THE PEDAGOGICAL CONTENT KNOWLEDGE OF TWO MIDDLE-SCHOOL MATHEMATICS TEACHERS

by

GOOYEON KIM

(Under the Direction of Jeremy Kilpatrick)

ABSTRACT

The purpose of this study was to investigate two middle-school mathematics teachers' pedagogical content knowledge in terms of how it was manifested in their classroom instruction and ultimately to identify its components. The study also explored the teachers' beliefs about mathematics, about teaching, and about learning mathematics and how those beliefs, as a backdrop, were related to their pedagogical content knowledge.

Two eighth-grade mathematics teachers participated in the study. Data were collected in the form of classroom observations, individual interviews, questionnaires, and documents. Data were analyzed using a strategy of case study and grounded theory methods.

Mr. Smith's pedagogical content knowledge included: (a) knowledge of mathematics, consisting of purposes of teaching mathematics, connections among

topics, concepts to teach, various ways of solving problems, and textbook knowledge; (b) knowledge of students' understanding, which involves particular students' understanding, students' errors and common misconceptions, and students' difficulties and confusions; and (c) knowledge of pedagogy as revealed in learning activities, attempts to motivate students, and realistic applications.

Ms. King's pedagogical content knowledge included: (a) knowledge of mathematics that involved purposes of teaching mathematics, understanding topics, and curricular knowledge or topic organization; (b) knowledge of students' understanding, consisting of students' learning styles, learning difficulties, and common errors and misconceptions; and (c) knowledge of pedagogy, which includes ways of representation, lesson planning and organization, and teaching strategies that involved designing learning activities, incorporating student presentations, using various teaching styles and realistic applications, and using textbooks and journals.

The two teachers had slightly different structures of pedagogical content knowledge. Mr. Smith was more dependent on his knowledge of mathematics and knowledge of students' understanding, and Ms. King on her knowledge of pedagogy and, to a lesser extent, knowledge of mathematics. Consequently, no single model fits the pedagogical content knowledge of both teachers, perhaps because they used their knowledge differently or focused it differently in teaching mathematics.

INDEX WORDS: Pedagogical Content Knowledge, Middle-Grade Mathematics Teachers, Teacher Beliefs, Teacher Knowledge, Mathematics Teaching

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B.S., Sogang University, Korea, 1991

M.Ed., Sogang University, Korea, 1995

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in

Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2004

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MATHEMATICS TEACHERS

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December 2004

DEDICATION

To my parents and God

ACKNOWLEDGEMENTS

My doctoral study could never have come to be without the people who held my hand along the way. I first would like to thank my major professor, Jeremy Kilpatrick, for his continuous support, patience, and inspiration. His insightful comments helped me to reflect on and evaluate my thinking so that I could produce a more solid product. I also thank Denise S. Mewborn for her encouragement, support, and insight. She always listened and gave me invaluable guidance throughout my doctoral process.

My gratitude goes to the members of my doctoral advisory committee, Shawn Glynn, Elliot Gootman, and Dorothy Y. White for giving freely of their time and expertise to help me make a worthwhile study. I am grateful for their guidance and encouragement throughout my doctoral studies.

I would like to thank the middle-school mathematics teachers for their willingness to open up themselves and their classrooms to me and allowed me to explore their knowledge and practice. I am grateful for the opportunity afforded by them to learn their knowledge and teaching practice.

My deep appreciation goes to my family and friends in Korea for their unconditional support and prayers throughout my doctoral studies. I could not have endured such a long and difficult time without their love and prayers. I also thank many new friends in Georgia who helped me with their prayers, warmth, and kindness.

Finally, I would like to thank God. Without the close relationship with Him, nothing would have been possible during the whole time. He restored my soul and guided me so I could make this happen.

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CHAPTER 1

THE PROBLEM AND ITS BACKGROUND

This study examined the pedagogical content knowledge of two mathematics teachers in an effort to conceptualize it in the contexts of teaching. Pedagogical content knowledge is "a unique kind of knowledge that intertwines content with aspects of teaching and learning" (Ball, Lubienski, & Mewborn, 2001, p. 448) and is commonly considered to be a transformation or offspring of two constituent knowledge domains: general pedagogical knowledge and subject matter knowledge (Gess-Newsom, 1999a; Marks, 1990a, 1990b). In introducing the term into the literature, Shulman (1986) argued that pedagogical content knowledge is specific to particular subject matter and that learning to teach requires not only understanding the subject itself but also developing a wide repertoire of pedagogical content knowledge. Ball et al. noted that the concept of pedagogical content knowledge in mathematics implies that teachers must not only know the content of mathematics in depth and conceptually and the connections among mathematical ideas and concepts, but also know the common representations of particular ideas or concepts for students. Marks (1990b) claimed, however, that pedagogical content knowledge is not a separate category of teacher knowledge; instead, it is inextricably tied to content knowledge.

Under any circumstances, teachers' knowledge helps teaching occur. What does the teaching of a mathematics teacher look like? What do teachers need to know in order to teach mathematics? How do they use their mathematical knowledge in teaching mathematics? These questions have led me to try to understand teachers' knowledge. Teaching, in a generic sense, refers to "action undertaken with the intention of bringing about learning in another. In this way, teaching is different from mere telling or showing how" (Robertson, 1985, p. 15). Teaching in mathematics classrooms is not composed only of lecturing or telling to present or pass on information (Raths, 1999). Rather, mathematics teaching is an integrated process that includes lecturing, discussing, questioning, responding by teachers and students (Gage, 1984). If teaching is different from mere telling or showing how, how does mathematics teachers' knowledge affect their teaching? When I began to work on this question, I had no doubt that a knowledge of mathematics must accompany mathematics teaching. I realized, however, that the knowledge to teach mathematics well includes more than knowing mathematics. Beyond having mathematical knowledge, teachers should be able to present mathematical knowledge to students in an understandable form.

Mathematics teachers' mathematical knowledge plays a key role in their teaching. It affects both what mathematics they teach and how they teach it (Ball, 1988, 1990, 1991, 2000; Ball & Bass, 2000; Ball & McDiarmid, 1990; Buchmann, 1984; Leinhardt & Smith, 1985; Ma, 1999; Shulman, 1987). As a student learning mathematics, I assumed that secondary teachers were experts in their subjects. That is, I had a tendency to consider mathematics teachers at the secondary level more or less as mathematicians. When I had trouble understanding the ways in which my mathematics teachers presented mathematical ideas or problems, I would blame myself for not being able to understand. Although teachers' mathematical knowledge influences how they represent the nature of knowing to their students (Grossman, 1995), I did not see that my teachers lacked any knowledge of how to teach mathematics. I still cannot tell whether they lacked general pedagogical knowledge or lacked what Shulman (1986) called pedagogical content knowledge, which comprises, as knowledge going beyond the knowledge of subject matter, "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

I believe that pedagogical content knowledge is essential for mathematics teachers to teach mathematics well. Pedagogical content knowledge in mathematics is much more than simple knowledge of mathematics (Grossman, 1990; Kahan, Cooper, & Bethea, 2003; Marks, 1990a, 1990b; Tamir, 1988). There have been relatively few studies, however, on the pedagogical content knowledge of mathematics teachers and how it might be characterized in their teaching practice. One reason for so few studies is the difficulties and ambiguities that, by its nature, pedagogical content knowledge contains

(Mewborn, 2000). Even though the concept of pedagogical content knowledge might be difficult to identify distinctly, such identification would be very useful because pedagogical content knowledge "represents a class of knowledge that is central to teachers' work and that would not typically be held by nonteaching subject matter experts or by teachers who know little of that subject" (Marks, 1990b, p. 9).

Pedagogical content knowledge strongly supports a perspective of teaching as a profession and teachers as professionals (Grossman, 1995). The teaching practices of mathematics teachers in the classroom are evidence of their pedagogical content knowledge. Nevertheless, we lack data that show how teachers' pedagogical content knowledge is manifested and how it can be characterized in classroom teaching. It is necessary to fill the gap in the literature by investigating the pedagogical content knowledge of mathematics teachers in the mathematics classroom. We can learn more about what constitutes mathematics teaching through the lens of pedagogical content knowledge. Also, revealing the characteristics of pedagogical content knowledge in teaching mathematics may help prospective teachers develop such knowledge during their teacher education programs and continue to elaborate it as practicing teachers in order to facilitate their students' understanding of mathematics.

Background

What is effective teaching? Teaching effectively is, in essence, teaching for understanding (Gess-Newsome, 1999b). As I claimed above, teachers' mathematical knowledge definitely affects their mathematics teaching. Knowing mathematics, even advanced mathematics, however, is not sufficient for teaching well. For their mathematics teaching to be effective, mathematics teachers should know and deeply understand mathematics, students as learners, and pedagogical strategies, as well as be capable of using that knowledge flexibly while teaching mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Mathematics teachers should also be continuously exposed to opportunities and resources in order to enhance and refresh their knowledge for effective teaching (NCTM, 2000).

What mathematics teachers know limits what is done in their classrooms and ultimately what their students learn. With an extensive knowledge of mathematical content, mathematics teachers can structure their mathematics teaching to encourage their students to continue to learn mathematics (Fennema & Franke, 1992). Without such knowledge, mathematics teachers cannot help their students appropriately.

Examining how mathematics teachers communicate the structure of the subject matter of mathematics is important because the ways in which students learn the subject matter shape their attitudes, values, and beliefs about learning. That is, students' learning mainly depends on the structure of the subject matter and the approach to teaching it (Shavelson & Stern, 1981). However, “very little attention has been paid to how knowledge of subject-matter is integrated into teachers’ instructional planning and the conduct of teaching” (p. 491).

Researchers need to examine the mathematical knowledge of teachers and how it might influence how they structure learning activities. To understand what a conceptual understanding of mathematics might be, one should consider the nature of mathematics itself as well as teachers' mental organization of their knowledge of mathematics (Fennema & Franke, 1992). According to Borko and Putnam (1996), many studies imply that

> teachers with greater subject matter knowledge tend to emphasize the conceptual, problem-solving, and inquiry aspects of their subject. Less knowledgeable teachers tend to emphasize fact, rules, and procedures and to stick closely to detailed lesson plans or the text, sometimes missing opportunities to focus on important ideas or connections among ideas. (p. 685)

Likewise, Ball and McDiarmid (1990) affirmed that knowledge of subject matter is an essential part of teacher knowledge: “If teaching entails helping others learn, then understanding what is to be taught is a central requirement of teaching” (p. 437). They also point out that prospective teachers’ lack of expertise and confidence in subject matter might be a serious issue in teacher education. It is, however, extremely important to recognize that “mathematical knowledge alone does not translate into better teaching” (Cooney, 1999, p. 166).

Shulman (1986) suggested that teachers’ understanding of their subject ought to be studied and that such understanding plays a key role in helping students develop their understanding of the subject. Subject matter knowledge is knowledge of the subject and has been referred to as content knowledge or substantive knowledge (Ball, 1991). Content knowledge or substantive knowledge refers to knowledge of the factual information and concepts in the field. In mathematics, this knowledge includes understanding of particular topics, procedures, and concepts, as well as the relationships among them.

Borko and Putnam (1996) argued that teachers ought to know more than just the facts, terms, and concepts of a subject. Teachers’ knowledge of the organizing ideas and knowledge growth within the subject is an important factor in how they will teach it. Teachers of mathematics must have deep and highly structured content knowledge so that they can retrieve it flexibly, efficiently, and effectively for their students (Sternberg & Horvath, 1995).

Further, effective mathematics teachers ought to be able to stimulate students to learn mathematics (NCTM, 1991) because “effective teaching conveys a belief that each student can and is expected to understand mathematics and that each will be supported in his or her efforts to accomplish this goal” (NCTM, 2000, p. 18).

Teachers’ beliefs also affect their teaching. Gess-Newsome (1999b) explained:

> Beliefs have both affective and evaluative functions, acting as information filters and impacting how knowledge is used, organized, and retrieved. Beliefs are also powerful predictors of behavior, in some cases reinforcing actions that are consistent with beliefs and in other cases allowing for belief compartmentalization, allowing for inconsistent behaviors to occur in different contexts. (p. 55)

Teachers have many untested presumptions that influence how they think about teaching and learning. Calderhead (1996) asserted that the assumptions teachers have about their students and how their students learn are likely to direct how they approach teaching tasks and how they interact with their students. Beliefs about teaching may be closely related to beliefs about learning and the subject itself. For instance, “if a mathematics teacher believes mathematics to be about the application of techniques, this may itself imply certain beliefs about how the subject is most appropriately taught and learned and what the role of the teacher should be” (p. 719).

Research with regard to teacher beliefs shows that student teachers' conceptions about learning to teach affect how they approach their professional learning and the aspects of their preservice education programs (Calderhead & Robson, 1991). Teachers tend to keep many beliefs about the learning and teaching of mathematics that they had as prospective teachers (Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Calderhead & Robson, 1991; Thompson, 1992).

Teachers' beliefs have a profound influence on all aspects of their teaching (Hashweh, 1987; Nespor, 1987). Moreover, teachers' beliefs about teaching and mathematics considerably influence their behavior because "the process of learning is fundamentally connected to how beliefs are structured, whether the beliefs are rooted in rationality or are the consequence of telling" (Cooney, 1999, p. 172). Mathematics teachers' beliefs and views about mathematics and its teaching, consciously or unconsciously, play a significant subtle role in shaping the teachers' characteristic patterns of instructional decisions (Cooney, 1985; Fennema & Franke, 1992; Ges-Newsom, 1999b; Thompson, 1984, 1992). As a consequence, teachers' professional development should include making their implicit belief systems explicit and thereby developing a language for talking and thinking about their own practice, questioning the sometimes contradictory beliefs underpinning their practice, and taking greater control over their own professional growth (Freeman, 1991).

Statement of the Problem

I wanted to examine the pedagogical content knowledge of mathematics teachers in terms of how it is manifested in their classroom instruction and thereby to identify its components. Such an examination would provide an example of what pedagogical content knowledge looks like in a particular subject domain. I also wanted to investigate teachers' beliefs about mathematics, about teaching, and about learning mathematics and how those beliefs, as a backdrop, are related to their pedagogical content knowledge because teachers' knowledge about mathematics is interwoven with their beliefs about mathematics and teaching mathematics (Sowder & Schappelle, 1995).

In this study, I investigated the pedagogical content knowledge of mathematics teachers by: (a) exploring how two middle-school mathematics teachers' beliefs are, by implication, related to their pedagogical content knowledge, (b) analyzing and articulating the teachers' pedagogical content knowledge in the context of their teaching practice, and (c) studying what elements contribute most to their pedagogical content knowledge in teaching mathematics. The research questions that guided the collection and analysis of data were the following:

1. What are middle-school mathematics teachers' beliefs about mathematics and about learning and teaching mathematics?
2. How is middle-school mathematics teachers' pedagogical content knowledge manifested in their instruction?
3. What are the components of the pedagogical content knowledge of middle-school mathematics teachers?

Definition of Terms

The purpose of this section is to provide explanations of some of the terms used in this study. Those terms include: knowledge of subject matter; knowledge of students' understanding; knowledge of pedagogy or instructional strategies; and pedagogical content knowledge.

Knowledge of subject matter is not only understanding concepts, algorithms, facts, procedures, and connections among concepts and algorithms but also understanding the structure of the subject matter (Ball, 1991; Leinhardt & Smith, 1985; Shulman, 1986). Knowledge of subject matter includes knowledge of purposes of instruction, justifications for learning a topic, important ideas for a topic, prerequisites, and typical school problems (Marks, 1999a, 1999b).

Knowledge of students' understanding involves knowledge of students' conceptions, misconceptions, common mistakes, difficulties, and confusions (Marks, 1990a, 1990b). Also, knowledge of students' understanding includes knowledge of students' learning processes, which refers to the ways students learn and come to understand a concept and to students' different learning styles, and it includes knowledge of particular students' understanding (Cochran, DeRuiter, & King, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Marks, 1990a, 1990b; Smith & Neale, 1989).

Knowledge of pedagogy or instructional strategies focuses on subject-specific instructional strategies such as capability to provide representations that facilitate students' understanding through explanations, metaphors, illustrations, examples, and analogies. In addition, it involves learning activities, lesson organization, and use of materials and textbooks (Grossman, 1990; Magnusson et al., 1999; Marks, 1990b; Smith & Neale, 1989).

As noted above (p. 3), pedagogical content knowledge refers to "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Thus, pedagogical content knowledge is the blending of content and pedagogy into an understanding of how particular topic are organized, represented, and adapted, and presented for instruction (Shulman, 1987). Pedagogical content knowledge involves "an understanding of what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9).

CHAPTER 2

PEDAGOGICAL CONTENT KNOWLEDGE

For over a decade, research studies in education have focused on teachers' knowledge. Scholars in the field have tried to characterize teacher knowledge to ascertain what knowledge is needed to teach effectively. In mathematics education, various research studies have, in general, focused on teachers' mathematical knowledge. Pedagogical content knowledge is a relatively new construct that requires more research; there are relatively few studies of the pedagogical content knowledge of mathematics teachers. This chapter begins with a review of how pedagogical content knowledge has been conceptualized, not limited to mathematics education, and how and in what context research studies have affirmed its existence. A discussion of how pedagogical content knowledge develops is followed by a review of its components as identified by various researchers and a description of the operational definition of pedagogical content knowledge used in this study.

Conceptualizations of Pedagogical Content Knowledge

Teaching is a very complex process that is influenced by many kinds of teacher knowledge (Ball, 1991; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter & Franke, 1996; Even, 1993; Even & Tirosh, 1995; Fernandez, 1997; Geddis & Wood, 1997; Leinhardt, 1986; Leinhardt, Putnam, Stein, & Baxter, 1991; Leinhardt & Greeno, 1986; Ma, 1999; Wilson, Shulman, & Richert, 1987). Although teaching is different from knowing, teaching any subject matter depends on knowing that subject matter. In other words, not only is teaching mathematics directly related to knowing mathematics, but neither knowing nor understanding mathematics is sufficient for being able to teach mathematics (Ball, 1999; 2000; Ball & Bass, 2000; Kahan, Cooper, & Bethea, 2003; Shulman, 1986, 1987). Teaching mathematics thus requires not only knowing mathematics but also knowing mathematics for teaching. Shulman (1986) redefined Dewey's (1969) notion that teaching a subject matter is a process of teachers' psychologizing the subject matter for teaching as *pedagogical content knowledge*.

Shulman (1986) proposed that pedagogical content knowledge is specific to a particular subject matter and that learning to teach demands not only understanding the subject itself but also developing a large body of pedagogical content knowledge. Various research studies on pedagogical content knowledge, not limited to those in mathematics education, have sought to reveal the components of pedagogical content knowledge; those studies have shown how pedagogical content knowledge varies across content (Cochran, DeRuiter, & King, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; Marks, 1990a, 1990b; Morine-Dershimer & Kent, 1999; Shulman, 1986, 1987; Smith & Neale, 1989; Tamir, 1988; Wilson et al., 1987). Shulman (1986) characterized pedagogical content knowledge as follows:

> The most useful forms of representation of [subject matter] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. Also pedagogical content knowledge includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and background bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

He later elaborated that pedagogical content knowledge is

> that special amalgam of content and pedagogy that is uniquely the providence of teachers, their own special form of professional understanding…. Pedagogical content knowledge…identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to diverse interests and abilities of learners, and presented for instruction.

> Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (Shulman, 1987, p. 8)

In sum, pedagogical content knowledge is a distinctive body of knowledge for teaching (Ball et al., 2001; Shulman, 1987) and thus is a teacher's understanding of how to help students understand mathematics (Gess-Newsome, 1999a; Magnusson et al., 1999; Morine-Dershimer & Kent, 1999; Wilson et al., 1987).

Shulman (1986) distinguished three categories of content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Content knowledge "refers to the amount and organization of knowledge per se in the mind of the teacher" (p. 9); it involves understanding the structures of the subject matter, that is, the substantive and the syntactic structure. It includes facts, how facts are organized, and how facts are generated and verified as acceptable within the subject matter. Further, content knowledge involves concepts, algorithmic operations, and connections among different algorithmic procedures, understanding of students' errors and misconceptions, and curricular presentation (Ball, 1991; Leinhardt & Smith, 1985). Pedagogical content knowledge refers to knowledge that allows the subject matter content to be taught. The concept of pedagogical content knowledge "was originally construed as a form of content knowledge composed of subject matter transformed for the purpose of teaching" (Munby, Russell, & Martin, 2001, p. 881). Hence, teachers must find ways to adapt and represent subject matter by reflecting on it in accordance with the needs of students. As a result,

> the idea of pedagogical content knowledge substantially improves our understanding of the knowledge required for teaching. The concept implies that not only must teachers know content deeply, know it conceptually, and know the connections among ideas, but also [they] must know the representations for and the common student difficulties with particular ideas. (Ball, Lubinski, & Mewborn, 2001, p. 449)

Curricular knowledge involves understanding the available curricular alternatives, familiarity with the curriculum materials, and ability to relate the content that was taught in the previous year and will be taught in later years in school (Shulman, 1986).

Although he considered pedagogical content knowledge to be a part of content knowledge, Shulman (1987) later outlined seven categories of teachers' professional knowledge base of teaching: subject matter knowledge; general pedagogical knowledge; pedagogical content knowledge; knowledge of learners and learning; curriculum knowledge; knowledge of educational contexts; and knowledge of educational philosophies, goals, and objectives. In particular, he saw pedagogical content knowledge as separate kind of knowledge; however, he identified it as a combination of knowledge of content, knowledge of pedagogy, and knowledge of students.

By extending Shulman's characterization of teachers' professional knowledge base through an investigation of teachers of English and by emphasizing the importance of pedagogical content knowledge as not being a subset of content knowledge and pedagogy, Grossman (1990) identified pedagogical content knowledge as including: conceptions of purposes for teaching subject matter; knowledge of students' understanding, including common misconceptions and difficulties; curricular knowledge; and knowledge of instructional strategies and representations. "The first component includes beliefs and knowledge about the purposes for teaching a subject" (p. 8) since the "conceptions are reflected in teachers' goals" (p. 8) for teaching the subject. Second, the knowledge of students' understanding provides knowledge about students' previous knowledge and difficulties about a concept or topic and enables teachers to appropriately explain and represent the concept or topic for their students. Third, curricular knowledge covers both knowledge of curriculum materials and knowledge about "both horizontal and vertical curricula for a subject" (p. 8). In the case of mathematics, teachers must be aware that "a mathematics curriculum is more than a collection of activities" (NCTM, 2000, p. 14) and thus, it should be coherent; furthermore, the different topical strands of mathematics are highly interconnected. Last, knowledge of instructional strategies and representations for teaching a concept or topic allows teachers to possess a collection of various explanations, metaphors, analogies, and activities.

Similarly, Smith and Neale (1989) characterized pedagogical content knowledge as consisting of four components: (a) knowledge of students' concepts, including students' typical errors and developmental paths; (b) knowledge of strategies for teaching content that enable students to conceptually understand a concept by eliciting students' preconceptions, asking for clarification and explanation, encouraging debate and discussion, and clearly presenting explanations; (c) knowledge of shaping and elaborating the content, which is revealed by using of examples, good explanations, metaphors, analogies, and representations; and (d) knowledge of particular curriculum materials and activities.

