PRESERVICE TEACHERS’ KNOWLEDGE OF CONTENT AND STUDENTS IN GEOMETRY

by

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(Under the Direction of Denise A. Spangler)

ABSTRACT

The purpose of this study was to explore what kind of knowledge of content and students preservice teachers have with respect to geometry and how they apply their knowledge to examining students’ work. The data were collected in several ways: written tasks, pre- and post-tests, interviews, and observations. The knowledge of content and student (KCS) tasks were designed to investigate preservice teachers’ knowledge of students’ thinking and misconceptions and were analyzed by applying two different frameworks: Radatz’s error analysis to describe causes of students’ errors and Shulman-Fischbein framework for analyzing preservice teachers’ subject matter and pedagogical content knowledge in mathematics. The pre- and post-tests were scored and analyzed to identify changes in preservice teachers’ knowledge and to verify the influence of the KCS tasks and the content and methods courses. The interviews and observations were used to identify preservice teachers’ backgrounds and opinions.

The analysis revealed that preservice teachers had strong subject matter knowledge in geometry. When analyzing students’ thinking and misconceptions, they sometimes tended to focus on the correctness of students’ final answers and to ignore details in students’ solutions.
Additionally, they tended to pay more attention to algorithmic and formal aspects of mathematical knowledge than to the intuitive aspect. The preservice teachers’ ability to propose instructional strategies to correct students’ misconceptions was narrow in that they did not apply a variety of teaching methods, activities, and manipulatives. The causes of students’ errors and misconceptions were multidimensional, whereas preservice teachers’ knowledge in this area was narrow in that preservice teachers interpreted students’ errors from only one or two perspectives. The results of the pre- and post-tests revealed that preservice teachers’ knowledge in geometry improved, which may be influenced by their course taking and the KCS tasks used in this study. These findings imply that teacher preparation programs need to provide opportunities for preservice teachers to investigate students’ thinking and misconceptions through content and methods courses in order to enhance their knowledge.

INDEX WORDS: knowledge of content and students, subject matter knowledge (SMK), pedagogical content knowledge (PCK), preservice teachers, geometry, misconceptions
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To my beloved husband, Eui-Cheol Shin, and family for their support and endless love
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CHAPTER 1
INTRODUCTION

When students do not understand mathematical concepts, who is responsible? Are teachers responsible for students’ lack of understanding or are students responsible for themselves? I have experienced two different teacher education cultures: One was in South Korea and the other was in the United States (U.S.). In South Korea, I believed that the best way to teach mathematics well was to know as much mathematics as possible. I believed that if I fully understood mathematical concepts, I could teach those concepts well. Thus, it was the students who were responsible for understanding what I taught. Although I considered how to explain concepts in easy ways, I was much more interested in equipping myself, as a teacher, with strong mathematical knowledge. On the other hand, after I studied in the US, my belief changed. In particular, the coexistence of content courses and their parallel methods courses in a preservice teacher education program made an impression on me and led me to realize that teachers must have knowledge about good teaching as well as mathematical content knowledge. I became aware that although I have strong mathematical knowledge, my students may not understand what I teach because my approach to the concepts may not be appropriate for the students. I must consider my students’ knowledge and skills, such as their abilities in mathematical reasoning and their previous knowledge, for effective teaching. With this perspective, I saw that it was teachers who were responsible for students’ misunderstanding of mathematics. Teachers should create an environment where students have meaningful learning experiences.
The National Council of Teachers of Mathematics’ (NCTM, 1991) *Professional Standards for Teaching Mathematics* emphasizes the standard about knowing students as learners of mathematics and the necessity that multiple perspectives on students as learners are provided throughout the preservice and continuing education program. The standards claimed the following:

Teachers need opportunities to examine children’s thinking about mathematics so that they can select or create tasks that can help children build more valid conceptions of mathematics. Developing multiple perspectives on students as learners of mathematics enables teachers to build an environment in which students may learn mathematics with appropriate support and acceptance. (p. 144)

The NCTM’s (2000) *Principles and Standards for School Mathematics* also emphasized the teaching principle; “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (p. 16). Because students’ learning depends on the experiences that their teachers provide, the standards recommend having students enhance their ability to use mathematical knowledge to solve problems and their confidence in mathematics. Additionally, teachers should “understand and be committed to their students as learners of mathematics and as human beings and be skillful in choosing from and using a variety of pedagogical and assessment strategies” (p. 17).

In an empirical study, Wilson, Cooney, and Stinson (2005) verified that teachers agree with the necessity to know students, including having personal knowledge of a student and having knowledge of students’ skill levels. In the study, teachers indicated that having solid knowledge of students is an important component for good teaching. It is experience that is the most fundamental way to establish teachers’ knowledge of students. Generally, experience would be acquired by teachers’ own teaching in a classroom. Therefore, as teachers’ teaching experience increases, their knowledge of students should increase.
However, preservice teachers do not have much opportunity to accumulate knowledge of students from their own teaching. One possible way for preservice teachers to enlarge their knowledge of students is throughout preservice teacher education programs, which is the second important means for developing good teaching according to Wilson et al. (2005). In fact, the mathematics education program of the University of Georgia provides a series of methods courses as well as content courses for prospective teachers. For example, the course, Teaching Geometry and Measurement in the Middle School is a methods course aligned with the content course Geometry and Measurement for Middle School Teachers. The courses in teacher preparation programs can allow preservice teachers to learn about students’ mathematical thinking by introducing studies and theories about students’ learning and thinking.

Although teachers can develop their know-how of students’ thinking through their teaching careers, beginning teachers do not have such a background. This lack does not mean that novice teachers’ failure to understand their students can be accepted. Therefore, opportunities to build knowledge of students’ learning and thinking need to be given to preservice teachers. I began this study with my belief that preservice teachers can develop their knowledge of content and students through their course taking and extra work with tasks that engage them in analyzing students’ thinking. I expected that preservice teachers could make sense of and interpret students’ thinking and misconceptions. Additionally, I expected that they could relate their knowledge to their future teaching. One of the contexts that was available to me in which to study teachers’ knowledge of students’ thinking was a pair of content and methods courses in geometry and measurement for middle grades teachers

According to Clements and Battista (1992), school geometry can be defined as “the study of those spatial objects, relationships, and transformations that have been formalized (or
mathematized) and the axiomatic mathematical systems that have been constructed to represent them” (p. 420), where spatial reasoning is composed of “the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated” (p. 420). The NCTM’s (2000) *Principles and Standards for School Mathematics* emphasized the importance of geometry: “Geometric modeling and spatial reasoning offer ways to interpret and describe physical environments and can be important tools in problem solving” (p. 41). Despite the significance of geometry, students’ performance in geometry has historically been unsatisfactory. Clements and Battista (1992) summarized studies that reported students’ poor performance in learning basic geometric concepts, proofs, and problem solving, and in dealing with properties of figures, visualization, and applications. They also indicated that plausible reasons for students’ poor performance were that teachers did not have enough time to teach geometry or that teachers had less opportunity to learn geometry than their opportunity to learn other areas of mathematics. Additionally, teachers’ unawareness of students’ status of thinking can result in low performance in geometry. For example, if students’ levels of thinking are such that they are unprepared to learn proofs at the secondary level, teachers may not be able to help them understand proofs. Thus, it is necessary for teachers to be prepared to teach geometry by considering that status students’ of thinking in order to improve the learning and teaching of geometry.

**Background**

There are studies that investigated teachers’ understanding of students’ thinking. Carpenter, Fennema, Peterson, and Carey (1988) focused on teachers’ knowledge of children’s thinking about mathematics and attempted to address the relationship between teachers’ pedagogical content knowledge (PCK) and their students’ achievement in the domain of addition
and subtraction problem solving. Their results showed that teachers were able to identify the critical distinctions between problems and children’s problem solving strategies, but teachers’ knowledge was not organized into “a coherent network that related distinctions between problems, children’s solutions, and problem difficulty to one another” (p. 398). Additionally, they suggested the following:

Teachers do not traditionally make instructional decisions based on the strategies that children use to solve different problems, whereas they do make decision about whether to include particular problems based on their assessment of whether the problem would be too difficult for their students. (p. 399)

This study was influential in research on teachers’ PCK, but it focused on arithmetic at the elementary school rather than geometry at the middle school level.

Carpenter, Fennema, Peterson, Chiang, and Loef (1989) examined teachers’ use of knowledge from research on children’s mathematical thinking in a particular content domain. They assumed that “knowledge about differences among problems, children’s strategies for solving different problems, and how children’s knowledge and skills evolve” (p. 500) influenced teachers’ instruction and that such knowledge influenced teachers’ ability to evaluate their students. Knowledge of students would help teachers design and implement better instruction that corresponded to students’ knowledge. Their results showed that using research on children’s thinking and problem solving to improve classroom instruction was helpful in developing teachers’ knowledge of students.

Manizade (2006) developed an instrument that assessed teachers’ PCK in geometry and measurement; in particular, decomposing and recomposing geometric objects in one-dimensional and two-dimensional space. She identified five components of teachers’ PCK: (a) knowledge of subject specific difficulties and common misconceptions; (b) knowledge of useful representations of the content; (c) understanding of appropriateness of student's proof;
justification or mathematical discourse; (d) levels of geometric development; and (e) connections among big mathematical ideas. She described the process of designing the instrument by applying the Delphi method, which consisted of three rounds of surveys that were sent to a diverse panel of 20 experts. The experts evaluated items and gave feedback, and Manizade modified the items based on the feedback. The instrument that she developed was the product of the final round of the survey. Although she suggested a possible way to assess teachers’ PCK, she did not actually implement her instrument with teachers. Thus, there was a need to investigate how teachers perform on the instrument and how teachers’ responses can be interpreted.

Hill, Ball, and Schilling (2008) addressed an effort to conceptualize mathematical knowledge for teaching and to develop measures of teachers’ knowledge of content and students by writing multiple-choice items, piloting, and analyzing the items. They defined the knowledge of content and students as a combination of subject matter knowledge and knowledge of students. They expected that teachers would employ mathematical knowledge and reasoning when they interpreted students’ thinking. They designed multiple-choice items that fall into four categories: common student errors, students’ understanding of content, student developmental sequences, and common student computational strategies. The results showed that their items were able to partially measure the domain of knowledge of content and students. They claimed:

Teachers “know” that students often make certain errors in particular areas, or that some topics are likely to be difficult, or that some representations often work well. But teachers also reason about students’ mathematics: They see student work, hear student statements, and see students solving problems. Teachers must puzzle about what students are doing or thinking, using their own knowledge of the topic and their insights about students (p. 396).

Although Hill et al.’s study had a significant influence on other studies about knowledge of content and students, there was a limitation of the multiple-choice format. They reported the
difficulty of writing answer choices; they sometimes were too obvious or too absurd. Thus, in order to examine teachers’ in-depth knowledge, there was a need to consider open-ended items.

**Research Questions**

The purpose of this study was to learn what kind of knowledge of content and students preservice teachers have with respect to geometry and how they apply their knowledge to examining students’ work. Knowledge of students should be a fundamental component of teacher knowledge so that teachers can choose and design tasks and activities to help children develop valid mathematical (NCTM, 1991). Hill et al. (2008) identified that teachers were able to interpret students’ thinking by using their mathematical knowledge. Two studies (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, et al., 1989) identified a positive effect of using students’ solutions in order for teachers to build their knowledge of students. Those studies led to my interest in studying preservice teachers’ knowledge of content and students in geometry. Additionally, I hypothesized that the courses provided by the teacher preparation program as well as doing extra work that was related to examining students’ thinking would influence the development of preservice teachers’ knowledge.

On the basis of the literature and consideration of teachers’ knowledge of content and students, this study focused on the following research questions:

1. What kinds of errors do middle school grades make, and what kinds of misconceptions do they exhibit with regard to particular geometry topics?
2. How do preservice middle grades teachers interpret students’ work in geometry, and how do they apply their knowledge to the interpretation?
3. How is preservice middle grades teachers’ pedagogical content knowledge influenced by their course taking and engaging with tasks that assess their knowledge of content and students?

Before conducting this study, I needed to clarify the use of the words *errors* and *misconceptions*. According to the dictionary, an error refers to “a mistake, inaccuracy, or misjudgment” or “the act or state of being wrong or making a misjudgment,” whereas a misconception is defined as “a false or mistaken view, idea, or belief.” There was no research on explicitly distinguishing between an error and misconception in the field of mathematics education. Hiebert and Carpenter (1992) introduced research on students’ misconceptions; there are some studies that used the word error, but others used misconception. Therefore, in this study, I did not differentiate those meanings of the words; I used them interchangeably.
CHAPTER 2
LITERATURE REVIEW AND THEORETICAL FRAMEWORK

It is a common belief that teachers’ mathematical knowledge plays an important role in students’ mathematical learning (Ball, 2003). To achieve the goal of improving students’ mathematical learning, many researchers have investigated the nature of teacher knowledge. The first part of this chapter introduces two streams of research on teacher knowledge. One stems from Shulman’s conceptualization of knowledge necessary for teachers to teach. Although Shulman was a pioneer in the realm of research on teacher knowledge, his conceptualization was not limited to mathematics education. The second stream of research was mathematical knowledge for teaching. Unlike Shulman’s conceptualization, studies within this stream were specific to mathematics. Then, this chapter introduces studies that investigated teacher preparation programs. Because this study was conducted along with the content and methods courses of a teacher preparation program, I assumed that those courses affected the preservice teachers’ knowledge. Lastly, this chapter introduces the frameworks that I used to analyze the participants’ work.

Teacher Knowledge

Shulman’s Dimensions of Content Knowledge for Teaching

Shulman (1986) established the theory of dimensions of content knowledge that are necessary for teachers. He divided content knowledge for teaching into three components: subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge. According to his conceptualization, SMK is “the amount and organization of
knowledge per se in the mind of the teacher” (p. 9). SMK requires both knowing beyond knowledge of the facts or concepts of a particular subject area and understanding the structures of the subject matter in substantive and the syntactic ways simultaneously. By *substantive* structures he meant how to organize the basic concepts and principles of a particular subject, say mathematics, whereas the *syntactic structures*, like grammar, are the set of rules on which the area of mathematics builds truth or falsehood, validity or invalidity (Shulman, 1986).

PCK refers to “subject matter knowledge for teaching” (p. 9) beyond subject matter knowledge per se. PCK includes the common topics in a particular subject, the efficient ways of representation of the ideas, and the powerful analogies, illustrations, examples, explanations, and demonstrations that make it easy for students to understand. PCK also involves understanding factors that make a specific topic easy or difficult to learn, conceptions and preconceptions that students of different ages and with different backgrounds have, and the understanding of student misconceptions and their influence on subsequent learning (Shulman, 1986).

Curricular knowledge is defined as

the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indication and contraindications for the use of particular curriculum or program materials in particular circumstances. (Shulman, 1986, p. 10)

Teachers should be able to choose tools of teaching that represent particular content well and tools that help them assess students’ achievements. Curricular knowledge also includes understanding alternative curricula for instruction and requires teachers’ to associate the content of a particular subject with ideas in other subjects (Shulman, 1986).
Mathematical Knowledge for Teaching

According to Ball, Thames, and Phelps (2008), mathematical knowledge for teaching (MKT) is “the mathematical knowledge needed to carry out the work of teaching mathematics (p. 395).” The research team for the Mathematics Teaching and Learning to Teach Project and the Learning Mathematics for Teaching Project conceptualized six domains for MKT. Based on two of Shulman’s dimensions of teacher knowledge, SMK and PCK, the research team divided each dimension into three domains, and consequently, there are six domains that comprise MKT. The three domains that compose SMK are common content knowledge, specialized content knowledge, and horizon content knowledge, whereas the three domains that constitute PCK are knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum (Ball et al., 2008).

Common content knowledge (CCK) is the mathematical knowledge and skills that are used in a variety of settings that are not unique to teaching. It includes correctly solving mathematics problems, recognizing students’ wrong answers and textbooks’ inaccurate definitions, and doing the work that is given to students. An example of CCK is to know the answers to questions such as “What is a number that lies between 1.1 and 1.11?”, “Is a square a rectangle?”, “Is 0/7 equal to 0?”, and “Are diagonals of a parallelogram perpendicular to each other?” Knowing the answers to these questions is not unique to teachers; people who know mathematics can find correct answers (Ball et al., 2008).

Specialized content knowledge (SCK) indicates the mathematical knowledge and skills that are unique to teaching. Because this kind of knowledge is not needed in other situations, Ball et al. called it “specialized.” SCK includes knowledge beyond what being taught to students and the use of decompressed or unpacked forms of knowledge. We expect that students can improve
their fluency with compressed knowledge and use sophisticated mathematical ideas and procedures. For this to happen, teachers must have decompressed knowledge and make features of particular mathematical content visible to and learnable by students (Ball et al., 2008).

Horizon content knowledge (HCK) is “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403). With HCK, teachers can see connections between present mathematical ideas and much later mathematical ideas. HKC would help teachers make decisions about how to talk about a mathematical concept. As Ball et al. mentioned, however, there is not yet much research on HCK, so it is not clear whether HCK is part of subject matter knowledge or may run across the other categories.

Knowledge of content and students (KCS) refers to the combination of knowing about students and knowing about mathematics. KCS includes anticipation of students’ thinking, their potential confusion, and knowledge of common student conceptions and misconceptions about particular mathematical content. When choosing an example and when assigning a task, teachers should be able to consider how to motivate and encourage students and predict what students will be able to do and not do (Ball, Thames, & Phelps, 2008). In follow-up research about KCS, Ball et al. suggested that it can be separated from knowledge of teaching, such as how to establish students’ mathematical thinking and remedy their errors, and from knowledge of curriculum materials. Additionally, KCS concentrates on teachers’ understanding of students’ learning of a particular topic. KCS is different from teachers’ SMK because a teacher who has a strong background of mathematical content may have weak knowledge of students’ learning or vice versa (Hill et al., 2008).
Knowledge of content and teaching (KCT) is the combination of knowing about teaching and knowing about mathematics. KCT includes planning instruction and instructional decisions as well as choosing and sequencing examples to use in instruction and tasks to take students deeper into the content. Teachers should be able to evaluate which representations used to teach a specific idea are good or bad and to identify which instructional methods and procedures are more efficient. All of these need “an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning” and “coordination between the mathematics at stake and the instructional options and purposes at play” (Ball et al., 2008, p. 401).

Ball et al. defined Shulman’s curricular knowledge as knowledge of content and curriculum (KCC) so as to locate it under PCK because they believed that definition is consistent with follow-up studies from Shulman’s research team. However, they stated that like HCK, it is not clear whether KCC is a part of the category of KCT, whether it runs across several categories, or whether it is a category in its own right (Ball et al., 2008). Figure 1 shows a visual depiction of Ball et al.’s conceptualization of mathematical knowledge for teaching.

Because my goal was to examine preservice teachers’ knowledge of content and students, I placed more emphasis on KCS than on other areas of MKT. However, I did not consider that the domains can be exclusively separated from one another. In particular, in my investigation of preservice teachers’ SMK I considered CCK, SCK, and HCK, and when I asked the teachers to suggest instructional strategies they revealed elements of KCT and KCC. Thus, despite the fact that my main focus was on KCS, I did not use an exclusive conceptualization of the subcategories of MKT.
Influence of Teacher Education Program

**Improvement of Preservice Teachers’ Knowledge of Students’ Conceptions**

Tirosh (2000) investigated preservice teachers’ PCK in the topic of division of fractions and found that it can be enhanced through mathematics methods courses. Before entering the course, preservice teachers attributed sources of students’ incorrect responses to “algorithmically based mistakes or reading-comprehension difficulties” (p. 21), whereas at the end of the course they were aware of various sources of incorrect responses including overgeneralization of properties of division of natural numbers, influences of the partitive intuitive model of division, and intuitive mistakes. Tirosh claimed that knowledge of students’ conceptions/misconceptions should be a part of teachers’ knowledge of students’ mathematics, that researchers need to analyze preservice teachers’ knowledge of students’ ways of thinking about particular topics, and
that teacher preparation programs can help preservice teachers acquire some components of such knowledge.

Sowder (2007) summarized the recommendations for better preparation of teachers of mathematics and science suggested by the National Research Council:

- Help prospective teachers to know well, understand deeply, and use effectively and creatively the fundamental content and concepts of the disciplines that they will teach;
- Unify, coordinate, and connect content courses in mathematics with methods courses and field experience;
- Teach content through the perspectives and methods of inquiry and problem solving;
- Present content in ways that allow students to appreciate the applications of mathematics;
- Provide learning experiences in which mathematics is related to and integrated with students’ interests, community concerns, and societal issues;
- Provide opportunities for prospective teachers to learn about and practice teaching in a variety of school contexts and with diverse groups of children (p. 200).

Sowder (2007) also reported and summarized several successful studies and projects that investigated the role of children’s thinking in teacher preparation program. Those studies verified that the opportunity to observe and examine children’s thinking about mathematics can serve as a productive component of teacher preparation programs as well as a powerful component of professional development. The Developing Mathematical Ideas (DMI) project was a program that allowed teachers to investigate how children understood a big mathematical idea. A study revealed that having preservice teachers examine children’s understanding of the big ideas using the DMI materials led to increased knowledge for the preservice teachers. The Integrating Mathematics and Pedagogy (IMAP) project conducted an experiment where each of five groups of prospective teachers was given one of five treatments providing opportunities to study children’s thinking by: (1) interviewing children, (2) watching videos of interviews of children, (3) visiting classrooms of exemplary teachers, (4) visiting classrooms of teachers unknown to the researchers, or (5) having no treatment (as a control group). The results showed that the groups
of preservice teachers who studied children’s mathematical thinking, whether through interviews or by watching videos, had higher scores than the other groups.

**Theoretical Framework**

*Error Analysis*

In the process of learning mathematics, errors occur as the result of complicated processes. They result from a variety of variables including teachers, curricula, environments, and interactions among those variables. Thus, it is difficult to identify causes of errors in a clearly separate way. However, students’ errors do not occur randomly; they are not only based on systematic rules but also generated from sensible origins (Radatz, 1979).

Radatz (1979) classified five possible causes of errors based on an information-processing theory across mathematical topics. The information-processing theory was to examine “the mechanisms used in obtaining, processing, retaining, and reproducing the information contained in mathematical tasks” (p. 164). The first type of error occurs because of language difficulties. The meaning and usage of a word that is used in mathematical concepts, symbols, and vocabulary is often different from that in everyday life, which results in a “misunderstanding of the semantics of mathematical text” (p. 165). For example, children often produce errors in word problem because of difficulty translating a semantic network in everyday language into a formal network in mathematical language.

The second type of error is due to difficulties in obtaining spatial information. Many mathematical tasks employ a number of iconic instructions, diagrams, and visualizations, whereas students’ spatial abilities and capacity sometimes are not sufficient for interpreting these representations, which can result in students’ errors. For example, difficulties in reading Venn diagrams sometimes cause the error shown in Figure 2. Students may answer that the shaded
region indicates the set A. They may make this error because they do not correctly read the boundary lines of sets and ignore irrelevant lines (Radatz, 1979).

Figure 2. Student’s error in reading Venn diagram (Radatz, 1979, p. 166).

The third type of error is due to deficient mastery of prerequisite skills, facts, and concepts including “ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of necessary concepts and symbols” (p. 166). For example, a student was given a problem that asked the student to double the smallest three-digit number and add the largest four-digit number. The student answered $111 + 111 = 222$ and $222 + 9999 = 10221$. The student may be able to add three-digit and four-digit numbers and to find the largest four-digit number. However, the student lacked the concept of the smallest three-digit number or possibly also the smallest $n$-digit number (Radatz, 1979).

The fourth type of error is due to incorrect associations or rigidity of thinking, which means that students habitually apply cognitive operations that are previously established to a new mathematical task whose fundamental conditions are different from previous ones. Radatz (1979) provided an example of this type of error about the basic operation (see Fig. 3). Looking at the left column of a student’s solution, we see that the student found the answers to the two blank cells by adding 7 to the numbers in the corresponding right column rather than subtracting 7
from the numbers. The student may incorrectly associate the rule for the 1st, 2nd, 4th, and 6th rows with the 3rd and 5th rows.

<table>
<thead>
<tr>
<th>Task</th>
<th>A student’s solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>+7</td>
<td>+7</td>
</tr>
<tr>
<td>In</td>
<td>Out</td>
</tr>
<tr>
<td>31</td>
<td>38</td>
</tr>
<tr>
<td>20</td>
<td>27</td>
</tr>
<tr>
<td>79</td>
<td>79</td>
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<td>42</td>
<td>49</td>
</tr>
<tr>
<td>68</td>
<td>68</td>
</tr>
<tr>
<td>45</td>
<td>52</td>
</tr>
</tbody>
</table>

*Figure 3. Example of errors due to incorrect associations or rigidity of thinking (Radatz, 1979, p. 168).*

The fifth type of error results from the application of irrelevant rules or strategies. Because of successful experiences where students apply comparable or irrelevant rules or strategies to other tasks, students make errors. Radatz (1979) provided a geometric example of this category. In a criterion-referenced test, students were asked to choose the object that resulted from rotating a given figure by 180°. Common student answers are given in Figure 4. Their strategy was to fold the square down instead of rotating the square. This strategy worked for Item 1 but did not work for Item 3 (Radatz, 1979).
Fischbein (1994) emphasized that mathematics is a human activity and is invented by human beings. As a human activity, mathematics has three components: the algorithmic, the formal, and the intuitive. The algorithmic aspect refers to knowledge of procedures such as rules and solving procedures. Although algorithmic knowledge is acquired by practical and systematic practice and training and is remembered by rote or as a routine skill, it involves understanding why the algorithms or the particular procedures work and when they are appropriate to use. This aspect of knowledge is important because we may not use mathematical concepts in problem solving situations without training in the skills that are relevant to the concepts even though we understand the concepts (Fischbein, 1994; Tsamir & Tirosh, 2008).

The formal aspect of knowledge indicates axioms, definitions, theorems, and proofs, which are essential components in the reasoning process. Students should be able to invent, learn, organize, check, and use those components as a human activity. This aspect of
mathematical knowledge also requires “a hypothetic-deductive construction, the feeling of coherence and consistency, the capacity to think propositionally, independently of practical constraints” (Fischbein, 1994, p. 232).

