see what is going on than drawing on the board by hand. She believed that *GSP* can help the students think of the idea or convince them that the sum of exterior angles is actually 360 degrees making a circle. Moreover, Jane was the only participant who considered aesthetic features of *GSP*. She used different colors for the students to see easily where the exterior angles or their corresponding angles are and why the theorem works (see Figure 10). Jane also said that *GSP* lets the students accurately and quickly construct figures, manipulate colors, measure angles, and calculate the sum of angles. Therefore, Jane thought that *GSP* is an appropriate tool to learn the exterior angle theorem.

Relationships between Beliefs and TPACK

I examined data from the belief, task-based, and performance interviews to identify the relationships between the participants' beliefs and TPACK. I used data from the performance interview to support or supplement the results of the task-based interview. Each participant had a unique belief classification (see Table 5) and displayed a unique set of TPACK levels in the task-based interview (see Table 6). In some cases, I was unable to classify the participants' beliefs in a single category because they seemed to have different beliefs simultaneously. Diane and Terry believed that mathematics is a set of rules and numbers. They simultaneously thought that mathematical knowledge is also connected. Diane, Terry, and Rebecca learned about how students should learn mathematics in their teacher education program, so they thought that the students can learn mathematics best when they actively construct their own mathematical knowledge. They still believed the students should learn mathematics through repetition, however, as they learned it in the schools. Although some of the participants' beliefs could not be classified into a single category, there seemed to be relationships between certain belief categories and certain levels of TPACK components. In particular, the participants' beliefs about the nature of mathematics seemed to be related to their CK, their beliefs about learning mathematics seemed to be related to their PCK, and their beliefs about the use of technology seemed to be related to their TCK.

Table 5

Classification of Participants' Beliefs

Name	Nature of Mathematics Ernest (1989a)	Learning Ernest (1989a)	Teaching Ernest (1989a)	Technology Goos et al. (2003)
Diane	Instrumentalist & Platonist	Passive & Active	Explainer	Servant
Terry	Instrumentalist & Platonist	Passive & Active	Facilitator	Partner
Rebecca	Instrumentalist	Passive & Active	Facilitator	Partner
Jane	Problem Solving	Active	Facilitator	Partner

Table 6

Participants' Levelsof TPACK Components in the Task-based Interview

Name	CK	PCK	TCK	TPCK
Diane	Intermediate	Intermediate	Intermediate	Beginner
Terry	Intermediate	Intermediate	Advanced	Intermediate
Rebecca	Advanced	Intermediate	Advanced	Intermediate
Jane	Advanced	Advanced	Advanced	Advanced

Content

I found a potential relationship between the participants' views of the nature of mathematics and their levels of CK. However, the participants' beliefs about mathematics were not aligned with their levels of CK, when taking into account the results only from the beliefs and task-based interviews. Considering the analysis of the performance interview data (see Table 7), however, it is reasonable to claim that there is a relationship between the participants' beliefs about the nature of mathematics and their levels of CK. Rebecca displayed an Advanced level of CK in the task-based interview, but she was unable to demonstrate her knowledge about the exterior angle theorem and its proof in the performance interview. Given evidence from both the task-based and performance interviews, Rebecca seemed to have an intermediate level of CK overall. Diane and Terry had both Instrumentalist and Platonist views of mathematics simultaneously, and Rebecca held an Instrumentalist view. Comprehensively, the three participants displayed an Intermediate level of CK considering both the task-based and performance interviews. In the belief interview, Jane held a Problem Solving view of mathematics and displayed an Advanced level of CK. Thus, the participants who held both Instrumentalist and Platonist views of mathematics displayed lower levels of CK, while the participant who held a Problem Solving view displayed a higher level of CK. For example, Diane, Terry, and Rebecca, who had traditional beliefs about mathematics, seemed to have fragmentary mathematical knowledge. They were unable to connect or reorganize what they already knew to see the relationships between pieces of knowledge or use them to develop the proofs of the theorems. Diane and Terry knew the properties of a rectangle and square, but they could not recognize the inclusion relation between them. Terry knew the sum of exterior and interior angles at one vertex of a triangle and the sum of interior angles of a triangle are both 180 degrees, but he could not prove the exterior theorem for a triangle. Although Rebecca already knew the Pythagorean theorem and knew all angles of a rectangle and square are 90 degrees, she was unable to use this knowledge to prove why the diagonals of a rectangle and square are congruent. On the other hand, Jane, who had constructivist-oriented beliefs about mathematics, was able to use her mathematical knowledge and reasoning to define the relationships between rectangles and squares and prove the theorems in the task-based and performance interviews. In other words, the participants who believed that mathematics just consists of numbers, formulas, and skills and is a static unified body of knowledge displayed lower levels of CK than the participant who viewed mathematics as a human creation that is a continually expanding field.

Table 7

Participants' Levels of the Performance Interview

Name	Content	Pedagogy	Technology
Diane	Intermediate	Beginner	Beginner
Terry	Beginner	Intermediate	Intermediate
Rebecca	Beginner	Intermediate	Intermediate
Jane	Advanced	Advanced	Advanced

Pedagogy

From my analysis of the participants' beliefs about teaching and learning mathematics and their levels of PCK, one possible relationship emerged. The participants' beliefs about learning mathematics and their PCK seemed to be connected. Diane, Terry, and Rebecca held both Passive and Active views of learning mathematics. Overall, they displayed an Intermediate level of PCK in the task-based and performance interviews. Jane had an Active view of learning and displayed an Advanced level of PCK. Thus, the participants who had both Passive and Active views of learning mathematics displayed lower levels of PCK, while the participant who held an Active view displayed a higher level of PCK. For instance, in the belief interview, Diane described how she was taught mathematics in middle and high school. When solving mathematical problems, she just followed her mathematics teacher's solution or procedures. Diane also believed that memorizing formulas, skills, and procedures to solve problems is important in learning mathematics. Such experience and beliefs seemed to be aligned with her PCK. In the task-based interview, Diane could show whether students' claims were correct or not using an example or counterexample, but she did not seem to know what questions to ask or what activities to use to help students develop their mathematical understanding beyond correcting their misconceptions. In the performance interview, Diane focused more on conveying the proof of the exterior theorem than providing the students with opportunities to explore many examples. In addition, Diane said that she would first explain her method of finding the proof of the exterior angle theorem for a triangle and then ask the students to apply her method to find the proof of the theorem for all polygons, asking direct questions and explaining procedures if the students encountered any difficulty. In Jane's case, her beliefs about learning mathematics were also aligned with her PCK. Jane believed that students can learn mathematics through meaningful mathematical activities and sharing and discussing their ideas with peers. In the task-based and performance interviews, using her PCK, Jane was able to develop appropriate tasks or activities to help students explore diverse cases, discover properties, and fully understand mathematical concepts. In addition, in the performance interview, Jane said that the students would participate in individual exploration and then, in a class discussion, share and examine their mathematical thinking and ways to prove the exterior angle theorem based on their individual exploration. Therefore, the participants who, in part, viewed that students can construct their own mathematical knowledge through active learning, but still had a Passive view that students learn mathematics by mastering skills and repeating the same procedures displayed lower levels of PCK than the participant who only viewed learning mathematics as an Active Construction of Knowledge.