Starting with Shulman and Sykes's (1986) categories of teachers' knowledge, Tamir (1988) organized and extended the categories by presenting *a framework for teachers' knowledge* (p. 100, italics mine). Tamir's eight categories consisted of general liberal education, personal performance, subject matter knowledge, general pedagogical knowledge, subject matter specific pedagogical knowledge, and evaluation. Although the two sets of categories have four subcategories that are the same (knowledge about students, knowledge of curriculum, knowledge of instruction, and knowledge of evaluation), Tamir distinguished between general pedagogical knowledge and subject-matter-specific pedagogical knowledge.

Lampert (1991) and Cochran, DeRuiter, and King (1993) claimed that Shulman described pedagogical content knowledge as a subset of what teachers need to know about content and about pedagogy as static knowledge. Cochran et al. proposed renaming pedagogical content knowledge as *pedagogical content knowing* to recognize its dynamic nature. They defined pedagogical content knowing as “a teacher’s integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning” (p. 266). According to this definition, pedagogical and subject matter knowledge should be developed in the context of teachers’ understanding of students and of the environmental context of learning. Consequently, Cochran et al. characterized pedagogical content knowing as follows: (a) teachers’ understanding of students, such as abilities, learning strategies, ages, developmental levels, attitudes, motivations, and prior conceptions of a subject; (b) teachers’ understanding of the environmental contexts of learning, which encompass social, political, cultural, and physical environmental contexts that affect teaching and learning; (c) pedagogical understanding, which involves knowledge of curriculum and knowledge of educational goals and purposes; and (d) knowledge of subject matter.

Investigating 8 fifth-grade teachers’ teaching of equivalence of fractions, Marks (1990a, 1990b) presented a structure of pedagogical content knowledge: (a) subject matter for instructional purposes; (b) students' understanding of the subject matter; (c) media for instruction in the subject matter (texts and materials); and (d) instructional processes for the subject matter. He included itemized subcategories in each component (Figure 1). He proposed three possible derivations of pedagogical content knowledge: an interpretation of subject matter knowledge, a specification of general pedagogical knowledge, and a synthesis of both general content knowledge and subject matter knowledge. Marks indicated that the categories of pedagogical content knowledge could be almost equally derived from subject matter knowledge and pedagogical knowledge; for instance, learning activities, understanding students' misconceptions, and the use of teaching strategies.

Magnusson et al. (1999) viewed pedagogical content knowledge as the transformation of several types of knowledge for teaching; therefore, pedagogical content knowledge is "a teacher's understanding of how to help students understand specific subject matter" (p. 96). More to the point, they adopted Grossman's (1990) work in pedagogical content knowledge as the result of transformation of content, pedagogical, and contextual knowledge. They modified Grossman's model to include five additional components for the case of science teaching: (a) orientations to teaching science, referring to teachers' knowledge and beliefs about the purposes and goals for teaching science; (b) knowledge and beliefs about science curriculum, consisting of mandated goals and objectives and specific curricular programs and materials; (c)

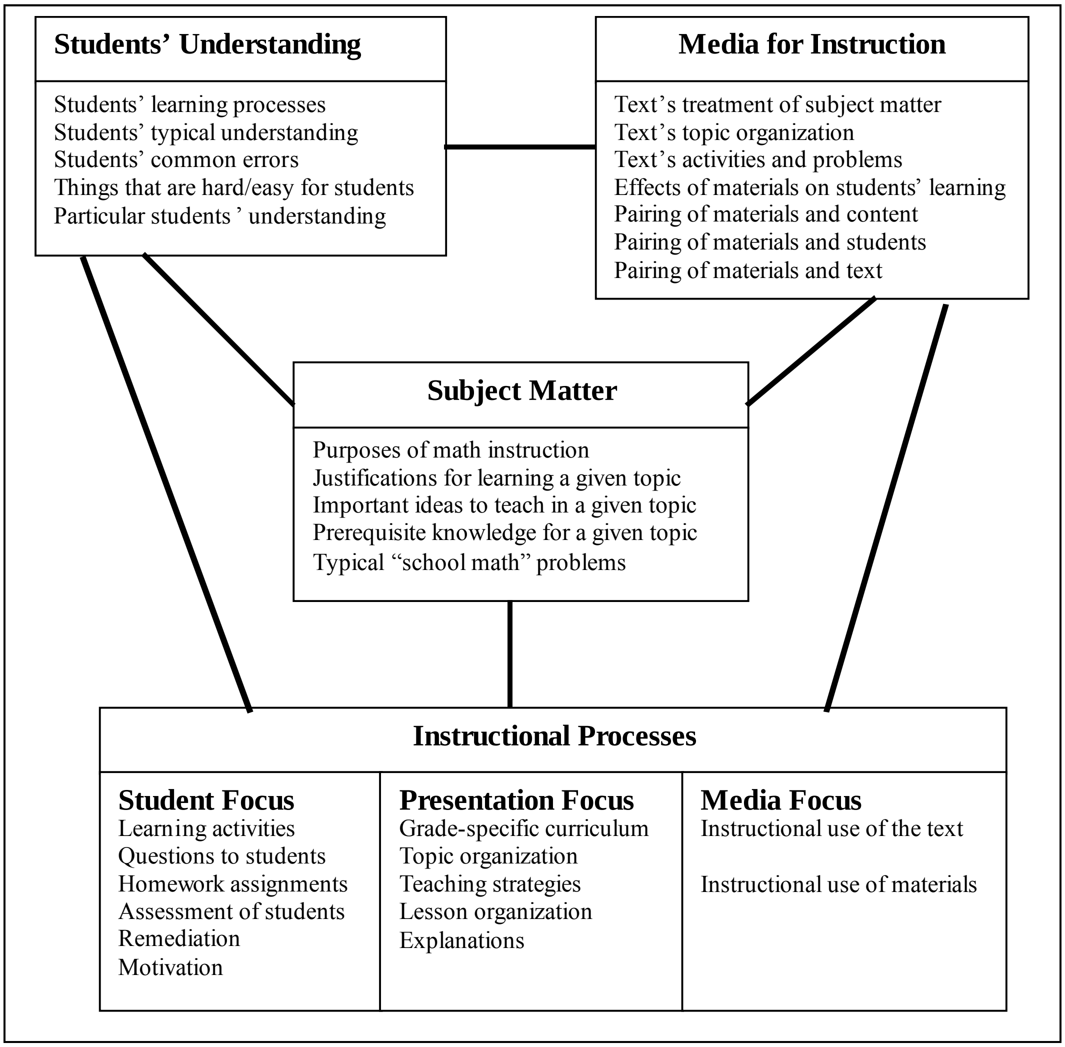

Figure 1. A structure for pedagogical content knowledge in fifth-grade equivalence of fractions (Marks, 1990b, p. 5).

knowledge and beliefs about students' understanding, including knowledge of requirements for learning and knowledge of areas of student difficulty; (d) knowledge and beliefs about instructional strategies for teaching science that contain knowledge of subject-specific strategies and knowledge of topic-specific strategies; and (e) knowledge and beliefs about assessment in science, as comprising knowledge of the dimensions of science learning to assess and knowledge of the methods of assessment.

Components of Pedagogical Content Knowledge

The preceding section considered pedagogical content knowledge with regard to its components as described in research reports. Those components are summarized in Table 1. Even though the reports vary somewhat in the terms they use to refer to the components, they focus on similar concepts: conceptions of or orientations to teaching the subject, subject matter knowledge, knowledge about students' understanding of the subject, knowledge of curriculum, knowledge of pedagogy or knowledge of instructional strategies, knowledge of assessment, and knowledge of context (see Table 1). Most of the researchers conclude that pedagogical content knowledge consists of knowledge of subject matter, knowledge of students' understanding, knowledge of curriculum, and knowledge of pedagogy or knowledge of instructional strategies. But there are some variations in their perspectives.

First, knowledge about subject matter is seen as a key contributor to pedagogical content knowledge, although with some qualifications. Tamir's (1988) characterization is based on a framework of teachers' knowledge rather than on pedagogical content knowledge. Both content knowledge and pedagogical content knowledge are seen as elements of teachers' knowledge. Likewise, Grossman's (1990) characterization is drawn

Table 1. Components of Pedagogical Content Knowledge Identified in the Literature

Researcher	Conceptions of or Orientations to Teaching the Subject	Subject Matter Knowledge	Knowledge of Students' Understanding	Knowledge of Curriculum	Knowledge of Pedagogy or Knowledge of Instructional Strategies	Knowledge of Assessment	Knowledge of Context
Shulman (1987)	*	*	*	*	*	*	*
Tamir (1988)		*	√	√	√		
Smith and Neale (1989)		√	√	√	√		
Grossman (1990)	√	*	√	√	√		*
Marks (1990a, 1990b)		√	√	√	√	**	
Cochran, DeRuiter, and King (1993) *(pedagogical content knowing)*		√	√	**	√		√
Magnusson, Krajcik, and Borko (1999)	√	*	√	√	√	√	

Note: A check mark indicates that the researcher identified the item as a component of pedagogical content knowledge. An asterisk indicates that the item was identified as separate from pedagogical content knowledge. Two asterisks indicate that the item was included in another item.

from a model of teacher knowledge that comprises four types of knowledge. Grossman regards pedagogical content knowledge as another area of teacher knowledge, and, as a result, separates pedagogical content knowledge from subject matter knowledge. Unlike the other researchers, Magnusson et al. (1999) combines knowledge of subject matter with conceptions of or orientations to the teaching the subject matter as *orientation to teaching science*; they see it as a component covering the other components of pedagogical content knowledge. In other words, orientation to teaching a subject matter shapes the rest such as knowledge of curricula, knowledge of students' understanding, knowledge of instructional strategies, and knowledge of assessment. Magnusson et al. define *orientations toward teaching science* as "teachers' knowledge and beliefs about the purposes and goals for teaching science at a particular grade level" (p. 97). Presumably, this definition shows that they adopted Grossman's viewpoint. What is more interesting in their characterization is that beliefs are given the same weight as knowledge. All the components they list explicitly include the terms *knowledge* and *beliefs*.

Smith and Neale (1989) identify subject matter knowledge as knowledge of shaping and elaborating content. This characterization can be contrasted with that of other researchers who include in knowledge of instructional strategies the way subject matter knowledge is represented. For instance, Marks (1990a, 1990b) considers subject matter knowledge as purposes of mathematics instruction, justifications for learning a given topic, important ideas to teach in a given topic, prerequisite knowledge for a given topic, and typical "school mathematics" problems. In contrast, Cochran et al. (1993) emphasize constructing an understanding of subject matter so that teachers can use it in classroom settings. All of the descriptions of subject matter knowledge pay attention to the understanding of content such as facts, concepts, procedures, and organizing ideas of a subject, as well as to the understanding of the subject as a discipline that includes knowledge about discourse of the subject matter (Ball, 1991).

Second, all the researchers emphasize the importance of knowledge about students' learning and understanding of a particular topic. On the one hand, the most common constituents of knowing about students are students' conceptions, misconceptions, typical errors, and difficulties with a given topic. On the other hand, the researchers list slightly different details of what teachers should know about students' understanding. Smith and Neale (1989), Cochran et al. (1993), and Magnusson et al. (1999) mention students' developmental levels, which is a more general knowledge of learners. In contrast, Grossman (1990) argues that knowledge of students' understanding is different from a general knowledge of learners. Grossman, Marks (1990b), Cochran et al. (1993), and Magnusson et al. (1999) underscore the teacher's knowledge of students' learning processes, which refer to the ways that students learn, use, and come to understand a concept, and to students' different learning strategies and styles. In addition, Marks categorizes knowledge of students' understanding:

students' learning processes, students' typical understanding, particular students' understanding, students' common errors, and things that are hard or easy for students to learn. These subcategories of knowledge of students' learning, however, seem closely connected and difficult to distinguish one from another. For example, knowledge of students' learning processes and knowledge of students' typical understanding might be the same in some contexts. Grossman, Cochran et al., and Magnusson et al. single out an understanding about students' knowledge and beliefs about prior knowledge of a subject. Cochran et al. stress understanding of students' abilities, motivation, and attitudes. Magnusson et al. differentiate the knowledge of students' understanding into two levels: knowledge of requirements for learning and knowledge of areas of student difficulties.

Third, all the researchers refer to knowledge of curriculum. Even though Cochran et al. (1993) do not categorize it as a component of pedagogical content knowledge, they include knowledge of curriculum in knowledge of pedagogy or knowledge of instructional strategies. Tamir (1988) characterizes knowledge of curriculum as both knowing prerequisite concepts for understanding a particular topic and being able to design a lesson. For Grossman (1990), it is knowledge about selecting and organizing content, knowledge of curriculum materials, and knowledge about both horizontal and vertical curricula. Magnusson et al. (1999) cite both knowledge of curricular goals and objectives involving national- or state-level curriculum and knowledge of specific curricular programs. In contrast, Smith and Neale (1989) and Marks (1990b) restrict knowledge of curriculum to knowledge of curricular materials and activities. In particular, Marks pays more attention than the other researchers to texts and materials by connecting texts and materials with students' learning; thus, the topic organization, problems, and activities in a text and its treatment of a topic or concept are included.

Finally, as it is conducive to pedagogical content knowledge, knowledge of pedagogy or knowledge of instructional strategies is mentioned by all of the researchers; all the reports focused on subject-specific instructional strategies rather than on general pedagogical ones. Tamir (1988) asserts that teachers ought to have knowledge and skills for instruction, including both teaching and management. For example, not only should science teachers have knowledge about how a laboratory lesson needs to be organized and what elements there are, but also they should have skills such as how to teach students to use a microscope during the laboratory lesson. In the context of science teaching, Smith and Neale (1989) list content-specific strategies such as capability to elicit students' preconceptions and predictions, to ask for clarification and explanation, to provide discrepant events, to encourage debate and discussion, and to present alternative scientific explanations. These skills seem more related to being able to *do* in science teaching than to having science knowledge alone. For Grossman (1990), pedagogical strategies are connected with representations that facilitate rich repertoires of metaphors, experiments, activities, and explanations. This view is developed further by Magnusson et al. (1999), who divide the knowledge of instructional strategies into two subcategories: knowledge of subject-specific strategies and knowledge of topic-specific strategies. Knowledge of subject-specific strategies is associated with orientations to teaching science. Knowledge of topic-specific strategies is connected with representations and activities of the topic that show teachers' abilities to invent representations and to judge whether and when a representation would be useful. In this view, representations incorporate illustrations, examples, models, or analogies. Marks (1990b) emphasizes knowledge of instructional strategies as instructional process, which entails learning activities, teachers' questions to students, lesson organization, explanations, and use of text and materials. In contrast, Cochran et al. (1993) delineate pedagogical knowledge in a more general sense. They describe it as knowledge of educational goals and purposes and knowledge of curriculum; therefore, knowledge of pedagogy or knowledge of instructional strategies is more related to representations during lessons. There is, in general, a thread of connections across the characterization of representations among those studies.

Beyond these components of knowledge of subject matter, students' understanding, curriculum, and instructional strategies, Tamir (1988), Marks (1990b), and Magnusson et al. (1999) include knowledge of assessment as a component of pedagogical content knowledge. Although he did not define knowledge of assessment as a component of pedagogical content knowledge, Marks categorized it under knowledge of students' understanding. Magnusson et al. (1999), in contrast, adopted Tamir's point of view and conceptualized it as consisting of knowledge of the methods of assessment. Cochran et al. (1993) regard knowledge of contexts as a distinctive component of pedagogical content knowledge. Cochran et al. propose a dynamic state of knowledge construction, which seems to result in an emphasis on teachers' understanding about the environmental contexts of learning where the processes of teaching and learning take place.

The above descriptions of pedagogical content knowledge have several implications for research. First, it should be pointed out that some researchers divide pedagogical content knowledge into many subcomponents, which results in describing teachers' knowledge in a broad sense rather than specifically describing pedagogical content knowledge. It is surely complex and complicated to identify pedagogical content knowledge. One subcategory of pedagogical content knowledge, however, is likely to share meaning with another subcategory; thus, it might become even more complicated to describe pedagogical content knowledge in teaching practice. Any effort to explain pedagogical content knowledge through empirical research studies should focus on the subject matter and on the students that teachers are to teach. The constituents of pedagogical content knowledge can be combined into key factors such as knowledge of subject matter, knowledge of students, and knowledge of instructional strategies. For instance, knowledge of assessment could be included in knowledge of students' learning since it is necessary to be aware of assessment in order to understand and facilitate students' learning. The knowledge of subject matter could involve knowledge of curriculum.

In her continuum of models of teacher knowledge, Gess-Newsom (1999a) presented a *transformative model* (Figure 2) and an *integrative model* (Figure 3) and placed them at opposite ends of a continuum. Some of the research studies mentioned above appear to have come out of the perspective that pedagogical content knowledge is a "transformation of at least two constituent knowledge domains: general pedagogical knowledge and subject matter knowledge" (p. 5). In this view, pedagogical content knowledge as the only form of knowledge that affects teaching is the synthesis of all knowledge needed. That is, pedagogical content knowledge is the transformation of subject matter, pedagogical, and contextual knowledge into a unique form, which is called the *transformative model*.

> The transformative model implies that these initial knowledge bases are inextricably combined into a new form of knowledge, pedagogical content knowledge, in which the parent domain may be discovered only through complicated analysis. The resulting amalgam is more interesting and powerful than its constituent parts. (p. 11)

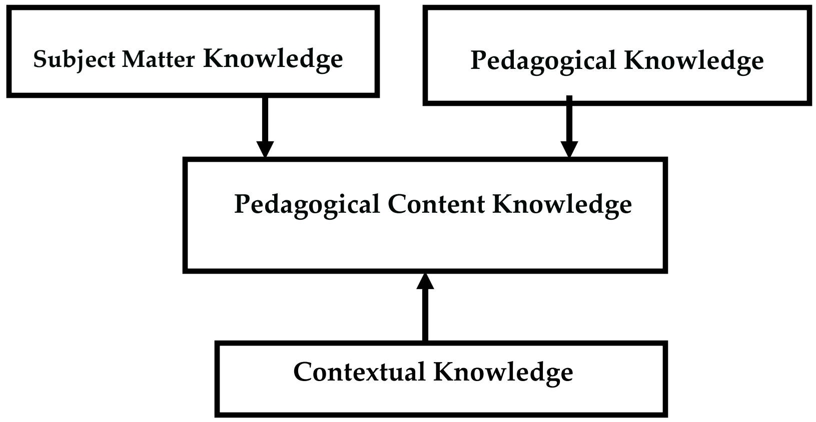

Figure 2. Transformative model (from Gess-Newsome, 1999a, p. 12).

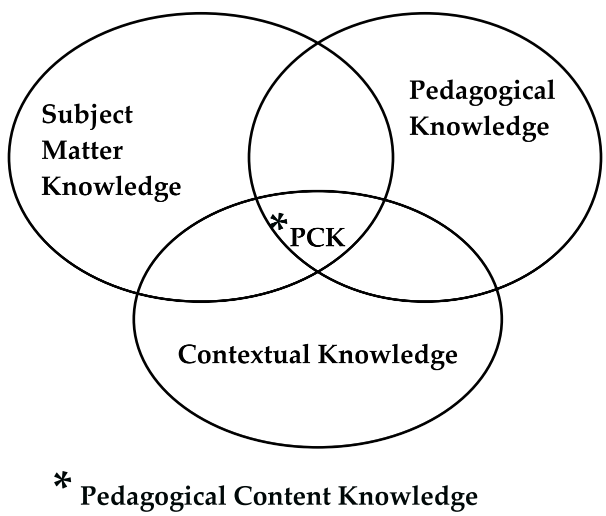

Figure 3. Integrative model (from Gess-Newsome, 1999a, p. 12).

Pedagogical content knowledge, therefore, should be "well structured and easily accessible" (p. 13) so that teachers can foster students' understanding of concepts and ideas in classroom instruction.

In the *integrative model*, in contrast, pedagogical content knowledge "does not exist as a domain of knowledge" (Gess-Newsome, 1999a, p. 11), with which I do not agree, and "can be most readily explained by the intersection" (p. 10) of subject matter, pedagogical, and contextual knowledge. Teaching, in this model, is "the act of integrating knowledge across these three domains" (p. 10). Pedagogical content knowledge in the integrative model can be compared with a mixture of elements from knowledge of subject matter, pedagogy, and context. As a result, teaching tends to be highly dependent on those three independent knowledge domains and on the capability to integrate them to facilitate learning.

In particular, concerning preservice teacher education, Grossman (1990) suggested four possible ways to develop pedagogical content knowledge: (a) apprenticeship of observation; (b) subject matter knowledge; (c) teacher education; and (d) classroom experience. Apprenticeship of observation, especially, provides prospective teachers with memories of strategies for teaching specific content. It affects prospective teachers' knowledge of students' understanding and their knowledge of the curriculum as well. Teachers' subject matter preparation contributes to decision making about the relative importance of particular content, selection and sequencing of curricula, and critiques of particular curricular materials. Professional coursework during teacher education programs such as subject-specific methods courses contribute to the development of the pedagogical content knowledge of prospective teachers. Finally, learning from teaching experience allows prospective teachers to test their knowledge, to learn about students' current knowledge and misconceptions about an idea or topic, and to identify effective strategies and representations.

An Operational Definition of Pedagogical Content Knowledge

My analyses of the descriptions in the literature led me to think through what pedagogical content knowledge means and how I would characterize it. First, pedagogical content knowledge is knowledge for teaching (Ball, 2000; Ball et al., 2001). Since it is not knowledge for just knowing, pedagogical content knowledge is meaningful only when it emerges from the contexts of teaching; hence, pedagogical content knowledge should be not only knowing but also being able to use that knowing in the real contexts of teaching practice (Ball, 1999; Ball & Bass, 2000; Ball et al., 2001).

I conclude that pedagogical content knowledge is knowledge for teaching that is a transformation of knowledge of subject matter, knowledge of students' understanding, and knowledge of pedagogy or instructional strategies. Of these three kinds of knowledge, knowledge of subject matter is likely the most significant constituent. For example, mathematics teachers need to know mathematics in a way that makes them able to teach mathematics conceptually as well as procedurally. Without a profound understanding of mathematics, mathematics teachers cannot help their students understand a mathematical concept by providing appropriate representations of it (Ball, 1990, 1991, 2000; Ball & Bass, 2000; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Hiebert & Carpenter, 1992; Kahan et al., 2003; Ma, 1999; National Research Council, 2001). What is more, only when mathematics teachers understand mathematics deeply are they capable of using and adapting their knowledge flexibly according to the needs of their students. Knowledge of mathematics enables mathematics teachers to provide ideas of how to represent a mathematical concept in a particular way and to explain why the representation would be effective (Ball, 1991; Ma, 1999).