Fischbein (1994) considered intuition as the third component of productive mathematical reasoning. According to him, intuitive cognition is “a kind of cognition that is accepted directly without the feeling that any kind of justification is required” (p. 232). Thus, a feature of intuitive cognition is self-evidence. Intuitive cognition is sometimes consistent with logically justifiable truths but sometimes incompatible with the truths. The latter case causes epistemological obstacles in the learning, solving, and invention processes.

The three components of mathematical knowledge sometimes converge and sometimes conflict with one another. Students sometimes apply their solving schema inappropriately because of superficial similarities without considering formal constraints. Intuitive interpretation sometimes prohibits the formal control and the requirements of the algorithmic solution from being activated because it has a strong basis of individual experience (Fischbein, 1994).

**Shulman-Fischbein Framework**

Tsamir and Tirosh (2008) attempted to combine two theories to contribute to evaluating mathematics teachers’ knowledge. Because Shulman’s theory did not reflect the distinctive characteristics of mathematical thinking, they looked for a theory dealing with mathematical ways of thinking. In an effort to associate the theory of teacher knowledge with mathematical ways of thinking, they incorporated two dimensions of Shulman’s theory, SMK and PCK, with three components of mathematical knowledge, algorithmic, formal, and intuitive, suggested by Fischbein. Each dimension of teacher knowledge is integrated into three components of
mathematical knowledge, which results in six cells (Fig. 5). In their study, they restricted PCK to teachers’ knowledge of students’ misconceptions.

<table>
<thead>
<tr>
<th>Components of Mathematical Knowledge (Fischbein’s theory)</th>
<th>Algorithmic</th>
<th>Dimensions of Teachers’ Knowledge (Shulman’s theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Algorithmic</td>
<td>SMK</td>
</tr>
<tr>
<td></td>
<td>Formal</td>
<td>PCK</td>
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<tr>
<td></td>
<td>Intuitive</td>
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<tr>
<td></td>
<td></td>
<td>Cell 1 Mathematical algorithmic-SMK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cell 4 Mathematical algorithmic-PCK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cell 2 Mathematical formal-SMK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cell 5 Mathematical formal-PCK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cell 3 Mathematical intuitive-SMK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cell 6 Mathematical intuitive-PCK</td>
</tr>
</tbody>
</table>

Figure 5. A schematic description of the Shulman-Fischbein framework (Tsamir & Tirosh, 2008, p. 863).

According to Tsamir and Tirosh (2008), the mathematical algorithmic-SMK refers to “teachers’ knowledge of solving procedures and supporting them by explicit justifications” (p. 863). An example of this category of knowledge is to know the addition algorithm for fractions and to be able to explain each step in the algorithm. The mathematical formal-SMK indicates “teachers’ knowledge of the core principles of the discipline of mathematics” (p. 863). An example of this type of knowledge is to know that one counterexample is sufficient to refute a universal statement but insufficient to refute an existential statement in the process of proof. The mathematical intuitive-SMK is related to the notion of secondary intuitions, which “are formulated when formal knowledge becomes intuitive” (p. 862). An example is to accept that multiplication does not necessarily make bigger without needing justification. The mathematical algorithmic-PCK refers to a “teacher’s knowledge of the most common incorrect algorithms that students apply when solving mathematical tasks and their possible sources” (p. 863). An example is to be aware of students’ incorrect use of the algorithm, 

\[-5x > 5 \rightarrow x > -1,\]
retaining the direction of the inequality despite dividing both sides by a negative number. The mathematical formal-PCK indicates “teachers’ knowledge of students’ common formal-related errors” (p. 863). An example is to recognize students’ conception that it is sufficient to consider one example, $2 + 4 = 6$, to verify that the sum of any two even numbers is an even numbers. The mathematical intuitive-PCK is “teachers’ awareness of students’ intuitive tendency” (p. 863). An example is the commonly reported overgeneralization that division as an operation always makes the result smaller.

**Figural Concepts**

Geometrical figures are mental entities that have both conceptual and figural characteristics. They are not mere formal concepts, because of their spatial representation and simultaneously not a mere image, because of their properties that are rigorously imposed by definitions in a certain axiomatic system. In this sense, geometrical figures are figures and concepts simultaneously, and Fischbein called them figural concepts. Figural concepts are abstract, general, ideal, universal, pure, logically determinable, and definition-dependent because of their conceptual aspects, and at the same time, reflect mental representations of spatial properties such as shape, position, and magnitudes. Ideally, we expect that both formal and figural constraints converge. However, the figural constraints are often liberated from the conceptual controls and lead to interpreting a concept in figurally consistent but conceptually inconsistent ways. This confliction may affect the flow of geometrical reasoning and cause misconceptions (Fischbein, 1993, 1994).
CHAPTER 3
METHODOLOGY

This study was designed to investigate the nature of preservice teachers’ knowledge content and students. Preservice teachers who were enrolled in the middle school teacher preparation program at the University of Georgia took both content courses and methods courses. While the preservice teachers were enrolled in the content course and methods course in geometry, I intended to provide them with the opportunity to examine students’ thinking about particular geometric topics. Before designing this study, I observed a few sessions of methods courses for preservice teachers, and the observation led me to understand the importance of teachers’ knowledge of students as well as knowledge of mathematical content. Because the preservice teachers did not have sufficient experience communicating with students, I assumed that their knowledge of students’ thinking was not deep. Although they had discussed students’ thinking and misconceptions in their methods and content courses, these kinds of discussions did not frequently occur. Additionally, the teachers’ knowledge of content and students was related to their instruction. Thus, in this study I intended to encourage the preservice teachers to start with identifying students’ errors and misconceptions and then to develop a plan to correct their misconceptions by inferring plausible causes of the misconceptions.

To accomplish these goals, I conducted a qualitative study because I was not interested in whether the preservice teachers’ indications of students’ misconceptions and instructional strategies were correct or not, but rather I was concerned with how they examined students’ problem solving abilities and students’ thinking and how they were able to connect instructional
strategies to what they knew and what they learned through their content and methods courses. Thus a qualitative study was more appropriate to help me gain an in-depth understanding of the preservice teachers’ performance. Additionally, I used a variety of data sources: a pretest and posttest, four written tasks, one oral task, interviews, and class observation. The results from the pretest and posttest were used to compare participants’ knowledge before and after taking the geometry content and methods courses and participating in the study. The preservice teachers’ performance on the tasks were the main source of data. The interview was intended to augment the written tasks. In the interview, I asked about participants’ backgrounds and their opinions of courses and the tasks and implemented the fifth task in an oral format. During data collection, I observed the content and methods courses in geometry. I attempted to triangulate those data sources in order to increase confidence in the results of this study (Bogdan & Biklen, 2007).

**Participant Selection**

This study was implemented with preservice teachers who were enrolled in geometry content and methods courses for preservice middle school teachers at the University of Georgia. Thirty-two students were enrolled in both courses; there were 1 male and 31 female students. Because I was a teaching assistant for the prerequisite course that all students already took, I was acquainted with them.

The process of participant selection was purposeful (Merriam, 1998). Because this study required extra time to perform a series of tasks that were not mandatory in the courses, recruitment depended on voluntary participation. This method of recruitment did not allow for the sample to represent the whole population of preservice middle school teachers.

Seven female preservice teachers voluntarily joined this study. After completing the first task, however, one preservice teacher wanted to quit the study. Thus, the pretest and the first task
were implemented with 7 participants, but the other three written tasks and one oral task, the interview, and the posttest were conducted with 6 participants.

**Data Collection**

Data collection was conducted in four phases in order to gain insight into preservice teachers’ knowledge of content and students. The pretest and posttest were used to diagnose the status of the participants’ knowledge in terms of content and pedagogy. The five tasks were used to investigate the preservice teachers’ knowledge of content and students. The participants were asked to respond to six questions about each task—solving the problem that students were given, identifying important mathematical knowledge needed to solve the problem, assessing whether or not students’ answers were correct, identifying students’ errors and misconceptions, considering plausible causes underlying misconceptions, and planning instructional strategies to correct and avoid misconceptions. The interview was conducted to inquire about the participants’ background and their experiences. The purpose of observation of the content and methods courses was to investigate the preservice teachers’ content knowledge and their learning trajectory in geometry.

**Pretest and Posttest**

At the beginning of the course and at the end of the course, the pretest and posttest were used to diagnose the preservice teachers’ knowledge of content and students in geometry. I employed the Diagnostic Mathematics Assessments for Middle School Teachers (DMAMST), which was developed by the Center for Research in Mathematics and Science Teacher Development (2005). According to the developers, one of the purposes of this assessment was to describe the extent of mathematics content knowledge, the effects of specific experience on teacher knowledge, or relationships among content knowledge that teachers have, teaching
practice, and students’ performance. This purpose coincided with my intention to identify a
change preservice teachers’ knowledge and to verify some influence of the tasks that were used
in this study and the content and methods courses.

The original DMAMST consisted of four content domains, Number/Computation,
Geometry/Measurement, Probability/Statistics, and Algebraic Ideas, and each domain of the
assessment contained 20 items, including 10 multiple-choice and 10 open-response items. I used
only the Geometry/Measurement assessment because this study focused on topics from
gometry. The assessment included four types of knowledge: Type I- Memorized Knowledge,
Type II- Conceptual Understanding, Type III- Problem Solving/ Reasoning, and Type IV-
Pedagogical Content Knowledge.

The validity of the assessment was established in three ways (Center for Research in
Mathematics and Science Teacher Development, 2005). The first way was that the breadth and
depth of mathematical content for middle school teachers was defined by national
recommendations, objectives of standardized tests, and research on misconceptions for middle
school students and teachers. The second way was that mathematicians, mathematics educators,
and middle school teachers cooperated to develop prototype and parallel assessments. The third
way was that national reviewers assessed the appropriateness of items that were created for the
six forms of the four assessments. Reviewers were asked to identify the mathematics content of
the item from a list of specific topics, to identify the four knowledge types, and to indicate
whether or not the item assessed important mathematical content for middle school teachers. The
DMAMST included only items that were validated.

In addition to validity, three types of reliability estimates were made: internal reliability,
equivalency reliability, and inter-scorer reliability (Center for Research in Mathematics and
Science Teacher Development, 2005). For the internal reliability, Cronbach’s alpha coefficient was computed, and it was .89 for the Geometry/Measurement domain. Equivalency reliability was estimated for six parallel forms of assessments. Equivalency reliability was tested by the Pearson product moment correlations for each pair of parallel assessments completed by the same groups of teachers. The Pearson product moment correlations measures the strength of association between the pretest and posttest. Parallel versions of the assessment were administrated to the same groups of teachers, and then the Pearson product moment correlation coefficients were computed. The coefficient of equivalency reliability for Geometry/Measurement was .69. The reliability of open-response items was establish as inter-scorer reliability by using percents of agreements among three graduate students who developed and used the scoring guides. The percent of agreement was calculated by adding the numbers of people who got the same scores from all graders, dividing by the total number of participants in the group, and multiplying by 100. The inter-scorer reliability coefficient for Geometry/Measurement was .90.

When I asked the developers to allow me to use the DMAMST items, they sent a pair of tests that were parallel. I conducted the pre-test before participants completed the first task in my study, and I administered the post-test after participants completed the fourth task. Participants’ answers were sent to and scored by the developers (Center for Research in Mathematics and Science Teacher Development, 2005).

Knowledge of Content and Students (KCS) Tasks

To identify the nature of preservice teachers’ knowledge of content and students, I designed five tasks. Each task included actual students’ solutions and was designed to have the participants investigate students’ solutions with regard to their thinking and misconceptions. The
problems used in this study came from two data sources: Open-Ended Assessment in Math
(Cooney, Sanchez, Leatham, & Mewborn, n.d.) and Elementary Mathematics Assessment
Project (EMAP) (Hickey, Mewborn, & Lewison, 2005-2008). Open-Ended Assessment in Math
was developed to encourage teachers to use open-ended questions and to save teachers’ trouble
in writing open-ended questions. Through field-testing, this project accumulated students’
answers as well as a variety of questions. Among those questions and students’ answers, for this
study I employed some items for middle school grade levels in the domains of geometry, spatial
sense, and measurement.

EMAP was 3-year project and was conducted by Indiana University and the University
of Georgia (Hickey, Mewborn, Lewison, 2005-2008). The goals of the project were to promote
students’ proficiency and to foster new ways of engaging mathematics learning by coaching and
supporting higher-level mathematics discourse. To achieve these goals, this project used open-
ended items that have more than one correct answer and solution strategy or open-middle items
that have multiple problem solving paths to a specific answer, and dealt with content in the
mathematics standards from a particular state (Yeo, 2007). Students who participated in this
project worked each item individually and in a group, and students’ written responses were
gathered after each period of implementation of a set of items. For this study, I selected some of
the student responses that showed particular errors or misconceptions.

Before implementing each task of this study, the participants were requested to solve the
same problem that students were given, the purpose of which was to identify whether or not
participants could provide the correct answer to the problem. Then participants were presented
with several students’ written solutions and were asked:
• To identify important mathematical ideas that the student might use to be able to successfully perform the item,
• To examine errors that the student made for each student’s solution,
• To identify underlying mathematical misconceptions or misunderstandings that might lead the students to the error presented,
• To infer plausible causes of the misconceptions or misunderstandings,
• To suggest ways to help each student to have him/her recognize and correct his errors and misconceptions, and
• To make a plan of instructional strategies or tasks to use during the next instructional period to address and avoid the students’ misconceptions.

The format of these questions initially came from items designed by Manizade (2006), who attempted to develop an instrument to measure teachers’ pedagogical content knowledge in geometry and measurement at the middle school level. Because the items did not include a variety of actual students’ solutions, I did not use her original items but her ways of questioning. The questions from Manizade’s work were also modified so as to be appropriate to the purposes of this study.

Each task was distributed to participants after they learned the relevant topics in their content and methods classes. The reason that the participants were given each task after they learned the topics was to avoid topics unfamiliar to them and to investigate how they were able to apply their knowledge from the courses that they were taking. The participants were given 1 or 2 weeks to complete all questions for a particular task on their own time, and then I gathered their written responses. Four tasks were implemented in a written format, and the participants spent a long time completing them, which I expected because they were required to apply their
knowledge in depth. For example, the participants may not have found it easy to identify plausible causes of misconceptions, and they needed to consider classroom activity in the content and methods courses in order to plan instructional strategies to correct students’ misconceptions. While collecting the participants’ responses to the first four KCS tasks, I began to wonder whether there was difference in conducting the tasks in written and oral format. I believed that there were participants who would prefer speaking to writing something and who might elaborate more if given the chance to explain orally. Thus, I decided to provide the fifth task in an oral form in the interview. The following are descriptions of each possible solution strategies for the problem based on what was addressed in the content course, and actual students’ solutions that were given to the participants.

I labeled students’ solutions to each task as Student A, Student B, Student C, and so on. However, Student A is not the same student in each task. Rather, the first student solution is always Student A and the second Student B and so on. Additionally, I referred to each student as he because all of the participants were female.

Task 1: The Area of a Triangle

This task (Figure 6) was first given to the participants right after they studied the concept of the area of a triangle in their content course. They had experience dealing with a variety of strategies to find areas of shapes and verifying the area formula of rectangles and triangles. Before seeing students’ solutions, the participants were asked to solve the area of a triangle problem. They spent a few minutes solving the problem, and their solutions were collected. After they submitted their solutions, the task that included students’ solutions was distributed. The participants’ responses to the task were gathered 1 or 2 weeks after distribution.
Anna has 2 corners of her yard that she could use for a garden, so she needs to decide which one has the larger area. Which one is bigger? How do you know?

Figure 6. The area of a triangle problem.

This problem came from EMAP (Hickey, Mewborn, & Lewison, 2005-2008). The content that the problem covered was the concept of areas of triangles and heights of triangles. The purposes of the problem were to identify whether or not students understand what area of triangles is and to diagnose students’ understanding of the concept of the height of a triangle. On the basis of their learning, middle school preservice teachers might be able to estimate the areas of the first triangle in a variety of ways (Figure 7). They might count unit squares inside the triangle, move small pieces to fill incomplete unit squares, move bigger chunks to create a rectangle, view as part of a bigger rectangle, and use the area formula for a triangle. Because the first triangle is a right triangle, both teachers and students might easily calculate the area by applying the area formula: $\frac{1}{2} \times 3 \times 6 = 9$. 
Figure 7. Three ways of estimating the area of a triangle

Unlike the first triangle (Figure 6), the exact length of the height was not given for the second triangle. Understanding the situation of the second triangle requires knowing what the height is. Mathematically, the height of triangles is defined as follows: For a given base, the associated height is perpendicular to the base and goes through the vertex that is not on the base. The side labeled 3 cm is not the height because it is not perpendicular to the base that is the side labeled 6 cm. With the same base, however, we can say that the height of the second triangle is less than that of the first triangle, which allows us to conclude that the area of the second triangle is less than that of the first triangle. When trying to estimate the area of the second triangle, students might use the length 2 cm as its height by depending on the picture shown. However, the original intention of the problem was to encourage students not to rely on the picture.

For this problem, seven students’ solutions were provided and four big questions were asked. Participants were asked to answer questions 2 and 4-(a) for each student’s solution.
• Student A

The first one is bigger because base times height divided by two gets you the answer.

• Student B

They are the same because you do \((3 \times 6) + 2 = A\) and since you have a triangle it's half a rectangle or square and that is why you divide it by two.

• Student C

They are the same because you have to do \(b \times h \times \frac{1}{2}\) and \(b \times 3 \times 1.8\)

• Student D

You make the triangles into rectangle then divide it by 2.

• Student E

We think the triangles are the same area but one of them is stretched out and one is its real size. We know this because they both have the area of 18. 1 side on each is 6 cm and the other is 3 cm.
1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. (a) What are the errors that those students are making?
   (b) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the error presented in this item?

3. How might the student have developed the misconception(s)?

4. (a) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
   (b) What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?

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**Task 2: Articulating the Relationship Between Area and Perimeter**

Task 2 (Figure 9) was the second given to the participants and was given after they had the experience of investigating change in areas of rectangles with a fixed perimeter and inducing the conclusion that there was no direct relationship between area and perimeter. The procedure of implementation of the second task was the same as that of the first task.
Rectangle I has a larger perimeter than rectangle II. Can you conclude that rectangle I also has a larger area than rectangle II? Why or Why not?

Figure 9. The relationship between area and perimeter problem.

This problem came from the Open-Ended Assessment in Math (Cooney, Sanchez, Leatham, & Mewborn, n.d.) project. The content that the problem covered was the concepts of area and perimeter, and their relationship in rectangles. The problem intended to examine the areas and perimeters of rectangles and how students can determine areas of rectangles with changing perimeter. In addition to students’ concept of area and perimeter, the problem required students’ ability to generalize. Students may believe areas of rectangles might increase as their perimeters increase, but that is not always true. It is possible that the area of a rectangle with a smaller perimeter is larger than that of a rectangle with a larger perimeter. Agreement with the statement requires proving it in a general sense. Providing a counterexample is sufficient to show disagreement with the statement. It is impossible to prove that the statement works for every rectangle, whereas it is easy to find an example that shows that it does not work. Thus, students were expected to provide examples that showed the statement did not hold.

Based on the instruction in their content course, middle school preservice teachers might be able to approach the problem by examining rectangles with the same perimeter and different area or examining rectangles with the same area and different perimeter. The situation where rectangles have the same perimeter but different areas can be introduced as designing the plan of building a fence with a fixed perimeter and the largest area for pets. Preservice teachers can determine the area by constructing a table with columns of length, width, perimeter, and area, and by making a graph with length and area as its x- and y- axes from the table. The situation where rectangles have the same area but different perimeters can be introduced as designing shelters where the floor is a rectangle with a fixed area and the shortest perimeter. This context is
reasonable because the shelters would be built with the smallest amount of wooden panels of walls with a constant height. Preservice teachers can determine the perimeter by constructing a table with columns of length, width, perimeter, and area, and by making a graph with length and perimeter as its $x$- and $y$- axes from the table (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006).

The content course introduced this concept by casting the following questions: Given the perimeter of a shape, can we determine its area? If not, what are all possible areas? To encourage an experiment, using a loop of string with a fixed length was allowed. The preservice teachers in the course investigated the questions through experimenting with string, drawing pictures on graph paper, and constructing a table of areas of rectangles in increasing order (Figure 10).

![Figure 10. Example of the table of areas of rectangles with the perimeter of 20.](image)

After implementing the first task, I modified the format of the second task. Because the first task had repetitive students’ responses, I attempted not to include responses that included similar errors. For the first task I gave all students’ responses and then all questions that participants were supposed to answer, whereas for the second task I provided all questions under each student’s response. For this problem, seven students’ responses were provided (Figure 11).
1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

   • Student A

   • Student B

   • Student C
• Student D

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?

Figure 11. Solutions to the Task of Articulating the Relationship Between Area and Perimeter.

Task 3: Construction of Triangles and Their Angle Properties

The third task (Figure 12) was given to the participants after they had experience investigating three types of angles (acute, right, and obtuse angles) and the property that the
interior angles in a triangle add to 180 degrees. The procedure for implementing the third task was the same as that of the previous two tasks.

Students in Ms. Smith’s class have drawn a variety of triangles. Here is what they drew:

- Roberto: I made a triangle with 3 acute angles.
- Sunnum: I made a triangle with 1 right angle.
- Alicia: I made a triangle with 2 obtuse angles.
- Raj: I made a triangle with no right angles.

Check each student’s work to see if it is correct. It may help to try drawing pictures.

Figure 12. The triangle construction problem.

This problem came from EMAP (Hickey, Mewborn, & Lewison, 2005-2008). The content that the problem covered was the concept of constructing triangles and the angle properties of triangles. The purposes of the problem were to examine students’ conceptions of three types of angles, to encourage students to think of and draw examples of triangles that fit the description, and to identify students’ ability to justify their answer with the angle sum property of a triangle.

Based on their course work, middle school preservice teachers should be able to distinguish among acute, right, and obtuse angles. They should also perceive when a triangle is called an acute, right, or obtuse triangle: An acute triangle has three acute angles, a right triangle has one right angle, and an obtuse triangle has one obtuse angle. With regard to the angle properties, they should understand why the sum of the angles in a triangle is 180 degrees. They should be able to convince their students why this property works in a variety of ways. The easiest way is to use the activity of tearing and attaching angles of a paper triangle: Draw a triangle on a piece of paper and cut out the triangle, and then tear off three corners of the triangle and arrange the corners on a straight line. They can see that three corners form a straight line by putting them together, which means the sum of the three angles of the corners is 180 degrees. A
walking and turning activity is less mathematically rigorous but easier for students to understand. The procedure of the activity is as follows: (a) Draw a triangle labeled A, B, and C on the floor with masking tape, and label a point P on the line segment between A and B; (b) pick two people, a walker who stands at P, facing B, and then walks all the way around the triangle until returning P, and a turner who stands at a fixed spot and faces or turns the same direction as the walker; and (c) note the full angle of rotation of the turner (see Figure 13). After the full angle of rotation, we can see that the turner rotates 360 degrees, which indicates the sum of angles $d$, $e$, and $f$; that is, $d + e + f = 360^\circ$. Because angles $a$ and $f$, $b$ and $d$, and $c$ and $e$ form straight lines, their sum would be three times 180 degrees; that is, $(a + f) + (b + d) + (c + e) = 180^\circ + 180^\circ + 180^\circ = 540^\circ$. The sum of interior angles in a triangle, $a + b + c$, can be obtained by subtracting $d + e + f$ from $(a + f) + (b + d) + (c + e)$; that is, $a + b + c = 540^\circ - 360^\circ = 180^\circ$ (Beckmann, 2008).

![Figure 13. Walking and turning activity.](image)

Justification of those two activities depends on a specific triangle, and we cannot examine the property for all possible triangles through the activities. The more mathematically rigorous way to prove that the sum of the interior angles of a triangle is to use the property of parallel
lines. For a given Triangle ABC, we can draw the line that is parallel to BC and passes through Point A according to the Parallel Postulate. Applying the property that the measures of alternate interior angles are equal leads to the conclusion that interior angles in a triangle add to 180 degrees (see Fig. 14). By applying the angle property of a triangle to this problem, we can recognize that Alicia’s triangle cannot be constructed (Figure 12), because the sum of two obtuse angles exceeds 180 degrees. Except for Alicia’s triangle, students might be able to find examples of Roberto’s, Sunnum’s, and Raj’s triangles.

![Diagram showing parallel lines and alternate interior angles]

*Figure 14. Proof of the sum of interior angles in a triangle.*

For this problem, five students’ solutions were provided using the same format as the second task where the questions were posed after each student solution (Figure 15).

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
- Student A

Robert is wrong because you can only set a acute angles instead of

- Student B

Roberto is correct because his triangle can be a triangle can and does have 3 acute angles. Alicia is correct because a triangle can't be a right angle unless it is a right angle triangle.

- Student C

Raj's triangle triangle won't work because every triangle has at least one right angle.
• Student D

Roberto: I made a triangle with 3 acute angles. Yes.
Sunnum: I made a triangle with 1 right angle. No.
Alicia: I made a triangle with 2 obtuse angles. Yes.
Raj: I made a triangle with no right angles. Yes.

Check each student's work to see if it is correct. It may help to try drawing pictures.

• Student E

Roberto is wrong because a triangle has only three sides and you cannot have 3 acute angles in a triangle. Sunnum is right. Alicia is wrong because there is no way for a triangle to have 2 obtuse angles because a triangle has only three sides. Raj is right.

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?

Figure 15. Solutions to the Task of the Construction of Triangles and Their Angle Properties.

Task 4: Similar Triangles

The fourth task (Figure 16) was given to the participants after the content course introduced the concept of similarity and the criteria for similar triangles. Unlike the previous three tasks, I did not ask the participants to solve the problem before they got students’ solutions. I included solving the problem as part of the take-home task because I believed the participants’ solution would not be influenced by students’ solution. This task provided only three students’ solutions, and their solutions were all incorrect. Thus, I thought it was unlikely that the participants would change their own solutions after reading students’ solutions.
Are triangle A and B similar triangles? Why or why not?

**Figure 16.** Similar triangles problem.

This problem came from the Open-Ended Assessment in Math project (Cooney, Sanchez, Leatham, & Mewborn, n.d.). The content that the problem covered was the concept of similarity in triangles. The problem was intended to identify students’ understanding of similar triangles by using a criterion for triangles to be similar or applying the scale factor.

