A Waiting Time Approach for a Disability Model

by

Hejiao Hu

(Under the Direction of Lynne Billard)

Abstract

The aim of this dissertation is to provide a waiting time focused method for calculating transition probabilities of a disability model. The challenge that the traditional approach, i.e, differential-difference approach faces is its computational difficulty. The waiting time approach described in the dissertation transforms the event that describes an insured state at time t given its initial state into an equivalent event that considerably eases the calculation burden. The dissertation is composed of five chapters: (1) Introduction, referring to the model, and the history of the investigation about the model, and a sketch of the contents from chapter 2 to 5; (2) Literature Review, reviewing the academic background for the investigation of compartment models and the waiting time approach; (3) Waiting time approach for disability models, deriving the transition probabilities with the waiting time approach for a disability model and their sensitivities to the parameters; (4) Application to insurance functions, investigating the sensitivities of some insurance functions of interest to the parameters; (5) Future work, discussing possible theoretical simulation work based on the results of chapter 3 and 4.

INDEX WORDS: Compartment models, Disability model, Waiting time approach, Transition probability, health insurance, holding time

A Waiting Time Approach for a Disability Model

by

Hejiao Hu

B.A., Hubei University of Technology, 2001

M.A., Zhejiang University, 2005

M.A., University of Georgia, 2011

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2016

A Waiting Time Approach for a Disability Model

by

Hejiao Hu

Approved:

Major Professors: Billard, Lynne

Committee: McCormick, William
Reeves, Jaxk
Sriram, T.N.
Seymour, Lynne

Electronic Version Approved:

Suzanne Barbour
Dean of the Graduate School
The University of Georgia
May 2016

Acknowledgments

I owe my gratitude to a lot of people. Without their support, I could not finish the dissertation forever. My deepest gratitude is to my advisor, Dr. Lynne, Billard. I truly feel fortunate and honored to be her Ph. D student. She gives me freedom to work before graduation, confidence to focus on the research whenever I feel confused, and patience whenever I have problems.

I also would like to thank Dr. Seymour and the admission committee for granting me transferring to the statistics department. I still feel that applying for transferring to the department is one of the most right decisions in my life and I am so lucky to be accepted.

I am grateful to Dr. Reeves for giving me chance to work in the Statistics Consulting Center and get a lot of precious experience.

Dr. McCormick and Dr. Sriram are two of best teachers in our department. I truly appreciate that you teach me professional knowledge and always respond instantly whenever I need help for the dissertation or the courses. I am also thankful to you being my committee.

I am grateful to the following former employee or students in the department for all their help and support during my graduate study– Kim Love-Myers, Yi Chen, Guannan Wang, Wei Zhang, Fei Liu, Xijue Tan, Wenbo wu, Lina Liao, Yuan Zhuang, and Nan Zhang.

My friends Shiyao Wang and He Ren were always there for me through these difficult years. Their support and encouragement help me focus on the study. I truly appreciate their friendship.

I would like to acknowledge my managers David Chenyao and Wenzhi Li in Citigroup for their kindness and considerateness.

Most importantly, I would like to express my heart-felt gratitude to my parents, my husband and my parents in law for their love and taking care of my son for the long time.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

The investigation of population behavior modeling can be traced back to the malaria model built by Ross (1911). With the mathematic theory derived by Kermack and McKendrick (1927), such deterministic models were widely used for predicting reaction mechanisms in chemistry such as Zilversmit and Fishler (1943), and Sheppard and Housholder (1951). Since Feller (1939) claimed the importance of building stochastic models for population behavior, a lot of papers have contributed to the development of stochastic models, such as Bartlett (1949) and Bartholomay (1958). A series of papers (e.g. Thakur et al. (1972), Purdue (1974a), Purdue (1974b), Purdue (1975) and Matis and Tolley (1979)) built stochastic theory behind their models. Matis and Hartley (1971) developed least squares methods for the parameter estimation of compartment models.

The basic disability model in our work is composed of the states, defined as healthy, disabled and death. An individual can move back and forth between healthy and disabled states. We unfold the process so that the process has $2m+1$ compartments, in which states $2i-1$ and $2i$ are, respectively, denoted as the ith time that an individual is healthy and disabled for $i = 1, 2, ..., m$. State $2m+1$ is the state death, and is an absorbing state. Also, individuals in any state other than death can go directly to state death for an unexpected reason.

In Chapter 2 of this dissertation, we review the development of the methods for investigating compartment system by introducing deterministic models as well as stochastic models, the theories behind the one compartment, two compartment and multi-compartment models, the general process for solving the differential-difference equation, and the parameter estimations for stochastic models. In addition, we go through the waiting time approach for an AIDS model and extract some general information about disability insurance.

In Chapter 3, based on the investigated disability model, we derive the general formula for transition probabilities. In addition, under the condition that an insurance period is fixed as one year, and the insured is either healthy or disabled in the end of the period, the transition probabilities are specified; Then the sensitivities of the transition probabilities to the waiting time rates are investigated.

In Chapter 4, based on the transition probabilities derived in Chapter 3, some functions of interest to insurance companies are derived and their sensitivities to the waiting time rates are investigated as well.

Future work is discussed in Chapter 5. We plan to compare how the different distributions of accidental waiting time for the first time, D_1, will affect transition probability to death for the long term.

References

Bartholomay, A. F. (1958). Stochastic Models for Chemical Reactions: I. Theory of The Unimolecular Reaction Process. *Bulletin of Mathematical Biophysics*, 20, 175-190.

Bartlett M. S. (1949). Some Evolutionary Stochastic Process. *Journal of the Royal Statistical Society Series B*, 11, 211-229.

Billard, L. and Dayananda, P. W. A. (2014). A Multi-Stage Compartmental Model for HIV-infected individuals: I - Waiting time approach. *Mathematical Biosciences*, 249, 92-101.

Feller, W. (1939). Die Grundlagen der Volterraschen Theorie des Kampfes ums Dasein in Wahrscheinlichkeithoeretischer Behandlung. *Acta Biotheoretica*, 5, 11-40.

Kermack, W. O. and McKendrick, A. G. (1927). A Contribution to the Mathematical Theory of Epidemics. *Proceedings of the Royal Society A*, 115, 700-721.

Matis, J. H. and Tolley, H. D. (1979). Compartmental Models with Multiple Sources of Stochastic Variability: The One-Compartment, Time Invariant Hazard Rate Case. *Bulletin of Mathematical Biology*, 41, 491-515.

Matis, J. H. and Hartley, H. O. (1971). Stochastic Compartmental Analysis: Model and Least Squares Estimation From Time Series Data. *Biometrics*, 27, 77-102.

Purdue, P. (1974a). Stochastic Theory of Compartments. *Bulletin of Mathematical Biology*, 36, 305-309.

Purdue, P. (1974b). Stochastic Theory of Compartments: One and Two Compartment Systems. *Bulletin of Mathematical Biology*, 36, 577-587.

Purdue, P. (1975). Stochastic Theory of Compartments: An Open, Two Compartment, Reversible System With Independent Poisson Arrivals. *Bulletin of Mathematical Biology*, 36, 269-275.

Ross, R. (1911). *The Prevention of Malaria* (2nd edition), London: Murray.

Sheppard C. W. and Housholder A.S. (1951). The Mathematical Basis of the Interpretation of Tracer Experiments in Closed Steady-State Systems. *Journal of Applied Physics*, 22, 510-520.

Thakur, A. K., Rescigno, A. and Schafer, D. E. (1972). On the Stochastic Theory of Compartments: I. A Single - Compartment System. *Bulletin of Mathematical Biophysics*, 34, 53-63.

Zilversmit, C. E. and Fishler M. C. (1943). Stochastic Compartmental Analysis: Model and Least Squares Estimation From Time Series Data. *Biometrics*, 27, 77-102.

Chapter 2

Literature Review

2.1 Deterministic Models

The simplest deterministic model governing the spread of infection over time is the Ross (1911) model for malaria, i.e., the simple epidemic model,

$$\frac{dX}{dt} = \lambda X(N - X + 1), \qquad t > 0, \tag{2.1}$$

where $X = X(t)$ is the number of malaria cases in a population of size N at time t with a constant infection rate λ. This is a logistic differential equation with solution

$$X = (N+1)[1 + e^{(N+1)(c-\lambda t)}]^{-1} \tag{2.2}$$

where c is a known constant given the initial number of people who are infected with malaria. This was followed by many different deterministic models, the most notable being the Kermack and McKendrick (1927) model, which model was revisited in Brauser (2005). There is a huge literature establishing deterministic models for a vast number of population models. For example, deterministic models were built to model chemical reactions in Zilversmit and Fishler (1943). Sheppard and Housholder (1951) extends deterministic models to n-compartment systems. Beauchamp and Cornell (1968) develop a two-step method for the estimation of parameters, a general partial total approach for the linear coefficients followed by a nonlinear method from Beauchamp and Cornell (1966) for exponential parameter estimation.

2.2 Stochastic Models

Feller (1939) reminded researchers that life was stochastic; i.e., models should be stochastic models rather than deterministic ones. Instead of assuming that the number of cases, i.e., infected people in the above example, is a function of time t, stochastic models consider that the number of cases at time t is a random variable, and has a distribution at each time point. In particular, the stochastic version of the simple epidemic model in equation (2.1) is

$$\begin{aligned} \frac{dP_x(t)}{dt} &= \beta[(N-x)(x+1)P_{x+1}(t) - x(N-x+1)P_x(t)], \qquad (2.3) \\ & \quad x = 1, 2, ..., N-1, \\ \frac{dP_N(t)}{dt} &= -\beta N P_N(t), \qquad X = N, \qquad (2.4) \end{aligned}$$

where $P_x(t)$ is the probability that the number of infected individuals at time t is $X(t) = x$ given that there is $X(0) = 1$ infected person in a population of size N

at time $t = 0$, and where β is the rate of infection. Typically, the deterministic equations are much simpler to solve than are their stochastic analogues which are usually non-trivial. For example, while the deterministic malaria model was solved by Ross (1911), it was not until Bartlett (1949) that a solution was found for the stochastic simple epidemic model of equation(2.3). Bartholomay (1958) compares the stochastic theory to deterministic theory based on a basic unimolecular chemical reaction, and concludes that the derived stochastic model is consistent in the mean with the deterministic model where the transition rate is linear in time. Our focus is on stochastic models.

Compartment analysis is widely used in many disciplines. It is assumed that the system is composed of at least one compartment. Compartment is also called state in stochastic models. Material will enter the system via different compartments, stay some time in some compartments, flow between compartments and finally leave the system. For each compartment, researchers are usually interested in the inflow and outflow rates at different time points so that the probability distribution that the material in each compartment at any time can be obtained, given its state at time 0. Next, we will review one compartment models, two compartment models, and extensions to general n compartment models, for $n \geq 2$.

Stochastic Models for One Compartment Systems

Let us first review stochastic models for one compartment systems. Let $X(t)$ denote the number of particles in the compartment, i.e., the system at any time t. This $X(t)$ describes a time continuous Markov process. Thakur et al. (1972) assumes that all particles are homogeneous and any one particle can enter the compartment with probability $f(t)\Delta t$ and leave the compartment with probability $\mu\Delta t$ in the time interval $(t, t+\Delta t)$. Also the probability that two or more particles enter or leave the system in the time interval $(t, t+\Delta t)$ is assumed to be an infinitesimal of higher order than Δt. The generating function of $X(t)$, $G_x(s,t)$, is calculated as

$$G_x(s,t) \quad = \quad g[1-(1-s)e^{-\mu t}]\exp[-(1-s)f(t)*e^{-\mu t}] \tag{2.5}$$

where, in equation (2.5), $g(s) = \sum_{i=0}^{\infty} p_i(0)s^i$, and the notation $f(t)*e^{-\mu t}$ is the convolution integral $\int_0^t e^{-\mu(t-\tau)f(\tau)}d\tau$. This $G_x(s,t)$ of equation (2.5) looks like the product of two new random variables $Y(t)$ and $Z(t)$, with their probability generating function $G_y(s,t)$ and $G_z(s,t)$ given by, respectively,

$$G_y(s,t) \quad = \quad g[1-(1-s)e^{-\mu t}], \tag{2.6}$$

$$G_z(s,t) \quad = \quad \exp[-(1-s)f(t)*e^{-\mu t}]. \tag{2.7}$$

If the number of particles, x_0, in the compartment at time 0 is known, then $G_y(s,t) = [1-(1-s)e^{-\mu t}]^{x_0}$, which indicates that $Y(t)$ has a binomial distribution. Purdue (1974a) assumes that there are an input process and an output process for the compartment. The number of particles in the compartment at time t, $X(t)$, will be the sum of those in the system at the starting time and still there at time t, denoted by $Y(t)$, and those who enter the system at some time in $(0,t]$, $Z(t)$, and are still in the compartment at time t. The input and output processes are characterized as time dependent intensity functions $\lambda(t)$ and $\mu(t)$, respectively. It can be shown that

$$Y(t) \quad = \quad \sum_{i=0}^{X(0)} B_i(t) \tag{2.8}$$

where the $B_i(t)$s are identically independent random variables with a Bernoulli distribution as follows

$$P[B_i(t) = 1] \quad = \quad \exp\{-\int_0^t \mu(\tau)d\tau\}, \qquad i = 1, 2, ..., X(0). \tag{2.9}$$

This is confirmative with the definition of $Y(t)$ in Thakur et al. (1972). The $Z(t)$ has been shown to have a Poisson process with intensity $h(t)$, i.e.,

$$P(Z(t) = k) \quad = \quad e^{-h(t)} \frac{[h(t)]^k}{k!}, \qquad k = 0, 1, ..., \tag{2.10}$$

where $h(t) = \int_0^t \lambda(x) \exp\{-\int_x^t \mu(\tau)d\tau\}dx$.

The population moment values of $X(t)$ are summarized as:

$$\begin{aligned} E[X(t)] &= E[(X_0)]E[B(t)] + E(Z(t)) && (2.11) \\ &= \mu_0[1 - F_0(t)] + h(t), && (2.12) \\ \mathrm{Var}[X(t)] &= \mathrm{Var}[Z(t)] + \mathrm{Var}[\sum_{i=0}^{X(0)} B_i(t)] \\ &= h(t) + E[X(0)]F_0(t)[1 - F_0(t)] + \mathrm{Var}[X(0)][1 - F_0(t)]^2 && (2.13) \end{aligned}$$

where $1 - F_0(t) = \exp\{-\int_0^t \mu(\tau)\,\mathrm{d}\tau\}$, and $h(t) = \int_0^t \lambda(x)[1 - F(x,t)]\mathrm{d}x$. Matis and Tolley (1979) summarizes the mean and variance of $X(t)$ for different sources of stochasticity.

Stochastic Models for Two or Multi-Compartment Systems.

Purdue (1974b) extends the one compartment stochastic theory to a two compartment system. The method to obtain the distribution for the number of particles in the first compartment at time t is the same as the situation of the one compartment system. The input process of the second unit will include the inflow from the first unit as well as from outside the system. Assume that $\lambda_{01}(t)$ and $\lambda_{02}(t)$ are the input rates of the two Poisson processes to compartments 1 and 2, respectively. Let $F_1(x,t)$ and $F_2(x,t)$ be the distributions of the residence time for particles that enter the first and second compartment at time x, respectively. For those particles that are in the first and second compartment initially, their residence time has distribution function $F_{10}(t)$ and $F_{20}(t)$, respectively. Let $\alpha(t)$ be the proportion of the particles from compartment 1 that will go to compartment 2. Then, we have that the number of particles in the compartments 1 and 2, $X_1(t)$ and $X_2(t)$, are, respectively,

$$X_1(t) = \sum_{i=0}^{X_1(0)} B_{1i}(t) + Z_1(t), \tag{2.14}$$

$$X_2(t) = \sum_{i=0}^{X_2(0)} B_{2i}(t) + Z_{20}(t) + \sum_{i=0}^{X_1(0)} B_{2i}^*(t) + Z_{21}(t) \tag{2.15}$$

where $B_{1i}(t)$ is the Bernoulli random variable with parameter $1 - F_{10}(t)$; the $Z_1(t)$ is a Poisson process with rate $h_1(t) = \int_0^t [1 - F_1(x,t)]\lambda_{01}(x)\mathrm{d}x$; $B_{2i}(t)$ is the Bernoulli random variable with parameter $1 - F_{20}(t)$; $Z_{20}(t)$ is a Poisson process with intensity $h_2(t) = \int_0^t [1 - F_2(x,t)]\lambda_{02}(x)\mathrm{d}x$; and $B_{2i}^*(t)$ is a random variable with possible values

1 and 0. If a particle stays in compartment 2 at time t given that it is in compartment 1 at time 0, then $B^*_{2i}(t)$ will be 1, else it will be 0; so, it is a Bernoulli random variable with parameter $\int_0^t \alpha(t)[1 - F_(x,t)]F'_{01}(x)\mathrm{d}x$. Finally, $Z_{21}(t)$ is also a Poisson process with parameter $h_{21}(t) = \int_{x_1=0}^t \int_{x_2=x_1}^t \alpha(x_2)\lambda_{01}(x_1)F'_i(x_1,x_2)[1 - F_2(x_2,t)]\mathrm{d}x_1\mathrm{d}x_2$.

The mean and variance of the two units of the system are given as follows:

$$
\begin{aligned}
E[X_1(t)] &= \mu_1[1 - F_{10}(t)] + h_1(t), & (2.16)\\
E[X_2(t)] &= \mu_2[1 - F_{20}(t)] + h_2(t) + h_{21}(t)\\
&\quad + \mu_1 \int_0^t \alpha(t)[1 - F_2(x,t)]F'_{10}(x)\mathrm{d}x, & (2.17)\\
Var[X_1(t)] &= \mu_1 F_{10}(t)[1 - F_{10}(t)] + \sigma_1^2[1 - F_{10}(t)]^2 + h_1(t), & (2.18)\\
Var[X_2(t)] &= \mu_2 F_{20}(t)[1 - F_{20}(t)] + \sigma_2^2[1 - F_{20}(t)]^2\\
&\quad + \mu_1[1 - B_{12}(t)][B_{12}(t)] + \sigma_1^2[1 - B_{12}(t)]^2 + h_2(t) + h_{21}(t) & (2.19)
\end{aligned}
$$

where $E[X_i(0)] = \mu_i$, and $\mathrm{Var}[X_i(0)] = \sigma_i^2$ for $i = 1, 2$.

Purdue (1975) further discussed a reversible two compartment system, where the particles can go back and forth between the two units of the system. Assume the λ_i and $F_i(t), i = 1, 2$, is the input rate from the ith unit and the distribution function of residence time in unit i, respectively. Let α_{ij} be the probability that a particle moves from unit i to j, $i, j = 1, 2$. Let $\alpha_i, i = 1, 2$, be the probability of a particle to leave the system from the ith unit. In addition, it is assumed that there is no particle in the system at time 0. To investigate the number of particles in the system in any time t, Purdue (1975) establishes that the system can be considered as a one compartment system, but with two types of particles with different input rates and residence distributions. The residence time distribution function of a particle on the condition that it is in the ith unit at the beginning, i.e., $G_i = P[S \leq t|E = i]$, for $i = 1, 2$, respectively, is as follows:

$$G_1(t) = (\alpha_1 F_1 + \alpha_{12}\alpha_2 F_1 * F_2) * \sum_{n=0}^{\infty} H^{(n)}(t), \tag{2.20}$$

$$G_2(t) = (\alpha_2 F_2 + \alpha_{21}\alpha_1 F_1 * F_2) * \sum_{n=0}^{\infty} H^{(n)}(t) \tag{2.21}$$

where $H(t) = \alpha_{12}\alpha_{21} F_1 * F_2(t)$. The number of particles in the system at time t will follow a Poisson process with mean $\Gamma(t) = (\lambda_1 + \lambda_2) \int_0^t [1 - G(x)] \mathrm{d}x$, where $G(t) = \frac{\lambda_1}{\lambda_1+\lambda_2} G_1(t) + \frac{\lambda_2}{\lambda_1+\lambda_2} G_1(t)$. It is also shown that the particles in the two units are independently Poisson distributed. The result is confirmative with that of Thakur et al. (1973) obtained from the generating function method.

The methods to investigate one way n-compartments models are not new. Assume that the input and output rate for each compartment are λ_{i0} and λ_{0i}, $i = 1, 2, ..., n$, the transition rate from compartment i to $i + 1$ is $\lambda_{i+1,i}$, $i = 1, 2, ..., n - 1$, μ_i, and σ_{ii}^2, $i = 1, 2, ..., n$, are the mean and variance of $X_i(0)$, and σ_{ij} is the covariance of $X_i(t)$ and $X_j(t)$.

2.3 Differential-Difference Equation Approach

For one way n-compartment models, one typical method to obtain the joint distribution for the number of particles in each unit given the initial states is to construct and solve the differential-difference equations with appropriate assumptions. Thakur et al. (1973) obtains the distribution of particles in each compartment at any time t assuming that the transition rates of each compartment are constant. More generally, Matis (1974) derived the joint cumulant generating function of $X_1(t), X_2(t), ..., X_n(t)$ at any time t for an n-comparment system with time dependent transition rates. Billard and Zhao (1994) extended this approach to general $m+1$ multiple-stage models, under the assumption that the infinitesimal transition rate is

$$P\{\mathbf{X}(t+h) = \boldsymbol{x} - \boldsymbol{e}_j + \boldsymbol{e}_{j+1} | \boldsymbol{X}(t) = \boldsymbol{x}\} \quad = \quad \lambda_j(\boldsymbol{x}; t)h + o(h) \tag{2.22}$$

where $\boldsymbol{e}_i$ is the $m+1$ component vector with its ith element equal to 1, and 0 for all other elements. Also, $o(h)/h \longrightarrow 0$ as $h \longrightarrow 0$. The forward differential-difference equation can be constructed as

$$\begin{aligned} \frac{d}{dt} P(\boldsymbol{x}\ ; t) \quad &= \quad -\sum_{j=0}^{m} \lambda_j(\boldsymbol{x}\ ; t) P(\boldsymbol{x}\ ; t) \\ &+ \quad \sum_{j=0}^{m} \lambda_j(\boldsymbol{x} + \boldsymbol{e}_j - \boldsymbol{e}_{j+1}; t) P(\boldsymbol{x} + \boldsymbol{e}_j - \boldsymbol{e}_{j+1}; t) \end{aligned} \tag{2.23}$$

where $P(\boldsymbol{x}\ ; t) = P\{\boldsymbol{X}(t) = \mathbf{x} | \boldsymbol{X}(0) = \boldsymbol{x}_0\}$. To solve the equation (2.23), Billard and Zhao (1994) shows that there exists a function of the coordinates of the point $\boldsymbol{x} = (x_1, x_2, ..., x_m)$, $k(\boldsymbol{i})$ such that $P(\mathbf{x}; t) \equiv Z_{k(\mathbf{i})}(t)$. After the transformation, the

$Z_{k(\mathbf{i})}(t)$ can be solved as the following

$$Z_{k(\mathbf{i})}(t) = \exp\left[-\int_0^t \Big\{\sum_{l=1}^m \lambda_l(\mathbf{i},u)\Big\}\mathrm{d}u\right]\left(a_{k(\mathbf{i})} + \int_0^t \Big\{\sum_{l=1}^m \lambda_l(\mathbf{i}-\mathbf{c}_l,u)Z_{k(\mathbf{i}-\mathbf{c}_l)}(\mu)\Big\} \right.$$
$$\left. \times\exp\left[-\int_0^u \Big\{\sum_{l=1}^m \lambda_l(\mathbf{i},v)\Big\}\mathrm{d}v\right]\mathrm{d}u\right) \tag{2.24}$$

where $\mathbf{c}_l$ is an m component vector with its lth element equal to 1, and 0 for others, and $a_{k(\mathbf{i})}$ is the $k(\mathbf{i})th$ element of $Z(\mathbf{0})$. While, in theory, this is an explicit solution for the equation (2.24), however, in practice, it is computationally difficult in solving extensive recursively such an algorithm.

2.4 Parameters Estimations for Compartment Models

Once compartmental models were built, a lot of papers discussed concern about the estimation. Beauchamp and Cornell (1968) developed a model that can apply to the biological mammillary and catenarry systems as follows:

$$Y_{ij} = \alpha_{i0} + \sum_{k=1}^n \alpha_{ik} e^{-\lambda_k x_j} + \varepsilon_{ij} \tag{2.25}$$

for $i = 1, 2, ..., n$, and $j = 0, 1, ..., N-1$. In the equation (2.25), Y_{ij} is the jth observation of the ith equation. In the catenary system, Y_{ij} is the amount of particles in the ith compartment at time j. The parameters $\alpha_{i0}, \alpha_{i1}, ..., \alpha_{in}$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are constants that we need to estimate. The estimation procedure includes two steps. The first step is to estimate the exponential parameters by the generalized partial total approach described by Cornell (1962). Beauchamp and Cornell (1969) discussed the generalized Spearman estimation approach originated from Johnson and Brown (1961) assuming that the independent variables in the model are equally spaced on a logarithmatic scale. The second step is to estimate the linear coefficients $\alpha_{11}, \alpha_{1n}, ..., \alpha_{nn}$ via least squares methods after substituting for the exponential estimates of $(\lambda_1, \lambda_2, ..., \lambda_n)$ obtained in the first step.