I did not find a relationship between the participants' beliefs about teaching mathematics and their levels of PCK. Terry and Rebecca displayed a lower level of PCK than Jane even though all of them viewed the role of the teacher as a Facilitator.

Technology

Examining the participants' beliefs about the use of technology and their levels of TCK and TPCK, I found one possible relationship. The participants' beliefs about the use of technology in the teaching and learning mathematics aligned with their levels of TCK. I categorized Diane's beliefs about technology as a Servant and her TCK level as a Beginner based on the task-based and performance interviews. However, Terry, Rebecca, and Jane, who had a Partner view of technology, displayed an Advanced level of TCK. Thus, the participant who viewed technology as a Servant tool displayed a lower level of TCK, while the participants who held a Partner view of technology displayed a higher level of TCK. For example, when I asked Diane what technologies are available for teaching mathematics, she said, "I think the technology is more effective... like calculator, Smart Board and then tools like... computer for email" (Interview 1). As I stated in her case section, Diane considered the Smart Board as another white board. Although Diane had experienced diverse technologies in her mathematics content and methods courses in college, she still viewed technology as a tool to calculate quickly and accurately, save teachers' or students' work, or communicate with students or their parents. She did not seem to view technology as a tool for facilitating or developing students' mathematical understanding. During the task-based and performance interviews, Diane seemed to feel more comfortable with writing on paper than using *GSP* when thinking about mathematical concepts or demonstrating her mathematical knowledge. Although Diane knew how to represent the exterior angle of one vertex of a figure on paper, she was unable to construct the extended line of the side of the figure using *GSP*. In the performance interview, because Diane valued that technology enables students to work quickly, she said that she would not use *GSP* for the exterior angle theorem when dealing with polygons that had many sides because it would be time-consuming work. There seemed to be a connection between Diane's beliefs about technology and her TCK. In the belief interview, Rebecca believed that technology could help students explore, facilitate their learning and discussion, and change the main agent of learning from the teacher to the students. Thus, unlike Diane, Rebecca was willing to use *GSP* rather than writing on paper during the task-based and performance interviews. She was able to construct mathematical figures using *GSP* and use diverse affordances of *GSP* such as dragging, measuring, or calculating features. In the performance interview, Rebecca tried to develop students' understanding of the exterior theorem by providing two different ways using *GSP*. She was able to use many features of *GSP* for students to explore many examples and make their own conclusion about the exterior angle measures. Similar to Diane's case, Rebecca's beliefs about technology were also aligned with her TCK. Thus, the participant who viewed technology as a fast and accurate tool that amplifies what one can do by hand displayed a lower level of TCK than the participants who believed that technology can facilitate students' understanding of mathematical concepts by providing the opportunity to explore different perspectives.

I did not find a relationship between the participants' beliefs about the use of technology in teaching and learning mathematics and their levels of TPCK. Although Terry and Rebecca held the same view of technology as Jane held, namely Partner, they displayed a lower level of TPCK than Jane.

Discussion

In this study, I investigated the relationships between the preservice secondary mathematics teachers' (PSTs') beliefs and their technological pedagogical content knowledge (TPACK). This study was guided by the following research questions:

1. What are PSTs' beliefs about the nature of mathematics, learning mathematics, teaching mathematics, and the use of technology in the mathematics classroom?
2. What levels of TPACK components do the PSTs have in the context of geometry?
3. How do the PSTs' beliefs (that is, their beliefs about the nature of mathematics, learning and teaching mathematics, and the use of technology in the mathematics classroom) relate to their TPACK components?

In this chapter, I respond to the three research questions based on my analysis and interpretation of the findings. Next, I describe how I identify the relationships between the participants' beliefs and TPACK components, and then I identify other possible relationships.

Preservice Secondary Mathematics Teachers' Beliefs

Overall, the PSTs' beliefs about the nature of mathematics and learning mathematics were more traditional than their beliefs about teaching mathematics and using technology in the mathematics classroom. With the exception of Jane, who had a problem-solving view of mathematics, all three PSTs had traditional beliefs about the nature of mathematics. Diane, Terry, and Rebecca commonly held a instrumentalist view of mathematics, and both Diane and Terry had a Platonist view of mathematics simultaneously. They viewed mathematics as just numbers and formulas and/or a static set of unified mathematical knowledge.

All PSTs believed that learning is an active construction of knowledge. However, only Jane held an active learning view solely. The other three PSTs simultaneously held both passive and active learning views. They knew that students do not merely receive mathematical knowledge from teachers, but they emphasized repetition or a trial-and-error process when explaining how to learn mathematics.

Only Diane viewed a mathematics teacher as an explainer who focuses on explanations of a unified body of mathematical knowledge. She also viewed technology as a servant that amplifies mental or pen-and-paper calculations. The other three PSTs believed that a mathematics teacher should be a facilitator who provides active learning environments. Terry, Rebecca, and Jane believed that technology is one of the beneficial tools that provides active and enriched learning environments. Thus, they viewed technology as a partner that enhances students' mathematics learning by serving as exploratory tools and providing visual representations of mathematics concepts.

Diane, Terry, and Rebecca's traditional beliefs about the nature of mathematics and learning mathematics seemed to be influenced by their experiences from their own past schooling (Raymond, 1997; Richardson, 2003). On the other hand, Jane had student-centered beliefs about mathematics and learning even though she also had the same experience the other three had in middle and high school. Jane said that in her teacher education program, she learned how to teach mathematics and had also observed that students learn better when actively engaged in the learning process. Terry also stated that he learned a similar lesson about how students learn mathematics. Terry's and Jane's active views of learning mathematics seemed to be affected by their experiences in their teacher education programs.

The results of this study are aligned with the results of Raymond's (1997) study. Raymond's participants stated that their prior school experiences were the main influence on their beliefs about mathematics, and the experiences from their own teaching and teacher education programs were the primary influences on their pedagogical beliefs.