Based on their course work, middle school preservice teachers should have been able to distinguish mathematical similarity from similarity in everyday language. For example, all triangles are similar in terms of everyday language because all triangles have three sides and three angles, whereas all triangles are not mathematically similar. In terms of mathematics, two shapes are similar if there is a number $k$, called a scale factor, such that each distance on the second shape is $k$ times the corresponding distance on the first shape. In addition to the definition, middle school preservice teachers should be able to approach this problem in two ways: using a similarity criterion of triangles and applying the scale factor. The preservice teachers studied three criteria to determine whether or not two triangles are similar: (a) all corresponding angles (or two pairs of corresponding angles) are the same, (b) all three pairs of corresponding sides have the same ratio, and (c) two pairs of corresponding sides have the same ratio and the corresponding angles between two sides are the same. The same ratio of three (or two) pairs of corresponding sides in the criteria, or the ratio of lengths of any two corresponding
sides in two similar figures, is called the scale factor. Therefore, by applying the first criterion of similar triangles, that all corresponding angles are equal, students might be able to recognize that the two triangles in the problem (Figure 16) are not similar. Moreover, with regard to the scale factor, the ratio between heights is 4/2, whereas the ratio between bases is 1/2, which means there is not a constant scale factor and the two triangles are not similar.

For this problem, three students’ solutions were provided (Figure 17).

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

   - Student A – “No, because A triangle is more long than B triangle. B triangle is very short and very wide, A triangle is very thin and very small space.”
   - Student B – “Yes, because both A and B have the same number of dots and that’s 6, and they both are triangles and they both have straight lines or edges.”
   - Student C – “Yes. Because bother have square corners. And also both could make a square and a rectangle if you just put a line like this:”

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?

*Figure 17. Solutions to the similar triangle task.*
**Task 5: The Volumes of Rectangular Prisms**

Task 5 (Figure 18) was the last task and was implemented in an oral format in an interview setting. During the interview, each participant was asked to respond the same questions as the previous tasks. However, unlike the other tasks, I did not ask them to solve the problem first. The reason that this task was orally given was that I wanted to compare the participants’ responses between written and oral formats in terms of thoroughness and justification.

Give the dimensions of a box that has four times the volume of the box below. Explain why your box has four times the volume of the given box.

![Figure 18. The volume of rectangular prisms problem.](image)

This problem came from the Open-ended Assessment in Math project (Cooney, Sanchez, Leatham, & Mewborn, n.d.). The content that the problem covered was the concept of the volumes of rectangular prisms. The problem aimed at examining students’ conceptions of the volume of rectangular prisms, their ability to find the volume, and their understanding of the volume formula. One thing that students should notice was that the problem did not restrict the shape to be similar to that of the original rectangular prism.

Based on their course work, middle school preservice teachers should be able to understand the concept of the volume of a rectangular prism as the number of unit cubes that it would take to fill the shape without gaps or overlaps. With this definition of volume, middle school preservice teachers should know why the volume formula for rectangular prisms works. For a rectangular prism with dimensions of \( l \) inches, \( w \) inches, and \( h \) inches (Figure 19), the volume formula is...
\[ V = l \times w \times h \text{ (in}^3\text{).} \]  

In the formula \( l \) times \( w \) indicates the number of unit cubes in the base of the rectangular prism, which is also the number of unit cubes in each layer. Because there are \( h \) layers in the rectangular prism, the total number of unit cubes that fill the prism is \((l \times w) \times h\).

![Rectangular Prism](image)

*Figure 19. Rectangular prism with dimensions \( l, w, \) and \( h \)*

To obtain rectangular prisms with volumes four times larger than the original prism, one can make a new prism with four times one of three dimensions or making two times two of three dimensions. This idea is related to the concept of factorization and the commutative law and associative law in algebra. That is, by the commutative and associative laws,

\[
4 \times l \times w \times h = (4 \times l) \times w \times h
\]

\[
= l \times (4 \times w) \times h
\]

\[
= l \times w \times (4 \times h).
\]

Because \( 4 = 2 \times 2 \) by the factorization,

\[
4 \times l \times w \times h = (2 \times l) \times (2 \times w) \times h
\]

\[
= l \times (2 \times w) \times (2 \times h)
\]

\[
= (2 \times l) \times w \times (2 \times h).
\]

Although there are many other possibilities to deal with dimensions to make the volume four times larger, here I considered those cases that I addressed above relating to the students’ solutions.
For this problem, four students’ solutions were provided.

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

   • Student (1)

   • Student (2)

   • Student (3)
• Student (4)

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?

*Figure 20. Solutions to the Volumes of Rectangular Prisms Task.*

*Interview*

At the end of the semester, I conducted an interview with each participant. Each interview took approximately 40 minutes to 1 hour. The interview consisted of four parts: (a) questions about personal information of the participants, (b) questions about the participants’ opinions of their content and methods courses, (c) questions about the participants’ opinions of the tasks, and (d) the fifth task. The purpose of my questions about personal information was to inquire about the participants’ background in course taking, field experiences, and familiarity with middle school students’ thinking. The purpose of the questions about the participants’ opinions of their content and methods courses was to hear how the participants thought the courses had influenced their knowledge and future teaching. The purpose of asking participants’ opinions of the tasks was to inquire about how the participants performed the tasks, about their opinions about the strengths and weaknesses of the tasks, and about their opinions about the influence of the tasks on their knowledge and teaching. Lastly the fifth task, the Volumes of Rectangular Prisms Task, was implemented during the interview. Because this task was given in oral format, unlike the other tasks, I included it in the interview. Although the fifth task was given in interview format, the questions and procedures were similar to those of the other four tasks. All interviews were videotaped and transcribed.
Observation

During the semester in which I implemented the written tasks, I observed the two content and methods courses in which the participants were enrolled. I attended every class session for both courses except for tests and examinations. The purposes of the observation were to investigate preservice teachers’ learning trajectory in geometry and to incorporate their learning into the tasks in this study. I followed their learning trajectory when I distributed each task and considered what the courses dealt with as the ideal level of their content knowledge. I did not focus on how the participants performed during each class session over the semester. During the observation, I took notes about what topics in geometry the courses dealt with and how the instructors introduced the topics. These notes functioned as a baseline of the participants’ knowledge of content and pedagogy. I allowed the participants to refer to their class notes and to apply what they learned in their courses when they completed the written tasks. Specifically, I followed the learning progress of the content course in order to determine the order of implementation of the tasks. I distributed each task after the participants had studied the topics that were relevant to the task.

Description of Participants

Common Features of Participants

All participants were juniors in the middle school teacher preparation program with mathematics as their area of specialization. In their program, they had taken the content course Arithmetic for Middle School Teachers and its parallel methods course, Teaching Number System in the Middle School, the semester prior to data collection. During data collection, all participants were taking the content course Geometry and Measurement for Middle School Teachers offered by the mathematics department and its parallel methods course, Teaching
Geometry and Measurement in the Middle School, offered by the mathematics and science education department. Those were the only geometry courses that participants took at the college level; before this, they had taken a geometry class in their high school. In addition to courses that specialized in mathematics, the participants had taken prerequisite courses such as educational psychology, special education, foundations of education, and teaching in the middle school. Even though the participants had a field experience during the semester in which I collected data, their experience was not necessarily in a middle school mathematics class as middle school majors have two areas of specialization and must complete a field experience in each. Their internship in teaching mathematics as a student teacher was scheduled for 2 semesters after my data collection.

Because of the nature and timing of the field experiences in the middle school program, not all the participants had direct experience on assessing students’ work in mathematics. Although they had investigated students’ work and made a rubric as an assignment in Teaching Number Systems in the Middle School, the content of the assignment was about algebra or arithmetic. There was no such assignment in the course Teaching Geometry and Measurement in the Middle School. Even though all participants had taken an educational psychology course, this did not provide a direct opportunity to investigate students’ mathematical thinking, particularly students’ misconceptions in geometry.

Eva

Eva’s two areas of specialization in the middle school education major were mathematics and science. She took three calculus classes and an instructional technology course in the “core” or first 60 hours of her degree program. During the semester of data collection Eva participated in a field experience in an 8th grade science classroom because science was her second
specialization. During this field experience, she taught a couple of science lessons. She did not have experience teaching mathematics or assessing students’ work in mathematics before participating in this study.

_Joy_

Joy took three calculus courses and an introductory higher mathematics course for preparation in mathematical reasoning and writing proofs as part of her core curriculum. In addition to a series of mathematics courses, she took an introductory statistics course. Joy’s field experience was in a 6th grade social studies classroom, and she observed the classroom 2 ½ hours per day for 34 days. She did not have any experience teaching mathematics; she only had experience observing a mathematics classroom.

_Liz_

Unlike the other participants, Liz began as a music major. She took courses in pre-calculus, calculus, mathematical modeling, and statistics. Liz spent her field experience in a 5th grade classroom at an elementary school. In this field experience, she had the opportunity to teach a few lessons after observing for 100 hours. Although she was in a mathematics classroom, the topic of instruction was mostly about fractions and not about geometry.

_May_

May did not take any mathematics courses except for Arithmetic for Middle School Teachers and Geometry and Measurement for Middle School Teachers in college because she had taken calculus classes in high school. Because her second major area was social studies, she also took several social studies education courses. May had a field experience in a 4th grade social studies classroom. During this field experience, she taught several lessons in social
studies. Even though she did not directly teach mathematics in this field experience, she had an opportunity to help with a mathematics lesson on 4th grade multiplication.

Pam

Pam had taken most of the mathematics courses that her high school offered including, Advanced Placement Calculus, and then she took two calculus courses in college. In addition to the education courses that were mentioned above, she also took a course in instructional technology. Because of her interest in science, she took a couple of courses offered by the science education department. Pam had a field experience in a 4th grade mathematics classroom. During the field experience, she had the opportunity to teach several lessons, which included a lesson on shapes, specifically identifying quadrilaterals.

Sam

Sam was in the middle school education program with science and mathematics as her areas of specialization. She took pre-calculus, two calculus courses, linear algebra, and a course about mathematical proofs. In addition to the education courses that were mentioned above, she also took a course in instructional technology. Sam did her field experience in a 7th grade science classroom. During this field experience, she had chances to teach several lessons on life science. Although she had served as a peer leader to help elementary school students study mathematics in her high school years, she did not have experience teaching mathematics.

Tia

Because Tia stopped participating after completing the first task, I did not interview her, which meant I was unable to collect background information about her. Her data about background and responses to the second task to the fifth task are not included in the analysis. I
added her written response to the first task to the analysis because her answers from the task were interesting.

**Description of Content and Methods Courses**

The University of Georgia provides three content courses and three methods courses in the areas of arithmetic and number theory, geometry and measurement, and algebra as part of the middle school teacher preparation program. Students who are enrolled in this program take a pair of courses (one content and one methods on the same topic) each semester. The content and methods courses played an important role in this study because I assumed that both courses would affect preservice teachers’ knowledge. Knowing what they had studied in the courses allowed me to understand the nature of preservice teachers’ knowledge of content and students.

*Content Course*

The content course Geometry and Measurement for Middle School Teachers aimed at enhancing preservice teachers’ understanding of concepts and ideas of geometry and measurement and their various applications. According to the instructor’s syllabus, the course emphasized four aspects in order to accomplish this goal:

- Understanding a variety of mathematical procedures and formulas and their underlying reasons,
- Clear communication in both oral and written forms,
- Investigating and explaining of mathematical phenomena, and
- Solving many problems in a variety of ways.

While I was collecting data for this study, the content course dealt with the following topics: visualization; angles; geometric shapes and their properties; constructions with straightedge and compass; transformation including reflections, translations, and rotations;
congruence; similarity; measurement including length, area, and volume; principles underlying formulas for areas and volumes; the relationship between area and perimeter; and the behavior of area and volume under scaling. Among those topics, this study focused on five topics that were used in the tasks: (1) the area of triangles, (2) the relationship between area and perimeter, (3) the properties of angles in a triangle, (4) similarity, and (5) volume of a rectangular prism. The KCS tasks for the study were administered after instruction on that topic in the content course had been completed.

The instructor of the content course attempted to use a variety of instructional strategies including lecturing on concepts and proving properties, having preservice teachers discuss concepts with others, and giving them hands-on manipulatives and related activities. In the interviews, the participants stated what they considered to be strong aspects of the content course:

- The course encouraged them to consider a variety of strategies to solve problems and gave them opportunities to discuss the strategies in a small group or with the whole class.
- The instructor introduced various manipulatives, applications, and activities and demonstrated them. For example, the instructor used Play-Doh to allow the preservice teachers to investigate the cross-sections of solids. When explaining why the sum of interior angles in a triangle is 180°, the instructor introduced the activity of walking and turning along a triangle mapped on the floor.
- The course emphasized explaining why a property works, which encouraged the preservice teachers to understand the property in depth.
Methods Course

The methods course, Teaching Geometry and Measurement in the Middle School, was designed to help prospective middle school teachers enhance the didactics of geometry including teaching methods, curriculum materials, psychological factors for developing geometric and measurement concepts, and assessment. According to the instructor’s syllabus, the methods course had 5 goals:

- Strengthening understanding of concepts and ideas in geometry and measurement in ways that middle school students can experience:
- Experiencing, investigating, interpreting, analyzing, predicting, concluding, and generalizing problems and situations in a mathematical (or geometrical) way:
- Learning about resources and tools that are used in a middle school mathematics classroom:
- Developing ideas of teaching methods that encourage students to explore mathematical concepts: and
- Collecting and developing problematic situations and related materials in order to use them in a classroom.

The course emphasized developing activities that would promote students’ conceptual understanding and higher order thinking.

During the time that I collected data for this study, the methods course focused on investigating geometry and measurement standards from the Georgia Performance Standards. At the beginning of the semester, the instructor spent a few hours having the preservice teachers practice using the software Geometer’s Sketchpad. Then, they, as a group, spent a few minutes discussing and exploring topics presented in the standards. One group of preservice teachers
designed a series of lessons and their relevant activities for a particular topic aligned with geometry and measurement standards. Another group studied concepts related to a particular topic in the geometry and measurement standards, and then introduced and taught them to the others. As the final project, the students had the opportunity to explore some geometrical topics beyond middle school mathematics such as fractals, coordinate geometry, taxicab geometry, and spherical geometry.

Although psychological factors in learning geometry and measurement were one of the course goals, the students in that semester did not have much time to discuss psychological aspects of learners for particular concepts of geometry and measurement. Unlike the content course, topics were not dealt with in a linear order; the instructor assigned a topic in the standards to each group, and some concepts from groups often overlapped. Furthermore, the five topics that were used in the tasks were not connected to the content that was covered in the methods course. The instructor led a discussion about the fact that the sum of the interior angles in a triangle is 180° and demonstrated that by tearing and attaching angles of a paper triangle. However, other topics related to my tasks (the area of triangles, the relationship between area and perimeter, similarity, and volumes of a rectangular prism) were not explicitly dealt with in the course. Some topics might have been discussed in a group’s work, whereas some topics were not introduced. For that reason, implementation of the tasks did not depend on the schedule of the methods course.

**Description of Content Knowledge**

Because the preservice teachers’ knowledge of content was influenced by what they learned in the course, this section describes what and how they learned from the content course with regard to the five KCS tasks.
Before introducing the area formula for triangles, the instructor developed the concept of area in a systematic way. First, she began with different kinds of units, including length, area, and volume units. Students spent a few minutes discussing differences among various kinds of units and differences among measurable attributes. Then, she introduced the area of rectangles in the primitive way of counting unit squares and progressively developed the idea of area through the moving and additivity properties. Those properties were applied to four strategies to find the area of a figure: the simply subdividing strategy, the take-away strategy, the moving and attaching strategy, and the combining multiple copies strategy. Based on these ideas for finding area, the concept of area of triangles was developed through progressively sophisticated ways to find area:

- Move small pieces and use definition of area,
- Move bigger chunks to create a rectangle,
- View as part of a bigger rectangle, and
- Use a take-away strategy.

These ways not only led to the area formula for triangles, half of base times height, but also gave students an idea of why the formula works. To justify the area formula for triangles, the instructor used three different kinds of triangles: right triangles, triangles with an altitude that is over the base, and triangles with an altitude that is not over the base. In addition, she spent a few minutes discussing what a base is and what a height is.

The Relationship Between Area and Perimeter

At the beginning of the semester, the instructor held a discussion of differences among measurable attributes including area and length. After dealing with finding areas of irregular
shapes, the instructor asked how perimeter and area are related. To find the answer to the question, the instructor had the preservice teachers investigate all possible areas of all rectangles of a given perimeter and all possible areas of general shapes with a given perimeter. In the activity, the instructor asked the preservice teachers to find areas of rectangles with the perimeter of 20 cm and pairs of natural numbers for width and length and to make a table that included the areas and the dimensions of the rectangles. Through this investigation, they were able to see that there is no direct relationship between area and perimeter.

*Angle Properties of Triangles*

The course dealt with a justification for an important property of triangles: why the sum of the interior angles in a triangle is $180^\circ$. Two approaches were used in explaining the property. First, the students experienced the activity of walking and turning, which was related to the fourth- and fifth-grade curriculum. Second, the preservice teachers investigated the property by using the property of parallel lines, which is a popular proof in geometry. Before discussing the proof, the instructor dealt with the Parallel Postulate and angle properties including vertical angles and alternate interior angles.

*Similarity*

The discussion of the concept of similarity began with differentiating the mathematical use of *similar* from the ways that *similar* is used in everyday language, and then scaling problems were provided. Scaling problems were to be solved through applying a scale factor and internal factor (or relative sizes). After exploration with general shapes, the instructor introduced the criteria for triangles to be similar. For the concept of similar triangles, the instructor offered a couple of applications, such as measuring distance by sighting and determining the height of a tree.
Volume of Rectangular Prisms

Solid figures were covered at the end of the semester, and the volume of solid figures was addressed. In the course, the volume of prisms and cylinders was defined as the number of unit cubes that it takes to fill the shape without gaps or overlaps. The relationship between prisms and pyramids was also discussed, and the volume of similar shapes was explored with an activity in which the preservice teachers made two similar rectangular prisms and counted the number of unit cubes needed to make them.
CHAPTER 4
DATA ANALYSIS

The purpose of this study was to learn what kind of knowledge of content and students preservice middle grades teachers have with respect to geometry and how they apply their knowledge to examining students’ work. As suggested by Ball et al. (2008), the domain of teacher knowledge includes a variety of subcategories, but in this study I paid attention to teachers’ knowledge of content, students, and teaching. The questions that I addressed were as follows:

1. What kinds of errors do middle grades students make, and what kinds of misconceptions do they exhibit with regard to particular geometry topics?

2. How do preservice middle grades teachers interpret students’ work in geometry, and how do they apply their knowledge to the interpretation?

3. How is preservice middle grades teachers’ pedagogical content knowledge influenced by their course taking and engaging with tasks that assess their knowledge of content and students?

I selected five topics in geometry (the area of a triangle, the relationship between area and perimeter, the angle properties of a triangle, similar triangles, and the volumes of rectangular prisms) for which there was little research that revealed errors students may make and misconceptions they may hold. Thus, before investigating the preservice teachers’ knowledge in terms of geometrical contents and students’ thinking, I examined research on students’ cognition underlying their reasoning shown in their solutions. Then, I attempted to understand the
preservice teachers’ PCK by looking at their interpretations of students’ errors and misconceptions and their instructional plans to correct and avoid students’ errors and misconceptions. Preservice teachers’ knowledge would be affected by their experiences, and taking courses and doing activities that are relevant to learning and teaching mathematics would be such an experience. Thus, assuming that those experiences influenced the preservice teachers’ knowledge, say PCK, I investigated how such an experience influenced the preservice teachers’ responses. Because there was no research to explore middle school students’ errors and misconceptions, and their causes, I began with students’ responses to the five KCS tasks.

**Students’ Errors and Misconceptions**

I analyzed students’ errors and misconceptions by applying Radatz’s categorization of causes of errors. In sum, Radatz (1979) proposed five causes of errors: errors due to language difficulties, errors due to difficulties in obtaining spatial information, errors due to deficient mastery of prerequisite skills, facts, and concepts, errors due to incorrect associations or rigidity of thinking, and errors due to the application of irrelevant rules or strategies. Because Radatz’s original work did not focus on the area of geometry, this section includes a brief example for each category.

The first type of errors was due to language difficulties. The mathematical meaning of the word *ray* is “the part of a line lying on one side of a point on the line” (Beckmann, 2010, p. 427), whereas a ray in everyday life means a narrow beam of light. If someone does not know the mathematical definition of the word, he or she may think that a ray is infinite in both directions or finite like a line segment.

The second type of error is related to difficulties in obtaining spatial information. A student was asked to determine which triangle has larger area in Figure 21. If the student answers
that one of the two triangles has a larger area than the other, he or she may have difficulty in obtaining spatial information. The shapes of the two triangles are different, but they have the same area by the definition of area and the area formula of triangles. Thus, the student may acquire the concept of area incorrectly.

Figure 21. Example of the second type of error.

The third type of error is caused by deficient mastery of prerequisite skills, facts, and concepts. A student might be given a problem about the Pythagorean Theorem that asks the student to find the length of the hypotenuse of the triangle in Figure 22. To solve this problem, the student needs to know the concept of square roots and how to deal with squares and square roots. Without that knowledge, he may not be able to calculate the length of the hypotenuse.

Figure 22. The Pythagorean Problem.

The fourth type of error is caused by incorrect associations or rigidity of thinking. Students are asked to find the perimeter of the shape in Figure 23. They may count the number of shaded squares because they have counted the number of all squares inside the figure when finding its area.
Figure 23. Finding the perimeter by counting the number of shaded squares.

The fifth type of error results from the application of irrelevant rules or strategies. Because of successful experiences where students apply comparable or irrelevant rules or strategies to other tasks, they may make errors. If students had the experience of finding the area and perimeter of the square with a side length of 4 by multiplying 4 by 4 or squaring 4, they may find the perimeter of the square with the side length of 5 by squaring 5. This error may occur because their successful experience of finding the perimeter of a particular square leads the students to an incorrect application of an irrelevant method.

Among Radatz’s (1979) categories, the second type of error is related to Fischbein’s (1993) figural concepts. In the course of a reasoning process, images and concepts interact intimately, but images and concepts usually are considered as belonging to distinct categories of mental entities. In the domain of geometry, Fischbein proposed assuming that “one has to do with a third type of mental objects which simultaneously possess both conceptual and figural properties” (p. 144). That is, geometrical figures are the objects of investigation, but their position is special, unlike other objects in mathematics. Geometric objects are images, and simultaneously ideal and abstract entities, because of their ideality and generality. A geometrical figure starts with its definition based on the axiomatic structure to which it belongs, and then its properties are imposed by a definition. This feature makes geometrical figures possess ideality, abstractness, universality, definition dependence, purity, and perfection, which are characteristics of concepts. At the same time, a geometrical figure is beyond a concept because it has a spatial...
representation. For example, consider a circle. We can imagine a circle as a round shape like a compact disc (CD). In geometry, a circle is defined as the collection of all points in a plane that are a constant distance away from a fixed point in the plane. All properties of a circle are derived from this definition. In this sense, geometrical figures are called figural concepts, and figural concepts have both spatial properties such as shape, position, and magnitude, and conceptual qualities such as ideality, abstractness, generality, and perfection (Fischbein, 1993, 1994).

According to Fischbein (1993), ideally, in a figural concept, the figural construct, as an image, is exhaustively controlled by logical rules and procedures, as a concept, in a certain axiomatic system. However, the figural and the conceptual features of a figural concept are influenced by their relative systems in psychological conditions, which leads to conflicts and errors in the process of geometrical reasoning. Thus, the figural constraints may often be separated from conceptual control, and attempts to interpret it in a figurally consistent way conflicts with conceptual constraints. For example, a student was asked to answer how many angles are shown in Figure 24. The student stated that the space between two lines is an angle, but answered that there is only one angle in both figures (a) and (b) because line 2 is the bisector of the angle formed by lines 1 and 3. This student’s error came from the fact that the concept of an angle did not control the figure and his interpretation of the figure depended on non-formal constraints.

Figure 24. Example of conflict of a figural concept.
In the next section, I describe students’ errors and analyze students’ solutions of the five KCS tasks using the frameworks of Radatz’s (1979) error analysis and Fischbein’s (1993) figural concepts. These are the tasks that the preservice teachers were given, and I use my analysis of the errors in students’ solutions as a yardstick for comparing the preservice teachers’ analyses to my own later in this chapter.

Task 1: The Area of a Triangle

- Student A

  ![Figure 25](image)

  Student A (Figure 25) correctly memorized the area formula of a triangle, but it was not obvious whether or not the student correctly understood what a base and its height meant.

- Student B

  ![Figure 26](image)

  Student B’s (Figure 26) reasoning was partly correct but insufficient. The calculation of area, \((3 \times 6) \div 2\), was appropriate only for the first triangle, which indicated that Student B might not fully understand how the area formula works and what the height of a triangle is. Student B also gave an explanation of why the product of height and base was divided by 2,
saying “a triangle is half a rectangle or square.” This comment was appropriate for the first triangle, and students might be able to draw a rectangle easily. However, students might have difficulty finding a suitable rectangle whose area is double the second triangle. Because Student B did not provide any visual explanation, we cannot confirm whether or not the student knew how to fit the triangle into a rectangle with twice the area of the triangle.

Student B may not understand what the height was, which might lead him to a wrong conclusion. This misunderstanding may be due to difficulty in understanding the word height because in everyday language height means the distance or length above the ground. In everyday language, height is not always specified to be perpendicular to the ground. The misconception of height may also be due to difficulties in obtaining spatial information because the student may think the height should be one of the sides in a triangle by looking at the height of right triangles and failing to visually discriminate the height of right triangles and the height of obtuse triangles.

In terms of figural concepts, his computation of the area was true for the first triangle, which implied that Student B had the figural concept of the height of a triangle and its role in the area formula. The figure of the second triangle showed two equal side lengths to the first triangle, which may liberate the figural component of the height from the formal constraint of the height.

Student B may also not understand how the area formula for triangles works, which may be due to deficient mastery of prerequisite skills, facts, and concepts. That is, when he applied the area formula, such as \( A = b \times h \div 2 \), Student B may not have considered what \( b \) and \( h \) indicated in the formula, which may have resulted from insufficient knowledge of necessary concepts and symbols about base and height. The student’s justifying division by 2 can be considered as due to incorrect associations or rigidity of thinking or the application of irrelevant rules or strategies for the second triangle. Student B may have easily succeeded in creating a
rectangle for the first triangle and so may have tried to associate what he did for the first triangle with the second triangle. In fact, fitting a triangle into a rectangle or a square is not an irrelevant strategy, but if a student is asked to draw a rectangle or a square fitted into the second triangle, it is possible for him to draw a rectangle as Student D (Figure 28) did. This situation can be considered an inappropriate application of the strategy.