For a general m-compartment system, where particles can transfer among all the states, it is assumed there are m^2 transition rates. Matis (1974) set up m random variables denoting the number of particles in each compartment at time t, i.e., $N_i(t)$, where $i = 1, 2, ..., m$. By assuming that the occurrence of more than two migrations in Δt is $o(\Delta t)$, the joint generating function of the number of particles for each compartment was derived, so was the joint probability function. As a result, $N_i(t)$, for $t = 1, 2, ..., m$, given t known, was interpreted as a mixture of multinomial distributions. Two propositions were derived in the paper as follows.

Proposition 2.4.1. *The expected value, $\mu_i(t)$, of the number of units in each compartment i of a stochastic m-compartment system is identically equal to its deterministic solution.*

Proposition 2.4.2. *Let the m-vector $\Delta(t)$ be defined by $\Delta^T(t) = [N_1(t), N_2(t), ..., N_m(t)]$. Also, let $\Gamma_i(t)$, where $\Gamma_i^T(t) = [\gamma_{1i}(t), \gamma_{2i}(t), ..., \gamma_{mi}(t)]$ for $i = 1, 2, ..., m$, be distributed as a multinomial distribution with parameters $N_i(0), p_{1i}(t), p_{2i}(t), ..., p_{mi}(t)$;*

i.e.,

$$Prob[\gamma_{1i}, \gamma_{2i}(t), ..., \gamma_{mi}(t)] = \frac{N_i(0)! \prod_{i=1}^{m} p_{ji}^{r_{ji}} [1 - \sum_{j=1}^{m}]^{N_i(0) - \sum_{j=1}^{m} \gamma_{ji}}}{\prod_{j=1}^{m} \gamma_{ji}! [N_i(0) - \sum_{i=1}^{m} \gamma_{ji}]!}.$$

Then, $\Delta(t)$ is distributed as the sum of the m independent variables $\Gamma_i(t)$, i.e.,

$$\Delta(t) = \sum_{i=1}^{m} \Gamma_i(t).$$

With Proposition (2.4.2), the estimation does not only allow for the correlation of observations over time but also enables the construction of specific functions between the covariance matrix and the estimated parameters to enhance the calculation efficiency. Under the condition that the covariance matrix is not specified, a two-fold procedure is proposed in the paper. The initial loop is to assume that the covariance matrix be the identity matrix to calculate the coefficient estimates as starting values to improve the estimates of the covariance matrix in the next cycle until convergence criteria are satisfied.

In a series of papers focusing on the HIV/AIDS process, Longini and Clark(1989) estimates the transition probabilities by formulating the likelihood function through the stages of infection for all the individuals as follows:

$$L(\lambda) = \prod_{j=1}^{n} (\prod_{k=0}^{m_j - 1} p_{y_{jk}, y_{jk+1}} (\tau_{jk+1} - \tau_{jk})) \tag{2.26}$$

where $p_{y_{jk}, y_{jk+1}}$ denotes the transition probability from state k to $k+1$ for the jth individual, and τ_{jk} is the time at which that jth individual is in the stage k. Then the maximum likelihood estimates of the parameters transition intensities as well as the estimated variance covariance matrix were obtained with the use of the derivative-free, pseudo-Gauss-Newton algorithm in the BMDP statistical package derived by Ralston (1985). Longini et al. (1992) extends the back calculation method for estimating the number of HIV infected individuals. The idea is to build the likelihood function for $X(t)$, $t = 1, 2, ..., n$, where $X(t)$ is the number of incidences of HIV at time t. Guihenneuc-Jouyaux et al. (2000) introduces Bayesian methods for the parameter estimation.

On the other hand, instead of focusing on the transition probabilities and intensities with a Markov process assumption, Datta et al. (2000b) directly estimates the stage-occupation probabilities with nonparametric methods. Datta and Satten (2000a) focus on estimating future stage entry and occupation probability for right-censored data. Datta and Satten (2001) also discuss the Aalen-Johansen estimators described by Aalen (1978) for stage occupation probabilities and Nelson-Aalen estimators described by Andersen et al. (1993) for integrated transition hazards for non-Markov models. Datta and Satten (2002) proposed nonparametric estimators of integrated transition hazards and stage occupation probabilities for non-markov models.

The above parametric models all require the solution of differential-difference equations. It is known that the solution, such as equation (2.24), can be very computationally difficult and intensive. We will review another approach called the waiting time approach.

2.5 Waiting Time Approach

Instead of focusing on building a traditional differential-difference equation, Billard and Dayananda (2014a, b) focus on the waiting time in each state until moving to the next state and includes the possibility of unexpected death. A four-state model is discussed, in which states 0, 1, 2, and 3 denote the susceptible status, the first and second stage of infection and death, respectively. Generally, an individual goes through the process successively from the susceptible state to death. In addition, at each state, other than death, an individual can move to death directly for any unexpected reason outside of the disease (AIDS) in their case. Assume that U_i denotes the waiting time in state i until going to state $i+1$, and V_i denotes the waiting time in state i until going to death for any AIDS-unrelated reason, for $i = 0, 1, 2$. Let $H_i = \min(U_i, V_i)$, and $W_i = U_i - V_i$; then $W_i > 0$ implies that disease happens without unexpected death. To obtain the transition probability $q_{ij}(t) = P(S(t) = j|S(0) = i)$, a theory is derived in the paper as follows:

$$\begin{aligned} q_{ij} &= P(\mathrm{S}(t) = j|\mathrm{S}(0) = i) \\ &= P(\mathrm{W}_k > 0, k = i, i+1, .., j-1)[P(\mathrm{Y}_{ij} > t|\mathrm{W}_k > 0, k = i, i+1, .., j-1) \\ &\quad -P(\mathrm{Y}_{i,j-1} > t|\mathrm{W}_k > 0, k = i, i+1, .., j-1)] \end{aligned} \tag{2.27}$$

where $Y_{ij} = \sum_{k=i}^{j} \mathrm{H}_k$. Hence, the problem is transformed to one where it is necessary to calculate the probability that $Y_{ij} > t$ conditional on no unexpected death.

2.6 Disability Insurance

Insurance companies are always interested in the premium they should collect and the payout they will have to make for the insured. Since this project is to investigate a disability model with a waiting time approach, we will describe some general concepts about disability insurance policy using Permanent Health Insurance policy as an example. Let x be the age of the insured when the policy is issued, and let $S(t+x)$ denote the status of the insured after a period of time t. Assume that an insurance company needs to pay out a continuous annuity of dt in the time interval of $(t, t+dt)$ on condition that the insured is disabled until he/she either recovers or dies. Let Y_x denote the present value of the payout by insurance companies; this can be expressed as

$$Y_x = \int_0^\infty \upsilon^u I_{\{S(x+u)=i|S(x)=a\}} \mathrm{d}u \tag{2.28}$$

where $I_{\{S(x+u)=i|S(x)=a\}}$ is an indicator function, and υ is the annual discount factor. In the actuarial area, $\phi(x,u) = P\{S(x+u) = i|S(x) = a\}$ is used to denote the probability that the insured is disabled given the initial state is active, i.e., not disabled. On the other hand, the net single premium, $\bar{a}_x^{ai}$, is defined as the expected payout, i.e.,

$$\bar{a}_x^{ai} = E(Y_x|S(x) = a) = \int_0^\infty \upsilon^u P\{S(x+u) = i|S(x) = a\} \mathrm{d}u. \tag{2.29}$$

The set of policy conditions is formally represented by a set of five parameters $\Gamma = (n_1, n_2, f, m, r)$ refering to Pitacco (1995), where (n_1, n_2) denotes the insured period, that is to say, an annuity would be payable if disability inception falls in this interval; f means the deferred period since a disability occurs, that is to say, an annuity would be payable after a disability lasts at least f units of time long; m is the maximum number of years of annuity payment since disability happens. In the end, r is the stopping time, that is to say, the number of years from the start of covering time to the time that the insured is not qualified for coverage any more. Another important term is the transition intensity defined as

$$\mu^{gh}(x) = \lim_{u \to 0} \frac{P\{S(x+u) = h | S(x) = g\}}{u}. \tag{2.30}$$

By assuming that one single insurance policy is independent of the others and every transition could be observed under the policy, The recovery rate ρ was estimated as the ratio of the observed number of recoveries that happened to the total time spent sick both in a valid policy time period by Water (1991). Similar results were derived for the mortality of sick intensities and sick intensities. Billard and Dayananda (2014b) list and run simulations about a series of functions of interest to insurance companies as follows:

1.) The net single premium for a t-year pure endowment, assuming that an individual was in stage i when he/she was covered. At time t, one dollar would be paid to him/her as long as he/she would be in some qualified state j. The present expected valued for the individual would be

$$\overline{E}_i(t) = \sum_{j=i}^{m+1} e^{-\delta t} q_{ij}(t) \tag{2.31}$$

where $q_{ij}(t)$ is the probability of moving into stage j given he started in stage i of equation (2.27) and $e^{-\delta u}$ is the present value of one dollar at time t.

2.) For continuous t-year life annuity policy, the present expected valued $\overline{a}_i(t)$ can be expressed as follows

$$\overline{a}_i(t) = \int_0^t e^{-\delta u} q_{ij}(u) du. \tag{2.32}$$

3.) On the other hand, under the continuous t-year insurance policy, the net single premium $\overline{A}_i(t)$ can be expressed as

$$\overline{A}_i(t) = \sum_{j=i}^{m+1} \mu_j^{'} \int_0^t e^{-\delta u} q_{ij}(u) du. \tag{2.33}$$

4.) The long-term annual premium rates $\overline{P}_i(\infty)$ defined was

$$\overline{P}_i(\infty) = \lim_{t \to \infty} \frac{\overline{A}_i(t)}{\overline{a}_i(t)}. \tag{2.34}$$

Further details were seen in Billard and Dayananda (2014b).

References

Aalen, O. O. (1978). Nonparametric Estimation of Partial Transition Probabilities in Multiple Decrement Models. *Annals of Statistics*, 5, 141-150.

Akkouchi, M. (2008). On the Convolution of Exponential Distributions. *Journal of the Chungcheong Mathematical Society*, 21, 501-510.

Andersen, P. K., Borgan, O., Gill, R.D., and Keiding, N. (1993). *Statistical Models Based On Counting Processes*. Springer, New York.

Bartholomay, A. F. (1958). Stochastic Models for Chemical Reactions: I. Theory of The Unimolecular Reaction Process *Bulletin of Mathematical Biophysics*, 20, 175-190.

Bartlett M. S. (1949). Some Evolutionary Stochastic Process. *Journal of the Royal Statistical Society Series B*, 11, 211-229.

Beauchamp, J. and Cornell, R. (1966). Simultaneous Nonelinear Estimation. *Technometrics*, 8, 319-326.

Beauchamp, J. and Cornell R. (1968). Simultaneous Estimation by Partial Totals for Compartmental Models. *Journal of the American Statistical Association* , 63, 573-583.

Beauchamp, J. and Cornell R. (1969). Spearman Simultaneous Estimation for a Compartmental Model. *Technometrics*, 11, 551-560.

Billard, L. and Zhao Z. (1991). Three-Stage Stochastic Epidemic Model: Application to AIDS. *Mathematical Biosciences*, 107, 431-449.

Billard, L. and Zhao Z. (1994). Multi-Stage Non-Homogeneous Markov Models for the Acquired Immune Deficiency Syndrome Epidemic. *Journal of the Royal Statistical Society Series B*, 56, 673-686.

Billard, L. and Dayananda, P. W. A. (2014a). A Multi-Stage Compartmental Model for HIV-infected individuals: I - Waiting time approach. *Mathematical Biosciences*, 249, 92-101.

Billard, L. and Dayananda, P. W. A. (2014b). A Multi-Stage Compartmental Model for HIV-Infected Individuals: II - Application to Insurance Functions and Health-Care Costs Approach. *Mathematical Biosciences*, 249, 92-101.

Brauer, F. (2005). The Kermack-McKendrick Epidemic Model Revisited. *Mathematical Biosciences*, 198, 119-131.

Cornell, R. (1962). A Method for Fitting Linear Combinations of Exponentials. *Biometrics*, 18, 104-113.

Datta, S. and Satten, G. A. (2000). Estimating Future Stage Entry and Occupation Probability in a Multistage Model Based on Randomly Right-Censored Data. *Statistics and Probability Letters*, 50, 89-95.

Datta, S. and Satten, G. A. (2001). Validity of the Aalen-Johansen Estimator of Stage Occupation Probabilities and Nelson-Aalen Estimators of Integrated Transition Hazards for Non-Markov Models. *Statistics and Probability Letters*, 55, 403-411.

Datta, S. and Satten, G. A.(2002). Estimation of Integrated Transition Harzards and Stage OccupationBeau Probabilities for Non-Markov Systems Under Dependent Censoring. *Biometrics*, 58, 792-802.

Datta, S., Satten, G. A. and Dattta, S. (2000). Nonparametric Estimation for Three-Stage Irreversible Illness-Death Model. *Biometrics*, 56, 841-847.

Feller, W. (1939). Die Grundlagen der Volterraschen Theorie des Kampfes ums Dasein in Wahrscheinlichkeithoeretischer Behandlung. *Acta Biotheoretica*, 5, 11-40.

Guihenneuc-Jouyaux, C., Richardson, S. and Longini, I. M. (2000). Modeling Markers of Disease Progression by a Hidden Markov Process: Application to Characterizing CD4 Cell Decline. *Biometrics*, 56, 733-741.

Johnson, E. A. and Brown, B. W. (1961). The Spearman estimator for serial dilution assays. Biometrics 17, 79-88.

Kermack, W. O. and McKendrick, A. G. (1927). A Contribution to the Mathematical Theory of Epidemics. *Proceedings of the Royal Society A*, 115, 700-721.

Longini, I. M., Bayers, R. H., Hessol, N. A. and Tan W. Y. (1992). Estimating the Stage-Specific Numbers of HIV Infection Using A Markov Model and Back-Calculation. *Statistics in Medicine*, 11, 831-843.

Longini, I. M. and Clark, W.S. (1989). Statistical Analysis of the Stage of HIV Infection Using A Markov Model. *Statistics in Medicine*, 8, 831-843.

Longini, I. M., Clark, W.S. Gardner, L. I. and Brundag, J. F. (1991). The Dynamics of $CD4^{+}$ T-Lymphocyte Decline in HIV-Infected Individuals: A Markov Modeling Approach. *Journal of Acquired Immune Deficiency Syndromes*, 4, 1141-1147.

Matis, J. (1974). On the Stochastic Theory of Compartments: Solution for n-Compartment Systems with Irreversible, Time-Dependent Transition Probabilities. *Bulletin of Mathematical Biology*, 36, 489-504.

Matis, J. H. and Hartley, H. O. (1971). Stochastic Compartmental Analysis: Model and Least Squares Estimation From Time Series Data. *Biometrics*, 27, 77-102.

Matis, J. H. and Tolley, H. D. (1979). Compartmental Models with Multiple Sources of Stochastic Variability: The One-Compartment, Time Invariant Hazard Rate Case. *Bulletin of Mathematical Biology*, 41, 491-515.

Pitacco, E. (1995). Actuarial Models for Pricing Disability Benefit: Towards a Unifying Approach. *Insurance: Mathematics and Economics*, 16, 39-62.

Purdue, P. (1974a). Stochastic Theory of Compartments. *Bulletin of Mathematical Biology*, 36, 305-309.

Purdue, P. (1974b). Stochastic Theory of Compartments: One and Two Compartment Systems. *Bulletin of Mathematical Biology*, 36, 577-587.

Purdue, P. (1975). Stochastic Theory of Compartments: An Open, Two Compartment, Reversible System With Independent Poisson Arrivals. *Bulletin of Mathematical Biology*, 36, 269-275.

Ralston, M. (1985) Derivative-free nonlinear regression in BMDP Statistical Software Manual, *University of California Press*, Berkeley, 305-329.

Ross, R.(1911). *The Prevention of Malaria* (2nd edition), London: Murray.

Sheppard C. W. and Housholder A.S. (1951). The Mathematical Basis of the Interpretation of Tracer Experiments in Closed Steady-State Systems. *Journal of Applied Physics*, 22, 510-520.

Thakur, A. K., Rescigno, A. and Schafer, D. E. (1972). On the Stochastic Theory of Compartments: I. A Single - Compartment System. *Bulletin of Mathematical Biophysics*, 34, 53-63.

Thakur, A. K., Rescigno, A. and Schafer, D. E. (1973). On the Stochastic Theory of Compartments: II. Multi-Compartment Systems. *Bulletin of Mathematical Biology*, 35, 263-271.

Thakur, A. K. and Rescigno, A. (1978). On the Stochastic Theory of Compartments: III. General Time-dependent Reversible Systems. *Bulletin of Mathematical Biology*, 35, 263-271.

Waters H. R. (1991). The analysis of permanent health insurance data. *Continuous Mortality Investigation Reports*, 12, 1-263.

Zilversmit, C. E. and Fishler M. C. (1943). Stochastic Compartmental Analysis: Model and Least Squares Estimation From Time Series Data. *Biometrics*, 27, 77-102.

Chapter 3

Waiting Time Approach for Disability Models

The waiting time approach will be applied to a disability model in this chapter. The description of the model will be given in section 3.1. Then the transition probabilities are derived with the waiting time approach by assuming the waiting time is exponentially distributed in section 3.2. In section 3.3, the sensitivities of transition probabilities to the change of parameters are investigated based on some specific health insurance policies.

3.1 The Description of the Disability Model

The description of the disability model

The disability model that is investigated in this chapter can be described clearly with reference to Figure 3.1. During a period of an effective insurance policy, an individual is assumed to be healthy (denoted as being in the state S_{11}) at the beginning of the period; after waiting for time X_1, the individual would become disabled (denoted as being in the state S_{21}); and then after staying disabled for time X_2, the individual comes back to a healthy state (denoted as being in the state S_{12}). Then, another cycle begins as described in Figure 3.1. The process iterates for m times, which assumes that the individual will go through the disability state m times. In addition, it is assumed that death could be attained independently from every healthy or disabled state.

To investigate the model more clearly, we unfold it so that it is formulated as a $2m+1$ multistage compartmental process as described in Figure 3.2. An individual could be healthy (by moving to state S_{1i}) or disabled (in state S_{2i}) at time t for the ith time, where $i = 1, 2, ..., m$. At each state, an individual can die (by going to state S_{2m+1}) with some probability. Overall, the state space will be represented by the states as $\{1, 2, ..., 2i, 2i+1, ..., 2m+1\}$, where the state $2i-1$ denotes the ith time of being healthy, the state $2i$ denotes the ith time of being disabled for $i = 1, 2, ..., m$, and the state $2m+1$ is considered as the state death.

Waiting time approach for the disability model

In this chapter, we will apply the waiting time approach for the calculation of transition probabilities. Referring to Figure 3.2, X_{2i-1} is defined as the waiting time in the ith healthy state until the individual goes to the ith disabled state; X_{2i} is defined as the waiting time in the ith disabled state until the individual goes to the $(i+1)$th healthy state, and D_{2i-1} is the waiting time in the ith healthy state until the individual dies instead of transferring to the ith disabled state. Similarly, D_{2i} is

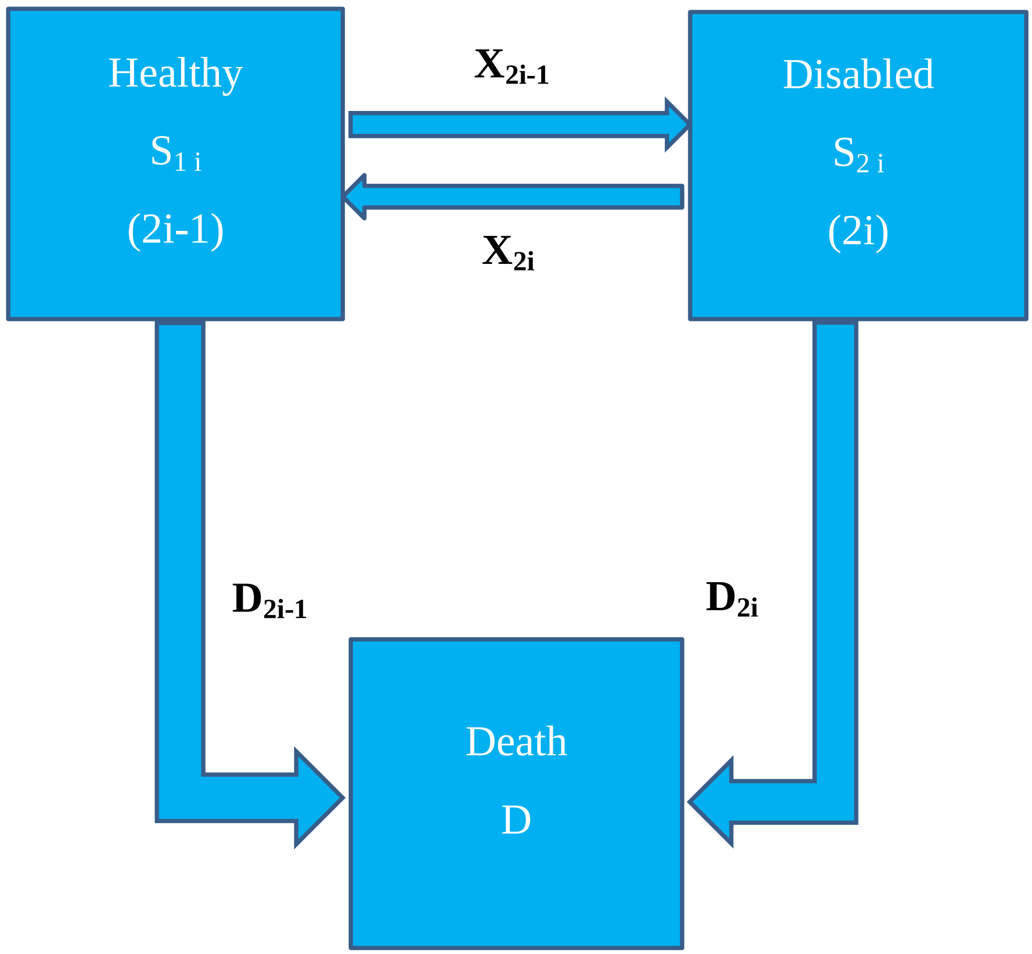

Figure 3.1: State transition diagram for a disabled model

defined as the waiting time in the ith disabled state until the individual dies without further transition back to the next healthy state, for $i = 1, 2, ..., m + 1$. Next, we need variables W_{2i-1},W_{2i}, H_{2i-1} and H_{2i} defined as

$$\mathrm{W}_{2i-1} = \mathrm{D}_{2i-1} - \mathrm{X}_{2i-1} \tag{3.1}$$

$$\mathrm{W}_{2i} = \mathrm{D}_{2i} - \mathrm{X}_{2i} \tag{3.2}$$

$$\mathrm{H}_{2i-1} = \min(\mathrm{X}_{2i-1}, \mathrm{D}_{2i-1}) \tag{3.3}$$

$$\mathrm{H}_{2i} = \min(\mathrm{X}_{2i}, \mathrm{D}_{2i}) \tag{3.4}$$

where $i = 1, 2, ..., m+1$. Here, $\mathrm{W}_{2i-1} > 0$ means that the individual moves from the ith healthy state to the next disabled state without going to the death state. On the other hand, $\mathrm{W}_{2i-1} < 0$ means that the individual in the ith healthy state moves to the state death without having a chance to go to the next disabled state. The variable H_{2i-1} corresponds to the length of time that the individual spends in the ith healthy state, and H_{2i} means the length of time that the individual spends in the ith disabled state. Let $\mathrm{S}(t)$ denote the state in which an individual stays at time t; the state space would be $\{S(t) = 1, 2, ..., 2m+1\}$. One of our goals is to calculate the transition probabilities that an individual is in stage j at time t given that the initial stage is i at time 0, i.e., $q_{ij}(t)$. This is expressed as follows:

$$\mathrm{q}_{ij}(t) = P(\mathrm{S}(t) = j|\mathrm{S}(0) = i) \tag{3.5}$$

for $i \leq j$, and $i, j = 1, 2, ..., 2m$.