In contrast to the findings of this study, the preservice middle-school mathematics teachers in the pilot study (Smith et al., 2015) held more student-centered beliefs about the nature of mathematics, learning mathematics, and teaching mathematics than beliefs about the use of technology in the mathematics classroom. Moreover, the majority of preservice middle-school teachers held a problem-solving view of mathematics as well as an active view of learning, which were more student-centered than those of the preservice secondary mathematics teachers. Regarding beliefs about technology use, more preservice middle-school teachers held servant views, which were more limited views than the preservice secondary teachers. These differences can be attributed to a few factors. First, the preservice secondary teachers focused on secondary mathematics, which is more difficult and abstract than middle school mathematics. Thus, they may have had more strict or traditional views of mathematics and the learning of mathematics. Another possible reason is the time period in which they participated in the study. Both preservice middle-school and secondary teachers were enrolled in the same pedagogy course in different years. The preservice middle-school teachers participated in the study in the middle of a semester when they were taking the pedagogy course. The preservice secondary teachers participated in the study after completing the pedagogy course. Thus, what the preservice secondary teachers learned from the course may have influenced their beliefs about the use of technology.

Preservice Secondary Mathematics Teachers' TPACK

The results from the task-based and performance interviews indicated that most of the PSTs' levels of CK and PCK were intermediate, and their TCK levels were advanced. The participants' technological knowledge to teach mathematics (TPCK) levels were the lowest among TPACK components.

Jane displayed an advanced level of CK, whereas the other three PSTs displayed lower levels. Terry and Rebecca, especially, had difficulties in justifying or proving why a mathematical statement or theorem is true or not. Usually, Diane, Terry, and Rebecca were aware of the properties of mathematical figures, rotation, or reflection, but they did not know exact definitions of them. They were often unable to make connections between properties.

In regard to PCK, Diane, Terry, and Rebecca were unable to design appropriate activities for students or ask meaningful questions. For example, Diane tended to give information or hints to students to lead students to the correct solution rather than lead them to developing their own solutions. In Terry's case, his lack of CK influenced his PCK. Terry was unable to suggest tasks or questions that helped students better understand in the areas where he lacked sufficient knowledge. In the performance interview, Rebecca said she would let students measure everything to find a pattern for the exterior angle theorem, but the instruction was too broad for students to find the pattern.

Unlike CK and PCK, the majority of PSTs displayed high levels of TCK. Terry, Rebecca, and Jane had technical knowledge of *GSP* and knew how to use diverse features of *GSP* to represent or explore mathematical concepts. However, Diane lacked knowledge about how to use *GSP* and did not use the dragging feature of *GSP*, which is a key feature of GPS.

Although most of the PSTs had high levels of TCK, they displayed low levels of TPCK. Only Jane displayed an advanced level of TPCK. Diane showed a beginner level of TPCK, and both Terry and Rebecca displayed intermediate levels of TPCK. Although they all knew how to use *GSP* to construct mathematical figures or test mathematical statements, they did not know how to use *GSP* to teach geometry or to remedy students' misconceptions.

Similar to the findings of this study, the preservice middle-school mathematics teachers in the pilot study (Smith et al., 2015) displayed the lowest levels of TPCK among all the TPACK components. However, in contrast with the preservice secondary teachers, the preservice middle-school teachers displayed higher levels of CK and PCK than TCK. Overall, the preservice middle-school teachers' levels of all TPACK components tended to be lower than those of the preservice secondary teachers. In particular, there was a major difference between the preservice middle-school and secondary teachers' technology-related knowledge (TCK and TPCK) levels. The level of mathematics they focus on and the influence of the pedagogy course might be possible factors explaining these differences.

Relationships Between Preservice Secondary Mathematics Teachers' Beliefs and TPACK

Given the limited number of participants and the specific geometry context, I cannot generalize nor assert that preservice secondary mathematics teachers with certain beliefs will have a certain level of TPACK or vice versa. However, I believe my findings provide evidence that the following relationships between preservice teachers' beliefs and TPACK components exist.

In this study, I found that the more sophisticated the beliefs about mathematics, learning, and technology, the higher the levels of knowledge of CK, PCK, and TCK respectively. First, the PSTs with traditional views of the nature of mathematics tended to display lower levels of CK, while the PST with a reformed or problem-solving view of mathematics displayed a higher level of CK. Thus, the PSTs who viewed mathematics as a set of numbers and skills and/or a static unified body of knowledge showed lower levels of CK than the PST who believed mathematics is a human creation and is continually changing. These results are similar to the findings of Kang's (2014) study. She found that preservice primary teachers with higher mathematics content knowledge are more likely to view the nature of mathematics as a process of inquiry than preservice primary teachers with lower mathematics content knowledge in the United States and some Eastern countries.

Second, the PSTs who partially held teacher-centered views of learning mathematics displayed lower levels of PCK, whereas the PST with a student-centered view of learning showed a higher level of PCK. Therefore, the PSTs who still believed that students learn mathematics and master skills through repetition tended to display lower levels of PCK than the PST who only believed that students acquire mathematics knowledge through active learning. Similarly, Chai and colleagues (2013) found that the constructivist-oriented pedagogical beliefs were significantly related to all TPACK components, including PCK, but the traditional beliefs were not associated with PCK.

Third, the PST who viewed technology as a supplement showed lower levels of TCK, whereas the PSTs who considered technology as a tool for students' learning tended to display higher levels of TCK. That is, the PST who believed that technology would be used for time-consuming processes displayed a lower level of TCK than the PSTs who believed that technology would be used to provide students with diverse representations of mathematics. This finding is consistent with the results of Abbitt's (2011a) study. He found that the preservice teachers' beliefs about their capability to effectively integrate technology into their teaching were strongly associated with technology-related knowledge (e.g., TPK, TCK, or TPCK). Unlike Abbitt's findings, however, there was no relationship between the PSTs' beliefs about technology and their TPCK in my study. In addition, Mudzimiri (2010) recorded similar results.

Preservice secondary mathematics teachers developed their TPK and TCK with changes in their beliefs about how to use technology. Thus, as they improved their technology-related knowledge, their beliefs about technology use became more student-centered.