- Student C

![Figure 27. Student C’s solution to the first task.](image)

Student C’s (Figure 27) solution was similar to Student B’s except for justifying multiplication by $\frac{1}{2}$. Thus, the same causes of the misconception can be applied to Student C.

- Student D

![Figure 28. Student D’s solution to the first task.](image)

Student D did not explicitly answer which triangle has the larger area (Figure 28), but his calculations of areas indicated that the first triangle is larger than the second triangle. Despite the correct answer, his estimation process had a couple of mistakes for the second triangle. Student D drew a rectangle with dimensions of 8 cm and 2 cm, but the area of the rectangle was not
double that of the triangle. Student D assumed that the height of the second triangle is 2 cm by relying on the visual representation. Based on his solution, although Student D might know the area formula of triangles with regard to the area of rectangles, he did not fully understand how a triangle can be fitted into a rectangle so that the triangle has half the area of the rectangle.

Student D’s computation of areas was based on creating rectangles in which triangles were embedded. For the second triangle, Student D added the unnecessary triangle and did not take it away after computing the area of the rectangle. The student did not take away two copies of the extra triangles when he calculated the area of the triangle. As mentioned in Student B’s misconception, this error may have been due to incorrect associations or rigidity of thinking or the application of irrelevant rules or strategies. Student D may associate the way to create a rectangle for the first triangle with that for the second triangle. Otherwise, he may inappropriately employ the strategy that was used in the first triangle in drawing a rectangle for the second triangle. The student used the strategy of dividing the area of the rectangle by 2 when he computed the area of the triangle. The error that occurred with the second triangle may be due to inappropriate application of this strategy. In Student D’s incorrect use of the base of the second triangle as 8 cm rather than 6 cm, he had a correct concept of the height, whereas he may not have a correct concept of the base of a triangle. This misconception may be due to difficulties in obtaining spatial information because he did not consider that the base of the triangle was different from the base of the rectangle. With regard to the figural concept, Student D had established the figural concept of the height that was perpendicular to its base, whereas his formal concept of the base was weak to control the figural component of the base. Thus, the student concentrated on the fact of the perpendicularity and considered the base 8 cm.
• Student E

Figure 29. Student E’s solution to the first task.

Student E (Figure 29) not only gave an incorrect answer but also provided the wrong explanation. The notable mistake was that Student E did not divide the result of multiplication of 3 and 6 by 2, which implied that Student E did not correctly know the area formula of triangles and he did not consider what each acronym indicates in the area formula. Moreover, Student E incorrectly thought that the areas of triangles with two equal side lengths are the same even though one was stretched out. This misconception may be due to deficient mastery of prerequisite skills, facts, and concepts. That is, he did not fully master how to compute the area of triangles. Since Student E considered that two triangles had the same area, he may have the misconception of the height, which was due to language difficulty or difficulties in obtaining spatial information.

• Student F

Figure 30. Student F’s solution to the first task.
Student F (Figure 30) gave the incorrect answer as well as incorrect reasoning. Depending on visual representation, Student F made incorrect conclusion, which may be due to difficulties in obtaining spatial information. The perimeter of the second triangle is longer than that of the first triangle because of the longer hypotenuse of the second triangle, whereas the area of the second triangle is not larger than that of the first triangle. Thus, his expression “more room and much width” may come from inappropriate obtaining spatial information. Student F may also have language difficulty using mathematically appropriate terms because his use of the words “room” and “width” was vague. In particular, the meaning of “width” was not clear; did he mean the perimeter of a triangle?

- Student G

![Diagram of triangles with measurements and calculations]

Figure 31. Student G’s solution to the first task.

Student G (Figure 31) made correct conclusion by using appropriate estimation strategy. Student G’s strategy was counting unit squares to estimate the areas of both triangles; he attempted to add up defective squares to form whole unit squares. Even though it was not clear how Student G added up defective squares for the second triangle, his strategy was appropriate.

- Participants’ misconception

Unlike other participants, Liz did not use the area formula of a triangle but applied the strategy of moving bigger chunks to the first triangle, which was appropriate strategy to find the
area of the triangle. However, her strategy used in the second triangle did not justify how she knew that the second triangle does not fit into the 3 cm by 3 cm square. This error may be due to deficient mastery of prerequisite skills of strategies to find the area of a triangle. She needs to fully understand three strategies that she learned in the content course and the area formula of triangles.

![Figure 32. Liz’s solution to the first task.](image)

Tia’s solution was similar to Student B and C, which indicated that Tia had the same misconception as Student B and C. Her misconception about the height of a triangle may be due to language difficulty about “height” and difficulties in obtaining spatial information. Tia’s incorrect reasoning based on the meaning of the area may be due to difficulties in obtaining spatial information because she believed two triangles took the same amount of unit squares although they did not. She did not try to estimate areas of two triangles in terms of the meaning of the area such as counting the unit squares inside the triangles. Her misconception may also be due to deficient mastery of prerequisite skills, facts, and concepts of the meaning of the area and the area formula. Even though she incorrectly knew the area formula of triangles, if she tried to count unit squares taken by the triangles as the application of the meaning of the area she might realize her computation was not correct. Thus, she may not fully understand the meaning of the area and the area formula.
Task 2: Articulating the Relationship Between Area and Perimeter

- Student A

Figure 33. Student A’s solution to the second task.

Student A (Figure 33) gave the incorrect answer and his reasoning was limited to a mere repetition of the problem. The student’s answer can be clarified as follows: If the area is not increased, the perimeter cannot be increased. This statement looks like the contraposition to the statement given in the problem, “If rectangle I has larger perimeter than rectangle II, then rectangle I has larger area than rectangle II.” Because the original statement was not true, however, the student’s reasoning was not true.

Student A’s misconception was that the area of a rectangle increases when its perimeter increases. This misconception may be due to difficulties in obtaining spatial information because his thinking about area and perimeter was limited. Even though area and perimeter are not directly related, the student assumed their relationship. Student A may see rectangles below (Fig. 34), and this experience may prompt restriction of the student’s spatial thinking. In terms of figural concepts, the student may have the figural concept of each perimeter and area of rectangles; he may know what area and perimeter are, what their properties are, and how to find the area and perimeter of rectangles. However, seeing the figures shown in the Figure 34 may cause the student’s figural concept not to be controlled by the conceptual component of the area and perimeter. Also, Student A’s misconception may be due to incorrect association or rigidity of thinking because he incorrectly assumed a connection between area and perimeter.
Figure 34. A student’s experience on seeing rectangles

- Student B

Figure 35. Student B’s solution to the second task.

Student B (Figure 35) also incorrectly answered because the student gave a pair of rectangles that satisfied his argument that adding length and width makes the area bigger. Student B had limited thinking because he did not consider another possibility to increase the area of a rectangle. He may assume that in order to increase the area he should make width and length of a rectangle longer, which would result in the perimeter also increasing. He was apparently not aware that the perimeter can be longer by lengthening one side length and shortening the other side length. For example, considering the first figure of the 3-by-2 rectangle, he needed to come up with the 5-by-1 rectangle. This misconception may be due to difficulties in obtaining spatial information about dimensions of rectangles and perimeters. Also, the misconception may be due to incorrect associations or rigidity of thinking because he incorrectly generalized that the area always increases as the perimeter does by depending on only a pair of examples that supported the statement but did not prove it. To solve the problem, Student B
found a pair of rectangles, but his example was not sufficient to verify his conclusion. Thus, he made an incorrect conclusion. This misconception can be considered as due to the application of irrelevant rules or strategies because his strategy of finding examples that make the conclusion valid was not appropriate to this problem. To show the statement is true, Student B must consider all possible rectangles. He may have successful experience on showing a property by using only a few examples. For example, when he was asked to show that the sum of angles in a triangle is $180^\circ$, he may be able to give a few triangles and their angle measurements. This experience may strengthen a tendency to check a few examples in a case where appropriate verification is required.

- Student C

Figure 36. Student C’s solution to the second task.

Student C (Figure 36) also reached an incorrect conclusion, and his reasoning was restricted. The student provided three examples of squares and then concluded that as the perimeter increases, so does the area. Although it is obvious the student’s examples supported his conclusion, he must consider all possible cases of rectangles rather than a few specific cases. Thus, the same causes of the misconception as Student B’s can be applied to Student C. Additionally, his drawing looked like rectangles, but the labeling of side lengths indicated squares. His drawing may result from difficulties in obtaining spatial information about shapes of
rectangles and squares because his labeling of the side lengths was not consistent with the shapes. Although the student knew that all squares are rectangles, he may not have sufficient ability to represent figures that he was supposed to draw. The misconception related to drawing may be due to deficient mastery of prerequisite skills, facts, and concepts because he may not have enough experience on drawing rectangles and squares.

- Student D

![Image of Student D's solution to the second task]

*Figure 37. Student D’s solution to the second task.*

Student D’s (Figure 37) answer was similar to Student C’s; based on specific examples of rectangles, the student concluded that the area of the rectangle increases as the perimeter of a rectangle increases. Unlike Student C, who used squares, Student D started with the 2-by-4 rectangle, and increased each side by 1 unit. Although his method looked more systematic than Student C’s method, Student D made the same mistake as Student C in that he did not consider all possible cases. The student failed to think of examples of rectangles that have longer perimeters but smaller areas. Thus, the same causes can be applied to this misconception.
Additionally, Student D thought both the width and the length of a rectangle should simultaneously lengthen in order to make its perimeter longer. In fact, Student D did not have to increase both side lengths; he could make a rectangle with a longer perimeter by decreasing one side length and increasing the other side length. This misconception may be due to difficulties in obtaining spatial information because he seemed not to understand how change in side lengths of rectangles affects their perimeter. In terms of figural concepts, the examples implied that the student had the figural concept of the perimeter of rectangles; what the perimeter indicates and how to find the perimeter of a rectangle. However, his dependence on the figural component of increasing lengths prohibited the figural component of area from being controlled by the formal component of the perimeter of rectangles. Consider a rectangle with the side lengths of $l$ and $w$. Student D figurally thought $(l+1)$ and $(w+1)$ as the side lengths of a new rectangle with a longer perimeter. In a formal sense, he did not consider that $l'+w'$ from the dimensions of a new rectangle should be greater than $l+w$ but one of $l'$ and $w'$ did not need to be greater than $l$ and $w$.

- Student E

![Image of a worksheet](image)

*Figure 38. Student E’s solution to the second task.*
Student E (Figure 38) correctly answered; the student was aware of cases where the statement holds (Example 1 in Student E’s response) or where the statement does not hold (Example 2 in Student E’s response). However, the student made a mistake in the second example when he wrote that the perimeter of the left-hand rectangle was 8, but it was actually 14, which means the example did not support his answer. This error may have been due to a deficient mastery of prerequisite skills, facts, and concepts; that is, he may be confused with how to calculate the perimeter of a rectangle.

- Participants’ misconception

Liz’s solution was similar to those of Students B, C, and D; her example of rectangles was similar to Student B’s example. Thus, her misconception may have been due to difficulties in obtaining spatial information of the concepts of areas and perimeters, to incorrect associations or rigidity of thinking about generalization, or to the application of irrelevant rules or strategies of proving a mathematical property.

Task 3: Construction of triangles and their angle properties

- Student A

Figure 39. Student A’s solution to the third task.
Student A (Figure 39) incorrectly answered without any clear explanation and provided unclear figures. He did not represent the symbol of a right angle, \( \angle \), for Sunnum’s triangle, and argued that Alicia’s triangle could be drawn. Even though he drew Raj’s triangle, it looked like a right triangle. The reasoning about Roberto’s triangle, “you can only get 2 acute angles instead of 3,” is incorrect because acute triangles always have three acute angles.

Based on his drawing and comment, Student A may have had language difficulties because he did not seem to understand what acute and obtuse angles are. Student A said that a triangle with only two acute angles, not three, was possible, whereas he drew Alicia’s triangle with two obtuse angles. He may not be familiar with the mathematical terms, **acute angle** and **obtuse angle**, which made him confused. Student A’s misconception may have been due to difficulties in obtaining spatial information because he seemed not to understand what acute and obtuse angles look like as well as their definitions. Student A’s drawing of Alicia’s triangle looked like an acute triangle with no obtuse angle, and his drawing of Raj’s triangle looked like a right triangle even though the problem asked not to have a right angle. Additionally, Student A’s misconception may have been due to deficient mastery of prerequisite skills, facts, and concepts. He did not fully understand the concepts of types of angles and the prerequisite property for this problem that the interior angles of triangles add to 180°.

- **Student B**

![Image of Student B's solution](image)

*Figure 40. Student B’s solution to the third task.*
Student B gave a correct answer (Figure 40) but only partly correct reasoning. He drew figures that satisfied each condition and provided an explanation for each figure. However, Student B did nothing but repeat the condition given in the problem as his explanation; he did not give exact examples that satisfied the conditions. Even though Student B correctly said that Alicia’s triangle is impossible, he did not try to apply the property of the sum of interior angles in a triangle. This neglect may have been due to a deficient mastery of prerequisite skills, facts, and concepts.

- Student C

![Figure 41](image)

Figure 41. Student C’s solution to the third task.

Student C also gave a partly correct answer and reasoning (Figure 41). For Roberto’s and Sunnum’s triangles, he drew figures without any written explanation, but the figure indicating Sunnum’s triangles did not look like a right triangle. Based on Student C’s response for Sunnum and Raj’s triangles, he did not have correct understanding of a right angle, which may have been due to language difficulties. The mathematical meaning of right is different from its everyday sense, which may have caused his confusion with the term. Based on his statement, “every triangle has at least one right angle,” this misconception may have been due to difficulties in
obtaining spatial information. Student C may not know how a right angle looks like and what big a right angle is. Also, his comment indicated that he may have had a limited understanding of triangles. According to his comment, Roberto’s triangle should not be a triangle. Thus, Student C may not have correct spatial information about triangles. In terms of figural concepts, the student may have figural concepts of three types of angles because he correctly attempted to draw triangles with three acute angles and with no right angle, and to show the impossibility of a triangle with two obtuse angles. However, his reasoning about Raj’s triangle was not compatible with his drawing, which implied that his conception of the figural component of the right angle conflicted with his formal and conceptual component. Like previous students, Student C did not recognize the property of the sum of interior angles of triangles, which may be due to deficient mastery of prerequisite skills, facts, and concepts.

- Student D

![Fig 42](image)

*Figure 42. Student D’s solution to the third task.*

Student D incorrectly answered with no reasoning. Based on only his drawing (Figure 42), Student D’s error may have been due to language difficulties. He drew Alicia’s triangle as an acute triangle, which indicated he may have been confused about the meaning of types of angles. Student D’s error may also have been due to difficulties in obtaining spatial information.
He said that Sunnum’s triangle was impossible although he drew a triangle that looked like a right triangle. He was not confident about his drawing of Raj’s triangle with no right angle. These responses, as well as the incorrect drawing of Alicia’s triangle, may have come from spatial information that he obtained before. Like previous students, Student D did not recognize the property of the sum of interior angles of triangles, which may have been due to deficient mastery of prerequisite skills, facts, and concepts.

- **Student E**

  Roberto is wrong because a triangle has only three sides and you cannot have 3 acute angles in a triangle. Sunnum is right. Alicia is wrong because there is no way for a triangle to have 2 obtuse angles because a triangle has only three sides. Raj is right.

*Figure 43.* Student E’s solution to the third task.

Student E offered a partly incorrect answer without any graphical representation (Figure 43). He said that Sunnum and Raj are right, but he did not provide any particular example or figure. Student E concluded that Roberto’s triangle is impossible based on insufficient reasoning. The part about a triangle having only three sides was correct, but saying that you cannot have 3 acute angles in a triangle was not correct. Additionally, he did not provide any reason why a triangle cannot have three acute angles. Student E’s belief that a triangle cannot have three acute angles may have been due to difficulties in obtaining spatial information about acute angles. He may have seen an equilateral triangle, but he was not able to incorporate this information with the idea of a triangle with three acute angles. With regard to figural concepts, the student may have the figural concept of an angle, but its figural component did not agree with its formal component in that he was figurally aware that there are more than three sides when a shape has two obtuse angles but was not formally aware that it is because the sum of obtuse angles becomes greater than 180 degrees.
For Alicia’s triangle his conclusion was correct, but his reasoning was not enough to support his conclusion. Student E wrote that “There is no way for a triangle to have 2 obtuse angles because a triangle has only three sides” as the reason. He might think that one cannot make a closed figure with two obtuse angles and three sides. Without any figure or clear explanation, however, it was uncertain that this guess accorded with what Student E intended. Like previous students, Student E did not try to use the property of the sum of the interior angles of a triangle, which may be due to deficient mastery of prerequisite skills, facts, and concepts.

- Participants’ misconception

For this task, there was no specific misconception. Eva was the only person who did not use the property of the sum of interior angles in a triangle, but she knew the property based on her indication in the response of identification of mathematical ideas.

Task 4: Similar triangles

- Student A

No, because A triangle is more long than B triangle. B triangle is very short and very wide, A triangle is very thin and very small space.

Figure 44. Student A’s solution to the fourth task.

Even though he made a correct conclusion, Student A’s (Figure 44) usage of terminology in the explanation was not mathematically rigorous. Student A described that Triangle A is long and thin whereas Triangle B is short and wide. However, it is not clear which component in the triangles is long, thin, short, or wide. For example, comparing their bases and heights, the base of Triangle A is shorter than that of Triangle B, whereas the height of Triangle A is longer than that of Triangle B. Additionally, Student A mentioned that Triangle A took up a very small space. If Student A meant the area of Triangle A, his reasoning was not correct because the areas of both triangles are actually the same. Thus, Student A should clarify the meaning of what he wrote in
the explanation. This misconception may have been due to language difficulties when he addressed his reasoning in mathematical language. His description of two triangles made sense in everyday language, but was not sufficient to justify why two triangles were not similar in mathematical sense.

Student A did not try to use any criteria for similar triangles, which may have been due to a deficient mastery of prerequisite facts and concepts about the definition or conditions of similar triangles. Considering two similar triangles, one of them is smaller than the other, which means the lengths of three sides are shorter than those of corresponding sides, and the area is also smaller than that of the other. Therefore, identifying whether or not two triangles are similar involves the concept of equal corresponding angles or the scale factor, but Student A did not apply either concept to the problem.

- Student B

Yes, because both A and B have the same number of dots and that’s 6, and they both are triangles and they both have straight lines or edges.

*Figure 45.* Student B’s solution to the fourth task.

Student B (Figure 45) considered that having the same number of dots was evidence of similarity, which may have been due to difficulties in obtaining spatial information about length and perimeter of a shape. Student B may believe that two triangles had the same perimeter because of the same number of dots, but that was not true, because the lengths of their hypotenuses were different. In terms of figural concepts, the student may have had the figural concept of lengths as the number of unit lengths by an experience with using a ruler or the geoboard. However, concentrating on tick marks on a ruler or dots on the geoboard may have caused the student to count the number of tick marks or dots rather than count the number of unit length between two tick marks or two dots.
Student B also did not try to apply the criteria of similar triangles, which may have been due to a deficient mastery of prerequisite skills, facts, and concepts. Assuming that Student B considered the number of dots as the length or perimeter of the triangle, he might attempt to think of the concept of the scale factor in terms of dots. However, he did not appropriately connect the concept of length to the concept of the scale factor or proportionality between lengths of corresponding sides. This misconception may have occurred because of an application of irrelevant rules or strategies; the total number of dots on the triangles was irrelevant to determining whether or not the two triangles are similar.

Additionally, he mentioned straight lines or edges as a condition of similarity, but the property of straight lines or edges was common to all triangles, not specific to similar triangles. Thus, these indicated Student B’s insufficient understanding of prerequisite facts and concepts of general triangles and similar triangles. This statement also seemed that he might have confused the mathematical meaning of similarity with the meaning in everyday language, which resulted from language difficulty.

- Student C

“Yes. Because bother have square corners. And also both could make a square and a rectangle if you just put a line like this.”

Figure 46. Student C’s solution to the fourth task.
Student C (Figure 46) also gave an incorrect answer and insufficient reasoning. In his reasoning, he used the term *square corners*, meaning the right angle, which suggests that he attempted to apply the criterion of equal corresponding angles. However, he did not develop this attempt in an appropriate way. The student may have considered having square corner the only criterion of similarity, which may have been due to a deficient mastery of prerequisite facts and concepts of the criteria of similar triangles.

Student C’s comment, “both could make a square and a rectangle,” seemed that he believed all squares and rectangles are similar. He might believe that rectangles and squares are similar because all of their angles are right angles, so that all right triangles would be similar because they are half of a rectangle or square and have right angles. This misconception may have been due to language difficulties; he may have confused mathematical similarity with similarity in everyday language.

- Participants’ misconception

Unlike other participants, Liz agreed that two triangles were similar based on two reasons: proportionality and areas. Even though she tried to apply the concept of proportionality or a scale factor in similar figures, her scale factor was not correct. This error may have been due to deficient mastery of prerequisite skills, facts, and concepts of similarity and proportionality. She was not aware that the ratio should be the ratio of “corresponding” sides. She may not fully understand how sides of two triangles were corresponded and how the ratio of corresponding sides looks. Also, she may be confused with the concepts of ratio and proportionality, which may affect her incorrect computation of proportionality. Liz incorrectly associated the concept of the area with similarity, which may have been due to incorrect associations or rigidity of thinking. She may remember the relationship between areas of two similar shapes, but she did not fully
understand this concept. She did not realize that the proportionality of the areas of the two triangles should be 1/4 if they are similar. Additionally, she did not understand that even though the proportionality of areas of two triangles is 1, it does not guarantee that the two triangles are congruent and that even though the proportionality of areas of two triangles is \( \frac{a^2}{b^2} \), it does not guarantee that the two triangles are similar with the scale factor \( \frac{a}{b} \). Like that of Student C, Liz’s misconception may have been due to language difficulties about similarity in everyday language versus similarity in mathematical sense.

**Task 5: Volumes of a rectangular prism**

- Student (1)

*Figure 47. Student (1)’s solution to the fifth task.*

Student (1) (Figure 47) provided an incorrect answer by squaring all dimensions without calculating the volumes. This misconception may have been due to difficulties in obtaining spatial information about the effect of change in dimensions on the volume of a rectangular prism. He may not have a sense of the volume; lengthening all dimensions makes the volume much bigger. Student (1) did not try to compute the volumes of his new prism and the original prism, which may have been due to deficient mastery of prerequisite skills, facts, and concepts of finding the volume. If he had computed the volumes of both rectangular prisms and compared them, he might have realized that his answer was not correct. Student (1)’s strategy of squaring
all dimensions may have been due to incorrect associations or rigidity of thinking and the application of irrelevant rules or strategies. He may have squared the dimensions because 4 is the square of 2, which meant that he applied an irrelevant rule. Squaring all dimensions rather than one dimension may have been done because of his idea about similar figures. However, the problem did not ask that the two rectangular prisms should be similar. This misconception may have arisen from an incorrect association between increasing the volume and similarity. Moreover, squaring dimensions cannot make similar shapes, which indicates that Student (1) may also have had a misconception about similarity.

- Student (2)

![Figure 48](image)

*Figure 48. Student (2)’s solution to the fifth task.*

Student (2) gave the wrong answer by multiplying all dimensions by four without any calculation of the volume (Figure 48), which may have been due to difficulties in obtaining spatial information of the volume. He may not have a sense of the volume, because he was not aware of how big the new rectangular prism was by making all dimensions 4 times longer. With regard to figural concepts, although the student may have the figural concept of the volume of a rectangular prism, his figural conception of making the volume 4 times larger was not controlled by the formal component of the volume such as the volume formula, how the formula works, and
the relationship between the formula and the dimensions of a rectangular prism. Like Student (1), Student (2) did not try to compute the volume of his new prism and the original prism, which may have been due to a deficient mastery of prerequisite skills, facts, and concepts of finding the volume. Student (2)’s strategy of multiplying all dimensions rather than one dimension may have been due to incorrect associations or rigidity of thinking about increasing the volume and similarity. If Student (2) multiplied all dimensions by 4 so as to make a similar rectangular prism, he also had a misconception about the relationships between the area and the volume of similar figures and the scale factor. He did not recognize that the volume became \(64 (= 4^3)\) times larger when he applied 4 as the scale factor.

- Student (3)

![Image](image1.png)

*Figure 49. Student (3)’s solution to the fifth task.*

- Student (4)

![Image](image2.png)

*Figure 50. Student (4)’s solution to the fifth task.*

Student (3) and Student (4) (Figure 49 and 50) answered correctly. Student (3) made the height, labeled 2 cm, four times as long, which became 8 cm, and left the other dimensions
unchanged. He knew that only one dimension needs to be four times larger in order to create a prism with four times the volume. Unlike Student (3), Student (4) doubled two dimensions, labeled 4 cm and 5 cm, which became 8 cm and 10 cm respectively. He understood that doubling two dimensions made the volume four times bigger because four can be factored as two two’s. Although neither student provided the process of calculating volumes, their explanations showed that they correctly knew how to calculate the volume of a rectangular prism.

**Preservice Teachers’ Subject Matter Knowledge**

For this study I regarded subject matter knowledge (SMK) as mathematical content knowledge, in particular, geometrical content knowledge. From Shulman’s (1986) perspective, teachers’ SMK should include both understanding mathematical facts and properties and knowing why the facts and properties are so, what guarantees that the facts and properties are true, and what conditions can make our belief in the process of justification weak. Additionally, Fischbein (1994; Tsamir & Tirosh, 2008) divided mathematical knowledge into three components: algorithmic, formal, and intuitive. Using the framework of Shulman-Fischbein as combined by Tsamir and Tirosh, in this section I address what kinds of SMK the preservice teachers had with regard to the five topics from KCS tasks. Their SMK was investigated by having them solve the problems that were given to students and having them identify important mathematical ideas needed to successfully solve the problems.

**Mathematical Algorithmic-SMK**

According to the combined framework of Shulman-Fischbein (Tsamir & Tirosh, 2008), mathematical algorithmic SMK includes teachers’ mathematical knowledge of solving procedures and supporting them with explicit justifications. According to Fischbein (1994), the algorithmic aspect of knowledge refers to problem solving techniques, skills, and strategies, and
these are acquired by practical and systematic training along with understanding underlying mathematical concepts. With regard to this study, this category involves teachers’ routine use of formulas and showing computations in a procedural way. Once they understand what a formula is and why it works, they can routinely apply the formula after solving practice problems that require applying the formula.