3.2 Waiting Time Approach for the Disability Model with Exponentially Distributed Waiting Time

Description of the exponential case

In this section, we would assume all the waiting time variables are independent to each other with exponential distributions. To be specific, we assume that X_1 is exponentially distributed with rate λ_{11}; $X_3, X_5, ..., X_{2i-1}$,..., and X_{2m-1} are exponentially distributed with rates $\lambda_3, \lambda_5, ..., \lambda_{2i-1}$,..., and λ_{2m-1}, respectively; $X_2, X_4, ..., X_{2i}$,...,and X_{2m} are exponentially distributed with rates $\lambda_2, \lambda_4, ..., \lambda_{2i}$,..., and λ_{2m}, respectively. On the other hand, we assume that D_1 is exponentially distributed with rate $\alpha(z)$, where $\alpha(z)$ is a function of age; $D_3, D_5, ..., D_{2i-1}$, ..., and D_{2m-1} are independently exponentially distributed with rates $\alpha_3, \alpha_5, ..., \alpha_{2i-1}$,..., and α_{2m-1}, respectively; $D_2, D_4, ...,$ D_{2i}, ..., and D_{2m} are independently exponentially distributed with rates α_2, α_4,...,α_{2i},...,and α_{2m} respectively. As we claimed in section 3.1, one of our goals is to calculate the transition probabilities defined in equation (3.5). Thus, we will show how the transition probabilities $q_{ij}(t)$s are derived for $i, j = 1, 2, ..., 2m$.

Distributions of H_{2i-1}, H_{2i}, W_{2i-1}, and W_{2i}

First, we start with H_{2i-1} and H_{2i} for $i = 1, 2, ..., m$. Because D_i and X_i are independent to each other for $i = 1, 2, ..., m$, the joint probability distribution function of D_i and X_i is the product of the two probability distribution functions. Based on the definitions of H_{2i-1} in equation (3.3) and H_{2i} in equation (3.4), we have that:

$$\begin{aligned}
P(\mathrm{H}_1 > t) &= P(\mathrm{X}_1 > t, \mathrm{D}_1 > t) \\
&= P(X_1 > t)P(D_1 > t) \\
&= e^{-\lambda_{11}t}e^{-\alpha(z)t} \\
&= e^{-(\lambda_{11}+\alpha(z))t}; \qquad (3.6) \\
P(\mathrm{H}_{2i+1} > t) &= P(\mathrm{X}_{2i+1} > t, \mathrm{D}_{2i+1} > t) \\
&= P(X_{2i+1} > t)P(D_{2i+1} > t) \\
&= e^{-\lambda_{2i+1}t}e^{-\alpha_{2i+1}t} \\
&= e^{-(\lambda_{2i+1}+\alpha_{2i+1})t}, \qquad i = 1, 2, ..., m; \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
P(\mathrm{H}_{2i} > t) &= P(\mathrm{X}_{2i} > t, \mathrm{D}_{2i} > t) \\
&= P(X_{2i} > t)P(D_{2i} > t) \\
&= e^{-\lambda_{2i}t}e^{-\alpha_{2i}t} \\
&= e^{-(\lambda_{2i}+\alpha_{2i})t}. \qquad i = 1, 2, ..., m. \qquad (3.8)
\end{aligned}$$

Next, we will calculate $P(W_{2i-1} > 0)$ and $P(W_{2i} > 0)$ for $i = 1, 2, ..., m$, based on the definitions of W_i in equations (3.1) and (3.2). We start with $P(W_1 > 0)$ as follows:

$$P(W_1 > 0) = P(D_1 - X_1 > 0) = P(D_1 > X_1). \qquad (3.9)$$

Since we have that D_1 and X_1 are independent to each other, also that, D_1 and X_1 are both exponentially distributed with rates $\alpha(z)$ and λ_{11}, equation (3.9) becomes that

$$\begin{aligned}
P(W_1 > 0) &= P(D_1 - X_1 > 0) \\
&= P(D_1 > X_1) \\
&= \int_0^{+\infty} \int_{x_1}^{+\infty} \lambda_{11}\alpha(z)e^{-(\lambda_{11}x_1+\alpha(z)d_1)}dd_1dx_1 \\
&= \int_0^{+\infty} \lambda_{11}e^{-\lambda_{11}x_1}(\int_{x_1}^{+\infty} \alpha(z)e^{-\alpha(z)d_1}dd_1)dx_1 \\
&= \int_0^{+\infty} \lambda_{11}e^{-\lambda_{11}x_1}e^{-\alpha(z)x_1}dx_1 \\
&= \frac{\lambda_{11}}{\lambda_{11}+\alpha(z)}. \qquad (3.10)
\end{aligned}$$

Also, based on the definition of W_{2i+1} in equation (3.1), we have that

$$P(W_{2i+1} > 0) = P(D_{2i+1} - X_{2i+1} > 0). \qquad (3.11)$$

We have assumed that D_{2i+1} and X_{2i+1} are independent to each other and that both are exponentially distributed with rates α_{2i+1} and λ_{2i+1}, for $i = 1, 2, ..., m$, equation (3.11) can be written as

$$\begin{aligned}
P(W_{2i+1} > 0) &= P(D_{2i+1} - X_{2i+1} > 0) \\
&= P(D_{2i+1} > X_{2i+1}) \\
&= \int_0^{+\infty} \int_{x_{2i+1}}^{+\infty} \lambda_{2i+1}\alpha_{2i+1} e^{-(\lambda_{2i+1}x_{2i+1}+\alpha_{2i+1}d_{2i+1})} dd_{2i+1} dx_{2i+1} \\
&= \int_0^{+\infty} \lambda_{2i+1} e^{-\lambda_{2i+1}x_{2i+1}} (\int_{x_{2i+1}}^{+\infty} \alpha_{2i+1} e^{-\alpha_{2i+1}d_{2i+1}} dd_{2i+1}) dx_{2i+1} \\
&= \int_0^{+\infty} \lambda_{2i+1} e^{-\lambda_{2i+1}x_{2i+1}} e^{-\alpha_{2i+1}x_{2i+1}} dx_{2i+1} \\
&= \frac{\lambda_{2i+1}}{\lambda_{2i+1} + \alpha_{2i+1}}, \qquad (3.12)
\end{aligned}$$

for $i = 1, 2, ...m$.

Similarly, based on the definition of W_{2i} in equation (3.2), we have that

$$P(W_{2i} > 0) = P(D_{2i} - X_{2i} > 0) \qquad (3.13)$$

We have assumed that D_{2i} and X_{2i} are independent to each other and that both exponentially distributed with rates α_{2i} and λ_{2i}, for $i = 1, 2, ..., m$. Thus, we have

$$
\begin{aligned}
P(W_{2i} > 0) &= P(D_{2i} - X_{2i} > 0) \\
&= P(D_{2i} > X_{2i}) \\
&= \int_0^{+\infty} \int_{x_{2i}}^{+\infty} \lambda_{2i}\alpha_{2i} e^{-(\lambda_{2i}x_{2i} + \alpha_{2i}d_{2i})} dd_{2i} dx_{2i} \\
&= \int_0^{+\infty} \lambda_{2i} e^{-\lambda_{2i}x_{2i}} (\int_{x_{2i}}^{+\infty} \alpha_{2i} e^{-\alpha_{2i}d_{2i}} dd_{2i}) dx_{2i} \\
&= \int_0^{+\infty} \lambda_{2i} e^{-\lambda_{2i}x_{2i}} e^{-\alpha_{2i}x_{2i}} dx_{2i} \\
&= \frac{\lambda_{2i}}{\lambda_{2i} + \alpha_{2i}} \qquad (3.14)
\end{aligned}
$$

for $i = 1, 2, ...m$. As we defined in equations (3.1) - (3.4), state H_i is the time an individual stays in state i before he/she goes to the next state. Next, we will calculate the conditional probabilities $P(H_i > t|W_i > 0)$ for $i = 1, 2, ..., 2m$, i.e., the probability that an individual stays in either a healthy or a disabled state longer than time t conditional on no accidental death. Based on these definitions, we have that

$$
\begin{aligned}
P(H_i > t|W_i > 0) &= \frac{P(min(X_i, D_i), W_i > 0)}{P(W_i > 0)} \\
&= \frac{P(D_i > X_i, X_i > t)}{P(D_i > X_i)}. \qquad (3.15)
\end{aligned}
$$

We will start with $P(H_1 > t|W_1 > 0)$. Based on the equation (3.15), we have that

$$P(H_1 > t|W_1 > 0) = \frac{P(D_1 > X_1, X_1 > t)}{P(W_1 > 0)}. \tag{3.16}$$

Since D_1 and X_1 are independently and exponentially distributed with rates $\alpha(z)$ and λ_{11}, we have that

$$\begin{aligned}
P(H_1 > t|W_1 > 0) &= K_1[\int_t^{+\infty}\int_t^{d_1} \lambda_{11}e^{-\lambda_{11}x_1}\alpha(z)e^{-\alpha(z)d_1}dx_1dd_1] \\
&= K_1[\int_t^{\infty} \alpha(z)e^{-\alpha(z)d_1}(\int_t^{d_1} \lambda_{11}e^{-\lambda_{11}x_1}dx_1)dd_1] \\
&= K_1[\int_t^{+\infty} \alpha(z)e^{-\alpha(z)d_1}(e^{-\lambda_{11}t} - e^{-\lambda_{11}d_1})dd_1] \\
&= K_1[\int_t^{+\infty} \alpha(z)e^{-\alpha(z)d_1}e^{-\lambda_{11}t}dd_1 - \\
&\quad \int_t^{+\infty} \alpha(z)e^{-\alpha(z)d_1}e^{-\lambda_{11}d_1}dd_1] \\
&= K_1[e^{-(\lambda_{11}+\alpha(z))t} - \frac{\alpha(z)}{\lambda_{11}+\alpha(z)}e^{-(\lambda_{11}+\alpha(z))t}] \\
&= e^{-(\lambda_{11}+\alpha(z))t} \qquad (3.17)
\end{aligned}$$

where $K_1 = [\lambda_{11}/(\lambda_{11} + \alpha(z))]^{-1}$. Similarly, since D_{2i+1} and X_{2i+1} are independent and exponentially distributed with rates α_{2i+1} and λ_{2i+1}, for $i = 1, 2, ..., m-1$, respectively, we have that

$$
\begin{aligned}
& P(H_{2i+1} > t | W_{2i+1} > 0) \\
= \; & \frac{P(D_{2i+1} > X_{2i+1}, X_{2i+1} > t)}{P(W_{2i+1} > 0)} \\
= \; & K_{2i+1}[\int_t^{+\infty} \int_t^{d_{2i+1}} \lambda_{2i+1} e^{-\lambda_{2i+1} x_{2i+1}} \alpha_{2i+1} e^{-\alpha_{2i+1} d_{2i+1}} dx_{2i+1} dd_{2i+1}] \\
= \; & K_{2i+1}[\int_t^{\infty} \alpha_{2i+1} e^{-\alpha_{2i+1} d_{2i+1}} (\int_t^{d_{2i+1}} \lambda_{2i+1} e^{-\lambda_{2i+1} x_{2i+1}} dx_{2i+1}) dd_{2i+1}] \\
= \; & K_{2i+1}[\int_t^{+\infty} \alpha_{2i+1} e^{-\alpha_{2i+1} d_{2i+1}} (e^{-\lambda_{2i+1} t} - e^{-\lambda_{2i+1} d_{2i+1}}) dd_{2i+1}] \\
= \; & K_{2i+1}[\int_t^{+\infty} (\alpha_{2i+1} e^{-\alpha_{2i+1} d_{2i+1}} e^{-\lambda_{2i+1} t} \\
& -\alpha_{2i+1} e^{-\alpha_{2i+1} d_{2i+1}} e^{-\lambda_{2i+1} d_{2i+1}}) dd_{2i+1}] \\
= \; & K_{2i+1}[e^{-(\lambda_{2i+1}+\alpha_{2i+1})t} - \frac{\alpha_{2i+1}}{\lambda_{2i+1}+\alpha_{2i+1}} e^{-(\lambda_{2i+1}+\alpha_{2i+1})t}] \\
= \; & e^{-(\lambda_{2i+1}+\alpha_{2i+1})t} \qquad (3.18)
\end{aligned}
$$

where $K_{2i+1} = [\lambda_{2i+1}/(\lambda_{2i+1} + \alpha_{2i+1})]^{-1}$. Likewise, since D_{2i} and X_{2i} are independently and exponentially distributed with rates α_{2i} and λ_{2i}, for $i = 1, 2, ..., m-1$, we have that

$$
\begin{aligned}
& P(H_{2i} > t | W_{2i} > 0) \\
= & \frac{P(D_{2i} > X_{2i}, X_{2i} > t)}{P(W_{2i} > 0)} \\
= & K_{2i}[\int_t^{+\infty} \int_t^{d_{2i}} \lambda_{2i} e^{-\lambda_{2i} x_{2i}} \alpha_{2i} e^{-\alpha_{2i} d_{2i}} dx_{2i} dd_{2i}] \\
= & K_{2i}[\int_t^{+\infty} \int_t^{d_{2i}} \lambda_{2i} e^{-\lambda_{2i} x_{2i}} \alpha_{2i} e^{-\alpha_{2i} d_{2i}} dx_{2i} dd_{2i}] \\
= & K_{2i}[\int_t^{\infty} \alpha_{2i} e^{-\alpha_{2i} d_{2i}} (\int_t^{d_{2i}} \lambda_{2i} e^{-\lambda_{2i} x_{2i}} dx_{2i}) dd_{2i}] \\
= & K_{2i}[\int_t^{+\infty} \alpha_{2i} e^{-\alpha_{2i} d_{2i}} (e^{-\lambda_{2i} t} - e^{-\lambda_{2i} d_{2i}}) dd_{2i}] \\
= & K_{2i}[\int_t^{+\infty} (\alpha_{2i} e^{-\alpha_{2i} d_{2i}} e^{-\lambda_{2i} t} - \alpha_{2i} e^{-\alpha_{2i} d_{2i}} e^{-\lambda_{2i} d_{2i}}) dd_{2i}] \\
= & K_{2i}[e^{-(\lambda_{2i}+\alpha_{2i})t} - \frac{\alpha_{2i}}{\lambda_{2i}+\alpha_{2i}} e^{-(\lambda_{2i}+\alpha_{2i})t}] \\
= & e^{-(\lambda_{2i}+\alpha_{2i})t} \qquad (3.19)
\end{aligned}
$$

where $K_{2i} = [\lambda_{2i}/(\lambda_{2i} + \alpha_{2i})]^{-1}$. From the equations (3.6) -(3.8), and (3.17)- (3.19), we have that for the exponential case, $P(H_i > t) = P(H_i > t | W_i > 0)$, for $i = 1, 2, ...,$ and $2m - 1$; i.e., the probabilities that the holding time H_i is greater than t would not change when the condition of no accidental death is added. This only applies to the exponential case, and does not apply to the general case.

Total holding time

Let Y_{ij} denote the total length of time that an individual stays where from the time point of entering the state i to the time of leaving from state j for the next stage.

Then, we have the following definition:

$$Y_{ij} = \sum_{k=i}^{j} \mathrm{H}_k \tag{3.20}$$

for $i \leq j$, $i, j = 1, 2, ..., 2m + 1$, where H_k is the holding time in state k. Based on the definition of H_k (see equations (3.3) and (3.4)), we know that the random variable Y_{ij} will be the sum of a series of independent random variables with exponential distributions. Akkouchi (2008) describes the probability density of the sum of n independent random variables having exponential distributions with different parameters as in equation (3.21). Assume that $A_1, A_2, ..., A_n$ are independently exponentially distributed random variables with rates β_1, β_2,..., β_n, respectively. The sum $\mathrm{S}_n = \sum_{i=1}^{n} \mathrm{A}_i$ will have the probability density function as follows:

$$f_{S_n}(t) = \sum_{i=1}^{n} \frac{\beta_1 \beta_2 ... \beta_n}{\prod_{j=1, j \neq i}^{n} (\beta_j - \beta_i)} \exp(-t\beta_i). \tag{3.21}$$

Applying equation (3.21) to Y_{ij} in equation (3.20), we will have that

$$f_{Y_{ij}}(t) = \sum_{p=i}^{j} \frac{\beta_i \beta_{i+1} ... \beta_j}{\prod_{k=i, k \neq p}^{j} (\beta_k - \beta_p)} e^{(-t\beta_p)} \tag{3.22}$$

where $\beta_i, \beta_{i+1}, ..., \beta_j$ would be the rate parameters for $H_i, H_{i+1}, ..., H_j$, respectively. Then, we will have that

$$P(Y_{ij} > t) = \sum_{p=i}^{j} \prod_{k=i,k\neq p}^{j} \frac{\beta_k}{\beta_k - \beta_p} e^{-t\beta_p} \quad (3.23)$$

for $i, j = 1, 2, ..., 2m$ and $i < j$.

Transition probability

Billard and Dayananda (2014a) proves that the transition probabilities q_{ij} of equation (2.27) satisfy

$$\begin{aligned} q_{ij}(t) &= P(S(t) = j|S(0) = i) \\ &= P(W_k > 0, k = i, i+1, .., j-1)[P(Y_{ij} > t|W_k > 0, k = i, i+1, .., j-1) \\ &\quad -P(Y_{i,j-1} > t|W_k > 0, k = i, i+1, .., j-1)]. \end{aligned} \quad (3.24)$$

Since in our case we have $P(H_i > t) = P(H_i > t|W_i > 0)$, for $i = 1, 2, ..., 2m-1$, the transition probabilities become

$$\begin{aligned} q_{ij}(t) &= P(S(t) = j|S(0) = i) \\ &= P(W_k > 0, k = i, i+1, .., j-1)[P(Y_{ij} > t) - P(Y_{i,j-1} > t)]. \end{aligned} \quad (3.25)$$

Then, based on the equation (3.23), we have the general transition probabilities

$P(\mathrm{S}(t) = j|\mathrm{S}(0) = i)$ as follows:

$$\begin{aligned} q_{i,j}(t) &= P(\mathrm{S}(t) = j|\mathrm{S}(0) = i) \\ &= P(W_k > 0, k = i, i+1, .., j-1)[P(Y_{ij} > t) - P(\mathrm{Y}_{i,j-1} > t)] \\ &= \Bigg[\sum_{p=i}^{j} \prod_{k=i,k\neq p}^{j} \frac{\beta_k}{\beta_k - \beta_p} e^{-t\beta_p} - \sum_{p=i}^{j-1} \prod_{k=i,k\neq p}^{j-1} \frac{\beta_k}{\beta_k - \beta_p} e^{-t\beta_p} \Bigg] \\ &\quad \times P(\mathrm{W}_i > 0) P(\mathrm{W}_{i+1} > 0) ... P(\mathrm{W}_{j-1}) > 0 \end{aligned} \tag{3.26}$$

where $i < j$, and $i, j = 1, 2, ..., 2m-1$. In addition, $\beta_1 = \lambda_{11} + \alpha(z)$, $\beta_{2i+1} = \lambda_{2i+1} + \alpha_{2i+1}$, and $\beta_{2i} = \lambda_{2i} + \alpha_{2i}$ for $i = 1, 2, ..., m-1$. The $P(W_i > 0)$ for $i = 1, 2, ..., 2m-1$ are calculated in equations (3.10), (3.12) and (3.14). When $i = j$, we have the special case of transition probabilities as follows:

$$\begin{aligned} q_{11}(t) &= P(S(t) = 1|S(0) = 1) \\ &= P(\mathrm{Y}_{11} > t); \end{aligned} \tag{3.27}$$

from the equations (3.20)and (3.6), we have

$$\begin{aligned} q_{11}(t) &= P(H_1 > t) \\ &= e^{-(\lambda_{11} + \alpha(z))t}. \end{aligned} \tag{3.28}$$

From the equation (3.5), we have that

$$q_{i,i}(t) = P(S(t) = i|S(0) = i). \tag{3.29}$$

The event that an individual still stays in state i at time t given that he/she stays in the state i at the beginning of the policy is the same as the event that the waiting time for the individual stays in state i before he/she goes to the next stage is longer than time t. In this case, then we have that

$$P(S(t) = i|S(0) = i) \quad = \quad P(Y_{i,i} > t). \tag{3.30}$$

Based on the equations (3.23), (3.29), and (3.30), we have the transition probabilities $q_{2i+1,2i+1}(t)$ and $q_{2i,2i}(t)$ as the following:

$$\begin{aligned} q_{2i+1,2i+1}(t) &= P(S(t) = 2i+1|S(0) = 2i+1) \\ &= P(Y_{2i+1,2i+1} > t) \\ &= P(H_{2i+1} > t) \\ &= e^{-(\lambda_{2i+1}+\alpha_{2i+1})t}, \qquad i = 1, 2, ..., m-1; \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} q_{2i,2i}(t) &= P(S(t) = 2i|S(0) = 2i) \\ &= P(Y_{2i,2i} > t) \\ &= P(H_{2i} > t) \\ &= e^{-(\lambda_{2i}+\alpha_{2i})t}, \qquad i = 1, 2, ..., m-1. \end{aligned} \tag{3.32}$$

So far, all the transition probabilities are derived in equations (3.26), (3.31) and (3.32). As we emphasized in the beginning of the chapter, the transition probabilities are important for some important insurance functions. In section 3.3, we will discuss the sensitivity of the transition probabilities to the parameter changes under specific insurance policies.

3.3 Discussion of the Times of Disease Occurrence

The equation (3.26) is general but has obvious drawbacks. For example, if $\beta_k = \beta_p$ for different k and p, the formula would be meaningless. In Section 3.3, we discuss three cases where the iteration time m is 1, 2 and 3, respectively. We assume that an individual covered by an insurance policy in the beginning of a policy period has a healthy status. Then, the insured person will go through m times of disease occurrences and recoveries. At the end of the policy, the insured will stay in either a healthy or a death state. In addition, since the policy spans only one year, we assume that $\lambda_{11} = \lambda_3 = ... = \lambda_{2m-1}$, $\lambda_2 = \lambda_4 = ... = \lambda_{2m}$, and $\alpha(z) = \alpha_3 = ... = \alpha_{2m-1}$, $\alpha_2 = \alpha_4 = ... = \alpha_{2m}$.

The case that $m = 1$

When $m = 1$, the transition diagram is as shown in Figure 3.3. It is assumed that an individual is healthy when he/she is first covered by a healthy insurance policy, then goes through one stage of disease and either recovers or dies. We will calculate the transition probabilities $q_{11}(t)$, $q_{12}(t)$, $q_{13}(t)$ and $q_{14}(t)$.

First, the transition probability $q_{11}(t)$ can be calculated based on equation (3.28) as follows:

$$q_{11}(t) = e^{-(\lambda_{11}+\alpha(z))t}. \tag{3.33}$$

Applying equation (3.26), we have $q_{12}(t)$ as follows:

$$\begin{aligned}
q_{12}(t) &= P(S(t=2)|S(0)=1)) \\
&= P(W_1 > 0)[P(Y_{12} > t) - P(Y_{11} > t)] \\
&= P(W_1 > 0)[P(H_1 + H_2 > t) - P(H_1 > t)] \\
&= P(W_1 > 0)[P(H_1 < t) - P(H_1 + H_2 < t)] \\
&= \frac{\lambda_{11}}{A}[\int_0^t Ae^{-Ah_1}dh_1 - \int_0^t \int_0^{t-h_1} Ae^{-Ah_1}Be^{-Bh_2}dh_2dh_1] \\
&= \frac{\lambda_{11}}{A}[1 - e^{-At} - (1 - \frac{Ae^{-Bt} - Be^{-At}}{A-B})] \\
&= \frac{\lambda_{11}}{B-A}[e^{-At} - e^{-Bt}]
\end{aligned} \tag{3.34}$$

where $A = \lambda_{11} + \alpha(z)$ and $B = \lambda_2 + \alpha_2$. The event that the individual would be healthy at the end of the policy on condition that the insured is healthy at the beginning of the policy, is the same as when the insured person smoothly goes through the state 1 and 2 without death and the total time that the individual stays in state 1 and 2 is less than time t, i.e.,

$$
\begin{aligned}
q_{13}(t) &= P(S(t)=3|S(0)=1) \\
&= P(W_1>0)P(W_2>0)[P(H_1+H_2<t|W_1>0,W_2>0)] \\
&= P(W_1>0)P(W_2>0)P(H_1+H_2<t) \\
&= \frac{\lambda_{11}}{A}\frac{\lambda_2}{B}\int_0^t\int_0^{t-h_1} Ae^{-Ah_1}Be^{-Bh_2}dh_2dh_1 \\
&= \frac{\lambda_{11}}{A}\frac{\lambda_2}{B}[1+\frac{A}{B-A}e^{-Bt}-\frac{B}{B-A}e^{-At}] \qquad (3.35)
\end{aligned}
$$

Since we have transition probabilities $q_{11}(t)$, $q_{12}(t)$ and $q_{13}(t)$, the transition probability $q_{14}(t)$, i.e., the probabilities that the individual is dead at time t given that he/she is in the state healthy at time 0 is

$$
q_{14}(t) = 1 - q_{11}(t) - q_{12}(t) - q_{13}(t). \qquad (3.36)
$$

As we discussed in section 2.6, transition probabilities are crucial for many insurance functions; so it is important to investigate the sensitivity of parameter values to these transition probabilities. First, with reference to Billard and Dayananda (2014a), we take the healthy death rate of D_1, i.e., $\alpha(z)$, as taking values 0.001, 0.0026, 0.0042, and 0.0057 corresponding to the average death rate for a 20, 30, 40 and 50 years old person. The disease rate, λ_{11}, is set at 0.0384, which means the average waiting time until getting sick for the first time is approximately half a year. Plots of $q_{11}(t)$

against time are shown in Figure 3.4 for these healthy death rates $\alpha(z)$. From Figure 3.4, we see that the transition probability $q_{11}(t)$ is not sensitive to different death rates, $\alpha(z)$. The reason is that the healthy death rate is relatively small compared to the rate of becoming sick.