The finding from the pilot study (Smith et al., 2015) were similar to those of this study, for instance, in terms of the relationships between preservice teachers' beliefs about the nature of mathematics and learning mathematics and their levels of CK and PCK, respectively. Thus, the findings of the pilot study indicated that the more sophisticated or student-centered the beliefs about mathematics and learning, the higher the levels of CK and PCK. However, in the pilot study, we also found two other potential relationships between preservice middle-school teachers' beliefs about teaching mathematics and technology use and their levels of PCK and TPCK, respectively. The preservice middle-school teachers' beliefs about learning and teaching mathematics were aligned with each other, and their beliefs about learning were related to their PCK levels. Thus, their beliefs about teaching were also related to their PCK levels. In addition, the preservice middle-school teachers' beliefs about technology use were more closely related to their TPCK levels than their TCK levels.

In addition to the participants' differences in relation to population, the instrument used in this study might explain the different results from the pilot study. Because I added one more interview (the performance interview) in this study to examine the preservice secondary teachers' TPACK components in greater detail, this may have resulted in differences in the findings.

Other Relationships Between Beliefs and TPACK

When I identified the relationships between the participants' beliefs and TPACK components, I focused on the relevant subjects in beliefs categories and TPACK components. For instance, I examined whether the participants' beliefs about mathematics related to CK, whether their beliefs about learning or teaching were associated with PCK, and whether their beliefs about technology were related to TCK or TPCK. In doing so, I found the potential relationships between the participants' beliefs about the nature of mathematics and their CK, between beliefs about learning mathematics and PCK, and between their beliefs about the use of technology in the teaching and learning mathematics and TCK. However, I can see the other relationships only if I focus on levels of categories without considering subjects.

First, there was a possible relationship between the participants' beliefs about the nature of mathematics and their levels of PCK. In other words, the participants who held an instrumentalist or a Platonist view, or both, of mathematics displayed lower levels of PCK, whereas the participants who held a problem-solving view displayed a higher level of PCK. Because Raymond (1997) argued that "deeply held, traditional beliefs about the nature of mathematics have the potential to perpetuate mathematics teaching that is more traditional" (p. 574), the participants' view of mathematics is also likely related to their knowledge of how to teach mathematics.

Second, the participants' beliefs about learning mathematics were aligned with their levels of CK. That is, the participants who had both passive and active views of learning mathematics displayed low levels of CK, whereas the participant who held an active view displayed a high level of CK. This result is consistent with that of Blömeke's (2012) and Kang's (2014) studies, which showed that preservice teachers' beliefs about learning mathematics are related to their mathematical knowledge. Thus, in this study, the participant who possessed a high level of CK viewed learning as an active construction.

Third, the participants' beliefs about teaching mathematics and their levels of TCK seemed to be connected. The participant who viewed a teacher as an explainer displayed a low level of TCK, whereas the participants who viewed a teacher as a facilitator displayed a high level of TCK. Many researchers have indicated that pedagogical beliefs and their use of technology are closely connected (e.g., Cope & Ward, 2002; Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012). In addition, in the ACOT project, Dwyer et al. (1991) attributed teachers' changes in teaching approaches to the integration of technology into the classroom. Thus, the participants' sufficient knowledge of technology and their awareness of the benefits from the use of technology may support their student-centered beliefs about teaching and vice versa.

CHAPTER 5

SUMMARY and CONCLUSIONS

Summary

I investigated preservice secondary mathematics teachers' beliefs about the nature of mathematics, learning mathematics, teaching mathematics, and the use of technology in teaching and learning mathematics and examined their technology, pedagogy, and content knowledge (TPACK) and how to use a dynamic geometry environment (DGE) to teach a specific geometric theorem in the virtual setting. The four preservice teachers who volunteered for my study were enrolled in the same secondary mathematics pedagogy course about learning and teaching various secondary mathematics topics (e.g., probability, geometry, and sequences and series) with an emphasis on the effective use of technology. Through investigation of these preservice teachers' views, I was able to categorize their beliefs, examine their levels of TPACK components, and identify possible relationships between their beliefs and TPACK components.

I used a beliefs interview to identify the preservice teachers' beliefs and a task-based interview and performance interview to identify their TPACK. I analyzed preservice teachers' beliefs based on Ernest's (1989a) classification of beliefs and Goos et al.'s (2003) categories of the use of technology and analyzed their TPACK based on Hollebrands and Smith's (2010) rubric. Through the analysis of the data, I found that the four preservice teachers' beliefs and knowledge varied in classification and level. In addition, there existed patterns among their various beliefs and TPACK components. First, the preservice teachers held more traditional views of the nature of mathematics (instrumentalist and Platonist views) and learning mathematics (a passive view) than their views of teaching mathematics (a facilitator view) and the use of technology for mathematics class (a partner view). Second, the preservice teachers displayed lower levels of mathematical content knowledge (CK) and pedagogical content knowledge (PCK) (a intermediate level) than levels of technological content knowledge (TCK) (an advanced level). Third, the preservice teachers displayed the lowest levels of technological pedagogical content knowledge (TPCK) among TPACK components even though they held a student-centered view of technology use and displayed a high level of TCK.

After classifying the preservice teachers' beliefs and TPACK, I noticed three possible relationships between them. First, the preservice secondary mathematics teachers' beliefs about the nature of mathematics was related to their levels of CK. The participants with both instrumentalist and Platonist views of mathematics displayed lower levels of CK than the participant with a problem-solving view of mathematics.

Second, their beliefs about learning mathematics were related to their levels of PCK. The participants with both passive and active views of learning mathematics displayed lower levels of PCK than the participant with an active view of learning mathematics.

Lastly, their beliefs about the use of technology in mathematics classes were related to their levels of TCK. The participant with a servant view of technology use in a mathematics class displayed a lower level of TCK than the participants with a partner view of technology.

However, there was no relationship directly including TPCK, indicating that there were combined effects of both beliefs and knowledge on TPCK.

Conclusions

Based on the findings from the multiple-case study, I derived several conclusions from the relationships between preservice secondary mathematics teachers' beliefs and TPACK.

Beliefs and Knowledge about Technology

The findings of this study indicated that preservice teachers' beliefs about and knowledge of the use of technology seem to have a strong influence on their level of TPCK. Based on the results of this study, I noticed that having similar beliefs and knowledge about mathematics and pedagogy may not ensure having the same level of technology-related knowledge. If preservice teachers have a limited view of technology use and lack of TCK, then they may demonstrate a lower level of TPCK than the others who had more student-centered view of technology use and higher level of TCK even though they hold similar beliefs and knowledge in mathematics and pedagogy categories. In addition, a limited view of technology use and low level of TCK seem to be associated with a preservice teacher's few or limited experiences with technology in mathematics classes. For example, Diane rarely used technology in her middle- and high-school mathematics classes. She used technology in her college mathematics classes, but it was for displaying content and communicating with or asking questions to professors, not for exploring mathematics concepts. Her lack of robust experiences with technology aligned with student-centered learning approaches may influence her limited beliefs about how to use technology to teach mathematics. This result is consistent with the findings of Ottenbreit-Leftwich et al.'s (2010) study that teachers' value beliefs related to technology use have an impact on how to use technology in their teaching. In addition, Chai et al. (2013) revealed that a lack of technology-related knowledge may be associated with a low level of TPCK. Because Diane did not show competence in using a DGE and tended to view technology as a supplementary tool (that is, for reducing time to work or displaying contents), she was unable to integrate a DGE into her imaginary teaching in student-centered approaches.