For the purpose of this study, I adopted Shulman's (1986, 1987) ideas and defined pedagogical content knowledge as the ways of representing and formulating the subject that make it comprehensible to others through the most useful forms of representation, examples, demonstrations, explanations, and analogies. In addition, by partly adopting Marks' (1990a, 1990b) categorization and the transformative model (Gess-Newsome, 1999a), I conceptualized pedagogical content knowledge as comprising knowledge of subject matter that included: (a) knowledge of curriculum, purposes of teaching the subject, ideas and topics to teach, and typical school problems of the subject; (b) knowledge of students' understanding consisting of particular students' understanding, students' learning processes, and students' common errors, misconceptions, and difficulties; and (c) knowledge of pedagogy involving teaching strategies, learning activities, using materials and textbook for instruction, lesson organization, and representations (Figure 4).

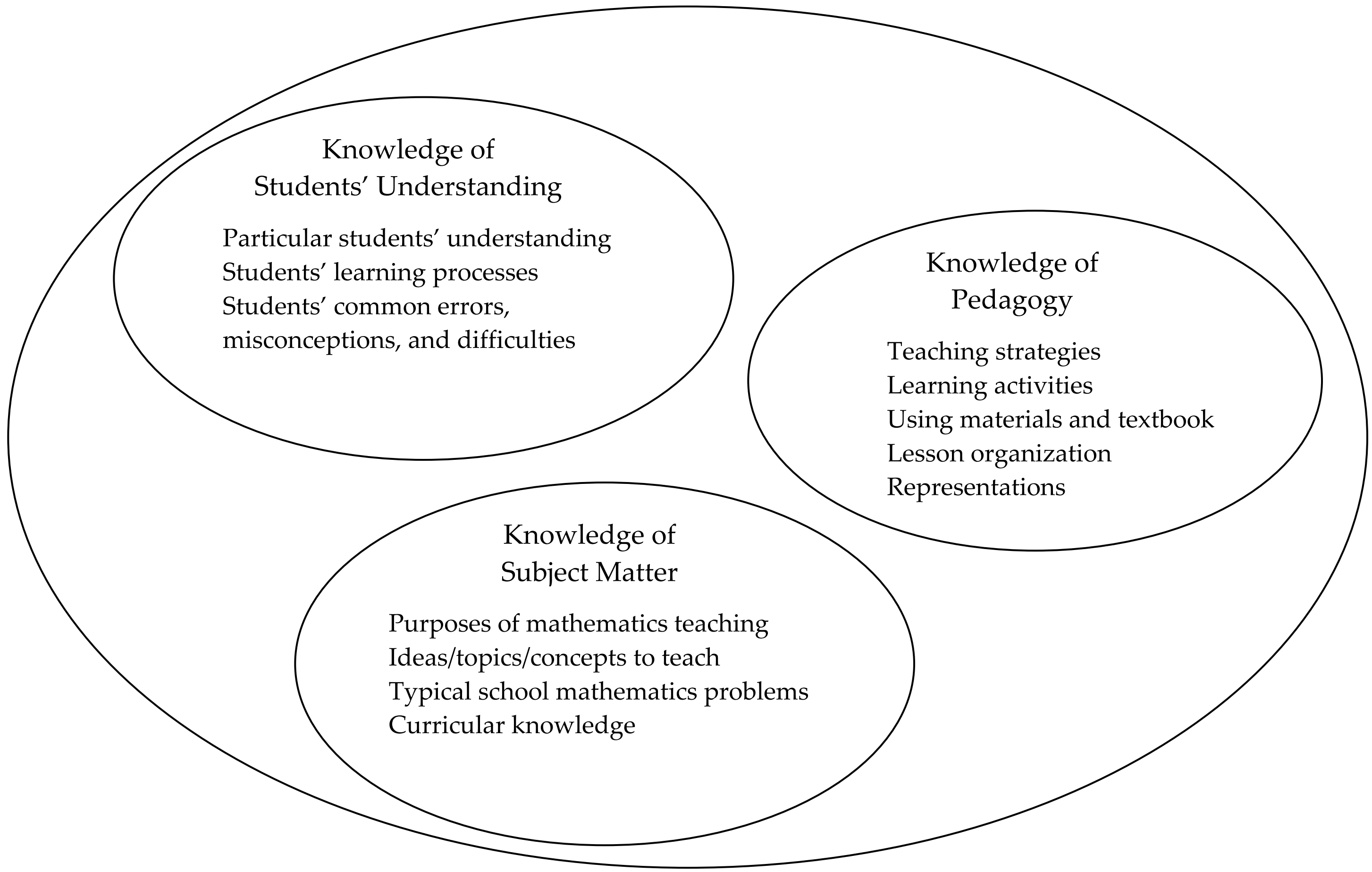

Figure 4. Theoretical framework of pedagogical content knowledge.

CHAPTER 3

METHODOLOGY

> Qualitative research is predicated upon the assumption that an "inner understanding" enables the comprehension of human behavior in greater depth than is possible from the study of surface behavior, from paper and pencil tests and from standardized interviews. (Rist, 1979, p. 20)

I used a qualitative research method to investigate the research questions. Qualitative research is designed to understand processes and describe poorly understood phenomena (Marshall & Rossman, 1995).

> Qualitative research is an inquiry process of understanding based on distinct methodological traditions of inquiry that explore a social or human problem. The research builds a complex, holistic picture, analyzes words, reports detailed views of informants, and conducts the study in a natural setting. (Creswell, 1998, p. 15)

Moreover, qualitative research is more concerned with process than with products or results (Bogdan & Biklen, 1994); the processes of qualitative research should be responsive to the ongoing data collection, analysis, and interpretation (Calderhead & Shorrock, 1997).

I studied the research questions within the framework of interpretivism, which aims "to understand and explain human and social reality" (Crotty, 1998, pp. 66-67). That is, the purpose of interpretative qualitative research design is “to understand the meaning people have constructed about their world and their experiences” (Merriam, 2002, p. 5). This perspective was an appropriate framework for the study, whose purpose was to understand, explain, and characterize the pedagogical content knowledge of two middle-school mathematics teachers in the context of teaching. The interpretivist approach “looks for culturally derived and historically situated interpretations of the social life-world” (Crotty, 1998, p. 67), which provide a rich and contextualized picture of what teachers do in the classroom.

This research used case study methods, which are appropriate “when how or why questions are being posed, when the investigator has little control over events, and when the focus is on a contemporary phenomenon within some real-life context” (Yin, 1994, p. 1). The case is a bounded and integrated system (Merriam, 1998; Stake, 1995). Case study methods are distinguished by their pursuit of intensive descriptions and analyses of single units such as an individual to acquire a deep understanding of the situation (Merriam, 1998). Beyond examining or illustrating one setting or a single subject (Bogdan & Biklen, 1998), a case study is a detailed exploration of a case of activity or an individual over time that is produced by collecting and analyzing comprehensive data that involve observations, interviews, audio-visual material, documents, and reports in context (Creswell, 1998). A case study research design is appropriate for gaining a rich description of what a mathematics teacher does while he or she is teaching to explore how pedagogical content knowledge is manifested in the context of teaching practice.

Participants

Mr. Smith

Mr. Smith (a pseudonym) was in his 40s and had taught mathematics in the middle grades—eighth grade primarily—at a school in a small Southern town for 13 years. There were about 230 eighth-grade students in the school; he was one of the two mathematics teachers in the grade at the school. He taught six groups of students every day; the students were homogeneously grouped. He was among about 20 teachers whom I contacted by e-mail. I was trying to find middle-school teachers with approximately 5 to 10 years of teaching experience in the middle grades; he was the only one to respond who was willing to participate in the study.

Ms. King

Ms. King (a pseudonym) was a teacher in her 50s who had participated in a summer institute that was held as a part of a professional development project at a large research university in the South. I wanted to find an eighth-grade mathematics teacher with approximately 10 years of teaching experience because I wanted to compare and contrast her case with that of Mr. Smith, which had been conducted the previous year.

Ms. King was the only eighth-grade mathematics teacher in the school system of a small city in the South. She had about 18 years of teaching experience in elementary and middle grades in the city, where she had been born and had grown up; however, it was her first year of teaching mathematics to eighth graders. She had started her teaching career as an elementary teacher and gradually moved up to middle-grades teaching. A primary criterion for participants was the teacher's confidence in his or her mathematical knowledge for teaching middle-grades mathematics. Ms. King appeared to meet the condition; she loved mathematics and had made good grades in mathematics as a student throughout her schooling and in college. She had been eager to help students learn and moreover had continued to learn mathematics herself. She seemed to enjoy teaching mathematics and believed that her calling was to help students learn the subject.

Contexts

The contexts of the schools where the two teachers taught were different. Although both teachers were teaching eighth-grade mathematics, the topics they taught during my observations were different. The way students were grouped in the two schools was different. Mr. Smith taught six groups of students who were homogeneously grouped. He taught Algebra to the most advanced group, Pre-Algebra to the next-most-advanced groups, and General Mathematics to the lowest group. My classroom observations focused on the advanced groups, the Algebra and Pre-Algebra classes. Mr. Smith was one of the two eighth-grade mathematics teachers in the school.

In contrast, Ms. King was the only eighth-grade mathematics teacher in the school and taught Pre-Algebra to all of the eighth graders, who were heterogeneously grouped. I observed all of the groups. Most important, the school where Ms. King taught eighth-grade mathematics was a Learning-Focused School, which encourages teachers to organize lessons in a certain way, such as starting with "essential questions."

The Learning-Focused Schools Model, developed by Learning Concepts and Learning-Focused Solutions professionals (Thompson & Thompson, 2004), is a school improvement model that provides exemplary practice strategies for learning and instruction within the framework of learning. This learning framework connects exemplary practice teaching strategies to teacher planning and instruction. Teachers plan and teach differently in schools when the focus is on learning. Learning Concepts and Learning-Focused Solutions provides professional development and resources to school districts and schools.

Data Collection

Mr. Smith

To build a detailed case, I collected multiple forms of data. The major sources of data were semi-structured interviews, observations of mathematics classes, and documents. I observed Mr. Smith's classroom teaching for 8 class periods over 2 weeks—there were tests on a couple of days when I did not observe. During the classroom observations, I took field notes that described almost everything that happened in the classes. The notes included Mr. Smith's introduction of new topics, his representations on the board or overhead, his questions for students, students' responses and questions, verbal and written explanations, and so on. The field notes were later typed. When possible, classroom observations were audiotaped. As classroom artifacts, copies of the chapters in the textbooks and pages from practice workbooks for algebra and pre-algebra were obtained. Additional materials that were gathered included copies of review sheets, quizzes from the previous years for a review or practice, and a test conducted during my observations.

I interviewed Mr. Smith twice using semi-structured questions to probe his beliefs about learning and teaching mathematics and about mathematics. The interviews were conducted after the observation of four classes and consisted of two parts: (1) questions designed to discover his background, conceptions about learning and teaching mathematics, and beliefs about mathematics; and (2) questions designed to elicit his goals for teaching certain topics, his views of common students' conceptions or misconceptions about the topics, and his reflections on the lessons that I observed. The interviews were audiotaped and later transcribed for analysis. In addition, a questionnaire was used to obtain more data. Mr. Smith audiotaped his answers to the questionnaire and sent the audiotape to me. That tape was later transcribed for analysis along with other data.

Mr. Smith taught three courses to eighth graders: Algebra, Pre-Algebra, and General Mathematics. I observed six Algebra and two Pre-Algebra class periods on consecutive days at the beginning of April 2003. During my observations of the Algebra class, he taught a chapter called "Radicals, Functions, and Coordinate Geometry." In the Pre-Algebra class, he taught the topic of theoretical probability in a chapter called "Probability."

Ms. King

Data were collected for Ms. King through: (a) initial interviews about her background; (b) interviews about her beliefs about learning and teaching mathematics and about mathematics itself; (c) observations of the classroom and post-observation interviews about lessons observed; and (d) documents including state curriculum objectives and classroom artifacts such as a copy of the textbook, handouts, worksheets, activities, and tests.

I conducted seven interviews with Ms. King; each interview lasted 30-60 minutes. Two semi-structured interviews were used to examine her background. The purpose was to develop rapport in observing the instruction and to get data about how Ms. King felt about and valued mathematics. She was asked to reflect on what she thought about mathematics and her goals and strategies when learning mathematics as a student. The main focus of the two interviews was to explore her personal background, motivation to teach, and ideas about teaching and learning mathematics.

The additional five interviews were intended to delve into Ms. King's beliefs about learning and teaching mathematics and about mathematics itself. Questions were designed to probe how her beliefs influenced and were related to her teaching of mathematics. These interviews were intended to inquire into Ms. King's understanding about mathematics and pedagogical issues in her teaching.

I observed 25 class periods of Ms. King's mathematics classroom during more than 4 weeks. She taught five groups of eighth graders every day; I observed all of the groups in order to explore how she taught different groups of students differently. The students in each class were heterogeneously grouped. The topics that she taught were solving equations with an unknown variable, probability, geometry, reviews of fractions and decimals, and algebraic expressions. While observing the lessons, I took descriptive field notes, focusing on how she explained and represented mathematical ideas, concepts, and topics in context. The lessons that I observed were all audiotaped and were later transcribed into a form of expanded field notes. In addition, when students were working in groups, not only did I closely observe their work but I also participated in their activities and helped those who needed assistance. Some students asked me to help with their work; others just wanted to know whether their answers were correct. In consequence, the classroom observations helped me gain insights into how Ms. King acted during instruction as well as how her knowledge and understanding about mathematics were presented to the students.

After the classroom observations, I asked Ms. King questions about her lessons concerning her goals for her students, her rationale for the selection of activities or assignments, the sources of those activities, the examples and explanations that she used in her lessons, and the lesson organization. The five post-observation interviews took place almost every other day of classroom observations and lasted 30 to 60 minutes each, depending on her availability.

Archival data related to Ms. King' teaching were collected. They included the textbook, the study guide workbook, handouts, worksheets designed for assignments and for in-class activities, problem sets designed for group work, tests, state curriculum objectives, lesson plans, review sheets, and her handwritten work for a class, Teaching Algebra for Students' Success, that she was taking through the state RESA (Regional Education Service Agency). Ms. King provided me with these documents.

Data Analysis

The data were analyzed using a case study strategy (Yin, 1994) and grounded theory (Glaser & Strauss, 1967). Yin's strategy of *relying on theoretical propositions* in analyzing a case study enables the researcher "to follow the theoretical propositions that led to the case study" (Yin, 1994, p. 103) because "the original objectives and design of the case study presumably were based on such propositions, which in turn reflected a set of research questions, reviews of the literature, and new insights" (p. 103). In addition, the theoretical propositions helped me to not only focus on certain ideas and discard other data but also to organize the case study and define alternative explanations (Yin, 1994).

Grounded theory uses the constant comparison method of data analysis (Glaser & Strauss, 1967). As implied in "constant comparison," this method guides the researcher to begin with a particular incident from interviews, field notes, or documents and compare it with another incident either in the same data source or another source (Merriam, 1998). These comparisons produce preliminary categories, which are constantly compared to other categories already derived from the data. Themes or categories emerge from these constant comparisons; these constant comparisons continue until a theory is formulated (Creswell, 1998; Merriam, 1998).

The data were analyzed using the constant comparative method. The process of data analysis began with finding keywords from the interview transcripts and categorizing those words into themes emerging from the interviews. I classified every problem, comment, question, representation, or symbol into categories derived from the theoretical framework, which included knowledge of subject matter, knowledge of students' understanding, and knowledge of pedagogy. Also, the field notes were analyzed according to the theoretical framework. The data were organized with regard to such characterizations. Each category consisted of subcategories that emerged from the data. I used the participants' terms in the data. After I had analyzed those data, I constantly compared the interview data with the field notes, the textbook, and other materials or classroom artifacts such as curriculum maps and worksheets.

Role of the Researcher

We don't see things as they are; we see them as we are. – Anaïs Nin

A researcher plays different roles (Stake, 1995), and those roles function, more or less, during the whole processes of research. Thus, "researchers should systematically identify their subjectivity throughout the course of their research" (Peshkin, 1988, p. 17). During the processes of this study, I played roles as an interviewer, observer, and aide.

My subjectivity during the entire processes of this study included being a mathematics teacher. I had been a mathematics major as an undergraduate and was a mathematics teacher of the middle grades (7-9) in Korea; I continually asked myself "What if?" questions while making the classroom observations in this study. That is, questions such as, "What I would do in situations like those the participant is encountering?" came to my mind as I observed. Also, I was able to start figuring out what made me frustrated and what I had not seen about students' learning as a mathematics teacher.

At the same time, I was a "foreigner" with excitement about being in the "real classroom" culture of middle-grades classrooms in the United States. I come from a different culture in which the mathematics curriculum is integrated; mathematics is not separated into algebra, geometry, and calculus. Thus, it was natural for both teachers and students in Korea to seek flow and connection among mathematical topics. The secondary school curriculum starts in the middle grades and continues to high school in Korea. Mathematics teachers in my country are certified as mathematics teachers and not as teachers who could teach any subject in the middle grades. They select mathematics or mathematics education as their major when entering college and are required to take a large number of mathematics courses during college.

I had seen many real or imaginary mathematics teachers represented through research, but had seen few U.S. teachers teaching mathematics. I value the teaching profession and the importance of the teacher's role. However, I found that many U.S. teachers are criticized and poorly represented in the research literature, which influenced me to attempt to see these teachers with an open and positive mind in order to keep myself away from such prejudices or misconceptions.

CHAPTER 4

THE CASE OF MR. SMITH

This chapter describes the case of a middle-school mathematics teacher, Mr. Smith, in terms of his beliefs and pedagogical content knowledge. I first present Mr. Smith's beliefs about mathematics and about learning and teaching mathematics that emerged from the data. Then I describe in detail how his pedagogical content knowledge was manifested in his mathematics teaching. Finally, I articulate components of his pedagogical content knowledge. I begin by describing Mr. Smith's background in becoming a middle-grades mathematics teacher.

Mr. Smith's Background

Mr. Smith earned a bachelor's degree in business and a master's degree in sports management from a large research university in the South. He worked in a business and then in an athletics department at the university for several years before returning to school to get his T4 and T5 teaching certificates (T4 is equivalent to an undergraduate degree in education and T5 to a master's degree). As soon as he finished the program, he was hired at a middle school. Later, he moved to another middle school in the same system.

Mr. Smith's parents, who were both educators, influenced him to choose the teaching profession. When he was a child, it was very meaningful to him that his parents were around a lot during afternoons and on weekends. They had a great influence on him and were very involved in his life as well. By choosing teaching, he would not be so busy that he had to work or be traveling all the time when he had a family. Another aspect of choosing the profession was that he was never really "driven by making a lot of money." Instead, he wanted to enter a profession in which he would feel that he had done something worthwhile with his life after investing 40 years. He thought the teaching profession was a very worthwhile profession and that he could influence other people.

Mr. Smith chose to teach in the middle grades because he loved that age group; teaching those grades provided him with a great opportunity to influence children. Middle-school students are at a very impressionable age; that is when students begin to formulate their own ideas. Children in the middle grades begin to develop values of their own, and as a teacher Mr. Smith thought he could help instill the importance of hard work, doing one's best, being honest, treating others with respect, and learning. Also, his experiences as a student influenced him to become a middle-school teacher. He had a middle-grades teacher who made learning fun for him; school was just a fun place to be, and learning was fun. He saw how his teachers showed respect and kindness as they interacted with custodians and the other teachers, which taught him the importance of such attitudes.

Mr. Smith's mathematical experiences had played a significant role in his becoming a mathematics teacher. As a learner, he had done very well in mathematics in the lower grades and had also found that mathematics was challenging. He had depended, however, upon teachers who would explain mathematics in a variety of ways, which influenced him to be the kind of teacher who challenges students who are really good at mathematics. On the other hand, he could "understand a student who does not like mathematics or who struggles [with] mathematics and identify with them" because he "was not the brightest math student" in his class and "was just an above average math student who had to work to make good grades."

What Did Mr. Smith's Beliefs Reveal?

Mathematics

Mr. Smith believed that mathematics is a "vital" area for every student to learn throughout his or her life; moreover, he thought that mathematics is a subject that is built on solving problems: "Everyone solves problems every day." He believed that problem solving is a process of analyzing a problem to figure out a solution. In his view, problem solving involves students' critical thinking skills and their abilities to select, analyze, and synthesize necessary and important information as well as to disregard unimportant information. Also, problem solving includes not only going through a series of steps but also having the skills or abilities to comprehend what a problem asks one to do and to evaluate whether the solution to the problem makes sense. He said that problem-solving skills should extend outside the mathematics classroom to everyday problems in real situations; hence, students will use mathematics in their daily life even though they may not be aware of how much mathematics is really involved and how much it is going to be implicated in everyday situations.

Learning Mathematics

Mr. Smith's beliefs about learning mathematics appeared to be directly related to his beliefs about learning in general (Cooney, 1985, 1999; Hashweh, 1987; Nespor, 1987; Thompson, 1984, 1992). He believed that learning is a process of problem solving, which parallels the notions of *Principles and Standards for School Mathematics* (NCTM, 2000), and that learning is a very important life skill; thus learning mathematics is helpful in many aspects of one's life. For Mr. Smith, the ultimate purpose of learning mathematics is to figure out things in life through the experience of figuring out how to solve mathematical problems.

Mr. Smith also believed that one learns mathematics to acquire a lifelong skill. The metaphors he chose among examples given for learning mathematics were that “learning mathematics is like building a house and a lot like putting together a jigsaw puzzle.” Learning mathematics begins with learning and mastering basic facts in the very early grades to build a solid foundation. Students continue to build upon the foundations and do many more things. He explained that to learn mathematics is to realize how little pieces of mathematics “fit together.” Furthermore, he argued that at first one piece of mathematics may not seem to mean a lot. Students later become aware of how two concepts or facts of mathematics fit together and are connected, and then those two can fit into much larger ideas of mathematics. This view seems to emphasize that students’ understanding of learning makes subsequent learning easier (NCTM, 2000; Skemp, 1976) and fosters a meaningful connection of new knowledge with previous knowledge (Schoenfeld, 1988).

Mr. Smith believed that students learn mathematics in different ways. In other words, “each student’s knowledge of mathematics is uniquely personal” (NCTM, 1991, p. 2). Students should be provided with various practices, examples, and activities because they think, learn, and make sense of mathematics differently. Some students are able to easily figure out some mathematical topics, some might not be interested in figuring them out, and some might struggle with mathematics.

Teaching Mathematics

Mr. Smith's beliefs about teaching mathematics were closely related to his goals for teaching mathematics, most of which focused on students. He said that teaching entails explaining a certain topic or concept in various ways; therefore, teachers should be willing to explain in more than one way so students can gain a better understanding (McDiarmid, Ball, & Anderson, 1989; NCTM, 1991; Wilson et al., 1987). His metaphor for mathematics teaching was "coaching" in that coaching has a lot of similarities with teaching. He himself had been a basketball coach at the school and had found that students are different in many ways. Students, like basketball players, respond differently, so teachers must use different techniques to address those differences. Some students might need more encouragement and praise than others. Mr. Smith would always pat them on the back and tell them how great they were doing. He gave direction when some seemed to want more of a push. Some students might expect the teacher or the coach to correct their mistakes, and so he corrected them. If students seemed to need to be motivated to perform, then Mr. Smith tried to motivate them. He gave just a yes-or-no response to those who might have just wanted to know whether an answer was right or wrong.