Next, we assume that $\lambda_{11} = 0.0384$, $\alpha(z) = 0.0026$, and the recovery rate, λ_2, takes values 0.0625, 0.125, 0.25 and 0.5 corresponding to an average sick time of 16 weeks, 8 weeks, 4 weeks and 2 weeks to get recovered, respectively. The transition probability $q_{12}(t)$ is plotted in Figure 3.5. The Figure 3.5 shows that the transition probability $q_{12}(t)$ is sensitive to the recovery rate λ_2. The smaller values the λ_2 is, i.e., the longer the sickness lasts, the larger the transition probability $q_{12}(t)$ is. In addition, as time passes, the transition probability $q_{12}(t)$ firstly increases, and then starts to decrease since the chance to transfer to the next healthy state becomes higher. Next, as Figure 3.6 shows, the $q_{13}(t)$ is also shown to be sensitive to λ_2. As λ_2 decreases, i.e., the sickness status lasts longer, the transition probability $q_{13}(t)$ decreases, i.e., the probability to recover becomes lower. In addition, since it is assumed that the individual would stay healthy at the end of a policy, the transition probability to be healthy keeps increasing.

The case that m=2

When $m = 2$, and the process is diagramed in Figure 3.7. In this case, we assume that an individual is covered by an insurance policy when he/she is healthy, and goes through a disabled state two times during the one-year policy period. In the end of the policy, the insured is either healthy or dead. We have that the transition probabilities $q_{11}(t)$ and $q_{12}(t)$ are the same as in equations (3.33) and (3.34). We will derive the formula for $q_{14}(t)$, $q_{15}(t)$,and $q_{16}(t)$. As we discussed, $P(W_1 > 0)$ and $P(W_2 > 0)$ have been derived in equation (3.10) and equation (3.14). Since H_1 and H_3 are identically independently exponential distributed with the parameter $\lambda_{11} + \alpha(z)$, the sum is a gamma distributed random variable with parameters 2 and $\lambda_{11} + \alpha(z)$. Furthermore, the distribution of the sum of H_1, H_2 and H_3 is the convolution of H_2 and the sum of H_1 and H_3. Thus, $P(H_1 + H_2 < t)$ and $P(H_1 + H_2 + H_3 < t)$ are calculated as follows:

$$
\begin{aligned}
P(H_1 + H_2 < t) &= \int_0^t \int_0^{t-h_1} Ae^{-Ah_1} Be^{-Bh_2} dh_2 dh_1 \\
&= \int_0^t -A(\int_0^{-B(t-h_1)} e^{-Bh_2} d(-Bh_2))e^{-Ah_1} dh_1 \\
&= \int_0^t -A(e^{-B(t-h_1)} - 1)e^{-Ah_1} dh_1 \\
&= \int_0^t -Ae^{(B-A)h_1} e^{-t(B-A)} dh_1 + \int_0^t Ae^{-Ah_1} dh_1 \\
&= -\frac{A(e^{(B-A)t} - 1)e^{-Bt}}{(B-A)} + 1 - e^{-At} \\
&= 1 - \frac{Ae^{-Bt} - Be^{-At}}{A-B} \qquad (3.37)
\end{aligned}
$$

where $A = \lambda_{11} + \alpha(z)$ and $B = \lambda_2 + \alpha_2$; and

$$
\begin{aligned}
&P(H_1 + H_2 + H_3 < t)\\
&= \int_0^t \int_0^{t-h_2} Be^{-Bh_2} A^2 h_{13} e^{-Ah_{13}} dh_{13} dh_2\\
&= \int_0^t Be^{-Bh_2} \left(\int_0^{-At+Ah_2} -e^{-Ah_{13}} Ah_{13} d(-Ah_{13}) \right) dh_2\\
&= \int_0^t (-(Ah_{13}+1)e^{-Ah_{13}-Bh_2} B|_{h_{13}=0}^{t-h_2}) dh_2\\
&= \int_0^t Be^{-Bh_2}(A(h_2-t)e^{A(h_2-t)} - e^{A(h_2-t)} + 1) dh_2\\
&= 1 + (-\frac{ABte^{-At+Ah_2-Bh_2}}{A-B} +\\
&+ \frac{BA(e^{(A-B)h_2-At}At + e^{(A-B)h_2-At}((A-B)h_2 - At) - e^{(A-B)h_2-At})}{(A-B)^2}\\
&- \frac{Be^{-At+Ah_2-Bh_2}}{A-B} - e^{-Bh_2})|_{h_2=0}^t\\
&= 1 + \frac{ABte^{-At}}{A-B} - \frac{A^2e^{-Bt} - 2ABe^{-At} + B^2e^{-At}}{(A-B)^2}
\end{aligned}
\tag{3.38}
$$

where $A = \lambda_{11} + \alpha(z)$, and $B = \lambda_2 + \alpha_2$.

Since H_1 and H_3 are identically independently exponentially distributed with parameter $\lambda_{11}+\alpha(z)$, and H_2 and H_4 are identically independently exponentially distributed with parameter $\lambda_2 + \alpha_2$, the probability density of $\sum_{i=1}^4 H_i$ would be the convolution of two gamma distributions, one with parameter 2 and $\lambda_{11} + \alpha(z) = A$, and another one with parameters 2 and $\lambda_2 + \alpha_2 = B$. Thus, we have

$$
\begin{aligned}
& P(\sum_{i=1}^{4} H_i < t) \\
= \;& \int_0^t \int_0^{t-h_{13}} A^2 h_{13} e^{-Ah_{13}} B^2 h_{24} e^{-Bh_{24}} dh_{24} dh_{13} \\
= \;& \int_0^t A^2 h_{13} e^{-Ah_{13}} (\int_1^{e^{-B(t-h_{13})}} ln(e^{-Bh_{24}}) d(e^{-Bh_{24}})) dh_{13} \\
= \;& \int_0^t (-(Bh_{24}+1) A^2 h_{13} e^{-Ah_{13}-Bh_{24}} |_{h_{24}=0}^{t-h_{13}}) dh_{13} \\
= \;& \int_0^t [A^2 h_{13} e^{-Ah_{13}} (-B(t-h_{13})-1) e^{-B(t-h_{13})} + 1] dh_{13} \\
= \;& 1 + (-\frac{A^2 Bt(e^{(-A+B)h_{13}-Bt} Bt + e^{(-A+B)h_{13}-Bt}((-A+B)h_{13}-Bt) - e^{(-A+B)h_{13}-Bt})}{(-A+B)^2} \\
& + \frac{1}{(-A+B)^3} (BA^2 (e^{(-A+B)h_{13}-Bt} B^2 t^2 + 2Bt(e^{(-A+B)h_{13}-Bt}((-A+B)h_{13}-Bt) \\
& - e^{(-A+B)h_{13}-Bt}) + e^{(-A+B)h_{13}-Bt}((-A+B)h_{13}-Bt)^2 - 2e^{(-A+B)h_{13}-Bt} \\
& ((-A+B)h_{13}-Bt) + 2e^{(-A+B)h_{13}-Bt})) - \frac{A^2(e^{(-A+B)h_{13}-Bt} Bt + e^{(-A+B)h_{13}-Bt})}{(-A+B)^2} \\
& - Ah_{13} e^{-Ah_{13}} - e^{-Ah_{13}}) |_{h_{13}=0}^{t} \\
= \;& 1 - \frac{A^2 Bte^{-Bt} + AB^2 te^{-At}}{(A-B)^2} - \frac{A^3 e^{-Bt} - 3A^2 Be^{-Bt} + 3AB^2 e^{-At} - B^3 e^{-At}}{(A-B)^3}. \qquad (3.39)
\end{aligned}
$$

Applying equation (3.37)and equation (3.38), we can show that $q_{13}(t)$ is calculated as follows:

$$
\begin{aligned}
q_{13}(t) &= P(S(t)=3|S(0)=1) \\
&= P(W_1>0)P(W_2>0)[P(Y_{13}>t)-P(Y_{12}>t)] \\
&= P(W_1>0)P(W_2>0)[P(\sum_{i=1}^{2} H_i<t)-P(\sum_{i=1}^{3} H_i<t)] \\
&= \frac{\lambda_{11}}{A}\frac{\lambda_2}{B}[\frac{Ae^{-Bt}-Be^{-At}+ABte^{-At}}{B-A} \\
&\quad -\frac{A^2e^{-Bt}-2ABe^{-At}+B^2e^{-At}}{(B-A)^2}].
\end{aligned} \tag{3.40}
$$

Similarly, applying equations (3.38) and (3.39), $q_{14}(t)$ can be calculated as follows:

$$
\begin{aligned}
q_{14}(t) &= P(S(t)=4|S(0)=1) \\
&= \prod_{i=1}^{3} P(W_i>0)[P(Y_{14}>t)-P(Y_{13}>t)] \\
&= \prod_{i=1}^{3} P(W_i>0)[P(\sum_{i=1}^{3} H_i<t)-P(\sum_{i=1}^{4} H_i<t)] \\
&= (\frac{\lambda_{11}}{A})^2\frac{\lambda_2}{B}[\frac{ABte^{-At}}{A-B}-\frac{A^2e^{-Bt}-2ABe^{-At}+B^2e^{-At}}{(A-B)^2} \\
&\quad +\frac{A^2Bte^{-Bt}+AB^2te^{-At}}{(A-B)^2} \\
&\quad +\frac{A^3e^{-Bt}-3A^2Be^{-Bt}+3AB^2e^{-At}-B^3e^{-At}}{(A-B)^3}].
\end{aligned} \tag{3.41}
$$

Likewise, the event that the individual would be healthy (in state 5) at the end of the policy on the condition that the insured is healthy at the beginning of the policy, is the same as the fact that the insured smoothly goes through the states 1, 2, 3 and

4 without death and the total time that the individual stays in states 1, 2, 3 and 4 is less than time t. Therefore, we have the the transition probability $q_{15}(t)$ as follows:

$$\begin{aligned}
q_{15}(t) &= P(S(t) = 5|S(0) = 1)) \\
&= P(\sum_{i=1}^{4} H_i < t|W_1 > 0, W_2 > 0, W_3 > 0, W_4 > 0) \prod_{i=1}^{4} P(W_i > 0) \\
&= P(\sum_{i=1}^{4} H_i < t) \prod_{i=1}^{4} P(W_i > 0) \\
&= (\frac{\lambda_{11}}{A})^2 \frac{\lambda_2}{B}[1 - \frac{A^2Bte^{-Bt} + AB^2te^{-At}}{(A-B)^2} \\
&\quad - \frac{A^3e^{-Bt} - 3A^2Be^{-Bt} + 3AB^2e^{-At} - B^3e^{-At}}{(A-B)^3}]; \qquad (3.42)
\end{aligned}$$

and hence

$$q_{16}(t) = 1 - q_{11}(t) - q_{12}(t) - q_{13}(t) - q_{14}(t) - q_{15}(t) \qquad (3.43)$$

Then the impacts of parameters λ_{11} and λ_2 on these probability transitions are shown from Figures 3.8 to 3.13. As Figure 3.8 shows, the disease rate λ_{11} is set up as 0.025, 0.033, 0.05 and 0.1, corresponding to the average waiting time from a healthy to a disabled state is 40, 30, 20 and 10 weeks. Figure 3.8 shows that the transition probability $q_{13}(t)$ firstly increases as time goes on, then decreases as the chance to transition to the next state increases. Thus, Figure 3.8 shows that $q_{13}(t)$ is sensitive to λ_{11}. Initially, the larger value the λ_{11} has, i.e., the shorter waiting time from a healthy to a disabled state, the larger the transition probability $q_{13}(t)$ will be. As time passes by, larger value in λ_{11} tends to result in lower value in the transition probability $q_{13}(t)$ considering that the chance to transfer to next stages increases. Figure 3.9 plots the transition probability $q_{13}(t)$ when the recovery rate λ_2 is set as 0.0833, 0.125, 0.25 and 0.5, corresponding to the average recovery time of 12, 8, 4 and 2 weeks, respectively. The Figure 3.9 shows that the transition probability is sensitive to λ_2 in the first half year. As λ_2 increases, $q_{13}(t)$ increases, i.e., as the disabled state lasts for a shorter time, the transition probability to return to the healthy state increases. It makes sense that as time passes, the transition probability $q_{13}(t)$ increases firstly, then starts to decrease as the chance to transition to the next states increases.

Figures 3.10 and 3.11 test the sensitivity of the parameters λ_{11} and λ_2 to the transition probabilities $q_{14}(t)$. In Figure 3.10, λ_{11} takes values 0.025, 0.033, 0.05, and 0.1, corresponding to the average waiting time until an individual becomes ill is 40, 30, 20 and 10 weeks, respectively. In Figure 3.11, λ_2 takes values 0.0833, 0.125, 0.25, and 0.5, which corresponds to the average time of 12, 8, 4 and 2 weeks. The two figures show that $q_{14}(t)$ is sensitive to both λ_{11} and λ_2. As λ_{11} increases, i.e., the average waiting time for a sickness occurrence decreases, the transition probability $q_{14}(t)$ increases. As time passes, the transition probability $q_{14}(t)$ decreases because the chance to transition to the next state increases. Also from the Figure 3.11, we see that $q_{14}(t)$ is sensitive to λ_2 as time passes. As λ_2 increases, $q_{14}(t)$ decreases. That is to say, as the duration of the disease becomes shorter, the chance to become sick decreases in the last half year.

Figures 3.12 and 3.13 test the sensitivity of the parameters λ_{11} and λ_2 to the transition probabilities $q_{15}(t)$, with λ_{11} taking values 0.025, 0.033, 0.05 and 0.1, and λ_2

taking values 0.0833, 0.125, 0.25, and 0.5, as in Figures 3.10 and 3.11, respectively. Both plots show that the $q_{15}(t)$ is sensitive to λ_{11} and λ_2. Since it is assumed that the individual will stay healthy when the policy ends, $q_{15}(t)$ increases as time passes. As either λ_{11} or λ_2 increases, i.e., the shorter the time that an individual stays in the interval states, the higher the chance the insured will be in the last state given time t.

The case that m=3

When $m = 3$, and transition diagrams are shown in Figure 3.14. In this case, we assume that an individual is covered by an insurance policy when he/she is healthy, and goes through the disabled state three times during the one-year policy period. At the end of the policy, the insured is either healthy or dead. The transition probabilities $q_{11}(t)$, $q_{12}(t)$, $q_{13}(t)$ and $q_{14}(t)$ will be the same as for the cases of $m = 1$ and $m = 2$. Similarly, the probability density of $\sum_{i=1}^{5} H_i$ would be the convolution of two gamma distributions, one with parameters 3 and A, and another one with 2 and B, where $A = \lambda_{11} + \alpha(z)$, and $B = \lambda_2 + \alpha_2$. The probability density of $\sum_{i=1}^{6} H_i$ is the convolution of two gamma distributions, one with parameter 3 and A, and another one with 3 and B. In this case, we have $P(\sum_{i=1}^{5} H_i < t)$ and $P(\sum_{i=1}^{6} H_i < t)$ as the following:

$$
\begin{aligned}
& P(\sum_{i=1}^{5} H_i < t) \\
= \;& \int_0^t \int_0^{t-h_{135}} \frac{1}{2} A^3 h_{135}^2 e^{-Ah_{135}} B^2 h_{24} e^{-Bh_{24}} dh_{24} dh_{135} \\
= \;& \int_0^t (-\frac{(Bh_{24}+1)A^3 h_{135}^2 e^{-Ah_{135}-Bh_{24}}}{2}|_{h_{24}=0}^{t-h_{135}}) dh_{135} \\
= \;& \int_0^t \frac{1}{2} A^3 [h_{135}^2 e^{-Ah_{135}} - Be^{-Ah_{135}-Bt+Bh_{135}} t h_{135}^2 \\
& + Be^{-Ah_{135}-Bt+Bh_{135}} h_{135}^3 - h_{135}^2 e^{-Ah_{135}-Bt+Bh_{135}}] dh_{135} \\
= \;& 1 + (-\frac{e^{-Ah_{135}} A^2 h_{135}^2}{2} - Ah_{135} e^{-Ah_{135}} - e^{-Ah_{135}} \\
& - \frac{1}{2(-A+B)^3} (A^3 Bt (e^{(-A+B)h_{135}-Bt} B^2 t^2 \\
& + 2Bt(e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) \\
& + e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt)^2 - 2e^{(-A+B)h_{135}-Bt}((-A \\
& + B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt})) \\
& + \frac{1}{2(-A+B)^4} (A^3 B (e^{(-A+B)h_{135}-Bt} B^3 t^3 + 3B^2 t^2 (e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} \\
& - Bt) - e^{(-A+B)h_{135}-Bt}) + 3Bt(e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt)^2 \\
& - 2e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt}) \\
& + e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt)^3 \\
& - 3e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt)^2 + 6e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} \\
& - Bt) - 6e^{(-A+B)h_{135}-Bt})) - \frac{1}{2(-A+B)^3} (A^3 (e^{(-A+B)h_{135}-Bt} B^2 t^2 \\
& + 2Bt(e^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A \\
& + B)h_{135} - Bt)^2 - 2^{(-A+B)h_{135}-Bt}((-A+B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt})))|_{h_{135}=0}^{t}
\end{aligned}
$$

$$
\begin{aligned}
&= 1-\frac{A^2B^2}{2(A-B)^2}e^{-At}t^2-\frac{A^3Be^{-Bt}t}{(A-B)^3}-\frac{AB^2e^{-At}t(3A-B)}{(A-B)^3}\\
&\quad -\frac{1}{(A-B)^4}(A^4e^{-Bt}-4A^3Be^{-Bt}+6A^2B^2e^{-At}-4AB^3e^{-At}\\
&\quad +B^4e^{-At});
\end{aligned}
\tag{3.44}
$$

and

$$
\begin{aligned}
& P(\sum_{i=1}^{6} H_i < t)\\
= & \int_0^t \int_0^{t-h_{135}} \frac{1}{2}A^3h_{135}^2e^{-Ah_{135}}\frac{1}{2}B^3h_{246}^2e^{-Bh_{246}}dh_{246}dh_{135}\\
= & \int_0^t (-\frac{(h_{246}^2B^2+2Bh_{246}+2)A^3h_{135}^2e^{-Ah_{135}-Bh_{246}}}{4}|_{h_{246}=0}^{t-h_{135}})dh_{135}\\
= & \int_0^t (\frac{A^3h_{135}^2e^{-Ah_{135}}}{2}-\frac{A^3B^2e^{-Ah_{135}-Bt+Bh_{135}}t^2h_{135}^2}{4}+\frac{A^3B^2e^{-Ah_{135}-Bt+Bh_{135}}th_{135}^3}{2}\\
& -\frac{A^3B^2e^{-Ah_{135}-Bt+Bh_{135}}h_{135}^4}{4}-\frac{A^3Be^{-Ah_{135}-Bt+Bh_{135}}th_{135}^2}{2}\\
& +\frac{A^3Be^{-Ah_{135}-Bt+Bh_{135}}h_{135}^2}{2}-\frac{A^3h_{135}^2e^{-Ah_{135}-Bt+Bh_{135}}}{2})dh_{135}\\
= & 1+(-\frac{e^{-Ah_{135}}A^2h_{135}^2}{2}-Ah_{135}e^{-Ah_{135}}-e^{-Ah_{135}}\\
& -\frac{1}{4(-A+B)^3}(A^3B^2t^2(e^{(-A+B)h_{135}-Bt}B^2t^2+2Bt(e^{(-A+B)h_{135}-Bt}((-A\\
& +B)h_{135}-Bt)-e^{(-A+B)h_{135}-Bt})+e^{(-A+B)h_{135}-Bt}((-A+B)h_{135}-Bt)^2\\
& -2e^{(-A+B)h_{135}-Bt}((-A+B)h_{135}-Bt)+2e^{(-A+B)h_{135}-Bt}))\\
& +\frac{1}{2(-A+B)^4}(A^3B^2t(e^{(-A+B)h_{135}-Bt}B^3t^3+3B^2t^2(e^{(-A+B)h_{135}-Bt}((-A
\end{aligned}
$$

$+B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) + 3Bt(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2$

$-2e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt)^3 - 3e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2 + 6e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt) - 6e^{(-A+B)h_{135}-Bt})) - \dfrac{1}{4(-A + B)^5}(A^3B^2(e^{(-A+B)h_{135}-Bt}B^4t^4$

$+4B^3t^3(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt})$

$+6B^2t^2(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2 - 2e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt}) + 4Bt(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^3$

$-3e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2 + 6e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)$

$-6e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^4 - 4e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt)^3 + 12e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2 - 24e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt) + 24e^{(-A+B)h_{135}-Bt})) - \dfrac{1}{2(-A + B)^3}(A^3Bt(e^{(-A+B)h_{135}-Bt}B^2t^2$

$+2Bt(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt)^2 - 2e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt}))$

$+\dfrac{1}{2(-A + B)^4}(A^3B(e^{(-A+B)h_{135}-Bt}B^3t^3 + 3B^2t^2(e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) + 3Bt(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2$

$-2e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt)^3 - 3e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt)^2 + 6e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt) - 6e^{(-A+B)h_{135}-Bt})) - \dfrac{1}{2(-A + B)^3}(A^3(e^{(-A+B)h_{135}-Bt}B^2t^2$

$+2Bt(e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) - e^{(-A+B)h_{135}-Bt}) + e^{(-A+B)h_{135}-Bt}((-A$

$+B)h_{135} - Bt)^2 - 2e^{(-A+B)h_{135}-Bt}((-A + B)h_{135} - Bt) + 2e^{(-A+B)h_{135}-Bt})))|_{h_{135}=0}^{t}$

$$
\begin{aligned}
&= 1 - \frac{A^2B^2t^2(Be^{-At} - Ae^{-Bt})}{2(A-B)^3} \\
&\quad - \frac{A^3Be^{-Bt}(A-4B)t}{(A-B)^4} - \frac{AB^3e^{-At}(B-4A)t}{(A-B)^4} \\
&\quad - \frac{A^3e^{-Bt}(A^2-5AB+10B^2)}{(A-B)^5} \\
&\quad + \frac{B^3e^{-At}(10A^2-5AB+B^2)}{(A-B)^5}.
\end{aligned} \tag{3.45}
$$

Applying equation (3.39) and equation (3.44), we can calculate $q_{15}(t)$ as follows:

$$
\begin{aligned}
q_{15}(t) &= [P(\sum_{i=1}^{5} H_i > t) - P(\sum_{i=1}^{4} H_i > t)] \prod_{i=1}^{4} P(W_i > 0) \\
&= [P(\sum_{i=1}^{4} H_i < t) - P(\sum_{i=1}^{5} H_i < t)] \prod_{i=1}^{4} P(W_i > 0) \\
&= (\frac{\lambda_{11}}{A})^2(\frac{\lambda_2}{B})^2\{\frac{A^2B^2}{2(A-B)^2}e^{-At}t^2 + \frac{A^3Be^{-Bt}t}{(A-B)^3} + \frac{AB^2e^{-At}t(3A-B)}{(A-B)^3} \\
&\quad + \frac{1}{(A-B)^4}(A^4e^{-Bt} - 4A^3Be^{-Bt} + 6A^2B^2e^{-At} - 4AB^3e^{-At} + B^4e^{-At}) \\
&\quad - \frac{A^2Bte^{-Bt} + AB^2te^{-At}}{(A-B)^2} \\
&\quad - \frac{A^3e^{-Bt} - 3A^2Be^{-Bt} + 3AB^2e^{-At} - B^3e^{-At}}{(A-B)^3}\}.
\end{aligned} \tag{3.46}
$$

Similarly, applying equation (3.44) and equation (3.45), the transition probability