Beliefs and Knowledge about Mathematics and Pedagogy

Another interesting finding is that preservice teachers' beliefs and knowledge about mathematics and pedagogy seem to affect their levels of TPCK. Although preservice teachers hold a student-centered view of the use of technology and display an advanced level of TCK, their TPCK levels may not be high if they have more traditional beliefs about mathematics and learning and low levels of CK and PCK. That is, as Polly et al. (2010) stated, the acquisition of technology-related knowledge does not always ensure successful technology integration. Kim et al. (2013) had similar findings—that is, teachers who held more student-centered pedagogical beliefs tended to integrate technology more seamlessly into their teaching than those with more teacher-centered pedagogical beliefs.

Consistent with the results of Choy, Wong, and Gao's (2009) study, preservice teachers had positive attitudes toward technology, expressed a willingness to use technology in their future teaching, and showed good technical knowledge of the DGE, but they did not have appropriate knowledge to ask meaningful questions or create tasks that facilitate students' conceptual learning.

The Importance of Beliefs and TPACK

As indicated by findings from my study and other studies, preservice teachers with similar beliefs and knowledge about mathematics and pedagogy may use technology differently to teach mathematics. Or, preservice teachers who have positive attitudes toward technology use in mathematics classes and strong technical knowledge may not know how to use technology effectively to teach mathematics. The findings of this study indicated that in order to be able to use technology effectively to teach mathematics, preservice teachers should develop all areas of content, pedagogy, and technology in their beliefs and knowledge.

Preservice teachers definitely need to improve their knowledge of mathematics, how students think about and learn mathematics with/without technology, and how to use technology to teach mathematics. Just having knowledge, however, would not be enough. The preservice teachers also need to view mathematics as a continually expanding field in which students can construct their own mathematics through active engagement, teachers can facilitate students' conceptual learning, and technology can support student-centered approaches.

Implications

Much previous research has focused on the relationships between current mathematics teachers' beliefs and their teaching practices with technology. Although such previous research is necessary to improve teachers' integration of technology into their mathematics teaching, research on preservice mathematics teachers' beliefs and TPACK is also critical and essential to provide teacher educators with the knowledge base to develop ways to teach preservice teachers to effectively use technology in their future mathematics teaching. However, there are a few studies that address both preservice mathematics teachers' beliefs and TPACK or the relationships between them. This multi-case study focused on the relationships between the beliefs and TPACK of preservice mathematics teachers and used qualitative research methods rather than the self-report measurement approach used in some studies (Abbitt, 2011a; Chai et al., 2013). Consequently, it allowed more targeted investigation of my research questions by providing thorough descriptions of preservice teacher's beliefs and TPACK and an identification of the relationships between them. By conducting the multi-case study, I could better understand what the preservice teachers experienced with respect to mathematics and technology during their early schooling and college and how they reflected on their experiences. The study also allowed me to observe in detail how they solved mathematical problems and what pedagogical and technological strategies they used to teach students during the interviews.

In addition, the findings of this study provide insight into what aspects mathematics teacher educators or researchers should consider to cultivate teachers who effectively teach mathematics using technology. First, the preservice teachers' beliefs about the nature of mathematics and learning mathematics were more traditional and inflexible, whereas their beliefs about teaching mathematics and technology use were more progressive. The preservice teachers' beliefs about teaching and technology use may be more amenable to change because they have had less experience with teaching and technology. Thus, teacher educators should place more emphasis on developing preservice teachers' student-centered beliefs about teaching and technology because these are the beliefs that exhibit a high degree of malleability.

Second, the preservice teachers displayed higher levels of TCK than their levels of CK and PCK. They were familiar with *GSP* but knew how to use it only for themselves, not for mathematics instruction. Therefore, in order to develop preservice teachers' TPCK, teacher educators should focus on harnessing preservice teachers' TCK as a connection between mathematics instruction and technology. In particular, preservice teachers' low levels of TPCK may be due to their low CK and PCK. The preservice teachers had rough ideas of geometric shapes (e.g., they roughly recognized an acute triangle is a triangle that has small angles or how an acute triangle looks) but tended not to know accurate definitions of those geometric shapes (e.g., they could not provide a precise definition of the acute triangle as a triangle in which all three angles are less than 90 degrees). For another example, the preservice teachers could locate an exterior angle of a triangle, but they could not provide a precise definition of the exterior angle. Their lack of content knowledge may be related to their lack of both pedagogical knowledge and technological pedagogical content knowledge, so teacher educators should focus on developing preservice teachers' mathematical knowledge and pedagogical content knowledge, even if neither type knowledge includes technology.

Third, this study provides evidence suggesting that preservice secondary mathematics teachers' beliefs (beliefs about the nature of mathematics, learning and teaching mathematics, and technology integration) and their knowledge (knowledge about mathematics, pedagogy, and technology) are all closely related. Thus, to develop preservice teachers' levels of CK, PCK, and TCK, their beliefs about mathematics, learning, and technology use should be developed in concert and with an orientation toward student-centered approaches. Moreover, both their beliefs and knowledge are crucial influences on their knowledge of how to effectively use technology to teach mathematics (TPCK). Therefore, to encourage and improve preservice teachers' student-centered technology use, it is important that mathematics teacher educators focus on both beliefs and knowledge regarding pedagogy and technology. Mathematics teacher educators should focus on the development of preservice teachers' mathematical, pedagogical, and technological knowledge by providing learning and teaching experiences with student-centered approaches and positive experiences with technology and training to integrate technology into mathematics instruction. In addition, mathematics teacher educators should help preservice teachers develop student-centered beliefs about the nature of mathematics, learning, teaching, and using technology by providing them with opportunities to examine and reflect on their beliefs about mathematics, teaching, learning, and technology (Choy, Wong, & Gao, 2009; Richardson, 2003).