The Area of a Triangle Task did not directly ask the participants to compute the areas of two triangles because it was sufficient to say that the first triangle has a larger area because of its greater height than that of the second triangle with the same length of the base. The participants tended to automatically compute the areas in order to compare the triangles. Joy, May, Pam, Sam, and Tia explicitly showed how they calculated the areas. Joy, May, Pam, and Sam employed the triangle area formula with a brief description such as \( A = \frac{1}{2} \times \text{base} \times \text{height} \). Tia briefly mentioned the area formula and attempted to describe the meaning of area. However, she considered that the heights of the two triangles were the same, and thus their areas were equal. Eva did not explicitly show the computation of the areas, but she described how the area of a triangle is computed on the basis of the two given triangles by saying that “in order to calculate the area of a triangle one must multiple [sic] one half of the base (which is 6 cm) by the height.” Therefore, these participants’ SMK about the area of a triangle would be described as algorithmic knowledge.

In the question that asked them to identify important mathematical ideas a student needed to possess to find the areas of triangles, all participants pointed out the area formula of a triangle. They did not attempt to show how the area formula can be induced. Only Pam responded that the area formula for triangles came from the area formula for rectangles by dividing the area of rectangles by 2, but she did not explain how the area formula for rectangles can be acquired. This
omission would indicate that these participants’ SMK for the area formula was memorized and that they routinely applied the area formula whenever they needed to compute the areas of triangles.

The Task of Articulating the Relationship Between Area and Perimeter was not intended to directly examine students’ ability to compute areas and perimeters of rectangles. However, the participants’ solutions included computations of areas and perimeters of rectangles. Eva, Joy, Liz, Pam, and Sam solved the problem by providing a pair of rectangles as an example. Although they did not clearly say what the area formula and perimeter formula are, they gave the area and perimeter of each rectangle. This response implies that those participants were able to routinely apply the formulas for area and perimeter of rectangles.

In the question that asked the participants to identify important mathematical ideas to solve the problem of the relationship between areas and perimeters of rectangles, on the other hand, only Sam mentioned the area and perimeter formulas for rectangles. Eva and May responded that students need to know the area and perimeter, but they did not say that students need to know “the area formula” and “the perimeter formula.” Given that they used the formulas as secondary knowledge rather than essential knowledge, the participants were aware of the priority order of mathematical knowledge to solve this problem; they may have believed that knowledge about the relationship between area and perimeter is more important than the area and perimeter formulas themselves.

To solve the volumes of rectangular prisms problem, like the relationship between area and perimeter problem, students did not have to compute the volume of the given rectangular prism by applying the volume formula. They could use the volume formula for rectangular prisms to confirm the dimensions. For the question about identifying important mathematical
ideas to solve the problem, Joy, Liz, and Sam explicitly stated the volume formula for rectangular prisms as multiplication of three numbers assigned to length, width, and height. Pam did not state the volume formula, but she mentioned that students need to understand how to find the volume of a rectangular prism. When they investigated students’ solutions, Liz, May, Pam, and Sam spent a few minutes computing volumes of the rectangular prisms that were provided by students. This activity implied that these participants were able to routinely remember and apply the volume formula for rectangular prisms whenever they needed to use it.

**Mathematical Formal-SMK**

The mathematical formal-SMK refers to teachers’ knowledge of the core principles of the discipline of mathematics (Tsamir & Tirosh, 2008). Formal knowledge includes axioms, definitions, theorems, and proofs, which need to be reflected in the reasoning process (Fischbein, 1994). With regard to this study, this category involves teachers’ abilities to use geometrical definitions, properties, and justifications that are relevant to the given tasks.

The Area of a Triangle Task aimed at examining the preservice teachers’ understanding of the concept of the height in a triangle and the meaning of the area of a triangle. Thus, the teachers should have correct knowledge about the concept of the height in a triangle and what area of a triangle is in order to solve the problem and to teach this problem to students. Because the two triangles have the same length of their bases, this problem can be solved by comparing the lengths of the heights of the two triangles. In this sense, solving this problem requires understanding the concept of the height, which involves knowing the mathematical definition of the height. Eva, Joy, May, Pam, and Sam clearly indicated that two triangles had the same base but different heights. In particular, May said that “the height must be perpendicular to the base” when she explained that the side with the length of 3 cm in the second triangle was not the
height. Joy also wrote that the height is perpendicular to its base. Eva, Joy, Liz, May, Pam, and Sam identified the concept of the height of a triangle as an important idea to solve the problem. Eva, Liz, and Sam indicated that the height of a triangle may not be its side length. Joy pointed out the perpendicularity of the height to its base. These observations implied that these participants had solid formal knowledge of the concept of the height.

For the triangle area task, Tia was the only participant who incorrectly solved the problem. She believed that two triangles had the same length of heights so the areas of the two triangles were equal. She referred to the area formula and the idea of shearing as important mathematical knowledge, but she did not consider the concept of the height. Her SMK in terms of the area of triangles influenced her investigation of students’ answers, so she judged that the students who answered that the two triangles had different areas were wrong whereas the students who answered that the two triangles had the same area were correct. This judgment serves as evidence that teachers’ SMK is related to their PCK.

An important strategy to solve the problem of Articulating the Relationship Between Area and Perimeter is to prove or disprove the statement that a rectangle with a larger perimeter has larger area. In this case, it is not plausible to prove this statement, but it is easy to find a pair of counterexample to disprove it. Thus, the strategy of providing counterexamples can be considered as formal SMK. Eva, Joy, Pam, and Sam provided appropriate counterexamples that supported their claim that the statement was not true. That is, they showed two rectangles where one had a shorter perimeter but larger area than the other. This implied that these participants recognized how to refute a statement by a counterexample.

In order to correctly answer the problem of Articulating the Relationship Between Area and Perimeter, students need to be aware that there is not a linear relationship between those
concepts. They need to know differences between areas and perimeter as well as what each concept means. For example, students need to know the perimeter is a one-dimensional concept, whereas area is a two-dimensional concept. Joy wrote of the meanings and properties of area and perimeter:

Perimeter is a measurement of the LENGTH around an object. It is a one-dimensional measurement.
Area is the amount of square units that can fit inside the shape without overlaps or gaps. It is a 2-D measurement.

Eva and May referred to understanding of the area and perimeter without describing what the area and perimeter mean. They also mentioned the difference between area and perimeter, but they did not say what the difference was. These responses are one example of participants’ superficial responses. Participants often suggested their ideas briefly, which made it hard to decide whether or not they had correct knowledge. Pam and Sam indicated that students should know there is no relationship between area and perimeter. Even though other participants did not identify the meanings and properties of area and perimeter, we cannot conclude that they did not know those meanings and properties. Other parts of the participants’ responses, such as computation of area and perimeter with their examples of rectangles and their use of units of area and perimeter, implied their status of knowledge of the meanings and properties.

The problem given in the Task of Construction of Triangles and Their Angle Properties was intended to examine students’ understanding of three types of angles (acute, obtuse, and right angles) and their ability to justify a conclusion by applying the property that the sum of the interior angles in a triangle is always 180 degrees. Based on their responses, all participants were aware of the definitions of acute, obtuse, and right angles. Although they did not write the exact definitions of those concepts on their answer sheets, their drawings or their examples indicated that they had this knowledge. For example, Eva, Liz, and Sam suggested 60 degrees as an acute
angle measure, and Pam suggested an acute triangle with angles of 80°, 80°, and 20°. Eva, Joy, Liz, May, Pam, and Sam marked the right angle with its symbol or clearly mentioned 90 degrees. In addition to the definitions of the three types of angles, Joy, Liz, May, Pam, and Sam justified the impossibility of Alicia’s triangle by applying the angle property of triangles; they claimed that a triangle cannot have two obtuse angles because the sum of two obtuse angles is greater than 180 degrees, which violates the property that three angles of a triangle always add to 180 degrees.

For the question that asked them to identify important mathematical ideas needed to solve the problem of the Task of Construction of Triangles and the Angle Property, Eva, Joy, May, Pam, and Sam indicated that the definitions or meanings of types of angles - acute, obtuse, and right - were important ideas. Among them, Joy described the definitions of those angles, but the others mentioned that the definitions of those angles were important without addressing what the definitions were. All the participants pointed out the property of the sum of angles in a triangle as an important piece of mathematical knowledge. Those participants’ responses implied that they had appropriate formal SMK for angles and the angle property in triangles.

To answer the question given in the Similar Triangles Task, the preservice teachers needed to employ one criterion for similar triangles: (a) all corresponding angles are the same, (b) all three pairs of corresponding sides have the same ratio (scale factor), or (c) two pairs of corresponding sides have the same ratio (scale factor) and the corresponding angles between two sides are the same. Joy, May, Pam, and Sam addressed two reasons that two triangles were not similar: (1) corresponding angles were not equal, and (2) corresponding sides did not have the same ratio. Eva explained one reason using the scale factor. In addition to their solutions, they identified equal corresponding angles and scale factor (or proportionality of corresponding sides).
as important mathematical ideas. These participants’ responses implied that they had the appropriate formal aspect of SMK for the concept of similar triangles.

Even though Liz attempted to use the concept of proportionality to solve the similar triangle problem, her knowledge about proportionality of similar triangles was not complete. She wrote “Yes because their side lengths are proportional, i.e. one is either 1/2 or equal the other side length.” Her incorrect formal SMK for similar triangles may have affected her investigation of students’ answers, which can serve as evidence that teachers’ SMK is related to their PCK.

**Mathematical Intuitive-SMK**

The mathematical intuitive-SMK refers to the mathematical knowledge, such as a notion, a theorem, or a solution that is directly accepted without the feeling that any kind of justification is required. An individual may believe that justification is not necessary because an idea is self-evident. This intuitive aspect of knowledge may sometimes agree with logically justified truths, but sometimes it conflicts with them. When an individual’s intuitive knowledge contradicts the mathematical truths, one’s intuition functions as an epistemological obstacle in the learning process (Fischbein, 1994). Thus, having appropriate intuitive SMK is important for teachers.

In the triangle area problem, the triangle figures were given on grid paper. Although the height of the second triangle was not labeled, its length looked approximately 2 cm. In fact, it did not have to be 2 cm; it could be any length less than 3 cm. If one relies on the given figure, one would use 2 cm as the length of the height without any justification or explanation about why the length becomes 2 cm. As I mentioned earlier, most participants computed the areas of the triangles. In their process of calculation, Joy and Sam regarded the height of the second triangle as 2 cm even though the original problem did not assign an exact number to the length of its height. This response may be interpreted as Joy and Sam answering this problem by relying on a
given visual representation, and that visual representation did not stimulate them to find it necessary to explain why it is 2 cm. Additionally, Eva considered the length of the height of the second triangle was approximately 1.5 cm. Unlike Joy and Sam, without depending on the visual representation, May and Pam did not restrict the length of the height to a particular number and clearly said that the height of the second triangle is shorter than that of the first triangle. This kind of intuitive SMK did not seriously conflict with mathematical truths, but it may be possible that intuitive SMK makes teachers’ mathematical thinking narrow.

A possible intuitive SMK for the problem of Articulating the Relationship Between Area and Perimeter was to believe that there is a linear relationship between the area and the perimeter. That is, one may consider that the area increases as the perimeter increases, or vice versa. This is intuitive thinking that contradicts a mathematical truth because there are a lot of rectangles that have smaller areas but greater perimeters. Among the participants, only Liz had this intuitive SMK; she suggested a 4-by-5 rectangle and 4-by-3 rectangle and concluded that a rectangle with a larger perimeter had a larger area than a rectangle with a smaller perimeter. However, she changed her mind when she worked on the task of identifying necessary mathematical ideas and students’ errors and suggesting instructional strategies to correct students’ errors. This change may have been because she was reminded of what she learned in her geometry class or because she recalled her thinking based on given students’ responses after she completed solving the problem. Although her initial thinking about the relationship between area and perimeter was incorrect, her thinking was flexible, and she was able to change it in a correct way.
Preservice Teachers’ PCK with Regard to Students’ Thinking and Instructional Strategies

According to Shulman (1986), teachers’ pedagogical content knowledge (PCK) includes not only knowledge about students, that is, students’ conceptions, preconceptions, and misconceptions, and the influence of students’ misconceptions on subsequent learning process, but also efficient representations, illustrations, examples, explanations, and demonstrations to make a subject comprehensible to students and to avoid students’ misconceptions. Among those domains of PCK, this study concentrated on preservice teachers’ PCK in mathematics with two different perspectives: (1) knowledge of students’ misconceptions, and (2) instructional strategies to correct students’ misconceptions. Additionally, PCK was incorporated with Fischbein’s three components of mathematical knowledge. This combined framework addressed how the preservice teachers interpreted students’ thinking and how they made a plan to correct and avoid students’ misconceptions. Their PCK was investigated by having them identify students’ errors and misconceptions and by having them suggest ways to correct and avoid students’ misconceptions.

Mathematical Algorithmic-PCK with Regard to Knowledge of Students’ Misconceptions

According to the combined framework, the mathematical algorithmic pedagogical content knowledge includes teachers’ knowledge of the most common incorrect or misused algorithms that students apply in their mathematical problem solving processes and their possible sources (Tsamir & Tirosh, 2008). Even though there was no problem that directly asked about algorithmic knowledge in this study, for this analysis I considered routine use of geometrical formulas as algorithmic knowledge because students tend to apply a formula as routinely as an algorithm once they understand and acquire the formula.
The Area of a Triangle Task included students’ solutions that applied the area formula of a triangle. Among them, Student E’s application of the area formula of a triangle was incorrect because he did not divide the product of two side lengths by 2. Liz and Tia were the only people who pointed out this error. They indicated that the student forgot to multiply by 1/2 and he computed the area of a rectangle. Even though the other participants did not point out this error, we cannot judge that they did not have the mathematical algorithmic-PCK of the area formula of a triangle because they were fully aware of the area formula, and they did not mention anything about correct computation of the first right triangle given by Students B, C, and D. Their failure to identify of this error may have been due to their limited perspective on students’ thinking. For Student E’s work, the participants focused more on the student’s incorrect use of side length as the height of a triangle. In the initial task, participants tended not to provide their various perspectives on students’ errors and misconceptions. Although Liz referred to this error about incorrect application about the formula, she did not explicitly write about the underlying mathematical misconception for this particular error and its plausible cause. She considered that Student E had a similar misconception as other students such as Students C and D. Like other participants, she may have focused more on other misconceptions that Student E had.

It is important for teachers to recognize students’ computational errors in their mathematical reasoning. In the Task of Articulating the Relationship Between Area and Perimeter, Student E (Figure 51) was on the right track in his reasoning, but he made a mistake in computing the perimeter of the rectangle shown in his second example. The dimensions of his rectangle were 6 and 1, and his computation of its perimeter was 8 rather than 14. Although Student E was wrong, it may not have been due to his misunderstanding of the concept of the perimeter of a rectangle because he correctly calculated the perimeters of other examples that he
provided. He may know how to find the perimeter of a rectangle or the formula of the perimeter of a rectangle, and be able to use the way to compute the perimeter routinely. This error looked like a minor error, but it was crucial because this rectangle did not support his conclusion.

![Figure 51. Student E’s computational error.](image)

Joy and Sam were the only participants who perceived Student E’s error. They were also able to indicate that this example was not appropriate to verify his conclusion that “perimeter increases as area decreases OR perimeter decreases as area increases.” Other participants focused more on Student E’s conclusion itself, so they were not aware of the error in the second example. Teachers should be careful when they assess their students’ reasoning. Even though students’ final answers might be correct, teachers need to examine in detail whether or not there is any error in students’ reasoning. In this study, the participants tended to focus more on identifying whether or not students’ answers or conclusions were correct. If students’ answers or conclusions were correct, the participants tended to assume that there was no error or misconception.

In the Volumes of Rectangular Prisms Task, one student squared all dimensions, and another student made all dimensions four times longer in order to make the volume of a rectangular prism four times larger. Eva, Joy, May, and Sam explicitly pointed out these misconceptions. In particular, Sam showed those two students’ solutions were incorrect by computing how many times larger their new prisms were.
Mathematical Algorithmic-PCK with Regard to Instructional Strategies to Correct and Avoid Students’ Misconceptions

Teachers must be aware of how they can help students correct their misconceptions as well as what kind of misconceptions their students have. In terms of instructional strategies to correct students’ misconceptions, mathematical algorithmic-PCK includes algorithmic representations, illustrations, examples, and explanations, and demonstrations provided by teachers (Shulman, 1986; Tsamir & Tirosh, 2008). When teachers suggest introducing a mathematical formula or algorithm without explaining the underlying reasons, and when teachers suggest practicing a formula or algorithm with several examples, these suggestions can be considered as their algorithmic-PCK. This category also includes teachers having students do routine behavior such as measuring a length of a side of a figure or an angle of a polygon.

For the Area of a Triangle Task, Joy proposed showing Student E how to find the area of a triangle in terms of its formula, \( \frac{1}{2}bh \), not just \( bh \). Although she did not point out Student E’s error in using the area formula in the previous part of the task, she was aware of that error and provided a way to help the student. Joy also suggested a similar strategy for dealing with Student G, who did not use the area formula and applied the counting strategy: explaining the area formula and showing how to find areas to Student G. However, she did not mention the area formula for the question about a general instructional strategy. Tia proposed revisiting the area formula with examples of several different triangles as a general instructional strategy. The other participants did not directly suggest teaching or practicing the area formula in a procedural or algorithmic way.

For the Task of Construction of Triangles and the Angle Properties, Liz proposed having students measure angles of a variety of triangles to address students’ misconceptions about the
types of angles. She suggested measuring angles as a general instructional strategy. May similarly recommended having students draw several acute and obtuse angles with a protractor and measure angles of several triangles. Pam mentioned that she would have Student D use a protractor to measure each angle in a triangle. If students have correct knowledge of the types of angles, the activity of measuring angles would play a positive role in facilitating students’ understanding of angles.

For the Volumes of Rectangular Prisms Task, Sam attempted to challenge Student (2)’s answer in a procedural way. Her strategy was to compare the results of computations of the volumes: One was the original volume and the other was the volume of the prism whose dimensions were 4 times longer than those of the original prism:

I would probably go through and have them see that the to make this box it actually 64 times bigger showing them like breaking them into 4 times 4, and this 2 times 4, and that 5 times 4, and by showing the.. it multiplying the original you had which has 40, from that two four five and then multiplying, I guess, like I do 40 times 4 for the first one, you get 160 times 4 again, you get 640, and then times 4 is 2560. So showing them each time that they made it bigger, they’re actually making the box this much, which is not just 4 times bigger because they do multiplication and see that 40 times 4 as 160.

She used consecutive multiplications, which implied she concentrated on an algorithmic method in order to have the student realize his misconception.

*Mathematical Formal-PCK with Regard to Knowledge of Students’ Misconceptions*

The mathematical formal-PCK refers to teachers’ knowledge of students’ common formal-related errors (Tsamir & Tirosh, 2008). Because the formal aspect of knowledge includes knowing axioms, definitions, theorems, and proofs, the mathematical formal-PCK incorporates teachers’ knowledge of students’ misunderstanding of axioms, definitions, theorems, and proofs.

The problem given in the task of constructing triangles and angle properties aimed at investigating students’ formal knowledge of the definitions of acute, right, and obtuse angles and
the property that the sum of interior angles in a triangle is always 180 degrees. Thus, the task included students’ responses that implied their understanding of the definitions and the property. Even though there was no student who explicitly wrote the definitions of three types of angles, their responses allowed the participants to examine their understanding of the concepts.

Student A had an incorrect understanding of acute and obtuse angles because he mentioned that a triangle can have only two acute angles rather than three and did not consider Alicia’s triangle impossible. For this response, all participants pointed out Student A’s misunderstanding of what an acute angle is. In addition, Eva and Joy were able to refute Student A’s claim by providing an equilateral triangle as a counterexample, which implied that they had formal knowledge of refutation as a kind of proof. Joy and Pam mentioned the sum of angle property to address the impossibility of Alicia’s triangle. This comment may indicate that they were able to identify Student A’s error and misconception based on their formal knowledge. Although other participants did not invoke the property when analyzing this particular student’s work, it did not mean that they did not know the property because they explicitly suggested the property as an important mathematical idea to solve the problem. They may have focused more on the incorrect response for Roberto’s triangle, and may not have clearly related the property to a criterion for identifying the student’s error and misconception.

In the Area of a Triangle Task, students B, C, and E showed evidence of incorrect understanding of the area formula of a triangle and the concept of the height. The triangle area formula involves multiplying a base of a triangle and its height. However, those students multiplied two side lengths rather than a base and its corresponding height. Multiplying two side lengths works for a right triangle, but it does not work for an acute or obtuse triangle. On the other hand, students may have correct knowledge of the area formula, 1/2 times base times
height, but an incorrect concept of the height of a triangle. We can choose any side as a base in a triangle, but its corresponding height must be perpendicular to the chosen base. In this task, students used one of the sides in a given triangle as its height without considering whether or not the height that they used was perpendicular to the base.

The participants Eva, Joy, Liz, May, Pam, and Sam referred to students’ incorrect use of the area formula or the height. For Student B, Eva, Joy, Liz, May, Pam, and Sam pointed out his misunderstanding of side length and height. Joy also indicated that students may not understand the area formula. For Student C, Eva, Joy, Liz, May, and Sam pointed out his confusion of the concept of the height, and Pam referred to his misunderstanding of the area formula. For Student E, Eva, Joy, Liz, May, and Sam pointed out his misunderstanding of the concept of the height. Pam indicated that he misunderstood the concept of shearing and thought stretching does not change the area. Based on the participants’ responses, they were able to identify students’ errors and misconceptions underlying the formal knowledge about the area formula of a triangle.

Unlike other participants, Tia’s content knowledge of the area formula was similar to the students’ knowledge; she computed the area of a triangle by multiplying two side lengths. That computation led Tia to conclude that Student B and C were correct and that there was no error or misconception. For Student E, she pointed out the error that he did not divide the product of two side lengths by 2. Tia’s responses suggest that a teacher’s content knowledge affects her knowledge of students in the sense that content knowledge affects her ability to interpret student work.

In the Area of a Triangle Task, Student G used a counting strategy to find the areas of triangles. Counting unit squares is a primitive way to find the area of a shape. A counting strategy can be considered a kind of formal knowledge of the concept of areas in that the area is
defined as how many of a chosen unit of area it takes to cover a given object without gaps or overlaps. Thus, Student G was on the right track to find the areas of triangles even though his strategy may not give the exact area of the second triangle. In addition, Student G’s counting was reasonable because he appropriately considered making up squares that were not fully filled. Joy, May, Pam, and Sam indicated that Student G’s strategy was appropriate. Joy added that Student G was not able to find the exact area because he did not use the area formula. However, Eva said that Student G’s estimation was wrong without any explanation of why he was wrong. Liz mentioned that he added an extra triangle but it was not clear what extra triangle she meant. She also said that counting unit squares is not accurate when finding the area of triangles. Although the problem did not acquire finding the exact areas of two triangles, some participants believed that students should provide exact numbers as an answer.

The error that Student E made in the Task of Articulating the Relationship Between Area and Perimeter can be interpreted in terms of the formal aspect. Student E’s overall idea was correct because he was aware that area can decrease as perimeter increases and attempted to provide an example that refute the statement given in the problem. Showing counterexamples is a typical way to refute a statement. However, Student E provided an inappropriate example, which made his refutation invalid. Participants, Eva, Joy, Pam, and Sam, solved this problem by giving counterexamples, which implied they had this kind of formal SMK of proof. Joy and Sam were able to identify that Student E’s example was wrong and the example did not support his conclusion. Other participants did not recognize this error in Student E’s reasoning. This failure may imply that even though preservice teachers have formal SMK they may not be careful in investigating students’ thinking and their reasoning.
The Similar Triangles Task included students’ errors and misconceptions when deciding whether or not two triangles are similar. In order to determine similarity of two triangles, students should apply one of criteria of similar triangles, but Students A, B, and C did not employ the property of similar triangles. Additionally, students B and C did not know the mathematical definition of similarity because they believed that two right triangles were similar although one triangle was not a scaled version of the other. To explain this absence of the concept of similarity in students’ thinking, Eva, Joy, May, Pam, and Sam, clearly stated that students may not understand the “mathematical” definition of similarity. Before working on this task, the participants had discussed the difference between mathematical similarity and similarity in everyday sense, and that discussion may have affected their analysis of students’ solutions.

Unlike the other participants, Liz believed that two triangles were similar based on her concept of proportionality. She knew that side lengths of two similar triangles have a constant proportion, but she was confused about how the proportion looks. Although the proportions of corresponding sides were different ($\frac{1}{2}$ and $\frac{2}{1}$), she regarded them as the same proportion. Liz had insufficient knowledge of similarity, but her interpretation of students’ responses was interesting. For Student B’s and C’s responses, she indicated that they were on the right track but their answers were not sufficient. Moreover, her responses were not compatible; for example, her computation of the ratios of side lengths showed that she did not fully understand the concept of scale factors (see Figure 52). She wrote about Student B’s misconception that “they did not know to compare lengths to length, width to width, and hypotenuse to hypotenuse.” She may not comprehend that those ratios should be the same for similar triangles.
Mathematical Formal-PCK with Regard to Instructional Strategies to Correct and Avoid Students’ Misconceptions

In terms of instructional strategies, mathematical formal-PCK includes teachers’ ideas about how to teach axioms, definitions, theorems, and proofs in a variety of ways. Many misconceptions that students have are often attributed to students’ difficulty understanding formal aspects of concepts. Thus, a good way to encourage students to correct their misconceptions or misunderstanding of concepts is to help them develop formal knowledge of the concepts.