$q_{16}(t)$ is calculated as follows:

$$
\begin{aligned}
q_{16}(t) &= [P(\sum_{i=1}^{6} H_i > t) - P(\sum_{i=1}^{5} H_i < t)] \prod_{i=1}^{5} P(W_i > 0) \\
&= [P(\sum_{i=1}^{5} H_i < t) - P(\sum_{i=1}^{6} H_i < t)] \\
&\quad \prod_{i=1}^{5} P(W_i > 0) \\
&= (\frac{\lambda_{11}}{A})^3 (\frac{\lambda_2}{B})^2 \{ \frac{A^2 B^2 t^2 (Be^{-At} - Ae^{-Bt})}{2(A-B)^3} + \frac{A^3 B e^{-Bt}(A-4B)t}{(A-B)^4} \\
&\quad + \frac{AB^3 e^{-At}(B-4A)t}{(A-B)^4} + \frac{A^3 e^{-Bt}(A^2 - 5AB + 10B^2)}{(A-B)^5} \\
&\quad - \frac{B^3 e^{-At}(10A^2 - 5AB + B^2)}{(A-B)^5} - \frac{A^2 B^2}{2(A-B)^2} e^{-At} t^2 \\
&\quad - \frac{A^3 B e^{-Bt} t}{(A-B)^3} - \frac{AB^2 e^{-At} t(3A-B)}{(A-B)^3} \\
&\quad - \frac{1}{(A-B)^4} (A^4 e^{-Bt} - 4A^3 B e^{-Bt} + 6A^2 B^2 e^{-At} \\
&\quad - 4AB^3 e^{-At} + B^4 e^{-At}) \}. \qquad (3.47)
\end{aligned}
$$

Similarly, the event that the individual would be healthy (in state 7) at the end of the policy on the condition that the insured person is healthy in the beginning of the policy, is the same as the fact that the insured smoothly goes through the state 1, 2, 3, 4, 5 and 6 without death and the total time that the individual stays in states 1, 2, 3, 4, 5 and 6 is less than time t. Then we have the transition probability $q_{17}(t)$ as follows:

$$
\begin{aligned}
q_{17}(t) &= P(\sum_{i=1}^{6} H_i < t)\prod_{i=1}^{6} P(W_i > 0) \\
&= (\frac{\lambda_{11}}{A})^3(\frac{\lambda_2}{B})^3[1 - \frac{A^2B^2t^2(Be^{-At} - Ae^{-Bt})}{2(A-B)^3} - \frac{A^3Be^{-Bt}(A-4B)t}{(A-B)^4} \\
&\quad -\frac{AB^3e^{-At}(B-4A)t}{(A-B)^4} - \frac{A^3e^{-Bt}(A^2-5AB+10B^2)}{(A-B)^5} \\
&\quad +\frac{B^3e^{-At}(10A^2-5AB+B^2)}{(A-B)^5}]; \qquad (3.48)
\end{aligned}
$$

and hence,

$$
q_{18}(t) = 1 - q_{11}(t) - q_{12}(t) - q_{13}(t) - q_{14}(t) - q_{15}(t) - q_{16}(t) - q_{17}(t) \quad (3.49)
$$

Similarly, we can see the sensitivities of $q_{15}(t)$, $q_{16}(t)$ and $q_{17}(t)$ to the parameters λ_{11} and λ_2 from the plots of Figures 3.15 - 3.18. As Figure 3.15 shows, the λ_{11} is set up as taking values 0.05, 0.1, 0.2 and 0.5, corresponding to the average waiting time of 20, 10, 5 and 2 weeks to become sick, respectively. Figure 3.15 shows that $q_{15}(t)$ is sensitive to λ_{11}. In general, the larger is the value of λ_{11}, i.e., the shorter waiting time to become sick, will tend to result in a lower chance to stay in state 5, which is a healthy state, although increasing λ_{11} increases $q_{15}(t)$ during the first short period because of the trend to transferring to state 5 increases. Figure 3.16 shows that $q_{15}(t)$ is sensitive to λ_2 as well, where we have taken values of λ_2 as 0.0833, 0.125, 0.25 and 0.5, corresponding to an average recovery time of 12, 8, 4 and 2 weeks, respectively. In general, the larger is the value of λ_2, i.e., the shorter waiting time to recover, will result in higher chance to stay in state 5, which is a healthy state, although in the late period, higher value in λ_2 results in lower transition probability $q_{15}(t)$ due to the transition to next stages. Figure 3.17 shows the sensitivity of $q_{16}(t)$ to λ_{11}, where λ_{11} takes values 0.05, 0.1, 0.2 and 0.5 corresponding to a waiting time to become sick the first time of 40, 30, 20 and 10 weeks, respectively. In general, the larger value in λ_{11}, i.e., the shorter waiting time to become sick, will result in a higher chance to stay in state 6, which is a disabled state, although as time passes, the transition probability $q_{16}(t)$ decreases with larger value λ_{11} due to the higher chance to transition to next stages. Figure 3.18 shows the sensitivity of $q_{16}(t)$ to λ_2, where λ_2 takes values 0.0833, 0.125, 0.25, and 0.5 corresponding the average waiting time until getting recovered 12, 8, 4 and 2 weeks. In general, the larger value of λ_2, i.e., the less waiting time to recover, will result in a lower chance to stay in state 6, which is a disabled state, although in the first short period, the transition probability $q_{16}(t)$ increases as λ_2 due to the increasing trend to transferring to the state 6.

Figures 3.19 and 3.20 show the sensitivity of $q_{17}(t)$ to λ_{11} as well as λ_2. Here, λ_{11} takes values 0.05, 0.1, 0.2 and 0.5 corresponding to an average waiting time to become ill the first time 20, 10, 5 and 2 weeks, respectively; and, λ_2 takes values 0.0833, 0.125, 0.25 and 0.5 corresponding to an average recovery time of 12, 8, 4 and 2 weeks, respectively. Since it is assumed that the individual will stay healthy when the policy ends, the transition probability $q_{15}(t)$ increases as time passes. As either λ_{11} or λ_2 increases, i.e., the shorter the time that an individual stays in the interval states, the higher the chance will be in the last state given time t.

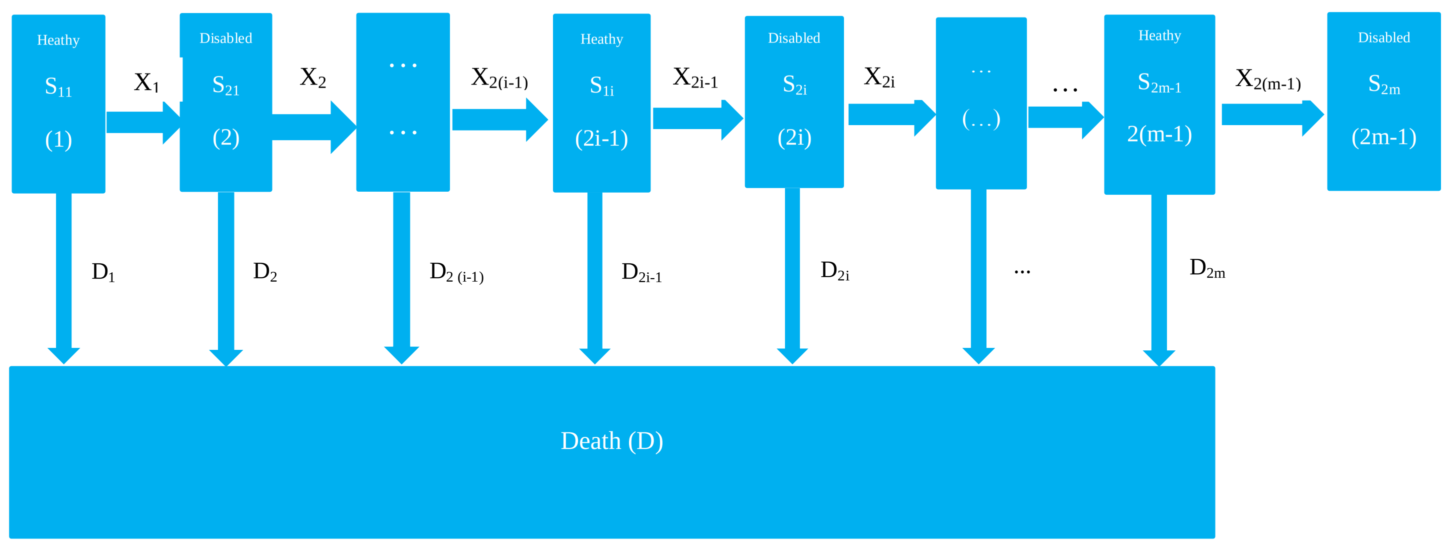

Figure 3.2: Unfolded state transition diagram for a disabled model

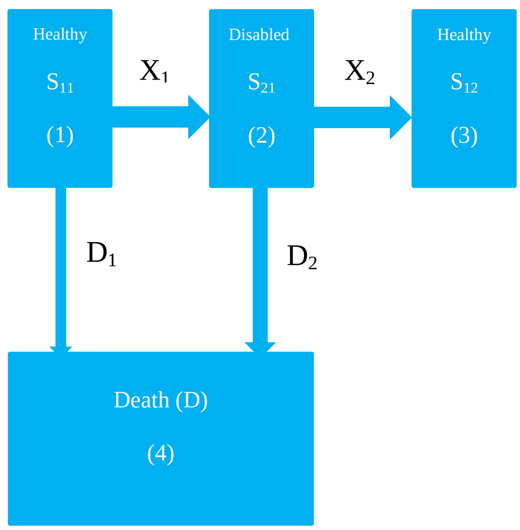

Figure 3.3: State transition diagram for a disabled model in which $m = 1$ ended with health or death.

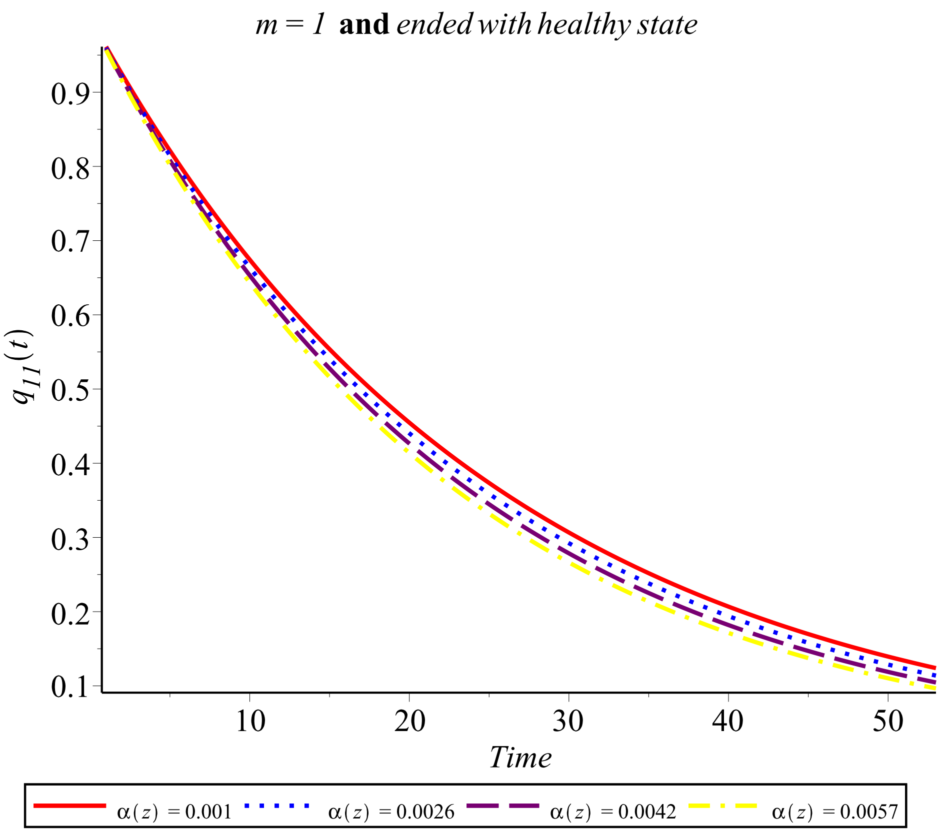

Figure 3.4: $P(S(t) = 1|S(0) = 1)$ for $\alpha(z) = (0.0001, 0.0026, 0.0042, 0.0057)$ in which m = 1 and ended with healthy state.

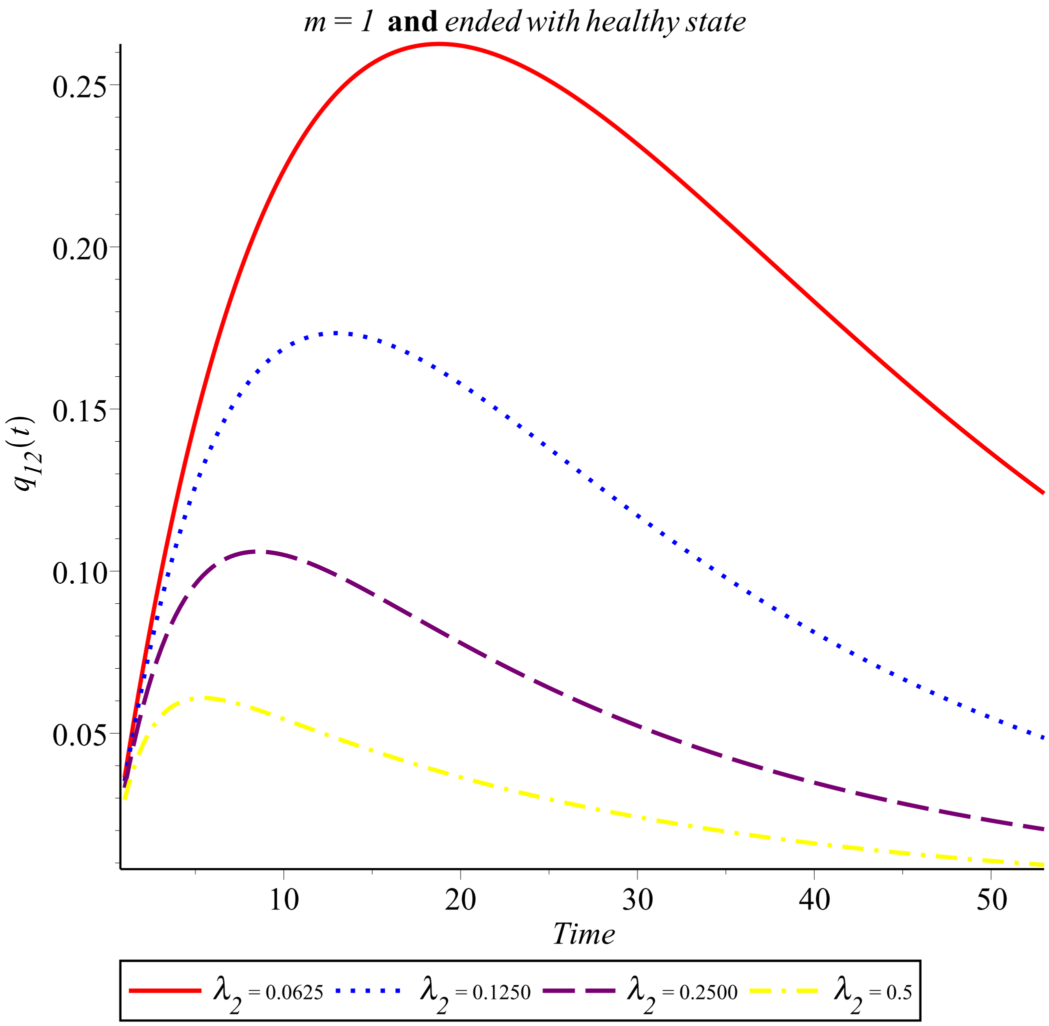

Figure 3.5: $P(S(t) = 2|S(0) = 1)$ for $\lambda_2 = (0.0625, 0.125, 0.25, 0.5)$ in which $m = 1$ and ended with healthy state.

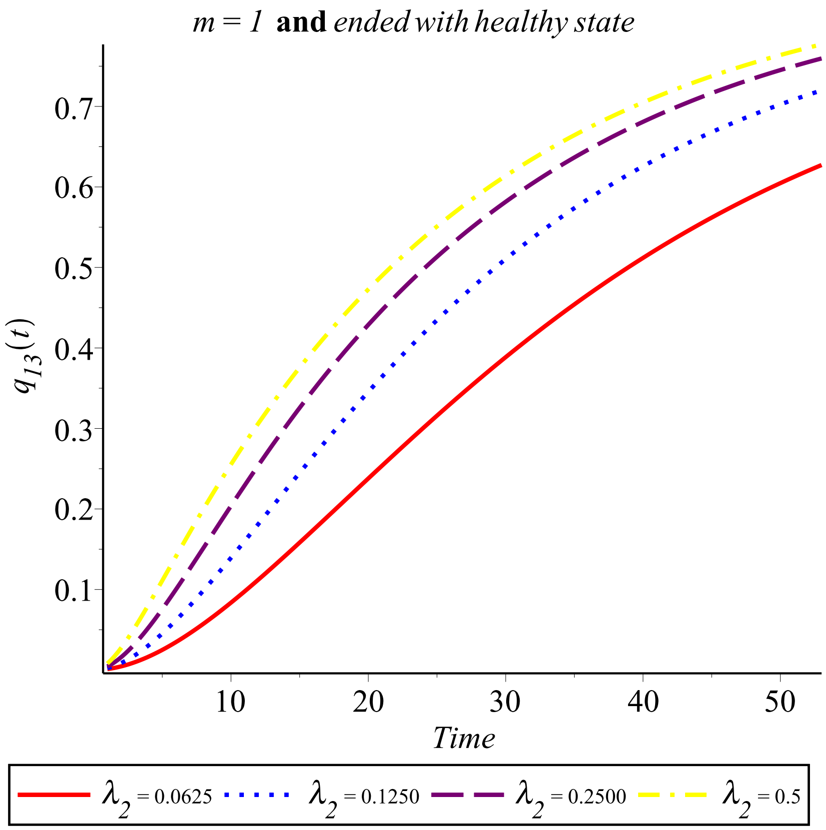

Figure 3.6: $P(S(t) = 3|S(0) = 1)$ for $\lambda_2 = (0.0625, 0.125, 0.25, 0.5)$ in which *m=1* and ended with healthy state.

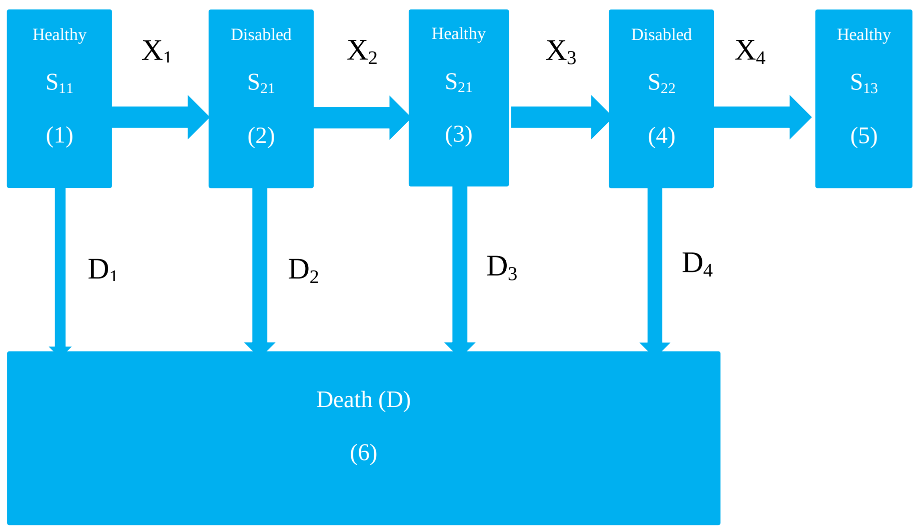

Figure 3.7: State transition diagram for a disabled model in which $m = 2$.

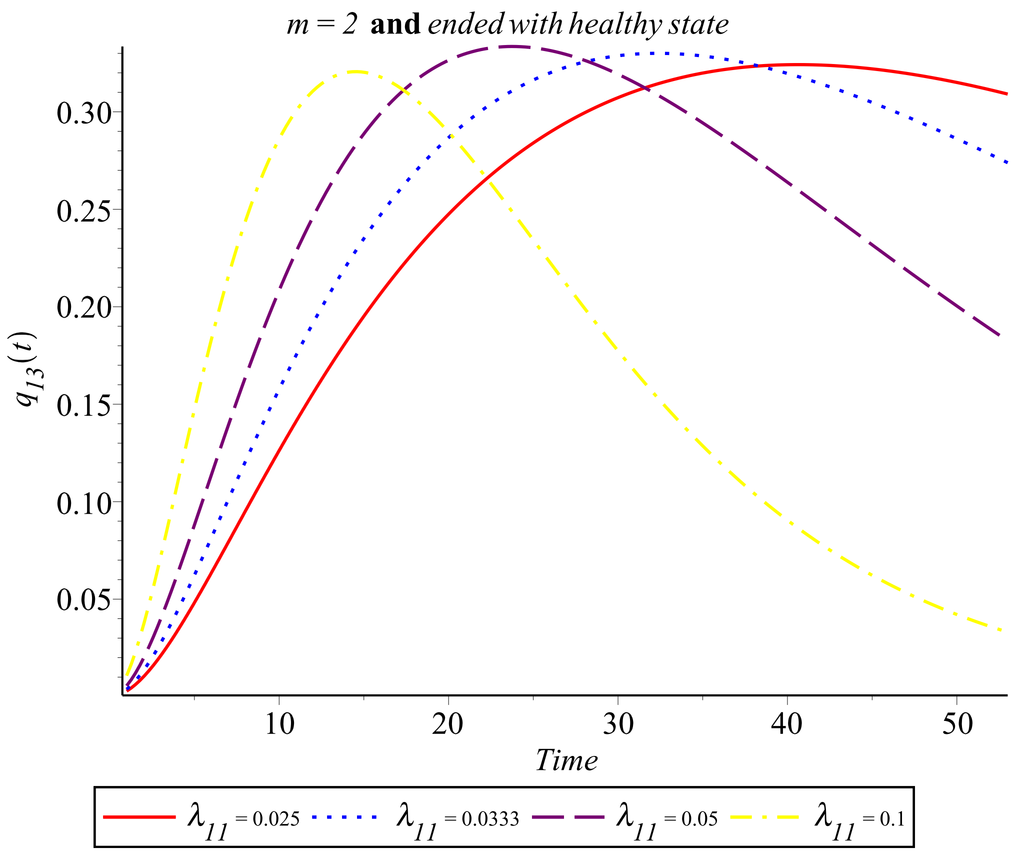

Figure 3.8: $P(S(t) = 3|S(0) = 1)$ for $\lambda_{11} = (0.025, 0.033, 0.05, 0.10)$ in which $m = 2$ and ended with healthy state.

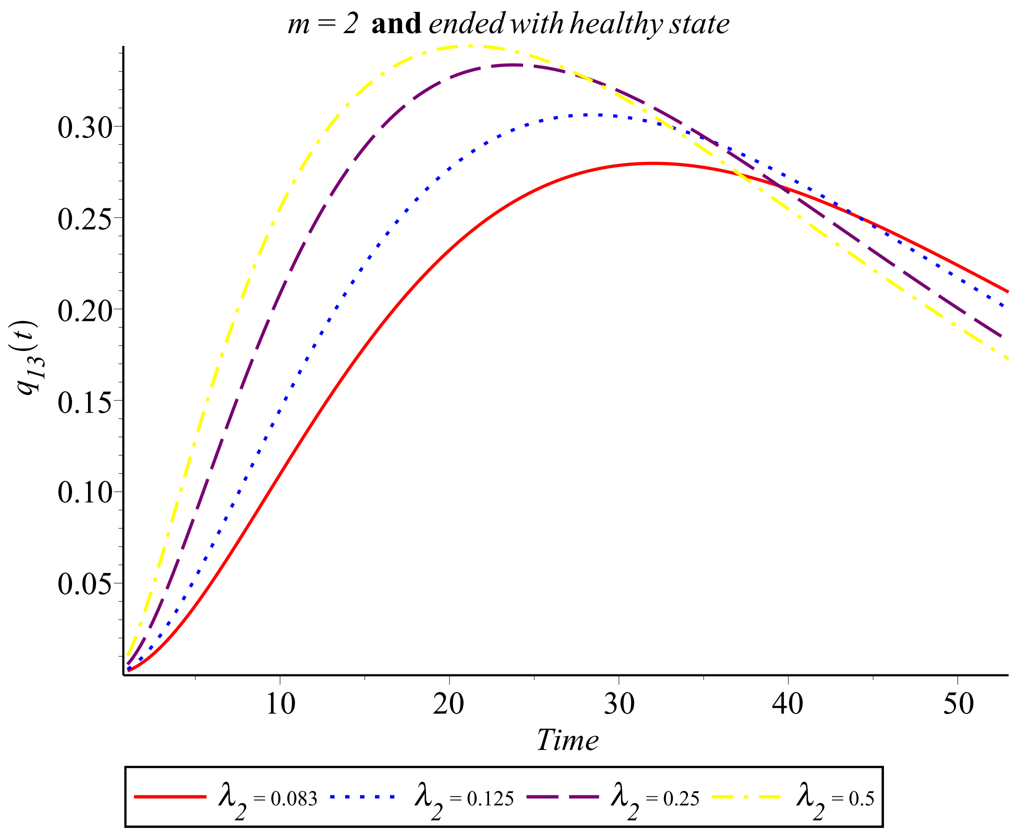

Figure 3.9: $P(S(t) = 3|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 2$ and ended with healthy state.