Future Research

Based on the findings of this study, I suggest several recommendations for further research. First, further studies could investigate randomly selected participants in different contextual settings, which could provide different results. In addition, further studies could be conducted in different subject areas of mathematics with diverse technology tools. It would help to measure and see what participants' TPACK is in overall mathematics rather than a specific mathematics content.

Second, in this study, I examined preservice teachers' TPACK. Although I could observe how they used *GSP* to teach a specific geometric theorem in their imaginary teaching, it was not teaching in their real classrooms. Thus, further longitudinal study that investigates these preservice teachers' teaching with technology in the classroom when they become mathematics teacher is needed. Third, my findings indicated that there are possible relationships between preservice teachers' beliefs about mathematics, learning and teaching, and the use of technology and their TPACK components. To investigate in depth how their beliefs are related to their TPACK components, further studies could subdivide the type of beliefs and then study the relationships between diverse types of beliefs and TPACK. For example, subdividing preservice teachers' beliefs into attitudes or motivations, further studies could research how each type of belief influences TPACK components.

Lastly, while conducting this study, the preservice teachers stated that their views on learning and teaching mathematics and using technology had been changed during their teacher education program courses. Therefore, further study could investigate what aspects of the teacher education program's courses influence the development of preservice teachers' beliefs.

Moreover, it could be studied whether or how preservice teachers' TPACK changes during the course of teacher education programs.

Concluding Remarks

In this study I attempted to investigate preservice secondary mathematics teachers' beliefs and TPACK and identify potential relationships between them. The four preservice secondary mathematics teachers in this study had unique beliefs and TPACK levels. Although they had similar experiences from past schooling, they have developed their own beliefs and knowledge with experience in college. I found the more student-centered the preservice teachers' beliefs, the higher their level of knowledge. To develop their knowledge, especially TPCK, having appropriate knowledge of technology is not enough; we should focus on preservice teachers' development of both beliefs and knowledge regarding mathematics, pedagogy, and technology. When preservice teachers have both appropriate beliefs and knowledge, they can develop their knowledge of how to effectively use technology to teach mathematics, and their beliefs and knowledge will be reflected in their future teaching practice with technology.

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APPENDIX A

Beliefs Interview

<u>Interview questions on beliefs about the nature of mathematics:</u>

a. When you hear the term mathematics, what do you think of? In other words, how do you define mathematics?

Possible questions to pose:

- Why do you think you view mathematics in this way?
- What other subject is mathematics most like? Least like?

b. Why do we need to learn mathematics?

Possible questions to pose:

- Could you describe how you are thinking about the need of mathematics in your everyday life?
- How can mathematics be useful in your everyday life?
- Could you give me some examples?

<u>Interview questions on beliefs about mathematics learning:</u>

a. How do you think students learn mathematics?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

b. How do you remember feeling about your mathematics experiences in middle school?

Possible questions to pose:

- How do you think about the way you have learned mathematics?
- What do you think was the hardest part about learning mathematics?
- Can you remember when you enjoyed learning mathematics?

c. What do you think is the most important aspect of mathematics that students should learn? In other words, what part of mathematics do you want students to be really good at?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

Interview questions on beliefs about mathematics teaching:

a. What do you think the role of mathematics teacher should be? You can give more than one role.

Possible questions to pose:

- Which one is the most important?
- Could you tell me why you think that way?
- Could you give me an example?

b. Could you describe your thoughts on your mathematics teachers in middle school and the instructional strategies they used to teach mathematics?

Possible questions to pose:

- Why do you think your mathematics teachers taught this way?
- Could you tell me why you think that way?
- Could you give me an example?

c. In order to be a good mathematics teacher, what do you think are the most important things for a teacher to do?

I will make a list of what you say.

Possible questions to pose:

- Could you rank these things most important to least important?
- Could you tell me why you think that way?
- Could you give me an example?

d. What do mathematics teachers need to know in order to be successful?

I will make a list of what you say.

Possible questions to pose:

- Could you rank these things most important to least important?
- Could you tell me why you think that way?
- Could you give me an example?

Interview questions on beliefs about the use of technology for learning and teaching:

a. In your mathematics classes in middle school, how often did you use technology?

Possible questions to pose:

- Could you give me an example of the way how you have used technology?
- What kinds of technology did your mathematics teachers use?
- How often did your mathematics teachers use it?

b. In your mathematics classes in high school, how often did you use technology?

Possible questions to pose:

- Could you give me an example of the way how you have used technology?
- What kinds of technology did your mathematics teachers use?
- How often did your mathematics teachers use it?

c. **In your mathematics classes in college, how often did you use technology?**

Possible questions to pose:

- Could you give me an example of the way how you have used technology?
- What kinds of technology did your mathematics teachers use?
- How often did your mathematics teachers use it?

d. **How do you think the use of technology affects students' mathematical thinking?**

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

e. **Are there any advantages or disadvantages in using technology instead of pen and paper?**

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example to illustrate how it helps or not?

f. **How do you think the use of technology to teach mathematics? Does using technology change the teacher's role in the classroom?**

Possible questions to pose:

- Could you tell me why you think that way?
- Could you describe the role of teacher when teaching mathematics using technology?

Interview questions on beliefs about the use of technology for their own teaching:

a. Describe your confidence in your ability to use technologies for mathematics instruction.

Possible questions to pose:

- Could you tell me which term of a scale indicates how you feel about your confidence among Very Confident, Confident, Not Confident, and Very Not Confident?
- Could you tell me why you think that way?
- Could you give me an example?

b. What technology has been available for you to use to teach mathematics?

Possible questions to pose:

- How do you use technology for the purpose of effective mathematics instruction?
- How do you think technology could be used for the purpose of assessment? Please provide examples.
- How do you think you could use technology for the purpose of communication? Please provide examples (colleagues, parents, etc).

c. When preparing lessons that incorporate technology, what do you take into account?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

d. What kinds of support would be most helpful in order to use technology more often in the mathematics classroom?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

e. What types of technology do you think you will need to better meet the needs of students when you become a teacher?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

f. What types of technology do you think you will need to better meet your needs as a teacher?

Possible questions to pose:

- Could you tell me why you think that way?
- Could you give me an example?

APPENDIX B

Task-based Interview

Task 1

Suppose students in your middle or high school mathematics class are studying rectangles and squares. They open a dynamic geometry sketch that contains a rectangle and a square, each of which have been constructed. Students are asked to consider properties of rectangles and squares, based on their exploration of the sketch. One pair of students has measured the diagonals and they have noticed they are always congruent. They claim, “quadrilaterals have congruent diagonals.”