As mentioned in the previous section, a primary misconception in the Area of a Triangle Task was confusion of the concept of the height and how the area formula works. Eva said she would have Students B, C, D, and E distinguish height from a side length with a variety of examples. Providing a variety of examples of triangles was her instructional strategy to avoid students’ misconceptions. Joy indicated that Student C and E need to know that the lengths of two shorter sides in a triangle are not always its base and height. She responded that, in her instruction, she would use a variety of different triangles in order to show that the height of a triangle is always perpendicular to a base but a base can be any side of a triangle. Liz proposed that she would ask Students B, C, E, and F to find the height of a triangle, but she did not elaborate on how she would do that in her instruction. May suggested having Students B, C, and E revisit the concepts of the height and base of a triangle, which may be different from side lengths. In her instructional strategy, she would have students label the height and base of
examples of triangles and compare side lengths as opposed to the base and height of triangles. Pam mentioned explaining the definition of the height to Student A. Eva, Joy, and May suggested dealing with several examples, which looked routine or procedural. However, understanding the formal meaning of the concept of the height is a prerequisite of their activities, and students should remember their understanding of the height whenever they find the height of a triangle.

In the Area of a Triangles Task, incorrect knowledge of the concept of the height influences students’ understanding of the area formula of triangles. With regard to this aspect, the participants pointed out students’ conception of the area of a triangle. Eva responded by saying she would discuss the meaning of the area with Student F. Joy indicated she would show Students A, B, E, F and G how area can be measured with different bases or how to find the area of acute and obtuse triangles and that she would tell Student D that a triangle is not a half of any rectangle. Her suggestion for Student D may come from incorrectly embedding the obtuse triangle into a rectangle because the area of the triangle was not half of the rectangle that the student drew. She proposed those points as her instructional strategy. Pam pointed out that she would explain that the area formula came from the area of a right triangle in order to help Student C. Although their strategies were slightly different in their ways to present the content, the participants attempted to suggest the formal aspects of the concepts of the height and the area of triangles.

For the Task of Articulating the Relationship Between Area and Perimeter, Eva proposed having Student A, B, C, and D discuss area, perimeter, and their differences. Even though she did not explicitly write what the discussion would be and what the differences between area and perimeter are, her idea seemed to come from emphasizing formal aspects of the concept of area.
and perimeter such as their definitions and features. Joy’s strategy was more concrete. She suggested having Student A explore the perimeter of rectangular shapes with a fixed area, having Students B and C explore areas of rectangular shapes with a fixed perimeter, giving Student C a counterexample to allow her to realize the statement did not work for all rectangles, and having Student D discuss the definitions of area and perimeter. She combined prescriptions for those students as her instructional strategy. In order to make the exploration easy, she suggested using the software package Geometers Sketchpad (GSP) and making tables of measurements. Liz proposed having students draw many different rectangles including a counterexample of the statement. May indicated that she would allow Students A, B, and D to explore the areas of rectangles with a constant perimeter. As her instructional strategy, she recommended allowing students to make charts to compare the perimeters and areas. Pam proposed providing Students A, B, C, and D with counterexamples, and this method was her instructional strategy. Sam suggested showing students three ways to increase perimeter and its effects on area: (1) when increasing both side lengths, the area would increase: (2) when increasing one side length and keeping other side lengths the same, the area would increase: and (3) when increasing one side length but decreasing other side lengths, the area can increase or decrease. The participants were aware that using counterexamples and exploring rectangles with a constant area or a constant perimeter are efficient ways to eliminate students’ misconceptions.

For the Task of Construction of Triangles and Angle Properties, Eva wrote that she would have Students A and E discuss the definition of an acute angle and Students C and D learn the differences among right triangles, isosceles triangles, acute triangles, obtuse triangles, and so on. Although she did not clearly state what the differences are, she appeared to be attempting to teach formal aspects of those figures such as their definitions and mathematical properties. As an
instructional strategy, she planned to allow students to investigate similarities and differences between side lengths and interior angles and the sum of those interior angles of a variety of triangles. Joy stated that she would have Students A and E describe and compare properties of various triangles. As an instructional strategy, she proposed having students know the properties of triangles including the property of the sum of interior angles in a triangle. Liz pointed out that she would have Student D learn definitions and properties of triangles, but she did not explicitly mention what the definitions and properties are and what the kinds of triangles would be. As an instructional strategy, she referred to the sum of angles property. May indicated that Student A needed to discuss the definitions of acute, right, and obtuse angles and their differences, Student D needed to learn how to classify the different types of angles, and Student E needed to discover differences between acute and obtuse angles. As an instructional strategy, she combined discussing differences among the different types of angles and exploring the angles with protractors. Pam’s way to help each student was to provide examples or counterexamples assuming that students knew the sum of angles property. Sam suggested teaching Students A and E the correct definitions of the different types of angles with examples and pictures, and providing Student C with counterexamples. Even though the participants did not always mention the sum of interior angles property as a way to correct each student’s misconceptions or as an instructional strategy, their responses implied they were aware of the property as important formal knowledge.

For the similar triangle task, Eva and Joy proposed a discussion about the mathematical definition of similarity as both a prescription for each student’s misconception and an instructional strategy. May suggested discussion of the criteria of similar triangles and the mathematical definition of similarity for Student A. Pam described how she would address part
of the criteria of similar triangles for Students A and C by showing examples of triangles with
different side lengths but equal corresponding angles. She thought that Student B needed to
know the mathematical definition of similarity. As an instructional strategy, she designed a
lesson in which students start with a variety of triangles including similar and nonsimilar
triangles, measure side lengths and angles with rulers and protractors, and then induce the
definition of similar figures. Her lesson may be problematic because a mathematical system
usually starts with the definition of a concept. Moreover, in order to accomplish her lesson,
students need to first know a criterion for similar triangles, but the mathematical definition of
similar shapes precedes the properties such as the criteria of similar triangles. Sam said she
would use discussion of the definition of similarity to address Student A’s misconception. For
this task, participants tended to emphasize the definition of similarity, whereas they focused less
on the criteria for similar triangles.

Mathematical Intuitive-PCK with Regard to Knowledge of Students’ Misconceptions

The mathematical intuitive-PCK refers to teachers’ awareness of students’ intuitive
tendencies (Tsamir & Tirosh, 2008). Students may directly accept mathematical concepts and
properties without the need for justification because they may think the concepts and properties
are self-evident. The concepts and properties that are intuitively accepted sometimes agree with
logically justifiable truths, but sometimes contradict mathematical truths. When the contradiction
occurs, students may encounter epistemological obstacles and make mistakes in their problem
solving. Therefore, teachers should be aware of students’ intuitive thinking that may lead them to
have misconceptions.

The Task of Articulating the Relationship Between Area and Perimeter included
students’ responses that suggested they were using their intuitive knowledge. Among them,
Student D’s reasoning was overgeneralized based on his intuition because he lengthened both dimensions of a rectangle to make a rectangle with a longer perimeter than the previous rectangle. It is evident that if we increase both dimensions of a rectangle both perimeter and area also increase. This may be the intuitive thinking that many students have. However, Student D was not aware of the possibility to lengthen the perimeter of a rectangle but decrease its area. For this particular student’s work, all participants recognized Student D’s error. Eva, Liz, May, and Pam attributed his misconception to his belief that there is a linear relationship between area and perimeter, whereas Joy and Sam attributed his misconception to the perception that we need to increase both length and width of a rectangle in order to increase its perimeter. Both misconceptions were reasonable, but the former misconception was broader than the latter. Student D may have the intuitive knowledge that perimeter is linearly related to area in rectangles, which may be due to his intuitive knowledge that both length and width should increase in order to lengthen perimeter. Thus, the latter thinking was more fundamental in this sense.

The Volumes of Rectangular Prisms Task included students’ intuitive thinking about the volume formula of a rectangular prism and the relationship between the formula and dimensions of a rectangular prism. Student A squared all dimensions to make a rectangular prism with the volume 4 times bigger than the given prism. Student B multiplied all dimensions by 4 to make the volume 4 times bigger. Both students dealt with ALL dimensions, that is, length, width, and height. They may think that they should do the same thing to all dimensions because we need to make a length 4 times longer by multiplying its length by 4. They may generalize this conception for one-dimensional to two-dimensional concepts such as area and three-dimensional concepts such as volume without justification. Thus, they may consider that treating all dimensions at the
same time with the same factor was necessary in order to make larger shapes. For those students’ responses, all participants were aware of students’ errors in treating all dimensions simultaneously. May, Pam, and Sam confirmed their incorrectness by computing the volumes of the original rectangular prism and the prisms with dimensions given by Students A and B. May stated that she initially thought like Student B, but she realized that the approach was not correct by calculating the volumes of the two prisms. This implied that preservice teachers’ errors that result from intuitive thinking can be resolved by algorithmic and formal knowledge and they depended more on the algorithmic and formal aspects of knowledge than on the intuitive aspect of knowledge.

The Task of Construction of Triangles and Angle Properties included a variety of students’ intuitive images of triangles. Student A mentioned that a triangle can have only 2 acute angles instead of 3. Student C said that every triangle has at least one right angle. Student E said that a triangle cannot have 3 acute angles because a triangle has only three sides. Those students did not provide any reason for their statements. Additionally, no students’ drawing included any particular angle measurements, which implied their drawings relied on their intuitive thinking about shapes of triangles. All participants that completed this task were able to identify these students’ intuitive images of triangles. They attempted to explain incorrect answers that resulted from students’ intuitive thinking by using formal knowledge such as the definition of acute, right, and obtuse angles and particular examples of triangles rather than examining each intuitive image of triangle. For example, Eva, Joy, May, and Sam assessed the incorrectness of Student A’s statement that a triangle can have only 2 acute angles and attributed this misconception to his insufficient acquisition of the definitions of acute, obtuse, and right angles. The participants may
interpret this misconception as the student having an intuitive image of triangles and consider its cause as his lack of understanding of formal knowledge.

Mathematical Intuitive-PCK with Regard to Instructional Strategies to Correct and Avoid Students’ Misconceptions

Understanding formal aspects of mathematical concepts is sometimes difficult, but an intuitive way of teaching often helps students understand those concepts. For example, it may be difficult for 5th grade students to prove that the angles opposite to the sides of the same length in an isosceles triangle are equal in a rigorous way. However, students can easily understand this property by using paper-folding; if they fold a paper isosceles triangle along the perpendicular bisector of its base, they can see that the angles are completely overlapped. Although such an intuitive approach cannot be a rigorous mathematical proof or justification, it often contributes to students’ better understanding.

For the Area of a Triangle Task, Pam’s approach used an intuitive aspect. She planned to have Student F cut pieces of paper to fit into the two triangles and see that the areas of the two triangles were not equal. As an instructional strategy, she suggested a similar method. She would allow students to take different shapes of pieces of paper and place them into the two triangles; then, they can see that the area is not the same because one triangle would use less paper than the other. Although she needed to give more explanation about how she would have students compare the amount of paper that was used to fill each triangle, her instructional plan can give students an opportunity to challenge their concepts of areas without knowing the area formula or the concept of the height of a triangle.

For the Similar Triangles Task, May and Sam proposed an intuitive instructional strategy. May had students manipulate triangles with GSP and zoom in and out in order to show similar
triangles. Sam suggested using the overhead projector to show an image and trace it on the board, and then move the projector forward and backward. When the projector is moved, students can see the properties of similar figures because the sizes of the images change but the corresponding angles are equal, and the proportions of corresponding sides are constant. These activities would enhance students’ understanding of similarity in mathematical sense.

**Preservice Teachers’ Knowledge with Regard to Causes of Misconceptions**

It is important for teachers to infer and understand the causes of errors that students make and misconceptions that students have because it allows teachers to make plans to help students correct their errors and misconceptions. Teachers’ instructional plans should consider why students make a particular error and where their error comes from in order to help students avoid the error. For this reason, I asked the participants to infer plausible causes underlying students’ errors and misconceptions. Radatz categorized students’ errors into five categories, and I analyzed students’ solutions with the categories in the previous section. Because participants in this study did not read the Radatz framework, their responses about identifying students’ causes of misconceptions did not exactly correspond with the five categories. Moreover, the participants’ responses were often brief, which made it hard to assign each response to a category.

Broadly speaking, the participants attributed students’ misconceptions to internal and external factors, whereas Radatz concentrated on students’ internal factors. Internal factor are causes that happen inside students, including their thinking, ability, experience, knowledge, and so on, whereas external factor are causes that occur outside students’ control, including the influence of teachers, standardized tests, students’ learning environment, and so on.
Internal Factor: Language Difficulties

With regard to internal factors, some participants pointed out students’ language difficulties, which were aligned with the first category from Radatz’s framework. In the Task about Articulating the Relationship Between Area and Perimeter, Pam attributed Student A’s misconception to the vague meaning of “larger,” noting that “They may developed [sic] this misunderstanding because they see length or thickness as being ‘larger,’ where that must be defined first.” Her inference of the cause of the misconception may be based on Student A’s statement about “making the object larger” because it was not clear what the student intended to make larger. Did the student intend to make the perimeter of the object larger, the area of the object larger, or the size of the object larger? In this sense, Pam may claim that the word “larger” needs to be defined. However, consider the student’s statement, “you cant [sic] increase the outer region, perimeter” and “when the object is larger so is the area.” Because the problem stated “a larger perimeter” and “a larger area,” the student may have used the word “larger” in a manner consistent with the problem statement and simply omitted “of perimeter” after the word “object.”

In the similar triangle task, Eva, Joy, May, Pam, and Sam pointed out this cause of misconceptions. Although the preservice teachers identified this misconception in the work of different students, all of them indicated that the students confused the mathematical meaning of similarity with everyday meaning of similarity.

Internal Factor: Difficulties in Obtaining Spatial Information

The participants did not identify errors based on obtaining spatial information although this category of causes of errors and misconceptions was noted in the literature as being common in geometry. In the Area of a Triangle Task, Liz mentioned that students may just memorize formulas. Although she did not say spatial information, she may consider that memorizing
formulas implies the students may not understand spatial information underlying the formulas such as height and base.

In articulating the relationship between area and formula task, Joy indicated that Student D did not understand the concepts of perimeter and area. It is difficult to know what she meant by “concepts,” but it is plausible that she considered that the concepts include both figural information, such as drawings, and formal information, such as definitions and formulas. Thus, if a student does not understand the concept, in Joy’s mind it may mean that he had difficulty with spatial information contained in the drawings.

In the Task of Constructing Triangles and Their Angle Properties, Pam pointed out Student A “did not do a calculation with the degree and so he just looked at an inaccurate drawing.” She may have meant that Student A did not relate angle measures to the drawing, which implies that the student did not acquire enough spatial information.

In the similar triangle task, Joy indicated that Student A made a conclusion based on the appearance of the triangles. Focusing on their appearance implies that Student A did not pay attention to the formal information about similar triangles that is necessary to solve the problem.

In the Volumes of Rectangular Prisms Task, Pam attributed the students’ misconceptions to their confusion of the conceptions of dimensions. She indicated how squaring and cubing affect surface area and volume and how multiplying dimensions affects side lengths and volumes. She stated that multiplying dimensions is related to one-dimension and volume is three-dimensional concept. Her inference implied the students lacked spatial understanding of the concepts of side length and volume.
Internal Factor: Deficient Mastery of Prerequisite Skills, Facts, and Concepts

Another category of internal factors that cause misconceptions is students’ limited experience and lack of prerequisite knowledge. In the Area of a Triangle Task, Eva pointed out that students may lack exposure to a variety of types of triangles. Similarly, May inferred that the students might not understand that triangles did not have to be right triangles. Even though Tia exhibited her own misconception about the area of a triangle, she mentioned that the students may only be comfortable with right triangles. Those participants’ suggested that students’ experience with dealing with triangles was restricted, which resulted in errors and misconceptions.

In the Task of Articulating the Relationship Between Area and Perimeter, participants attributed errors and misconceptions to students’ lack of experience or deficient mastery of prerequisite knowledge. For Student A’s misconceptions, Joy pointed out that the student’s experience may be limited to seeing examples where the area gets larger as the perimeter gets longer rather than working with different rectangular shapes with a constant area. Sam described a plausible scenario about how the student built the misconception. She claimed that Student A tried to relate the problem to the changes seen in everyday life such as an activity with Play-Doh. He may have played with Play-Doh, making the perimeter of a rectangular shape bigger and seeing its area increase as well, as long as the shape increased equally on all sides.

For Student B’s misconceptions, Joy referred to three types of limited experience that the student may have had: (a) not having various visualizations of changes in perimeter and area of a rectangle, (b) working with manipulatives or visualizations of rectangles of only a few sizes, and (c) not seeing what happens to the perimeter and area of a rectangle when we increase or decrease its length and width. Her first statement about the causes of the misconceptions can be
interpreted as follows: Student B may have experiences where the only way to increase the perimeter of a rectangle was by adding to both the length and width. This limited experience of visualization may have led his insufficient solution. Liz stated that the student’s misconception could have come from a lack of exposure to a variety of different rectangles.

Joy attributed Student C’s misconceptions to reasons (a) and (c) given above by Joy. Liz responded that the misconceptions may be formed through students’ experiences with area and perimeter, but she did not describe the kinds of experiences with area and perimeter she had in mind. Liz also attributed the misconceptions to the student’s lack of exposure to a variety of rectangles. May referred to Student C’s limited experience of testing similar rectangles rather than comparing different widths and lengths.

Liz suggested that Student D’s misconceptions were caused by the student’s lack of exposure to a variety of different shapes of rectangles and his experience with rectangles that supported the statement given in the problem. May pointed out Student C’s limited experience with similar rectangles. For Student E’s misconceptions, Joy referred to the student’s insufficient experience with rectangles such as working with rectangles with a constant perimeter. Overall, participants’ responses attributed students’ misconceptions to insufficient experiences with rectangles that would allow them to understand and investigate the relationship between area and perimeter.

In the Task about the Construction of Triangles and Their Angle Properties, Eva mentioned that Student A confused the definitions of acute and obtuse angles, which implied his lack of prerequisite knowledge of the definitions of those angles. Joy indicated that Student A’s thinking that a triangle cannot have more than 2 acute angles may be because he did not know much about the properties of triangles, but she did not describe what the properties were. Liz
attributed the misconception to Student A’s experience, but she did not mentioned what kind of experience reinforced the student’s misconception. May stated that Student A did not understand how to classify those angles, which implied that he believed the student lacked knowledge of the three types of angles.

Although Student B answered correctly, Liz made the same statement as she made about Student A—that the student lacked experience. Student B’s original answer was that Alicia’s triangle is not correct, but Liz indicated the part “where she says that Alicia is correct” was incorrect, which seemed that she misread Student B’s answer. She may not read Student B’s answer carefully, or pay much attention to his drawing for Alicia’s triangle. This may lead Liz to believe that Student B had the misconception of what an obtuse angle looks like and this misconception resulted from his experience.

For Student C’s misconceptions, Eva pointed out the student’s limited experience from seeing only right triangle examples. Joy indicated both the student’s limited experience and deficient mastery of prerequisite knowledge, namely: (a) the student did not have much experience working with a variety of types of triangles, and (b) the student did not know the property that all angles of a triangle add up to 180 degrees. Liz stated that the student’s experience with triangles may be mostly with right triangles.

Eva attributed Student D’s misconception to his experience of seeing pictures of only equilateral triangles. Joy inferred that Student D’s misconception may come from not knowing correct definitions of different types of angles, which implied that the student had deficient understanding of prerequisite knowledge. Liz referred to Student D’s experience without explaining what kind of experience it was. May attributed Student D’s misconception to his lack of understanding of the concept of right, acute, and obtuse angles, which were required to solve
the problem. Pam stated that Student D did not carefully consider a combination of three angle measures that makes 180 degrees, and he may be confused about the concept of obtuse angles. Her inference implied that Student D did not fully understand the prerequisite knowledge of the interior angle sum property of triangles.

For Student E’s misconceptions, Joy stated that the student did not have experience comparing triangles and their properties with other shapes. It sounded like she was claiming that Student E lacked understanding of prerequisite knowledge, but it was not clear what she meant by properties and other shapes. Liz attributed Student E’s misconceptions to misunderstanding the concept of an acute angle.

In the similar triangle task, participants attributed students’ misconceptions to their lack of experience and insufficient understanding of prerequisite knowledge such as not knowing the criteria for similar triangles. Joy indicated that Student A may not have grasped the concepts of scale factor and proportions, which were prerequisite concepts to solve the problem. For Student B, Joy pointed out his lack of experience with similar or non-similar shapes; the student may have dealt with only similar shapes. To explain Student C’s misconceptions, May referred to his insufficient understanding of the criteria for similar triangles: “The student may have heard that the angles matter when dealing with similarity but did not realize that all three angles must be the same to have similar triangles.” Pam also indicated Student C’s lack of knowledge of the criteria of similar triangles as a reason for the error: “He quickly sees that one angle is the same so begins to think they are similar. He then looks for other ‘things’ that are the same like dots, but he should not compare this because lengths can be different.”

In the Volumes of Rectangular Prisms Task, Eva attributed the misconception to students’ confusion of volumes and area. She stated that the students “are increasing the length
of it or the different dimensions, but they’re not actually increasing the volume of all of the stuff inside of it.” Her statement implied that the students did not understand the concepts of lengths and volumes that are necessary knowledge to solve the problem. May inferred that the students may not understand what the volume is and the volume formula.

**Internal Factor: Incorrect Associations or Rigidity of Thinking**

This category of the cause of misconceptions was not frequently used by the participants. The Area of a Triangle Task was the only task that led some participants to suggest incorrect associations as the cause of misconceptions. Liz stated that the way to find the area of rectangles played a role in developing students’ misconceptions. She implied that students may associate the way to find the area of triangles to that of rectangles according to their learning trajectory. They may start with finding the area of right triangles by taking a half of the area of the associated rectangle. Because of this experience, students may attempt to use this method with non-right triangles. Pam also indicated that students did not understand the relationships between right triangles and rectangles.

**Internal Factor: Application of Irrelevant Rules or Strategies**

This category of the cause of misconceptions was not explicitly mentioned by the participants. In the Volumes of Rectangular Prisms Task, Joy stated that Student (1) squared all dimensions and Student (2) multiplied all dimensions by 4, which were to apply an incorrect rule because they should multiply only one dimension of a given rectangular prism by 4. With Joy’s perspective, Student (1) and (2) used irrelevant strategies. Sam indicated that the students may believe the figures should keep the same shape like similar shapes; “misconception about how to scale something to make it bigger that it has to go by the same amount.” Her indication implied that the students applied the strategy that they used to similar shapes.
External Factors

Radatz did not include factors external to students in her framework, but my participants stated that students’ misconceptions were developed by teachers or other external factor such as a standardized test. In the Area of a Triangle Task, Joy mentioned that the teacher may have only shown students how to find the area of right triangles rather than giving them a variety of triangle area problems. Liz referred to bad teachers as a cause of students’ misconceptions but did not elaborate on what she meant by “bad teachers.” In the Task of Articulating the Relationship Between Area and Perimeter, Sam attributed Student B, C and D’s misconceptions to teachers demonstrating changing the figure by increasing both side lengths. In the Task of Construction of Triangles and Their Angle Properties, May inferred that Student C’s teacher may have always draw right triangles as examples, and Student E’s teacher may have always drawn the same triangles when the teacher introduced concepts. Similarly, Sam indicated that Student C’s teacher may have tended to only draw right triangles as examples of triangles. In the Similar Triangles Task, Liz mentioned that teachers reinforced students’ misconceptions, but she did not explain how the teachers may play a role in developing students’ misconceptions. For this task, Pam pointed that standardized tests could cause Student B’s misconception by having him/her practice with typical shapes.

Results and Interpretation of Pretest and Posttest

Because there was not the sufficient number of participants to conduct a statistical analysis, I did not employ any statistical test. Rather than looking at the mean and standard deviation of the participants’ scores, I considered pretest and posttest scores of each participant in order to identify how each participant’s performance changed. Although items on the pretest
and posttest were different, each pair of items was parallel; that is, the topic of the items was slightly different, but the format and the purpose of items were the same for each pair.

Overall, participants’ performances improved (See Table 1). Although there was individual variation in their improvement, all participants’ total scores in the posttest were higher than those in the pretest. For areas other than Type IV, pedagogical content knowledge (PCK), some participants’ performances improved, whereas the others’ performances did not. However, there were no participants whose scores in Type IV got worse. When each pair of items were considered, there was no participant whose score on an item in the pretest got worse than that of its parallel item in the posttest for Type IV knowledge.

In the pretest, the first item about PCK asked about the concept of similar triangles and required participants to identify a student’s misconception about overgeneralizing similarity because of specific cases and to provide a way to help the student understand errors in his thinking. Only Eva clearly mentioned the angle criterion of similar triangles and suggested an appropriate way to address the student’s misconception. All participants, except for Eva, referred to side lengths, or the same side lengths, rather than the proportion of lengths of corresponding sides, which is an important criterion of similar triangles. Additionally, Liz said that an isosceles triangle has all equal angles, not sides, which suggested that she confused an isosceles triangle and equilateral triangle at that moment. Pam pointed out angles, but she did not specify how the angles should be configured in similar triangles. Based on the participants’ responses on the pretest, they did not have a solid foundation for the concept of similar triangles and they seemed to confuse the criteria for similar triangles and the criteria for congruent triangles because many of them mentioned the same side lengths.
Table 1

Results of Pretest and Posttest

<table>
<thead>
<tr>
<th>Knowl-edge</th>
<th>Total (40)</th>
<th>Type I (10)</th>
<th>Type II (10)</th>
<th>Type III (10)</th>
<th>Type IV (10)</th>
</tr>
</thead>
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<td>Post</td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
</tr>
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<td>Eva</td>
<td>25</td>
<td>33</td>
<td>8</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>Joy</td>
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<td>6</td>
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<td>6</td>
</tr>
<tr>
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<td>19</td>
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<td>4</td>
<td>5</td>
</tr>
<tr>
<td>May</td>
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<td>28</td>
<td>5</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Pam</td>
<td>25</td>
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<td>6</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Sam</td>
<td>28</td>
<td>29</td>
<td>9</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

*Note. The types of knowledge were categorized by the developer (I = memorized or factual knowledge, II = conceptual understanding, III = reasoning or problem solving, IV = pedagogical content knowledge). The numbers in the parentheses are maximum possible scores.*

In the posttest, the item changed the concept into congruent triangles. All participants except for Joy got full credit for this item. Joy did not clearly indicate why given triangles are not congruent even though her suggestion to help the student correct the misconception was appropriate. Although the concept of congruence seemed easier to the participants than the concept of similarity, I expected that their studying about those two concepts in their courses would affect the expansion of their knowledge by clarifying and elaborating the definitions and properties of congruence and similarity and by having them experience activities related to those concepts.