Mr. Smith's teaching practice was consistent with his beliefs about learning and teaching mathematics in that he asked the students to think through the processes they used to get an answer at times during his lessons. He frequently asked them to think and to learn different approaches if there were different ways to solve a problem. He stressed that the students did not all have to work in the same way. His emphasis was on the students' understanding of the different ways of solving a problem and what they were doing by asking them to show and explain their solutions. It appeared that he made efforts to show a second solution for each problem and did not just stick to one way, neither his nor a student's, of solving it. After presenting a solution, he either asked the students whether they had a different solution or showed them another solution.

Mr. Smith often appeared to focus on whether the students got the right answer. For instance, the words "make sure," "you need to double check," "check," "verify answers," and "make sure the answers [you've got] are correct" were repeated often during his lessons. Such statements seem to come from his beliefs about learning mathematics as primarily problem solving in that problem solving is more about processes and concepts than products and procedures.

How Did Mr. Smith Teach Mathematics?

Knowledge of Mathematics

During my observations Mr. Smith was teaching radicals and radical equations in the Algebra class and probability in the Pre-Algebra class. He said that the topics of radicals and radical equations are pieces of the puzzle in Algebra I and that the purpose of his teaching those topics was to expose students to abstract thoughts or concepts of mathematics. He held the view that the students in Algebra I must understand the concepts of what squaring a number is, what square root means, how to take a square root, what the square of a number is, and how to get rid of a square root sign. One of the goals of his instruction on radicals was to "bridge the gap" between taking the square root of any positive integer and applying that process to taking the square root of a squared number. Second, he wanted the students to realize that the radical sign was going to disappear when squaring a number that had a radical sign and that they would be able to solve the resulting equation without the radical sign.

His illustration that solving equations with radicals would mean that the students could apply the concepts of radicals to equations with radicals reveals his concerns for the students' deeper understanding of those concepts (Hiebert & Carpenter, 1992; NCTM, 2000). The students should be able to apply the knowledge of how to take the square root not only to a number but also to variables and terms in equations. In other words, the students must be able to transfer the concept of radicals to radical equations. In addition, he said that there were a variety of ways of solving radical equations; there is always one more tool or method to solve the equation. Mr. Smith's knowledge of mathematics enabled him to explore a problem in several different ways (Sowder & Schappelle, 1995).

Mr. Smith also declared that concepts of probability are fairly practical, and that they help people see different options and possibilities, which supports the assertion of *Principles and Standards* (NCTM, 2000) that "teachers should give middle-grades students numerous opportunities to engage in probabilistic thinking about simple situations from which students can develop notions of chance" (p. 253). To do this, Mr. Smith engaged his students in an activity of rolling dice and flipping coins that allowed them to recognize how to get a probability. Learning probability can take learners through various processes; for example, they sometimes have to make diagrams or chart data. In Mr. Smith's explanation, a basic understanding of probability enabled him and his students to think through decisions and apply their knowledge to realistic situations.

Importantly, Mr. Smith regarded himself as still developing his knowledge about mathematics:

> As I have taught more, as I've read more, I think I've also been challenged by students more, and I just think I increased my knowledge through courses that I've taken. This is my, I think, thirteenth year of teaching, and [I] certainly am just challenged and grow in my math abilities each and every year. I think staff development [activities] are another way that I can become more confident in my knowledge of mathematics.

On the other hand, it was interesting to note that Mr. Smith also saw himself as having "limited knowledge" or "more of a textbook knowledge" of the concept of radicals; he described that knowledge as follows:

> I guess I would explain that to some degree like a history buff who all of his life has loved history and researched history, gone places, been a world traveler, and he teaches history; he's got a great knowledge to tap into, a great depth [of] knowledge. Another teacher who may be asked to teach one year, who maybe teaches science, for example, who [is] asked to teach history. Their knowledge is probably limited with just what they have been exposed to. They don't have the background knowledge or the depth of knowledge. And I think that I feel that way with some of my content. Radicals would probably be that way. I know what I have read from a book, and I know that I've never been taught as a student myself. But probably having a great depth of knowledge of how radicals can be used in research or the scientific world or in different areas of development [is needed]. I just feel that I would be confident in saying that I'm limited in that.

Purposes of Teaching Mathematics

Mr. Smith's teaching of mathematics aimed to identify his students' skill levels, to reassure the students, to help students build a foundation, and to motivate students, which shows a connection with his beliefs about learning mathematics. He hoped to help the students "investigate, be creative, be challenged enough, and to instill in them the ability" to figure things out. To be successful in mathematics, he believed that the students ought to have a foundational level of skills and background understanding and a strong grasp of mathematical concepts. As a mathematics teacher, pursuing those purposes also led him to help his students pass tests and to cover the objectives required by state law. The students needed to realize that mathematics was going to help when taking second-year algebra, going to college, or getting a job in the future. Furthermore, to achieve the goals, he "served the role" of preparing the students to do advanced mathematics in the higher grades and to build a foundation of mathematical concepts that would allow them to perform successfully in fields or occupations such as engineer, astronaut, or college professor of mathematics.

Knowledge of Students' Understanding

In general, Mr. Smith appeared to emphasize to his students that there are a variety of ways to get an answer to a mathematics problems. This emphasis came from his beliefs about mathematics. He repeatedly told the students that they needed to learn different approaches to solving a problem and to understand those approaches. Although he did not ask them to try a different way for each problem, he seemed to try to draw various methods of solution from the students. He would get a student to show his or her own way. Sometimes students would volunteer to talk or present their approach on the board if they had a different one from the teachers' presentation or from their peers' methods. By explicitly pointing out that students did not always have to work in the same way, Mr. Smith urged them to look for different methods and be able to find a lot of ways to arrive at an answer.

I noticed that Mr. Smith, on many occasions, anticipated what the students would feel confused about and what they would find difficult about the topic. In particular, when explaining a topic, he always mentioned, "One student may do it in this way," "Another student might do it using cross cancel later," or "One student may confuse it with cross cancel later." Confirming that students had different approaches, he encouraged them to present their solutions on the board and explain what they had done in case anyone had done it differently, which revealed his effort to understand particular students' understanding. Again and again, he emphasized that the students needed to learn different approaches and needed to find the easiest way to get the answer to a problem.

Mr. Smith did not, however, make a connection between his claims that there were many approaches to an answer and the practice of checking or verifying answers. Although he stressed that students should make sure that the answers were correct, he did not go further than that. He could have guided them to use a different way from the one that they already tried as a means of verifying an answer to a problem. Then the students could have experienced how different approaches would work.

When explaining how to simplify radicals, Mr. Smith made a connection to the concept of fractions and used various examples, some of which were rule based. For instance, he asked the students to think about what they did when learning about simplifying fractions and what the rules were in doing so. He showed the “rules” for fractions, using examples of $\frac{2}{4}=\frac{1}{2}$, $\frac{4}{2}=2$, and $\frac{7}{3}=2\frac{1}{3}$. Further extensions of the multiplication of fractions for radicals were given with examples like $\frac{2}{4}\times\frac{1}{2}$ and $\frac{16}{30}\times\frac{10}{12}$; doing cross cancellation first and multiplying later—$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, $\frac{4}{3}\times\frac{1}{3}=\frac{4}{9}$—or multiplying fractions first and doing cross cancellation later—$\frac{2}{8}=\frac{1}{4}$. He explained the possible processes that students might perform according to how comfortable they were with each.

Mr. Smith went through the same procedure with the radicals. He stressed simplifying and checking whether it was possible to break down the radicals when they were multiplied. He selected problems from the review he designed that students found difficult, and he explained how they might use different approaches to solve the problems. For example, the problem $\sqrt{10}\cdot\sqrt{6}$ could be approached in different ways. One approach is first to check whether the product has a perfect square factor: 60 is 4 times 15, and 4 is a perfect square, thus $2\sqrt{15}$. An example of another way was presented as follows: "A student may do, $\sqrt{10} = \sqrt{2} \cdot \sqrt{5}$, $\sqrt{6} = \sqrt{2} \cdot \sqrt{3}$ and multiply first the two $\sqrt{2}$ and then perform $\sqrt{5} \cdot \sqrt{3}$, which leads to the same answer $2\sqrt{15}$." Then he moved on to fractions under the radical sign. He further probed examples from among the problems of the review that he provided. After that, he showed two problems with different approaches. First, he treated one as a normal fraction: $\sqrt{\frac{64}{16}} = \sqrt{4} = 2$. Second, he factored the reduced fraction and got a perfect square:

$$\begin{aligned}
&\sqrt{\frac{96}{2}} \\
&= \sqrt{48} \\
&= \sqrt{4} \cdot \sqrt{12} \\
&= \sqrt{4} \cdot \sqrt{4} \cdot \sqrt{3} \\
&= 4\sqrt{3}
\end{aligned}$$

Mr. Smith often asked questions to elicit the students' thoughts, misconceptions, and errors as examples. After he had given a test on radicals, he went through common errors that the students made on the test. He picked out examples from the test, and the students asked questions. For instance, he wrote $\frac{4+\sqrt{20}}{2} = \frac{4}{2} + \frac{\sqrt{20}}{2}$ on the board, explaining that simplifying and factoring skills were required. The students were confused when simplifying radicals in this form. When a student asked if her way would be okay, he examined it without implying whether her solution was right or wrong: $\frac{4+\sqrt{20}}{2} = 2+\sqrt{20} = 2+2\sqrt{5} = 4+\sqrt{5}$. After going over her solution, he asked, "Is that, $2+\sqrt{20}$, possible?" to draw out the student's misconceptions. He presented an example of a fraction in this form: $\frac{2+6}{2}$. He demonstrated the solution in two ways: first, obtaining 6 by canceling out the 2s in the numerator and denominator; and $\frac{2}{2}+\frac{6}{2} = 1+3 = 4$ by asking what 2 + 6 is. Then he showed the problem in this way:

$$\begin{aligned} &\frac{4+\sqrt{20}}{2} \\ &= \frac{4+2\sqrt{5}}{2} \\ &= \frac{4}{2}+\frac{2\sqrt{5}}{2} \\ &= 2+\sqrt{5} \end{aligned}$$

It was apparent that he was using his knowledge about what the students would feel confused about and what was difficult about the topic. Extending the example above, he demonstrated another problem $\frac{2\sqrt{3}+2\sqrt{7}}{2} = \frac{2(\sqrt{3}+\sqrt{7})}{2} = \sqrt{3}+\sqrt{7}$ (he then crossed out the 2s in the numerator and the denominator). One of the students said, "It is confusing." Mr. Smith acknowledged that it was confusing. Then he presented several more examples that might be confusing to the students. He revisited fractions and showed a wrong solution. Also, when a student asked if an answer to a radical equation problem $\sqrt{12-x} = x$ could be -4 as well (the students already knew that 3 was an answer), he posed the question "Can you have a negative value of a square root?"

showing $\sqrt{12-(-4)}=\sqrt{16}=4\neq -4$. This provided the student with an opportunity to see and reflect upon his reasoning and his confusion about the concept.

Knowledge of Pedagogy

Motivation

To motivate the students' understanding of a certain topic, Mr. Smith tried to use examples of problems that had applications in the real world. When introducing the Pythagorean theorem, he posed a problem: A teenager gets a 5-foot pole at a store and has to take a bus to get home. But it is only permissible to carry a maximum of a 4-foot-wide object on the bus. How could the teenager resolve the problem? Although no students got the answer, the students seemed to really be engaged in working on the problem. He appeared to expect them to get into the Pythagorean theorem by being interested and excited, working on the example, and seeing how mathematics would be applied in the real world.

Use of the Textbook

For teaching the mathematical topics, Mr. Smith, on the whole, followed the organization of the textbook. The textbook chapter "Radicals, Functions, and Coordinate Geometry" contained eight topics: (1) operations with radicals; (2) square-root functions and radical equations; (3) the Pythagorean theorem; (4) the distance formula; (5) geometric properties; (6) the tangent function; (7) the sine and cosine functions; and (8) introduction to matrices. He usually spent one class period to cover each topic. In the case of probability in the Pre-Algebra class, he spent two class periods to cover theoretical probability.

The organization of each lesson at times did not correspond to that suggested in the textbook. Although Mr. Smith used example problems and practice problems from the textbook, he reorganized the concepts in a topic or made connections between the concepts and the students' previous knowledge. For example, he used a handwritten handout for the lesson on operations with radicals that included a brief explanation of simplifying "rules" and a list of problems. Some of the problems were from the textbook, but others were not. He demonstrated his solution to a couple of problems on the handout:

> Discussing simplifying of radicals, the textbook says the following:
>
> Because the radical sign designates the principal square root, the value of $\sqrt{x^2}$ must be positive. Use the absolute value sign to indicate this when the exponent of a variable in the radical is **even** and the simplified exponent outside of the radical is **odd**.

In class, Mr. Smith extended the examples in the textbook and illustrated them on the board:

$$\sqrt{x^2} = |x|$$
$$\sqrt{x^3} = x\sqrt{x}$$
$$\sqrt{x^4} = x^2$$
$$\sqrt{x^5} = x^2\sqrt{x}$$
$$\sqrt{x^6} = |x^3|$$
$$\sqrt{x^7} = x^3\sqrt{x}$$
$$\sqrt{x^8} = x^4$$
$$\sqrt{x^9} = x^4\sqrt{x}$$
$$\sqrt{x^{10}} = |x^5|$$

The textbook did not go beyond the explanation; he presented the examples above on the board and kept them there for the whole lesson. He did not, however, pay attention to some simplifying problems in the textbook such as $\sqrt{a^2b^{10}}$ and did not make a connection with the problems $\sqrt{400}$ and $\sqrt{72m^2n^5}$ in the textbook. During his instruction on the Pythagorean theorem and the distance formula, Mr. Smith tried to get his students to find relationships, rules, or formulas by working on many problems. Although the textbook did not cover the topic, he used equilateral triangles of different sizes to let the students see how the Pythagorean theorem could be applied to find the height of each one. Although it tended to be a step-by-step procedure, he let the students try to discover the distance formula from the Pythagorean theorem as well. In introducing probability in the Pre-Algebra course, Mr. Smith's lesson was based on an activity in which the students rolled a die to experiment with theoretical probability.

In contrast, Mr. Smith sometimes appeared to disregard an important concept or a fact that should have been explained to the students and was presented in the textbook. He neither mentioned $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ nor explained it until a student raised a related question. The textbook clearly presented the concept with problems showing counterexamples. Among the problems used in lessons or the review materials, the problems on simplifying radicals mostly focused on numbers rather than variables. On the topic of square-root functions, neither the textbook nor Mr. Smith thoroughly explained graphing square-root functions or connected graphing and solving radical equations. He clearly covered the topic of how to graph with calculators, but his explanation on graphing the functions was very cursory. The graphs of examples of radical functions that Mr. Smith presented were too hard for the students to distinguish, which might have come from his "limited knowledge" of the topic. The graphs looked alike, and he did not make clear how they differed from one another.

Mr. Smith's Pedagogical Content Knowledge

This section attempts to portray the components of Mr. Smith's pedagogical content knowledge. While teaching mathematics in eighth grade, Mr. Smith's pedagogical content knowledge consisted of: (a) knowledge of mathematics; (b) knowledge of the students' understanding; and (c) knowledge of pedagogy. Based on the theoretical framework in chapter 2, I could discern the following components under each category.

First, Mr. Smith's knowledge of mathematics included his purposes of teaching mathematics, understanding concepts to teach, connection among topics, various ways of solving problems, and textbook knowledge. His goals for teaching mathematics were for students both to understand abstract concepts of mathematics and to apply them to realistic applications. Ms. Smith seemed to have strong in knowledge of the topics he taught. Although all the researchers cited in chapter 2 identified knowledge of curriculum as a contributing factor, curricular knowledge did not emerge in Mr. Smith's case during my observations, perhaps because of the limited time spent in the data collection. However, he appeared to be concerned about covering topics such as radicals and radical equations because the students would build new knowledge on them in high school mathematics.

Second, Mr. Smith's knowledge of students' understanding involved particular students' understanding and students' misconceptions, common errors, difficulties, and confusions about a topic or concept. Although he gained and used his understanding about the students' understanding in his instruction, his knowledge of assessment was not explicitly revealed. He frequently used his understanding about students' common mistakes and confusions in simplifying fractions and, further, connected such mistakes in working on fractions to mistakes with radicals. Students were always encouraged to try various approaches to solving a problem; Mr. Smith incorporated students' methods into his instruction.

Finally, Mr. Smith's knowledge of pedagogy was revealed in the form of both his efforts to motivate his students by providing realistic applications of mathematical topics and his use of the textbook and materials. He tried to find problems and stories to motivate his students. He took problems from the textbook and used them in his instruction and for students' homework. He did not use many materials for teaching radicals and radical equations except using a calculator for graphing radical equations.

CHAPTER 5

THE CASE OF MS. KING

This chapter describes the case of Ms. King in terms of her beliefs and pedagogical content knowledge. I present her beliefs about mathematics and about learning and teaching mathematics as they emerged from the data. Then I describe in detail how Ms. King's pedagogical content knowledge was manifested in her mathematics teaching. This description results in an articulation of the components of her pedagogical content knowledge. I begin by describing Ms. King's background in becoming a middle-grades mathematics teacher.

Ms. King's Background

Ms. King was an eighth-grade mathematics teacher in a small city in the South. It was her first year of teaching eighth graders. They were the same group of students she had taught the previous year; the principal of the school had asked her to move up with them. Ms. King taught all of the eighth-grade students in the school. She taught five classes of Pre-Algebra every day.

It was understood in Ms. King's family that the children would go to college. She did not know what she wanted to do, however, when she entered college. She thought, "I might be a physical education teacher just to be something since I was good at sports." Then she took a class called Introduction to Education, and one of the requirements was to go into schools. She went to a first-grade classroom and realized that she wanted to be teaching reading, writing, and arithmetic, not sports. It was at that point in 1973 that she knew she was being called to be a teacher. She never had any subsequent doubt about that.

Ms. King earned a bachelor's degree in education, which certified her to teach any grade from kindergarten through 8. She went back to school and got a master's degree, for which she had to choose between early childhood and middle grades. She chose to get a master's degree in early childhood because she was teaching first grade at the time. Later she got middle-grades certification. In 1975, she started her teaching career by teaching first grade. She taught 13 years in the elementary grades and 5 in the middle grades—seventh grade for 4 years and eighth grade for 1 year. She moved from teaching first grade to second grade, then third grade to fourth grade, and fifth grade, and finally seventh grade and eighth grade.

When she had her youngest child, she took a break from teaching. While being an at-home mother, Ms. King began to tutor students who were having trouble with mathematics. That tutoring sparked her desire to teach mathematics. While tutoring, she found that her tutees were not able to understand mathematics in the way that their teacher taught it. They seemed to do better when Ms. King worked with them one on one and taught them her strategies. As her own children moved into the middle grades, she started to do more tutoring with middle-grades students, which led her to start teaching in the middle grades. When Ms. King came to the school system, there was an opening for a seventh-grade mathematics teacher, which is another reason she ended up teaching middle grades.

Ms. King claimed that her "mind likes to create things." She would get bored if she kept working from the textbook day in and day out. She said she was the one who had to move the furniture around in her house every so often and had to paint and wallpaper just to make it different. In addition, she realized that she was "talented" and "unique" in that she was so eager to see her students learn by trying to come up with strategies to help them be successful. It was frustrating for her when they did not learn the way she expected. She said that the students liked her and her ideas, and they liked doing the games and the strategies that she provided as class activities. Creating those activities took a lot of her time. However, Ms. King was willing to work and put in extra time to create and develop activities. Although she knew that it would be easier to use and work out of the textbook every day, it would have been a "disservice" to her students. She described her approach as follows:

> Things just come to my mind, and so I create them. And that's why I'm usually at the school longer than anybody. And I'm up earlier because I'm working on the computer trying to create and organize, as you can tell [from] my files. I'm trying to acquire a packet of activities that I can use when I want to teach a particular objective. So that's why I say that I'm unique. A lot of teachers wouldn't go to all this extra work, because it *is* extra work. And sometimes … it's exhausting to say the least.

Furthermore, Ms. King acknowledged that she was "gifted in that she is creative" and was "not the typical sit-at-the-desk-and-write-problems-and-work-problems-on-paper" kind of teacher. Rather, she considered that group work, oral discussion, and peer tutoring would more effectively facilitate students' learning.

Mathematics was a challenge for Ms. King, but she did not fear doing it. If she was given a problem, she tended to keep at it until she could solve it. Mathematics had been easy for her when she was in high school, where she took geometry and algebra and did very well in both courses. In college, however, she did not take a lot of mathematics courses; she took only the basic core. Above all, she said she always enjoyed learning mathematics and wanted to pass on her love and enjoyment of mathematics to students so that they could "catch a glimpse that learning can be fun" and understand that they too could enjoy mathematics.

At the time of the study, Ms. King was taking a class called Teaching Algebra for Students' Success once a week for the semester. She learned new teaching strategies through the class. She said she had begun to become more aware "how algebra does connect to geometry" and that "we do need to make the correlation" between algebra and geometry and "not treat them as two independent mathematics [subjects]."

What Did Ms. King's Beliefs Reveal?

Mathematics

Ms. King viewed mathematics as "just so much a part of life," saying that we can survive without knowing science or social studies, but not without reading and mathematics. "Mathematics is working with numbers and is such an active part of our lives." We constantly use numbers to solve problems in daily situations such as time management and money management. Although we use it often, we just do not think about it. She gave the following example:

> If you only have a hundred dollars, and you go to the grocery store, you can't spend more than a hundred dollars. So you are basically trying to think of what groceries I can get for less than a hundred dollars without going over.

She elucidated that such an experience brings in estimation skills because the amount spent needs to include tax. Thus, mathematics is everywhere. She gave another illustration:

> When you go to sports, you are working with numbers: when you gain yardage, when you lose yardage, [and] when you play golf [and] you are under par. When you cook and follow recipes, that's numbers. When you have money, that's numbers. When you balance a checkbook, that's numbers. When you are driving a car, [and] you want to see how many miles you get to the gallon, that's numbers. Depending on what job fields you are in, there is bound to be some kind of relation with numbers. I just think numbers are out there, and that's just such a daily part of your life.

To Ms. King, "mathematics is just out there," and being able to read and work with numbers was indispensable to surviving in the world.

Mathematics also involved problem solving. Ms. King viewed problem solving as involving an "ability to take a problem, set it up, and follow the legal steps or whatever and come up with a correct answer." The more she got students to think, the more she helped them develop their problem-solving ability. "If we do all the thinking for them, then it is not teaching them to think for themselves." To Ms. King, a problem was one that people dealt with in daily life:

> If there are so many hours in a day and you've got this, this, and this that you've got to accomplish, you've got to balance your time. … It's something as simple as if you wanted to paint a room, you've got to know how much paint.

She concluded that "right there is mathematics" that involves problem solving. Moreover, problem solving forces the students to think and develop their creative powers, which, in turn, helps their problem solving.

Learning Mathematics

Ms. King viewed learning mathematics as like a puzzle in which the "pieces fit together" and a complete picture can be made. She also saw learning mathematics as at times like conducting experiments in science. In mathematics, when we try to justify an algorithm or prove one of the properties of a system, "students are taught that doing an experiment one time is not enough to justify [a conclusion]." Thus, they are asked to "conduct an experiment several times." She argued that we look for patterns in mathematics; we repeat an experiment several or many times so as to determine that pattern, which is similar to conducting an experiment.