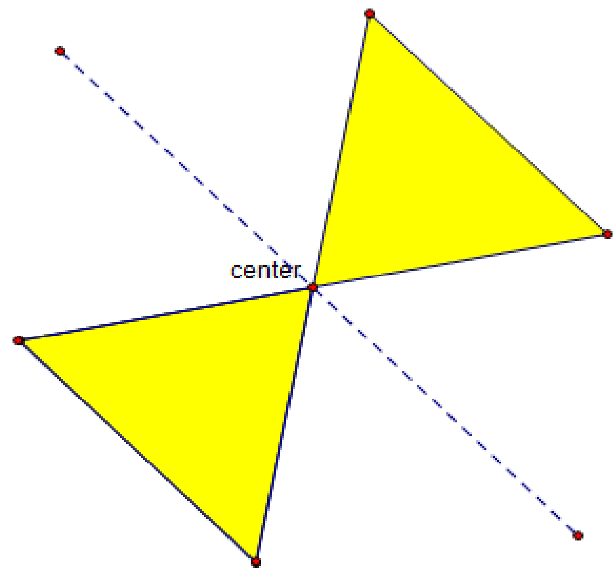
a. Is this claim always true, sometimes true, or never true? Explain.

b. How would you characterize their current level of geometric understanding?

c. Create a sketch using a dynamic geometry environment that you would like students to use to explore diagonals of quadrilaterals. Be sure to include directions and/or questions you would provide to students as they use this sketch.

Task 2

After studying rotations, reflections, and translations using a dynamic geometry tool a student is playing around with rotations through an angle of 180 degrees and reflections. After some time the student claims: “A rotation through 180 degrees is the same as a reflection!” The student includes a screen capture that looks similar to the picture below. They explain, “when I reflect the triangle on the right and when I rotate the triangle on the right, I get the same thing.”



a. Is the statement “A rotation through 180 degrees is the same as a reflection” true? Explain how you arrived at that conclusion.

b. What does the student understand about rotations and reflections?

c. What question or task using technology would you pose to the student to learn more about how they are thinking about rotations and reflections? Explain.

Task 3

Next week you are teaching a lesson on triangle centers and you are considering the following task.

Draw a large acute triangle on a sheet of paper. Fold the paper to form creases representing the perpendicular bisectors of each side of the triangle. What conclusions can you reach regarding the three perpendicular bisectors of the sides of the triangles?

a. Use the blank sheet of paper to complete the task. Describe what you notice.

b. Explore the same task using GSP. Describe what you do with the technology.

c. How would you extend the original task to take into consideration what you learned in part b?

d. How would you modify the original task to use technology with students? Give a restatement of the task. What pedagogical decisions and technological decisions did you make when redesigning this task?

Task 4

When using the sketch of a constructed rectangle in a dynamic geometry program a student, Mary, drags a vertex of the rectangle so that it becomes a square. Mary claims that “a rectangle is a square.”

a. How would you characterize the Mary’s current mathematical understanding? How might Mary have developed this understanding?

b. What important mathematical ideas does a student need to understand to know about relationships between rectangles and squares?

c. What instructional strategies and/or tasks would you use next with Mary? Why?

APPENDIX C

Rubric for Task-based Interview

Task 1	Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
	(A) Responds that the claim is sometimes true. (B) Knowledge that there exists at least one quadrilateral for which the diagonals are not always congruent. (C) States that for at least the rectangle and square the diagonals are always congruent. (D) Provides a correct mathematical justification for why the statement is sometimes true using proofs that involve triangles or other properties. Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Identifies that the student is able to notice that for a square and a rectangle that the diagonals are always congruent based on their measures. (B) Identifies that the student is at level 2 (descriptive) but probably not at level 3. (C) Has students consider at least one counterexample of a quadrilateral that has congruent diagonals. (D) Asks students to consider at least one example of a quadrilateral that has congruent diagonals. Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Accurately constructs or draws a quad using a DGE that is a counter-example. (B) Uses measures to find the lengths of the diagonals. (C) Drags to create multiple examples in a DGE. (D) Accurate constructions of the 2 of the following quad: • Square • Rectangle • Parallelogram • Rhombus Emergent: 0-1 of A – D or no response. Beginner: 2 of A – D Intermediate: 3 of A – D Advanced: All of A – D	(A) Uses the DGE technology to focus students on properties of different quadrilaterals and their relationships to the diagonals in the task. (B) Creates more than a single example using DGE technology to show the student that they are incorrect in the task. (C) Designs an exploration for students by creating accurate constructions and utilizing the measurement and dragging features Emergent: 0 of A – C or no response. Beginner: 1 of A – C Intermediate: 2 of A – C Advanced: All of A – C

Task 2	Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
	(A) Knowledge that a 180 degree rotation is never the same as a reflection when the domain and range are defined as all points in the plane. (B) Uses reasoning about orientation, such as a rotation preserves orientation and a reflection reverses orientation to explain why a rotation and reflection are different. (C) Understands that the images of symmetric polygons under a reflection and rotation of 180 degree may appear to look the same. (D) Understands images will align only when line of reflection is perpendicular to a line of symmetry and when the center of rotation is strategically placed on the line of reflection. (E) Knowledge of properties of reflection or rotation or how to perform a reflection and rotation Emergent: 0 or no response. Beginner: 1 of A – E Intermediate: 2 of A – E Advanced: 3 of A – E	(A) Displays knowledge about why students might think a rotation and reflection are the same. (B) Designs task that helps students see differences between rotation and reflections (uses labels for points, non-symmetric figure, matrices, etc) (C) Task or questions leads students to discover properties of reflections and rotations (D) Describes what students know about reflections and rotations Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Understands how to perform a rotation using the technology by marking a center of rotation, indicating an angle of rotation, selecting the preimage polygon and labeling the preimage and image. (B) Understands how to perform a reflection using the technology by marking the mirror line, selecting the preimage and labeling the preimage and image polygon. (C) Demonstrates a knowledge of how to label points (D) Uses dragging Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Creates a task using an appropriate figure to highlight the differences between rotations and reflections (non-regular polygon). (B) Considers lines of reflection that are not parallel to a side of the preimage in the task. (dragging) (C) Focuses on the labeling of points to illustrate differences in orientation in the task. (D) Considers other locations of the point of rotation that are not on the line of reflection in the task (dragging point). Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D