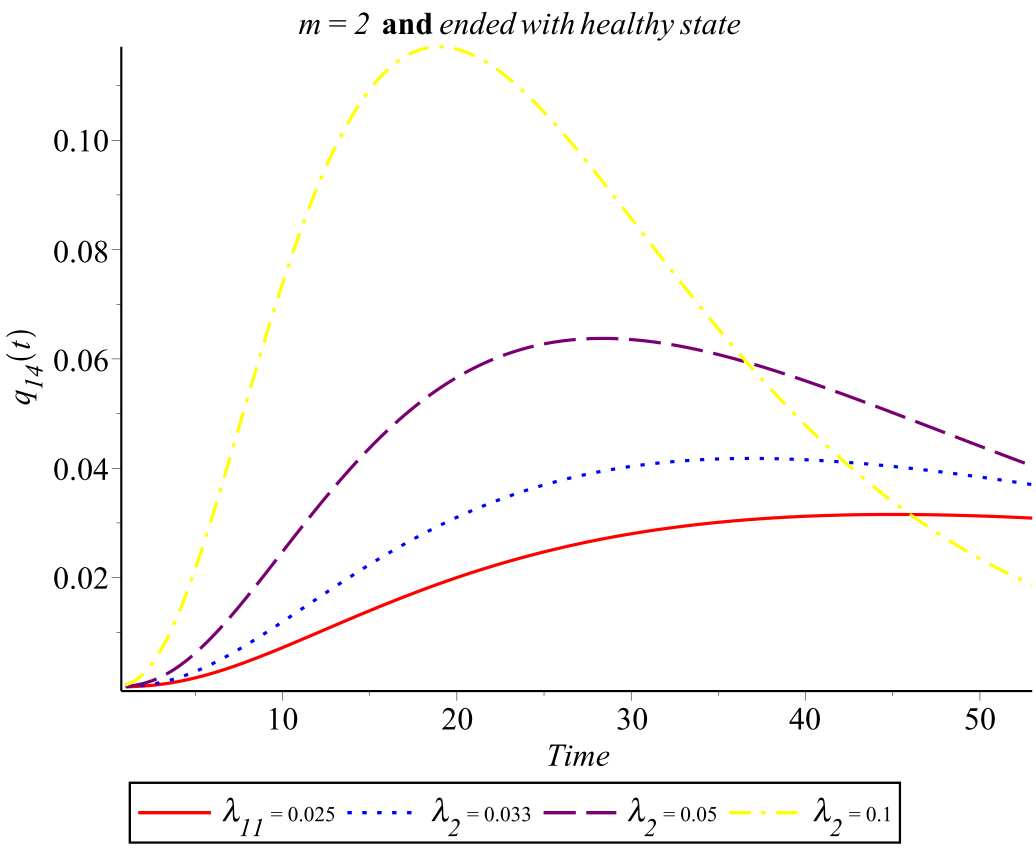

Figure 3.10: $P(S(t) = 4|S(0) = 1)$ for $\lambda_{11} = (0.025, 0.033, 0.05, 0.1)$ in which $m = 2$ and ended with healthy state.

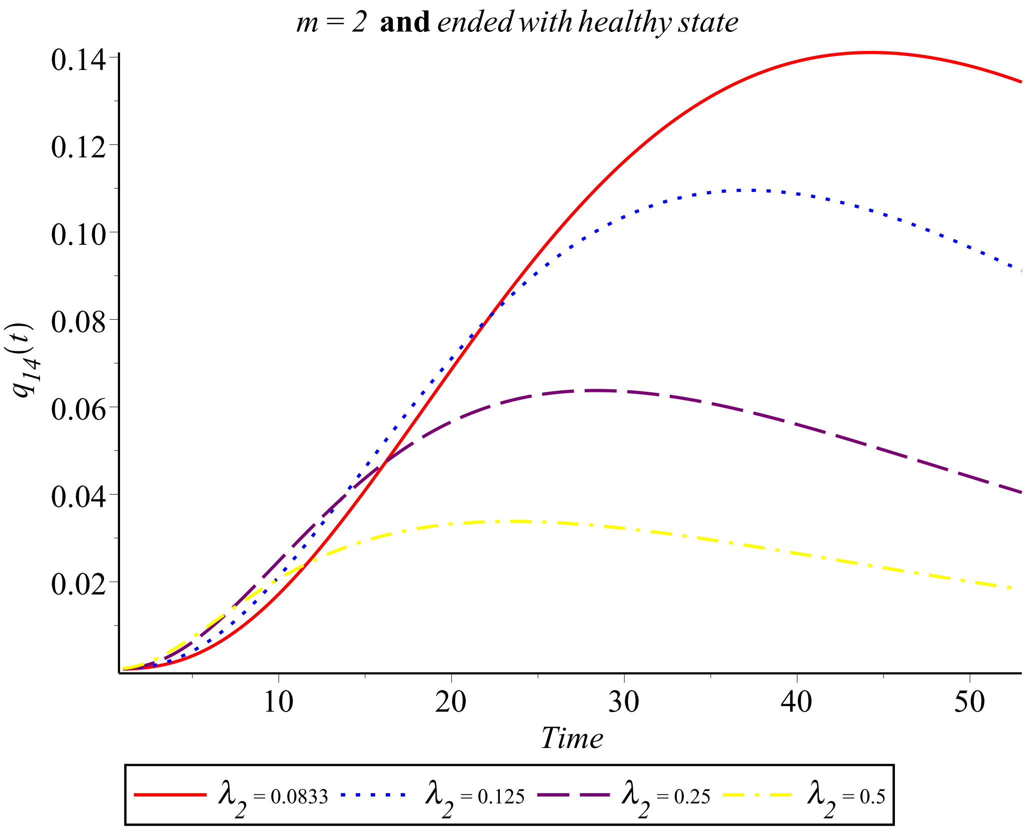

Figure 3.11: $P(S(t) = 4|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 2$ and ended with healthy state.

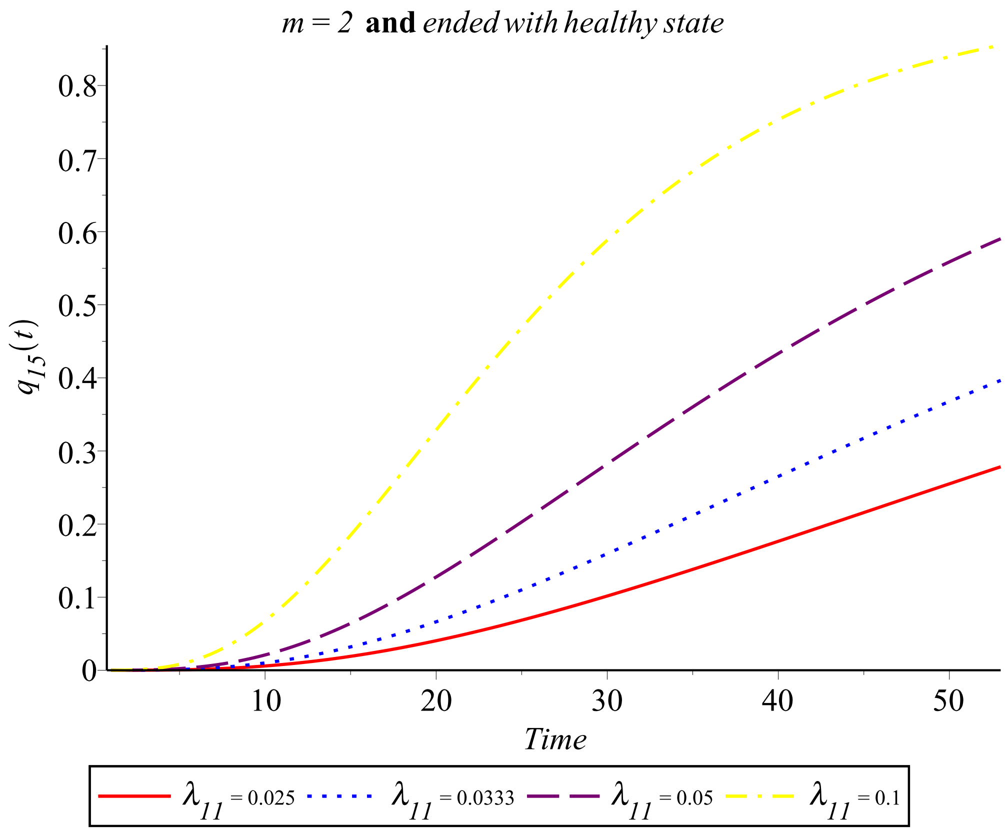

Figure 3.12: $P(S(t) = 5|S(0) = 1)$ for $\lambda_{11} = (0.025, 0.0333, 0.05, 0.1)$ in which $m = 2$ and ended with healthy state.

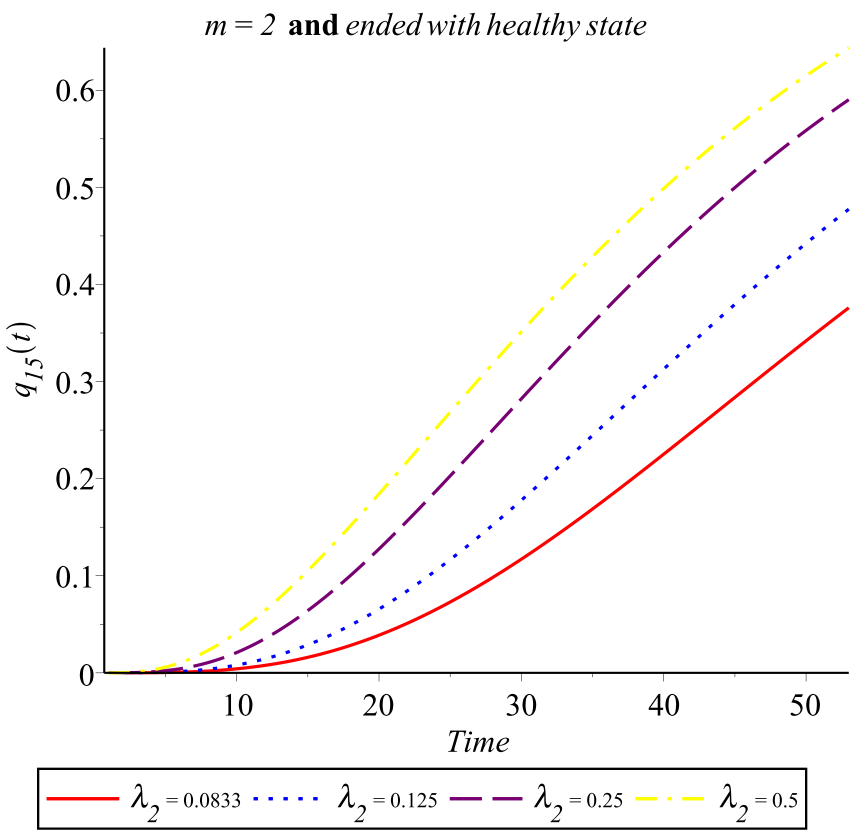

Figure 3.13: $P(S(t) = 5|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 2$ and ended with healthy state.

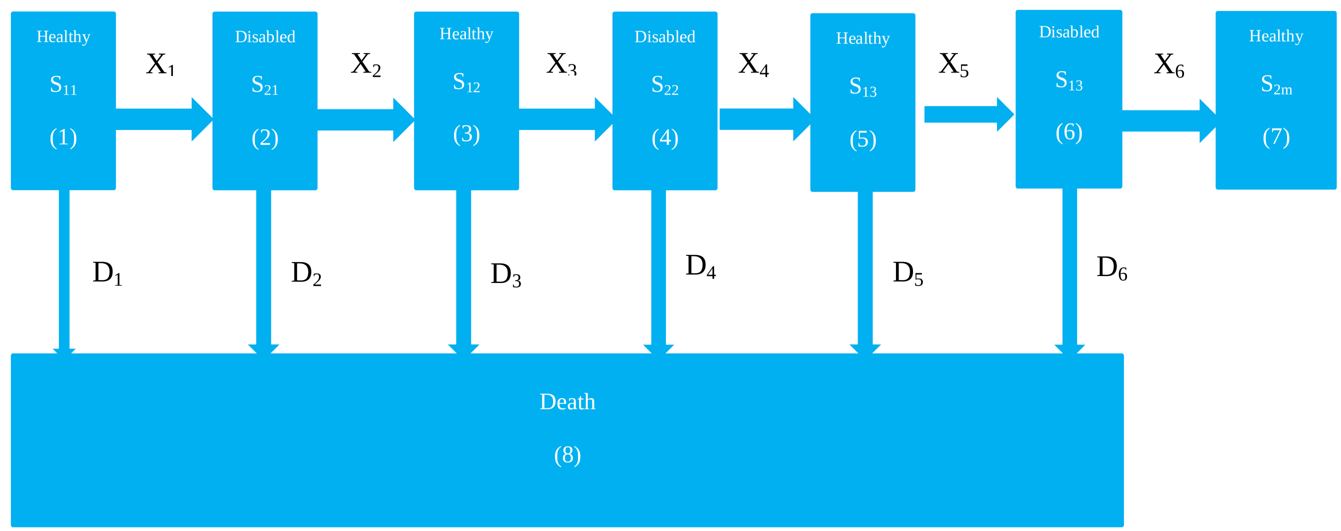

Figure 3.14: Unfolded state transition diagram for a disabled model

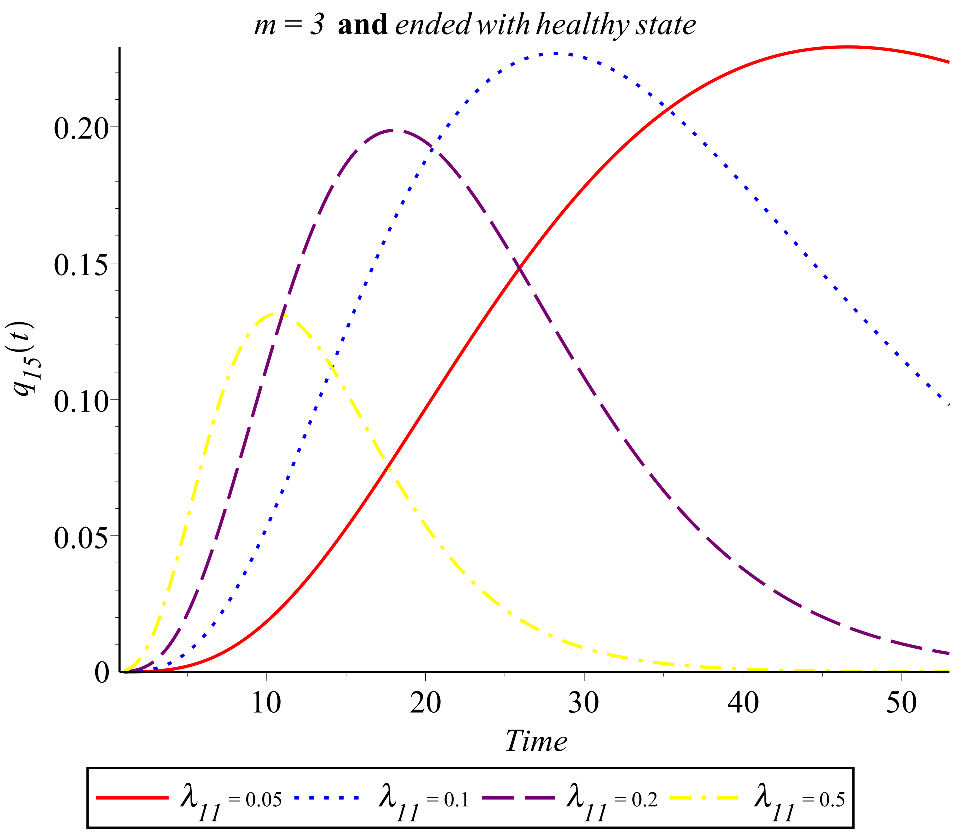

Figure 3.15: $P(S(t) = 5|S(0) = 1)$ for $\lambda_{11} = (0.05, 0.1, 0.2, 0.5)$ in which $m = 3$ and ended with healthy state.

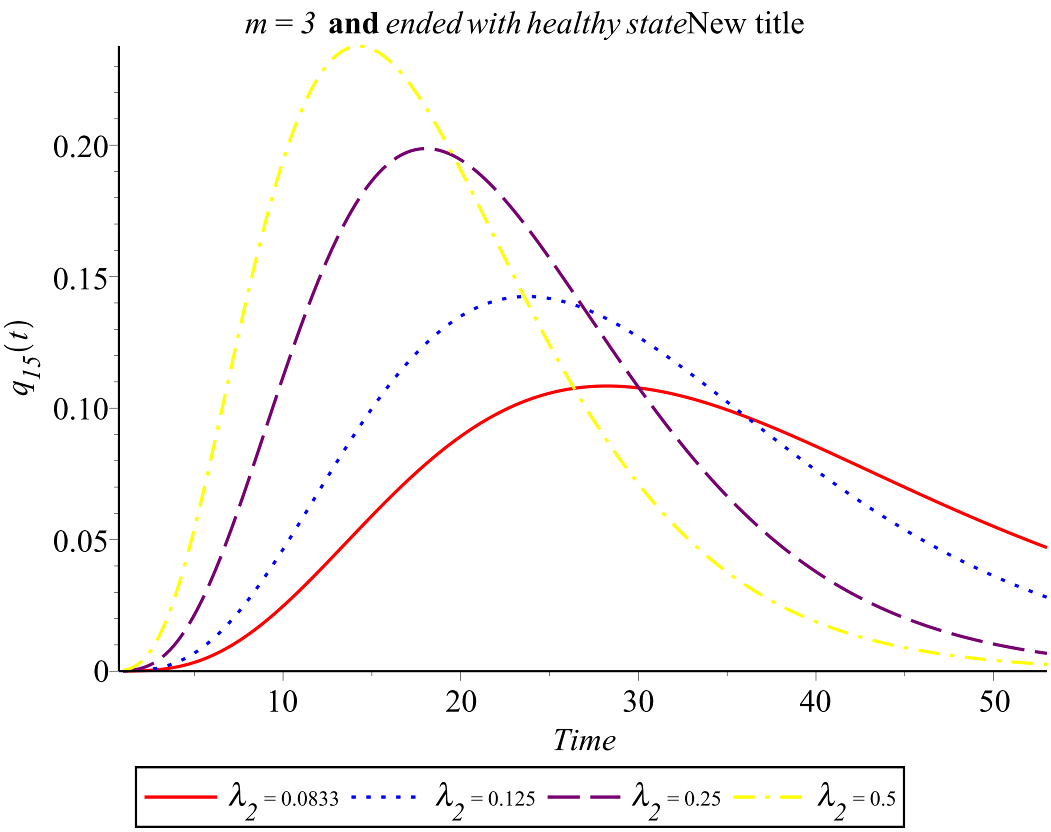

Figure 3.16: $P(S(t) = 5|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which m = 3 and ended with healthy state.

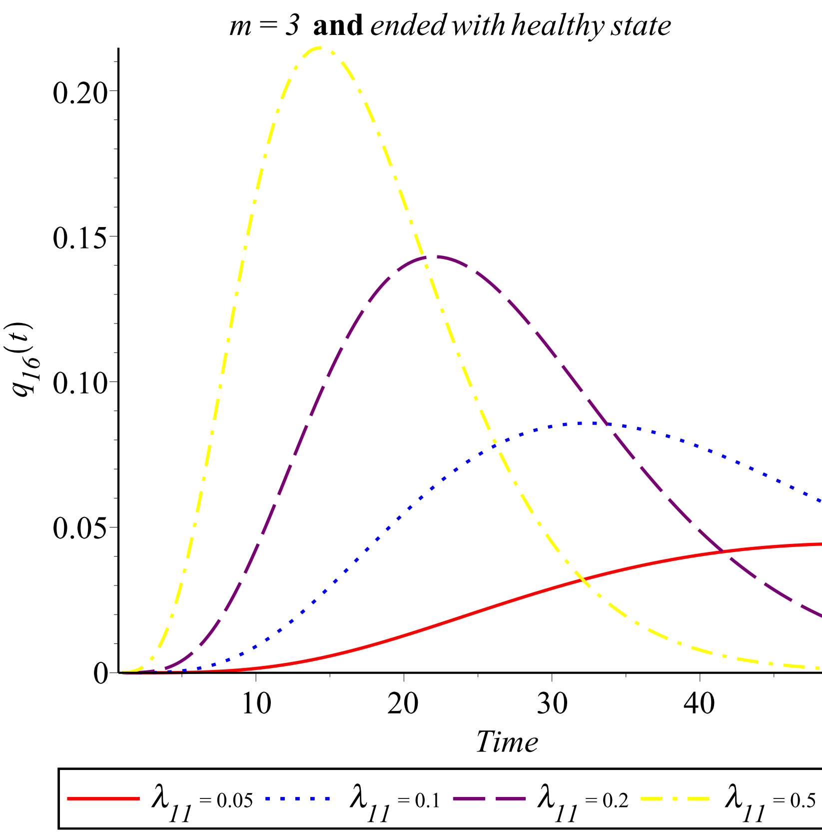

Figure 3.17: $P(S(t) = 6|S(0) = 1)$ for $\lambda_{11} = (0.05, 0.1, 0.2, 0.5)$ in which $m = 3$ and ended with healthy state.

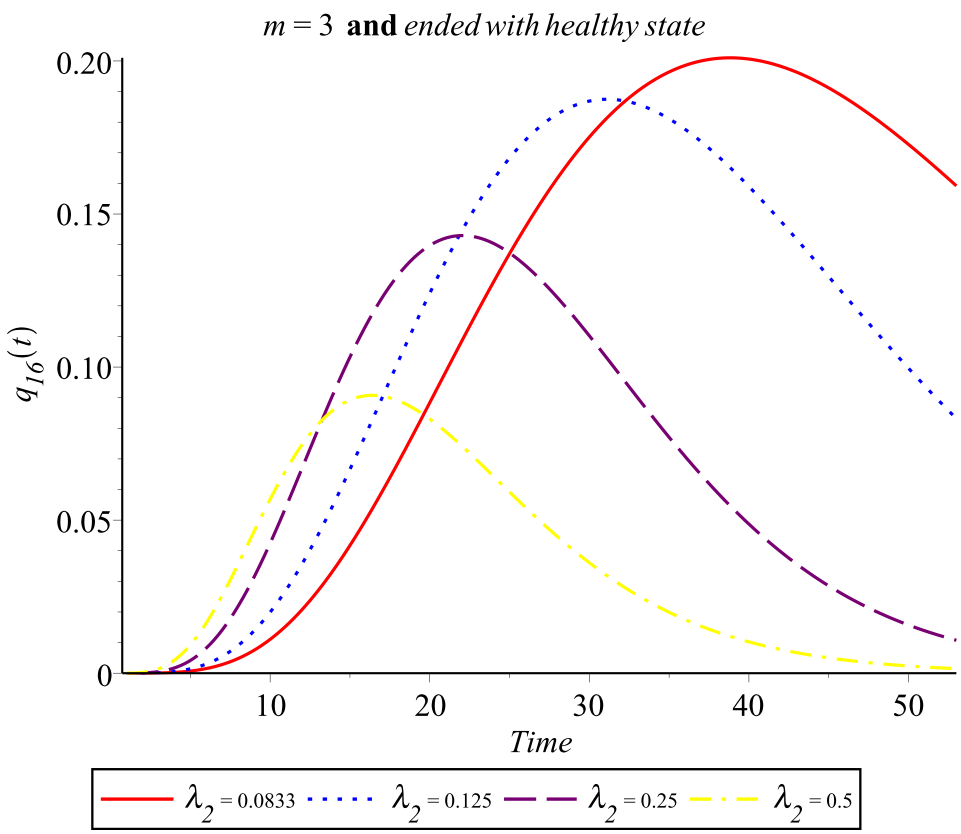

Figure 3.18: $P(S(t) = 6|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 3$ and ended with healthy state.