The second item aimed at identifying a student’s limited thinking in identifying cross-sections and describing how to help the student with the difficulty. The participants’ performances on this item were the best among the five items that assessed pedagogical content knowledge. In the pretest, May and Sam failed to explain the student’s limited thinking and how to help the student. May mentioned that the student saw the 3D object as a 2D object but did not explicitly refer to the concept of cross sections. Also, her way to help the student was talking about cross sections with a pyramid, which was ambiguous. Sam wrote that the student did not
think of cutting through all of the faces of the solid, but she did not indicate what was wrong in
the student’s answer. Joy, Liz, and Pam suggested using clay or Play-Doh in their instruction.

In the posttest, all participants got full credit on the parallel item. All of them clearly
indicated what the student’s limited thinking was and provided appropriate instructional method.
Many of the participants said they would use a clay or Play-Doh activity in their instruction to
show what a cross section looks like. This may be because they did a Play-Doh activity for
learning cross sectioning in their content course. Perhaps the reason that many participants
performed well on this item in both tests was that they had the opportunity to learn this concept
before taking the pretest. Although the pretest was conducted at the beginning of the semester,
the participants had already attended one or two class sessions, and they experienced a Play-Doh
activity in the first class session of their content course. This activity appeared to make an
impression on the participants because they often mentioned and used Play-Doh in their
responses.

The third item was about addressing a student’s misconception about naming edges,
vertices, and faces and suggesting how to help the student correct the misconception. In the
pretest, Pam was the only person who clearly gave both the student’s misconception and a way
to help the student correct his misconception. The other participants, Eva, Joy, Liz, and May
recognized the student’s confusion, but they did not sufficiently suggest how to help the student.
Sam wrote an equation, \[ \frac{\# \text{edges} + \# \text{faces}}{2} = \# \text{vertices} \], as a pattern that she saw. However, her
equation did not work for all shapes given in the problem.

In the posttest, May and Pam got full credit on the parallel item, but the others did not.
Eva and Joy’s responses did not show memorized/factual knowledge (Type I) or conceptual
understanding (Type II) in the pretest, whereas their responses were improved in the posttest in
terms of these types of knowledge. Liz did not provide a way to help the student correct misconceptions in either test. Interestingly, Sam also gave an equation,

\[ \# \text{ edges} = \frac{\# \text{ faces} + \# \text{ vertices}}{2} - 1, \]

as a pattern that she observed. Unlike the equation in the pretest, her equation in the posttest worked for all cases given in the problem. In both tests, the participants’ performance was not good. Additionally, the scores of memorized/factual or conceptual knowledge (Type I or Type II) in the posttest did not increase significantly, which may be because their content and methods courses did not explicitly deal with this topic.

The fourth item had participants address a student’s misconception about over-generalizing lines of symmetry based on particular examples and describe how to help the student correct the misconception. In the pretest, Sam was the only person who got full credit. Eva, Joy, Liz, and May’s responses were scored as inappropriate with respect to memorized/factual knowledge (Type I) and conceptual understanding (Type II), and they got partial credit in the pretest. Comparing those participants’ answers with Sam’s answer, those participants stated that they would deal with a variety of examples, whereas Sam suggested that she would provide a counterexample to refute the statement’s incorrectness.

In the posttest, Sam answered appropriately, and Eva and Pam provided improved answers. However, Joy, Liz, and May’s levels of performance were similar to their performances on the pretest. Because the student’s misconception came from his overgeneralization, that is, applying the conclusion from several special examples to a general case, an efficient way to perturb the student’s thinking is to show a counterexample. However, Joy, Liz, and May did not seem to be aware of this aspect of the problem. Interestingly, Joy suggested the same way as that of the pretest to address the student’s misconception; showing the Venn-diagram about relationships among shapes. Although the participants had some knowledge of symmetry, this
topic was not explicitly covered in the participants’ content and methods courses. This may explain their low performance on this item.

The fifth item aimed at proposing instructional strategies to address and correct the misconception about estimating the area of a shape. In the pretest, Joy was the only person who gave an appropriate answer. Pam was on the right track, but she did not explicitly mention the student’s misconception. May’s answer was about a general case rather than specific to the situation of this item, and Eva and Sam did not give an instructional activity.

In the posttest, Joy appropriately answered like in the pretest. Additionally, Pam and Sam provided improved answers. Eva and May’s responses were also improved, but their responses were not sufficient to get the full credit. Liz got only one point in the area of Type I and II knowledge in the pretest, whereas she got only one point in the area of Type III and IV knowledge in the posttest. In the posttest, all participants recognized that the perimeter of a shape cannot determine its area, which was a key point to explain the student’s misconception. Their content course covered the topic of the relationship between area and perimeter of shapes, and one of the KCS tasks in this study was about investigating students’ misconception of the relationship between area and perimeter. I expected that the participants’ answers would be influenced by these experiences when they took the posttest.

Although all items offered concrete contexts, the participants tended to give general responses. Their answers were often broad rather than focused on a particular example given in the problem. In particular, they talked about the way to help a student correct his misconception in a general way. For example, one item was about suggesting an activity to address the student’s misconception that shapes with the same perimeter have the same area. One participant proposed having the student discover the difference between perimeter and area, which was too broad to
explain the particular misconception that the student had. Even though the item gave a particular
example of a shape, she did not use the shape.

An interesting result of the tests was that the participants’ scores on the multiple choice
part of the posttest slightly decreased. This result was obvious for memorized/factual knowledge
(Type I). Of the five items, participants’ posttest scores were lower on four items although the
participants’ courses explicitly dealt with the topics of those items. However, when they were
asked to provide reasoning or explanation, their scores were improved in the area of
memorized/factual or conceptual knowledge as well as the reasoning, problem solving, and
pedagogical content knowledge. I believe that this result may be because their content and
methods courses emphasized the aspects of conceptual understanding rather than procedural
understanding. They may have become familiar with showing their thinking and reasoning
during taking their courses.

Discussion

Features of Preservice Teachers’ SMK: Strong SMK

Overall, the participants had strong SMK for the five topics of the area of a triangle, the
relationship between area and perimeter, the three types of angles and the sum of angles of
triangles, the similar triangles, and the volume of a rectangular prism. Except in a few cases, the
participants were able to solve the problems and to identify important mathematical ideas to
solve each problem. Among the three components of mathematical knowledge, the participants’
ability to use the algorithmic and formal aspects was remarkable. The participants attempted to
solve the problems by applying formulas or formal properties rather than depending on their
intuitive thinking. They also suggested mathematical formulas or properties as important ideas.
The participants sometimes mentioned a formula or property without explaining what it was or
why it worked, but this may not be because that they did not understand the formula or property. I believe that it is desirable for the preservice teachers to have well-established algorithmic and formal mathematical knowledge because taking an intuitive approach may result in misconceptions even though intuitive knowledge can help students understand concepts. When students’ misconceptions happen because of their intuitive thinking, we can help students correct their misconceptions by taking algorithmic and formal approaches. When the participants’ knowledge was intuitive, their intuitive constraints did not conflict with the formal constraints. In the Area of a Triangle Task, some participants assumed that the height of the second triangle was 2 cm. Although it was not restricted to 2 cm, considering the height as 2 cm did not cause any conflict. Based on their performance on the KCS tasks, therefore, the participants showed that they had strong SMK.

*Features of Preservice Teachers’ SMK: Difficulty Changing Deeply Rooted Incorrect SMK*

Because each task was given to the participants after they learned a relevant topic in their content course, I expected that the participants would demonstrate accurate mathematical knowledge. If they had erroneous knowledge before learning the topic, I expected that they would correct their errors by participating in the content course. Additionally, I expected that they would be able to realize their incorrectness by looking at students’ wrong solutions. However, this sometimes did not occur; the preservice teachers may have adhered to their incorrect SMK. In the Task of the Area of a Triangle, Tia’s knowledge of the area formula of a triangle was partly incorrect. She multiplied the length of a base by its adjacent side length rather than by the length of its corresponding height. Although the content course dealt with finding the corresponding height when picking a base of a triangle, her concept of height did not change after the instruction, which may have affected her understanding of the area formula of triangles.
Additionally, students’ solutions did not influence her incorrect conception. In the similar triangle task, Liz’ incorrect knowledge of similar triangles did not change after performing the task. Although the content course dealt with the criteria for similar triangles and gave preservice teachers the opportunity to find the scale factors of similar shapes, her concepts of similar triangles and the scale factors were not accurate. She also judged the correctness of students’ solutions based on her knowledge rather than changing her mind after reading students’ solutions. Therefore, those cases demonstrated that it was difficult to correct deeply rooted SMK that the preservice teachers established.

*Features of Preservice Teachers’ PCK: Focusing on Correctness*

The participants tended to focus on the correctness of students’ solutions. If students formed a correct conclusion, the participants tended to decide there was no misconception in students’ solutions. In the Task of Articulating the Relationship Between Area and Perimeter, Student E concluded that perimeter can decrease as area increases, but his reasoning included a critical error. Some participants judged that there was no misconception in the student’s answer. Those participants did not closely examine the student’s reasoning but only concentrated on his conclusion. When a student’s solution was correct, the participants tended not to make a deep investigation in to the student’s work. In the Task about the Construction of Triangles and the Angle Properties, Student B’s solution was on the right track, but his reasoning for Alicia’s triangle was not sufficient to show why it was impossible. Some participants did not consider how to make the student’s reasoning convincing by applying the property of the sum of interior angles in a triangle. Therefore, this tendency of focusing on correctness sometimes prohibited the participants from deeply understanding students’ thinking.
Features of Preservice Teachers’ PCK: Not Considering Details

Some participants tended not to pay attention to small differences between students’ solutions. This feature was similar to the previous feature, focusing on correctness. Because they considered that the small differences were not serious, they often responded that students’ misconceptions were the same as those of previous students. In the Area of a Triangle Task, a participant thought that Student D’s misconception was the same as that of Student C and E. Another participant responded the misconceptions of all students except for Student A were the same. These participants did not pay attention to Student D’s incorrect use of the base of the second triangle and Student E’s incorrect use of the area formula of triangles. If teachers ignore details, they may miss a serious misconception that their students have. It is also difficult to plan instruction that is responsive to students’ misconceptions and errors if the misconceptions and errors are not investigated in detail.

Features of Preservice Teachers’ PCK: Relationship between SMK and PCK

Although researchers have attempted to discriminate PCK from SMK, SMK may affect PCK or vice versa. Tia’s incorrect SMK of the area of a triangle or the concept of the height of a triangle served as a standard of judgment when she identified students’ errors and misconceptions. Thus, she rated incorrect solutions as correct and correct solutions as wrong. This case shows how a teacher’s SMK affects her PCK and implies that it is important for teachers to establish accurate SMK.

Features of Preservice Teachers’ PCK: Less Focusing on Intuitive Aspects of Mathematical Knowledge in Instructional Strategies

Intuitions can facilitate students’ understanding in the instructional process when they accord with logically justifiable truths. However, teachers need to keep in mind that
epistemological obstacles may occur when students’ intuitive thinking contradicts the truths (Fischbein, 1994). In the parts of the tasks that asked preservice teachers to suggest ways to correct students’ misconceptions and propose instructional strategies to avoid students’ misconceptions, the participants tended to focus more on algorithmic and formal aspects of mathematical knowledge than on intuitive aspects. Because many students’ misconceptions came from insufficient understanding of mathematical algorithms, formulas, definitions, properties, theorems, and proofs, or from incompatible intuition with them, the participants’ approach was appropriate. For students’ misconceptions, the participants suggested telling students mathematically correct definitions of the height of a triangle, acute, right, obtuse angles, and similar shapes, the area formula of triangles, the volume formula of rectangular prisms, the properties of the area and the perimeter of rectangle, and the theorem of the sum of interior angles in a triangle into students’ minds.

There were three cases where the participants attempted to introduce the concepts in an intuitive way. In the Area of a Triangle Task, a participant proposed showing that the areas of two triangles differed by cutting two triangles out of paper and comparing the sizes of the two paper triangles. Her suggestion was not enough to convince students of the difference in the areas because she did not explain how to compare their sizes. However, her approach may help students who have difficulty understand formal aspects of the areas of triangles. In the similar triangle task, two participants attempted to explain the concept of similar shapes in intuitive ways. One referred to using the GSP to introduce the concept of similar shapes by drawing a triangle and having students zoom in and out. The other suggested using the overhead projector to show an image with the overhead projector, trace the image on the board, and move the
overhead projector back and forth. These two intuitive approaches can help students understand that similar figures keep the same shapes but their sizes are different.

*Features of Preservice Teachers’ PCK: (In)consistency of Diagnosis and Prescription*

The KCS tasks aimed at giving preservice teachers opportunities to diagnose students’ errors and misconceptions and prescribe ways to correct and avoid the errors and misconceptions. I expected that preservice teachers would be able to propose different methods to help students who had different errors and misconceptions. I expected that the participants’ responses about students’ misconceptions, plausible causes of misconceptions, and instructional strategies to correct the misconceptions would be consistent. Overall the participants’ thought processes were consistent, but sometimes they were not. The participants who attributed the misconception about confusing the height and side length of a triangle to limited experience dealing with only right triangles prescribed providing students with a variety of examples of triangles, including acute and obtuse triangles. When participants attributed students’ misconception about a linear relationship between area and perimeter of rectangles to their limited experience of seeing only a few rectangles, the prescription was to provide a variety of rectangles, including counterexamples. Participants who attributed the misconception about similarity to students’ confusions between the mathematical definition of similarity and the everyday meaning of similarity prescribed having students distinguish between and discuss the mathematical use and everyday use of similarity. Those were examples of consistent diagnosis and prescription.

On the other hand, a participant (Sam) who identified the same misconception and cause of confusing a side of a triangle and its height for the area task prescribed showing students the areas by counting the blocks. This solution was not compatible in that her prescription was
appropriate to correct misconceptions about areas rather than misconceptions about heights. A participant (Liz) pointed out lack of awareness of the difference between perimeter and area as a plausible cause of the misconception that area increases as perimeter of a rectangle increases. However, she did not discuss how to correct this misconception. A participant (Eva) referred to a student’s misconception that every triangle has a right angle and attributed it to the student’s experience of seeing only right triangle examples. However, the student drew non-right triangles in his solution, which implied that the participant’s diagnosis was not compatible with the evidence presented. Although the participants attempted to answer consistently in general, those examples showed inconsistency from diagnosis to prescription.

*Features of Preservice Teachers’ PCK: Weak Application of Knowledge to Instruction*

When I prepared the KCS tasks, I expected that the participants would apply their knowledge from a variety of sources. In particular, I expected that they would apply what they learned in their courses to suggest instructional strategies because their courses attempted to introduce concepts with various methods including hands-on activities, manipulatives, and computer software such as GSP. The participants sometimes suggested using materials that they experienced in their courses, such as Play-Doh and GSP. Some participants planned to employ activities that they did in the courses including making tables of areas of rectangles with a constant perimeter, the overhead projector activity for similar triangles, and counting unit cubes to find the volume of rectangular prisms. However, these cases were not common. Many participants tended to propose discussing or showing the concepts for which students exhibited confusion. For example, they often mentioned discussing what the height of a triangle is, showing how to find the area of a triangle, teaching the area formula, discussing differences between area and perimeter, showing counterexamples or examples, discussing or showing three
types of angles, having students measure angles, and discussing the mathematical definition of similarity. In their class, the participants had experienced four strategies for finding the area of a shape and their application to a triangle, the activity of investigating the areas of rectangles with a fixed perimeter, the walking and turning activity, the proof of the sum of interior angles in a triangle by using parallel lines, the investigation of the volume formula of a rectangular prism by using unit cubes and counting the number of unit cubes, and so on. However, the participants’ responses about how they would help correct students’ misconceptions and errors did not include these activities. This may imply that the participants did not fully understand activities that they experienced in the courses or that they had difficulty applying their knowledge to their teaching plan.

Features of Preservice Teachers’ Interpretation of Causes of Misconceptions: Confusion of Causes of Misconceptions with Misconceptions

Participants sometimes described misconceptions or explained what students did rather than stating plausible causes of the misconceptions. For example, in the Area of a Triangle Task, Liz referred to misunderstanding the difference between side lengths and heights, which can be considered a misconception that students had. In this study, participants were encouraged to think about why students did not know the difference between side lengths and heights.

In the Task of Articulating the Relationship Between Area and Perimeter, Eva indicated that Students A, B, and D forgot that the area can also decrease when perimeter increases. Her explanation was about what the students did not know rather than why they thought the area increases whenever perimeter increases. Eva also stated that Student C developed the misconception by considering only square rectangles, which was closer to a misconception that the student had than its cause. Moreover, she pointed out that Student C’s misconception was his
misunderstanding of rectangles and perimeter and area of rectangles, which did not seem to be compatible with the cause that Eva presented. Liz indicated that Student A and B’s misconceptions came from not knowing the difference between area and perimeter, but investigating whether or not there is relationship between area and perimeter was the purpose of the problem. Thus, not knowing the difference between area and perimeter was a negative result after performing the problem, not a cause for why the students believed the area increases as the perimeter increases. Liz mentioned that Student D’s misconception resulted from a belief that there is an ideal rectangle, and a skinny, wide rectangle is not a rectangle. Even though one could consider Liz’s claim as a cause of misconceptions, this study aimed at deeply thinking about why and how the student established the belief, which Liz did not do. Like Liz, May stated that the cause of Student A and D’s misconceptions was misunderstanding the relationship between area and perimeter. May also indicated that Student B’s misconception was developed by testing several examples, which seemed to describe what the student did, not why he did it. May needed to think about why the student tested only a few rectangles and why the student used particular rectangles to test. Pam also suggested that Students B’s and E’s misconceptions arose from thinking that there was a linear relationship between area and perimeter. She explained that Students C and D found examples that satisfied their conclusion, which was a description of what the students did rather than a cause of their misconceptions.

For the Task of Constructing of Triangles and the Angle Properties, Pam stated that “not every triangle is a right triangle” for Student C’s misconception. The purpose of her statement was to correct the student’s erroneous claim rather than to infer plausible causes of why the student believed every triangle has at least one right angle. These participants’ responses suggest that they sometimes have difficulty providing plausible causes of students’ misconceptions. This
may be because the students’ solutions were too short to figure out what the students knew or did not know. Additionally, because they were in a preservice teacher program, the participants may not have had enough experience teaching students to have the opportunity to investigate students’ thinking and to examine students’ problem solving practices. In other words, they lacked a repertoire of pedagogical content knowledge from which to draw in analyzing the student work that was presented to them.

*Features of Preservice Teachers’ Interpretation of Causes of Misconceptions: Narrow Perspectives*

I wanted the participants to suggest as many plausible causes of misconceptions as possible because this can be an opportunity for preservice teachers’ to broaden their knowledge of students’ thinking. However, most participants referred to only one or two causes. Even when participants suggested several causes, their responses were sometimes repetitive. In a few cases participants articulated the similar but not identical misconceptions and causes for students’ solutions. For example, although Student C and D’s solutions were different from Student A’s solution to the Task of Articulating the Relationship Between Area and Perimeter in that Students C and D drew rectangles, Eva considered their misconceptions and causes the same. Participants sometimes did not detect slight differences among students’ solutions.

*Features of Preservice Teachers’ Interpretation of Causes of Misconceptions: Brief Descriptions*

Sometimes participants’ descriptions of plausible causes were quite brief. For example, Liz wrote that students’ previous experience reinforced their misconceptions, but she did not explain what she meant by *experience*. The word *experience* has broad meaning, so it was not possible for me to infer what she meant. The task of identifying plausible causes of misconceptions was intended to help participants as future teachers think about how to design
instruction to avoid students’ misconceptions. However, in order to encourage students to correct and avoid misconceptions, the word “experience” needs to be specified.

**Preservice Teachers’ Opinions of Courses**

Most of participants agreed that doing activities and application problems in their content and methods courses helped them enhance their knowledge for future teaching. They believed that the activities such as the walking and turning activity for the sum of interior angles in a triangle and the Play-Doh activity for cross sections of three dimensional shapes were good examples that can be applied in their mathematics classes. Some participants pointed out that the approach allowed them to understand geometrical content more deeply than what students learn in the middle school and saw that as a beneficial aspect of the content course. The participants expected to learn practical applications, teaching methods, and technology such as the GSP in their content and methods courses because they wanted to directly apply what they learned to their future teaching.

Although neither the content nor the methods course instructor spent much time having the preservice teachers investigate students’ mathematical thinking, the instructor of the content course sometimes started lessons with examples of students’ common misconceptions of a particular topic. Some participants stated that these introductory activities allowed them to become familiar with students’ thinking and served to stimulate brainstorming on the topic. The content course instructor often attempted to introduce several ways to explain a geometrical concept. For example, the preservice teachers proved the Pythagorean Theorem in several ways. A participant (Pam) indicated that knowing different ways to explain a particular topic helped her consider alternative ways when a student did not understand the concept in a particular way.
Preservice Teachers’ Opinions of the KCS Tasks

When the participants inferred students’ misconceptions, they pretended to be middle school students. They looked back on their personal experience and remembered their previous knowledge. When the participants answered the questions for the KCS tasks, in particular when they proposed instructional ways to correct and avoid misconceptions, they sometimes applied their knowledge from the content course. Most participants agreed that these tasks of investigating students’ misconceptions and suggesting instructional strategies to correct misconceptions encouraged them to broaden their perspectives. A participant explicitly indicated that she applied the students’ misconceptions from the KCS tasks to her homework and tests in the content course because considering misconceptions allowed her to think about how to explain the relevant mathematical concepts. Some participants stated that they had difficulty inferring misconceptions or plausible causes of misconceptions and suggesting different activities to correct a misconception.

The participants’ preference between written and oral formats for the KCS tasks depended on their tendencies. If participants would rather talk than write, they preferred the oral format. If participants would rather carefully think and write than talk, they preferred to write their responses to organize their thinking in their mind. Thus, I did not find systematic differences in participants’ responses as a result of the administration format.
CHAPTER 5

CONCLUSIONS

Summary

While they are in a teacher preparation program, preservice teachers need to improve various knowledge and skills to prepare them for effective teaching practice in the future. An important component of knowledge for teaching is to understand students’ thinking and misconceptions. Thus, preservice teachers need to develop their knowledge of students through taking courses, field experience, and extra activities that are related to the preparation of teaching practice. With this study I aimed at investigating the nature of preservice teachers’ knowledge of content and students (KCS) in five geometrical topics. In particular, I intended to answer the following research questions:

1. What kinds of errors do middle grades students make and what kinds of misconceptions do they exhibit with regard to particular geometry topics?

2. How do preservice middle grades teachers interpret students’ work in geometry, and how do they apply their knowledge to the interpretation?

3. How is preservice middle grades teachers’ pedagogical content knowledge influenced by their course taking and engaging with tasks that assess their knowledge of content and students?

To address these research questions, I prepared five KCS tasks regarding five geometric topics: the area of a triangle, the relationship between area of perimeter, the concepts of three types of angles and the angle sum property of triangles, similar triangles, and the volume of a
rectangular prism. Each task included three to six students’ solutions, and each solution had incorrect parts. The participants were asked to respond to seven parts:

- Solving the same problem that students were given in order to identify whether or not participants could provide the correct answer to the problem,
- Identifying important mathematical ideas that the student might use to be able to successfully perform the item,
- Examining whether the student’s work was correct or not,
- Identifying underlying mathematical misconception(s) or misunderstanding(s) that might lead the students to the error presented,
- Inferring plausible causes of the misconception(s) or misunderstanding(s),
- Suggesting ways to help each student to have him recognize and correct his errors and misconceptions, and
- Making a plan of instructional strategies and/or tasks to use during the next instructional period to address and avoid the students’ misconception(s).

The first two questions were used to identify the preservice teachers’ subject matter knowledge (SMK), and the others were use to investigate their pedagogical content knowledge (PCK). I distributed each task to the participants after they learned the relevant topics in their teacher education courses and gathered their written responses 1 or 2 weeks after. The fifth task was implemented as an oral interview in order to compare the participants’ written performance with their oral performance.

I administered a pretest before giving the participants the first task and a posttest after they completed the fifth task. I used a portion of the Diagnostic Mathematics Assessments for Middle School Teachers (DMAMST), which was developed by the University of Louisville
Center for Research in Mathematics and Science Teacher Development (Center for Research in Mathematics and Science Teacher Development, 2005). This set of assessments was appropriate for this study because the purpose of DMAMST was to describe the extent of mathematics content knowledge, the effects of specific experience on teacher knowledge, or relationships among content knowledge that teachers have, teaching practice, and students’ performance. I intended to examine preservice teachers’ SMK and PCK, to identify changes in preservice teachers’ knowledge, and to verify some influence of the tasks that were used in this study and the content and methods courses.

In addition to the KCS tasks and the pretest and posttest, I interviewed all participants and observed the content and methods courses while collecting my data. In the interviews I asked four different types of questions:

- Questions about personal information of the participants such as the participants’ background in terms of course taking, field experience, and familiarity with middle school students’ thinking,
- Questions about the participants’ opinions of their content and methods courses in order to hear how the participants thought the courses influenced their knowledge and future teaching,
- Questions about the participants’ opinions of the tasks such as how the participants performed the tasks, their opinions about the strengths and weaknesses of the tasks, and their opinions about the influence of the tasks on their knowledge and teaching, and
- The fifth task with the same questions as the other tasks.

I also observed each class period of the participants’ content course and methods course on geometry and measurement. The purpose of my observations was to explore the preservice
teachers’ learning trajectory in geometry and to connect their geometry learning to the tasks in this study. I took field notes during the observation and used them as a gauge of an ideal level of the participants’ knowledge of content and pedagogy.

The KCS tasks were analyzed by applying two different frameworks: Radatz’s (1979) error analysis and the Shulman-Fischbein framework (Tsamir & Tirosh, 2008). The students’ solutions from the KCS tasks were analyzed using Radatz’s (1979) categorizations of sources of errors: language difficulties; difficulties in obtaining spatial information; deficient mastery of prerequisite skills, facts, and concepts; incorrect associations or rigidity of thinking; and the application of irrelevant rules or strategies. Because all topics of the KCS tasks came from geometry, all misconceptions were in some way relevant to difficulties in obtaining spatial information. And misconceptions from difficulties in obtaining spatial information were related to the notion of figural concepts described by Fischbein (1993). Geometric figures are images and concepts simultaneously because they have figural constraints as well as formal constraints. When figural constraints are liberated from control of formal components, students can experience epistemological obstacles, that is, misconceptions.