Task 3	Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
	(A) Knowledge that the circumcenter is equidistant from the vertices of the triangle (B) Knowledge that the perpendicular bisectors are concurrent – that there is a point of intersection (C) Knowledge that the circumcenter of a triangle is the center of a circle the circumscribes the triangle (names circumcenter) (D) Demonstrates knowledge about the location of the circumcenter (Inside for acute, on for right, and outside for obtuse). Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Asks students to consider the distance the circumcenter is from each of the vertices (B) Considers what students may have already done in class when modifying the tasks (C) Has students consider different types of triangles (D) Asks students to create a circle using the circumcenter and a vertex of the triangle. Emergent: 0 of 4 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Constructs the perpendicular bisectors to locate the circumcenter. (B) Uses the measurement tool in an appropriate manner. (C) Uses dragging to modify the original triangle and examine different locations of the circumcenter (D) Uses the circle tool to create a circumcircle Emergent: 0or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Gives an equivalent restatement of the task using technology so students are still considering circumcenters. (B) Creates more than a single example to show that the relationships hold for all triangles (C) Constructs a figure that will enable students to discover relationships of a circumcenter. (D) Makes appropriate use of multiple features of the tool such as dragging, measures, constructing, etc. Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D

Task 4	Content Knowledge	Pedagogical Content Knowledge	Technological Content Knowledge	Technological Pedagogical Content Knowledge
	(A) Recognizes that a rectangle is not a square (B) Recognizes that a square is a rectangle (C) Uses knowledge of differences between a rectangle and square to justify why a rectangle is not a square (which includes the following properties of a square) • 4 congruent sides • Perpendicular diagonals • Diagonals are angle bisectors • Diagonals create 4 congruent right triangles (D) Uses knowledge of rectangles and squares to justify why a square is a rectangle (includes the following properties common to both) • 4 right angles • Opposite sides congruent • Congruent diagonals Emergent: 0 or no response Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) State's student misconception (B) Understands where the student's misconceptions may have come from and relate them to technology or van Hiele levels (C) Uses knowledge of properties of squares and rectangles and differences between these two figures to pose questions to the students (D) Task or questions leads students to understand that squares are always rectangles, but rectangles are not always squares Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Understands the drag feature in DGE and how it maintains the properties of the original construction (B) Uses measures to show that a rectangle is not a square since all sides are not congruent (C) Constructs a square and drags it to show that a square can never be a rectangle (D) Constructs a square and a rectangle Emergent: 0 or no response. Beginner: 1 of A – D Intermediate: 2 of A – D Advanced: 3 of A – D	(A) Describes a technological sketch that can help with student's misconceptions and justifies its appropriate use (Does not focus necessarily on properties, focuses on figures) (B) Designs an appropriate activity with the technology that assists students in learning the relationships between squares and rectangles by focusing students on the properties of each figure. (C) Makes appropriate use of multiple features of the tool such as dragging, measures, constructing, etc. Emergent: 0 or no response. Beginner: 1 of A – C Intermediate: 2 of A – C Advanced: 3 of A – C

APPENDIX D

Rebecca's Worksheet for the Performance Interview

Exterior Angles in a Polygon

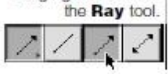
Name(s): ____________________

An exterior angle of a polygon is formed when one of the sides is extended. Exterior angles lie outside a convex polygon. In this investigation, you'll discover the sum of the measures of the exterior angles in a convex polygon.

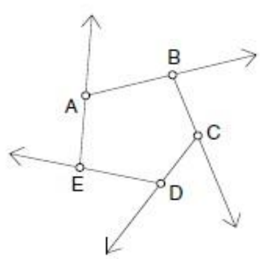
Do this investigation with a triangle, a quadrilateral, or a pentagon. Plan together with classmates at nearby computers to investigate different polygons so that you can compare your results. The activity here shows a pentagon. Don't let that throw you if you're investigating a triangle or a quadrilateral—the basic steps are the same.

Sketch and Investigate

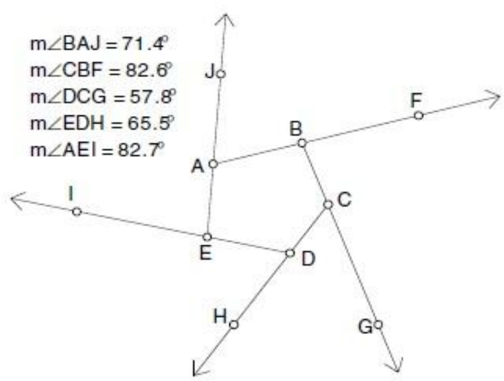
Hold down the mouse button on the **Segment** tool, then drag right to choose the **Ray** tool.



1. Use the **Ray** tool to construct a polygon with each side extended in one direction. Be sure to construct the polygon without creating any extra points. Your initial sketch should have the same number of points (vertices) as sides. If your polygon didn't end up convex, drag a vertex to make it convex.



Step 1



Steps 2 and 3

To measure an angle, select three points, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

2. Construct a point on each ray outside of the polygon so that you'll be able to measure exterior angles.
3. Measure each exterior angle. Be careful to measure the correct ones!

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

4. Calculate the sum of the exterior angles.
5. Drag different vertices of your polygon and observe the angle measures and their sum. Be sure the polygon stays convex.
6. Compare your observations with those of classmates who did this investigation with different polygons.

Exploring Geometry with The Geometer's Sketchpad

Chapter 5: Polygons • 109

Exterior Angles in a Polygon (continued)

Q1 Write a conjecture about the sum of the measures of the exterior angles in any polygon.

Double-click a point to mark it as a center.

In the Edit menu, choose **Select All**. Then click on each measurement to deselect it.

Hold down the mouse button on the **Arrow** tool, then drag right to choose the **Dilate Arrow** tool.

Follow the steps below for another way to demonstrate this conjecture.

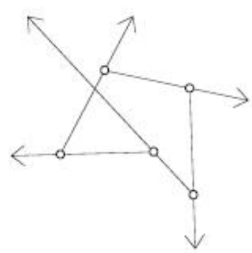
7. Mark any point in the sketch as a center for dilation.
8. Select everything in the sketch except for the measurements.
9. Change your **Arrow** tool to the **Dilate Arrow** tool and use it to drag any part of the construction toward the marked center. Keep dragging until the polygon is nearly reduced to a single point.

Q2 Write a paragraph explaining how this demonstrates the conjecture you made in Q1.

Explore More

In the Edit menu, choose **Preferences** and go to the Units panel. In the Angle Units pop-up menu, choose **directed degrees**.

1. Investigate the sum of the exterior angle measures in concave polygons. For this investigation, you may want to measure angles in directed degrees. The sign of an angle measured in directed degrees depends on the order in which you select points.



110 • Chapter 5: Polygons

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