Learning mathematics, according to Ms. King, is "a culmination of so much." She always attempted to tell her students that "learning is in a form of a spiral." In other words, what the students had learned in kindergarten was built upon in first grade, second grade, third grade, and so on. Thus, if students do not learn and understand basic concepts of mathematics in certain grades, then they come to have "learning gaps."

Her belief that mathematics is everywhere led her to conclude that students should learn mathematics because they would be involved in using numbers whether or not they were actively aware that they were doing mathematics. Even if a student were to become a painter, he or she would have to able to calculate how much paint would be needed on the job; it would be "working with numbers."

Teaching and Being a Teacher

Teaching Mathematics

Ms. King considered teaching mathematics to be similar to being a coach in that both involve teaching skills. Once students learn skills, they can put them all together, see what they have created, and apply them to new situations. Just as coaches teach by breaking skills into steps for skill building, in teaching mathematics, teachers "present the whole problem, and then you break it down." Having served as a softball coach at the school, Ms. King recognized that breaking skills down was very important.

Ms. King viewed teaching mathematics as a challenge as well and said that teachers should be entertainers:

> Teaching school is a challenge because of all the different personalities [they] are seeing. Some [students] who are aggressive in their learning, they desire to learn, and you see that. The hands go up. They are going to ask, and they are going to keep asking until they understand. Those are the kind of students [teachers] delight in. And then [there are] those who are sitting back, bored, could care less, misbehaving—how do you reach them? [Teachers] can try to catch them with an interesting question. [A teacher] can only be an entertainer so much.

For Ms. King, teaching mathematics was letting students see what mathematics is like. Thus, she tried to think of interesting ways to involve the students and make them want to participate actively.

Her Role As a Teacher of Mathematics

Ms. King saw her role as a compound of a "facilitator," "teacher or instructor," and "guidance counselor," depending on the teaching situation. When introducing a concept, she tried to present it in the form of a problem and let the students try to figure it out through some strategy of their own. At that point, she played the role of facilitator. When she let the students engage in an activity, for instance, she wanted them to actively think, talk, and create. Her "job was to be there to assist," "advise," and "give" them some direction when they got confused. Sometimes, as a teacher, she might know a method that would work better; their method might have worked only for that particular problem. Then she would regard it as her role as to present methods, possibly more than one method, for finding the solution. In doing so, she emphasized not only that there is more than one way to arrive at the same answer, but also that "it is not that somebody else is wrong and somebody else is right." Also, she regarded herself as a guide in that she led the students in their explorations.

Ms. King believed that it was important to know the "why" behind skills, procedures, formulas, and algorithms. If students have been taught only a formula, they can easily forget it and not know what to do. She emphasized the importance of knowing the why by asking students to prove their answers. She described examples from her lessons:

> If you don't remember that perimeter [of a square] is four times *s*, [you cannot know what to do]. But if you remember perimeter is the distance around it, 'take a walk and just start adding it up,' which is what one boy was saying. Same thing with area. If you understand what area means, that it is the area covered, I think that will help them.

Furthermore, she believed that if the students were able to explain "why," then they had an understanding. The reason that she encouraged proving was to "reinforce" how the students arrived at their answers.

> All of that is … to reinforce. And as they teach it back, any time you are the teacher, you learn it better. So by saying "prove it," that puts them in

the teacher position. And they go back and say how they arrived at it. So it now only helps them, it helps those who maybe didn't get the answer. And they can see how they should have arrived at it.

Her Strengths and Weaknesses in Teaching Mathematics

Ms. King said that she enjoyed teaching the mathematics in which she felt confident. She felt confident in using the "Hands-On Equation," which is a kind of manipulative device for teaching algebra. She was confident in solving algebraic equations. She said, "There would be no hesitation because I know that backwards and forwards, even if it is multi-step." Also, she acknowledged her confidence in teaching the divisibility rules; how to factor; finding the greatest common factor; prime and composite numbers; commutative, associative, and distributive properties; and classifying angles. She felt confident and comfortable with those topics since she had taught them for 4 years: "The more you teach it…that's where you feel strong."

The more she taught, the stronger she felt about her teaching. With experience, she came to learn new strategies. When she taught a concept repeatedly, she might realize that it had not gone over too well and would make an effort to come up with a different method or another style, which would result in her feeling stronger and better about her teaching. Also, her active involvement in teaching on a regular basis contributed to her confidence in mathematics. She expanded that view as follows:

> If you don't use it, you lose it. Well, I'm not going to lose it, because I'm using it. And in talking to my other team members, they don't feel confident in math. And I think the reason is, is they don't teach it. They don't use it on a daily basis, and so they are not working with those numbers. And so they begin to think they can't do math. The more you use it, the better you get at it.

She did not, however, consider herself an expert in mathematics. She said she would never teach advanced mathematics courses such as calculus or trigonometry, because she "would not feel comfortable" teaching those courses. She rationalized, "I don't use that enough in daily life. I would have to teach myself and re-teach myself that stuff. I could do it. I mean, I took trig in college and made an A, but I had to study." Therefore, she would not want to teach high school mathematics. Further, she did not think that her expertise lay in teaching high school mathematics; she judged that her expertise lay in working with middle-grades students.

Ms. King felt, she said, weaknesses in teaching "the form of word problems" and how to solve problems with percentages. A weakness meant "something [she] cannot step up and teach immediately without review." She related these weaknesses to being uncomfortable, saying, "a lot of teachers feel uncomfortable with word problems, and that's why they avoid them." Even though she felt weaknesses in understanding or teaching every word problem, she did not avoid doing word problems in her teaching.

She tried to help the students realize that "it is a story made up taking two numbers and requires [that you] either add, subtract, multiply or divide those two numbers. That's really all a word problem is." In her view, "somebody has made up a story to take those numbers, and now we take the information they gave you and work it out."

Despite the weaknesses she saw in her teaching, Ms. King was certain that she did "a very good job by talking," putting visuals on the board, and creating an environment in which the students can manipulate with the hands-on materials. She claimed that she made strong efforts to address all the learning styles—to accommodate each student's learning style and not just lecture. Thus, she "would not probably change [her style of teaching]."

Students

Ms. King was confident that she had a good rapport with students. She valued her students' understanding of mathematics. Mathematics was a subject that the students should not fear; they could "master" it. They could learn mathematics; they just had to learn how to manipulate it. "A good teacher pulls [her] students into wanting to learn, to participate," unlike an expert teacher, who just knows it very well. Ms. King said that experts are people who might know the material themselves but have not presented mathematics in such a way that the students could understand and would enjoy. In contrast, a good teacher is one who makes students feel as if they have learned something when they leave the class. Ms. King recounted her experience:

> When somebody asked the question about "how many outcomes [are] possible?" and you heard the different answers [of the students]. "Well, which one is it?" "I don't know, you go home and figure it out." You leave them hanging where they want to go investigate, and you will have some who are willing. Like I had one who came back and said, "I think it is this," and he explained why.

For Ms. King, middle-grades students in general seem to be difficult to deal with. "This is a difficult age"; not only do the students go through changes, but also they do not figure out what they want. Middle-grades students are not motivated all the time, whereas younger children are eager to learn. Having good rapport with the students, however, she was convinced that she was capable of engaging them in activities. She understood "activity levels" for the students and would not have the students sit the whole period, but would have them involve in activities. Also, doing activities as a group allowed students both to interact with the topic at hand and to socialize with one another, which is important to them at this age.

How Did Ms. King Teach Mathematics?

Knowledge of Mathematics

Purposes of Teaching Mathematics

Ms. King's goal for teaching mathematics was to teach students to be successful in society: being able to "balance a checkbook" and knowing "how to read a recipe," which are important to survive. She supposed that students who go into special fields, like engineers, would need "higher mathematics." Those students who are going into mathematics should know certain concepts more than others who have no desire; however, the majority of her students would not need higher mathematics. One of her objectives in teaching mathematics therefore was to make the average students successful. In other words, she aimed to get them to learn how to master their multiplication, division, addition, and subtraction facts so that they could balance their checkbooks, handle money, and follow a recipe and cook. Ms. King's ultimate goal was for the students to develop problem-solving skills through working in groups, which could only help them in life; "life is full of problems," and there is one problem-solving episode after another.

The objectives that Ms. King deemed she should teach came from the curriculum map, which was derived from the Quality Core Curriculum (QCC), the state curriculum standard for the eighth grade. The mathematics teachers in the school system, together with expert in mathematics from the RESA, had recently designed the map. They identified the standards of the QCC as *Essential*, *Important*, *Compact*, or *BM* (benchmark). What had already been taught and thus did not need to be "touched" again was identified as *Compact*. *BM* (benchmark) was used to identify a standard that needed to be revisited and further developed in a later grade. The team designated the quarter of the year in which they would address these objectives. Ms. King regarded their efforts to select the objectives that would flow smoothly into other objectives in planning the curriculum map as worthwhile and practical. Using the curriculum map, she decided what she would teach and sought to cover those objectives.

Curricular Knowledge

Ms. King realized that many topics and concepts in eighth grade were repeated from seventh grade. When a topic was repeated, it should take what the students had previously learned and add something new.

> Like what I gave them today: All of that was introduced and exposed to them last year, and so what I was trying to do was move to a quick review of your formulas.

She illustrated how she organized topics and concepts in geometry from the review of formulas:

> I was going to move more … into the triangles, and then especially the right triangles, and then moving into the Pythagorean theorem. I just felt like that flowed starting with the formulas. … We were talking about

> triangles, where they were classifying their triangles as right, acute, obtuse. And we talked about a triangle being equal to one hundred and eighty degrees. And then I wanted to move into how you can measure the lengths of right triangles, the hypotenuse, if you are given the legs. I was trying to stay in a circle with triangles. And then I'll move back out probably to the intersecting lines, and how they intersect. And you have alternate interior angles and alternate exterior angles.

Ms. King appeared to try to organize topics and concepts to teach according to the connections among them. Also, she attempted to arrange topics to build new knowledge on previous knowledge. For example, basic properties of triangles were covered in the previous year, and thus she wanted to revisit them and move on new knowledge.

Understanding Topics to Teach

Ms. King said that she did not have any trouble teaching the factoring of polynomials and multiplication of binomials in Algebra I. She declared that she could solve any algebraic equation without "hesitation," defining an algebraic equation as "a math problem, a mathematical sentence that contains an equal sign, but it also contains variables, numbers, and at least one operation." She understood that "formulas are really equations." When she started with an expression like $4s$ (an expression for the perimeter of a square), she should be given the value of s in order to find it. When teaching algebraic expressions, she tied them in with formulas for perimeter and area. She explained, “Those formulas are just algebraic expressions or algebraic equations.” Further, she said, “If the values of the variables were provided, then you [could] substitute them and find the answers” in the lesson. As a result, she believed that she “took away” the students’ fear of the formula because they just understood it to be just an algebraic equation, and they knew how to deal with that.

Ms. King explained that in her instruction, “algebra is just solving for the unknown.” Algebra is basic mathematics that is spread throughout the grades from K to 12. Even in kindergarten and first grade, children are “given a box or a question mark” and asked to solve “five plus question mark equals eight.”

> Then it became five plus a box equals eight. They were taught to find the missing addend. Algebra just replaces that box or that question mark with a letter, and then it becomes five plus *n* equals eight. So, to me, algebra is no different than what they were really being taught in first and second grade, but it is just using variables. And for many students, all of a sudden, they think it is something they can’t do. And I actually have taken it all the way back to kindergarten and first grade, and I showed them, “Do you remember where it was five plus a box equals eight?” “Yeah.” And I took them step by step and showed them, “Well, all they’ve done is replace that box with a variable, and you are still doing the exact [same] thing.” And I

> said, "What are you going to do find the value of *n*?" And they said, "I'm going to subtract eight minus five." And I said, "That's exactly right. That's all algebra is."

She would also use tables in which the students could begin to see patterns as a way to help them develop algebraic concepts. Those tables yielded *x*s and *y*s, and students could plot them on a coordinate plane, which led to ideas of slope and linear equations.

Regarding rational numbers, Ms. King understood that the concepts of number system, properties, compare, convert, simplify, evaluate, graph, and proportions are all related to rational numbers. That insight guided her to create "a wheel of rational numbers" (Figure 5). She used the wheel to help the students see "where the proportions came in, the graphing, evaluating the equations and inequalities, simplifying fractions, and converting fractions to decimals to percents."

After teaching rational numbers according the wheel of rational numbers, Ms. King moved on to probability. She said, "Probability is a ratio." "It is expressed as a ratio, and a ratio is basically a fraction." She thought that it would be a smooth transition into teaching probability. "Just like decimals and percents are equivalent expressions of fractions," she was inclined to teach "the connectivity" of fractions and probability. To help students understand the concept of probability, she sought to show that it was to "something they use more often than they think they do."

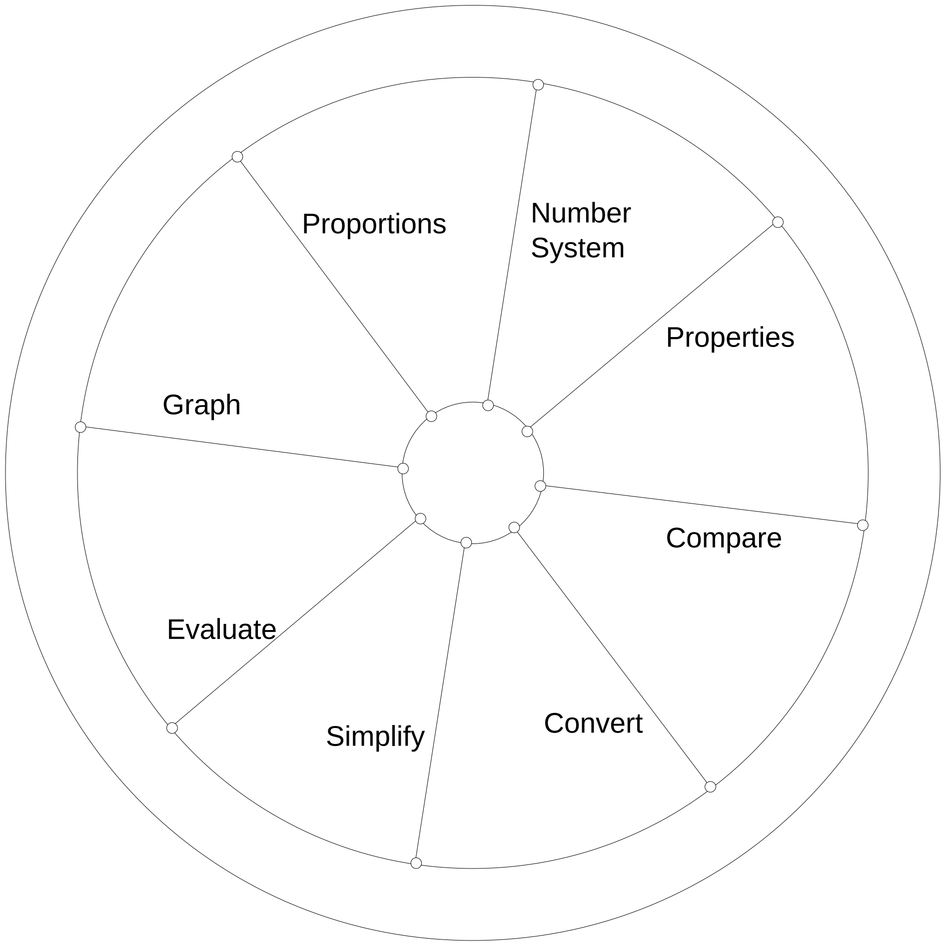

Figure 5. Ms. King's wheel of rational numbers.

Ms. King saw geometry in real-life situations. People encounter squares, rectangles, triangles, and circles in everyday life and are exposed to problems such as "covering [the] kitchen floor" and "putting [a] border in the mathematics classroom." Geometry is involved in real-life problems and requires using numbers and measurements dynamically, which is why she attempted to connect these concepts with geometry for the students to increase their understanding. Using word problems applied to realistic situations allows students to grasp "how geometry is a real problem and a life problem." This approach is more meaningful than just giving them a triangle and having them find its area. In addition, she saw that geometry employed algebra to solve problems involving shapes.

In the same manner, Ms. King sought to associate integers with realistic situations. She gave an example:

> We talked about negatives, and we talked about the three basic places where we actually saw negatives were. In football, the boys could definitely understand loss of yardage—they understood that to be a negative. And then we talked about thermometers—temperature—they could easily understand that one. [And] being in debt.

She tried to connect the students' real lives to the concept of negative integers so that the students could appreciate that "it is not an isolated concept but real" and that "they are using it" in their lives.

Knowledge of Students' Understanding

Ms. King thought that students should be not passive, but actively participating and discussing while learning. Learning occurs more when the students get actively involved than when they passively sit and "soak up [learning] like a sponge." The more the students dynamically engaged in learning activities in class, the greater the chances of their learning and remembering would be.

Different Learning Styles

Ms. King believed that all students would enjoy working with numbers. However, students are different in the way they learn; in any class there are "concrete learners" and "abstract learners." Teaching mathematics may well involve the abstract, and some students might not be able to grasp it yet. To motivate them, she used various manipulatives, visuals, and hands-on activities. Some of the students were prepared to move on; so they could "picture the abstract."

Considering the many different learning styles of her students, Ms. King sought to have a variety of ways for her students to actively participate in her classroom:

> Basically, they can use their learning style. Because you've got some who are talkers; they learn by hearing, so maybe they are my talkers in the group. And then you have those who are writing it down; they are your kinesthetic, or they could also be your visual learners, who like to see it in writing. I don't limit.

Ms. King did not force those students who were very quiet and diligent in their note taking to make responses. She recognized such students and respected their style of learning. At the same time, she did not mind calling on them to see whether they understood. Also, when putting the students in groups, she did not force students who do not want to play a part to work in a group. Instead, she let them work alone provided that they made an effort.

Similarly, Ms. King realized that there existed motivational differences among the students; students who like mathematics, are good at mathematics, and like to work with numbers will learn anything that teachers put out in front of them. Others, who accomplish "on a minimum scale only just to get by," have little or no desire to learn mathematics. Ms. King explained that if a student likes mathematics, then he or she is willing to learn mathematics that teachers present. Moreover, in some cases, it is like a challenge for them. Those who have developed a good number sense will take a given problem as a challenge and attempt to solve it; they enjoy being challenged. They are more interested in word problems.

> Some kids interacting—the ones you heard who are doing the most interacting, asking questions, and talking—are my good students. And they are going to work on it one way or another. And nine times out of ten, they are going to get the right answer.

In contrast, others neither like mathematics nor feel confident with it. They could not care less about mathematics, because they are frustrated, cannot solve problems, and do not understand mathematics.

Learning Difficulties

Ms. King claimed that eighth-grade students should know basic facts; however, they did not. The students should have a basic understanding of fractions and decimals as well. She recognized that some of her students were not capable of working on equations with fractions and decimals. Many of them apparently had not worked enough with fractions and decimals, which caused them to feel uncomfortable with those concepts. Ms. King supposed "they probably were not given manipulatives when they were working with fractions." Moreover, she was aware that they did not understand that fractions are part of a whole. She assumed that enough time was not spent in helping them develop the concept of fractions in the previous grades. Their teachers might have felt pressure to teach students how to add and subtract fractions and might not have attempted enough to help students understand what fractions mean. Therefore, she attempted to use manipulatives. Some of students, however, said that "it was more confusing than just applying the rules."

Ms. King realized that students had difficulties in working on inequalities having a negative sign with a variable. During the lesson, she just told them, "If you have a negative with a variable, you must reverse the inequality symbol in order to graph the solution correctly."

Common Errors and Misconceptions

Ms. King noticed the common mistakes students made when they worked with mixed numbers. In particular, the students tended to leave mixed numbers as they were and then try to subtract a fraction from the fractional part even when they needed to regroup. They appeared not understand why they needed to take one of those whole number units and by regrouping rewrite it as a fraction, which, she recognized, was confusing to them. In contrast, there was not much confusion with the addition of mixed numbers. The students were aware that if they got an improper fraction, then they could easily rewrite it as a mixed number, adding the whole number parts together and the fractional parts, respectively. However, she thought that approach was likely to confuse them. Instead, she used the approach of turning all mixed numbers into improper fractions to assist the students who could not grasp the concept. For instance, she had her students convert $4\frac{1}{3}-2\frac{4}{7}$ into the form $\frac{13}{3}-\frac{18}{7}$ rather than to do subtraction of whole numbers first, 2 from 4, and then solve $\frac{1}{3}-\frac{4}{7}$. The result of subtraction of fractions is negative, but the students tended to ignore the negative sign. Indeed, they just combined 2 and $\frac{5}{21}$, and thus, got $2\frac{5}{21}$, instead of $1\frac{16}{21}$.

Ms. King pointed out that some students acted as if they had no common sense. For instance, when reviewing the computation of rational numbers, Ms. King described the students' mistakes on one of problems in the previous test. The problem was to find the value of x in $x - \frac{9}{11} = \frac{1}{11}$ and some students put $\frac{8}{11}$ instead of the right answer $\frac{10}{11}$. It was apparent to Ms. King that the students tended not to carefully read what a problem asked them to do. Instead, they just performed procedures.

Likewise, she found similar errors when they worked the following problem: "Write a proportion to solve for the variable. Then solve: 25 tablets for \$2.69, 150 tablets for \$*x*." By using common sense, the students should have guessed that the answer must be more than \$2.69 because 25 tablets, which are fewer than 150 tablets, cost \$2.69. She was certain that the students just "went through a series of steps, put a number, and did not even check to see if they've got a reasonable answer."

The students had a misconception about using exponents. Ms. King said that the students had a tendency to multiply a base by an exponent "instead of multiplying the base times itself." Further, the students seemed to find it difficult to work with negative exponents as in the expression $x^{-5} \cdot x^{-5}$. By asking the students to think about how they could rewrite it, Ms. King encouraged them to connect the process with what they had done with $x^5 \cdot x^5$, $x^5 = x \cdot x \cdot x \cdot x \cdot x$. She went over the process as follows:

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = 2$$

$$2^0 = 1$$

$$
\begin{aligned}
2^{-1} &= \frac{1}{2} \\
2^{-2} &= \frac{1}{2^2} = \frac{1}{4} \\
2^{-3} &= \frac{1}{2^3} = \frac{1}{8} \\
&\vdots \\
2^{-10} &= \frac{1}{2^{10}}
\end{aligned}
$$

Students' prior knowledge affected their understanding of the area of parallelogram. When $b \times h$ (base times height) was presented on a card, the students immediately said it was the area of a triangle. When Ms. King showed the students another card with the expression $\frac{1}{2}(b \times h)$ written on it, they realized that both could not be triangles.

> Last year, I remember they frequently did base times height, but they failed to divide by two. So, there was not total understanding of the formula for a triangle. Because I tried it with a parallelogram, and showed them a parallelogram, and then said, "When you take the diagonal, you have a triangle, and so that is what you have to remember to do. Go ahead and find the area of the parallelogram and then divide by two, and you have found the area of the triangle." Maybe showing how the two go together—I don't think last year I did that—so maybe that will clear [up] some misconceptions.