The preservice teachers’ interpretations of students’ solutions were analyzed using two dimensions of knowledge—subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Each dimension was divided into three domains of algorithmic, formal, and intuitive aspects (Tsamir & Tirosh, 2008). The analysis of preservice teachers’ SMK revealed that the preservice teachers had strong knowledge of content in geometry based on their responses to the questions that asked them to solve the problems from the KCS tasks and to identify important mathematical ideas needed to solve the problems. They were well aware of geometrical formulas such as the area of a triangle and the volume of a rectangular prism as a procedure. They well
understood formal concepts of the area of a triangle, the concept of heights, no relationship between area and perimeter of rectangles, the three types of angles, the interior angle sum property of triangles, similar triangles, and the volume of rectangular prisms. The analysis also indicated that it was sometimes hard to change deeply rooted SMK. The preservice teachers had the same misconceptions as students. Even after receiving instruction about these concepts, some preservice teachers did not correct these misconceptions.

The analysis of the preservice teachers’ PCK showed that preservice teachers sometimes tended to focus on whether or not a student’s final answer or conclusion was correct and to ignore details in students’ solutions. Because their PCK was influenced by their SMK, preservice teachers who had erroneous SMK tended to evaluate students’ solutions incorrectly. The preservice teachers paid little attention to intuitive aspects of mathematical knowledge, perhaps because they considered algorithmic and formal aspects of knowledge more important. It is important for teachers to recognize intuitive aspects of knowledge because although intuitive aspects can help students understand a difficult mathematical concept, intuitive thinking can also cause misconceptions. Although preservice teachers often diagnosed students’ misconceptions and prescribed ways to correct the misconceptions coherently, this ability needs to be emphasized further in teacher education courses. If the diagnosis of a misconception is not consistent with its prescription, the misconception may not be eliminated. Preservice teachers’ PCK with regard to instructional strategies was quite narrow although the content course introduced a variety of ways to deal with a particular concept including hands-on activities, manipulatives, and technologies as well as discussion and lecture. They often relied on direct discussion of topics in order to have students correct and avoid their misconceptions.
### Table 2

**Summary of Analysis of Preservice Teachers’ Knowledge**

<table>
<thead>
<tr>
<th>Component of Mathematical Knowledge</th>
<th>SMK</th>
<th>PCK</th>
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<tbody>
<tr>
<td><strong>Algorithmic</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1. Mathematical algorithmic-SMK</td>
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<tr>
<td></td>
<td>● Teachers’ knowledge of solving procedures and supporting them by explicit justifications</td>
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<tr>
<td></td>
<td>● Teachers’ routine use of formulas and showing computations in a procedural way</td>
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</tr>
<tr>
<td></td>
<td>✓ Able to apply the area of a triangle formula (Eva, Joy, May, Pam, Sam, and Tia)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>✓ Able to compute areas and perimeters of rectangles (All)</td>
<td></td>
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<tr>
<td></td>
<td>✓ Able to apply the formula of the volumes of rectangular prisms (Joy, Liz, May, Pam, and Sam)</td>
<td></td>
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<td></td>
<td>4. Mathematical algorithmic-PCK</td>
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<tr>
<td></td>
<td>● Teachers’ knowledge of the most common incorrect algorithms that students apply when solving mathematical tasks and their possible sources</td>
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<td></td>
<td>● Teachers’ awareness of students’ incorrect use of geometrical formulas</td>
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<td></td>
<td>✓ Knowing students’ incorrect application of the area formula of a triangle (Liz and Tia)</td>
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<tr>
<td></td>
<td>✓ Aware of students’ incorrect computation of areas and perimeters of rectangles (Joy and Sam)</td>
<td></td>
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<tr>
<td></td>
<td>✓ Aware of students’ incorrect association of squaring or quadrupling all dimensions to make the volume of a rectangular prism 4 times larger (Eva, Joy, May, and Sam)</td>
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<tr>
<td></td>
<td>● Teachers’ knowledge of algorithmic representations, illustrations, examples, and explanations, and demonstrations</td>
<td></td>
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<tr>
<td></td>
<td>✓ Teachers’ ability to teach the area formula of triangles (Joy)</td>
<td></td>
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<td></td>
<td>✓ Teachers’ ability to teach the types of angles by having students measure angles (Liz, May, and Pam)</td>
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<td></td>
<td>✓ Teachers’ ability to have students recognize their misconception by computing the volumes of rectangular prisms (Sam)</td>
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<td></td>
<td>2. Mathematical formal-SMK</td>
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<tr>
<td></td>
<td>● Teachers’ knowledge of the core principles of the discipline of mathematics</td>
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<td></td>
<td>● Teachers’ abilities to use geometrical definitions, properties, and justifications</td>
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<td></td>
<td>✓ Understanding the concept of the height of a triangle (Eva, Joy, Liz, May, Pam, and Sam)</td>
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<td></td>
<td>✓ Able to refute a claim by providing a</td>
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<td></td>
<td>5. Mathematical formal-PCK</td>
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<td></td>
<td>● Teachers’ knowledge of students’ common formal-related errors</td>
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<tr>
<td></td>
<td>✓ Teachers’ awareness of students’ incorrect understanding of the concept of the height of a triangle (Eva, Joy, Liz, May, Pam, and Sam)</td>
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<tr>
<td></td>
<td>✓ Teachers’ ability to judge whether or not students’ examples are appropriate (Joy and Sam)</td>
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<tr>
<td>Intuitive</td>
<td>counterexample (Eva, Joy, Pam, and Sam)</td>
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</tr>
<tr>
<td>✓</td>
<td>Understanding the concepts of area and perimeter and their differences (Eva, Joy, and May)</td>
<td></td>
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<tr>
<td>✓</td>
<td>Understanding of the concepts of acute, right, obtuse angles (All)</td>
<td></td>
</tr>
<tr>
<td>✓</td>
<td>Knowing that interior angles of a triangle add up to 180 degrees (All)</td>
<td></td>
</tr>
<tr>
<td>✓</td>
<td>Understanding of the criteria of similar triangles (Joy, May, Pam, and Sam)</td>
<td></td>
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</tbody>
</table>

- Teachers’ awareness of students’ misunderstanding of the definitions of acute, right, obtuse angles (All)
- Teachers’ awareness of students’ incorrect understanding of similar triangles (Eva, Joy, May, Pam, and Sam)

- Teachers’ ideas about how to teach axioms, definitions, theorems, and proofs in a variety of ways
- Teachers’ ability to teach the concepts of the height and area by using examples (Eva, Joy, Liz, May, and Pam)
- Teachers’ ability to teach the area and perimeter of rectangles by discussion and by exploring areas of rectangles with a fixed perimeter (Joy, May, and Sam)
- Teachers’ ability to teach three types of angles by discussion of their definitions (Eva, Joy, Liz, May, and Sam)
- Teachers’ ability to teach the concept of similarity by discussion of mathematical definition and by using examples (Eva, Joy, May, Pam, and Sam)

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<tr>
<td>● Teachers’ mathematical knowledge that is directly accepted without the feeling that any kind of justification is required</td>
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<tr>
<td>● Self-evident</td>
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<tr>
<td>✓ Dependence on the figures (Eva, Joy, and Sam)</td>
<td></td>
</tr>
<tr>
<td>✓ Assuming that the area increases as the perimeter increases, or vice versa (Liz)</td>
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</table>

- Teachers’ awareness of students’ intuitive tendency
- Teachers’ recognizing that students tend to lengthened both dimensions of a rectangle to make a rectangle with a longer perimeter than a given rectangle (Joy and Sam)
- Teachers’ awareness of students’ dependence on figures (All)
- Teachers’ awareness that students tend to deal with all dimensions to make the volume of rectangular prisms 4 times larger (All)

- Teachers’ ability to teach content by using an intuitive way
- To have students recognize different areas by comparing paper triangles (Pam)
- To teach similar shapes by using zoom-in and –out of GSP or by using the overhead projector (May and Sam)
The preservice teachers’ inferences of causes of misconceptions also showed that they had a narrow perspective. They referred to only one or two causes with brief descriptions. They often attributed students’ misconceptions to a deficient mastery of prerequisite skills, facts, and concepts. This is a convincing cause of students’ misconceptions, but the misconceptions can be interpreted with different perspectives. Because students’ solutions in the KCS tasks were brief, the preservice teachers may have had difficulty identifying the causes. However, I believe that teachers need to consider students’ misconceptions from a variety of perspectives because students’ misconceptions may be intertwined, and the same misconceptions may result from different causes.

The preservice teachers’ interpretation of students’ work was often based on their learning from the content course. For example, when a student confused the height of a triangle with a side length, the preservice teachers pointed out the incorrectness by referring to the meaning of the height and how to find the area of a triangle, which were topics with which the content course dealt. They often depended on their mathematical knowledge from the content course when they assessed students’ solutions and suggested instructional plans to correct and avoid students’ misconceptions. This implied that the content course affected preservice teachers’ PCK as well as SMK.

The pretest and posttest results showed that the participants’ knowledge was improved. In particular, their PCK at the end of the semester was better than at the beginning of the semester. These results implied that the preservice teachers’ mathematical knowledge for teaching was affected by their course taking and other activities that were relevant to teaching such as the KCS tasks.
Conclusions and Implications

I identified four features of the nature of preservice teachers’ SMK and PCK with regard to geometry. First, the causes of students’ errors and misconceptions were multidimensional, whereas preservice teachers’ knowledge in this domain was not. Although Radatz categorized the causes of errors in an exclusive way, I attempted to analyze them by applying different categories and I believed this way was reasonable. I interpreted students’ errors and misconceptions and identified multiple potential causes of these errors and misconceptions. This analysis served as a basis for developing instructional prescriptions for resolving or preventing misconceptions because it is difficult to find a solution to a problem without knowing its cause. If we know the cause of a misconception, we can plan to correct the misconception. Therefore, clarifying misconceptions and their causes is significant and needs to be examined in follow-up studies. However, the preservice teachers superficially described the causes and their descriptions were brief. This may be due to preservice teachers’ insufficient experience working with students’ thinking. Additionally, the preservice teachers in this study did not have opportunities to learn cognitive models of children’s mathematical thinking and learning. Encouraging preservice teachers to study those models would help them understand students’ thinking and misconceptions.

Second, the preservice teachers had strong SMK, whereas they had relatively weak PCK. Their strong SMK was revealed by their performance on solving the problems and identifying important mathematical ideas needed to solve the problems. Most participants responded to these questions appropriately. However, their performance on questions about PCK was weaker than on those about SMK. The preservice teachers sometimes ignored small differences between students’ solutions, neglected errors when students’ conclusions were correct, and revealed
narrow application of their knowledge from the content course to instructional strategies. This implies that content courses need to cooperate closely with methods courses. Some participants stated that they expected to learn how to teach mathematical concepts throughout the methods courses. Thus, cooperation between instructors of content and methods courses would promote preservice teachers’ PCK as well as SMK simultaneously.

Third, I employed the framework of algorithmic, formal, and intuitive aspects of mathematical knowledge with regard to SMK and PCK. The analysis exhibited that participants had appropriate algorithmic and formal components of geometrical knowledge, whereas they rarely showed their mathematical knowledge in an intuitive way. It is not desirable to rank the importance of those three aspects of knowledge because each aspect is important. It is good that teachers have appropriate algorithmic and formal knowledge, but we should not neglect the intuitive aspect. Intuition can help promote learning in that self-evident features can be naturally accepted by children without justification. However, teachers should be careful in using intuitive thinking because it is possible to result in epistemological obstacles when intuition contradicts with logically justifiable truths. Therefore, preservice teachers need to consider how to balance those three aspects of mathematical knowledge in instruction.

Fourth, although this study did not prove the relationship between course taking and preservice teachers’ knowledge in a statistical way, their use of knowledge from the content course in completing the tasks indicated that preservice teachers’ knowledge for teaching is affected by their course taking. This implies that content and methods courses in teacher preparation program are responsible for improving preservice teachers’ mathematical knowledge. As a way to promote preservice teachers’ knowledge of students’ thinking and misconceptions, I suggest providing opportunities for preservice teachers to evaluate students’
work as was done in this study with the KCS tasks. The participants agreed that looking at students’ actual solutions was helpful to broaden their understanding of students. Thus, content and methods courses need to diversify activities in order to promote preservice teachers’ SMK and PCK.

While the preservice teachers in this study had taken content and methods courses, their experience in making sense of students’ thinking and misconceptions was limited. The results of this study suggest that preservice teachers need to examine students’ written work as soon as they begin the teacher education program. Although teachers can accumulate their knowledge of students as their teaching career lengthens, without explicit attention to students’ mathematical thinking in their preservice preparation program, they will not have a basis for making sense of students’ thinking during their first year of teaching.

**Limitations of the Study**

This study has three limitations. The first limitation was the written format of the tasks. The second limitation was the participants’ limited experience with investigating students’ mathematical thinking. The third limitation was a lack of studies investigating students’ errors and misconceptions in geometry on which I could build.

The biggest limitation of this study was the format of the tasks. Four of the KCS tasks were conducted in a written take-home format. Although I used this format because I intended to provide enough time for the preservice teachers to thoughtfully consider their answers, the format sometimes functioned as restriction. Because participants sometimes described their ideas only briefly, I would have gotten more and better data if I could have asked follow-up questions in an interview setting. For example, a participant said that a student’s previous experience reinforced his misconception. Because she did not describe what the experience was, I needed to
ask what she meant by the term “experience” in order to comprehend her response. I attempted to ask her to clarify this term during the interview at the end of the semester, but the participant did not remember the specific response because of the elapse of time. If I had conducted clinical interviews, I could have immediately asked why the participants answered and thought as they did. However, the written format of tasks prohibited me from giving immediate feedback and interacting with the participants, which restricted my understanding of the participants’ thinking and knowledge.

The participants in this study were a group of preservice teachers. Although I intended to provide preservice teachers with an opportunity to enhance their knowledge of students’ thinking, the tasks that I asked the participants to complete required them to infer how students developed their thinking and why their misconceptions occurred. The preservice teachers who volunteered for this study had no practical experience of interacting with children in geometry. They had limited field experience at this point in their teacher education program, and their experiences were not in mathematics classrooms, nor had they ever had experience with assessing students’ mathematical thinking. Thus, the participants performed these tasks as if they were middle school students by looking back on their middle school mathematics experiences. The limited experiences of the participants resulted in narrow perspectives and brief descriptions in their responses.

At the beginning of designing this study, I tried to search for studies that investigated students’ errors and misconceptions in geometry. There were several studies on students’ errors and misconceptions in arithmetic or algebra (Brown & Vanlehn, 1982; Matz, 1980; Resnick, Leonard, Magone, Omanson, &Peled, 1989), but it was hard to find a study in geometry. Because of that, I had to create tasks for the preservice teachers using student work that was not
verified by research that had already been published. The two sources of students’ work that I employed in this study did not publish any analysis of students’ errors and misconceptions, which is why I began the results section by analyzing students’ work.

**Suggestions for Further Studies**

At the stage of designing the KCS tasks, I became aware that there was little research on students’ thinking and misconceptions in geometry. Although van Hiele established the theory of the levels of geometric thinking (Burger & Shaughnessy, 1986; Mayberry, 1983), studies on van Hiele’s theory have focused more on developmental stages in students’ geometrical thinking than students’ errors and misconceptions of particular geometric topics. Moreover, there were studies on students’ misconceptions in arithmetic or algebra but not geometry. Hiebert and Carpenter (1992) introduced and summarized those studies on students’ misconceptions in the area of algebra and science education. Because investigating students’ thinking and misconceptions was not the primary purpose of this study, I focused less on analyzing students’ thinking and misconceptions. However, I believe that research on students’ geometrical thinking and misconceptions can promote follow-up studies such as the influence of using students’ thinking on teachers’ pedagogical knowledge.

The participants in this study were preservice teachers. Because of preservice teachers’ lack of experience interacting with students, their knowledge of students was restricted. I assume that in-service teachers’ knowledge of content and students would be different from that of preservice teachers. In-service teachers have much experience with interacting students and have developed their own teaching know-how, which may result in differences between preservice and in-service teachers’ knowledge. Thus, I believe that comparing features of preservice teachers’ knowledge of content and students with those of in-service teachers would be an
interesting topic for future research. Additionally, comparing the knowledge of these two groups of teachers with an emphasis on positive aspects of in-service teachers’ knowledge would help to establish guidelines about the knowledge that preservice teachers need to establish.

Watching a video of students’ performance and investigating students’ thinking and misconceptions is a good way to improve teachers’ knowledge of students. Videos can help make the study of students’ mathematical thinking come alive and show developmental sequences in learning. Additionally, videos can demonstrate how to assess students’ developmental stages with a particular analytical framework for students’ learning (NCTM, 1991). In this study, students’ written solutions sometimes were brief, which often prohibited the participants from assessing what the students really did not know. If students’ problem solving processes were videotaped, it might be easier to understand why and how the students solved the problems. Because of the lack of video data source of students’ work in geometry, however, I could not use this method. Therefore, I suggest that future research on teachers’ knowledge of content and students use videos of students’ problem solving performance with the same format as the KCS tasks.

**Concluding Remarks**

Knowing students as well as content is an essential requirement for effective teaching. When teachers make instructional decisions, their knowledge of content and students is important because they need to consider students’ learning pace. Additionally, teachers have some responsibility for students’ misconceptions because their instruction may affect the progress of students’ thinking. When misconceptions occur, teachers are responsible to recognize and correct the misconceptions. Thus, teachers need to develop PCK, including how to teach on the basis of students’ thinking, as well as SMK, such as what to teach. Preservice teachers can
develop both SMK and PCK in their content and methods courses in their teacher preparation program. The content and methods courses should consider needs from preservice teachers in order to enhance their knowledge. Because preservice teachers do not have sufficient experience interacting with students, those courses need to provide opportunities to assess students’ thinking and misconceptions so as to improve their knowledge of students. In this study I attempted to make sense of preservice teachers’ knowledge of content and students by having them complete KCS tasks and using their responses to investigate the nature of their knowledge. I cannot claim that the KCS task directly improved preservice teachers’ knowledge, but I believe the KCS tasks in cooperation with the content and methods courses played a positive role in expanding preservice teachers’ knowledge and experience.
REFERENCES


APPENDIX A

The KCS Tasks

Name: __________________________

Start Date: ____________________

Submit Date: ____________________

**Directions for completing the task:**
Please answer all questions in as much detail as possible. Show all your thoughts and opinions in responding to questions. This task is take-home, but do not discuss this task with others. Please submit your answers in a week. Thank you very much for your time.
• The Area of a Triangle
Anna has 2 corners of her yard that she could use for a garden, so she needs to decide which one has the larger area. Which one is bigger? How do you know?

![Image of two triangles](image)

Several students answered as follows:

<table>
<thead>
<tr>
<th>Student</th>
<th>Student’s Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>The first one is bigger because base times height divided by two gets you the answer.</td>
</tr>
<tr>
<td>B</td>
<td>They are the same because you do ((3 \times 6) \div 2 = A) and since you have a triangle, it is half a rectangle or some other that is why you divide it in half.</td>
</tr>
<tr>
<td>C</td>
<td>They are the same because you have to do (b \times h \times \frac{1}{2}) and (6 \times 3 \times \frac{1}{2} = 9).</td>
</tr>
</tbody>
</table>
You make the triangles into rectangles then divide it by 2.

We think the triangles are the same area but one of them is stretched out and one is its real size. We know this because they both have the area of 18. One side on each is 6 cm and the other is 3 cm.

It has more room and movement.
1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?
2. (a) What are the errors that those students are making?

<table>
<thead>
<tr>
<th>Student</th>
<th>Errors</th>
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<tbody>
<tr>
<td>A</td>
<td></td>
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<tr>
<td>B</td>
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<td>C</td>
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<td>F</td>
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<td>G</td>
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</tbody>
</table>
(b) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the error presented in this item?

<table>
<thead>
<tr>
<th>Student</th>
<th>Misconceptions or Misunderstandings</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
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<td>C</td>
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<td>G</td>
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</table>
3. How might the student have developed the misconception(s)?

4. (a) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

<table>
<thead>
<tr>
<th>Student</th>
<th>Ways to help a student correct errors and misconceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
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<td>B</td>
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<td>C</td>
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</table>
(b) What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?
• Articulating the Relationship between Area and Perimeter

**Rectangle I has a larger perimeter than rectangle II. Can you conclude that rectangle I also has a larger area than rectangle II? Why or Why not?**

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
Student A - “yes, b/c you can't increase the outer region, perimeter, w/o making the object larger and when the object is larger so is the area.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
• Student B - “yes! Because you are adding length and width which means there is more area.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
| (3) | How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s). |
| (4) | How can you help each student to have him/her recognize and correct his/her errors and misconceptions? |
Student D – “Sandra’s claim is correct, the perimeter of a rectangle increases, then the area of the rectangle also increases. So yes if you increase the number of the perimeter then the area of the rectangle will also increase.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
• Student E – “Sometimes, but sometimes not. / Perimeter increase as area increases. / Perimeter increases as area decreases OR perimeter decreases as area increases.”

(1) Is a student’s answer correct? If not, what part is incorrect?
(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).
(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?
Students in Ms. Smith’s class have drawn a variety of triangles. Here is what they drew:

Roberto: I made a triangle with 3 acute angles.
Sunnum: I made a triangle with 1 right angle.
Alicia: I made a triangle with 2 obtuse angles.
Raj: I made a triangle with no right angles.

Check each student’s work to see if it is correct. It may help to try drawing pictures.

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:

   (1) Is a student’s answer correct? If not, what part is incorrect?

   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
• Student A – “Roberto is wrong because you can only get 2 acute angles instead of 3.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
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<tr>
<td>(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).</td>
<td></td>
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<tr>
<td>(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?</td>
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</table>
• Student B – “Roberto is correct because his triangle can and does have 3 acute angles. / Sunnum is correct because his triangle can and does have a right angle. / Alicia is not correct because her triangle can have 2 obtuse angles and have it be a triangle. / Raj is correct because a triangle always has no right angle unless it is a right angle triangle.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
- Student C – “Raj’s triangle does not work because every triangle has at least one right angle. / Alicia’s triangle won’t work because her “triangle” won’t look like a triangle.”

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
Student D

Roberto: I made a triangle with 3 acute angles. Yes
Sunnum: I made a triangle with 1 right angle. No
Alicia: I made a triangle with 2 obtuse angles. Yes
Raj: I made a triangle with no right angles. Yes

Check each student's work to see if it is correct. It may help to try drawing pictures.

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
Student E – “Roberto is wrong because a triangle has only three sides and you cannot have 3 acute angles in a triangle. Sunnum is right. Alicia is wrong because there is no way for a triangle to have 2 obtuse angles because a triangle has only three sides. Raj is right.”

<table>
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<tr>
<th>(1)</th>
<th>Is a student’s answer correct? If not, what part is incorrect?</th>
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<th>(2)</th>
<th>What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?</th>
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</table>

| (3) | How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s). |
(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?
Part I: Solve the problem below.

Are triangles A and B similar triangles? Why or why not?
Part II: Answer the following questions.

1. What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

2. Here are several students’ responses. Discuss each student’s response with regard to:
   (1) Is a student’s answer correct? If not, what part is incorrect?
   (2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
   (3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).
   (4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
• **Student A** – No, because A triangle is more long than B triangle. B triangle is very short and very wide, A triangle is very thin and very small space.

<table>
<thead>
<tr>
<th>(1) Is a student’s answer correct? If not, what part is incorrect?</th>
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<tbody>
<tr>
<td><strong>Is</strong> a student’s answer <strong>correct</strong>? If not, what part is incorrect?</td>
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<tr>
<th>(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?</th>
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<tbody>
<tr>
<td>What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?</td>
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<th>(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).</th>
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<tr>
<td>How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).</td>
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</tbody>
</table>
(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

- **Student B** – Yes, because both A and B have the same number of dots and that’s 6, and they both are triangles and they both have straight lines or edges.

(1) Is a student’s answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
- **Student C** - Yes. Because bother have square corners. And also both could make a square and a rectangle if you just put a line like this:

![Diagram showing a square and a rectangle formed by adding a line to a shape with square corners.](image)

(1) Is a student's answer correct? If not, what part is incorrect?

(2) What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?
(3) How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

(4) How can you help each student to have him/her recognize and correct his/her errors and misconceptions?
3. What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?
The Volumes of Rectangular Prisms

Give the dimensions of a box that has four times the volume of the box below. Explain why your box has four times the volume of the given box.

(1)

My box has four times the volume because I squared the regular dimensions.

(2)

my box has four times the volume because I multiplied everything by 4.
What are some of the important mathematical ideas that the student might use to be able to successfully perform the item?

Is a student’s answer correct? If not, what part is incorrect?

What underlying mathematical misconception(s) or misunderstanding(s) might lead the students to the incorrect response presented in this item?

How might the student have developed the misconception(s) or misunderstanding(s)? That is, describe plausible causes of the misconception(s) or misunderstanding(s).

How can you help each student to have him/her recognize and correct his/her errors and misconceptions?

What instructional strategies and/or tasks would you use during the next instructional period to address the students’ misconception(s)? Why?
APPENDIX B

Interview Protocols

• **Personal background**
  1. How many mathematics courses have you taken? What courses?
  2. How many methods courses have you taken? What courses?
  3. Have you ever had field experience? When? How long? What was your role in the field experience? (What did you do during field experience?)
  4. How much are you familiar with middle school students’ thinking? Why do you think like that?

• **Opinions of the content or methods courses**
  1. What aspects of Dr. Beckmann’s class/ Dr. Oppong’s class would help your future teaching geometry?

  2. How did Dr. Beckmann’s class/ Dr. Oppong’s class affect your knowledge of students (learners)?

  3. What did you expect to learn in Dr. Beckmann’s class/ Dr. Oppong’s class that will contribute to your knowledge of content/ knowledge of students?

  4. What do you want to further learn about and discuss more in Dr. Beckmann’s class/ Dr. Oppong’s class? (What should be added? What should be eliminated?)
• **Opinions of the tasks**
  1. When you were doing the tasks, how did you do the tasks?
  
  2. How do you figure out students’ thinking and misconceptions?
  
  3. What do you think of the strengths and weaknesses of the tasks?
  
  4. After receiving feedback, was feedback helpful? Did feedback influence performing the next task? If yes, how did feedback affect your performance?
  
  5. How do the tasks influence your knowledge of students?
  
  6. When you were doing the tasks, was there anything interesting in terms of students’ mathematical thinking or misconceptions?
  
  7. What part in the tasks did you have difficulty answering?