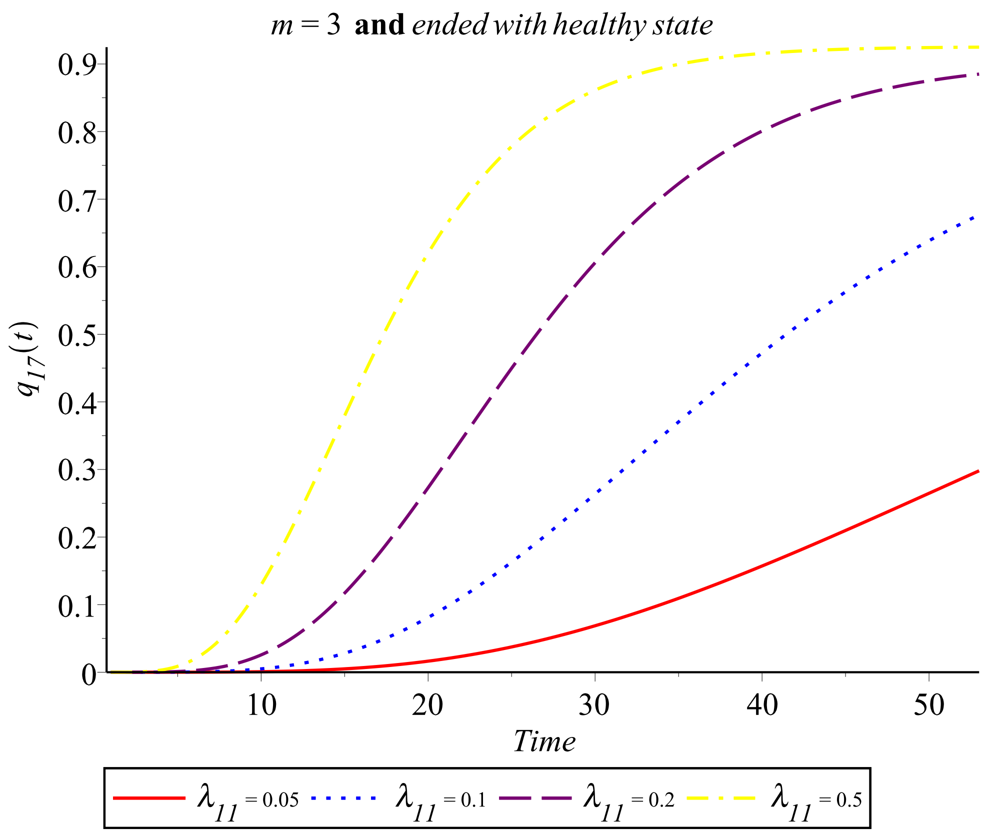

Figure 3.19: $P(S(t) = 7|S(0) = 1)$ for $\lambda_{11} = (0.05, 0.1, 0.2, 0.5)$ in which $m = 3$ and ended with healthy state.

Table 3.1 Summarizes the sensitivity of the transition probabilities to the parameter values.

Figure Number	Parameter	m	Transition Probability	Parameter vector	Description
3.4	$\alpha(z)$	1	$q_{11}(t)$	λ_{11}= 0.0384 $\alpha(z)$= (0.001, 0.0026, 0.0042, 0.0057) λ_2= NA α_2= NA	Insensitive
3.5	λ_2	1	$q_{12}(t)$	λ_{11}= 0.0384 $\alpha(z)$= 0.0026 λ_2= (0.0625, 0.125, 0.250, 0.5) α_2= 0.0052	Sensitive

3.6	λ_2	1	$q_{13}(t)$	λ_{11}= 0.0384 $\alpha(z)$= 0.0026 λ_2= (0.0625, 0.125, 0.250, 0.5) α_2= 0.0052	Sensitive
3.8	λ_{11}	2	$q_{13}(t)$	λ_{11}= (0.025, 0.0333, 0.50, 0.10) $\alpha(z)$=0.0026 λ_2= 0.25 α_2= 0.0052	Sensitive
3.9	λ_2	2	$q_{13}(t)$	λ_{11}= 0.05 $\alpha(z)$= 0.0026 λ_2= (0.0833, 0.125, 0.25, 0.5) α_2= 0.0052	Sensitive
3.10	λ_{11}	2	$q_{14}(t)$	λ_{11}= (0.025, 0.033, 0.05, 0.1) $\alpha(z)$= 0.0026 λ_2= 0.25 α_2=0.0052	Sensitive
3.11	λ_2	2	$q_{14}(t)$	λ_{11}= 0.05 $\alpha(z)$= 0.0026 λ_2=(0.0833, 0.125, 0.25, 0.5) α_2= 0.0052	Sensitive
3.12	λ_{11}	2	$q_{15}(t)$	λ_{11}= (0.025, 0.0333,0.05, 0.10) $\alpha(z)$= 0.0026 λ_2= 0.25 α_2= 0.0052	Sensitive
3.13	λ_2	2	$q_{15}(t)$	λ_{11}= 0.05 $\alpha(z)$= 0.0026 λ_2= (0.0833, 0.125, 0.25, 0.5) α_2= 0.0052	Sensitive
3.15	λ_{11}	3	$q_{15}(t)$	λ_{11}= (0.05, 0.10, 0.20, 0.50) $\alpha(z)$=0.0026 λ_2= 0.25 α_2= 0.0052	Sensitive
3.16	λ_2	3	$q_{15}(t)$	λ_{11}= 0.2 $\alpha(z)$= 0.0026 λ_2= (0.0833, 0.125, 0.25, 0.5) α_2= 0.0052	Sensitive
3.17	λ_{11}	3	$q_{16}(t)$	λ_{11}= (0.05, 0.1, 0.2, 0.5) $\alpha(z)$= 0.0026 λ_2= 0.25 α_2= 0.0052	Sensitive
3.18	λ_2	3	$q_{16}(t)$	λ_{11}= 0.2 $\alpha(z)$= 0.0026 λ_2= (0.00833, 0.125, 0.25, 0.50)	Sensitive

				α_2= 0.0052	
3.19	λ_{11}	3	$q_{17}(t)$	λ_{11}= (0.05, 0.1, 0.2, 0.5) $\alpha(z)$= 0.0026 λ_2=0.25 α_2= 0.0052	Sensitive
3.20	λ_2	3	$q_{17}(t)$	λ_{11}= 0.2 $\alpha(z)$= 0.0026 λ_2= (0.0833, 0.125, 0.25, 0.5) α_2= 0.0052	Sensitive

Table 3.1: Sensitivity of transition probabilities to parameter values

All of the above discussion on the transition probabilities $q_{ij}(t)$ is based on the assumption that at the end of the policy, the insured will be healthy or dead. Next, we assume that in the end of a policy, the insured will be either disabled or dead. The relevant unfolded state transition diagram is described in Figure 3.21. When $m = 1$, the transition diagram is shown in Figure 3.22.

Similarly, $q_{11}(t)$, $q_{12}(t)$, and $q_{13}(t)$ are derived as follows. First, $q_{11}(t)$ is the same as in equation (3.33). Secondly, based on Figure 3.22, the event that an insured is in state 2 given he/she is in the state 1 is the same as the fact that he/she smoothly transfers to state 2 and the time he/she spends in state 1 is less than time t. Then we have the following equation:

$$\begin{aligned} q_{12}(t) &= P(S(t) = 2|S(0) = 1) \\ &= P(W_1 > 0)P(H_1 < t|W_1 > 0) \\ &= P(W_1 > 0)P(H_1 < t) \\ &= \frac{\lambda_{11}}{\lambda_{11} + \alpha(z)}[1 - e^{-(\lambda_{11}+\alpha(z))}]; \end{aligned} \tag{3.50}$$

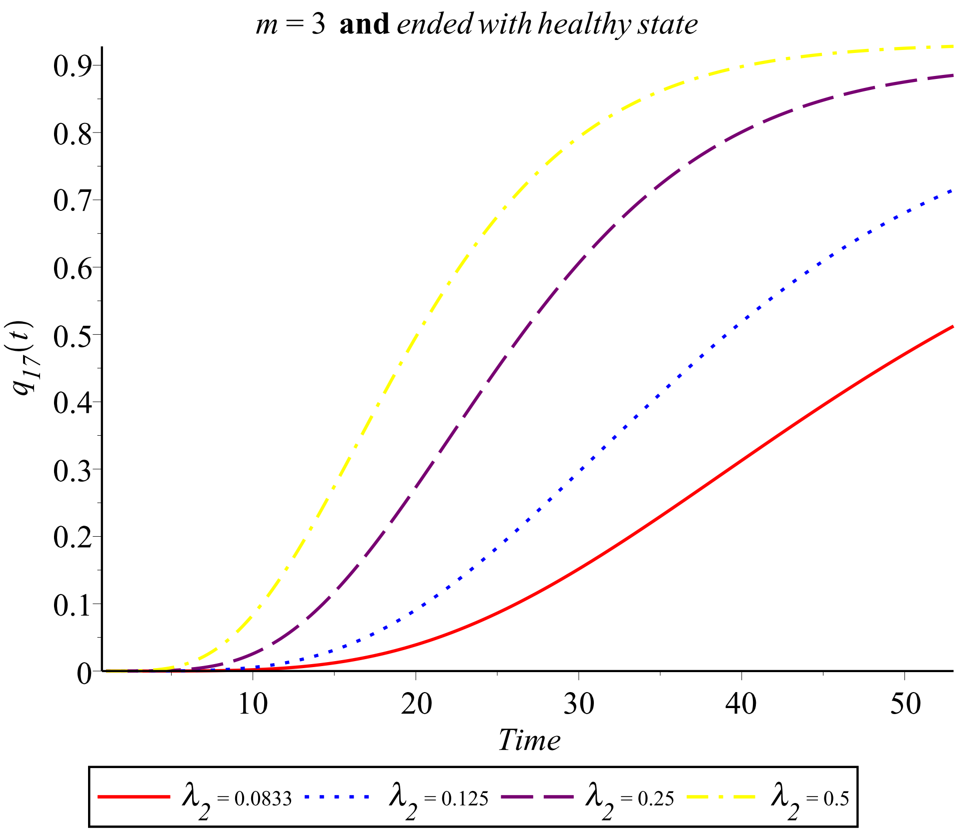

Figure 3.20: $P(S(t) = 7|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which m = *3* and ended with healthy state.

and

$$\begin{aligned} q_{13}(t) &= P(S(t) = |S(0) = 1) \\ &= 1 - q_{11}(t) - q_{12}(t). \end{aligned} \tag{3.51}$$

The sensitivity of $q_{12}(t)$ to parameter $\alpha(z)$ is shown in Figure 3.25. We fix the average disease rate at 0.0384 corresponding to the average waiting time for an disease occurrence as half a year; and we take the death rates of D_1, i.e., $\alpha(z)$, as 0.001, 0.0026, 0.0042 and 0.0057 corresponding to the average death rate for 20, 30, 40 and 50 years old persons. The plots in Figure 3.25 show that $q_{12}(t)$ is not sensitive to $\alpha(z)$. This does make sense since the death rate $\alpha(z)$ is relatively small compared to the disease rate λ_{11}.

When $m = 2$, the transition diagram is shown in Figure 3.23. The transition probabilities $q_{11}(t)$, $q_{12}(t)$, $q_{13}(t)$, $q_{14}(t)$, and $q_{15}(t)$ are derived as follows. First, $q_{11}(t)$ is the same as equation (3.33). Second, $q_{12}(t)$ and $q_{13}(t)$ are the same as in equation (3.34) and equation (3.40). Secondly, based on Figure 3.23, the event that an insured individual is in state 4 given he/she is in the state 1 is the same as the fact that he/she smoothly transfers to state 4 and the time he/she spends in state 1, 2, and 3 is less than time t. Then, we have the following equation:

$$\begin{aligned} q_{14}(t) &= P(S(t) = |S(0) = 1) \\ &= \prod_{i=1}^{3} P(W_i > 0)[P(\sum_{i=1}^{3} H_i < t | W_1 > 0, W_2 > 0, W_3 > 0)] \\ &= \prod_{i=1}^{3} P(W_i > 0)[P(\sum_{i=1}^{3} H_i < t)]. \end{aligned} \tag{3.52}$$

Applying equations (3.17), (3.18), (3.19)and (3.38), we have that

$$\begin{aligned} q_{14}(t) &= \prod_{i=1}^{3} P(W_i > 0)[P(\sum_{i=1}^{3} H_i < t)] \\ &= (\frac{\lambda_{11}}{A})^2(\frac{\lambda_2}{B})(1 - \frac{ABte^{-At}}{B-A} - \frac{A^2e^{-Bt} - 2ABe^{-At} + B^2e^{-At}}{(B-A)^2}); \quad (3.53) \end{aligned}$$

and

$$q_{15}(t) = 1 - q_{11}(t) - q_{12}(t) - q_{13}(t) - q_{14}(t). \quad (3.54)$$

The sensitivities of $q_{14}(t)$ to parameters λ_{11} and λ_2 are plotted in Figures 3.26 and 3.27. In Figure 3.26, λ_{11} takes values 0.025, 0.033, 0.05 and 0.1, corresponding to average waiting time to become sick 40, 30, 20 and 10 weeks, respectively; and in Figure 3.27, λ_2 takes values 0.0833, 0.125, 0.25 and 0.5, which correspond to an average recovery time of 12, 8, 4 and 2 weeks, respectively. Figures 3.26 and 3.27 both show that as time passes by, the chance to transition to state 4 will keep increasing. On the other hand, it is assumed that state 4 is one of the states that the insured will stay in at the end of the policy; so, increasing either λ_{11} or λ_2, i.e., decreasing the waiting time on either a healthy or a disabled state will increase the transition probability $q_{14}(t)$.

When $m = 3$, the transition diagram is shown in Figure 3.24, and the transition probabilities $q_{11}(t)$, $q_{12}(t)$, $q_{13}(t)$, $q_{14}(t)$, $q_{15}(t)$, $q_{16}(t)$ and $q_{17}(t)$ are derived as follows. First, $q_{11}(t)$, $q_{12}(t)$, $q_{13}(t)$, $q_{14}(t)$, and $q_{15}(t)$ are the same as in equations (3.33), (3.34), (3.40), (3.41) and (3.46), respectively. For $q_{16}(t)$, based on the Figure 3.24, we have the event that an insured is in state 6 given he/she is in the state 1 is the same as the fact that he/she smoothly transfers to state 6 and the total time he/she spends in state 1, 2, 3, 4, 5 and 6 is less than time t. Then, we have the following equation:

$$
\begin{aligned}
q_{16}(t) &= \prod_{i=1}^{5} P(W_i > 0)[P(\sum_{i=1}^{5} H_i < t | W_1 > 0, W_2 > 0, ..., W_5 > 0)] \\
&= \prod_{i=1}^{5} P(W_i > 0)[P(\sum_{i=1}^{5} H_i < t]. \qquad (3.55)
\end{aligned}
$$

Applying equations (3.17), (3.18), (3.19) and (3.44) tp equation (3.55), we have that

$$
\begin{aligned}
q_{16}(t) &= \prod_{i=1}^{5} P(W_i > 0)[P(\sum_{i=1}^{5} H_i < t] \\
&= (\frac{\lambda_{11}}{A})^3(\frac{\lambda_2}{B})^2(1 - \frac{A^2B^2}{2(A-B)^2}e^{-At}t^2 - \frac{A^3Be^{-Bt}t}{(A-B)^3} - \frac{AB^2e^{-At}t(3A-B)}{(A-B)^3} \\
&\quad - \frac{1}{(A-B)^4}(A^4e^{-Bt} - 4A^3Be^{-Bt} + 6A^2B^2e^{-At} - 4AB^3e^{-At} \\
&\quad + B^4e^{-At})). \qquad (3.56)
\end{aligned}
$$

The sensitivities of $q_{16}(t)$ to parameters λ_{11} and λ_2 are plotted in Figures 3.28 and 3.29. Thus, in Figure 3.28, λ_{11}, takes values 0.05, 0.1, 0.2 and 0.5 corresponding to an average waiting time until the first time of being ill of 20, 10, 5 and 2 weeks, respectively. In Figure 3.29, λ_2 takes values 0.0833, 0.125, 0.25, and 0.5 corresponding to an average recovering time 12, 8, 4 and 2 weeks, repectively. Figures 3.28 and 3.29 both show that as time passes by, the chance to transition to state 6 will keep increasing. On the other hand, it is assumed that state 6 is one of the states that the insured will stay in at the end of the policy; so, increasing either λ_{11} or λ_2, i.e., decreasing the waiting time on either a healthy or a disabled state will increase the transition probability $q_{16}(t)$.

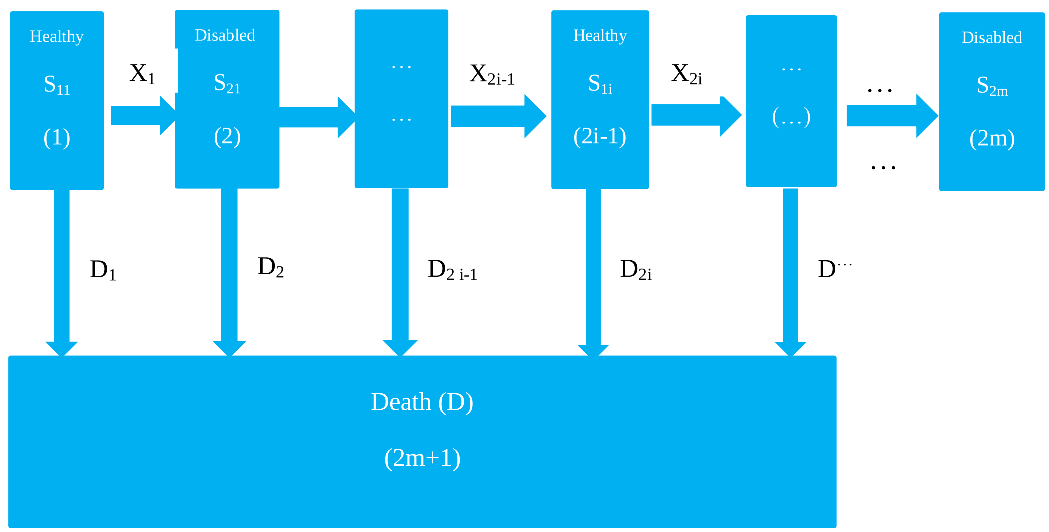

Figure 3.21: Unfolded state transition diagram for a disabled model ended either disabled or dead.

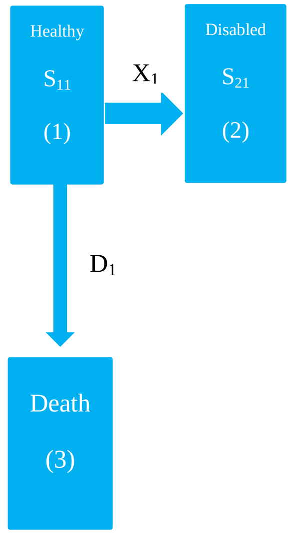

Figure 3.22: Unfolded state transition diagram for a disabled model ended either disabled or dead in which *m=1*

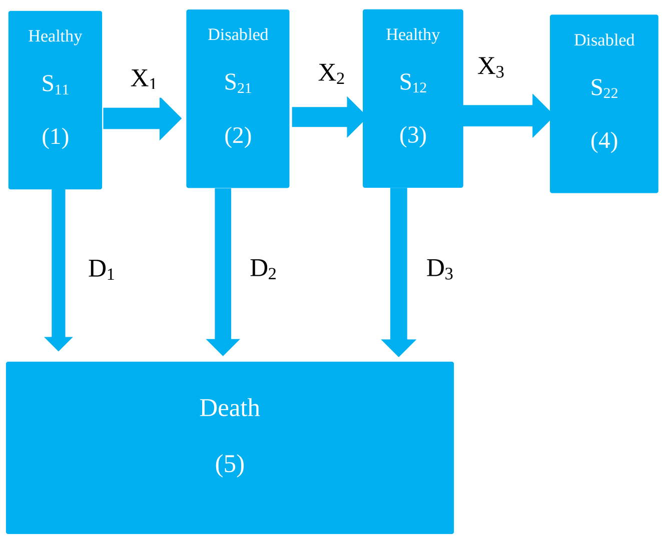

Figure 3.23: Unfolded state transition diagram for a disabled model ended either disabled or dead in which *m=2*

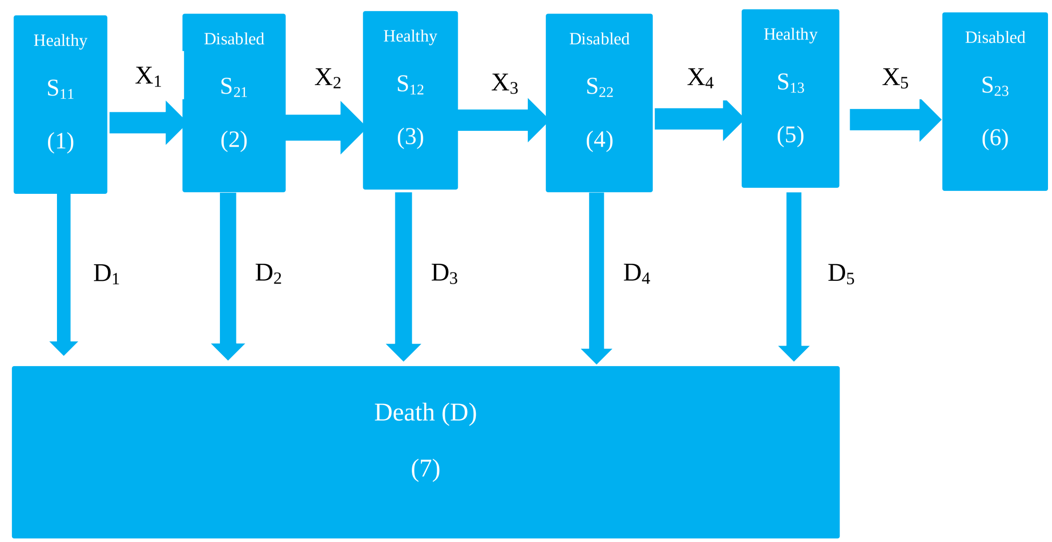

Figure 3.24: Unfolded state transition diagram for a disabled model ended either disabled or dead in which m=3

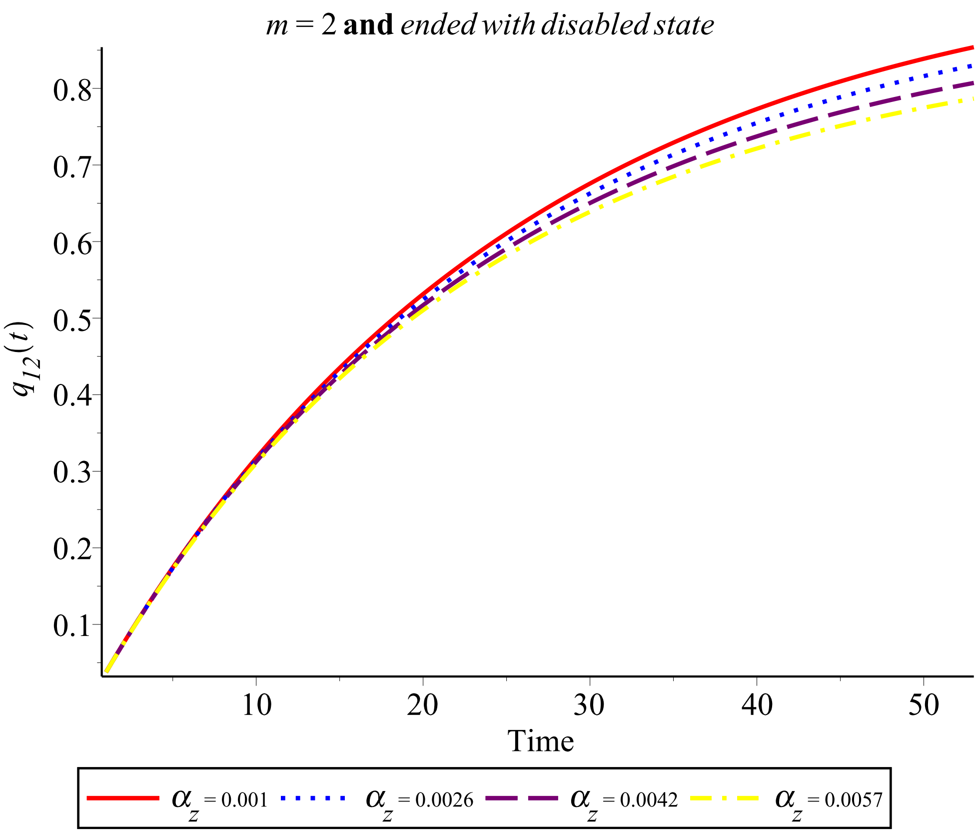

Figure 3.25: $P(S(t) = 2|S(0) = 1)$ for $\alpha(z) = (0.0001, 0.0026, 0.0042, 0.0057)$ in which m = *1* and ended with disabled or death.