The students also found it hard to find the height of a triangle or of a parallelogram when it was not given. Ms. King showed, using different triangles and parallelograms, how the height could be calculated.

Knowledge of Pedagogy

Representations

Ms. King had various ways of representing mathematical ideas and concepts during the lessons. The representations varied in explanations, examples, counterexamples, and demonstrations conveyed by symbols, words, and pictorial forms. The representations were combined with one another most of the time. In teaching geometry, she provided drawings along with verbal explanations when defining figures. For instance, she drew a triangle, square, and rectangle and showed what *perimeter* means by giving the verbal description “distance around” (Figure 6). She explained why the formulas $P = 4s$ and $P = 2l + 2w$ (s: side, l: length, w: width) work using drawings. To make the concept clear, she gave an example of finding the perimeter a polygon (Figure 7). *Perimeter* and *area* were verbally illustrated as a “border of a kitchen floor” and “how much paint

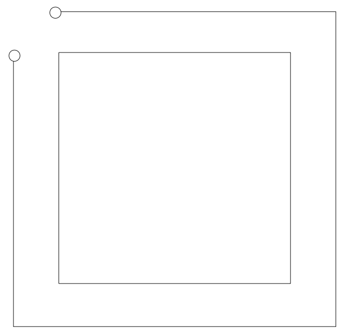

Figure 6. Perimeter of a square as “distance around.”

[or wallpaper] is needed to cover” a wall, respectively. *Parallelogram* was described as “leaning rectangle.” She used “chair [ ],” instead of , to indicate *perpendicular*.

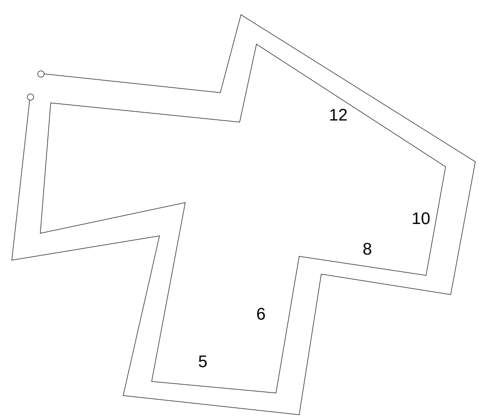

Figure 7. Perimeter of a polygon.

Ms. King attempted to find many examples of geometric figures in real-life situations. They included a shoebox for a rectangular prism; the edges of a wall and the floor of the classroom for skew lines that do "not cross, [are] not parallel, and [are] not in the same plane" with a drawing (Figure 8); railroad tracks for parallel lines; a baseball diamond for right triangles and the Pythagorean theorem; and a candy jar for a cylinder. When teaching the concept of probability, she exemplified it by referring to a weather forecast. She employed word problems for almost every concept and topic.

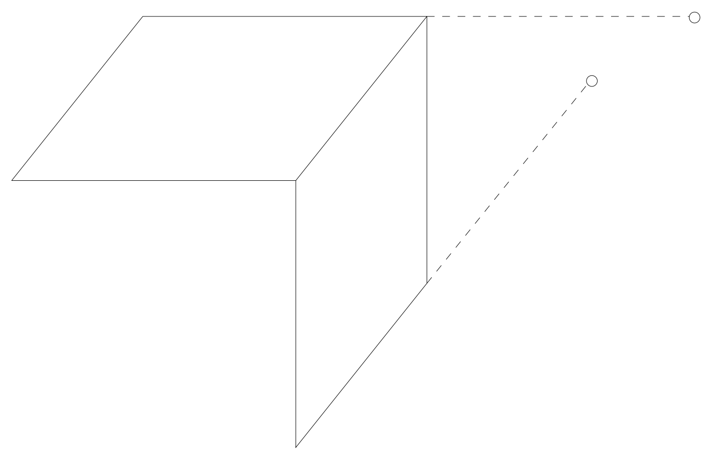

Figure 8. Skew lines.

In addition, Ms. King presented more than one example for a concept. For a sample space of probability, Ms. King asked the students to list all the possibilities when rolling a die, choosing a letter from the alphabet, selecting a student from among those in the fourth-period class, and flipping a coin. When the students confused finding the height of a triangle with finding a "diagonal," she offered various triangles (Figure 9) and extended the concept to the height of a parallelogram (Figure 10). While teaching the Pythagorean theorem, she showed several ways in which it can be applied to find the third side of a triangle (Figure 11). A nonexample was given to her students to challenge or confirm their grasp of a diameter (Figure 12).

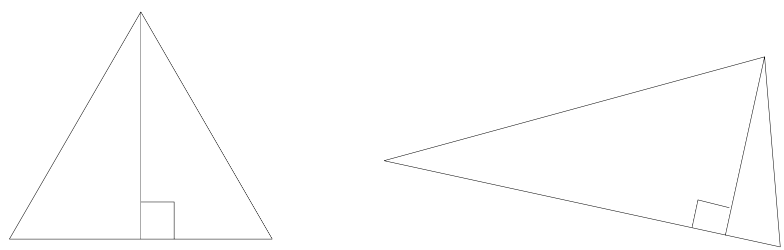

Figure 9. Height of triangles.

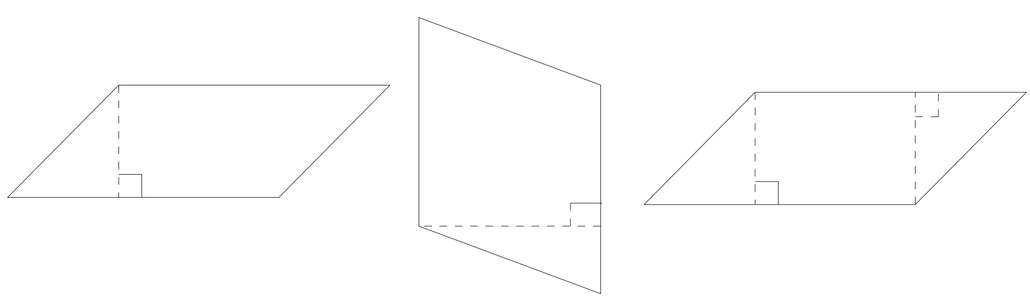

Figure 10. Height of parallelograms.

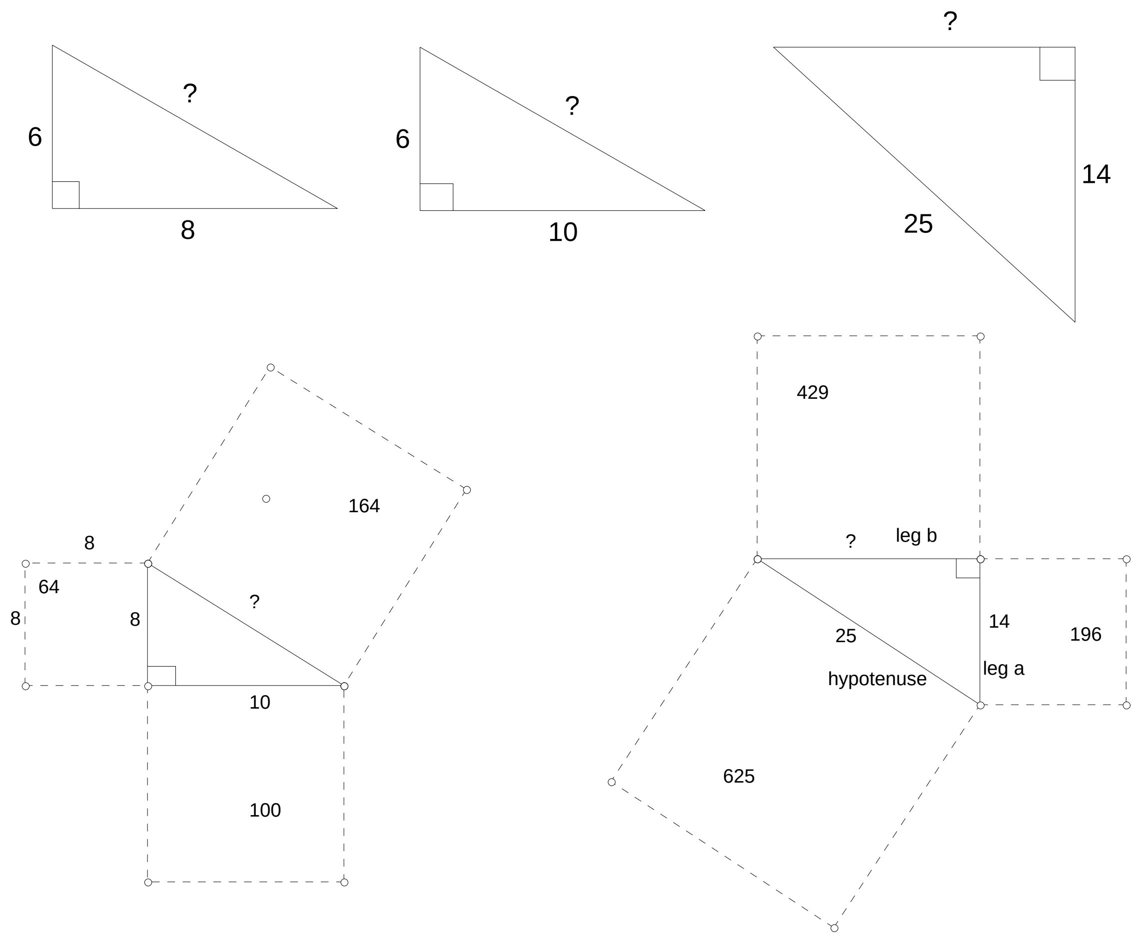

Figure 11. Right triangles used in applying the Pythagorean theorem.

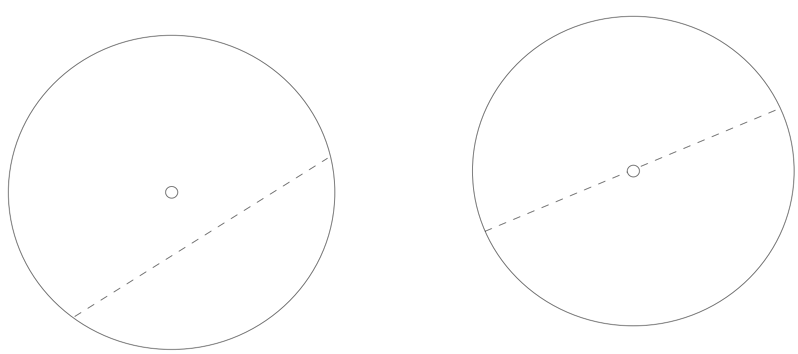

Figure 12. Diameter of a circle: Nonexample and example.

In demonstrating that the sum of the interior angles of a triangle is 180 degrees, Ms. King cut off the three vertices of a triangle and put them together to show they make a straight angle of 180 degrees. She also used a ¼ inch-grid paper to demonstrate how the Pythagorean theorem works. Ms. King drew 3-, 4-, and 5-inch squares at the right corner of the grid paper and cut them off the paper. Then, she glued 3-inch and 4-inch squares and then placed 5-inch square to touch the corners of them (Figure 13), showing

$$3^2 + 4^2 = 5^2 (?)$$

$$9 + 16 = 25 \quad (\surd)$$

All the students were given a ¼ inch-grid paper, scissors, and glue and followed the steps when Ms. King was doing.

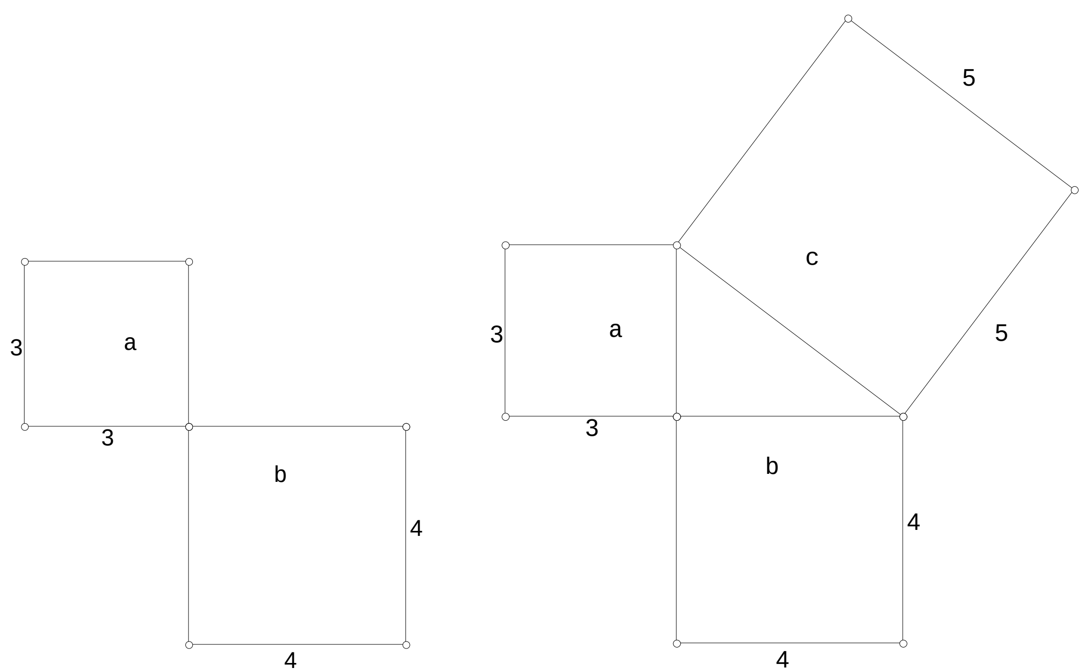

Figure 13. Ms. King's demonstration of the Pythagorean theorem.

Lesson Organization and Planning

In organizing her lessons, Ms. King consulted the curriculum map. The curriculum map attempted to create a flow that would lead from one objective to the next objective. When designing the map, the teachers had tried to identify “a good starting idea” in each grade by considering where students finish up in the previous grade. Finding one concept guided them into the next concepts, which affected Ms. King’s organization of her lessons. Ms. King believed that there was a “natural flow” in that certain concepts and topics needed to be taught first. Thus, she sought a flow that seemed to be natural and to move in that order. For instance, she said that she would not teach geometry before the students had learned the order of arithmetic operations and had worked on evaluating algebraic expressions, because they might not understand what you mean by algebraic expressions and that they have to “substitute the values in and evaluate them.” When the students understood algebraic expressions and could substitute the value in and use the order of operations, the problems with algebraic expressions might become simple problems that they could easily solve. So, they could work with geometric figures in figuring out perimeter, area, height, length, or width of a figure on a given condition by solving algebraic equations. For example, the area of a rectangle is expressed as $A = lw$ where l: length and w: width, which is an algebraic expression. Given *l* and *w*, finding the area of the rectangle is solving an algebraic equation.

Ms. King organized her lesson so that she started with “an essential question,” which was always written on a paper on the easel at the front of her classroom so that the students could read and copy it in their journals. She would then present a problem to interest the students, get them to work on the problem, and lead them back into the discussion of the essential question. In the lesson on probability, for example, she began with the question, “How can outcomes of events be predicted and described?” She wrote: (a) What is probability? (b) What is a sample space? (c) ______________________ [The range of probability?]. The essential questions were directly related to a key concept of the lesson. She said that she liked to use the essential question because it kept her focused throughout the lesson; she would not change that technique. However, “essential questions” varied in form: They were questions, key concepts, topics, terms, or formulas, depending on the topic of the lesson. The essential questions were continuously displayed for the duration of the lesson, including review lessons.

When planning a lesson, Ms. King acquired the objectives from the curriculum map.

> I’m always constantly looking back at this [curriculum map] when planning lessons. And I do try to make it make sense. When I choose

> something, I try to choose the combination that works well together. Like I wouldn't try to teach fractions and then jump to geometry or jump to probability. Now, the reason we tied in probability… so that's why I went into probability. But see this was actually review, trying to teach them that we had just finished studying integers, and rational numbers is the next subset. And it is the largest subset that includes all of these, and then we used the same properties with rational numbers as we do with integers and whole numbers. Just trying to show them how it all works together.

Ms. King used the textbook "as a resource" to consult what she would cover under an objective. She might come up with her own ideas or might adopt some of the activities out of the book. When planning, she declared that she designed her lesson plan by browsing through some of the textbook's activities to see how the textbook presented ideas and topics.

Ms. King usually made up most of the questions, especially the essential questions, and the classroom activities; she took word problems out of the textbook at times. In doing so, she always thought, "What is it that I want them to learn from this?" This question allowed her not only to concentrate on the topic or concept but also to see exactly where the lesson would go. Also, the questions were intended to facilitate the classroom discussion; she explained, "When you ask a question, by their answering, you can tell if they have understood. The question gets the discussion going."

Teaching Strategies

During her lessons, Ms. King employed activities and games through which the students might come to understand a concept. She said that she tried to acquire "a packet" of activities that she could use for teaching a particular objective. In particular, she loved activities and games because those made learning fun, which influenced the students to enjoy learning mathematics. If the lesson was boring, the students would not be motivated or pay attention, which is why she tried to create activities and worksheets, to make it interesting and to help and reach every child. However, she admitted that there was "a time for paper and pencil" as well during her lesson; there were needs to balance. By doing games and activities, the students would accomplish the goal, which was to understand a concept and be able to apply it to different situations. Then she would go to paper and pencil. Further, she affirmed that she was unusual in that she was "not like some teachers who just open a book and assign pages"; she also attempted to design worksheets for both in-class tasks and homework.

For the lesson on probability, Ms. King prepared eight tasks in plastic bags for the students to work on in groups. She chose a couple of activities from the textbook and invented the rest of them. Each package contained an index card on which a problem was written and some material to use in solving a problem. For example, the Deck of Cards instructions said, "Use the given deck of cards—36 cards, 18 red and 18 black, 6 face cards of each color." The Coins instructions said, "Use the given coins –24 coins of 2 quarters, 5 dimes, 10 nickels, and 7 pennies." An example of each task was shown on each index card; for instance, P (choosing a quarter) $= \frac{2}{24}$. Each group had to create 10 probability questions that should include all the probability of "an impossible," "a certain," and "a 50-50 chance," that is, 0, 1, and ½, and find answers to those 10 questions. Later, after this group activity was done, each group had a presentation for the following two class periods that was called "student teaching or presentation." In the student presentation, a group of two or three students stood in front of the whole class and presented the questions about probability that they created for their activity in the previous class period. While one of the students read their questions, the other students wrote both a sample space for the activity and their probability questions. After reading all of the questions, the presenters played a role of teacher by going over the questions and interacting with their classmates to get an answer. The student teachers encouraged their classmates to come up with answers.

While preparing these materials, Ms. King tried to get enough probability items so that she could have small groups to foster the students' active involvement in the activity. If the students were to work in larger groups, then one or two of them might never do anything. If they were in a group of two or three, however, then they would be forced to participate. In making groups, she purposefully assigned the students to insure that she had a high-level child mixed with a low-level child in each group. She pursued such combinations because she wanted to keep the low-level students from only trying to come up with an answer. Varied abilities in each group allowed students to benefit from one another. For that purpose, Ms. King designed and made use of a seating chart. She also tried to bring about discussion in each group. She believed that the discussion would help those students who had trouble analyzing to develop that skill by listening to their peers analyzing.

To address the different learning styles of the students, Ms. King tried not to stick to one teaching style. She said,

> If it is a visual learner, I try to be sure I put the information on the board. Auditory, I'm saying it so they can hear it. And then like if we do this activity … that would be a kinesthetic [opportunity] for the ones who actually want to get the hands-on stuff. … [I'm] trying to involve all the senses because there is greater retention when you involve more senses than when you just involve one. … I just basically use all the same strategies for all of them so that I'm reaching all the learning styles when I do that.

While teaching, Ms. King spoke loud enough for all the students to hear, wrote almost everything she said in an orderly way, and prepared materials for students to work with on the topic for each lesson. Instead of giving the students the answers to questions right away, she attempted to put the answers on the board after they had discussed and then had drawn a conclusion. In a review session, all the students had their own boards and markers, which they worked with. They shared their work with their peers and Ms. King, which also satisfied the students' desire to go to the board and share their work. The purpose of having the students use the boards and markers was to keep them all engaged. Ms. King explained that using boards and markers helped her easily see if they got a correct answer, and they could also see what their peers did.

Ms. King appeared to try to associate mathematics with realistic applications, which reflected her beliefs about mathematics and mathematics learning and teaching. She thought learning how to work with fractions, decimals, percents, negative integers, positive integers, and solving word problems was "real." She gave examples of how certain mathematical concepts could be applied to the world; for instance, weather forecasting and sports for probability, carpeting the kitchen floor for finding areas and perimeters, and shapes in real life for geometric figures and terms. Such efforts also resulted in frequent work on word problems. When teaching the Pythagorean theorem, she wrote a word problem on the board before the class began and kept it up during the lesson: "A baseball diamond is actually square. The distance between bases is 90 feet. When a runner on first base tries to steal second, a catcher has to throw from home to second base. How far must the catcher throw pick off the runner?" Along with the problem, she included a picture of a baseball diamond. As an application of the Pythagorean theorem, she let the students go to the field outside the classroom with protractors and measuring tapes to figure out how the Pythagorean theorem works in the real world.

Use of Materials

As stated above, Ms. King regarded the textbook as a resource or a manual and not as her “bible.” Although she referred to the textbook, she would not “go by the flow of the book.” She did not think that the book was “set up in such a way” that she needed to follow the order of its chapters. To prepare a lesson on a particular concept, she said that she studied that concept and would look through some of the examples in the book. Then she sought to think of what else she could use to help teach the concept. In other words, she took ideas from the textbook but also let her mind explore beyond the textbook. During the lessons that I observed, she never took out the textbook, even when she presented problems from it.

Ms. King looked through the textbook after she got objectives from the curriculum map in order to see how the textbook taught and presented those ideas in the objectives. She said that she sometimes applied the textbook’s suggestions to her lessons, but she usually added her own thinking and considerations to it. If “a creative idea” came to her, then she would design her own plan. After she taught, demonstrated, and worked on it with her students, the students could “go to the textbook” and attempt the problems out of the textbook. In addition, the textbook was used to find problems, formulas, and word problems for instruction and for homework. Further, she guided her students to use the textbook as a resource by saying, "That's where you go to find formulas; that's where you go to study examples." Thus, to her, the textbook was "just a manual" that she relied on to see the direction that it led.

On the other hand, she on occasions found some situations in the textbook to be confusing to the students. Some of the students said, "We tried to follow the examples in the book, and we didn't know what they were doing." She realized that certain procedures in the textbook were confusing; she would avoid teaching those procedures.

Along with the activities, Ms. King used materials consisting of worksheets, markers and boards, protractors, rulers, calculators, calendars, coins, cards, number tiles, letter tiles, measuring tapes, grid papers, scissors, and "Hands-On Equations." All these materials were part of her resources. Some of the materials, such as play money, number tiles, games, and letter tiles, were brought from home. She made every effort to find materials that could help the students develop a certain concept. All of these materials that belonged to one activity were grouped and stored in her cabinet in the classroom. With few exceptions, she distributed the materials to the students and gathered and put them back in the cabinet after those activities were finished. Everything was under her control.

Above all, a mathematics journal was a very important material for the students' learning in Ms. King's class. Every student had to have his or her own mathematics journal as a resource and future reference. Her motto was "If I put it on the board, you put it in your book." The mathematics journal was a manual that the students could use. She said,

> It is a manual, and that's what I've tried to explain to them. Just like a mechanic doesn't know everything; he has a book that he can go to. If it is a particular job that he hasn't done in a long time, he might need a little reminder of how to do this particular job. Same with an engineer. He doesn't know everything, but he's got a manual or a resource that he can go to. And what I've tried to emphasize to the kids is that this is just a resource so that [if] by chance you have forgotten this particular concept, you can go into your resource and look at it and see what it means and see an example or two. And then maybe that will be "Oh, that's how you do it."

The textbooks were never out on the students' desks during my observations. In contrast, the students took their mathematics journals out to take notes as needed in every lesson. What the students kept in their journals included the essential question, answers to the question, examples, properties, formulas, tables, and charts. For instance, when teaching about the Pythagorean theorem, Ms. King got the students to open their journals and make another triangle that works with natural numbers of a given length. At times, the students "transferred" items from worksheets when the concepts or problem types were important. What is more, the students were encouraged to consult their mathematics journals when taking tests and exams, which was another purpose Ms. King used to encourage the students to keep their journals.

Ms. King's Pedagogical Content Knowledge

The purpose of this section is to clarify the components of Ms. King's pedagogical content knowledge that emerged from the results in the previous sections. While teaching eighth-grade mathematics, Ms. King's pedagogical content knowledge consisted of: (a) knowledge of mathematics; (b) knowledge of the students' understanding; and (c) knowledge of pedagogy. Using the theoretical framework in chapter 2, I identified the following components under each category.

First, Ms. King's knowledge of mathematics included her purposes for teaching mathematics, understanding mathematical topics, and knowledge of curriculum. Her purpose of teaching mathematics was to help students master basic concepts of addition, subtraction, multiplication, and division to be successful to live in society. She ultimately aimed to help the students develop problem-solving skills because they faced problems to solve in everyday situations. This aim was related to her beliefs about mathematics: Mathematics is not isolated from real life situations, but a part of our lives. This aim was also connected with Ms. King's efforts to provide realistic examples and situations in her instruction.

Ms. King’s curricular knowledge seemed to focus on following and covering objectives of the curriculum map derived from the state standard curriculum for the eighth grade. Using the curriculum map, she selected topics and concepts to teach and organized them by considering the flow and connections with previous topics or concepts. Although Ms. King said that she was trying to help her students to build new knowledge on their previous knowledge, she often appeared to just repeat what she had done before in the form of review rather than to add something new. For instance, when teaching probability, Ms. King might have covered compound probability; however, she did not go beyond the basic concept of probability.

Ms. King appeared to have a somewhat deep understanding about topics she taught. The wheel of rational numbers is a good example that reveals her insight and understanding of rational numbers. In addition, she understood that algebra is not limited in upper grades but spread throughout the grades K to 12; young children in early grades such as kindergarten and first grade do algebra (Lodholz, 1990; NCTM, 2000; Philip & Schapelle, 1999; Usiskin, 1988). Further, she recognized patter recognition as a way to help develop algebraic reasoning (NCTM, 2000). Even though she admitted that she felt weak in solving word problems, Ms. King never seemed to be unwilling to work on word problems with students and planned good word problems all the time. In addition, she came to realize connections between algebra and geometry and seemed to apply those connections to her lesson planning and organization.

Second, Ms. King's knowledge of the students' understanding was composed of her knowledge of the students' different learning styles, a realization of their difficulties and confusions in understanding particular concepts, and a recognition of their misconceptions and common errors in the process of making sense of certain ideas. She saw the purpose of students' learning mathematics as to master basic facts and to manipulate mathematics to solve problems, which was reflected in her classroom instruction by engaging students in various activities. She regarded a good teacher as one who can present mathematics for students to foster understanding and enjoyment: Her attention to students was connected with the ways she used instructional strategies.

Ms. King realized that students learn in different ways; however, her realization seemed not to be associated with subject- or topic-specific knowledge. Rather, her dealt with students' different learning styles broadly by using auditory, visual, and kinesthetic approaches. She appeared, however, neither to recognize the processes students used to make sense nor to pay attention to particular students' understanding of a mathematical concept.

Ms. King recognized students' difficulties in learning and understanding fractions, decimals, and inequalities. Also, she was aware of students' common mistakes and confusions when working on addition, subtraction, multiplication, and division of fractions and decimals, on solving algebraic equations and inequalities, on using exponents, and on understanding the area and perimeter of polygons. Ms. King sometimes gave alternative explanations and examples to help the students understand and correct such confusions and misconceptions; at other times, she provided counterexamples and realistic situations, which showed her knowledge of pedagogy.

Finally, her knowledge of pedagogy included her ways of representing ideas, her ways of lesson planning and organization; her teaching strategies; and her use of the textbook, materials, and journals. To effectively represent mathematical ideas and concepts, she employed examples, explanations, and demonstrations in the form of words, symbols, and drawings. Her representations were very clear and precise. In particular, in writing algebraic equations, she not only wrote the equal sign appropriately but also stressed verifying actions in every step.

Ms. King's teaching strategies comprised using a variety of activities and games for groups; adopting student presentation into her lessons; employing visual, auditory, and kinesthetic approaches to match the students' different learning styles; and finding realistic applications. By engaging students in learning activities, she tried to motivate them and help them pay attention to learning mathematics. Word problems were given to students all the time in the form of individual or group activities that expressed her beliefs about mathematics as involving practical applications in the real world. By having students engage in group activities, Ms. King provided increased opportunities for students to work together to promote their active learning (Gess-Newsome, 1999b). In particular, having the students keep their own journals was noteworthy. What is more, she encouraged them to use their journals even when taking tests and exams so that they did not have to memorize formulas without understanding the concepts.

CHAPTER 6

SUMMARY AND CONCLUSIONS

The purpose of this study was to investigate how the pedagogical content knowledge of two middle-school mathematics teachers was manifested in their classroom instruction. Also, the study investigated how the teachers' beliefs were related to their pedagogical content knowledge. I collected data in multiple forms: classroom observations, individual interviews, a questionnaire, and documents. I analyzed those data using grounded theory methods (Glaser & Strauss, 1967).

Mr. Smith and Ms. King were teaching eighth-grade mathematics in different cities in the South. Whereas Mr. Smith had taught mathematics in the middle grades, primarily eighth grade, for 13 years, Ms. King had taught in the elementary grades for 13 years, resuming her teaching career in the middle grades. It was her fifth year in the middle grades and her first year teaching eighth-grade mathematics.

Mr. Smith showed consistency between his beliefs and his actions while teaching eighth graders in that he emphasized that there are various approaches to solving a problem, and he encouraged his students to use different methods. Mr. Smith's teaching was influenced by his view that mathematics is closely related to everyday life and that mathematics is problem solving that involves critical thinking. He strongly believed that problem-solving skills must be connected with and applied to real situations. His beliefs about mathematics guided his beliefs about learning mathematics: that learning mathematics is a process of problem solving, of figuring things out in real life, and of acquiring lifelong skills. Thus, the students' mathematics learning should be a process of realizing how mathematical ideas and concepts fit together within mathematics. This view was also linked to his beliefs about teaching mathematics: Teachers should be willing and able to explain problems in various ways to promote students' understanding (Calderhead, 1996; Cooney, 1985; Fennema & Franke, 1992; Thompson, 1984, 1992). He was fully aware that students learn in different ways and that they require different treatment according to their needs; his teaching practice showed such awareness.

Ms. King believed that mathematics is working with numbers and is indispensable to surviving in the world and to solving problems that one deals with every day. This view influenced her not only to find realistic applications for her students but also to emphasize basic concepts that were applicable to real-life situations. She believed that learning mathematics is a process of making pieces fit together and is similar to conducting experiments in science. This view appeared to support her emphasis on the connection between algebra and geometry and her use of instructional tasks that enabled the students to learn the concepts of probability and work with formulas about polygons. Further, Ms. King thought that teaching by breaking down steps for skill building, rather than for students' knowledge construction, was important. Although she articulated her concerns about students' understanding, her beliefs about teaching mathematics seemed to result in her teaching in a comparatively procedural way.

The two teachers' pedagogical content knowledge was characterized by knowledge of mathematics, knowledge of students' understanding, and knowledge of pedagogy. In the case of Mr. Smith, knowledge of subject matter included: (a) purposes of teaching mathematics; (b) connections among topics; (c) concepts to teach; (d) various ways of solving problems; and (e) textbook knowledge. His knowledge of students' understanding involved: (a) particular students' understanding; (b) students' errors and common misconceptions; and (c) students' difficulties and confusions. Mr. Smith's knowledge of pedagogy was revealed in the form of: (a) learning activities; (b) attempts to motivate students; and (c) realistic applications.

Ms. King's knowledge of mathematics included: (a) purposes of teaching mathematics; (b) understanding topics; and (c) curricular knowledge or topic organization. Her knowledge of students' understanding consisted of knowledge of students': (a) different learning styles; (b) learning difficulties; and (c) common errors and misconceptions. The most salient features of her knowledge of pedagogy involved:

(a) ways of representation such as explanations, examples, demonstrations, and illustrations; (b) lesson planning and organization; (c) teaching strategies that included designing learning activities, student presentations, and various teaching styles matching the students' different learning styles, and using realistic applications; and (d) using materials such as the textbook and journals.

Conclusions

The findings showed similarities in the two teachers' pedagogical content knowledge. First, they appeared to be influenced by knowledge of mathematics that appeared to have similar subcomponents, such as purpose of teaching mathematics, connections among topics, and understanding concepts to teach. Mr. Smith applied his knowledge about fractions to teaching radicals and radical equations. Likewise, Ms. King's understanding of rational numbers and the connections between them was reflected in her decisions about the order of topics to teach.

Mr. Smith saw himself not only as developing his knowledge but also as having limited knowledge about mathematics. Mr. Smith saw himself especially having limited knowledge of radicals and graphing radical equations because he did not take any courses in relation to those topics in college. In contrast, Mr. Smith's college professor of statistics definitely influenced the way he taught probability and increased his confidence in his knowledge of probability. During that college class, Mr. Smith came to understand what probability meant and how it was applicable to the real world. Similarly, although she expressed confidence in her knowledge of mathematics, Ms. King did not consider herself as an expert in mathematics.

Second, Mr. Smith and Ms. King seemed to focus very strongly on students' understanding. Their knowledge of students' understanding consisted of similar elements: knowledge of students' common errors and misconceptions and of students' difficulties and confusions. In particular, both teachers showed their understanding of students' misconceptions of the multiplication and division of fractions, which hindered the students in understanding new concepts and in solving problems.

Third, by using realistic examples and applications, Mr. Smith tried to motivate the students and to promote their thinking, which illustrated his knowledge of pedagogy and simultaneously demonstrated his knowledge of mathematics. Ms. King also tried to use realistic examples and applications. Although she felt uncomfortable solving word problems herself, she frequently incorporated word problems into her lessons.

Fourth, both teachers tended to emphasize the practical aspect of mathematics rather than its theoretical and abstract aspects. Mr. Smith saw the importance of developing reasoning and creative thinking through mathematics and emphasized these aspects in his lessons. Although Ms. King said that creative thinking and problem solving were important in mathematics learning, she appeared to put more value on understanding the basic concepts of mathematics than on developing thinking processes and reasoning in mathematics.

By organizing their lessons in their own way, which differed from that of the textbook, Mr. Smith and Ms. King tried to connect and extend certain ideas and concepts to students' previous knowledge and to new ideas and concepts. Finally, I could not determine how Mr. Smith or Ms. King used their knowledge of assessment during my observations and in my interviews. Although there were tests and quizzes during my observations, which made it possible for the two teachers to realize their students' difficulties, confusions, misconceptions, and common errors, their knowledge of assessment was not explicitly revealed.

The two teachers showed differences in their manifestation of pedagogical content knowledge. First, a knowledge of students' understanding played a key role in Mr. Smith's teaching. His major concerns in his teaching lay in the students' learning for both academic and practical purposes. In his instruction, he appeared to apply his knowledge about students' learning processes and understanding, students' common errors and difficulties, and particular students' topic-specific understanding. For instance, by paying attention to his students' common mistakes and difficulties in simplifying radicals, he used his knowledge to refine his explanations and to elicit the students' thinking so that the students would better understand the topic and correct themselves. His knowledge made it possible for him to improve his students' thinking and to develop appropriate activities for them (Ball, 2000; Ball & Bass, 2000; Franke & Kazemi, 2001; Llinares, 2000). Furthermore, he used many examples obtained from the students' solutions or approaches to problems.

In contrast, Ms. King's knowledge of students' understanding seemed general rather than topic specific; she was aware of how students learn differently, but her awareness was related to visual, auditory, and kinesthetic learning styles, not to subject-specific knowledge. Although Ms. King had more opportunities than Mr. Smith to obtain knowledge of students' understanding because her lessons were organized in a greater variety of pedagogical styles, she did not appear to apply much of that knowledge to her lessons. She responded to and corrected errors according to the students' needs when they expressed their confusions and misconceptions and made mistakes while solving problems; however, she did not use that knowledge in her instruction.

Most important, Mr. Smith's mathematics teaching was based on his knowledge of mathematics; he appeared to have a profound understanding of some concepts or topics so that he was able to use alternative approaches, give counterexamples, and make connections within the algebra chapter of the book and across chapters and grade levels. Also, his decisions about whether a particular student's answer and approach were correct were rooted in his knowledge of mathematics (Ball, 1999, 2000; Even & Tirosh, 1995; Fernandez, 1997).

Whereas Mr. Smith's pedagogical content knowledge appeared to be relatively dependent on his knowledge of students' learning and his knowledge of mathematics, Ms. King's pedagogical content knowledge relied heavily on her knowledge of pedagogy. Unlike Mr. Smith, whose use of instructional strategies appeared to be rather narrow during my observations, Ms. King seemed to depend on a variety of pedagogical techniques. Although both of them considered realistic applications important in their teaching, they used them differently. Mr. Smith used realistic examples and problems to motivate students; in contrast, Ms. King used them to help students better understand concepts and practice solving problems. She also used various instructional tasks, including word problems; hands-on activities; games; auditory, visual, and kinesthetic presentations; and student presentations. She either created or adapted an activity for almost every lesson, which seemed to be rooted in her beliefs that students learn better when they are actively involved. Whereas Mr. Smith covered most of the topics of a chapter in the textbook, Ms. King omitted some topics. This omission occurred because she used the textbook as a resource rather than as a curriculum guide. It might be related to her feeling of weakness and discomfort with certain topics of mathematics.

Ms. King's use of journals was significant. She had students keep journals to facilitate their learning. They could revisit their journals to see what they had done. I assumed that her use of journals was related to her knowledge of students' understanding, but she apparently did not use much of the knowledge she obtained from reading her students' journals. By assessing student presentations, she became aware of students' difficulties, confusions, common errors, and misconceptions; unlike Mr. Smith, however, Ms. King did not apply much of that knowledge to her lessons.

Ms. King's lesson planning and organization appeared to come from her knowledge of mathematics. When planning her lessons, she considered whether a topic could be connected to previous topics. This consideration showed her sound understanding of certain concepts such as rational numbers. Although she could see connections among topics in algebra, she apparently did not see the connections between algebra and geometry until she participated in a professional development program. She did not consider herself an expert in mathematics. She felt comfortable in some areas of mathematics teaching and uncomfortable in other areas.

In conclusion, this empirical study of the pedagogical content knowledge of the two eighth-grade mathematics teachers suggested slightly different structures of pedagogical content knowledge (Figures 14 and 15). Mr. Smith's pedagogical content knowledge sometimes appeared to be a combination of his knowledge of mathematics and knowledge of students' learning, and at other times a combination of his

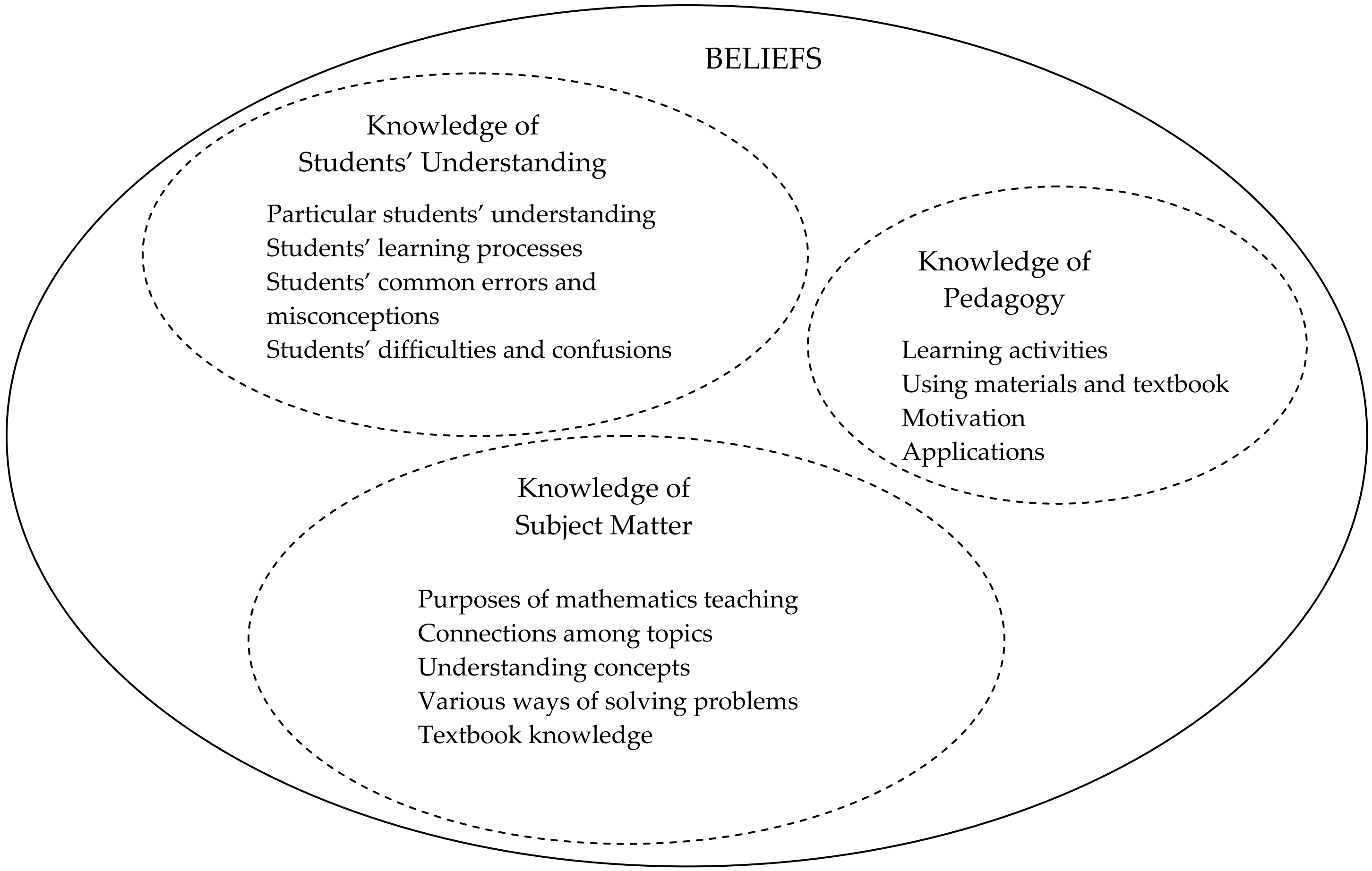

Figure 14. Salient elements of Mr. Smith's pedagogical content knowledge.

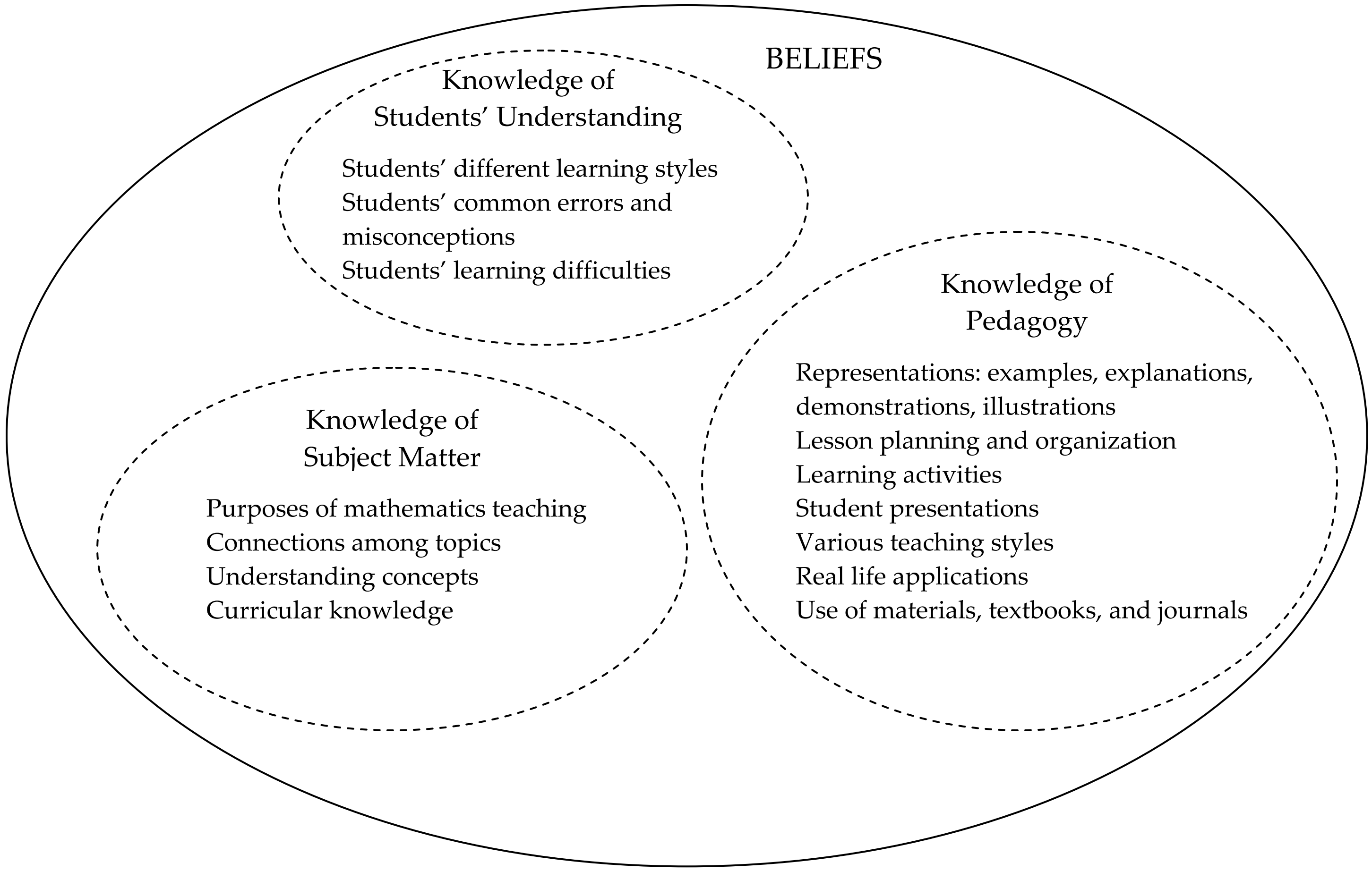

Figure 15. Salient elements of Ms. King's pedagogical content knowledge.

knowledge of mathematics and knowledge of pedagogy, or a combination of all the components. Ms. King's pedagogical content knowledge at times was manifested as a union of knowledge of pedagogy and knowledge of mathematics or of all components and on occasion as a combination of knowledge of students' understanding and knowledge of pedagogy. Mr. Smith was more dependent on a knowledge of mathematics and a knowledge of students' understanding, whereas Ms. King was more dependent on a knowledge of pedagogy and, to some degree, a knowledge of mathematics. Consequently, no single model fits the pedagogical content knowledge of both teachers, perhaps because they used their knowledge differently in teaching mathematics. Or they might have focused their knowledge differently in teaching. I cannot say whose knowledge was stronger. I did not measure that knowledge, since "who is better" was not the research question. This study did not explore the relationships among the components of pedagogical content knowledge. The results, however, implied that that the components are interactive and interrelated; each component affects the others.

Finally, it was continually obvious that pedagogical content knowledge is dynamic (Cochran et al., 1994). The components of pedagogical content knowledge seemed to be subsets of a whole; pedagogical content knowledge could not exist without its various components, and the union of those components constitutes pedagogical content knowledge.

Based on the findings from this study, I attempted to draw a model of the structure of pedagogical content knowledge (Figure 16) that explains its dynamics and shows that it is sometimes formed by several types of knowledge. They include knowledge of subject matter, knowledge of students' understanding, and knowledge of pedagogy or instructional strategies; I listed components of each category to show that those components are revealed in combination and not in isolation in teaching practice. The connections are several instances of such combinations. Moreover, it was evident that the teachers' beliefs in this study influenced their pedagogical content knowledge. Their beliefs seemed to lead them to emphasize realistic applications, motivation, and purposes of teaching mathematics and to motivate students by using various learning activities.

Limitations of the Study

This study has some limitations in the findings from the two teachers. Some limitations came from the research design. Other limitations derived from the research questions, which attempted to understand the components of pedagogical content knowledge in the two teachers' instruction.

Some limitations arose from the difference in time spent in collecting data. I did not investigate the two teachers at the same time; instead, the data were collected one

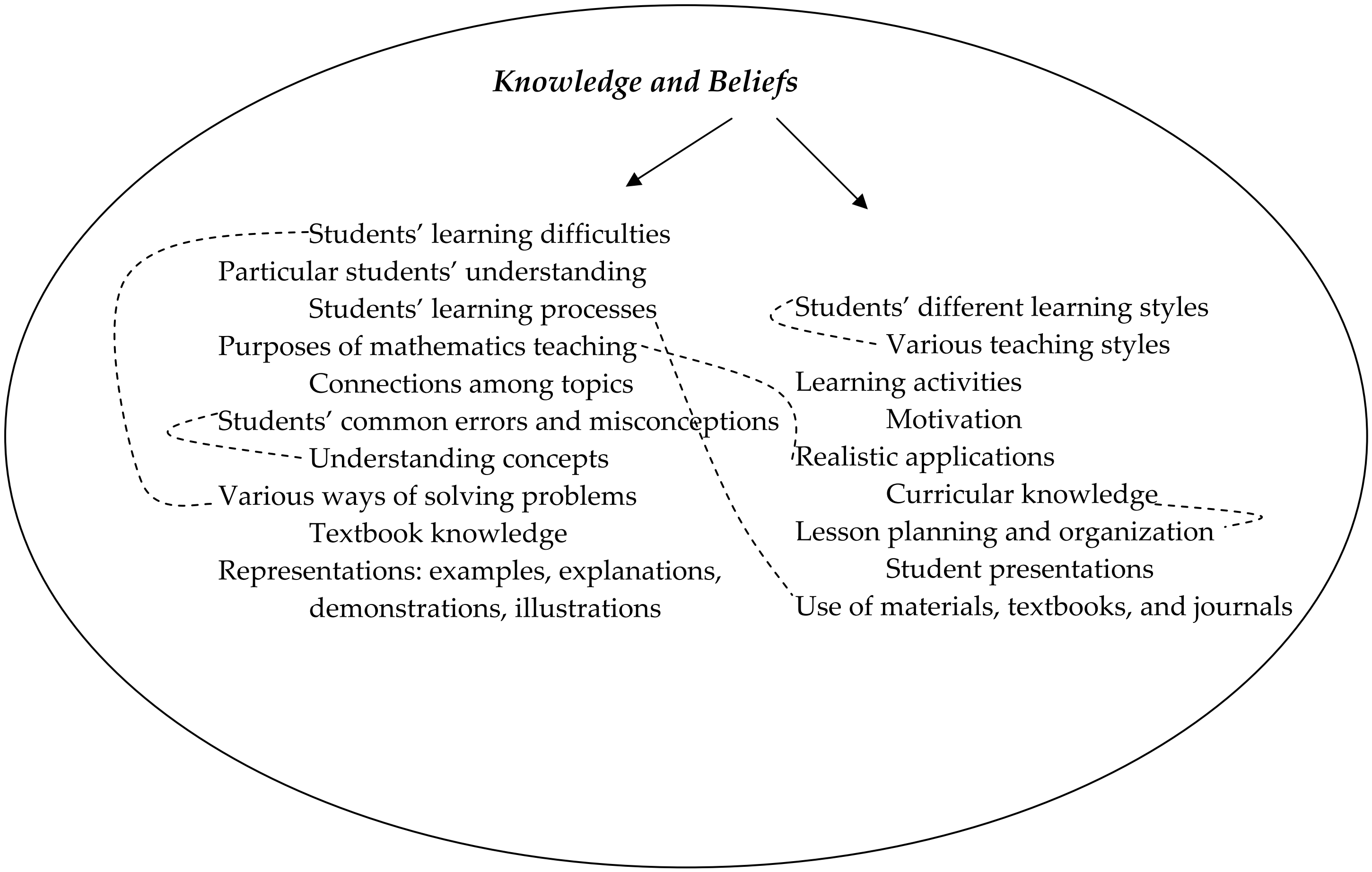

Figure 16. A model of the structure of pedagogical content knowledge.

year apart. I conducted a pilot study with Mr. Smith; I did not spend as much time with him as with Ms. King. I observed his classroom instruction for a week and interviewed him twice; however, I observed Ms. King's classroom for over a month and interviewed her seven times. Thus, I could not probe many things in interviewing Mr. Smith; I used questionnaires to compensate. Because of the short time in classroom observations, I might have missed important evidence about his pedagogical content knowledge such as his knowledge of curriculum and of assessment.

Other limitations arose while analyzing the data and might come from the characteristics of this study. Some difficulties arose in distinguishing various types of knowledge that occurred during the processes of data analysis. In addition, even though it was necessary to separate knowledge from actions, that was quite hard to do. Most of the knowledge I obtained about the two teachers was derived from their teaching actions rather than from measuring their knowledge by using instruments. Thus, the components of the teachers' pedagogical content knowledge might be seen as actions rather than as knowledge. Also, some of the components might not be considered knowledge.

Implications

The findings of this study show that the two teachers' knowledge of mathematics enabled them to teach mathematics differently (Ball, 1991; Ball et al., 2001; Even &

Tirosh, 1995; Fernandez, 1997; Gess-Newsome, 1999b; Leinhardt et al., 1991; Ma, 1999; Raths, 1999). The teachers not only realized their strengths, weaknesses, and limitations in their knowledge of mathematics but also saw themselves as developing it through teaching experiences (Hiebert & Carpenter, 1992).

The findings of this study suggest that pedagogical content knowledge varies by teachers, student groups, and topics. As shown in the previous chapters, the two teachers' pedagogical content knowledge did not look the same; further, the components comprising the pedagogical content knowledge were manifested in different ways. The teachers were more dependent on some types of knowledge than on others. Also, not only did their pedagogical content knowledge vary according to whom they taught but it also was differently revealed by the topics they taught.

In an attempt to characterize pedagogical content knowledge, this study supports the view that pedagogical content knowledge is quite dynamic (Cochran et al., 1993). That pedagogical content knowledge varied by teachers, student groups, and topics proves its dynamics. Pedagogical content knowledge is neither static nor unchangeable. The teachers seemed developing their pedagogical content knowledge through their own learning. Their increase in one type of knowledge influenced their pedagogical content knowledge. What is more, the findings suggest that the components of pedagogical content knowledge are combined rather than isolated, and interact with one another. In their instruction, the teachers' pedagogical content knowledge was often manifested in a combination of its components.

Finally, the results of this study verify that teachers' beliefs influence their teaching and use of knowledge (Fennema & Franke, 1992; Gess-Newsome, 1999b; Hashweh, 1987; Nespor, 1987; Cooney, 1985, 1999; Thompson, 1984, 1999). The two teachers' beliefs about mathematics and about learning and teaching mathematics were closely connected to their pedagogical content knowledge such as knowledge of instructional strategies. Their beliefs shaped their instructional decisions on planning, organization, and actions in instruction.

Recommendations for Further Research

This study has suggested what the pedagogical content knowledge of middle-grades mathematics teachers might look like as manifested in their instruction. I hope that this study will launch further efforts to reveal the pedagogical content knowledge of mathematics teachers from kindergarten to grade 16. The participants in this study had a comparatively strong knowledge of mathematics. Therefore, a replication of this study with teachers with a weak knowledge of mathematics might help determine the basic components of pedagogical content knowledge and how they influence one another.

A longitudinal study could provide more useful information on the pedagogical content knowledge of mathematics teachers. Characterizing pedagogical content knowledge in general would require a great deal of time. Additional components, such as knowledge of assessment, might be revealed as part of pedagogical content knowledge in longitudinal studies. In addition, one could explore the development of pedagogical content knowledge through longitudinal studies. Such longitudinal studies with beginning teachers will be especially helpful in understanding how teachers develop their pedagogical content knowledge. Longitudinal studies will also make it possible to study pedagogical content knowledge across topics.

Future studies of pedagogical content knowledge could explore the relationships among its components for particular mathematics teachers. Within the framework of a particular mathematics teacher's pedagogical content knowledge, the association between the components would explain how each affects the others. In other words, one could investigate how a teacher's improved knowledge of mathematics might influence his or her knowledge of students' understanding and of pedagogy.

The limitations of this study also suggest the need for further studies that would investigate what teachers' manifestation of pedagogical content knowledge looks like when several teachers teach the same topic. In addition, further studies could explore how one teacher's pedagogical content knowledge is manifested in different topics; how it is different when teaching topics about which he or she feels confident or uncertain;

how it is manifested by teachers in the same grade and teachers in different grades in the same school; what a beginning teacher's pedagogical content knowledge looks like, and how it is different from that of experienced teachers.

In conclusion, more research studies are needed not only to identify exemplary models of pedagogical content knowledge for specific mathematics topics but also to examine its influence on teachers' practice. Attempts to conceptualize pedagogical content knowledge might be powerful stimuli for understanding mathematics teaching—to understand what is involved and how various kinds of knowledge are used in teaching mathematics. This work might help build consensus on what teachers should know for effective teaching.

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APPENDIX

INTERVIEW PROTOCOL

- Would you tell me about your background?
- Why did you choose to go into teaching?
- Learning mathematics is like:

  working on an assembly line   watching a movie
  cooking with a recipe   picking fruit from a tree
  working a jigsaw puzzle   conducting an experiment
  building a house   creating a clay sculpture
  Other ___________________

  Which simile best describes learning mathematics? Why?

- A mathematics teacher is like a:

  news broadcaster   entertainer   doctor
  orchestra   conductor   gardener
  coach   missionary   social worker
  Other_________________

  Which simile best describes mathematics teaching? Why?

- What did you think about mathematics as a student?
- What do you think mathematics is about?
- What did you learn about teaching math from your time in the schools?
- What successes or frustrations did you have?

- What is your goal for teaching probability/algebraic equations/fractions/geometry/review session?
- Why did you teach probability after fractions?
- What do you consider when planning lessons?
- Would you describe your typical plan for your mathematics?
- What do you think the teacher's role is in the mathematics classroom?
- Describe your students' role in your classroom.
- How do you assess whether your students learn and understand?

SAMPLE OF FIELD NOTES

Field Notes 2

King	Textbook, journal, HW on your desk. Dates on your journal; it's your responsibilities Rearrange students' seating: mixed abilities Bring your homework tomorrow. Call several students who did not do their homework to finish outside the classroom. Have them read their journal since Feb. 10 for a test on the following day. 40 questions in 40 minutes. I will allow you to use a calculator tomorrow [for the test]. Anything in Chap 5 & 6 Did we do any addition or subtraction on that problem?	
Students	No.	
King	No. If you did, you broke every legal move in the book or you combined variables and values. The two did not look anything alike. Answers checked: 12) *y* is less than < 0.5 14) *a* is greater than or – 1.5	
Student	–1 ½	
King	1 ½, or	
Student	-3	
King	No.	
Student	-3/2	

King	You need to rewrite a mixed number as a decimal. You cannot have two totally different numbers. 16) *t* =	
Student	-2	
King	-2. That is correct. 18) [pointed to a student]	
Student	$r \leq 1.6$	
King	20) *b* is [called a student]	
Student	[Gave wrong answer.]	
King	No. What operation did you do?	
Student	Add	
King	Woo… We did talk about that. There is no addition and subtraction sign. [Pointed to another student.]	
Student	$b = (-2.1)$	
King	$b = (-2.1)$ I am gonna show you that addition and subtraction brought every legal move on that problem. 22) *d* equals	
Students	-1.08	
King	Thank you. That is correct. -1.08 24) $y > 2.4$ or $2\frac{2}{5}$ 26) *h* is less than < -2.3 Did you put less than < or did you put greater than >? Why did you put that sign when you look at and read a problem with a sign you deal with? It's very important. Called a student for #28.	
Student	-9	
King	Well, *x* what?	
Student	Greater than (and less than) -9	
King	You've got a different sign than you should be. What did you do? What is your operation? 28) $x > -9$	
Student	[Explained what she did.] [cannot hear]	
King	So, you multiplied. When you multiply,	

	3.2 × 2.5, you've got 9?	
Student	I wrote down a wrong number. It's -8.	
King	$x \geq -8$, It should be –8. I want you to notice. Look at #26 and 28. What kind of number did you notice with the variable?	
Students	Negative.	
King	We talked about that we studied.	
Student	Forgot.	
King	Please turn in your journal put that down as "reminder" for solving inequalities. "If there is a (-) sign on the same side as the variable, we must reverse, must reverse that inequality symbol." Put this example somewhere. e.g., $-2h > 4.6$ What's another word that means negative?	
S	Opposite	
	Negative means the opposite, then we have to take the opposite symbol. What you really need is, you need to immediately rewrite the problem reversing the inequality symbol. Reverse. Now you're gonna do. Let's go back. $-2 \times h$ So, I do the inverse operation. What can I do when it says $-2 \times h$? What can I do with h by itself?	
S	Divide	
King	Divide by –2. We get down to 1, which is what we're trying to do, what we always try to find is find the value of 1. And then we divide. Now, because I am allowing you to use a calculator tomorrow, which one of these numbers are you going to include first in your calculator?	

S	4.6. 4.6 divide by 2	
King	Top to bottom. When you use a calculator, you go top to bottom. When you go paper pencil, you go up into [top]	
	$\frac{-2h}{-2} < \frac{4.6}{-2}$ $1 \cdot h < -2.3$ Calculators don't do (-) signs for you. You have to remember the rules for multiplying and dividing with negative signs. So, you don't need a calculator telling negative [signs]. What's the whole purpose of using calculators? What you should know how to do. You know how to divide 4.6 by –2. To speed up! What you're already supposed to know. Let's go back to the problem. We have $3x + 4 = 13$ Hands-on equations *xxx* [4] [13] [9] (cancel out 4 and 9) *xxx* [9] $3x = 9$ What's the value of each *x*?	
S	Divide	
King	$\frac{3x}{3} = \frac{9}{3}$ x = 3 you have star, ** [4.6] how do you find each star? S: divide Divide by 2. Remember stars mean. You're gonna get one star; one * means –*x*; * = -*x*. ** [4.6] * is 2.3. Right? So two *'s and then star is negative because * is the opposite of *x*. That's why you've got to remember. You've got opposite, negative with the variables you must reverse the inequality symbol.	

* is always the opposite of *x*.

Because you began with *x*'s, not –*x*.
* is negative *x*.

Legal move is it's legal to add the same value both sides, it's legal to subtract the same value both sides, it's legal to multiply the same value both sides because we're always trying to *keep the scale balanced*. As long as you do to the left side and the right side the same value, you keeping your scale balanced. All idea was divide both sides by 2 to keep the scale balanced.

S: Hang on.

Pick a different problem. What would you like to see? Let's do 28 because there is a negative (-) sign.

Let's go get this one. All right.
#1.
I hope you wrote down that to remind me, I hope you put in that highlight; if there is a negative with the variable, I need to immediately *reverse* (pause) the symbol. Why I rewrite?

$\frac{x}{-2.5} \leq 3.2$

$\frac{x}{-2.5} \leq \frac{3.2}{1}$ immediately reverse the sign

cross multiplication
now you're gonna get
$x \cdot 1 \geq (-2.5)(3.2)$
$x \geq (-8.\underline{\square}\ \underline{\square})$
So, what operation are we gonna do now?
Sts: multiplication
Do I need to use calculators to multiply *x* times 1?

	Sts: No, already *x*. We have to do that much easier. Now, would calculators be good here? Sts: Yes. Yes, because by now you should know how to calculate decimal points all I do was count numbers … you punch 2.5 × 3.2 Sts: 8.00. it is negative. What property? Let's see if we can remember property. What property allows me to reverse? Sts: commutative	
King	The commutative property of multiplication says *a* times *b* is, equals to *b* times *a*: doesn't matter order of multiplication. We just get a habit of working left to right, left to right. What is your problem having a trouble you like to see?	
S	14, 18, 24	
King	5*a* > -7 ½ First thing I do is rewrite a problem. Rewrite the problem. I just told you. I hope you understood what I've just said. You could have done every problem as a proportion. I put everything over the 1. In this case, I don't need to put over 1, because I don't have numerator and denominator. Turn that into an improper fraction. Oh oh oh, I said you had a negative sign with the variable, negative sign with the same side variable. Do not, do not, do not reverse the symbol.	
King	$\frac{5a}{1} > \frac{-15}{2}$ What are you gonna do next? S: cross multiply Cross multiply, what are you gonna do?	

	S: $5a \cdot 2 > 1 \cdot (-15)$ Perfect. Then what's your next line? S: $10a > -15$ This point, there is no addition and subtraction. This is a cube sitting on the right side and this is a variable, 10 of them in fact. You can't combine variables and values. They don't look anything alike. So, if I see multiplication, what am I gonna do? S: Divide Divide Before, we've got three *x*'s equals 9. Here, we've got ten *a*'s greater than –15. We're gonna divide, the legal move says divide by the same value, $\frac{10a}{10} > \frac{-15}{10}$ $a > -1\frac{5}{10}$, which can be simplified -1 ½	
Student	Could you have done? $5a > -7.5$ $\frac{5a}{5} > \frac{-7.5}{5}$ $a > -1.5$	
King	You could've done. There is more than one approach. If you know what this is as a decimal, everybody should know 1/2 is equivalent to .5; ¾ is equivalent to .75. Right? ¼ is equivalent to .25. If you feel like you can set it up like that, oh yeah, I can now divide both sides by 5 because we always try to find the variable by itself. Is that a different answer? No, it's still the same answer. It's just one expressed as a mixed number, one expressed as a decimal. But they still have the same value. That's what we are talking about right here. Equivalent expression: Every fraction has equivalent decimal and equivalent percent.	

	You've just got to know which one you want to work with. So you begin to realize fractions aren't hard after all. You can turn everything into decimal unless you see you're beginning to work more comfortable. What's another one, Lencia? S: #18	
King	#18. The only thing I would do is rewrite and put the *r* in the numerator. Lencia, I will put the *r* in the numerator. I will put 1.4 over 1. Now you keep working. I'll give 1 minute. $7r \cdot 1 \leq 8 \cdot 1.4$ $\frac{7r}{7} \leq \frac{11.2}{7}$ $r \leq 1.6$ $-1r \cdot 4 = 8 \cdot 1$ $-\frac{4r}{4} = \frac{8}{4}$ $r = (-2)$	
Student	$\frac{7}{8}r \leq 1.4$ $\frac{7}{8}r \leq \frac{1.4}{1}$ $-\frac{1}{8}r = \frac{1}{4}$ rewrite $-\frac{1r}{8} = \frac{1}{4}$	
King	Solve these two problems. $7r \cdot 1 \leq 8 \cdot 1.4$ $\frac{7r}{7} \leq \frac{11.2}{7}$ $r \leq 1.6$ $-1r \cdot 4 = 8 \cdot 1$	

	$-\frac{4r}{4}=\frac{8}{4}$ $r=(-2)$ I don't mean to frustrate, but you do the same legal move that you've been doing. Fractions. You may use calculators. But you can't use calculators until you set it up. So if you're given two numbers to multiply, two fractions, or let's say we have: $3\frac{4}{5}\cdot\frac{2}{9}$ Let's say we don't know what to do first of all. What should you do to multiply? S: Common denominator S: Cross multiply No! S: Multiply numerator times numerator and denominator times denominator. That's what I am talking about. The journal is gonna be used. We do not gotta do a problem without looking back at the journal. That's what I am talking about. I heard three different choices and two of them were wrong. Where is the book open so I can see how to multiply fractions? Numerator times numerator. Denominator times and denominator. We don't find a common denominator. Go back and read your chart. S: 5 times 9 is 45; that is a common denominator. We don't find a common denominator. We multiply denominators. Don't say that. $\frac{19}{5}\cdot\frac{2}{9}=\frac{38}{45}$ cross multiplication And then we simplify if we can. Your journal better be out tomorrow.	

	So, let's say… Divide by fractions: S: $3\frac{4}{5} \div \frac{2}{9}$. No, you have to change? No. One legal move per line. You're right. Kevin, what would you say when we divide by fractions on your journal? Kevin: Dividing by fractions means … It's a chart. Kevin: multiply … Multiply what? Kevin: dividend Multiply dividend means times the Sts: reciprocal Which means what? S: flip flop, inverse Multiplicative inverse. Now, multiply. $\frac{19}{5} \div \frac{2}{9}$ $\frac{19}{5} \div \frac{2}{9} = \frac{171}{10}$ Now, what do I do? = 17 1/10	
Student	How did you do it?	
King	7/8 = 0.875 "L[eft]R[right]T[op]B[ottom]" on paper (long division), but on calculators, Top to Bottom. Listen, if you're given a proper fraction, proper fraction is greater than 0 and less than 1. It's a proper fraction. When you ate 7/8 of a pizza, did you eat a whole thing? Sts: No, 7/8. You ate 7/8. That means you ate less than a whole, right? Proper fraction's always less than 1, less than a whole.	