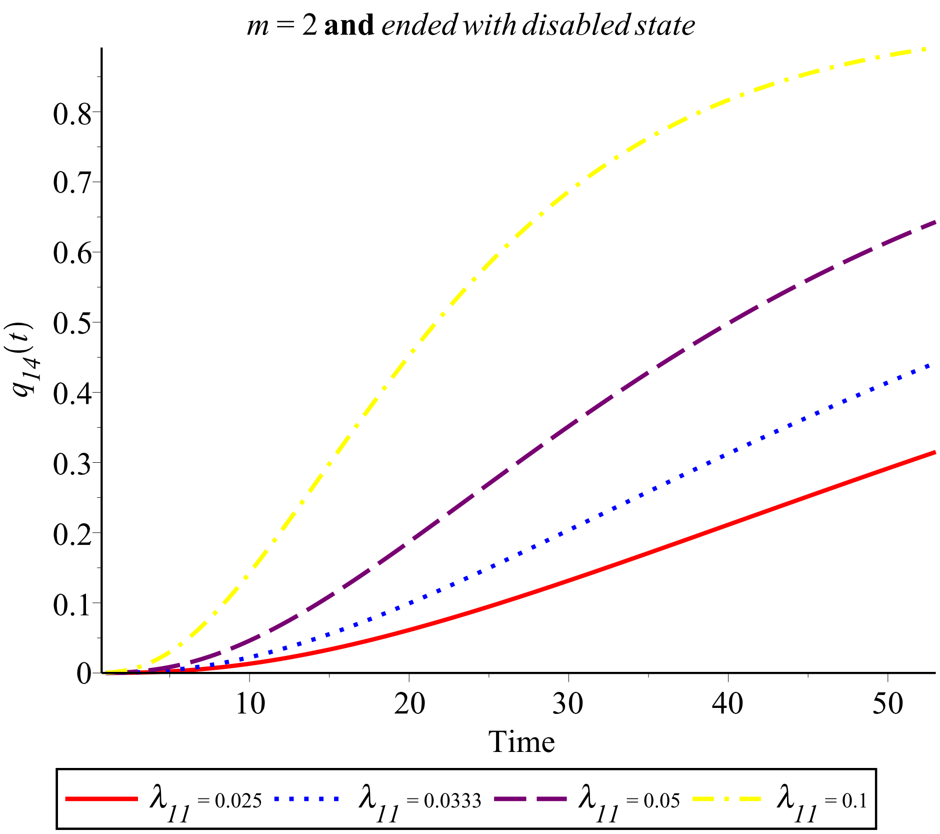

Figure 3.26: $P(S(t) = 4|S(0) = 1)$ for $\lambda_{11} = (0.025, 0.033, 0.05, 0.1)$ in which $m = 2$ and ended with disabled or death.

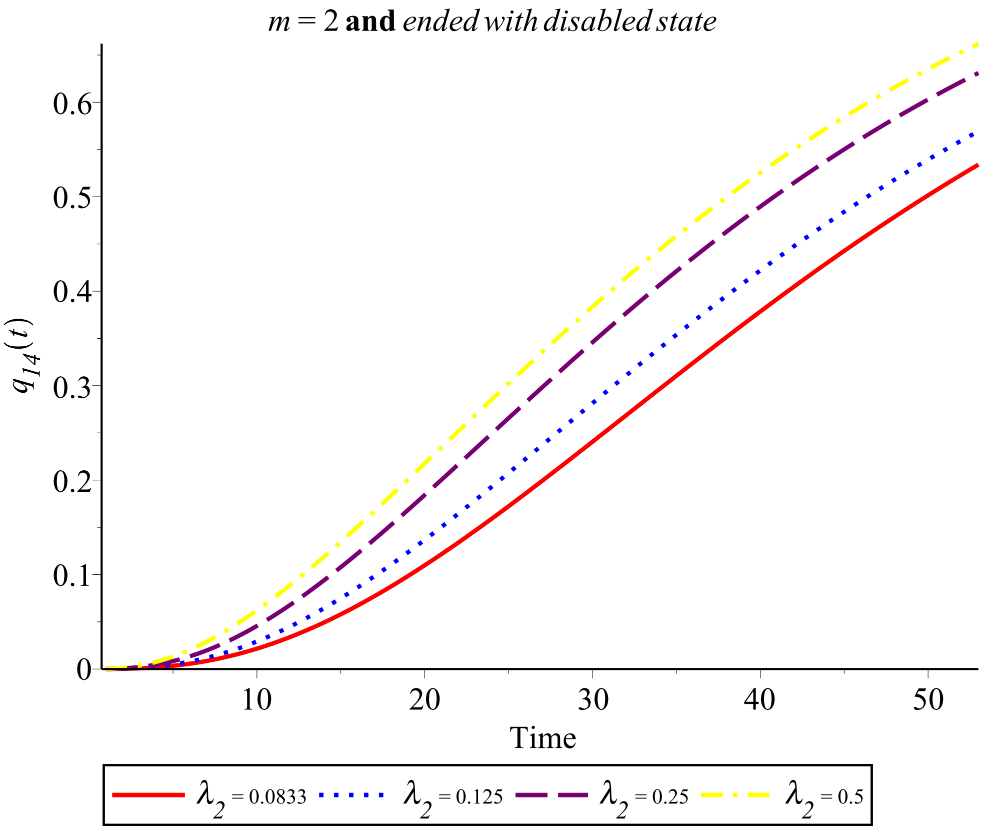

Figure 3.27: $P(S(t) = 4|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 2$ and ended with disabled or death.

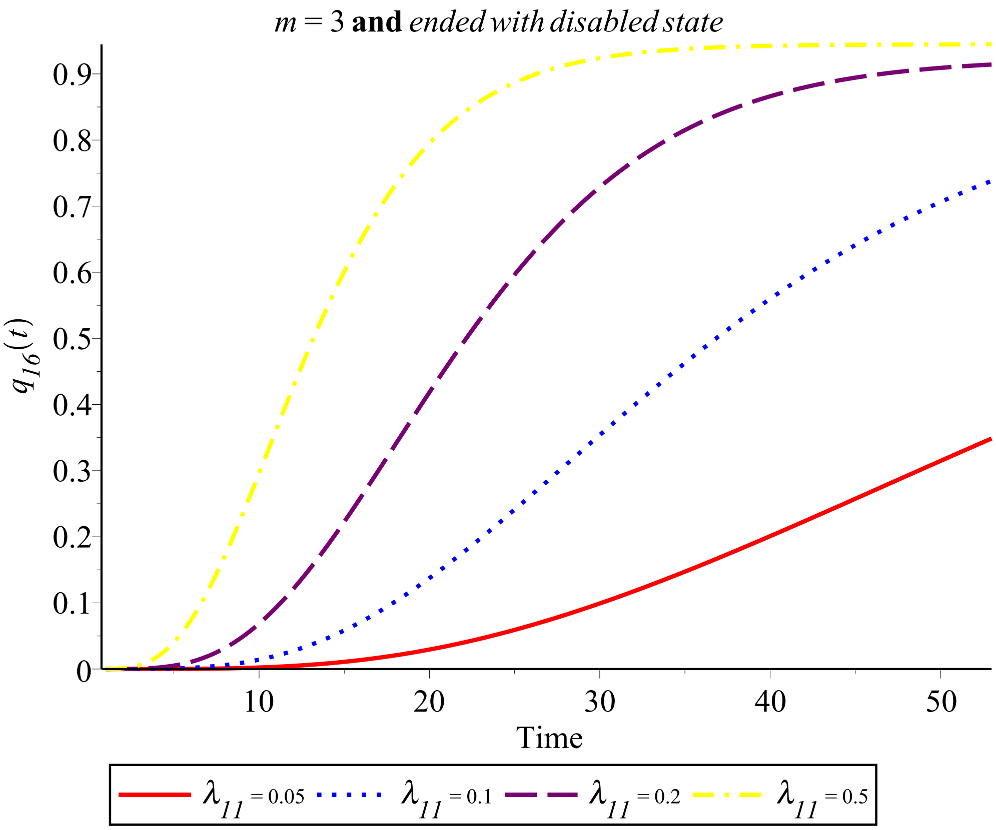

Figure 3.28: $P(S(t) = 6|S(0) = 1)$ for $\lambda_{11} = (0.05, 0.1, 0.2, 0.5)$ in which $m = 3$ and ended with disabled or death.

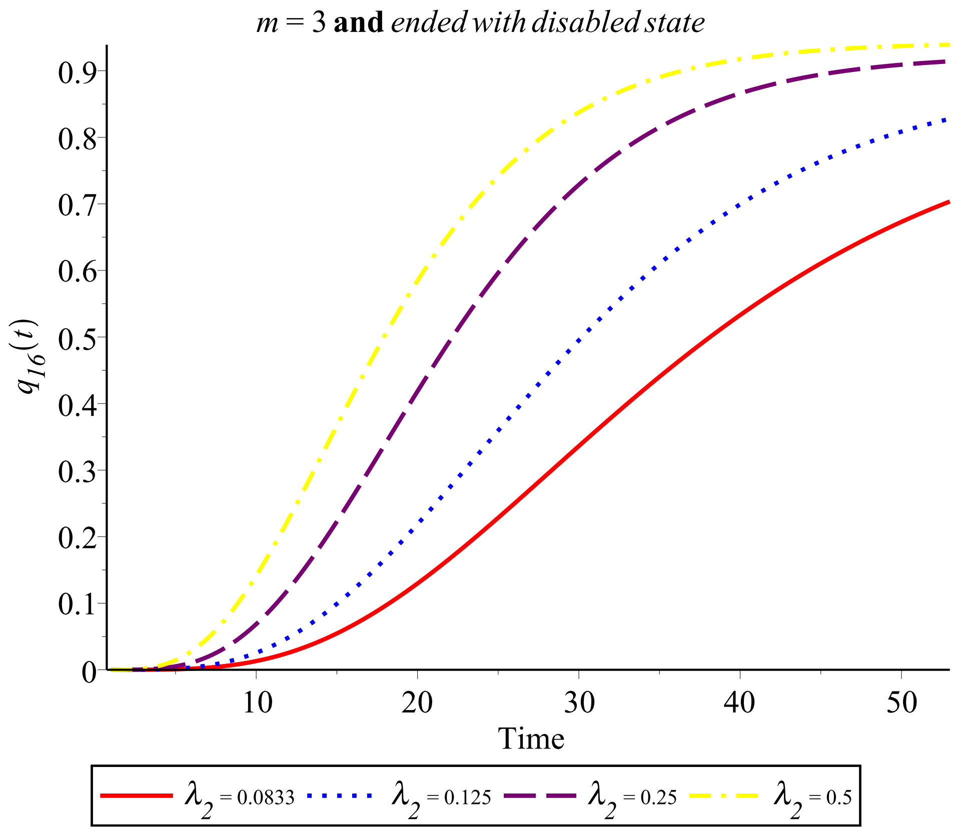

Figure 3.29: $P(S(t) = 6|S(0) = 1)$ for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ in which $m = 3$ and ended with disabled or death.

References

Akkouchi, M. (2008). On the Convolution of Exponential Distributions. *Journal of the Chungcheong Mathematical Society*, 21, 501-510.

Billard, L. and Dayananda, P. W. A. (2014a). A Multi-Stage Compartmental Model for HIV-infected individuals:I - Waiting time approach. *Mathematical Biosciences*, 249, 92-101.

Billard, L. and Dayananda, P. W. A. (2014b). A Multi-Stage Compartmental Model for HIV-Infected Individuals: II - Application to Insurance Functions and Health-Care Costs Approach. *Mathematical Bioscience*, 249, 102-109.

Waters H. R. (1991). The analysis of permanent health insurance data. *Continuous Mortality Investigation Reports*, 12, 1-263.

Pitacco, E. (1995). Actuarial Models for Pricing Disability Benefit: Towards a Unifying Approach. *Insurance: Mathematics and Economics* , 16, 39-62.

Chapter 4

Application in Insurance Functions

As discussed in section 2.6, transition probabilities are vital for many insurance functions of interest. In this chapter, we will focus on premium, expected payout and annual premium rate functions based on specific insurance policies described in Billard and Dayananda (2014b).

4.1 Some Insurance Functions of Interest

A t-year pure endowment policy

Under a t-year pure endowment policy, the insured is paid if and only if he/she survives to the end of the policy by definition. In addition, the payout will happen at the end of the policy. For calculation convenience, payout and premium rates will be assumed to be one dollar. Under a t-year pure endowment policy, an individual at stage i at initial time, will receive one dollar if he/she is still alive at time t. Also, one dollar at time t will have the present value $e^{(-\delta t)}$ (see, e.g., Billard and

Dayananda, 2014b). Then, the expected present value, denoted as $E_i(t)$, that an insurance company should pay out will be $E_i(t) = \sum_{j=i}^{2m} e^{-\delta t} q_{ij}(t)$. With the general expression of $q_{ij}(t)$, in equation (3.26), for $E_i(t)$, we have that,

$$\begin{aligned} E_i(t) &= \sum_{j=i}^{2m} e^{-\delta t} q_{ij}(t) \\ &= \sum_{j=i}^{2m} \Bigg[\sum_{p=i}^{j} \prod_{k=i,k\neq p}^{j} \frac{\beta_k}{\beta_k - \beta_p} e^{-t(\beta_p+\delta)} - \\ &\quad \sum_{p=i}^{j-1} \prod_{k=i,k\neq p}^{j-1} \frac{\beta_k}{\beta_k - \beta_p} e^{-t(\beta_p+\delta)} \Bigg] \\ &\quad \times P(\mathrm{W}_i > 0) P(\mathrm{W}_{i+1} > 0) ... P(\mathrm{W}_{j-1} > 0) \qquad (4.1) \end{aligned}$$

where

$$\beta_1 = \lambda_{11} + \alpha(z), \qquad (4.2)$$

$$\beta_{2i+1} = \lambda_{2i+1} + \alpha_{2i+1}, \qquad (4.3)$$

$$\beta_{2i} = \lambda_{2i} + \alpha_{2i}, \qquad (4.4)$$

for $i = 1, 2, ..., m-1$, and the $P(W_i > 0)'s$ were calculated in equations (3.10), (3.12) and (3.14) for $i = 1, 2, ..., 2m-1$.

Continuous t-year Life Annuity Policy

A continuous t- year life annuity is a contract in which the insured is provided continuous payment at 1 per unit time for t years. Assuming that the starting and ending time for the payment is 0 and t, respectively, the expected present value of the payout, denoted as $A_i(t)$, has the following expression

$$\begin{aligned} A_i(t) &= \sum_{j=i}^{2m} \int_0^t e^{-\delta u} q_{ij}(u) du \\ &= \sum_{j=i}^{2m} \Bigg[\sum_{p=i}^{j} \prod_{k=i,k\neq p}^{j} \frac{\beta_k}{\beta_k - \beta_p} \frac{1 - e^{-(\beta_p+\delta)t}}{\beta_p + \delta} - \\ &\quad \sum_{p=i}^{j-1} \prod_{k=i,k\neq p}^{j-1} \frac{\beta_k}{\beta_k - \beta_p} \frac{1 - e^{-(\beta_p+\delta)t}}{\beta_p + \delta} \Bigg] \\ &\quad \times P(\mathrm{W}_i > 0) P(\mathrm{W}_{i+1} > 0) ... P(\mathrm{W}_{j-1} > 0) \end{aligned} \tag{4.5}$$

where β_1, β_{2i+1} and β_{2i} are denoted in equations (4.2), (4.3), and (4.4). The expected net single premium based on the same policy is defined as the present value of the future death benefit. It assumes that the premium would be paid in the beginning of the policy. Assuming that one dollar would be paid to the insured at the end of the policy if death happened, the net single premium is calculated by the present value of this one dollar based on the death rate in each state and the continuous t-year life policy. It will be denoted as $B_i(t)$, and calculated as follows

$$
\begin{aligned}
B_i(t) &= \sum_{j=i}^{2m} \theta_j \int_0^t e^{-\delta u} q_{ij}(u) du \\
&= \sum_{j=i}^{2m} \left\{ \theta_j \left[\sum_{p=i}^{j} \frac{\beta_i ... \beta_{p-1}\beta_{p+1}...\beta_j}{\prod_{k=i,k\neq p}^{j} (\beta_k - \beta_p)} \frac{1 - e^{-(\beta_p+\delta)t}}{\beta_p + \delta} \right.\right. \\
&\quad \left. - \sum_{p=i}^{j-1} \frac{\beta_i ... \beta_{p-1}\beta_{p+1}...\beta_{j-1}}{\prod_{k=i,k\neq m}^{j-1} (\beta_k - \beta p)} \frac{1 - e^{-(\beta_p+\delta)t}}{\beta_p + \delta} \right] \\
&\quad \left. \times P(\mathrm{W}_i > 0) P(\mathrm{W}_{i+1} > 0) ... P(\mathrm{W}_{j-1} > 0) \right\} \qquad (4.6)
\end{aligned}
$$

where θ_j is the death rate at state j, i.e., $\theta_1 = \alpha(z)$, $\theta_{2i} = \alpha_{2i}$, and $\theta_{2i+1} = \alpha_{2i+1}$ for $i = 1, 2, ..., m$. Having the payout $A_i(t)$ in equation (4.5) and the net premium $B_i(t)$ in equation (4.6), we will be able to find the annual premium rate $P_i(t)$, i.e., the ratio of $B_i(t)$ to $A_i(t)$, as

$$
P_i(t) = B_i(t)/A_i(t) \qquad (4.7)
$$

To estimate their cost, insurance companies are often interested in the long term situation, as time goes into infinity. Then, for both $A_i(t)$ and $B_i(t)$ in equations (4.5) and (4.6), as time goes to infinity., i.e., $t \to \infty$, we have that

$$
\begin{aligned}
\lim_{t\to+\infty} A_i(t) &= \lim_{t\to+\infty} \sum_{j=i}^{2m} \Big[\sum_{p=i}^{j} \prod_{k=i,k\neq p}^{j} \frac{\beta_k}{\beta_k-\beta_p} \frac{1-e^{-(\beta_p+\delta)t}}{\beta_p+\delta} - \\
& \sum_{p=i}^{j-1} \prod_{k=i,k\neq p}^{j-1} \frac{\beta_k}{\beta_k-\beta_p} \frac{1-e^{-(\beta_p+\delta)t}}{\beta_p+\delta} \Big] \\
& \times P(\mathrm{W}_i>0)P(\mathrm{W}_{i+1}>0)...P(\mathrm{W}_{j-1}>0) \\
&= \sum_{j=i}^{2m} \Bigg\{ \Big[\sum_{p=i}^{j} \frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_j}{(\beta_p+\delta)\prod_{k=i,k\neq p}^{j}(\beta_k-\beta_p)} \\
& - \sum_{p=i}^{j-1} \frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_{j-1}}{(\beta_p+\delta)\prod_{k=i,k\neq m}^{j-1}(\beta_k-\beta p)} \Big] \\
& \times P(\mathrm{W}_i>0)P(\mathrm{W}_{i+1}>0)...P(\mathrm{W}_{j-1}>0) \Bigg\}; \qquad (4.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{t\to+\infty} B_i(t) &= \sum_{j=i}^{2m}\Bigg\{\theta_j\Bigg[\sum_{p=i}^{j}\frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_j}{\prod\limits_{k=i,k\neq p}^{j}(\beta_k-\beta_p)}\frac{1-e^{-(\beta_p+\delta)t}}{\beta_p+\delta} \\
&\quad -\sum_{p=i}^{j-1}\frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_{j-1}}{\prod\limits_{k=i,k\neq m}^{j-1}(\beta_k-\beta p)}\frac{1-e^{-(\beta_p+\delta)t}}{\beta_p+\delta}\Bigg] \\
&\quad \times P(\mathrm{W}_i>0)P(\mathrm{W}_{i+1}>0)...P(\mathrm{W}_{j-1}>0)\Bigg\} \\
&= \sum_{j=i}^{2m}\Bigg\{\theta_j\Bigg[\sum_{p=i}^{j}\frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_j}{(\beta_p+\delta)\prod\limits_{k=i,k\neq p}^{j}(\beta_k-\beta_p)} \\
&\quad -\sum_{p=i}^{j-1}\frac{\beta_i...\beta_{p-1}\beta_{p+1}...\beta_{j-1}}{(\beta_p+\delta)\prod\limits_{k=i,k\neq m}^{j-1}(\beta_k-\beta p)}\Bigg] \\
&\quad \times P(\mathrm{W}_i>0)P(\mathrm{W}_{i+1}>0)...P(\mathrm{W}_{j-1}>0)\Bigg\}. \qquad (4.9)
\end{aligned}
$$

Based on the above two equations (4.8) and (4.9), we can calculate the long-term annual premium rate as follows

$$
\lim_{t\to+\infty} P_i(t) = \lim_{t\to+\infty} B_i(t)/\lim_{t\to+\infty} A_i(t). \qquad (4.10)
$$

Life Expectancies

The life expectancy for an individual in stage i at time 0, L_i, is defined as equation

(4.11). Once we have have $q_{ij}(t)$, we can calculate the life expectancy as

$$\begin{aligned} L_i &= \int_0^{+\infty} \sum_{j=i}^{2m} \theta_j t q_{ij}(t) dt \\ &= \sum_{j=i}^{2m} \theta_j \Bigg\{ \Bigg[\sum_{p=i}^{j} \frac{\beta_i ... \beta_{p-1} \beta_{p+1} ... \beta_j}{\beta_p^2 \prod_{k=i,k\neq p}^{j} (\beta_k - \beta_p)} \\ &\quad - \sum_{p=i}^{j-1} \frac{\beta_i ... \beta_{p-1} \beta_{p+1} ... \beta_{j-1}}{\beta_p^2 \prod_{k=i,k\neq p}^{j-1} (\beta_k - \beta p)} \Bigg] \\ &\quad \times P(\mathrm{W}_i > 0) P(\mathrm{W}_2 > 0) ... P(\mathrm{W}_{j-1} > 0) \Bigg\} \end{aligned} \tag{4.11}$$

where β_1, β_{2i+1} and β_{2i} are denoted in equations (4.2), (4.3), and (4.4). In addition, the $P(W_i > 0)'s$ were calculated in equations (3.10), (3.12) and (3.14) for $i = 1, 2, ..., 2m-1$.

4.2 Health Care Discussion for Disability Model

In chapter 3, we discussed the transition $q_{ij}(t)$ for the cases that m equals 1, 2 and 3. In this section, we assume that a certain amount of money C, needs to be paid if the insured is disabled. The expected health cost until time t for the insured who is in state i initially assuming that the insured will go through m times of disease will be denoted as $H(t, i, m)$ and can be calculated as follows:

$$H(t, i, m) = \sum_{k=i}^{j} \int_0^t C e^{-\delta u} q_{ik}(u) du \tag{4.12}$$

where $i \leq k \leq j$ and k denote all the disabled states spanning from state i to state j. Next, we will discuss the expected cost described in equation (4.12), where $m =$ 1, 2 and 3.

The case where $m = 1$

When $m = 1$, we know that state 2 would be the only disabled state that the insured would go through assuming the insured will be either healthy or dead at the end of the policy. Applying equation (3.34), we have that

$$\begin{aligned} H(t,1,1) &= \int_0^t Ce^{-\delta u} q_{12}(u)du \\ &= \int_0^t Ce^{-\delta u} \frac{\lambda_{11}}{B-A}(e^{-Au} - e^{-Bu})du \\ &= \int_0^t C \frac{\lambda_{11}}{B-A}(e^{-(A+\delta)u} - e^{-(B+\delta)u})du \\ &= C\frac{\lambda_{11}}{B-A}[\frac{1}{\delta+A} - \frac{1}{\delta+B} - (\frac{1}{\delta+A}e^{-(\delta+A)t} \\ &\quad -\frac{1}{\delta+B}e^{-(\delta+B)t})]. \end{aligned} \tag{4.13}$$

The sensitivity of expected cost $H(t, 1, 1)$ to the interest rate is plotted in Figure 4.1. The weekly interest rate δ is taken at values 0.000189, 0.000377, 0.000755, and 0.001511, corresponding to the annual interest rate 0.01, 0.02, 0.04 and 0.08. The disease rate λ_{11}, is set as 0.0384, the death rate of D_1, i.e., $\alpha(z)$, takes the value 0.0026, the disabled rate, λ_2 has the value 0.125 and the recovery rate, α_2, is given the value 0.0052. The plots in Figure 4.1 show that the expected payout is not sensitive to the interest rate, since the plots are essentially the same over time regardless of the value of δ. It does make sense since the interest rate is relatively small to other parameters. Also, this is only one year policy.

The case where $m = 2$

When $m = 2$, we know that states 2 and 4 would be the two disabled states that the insured would go through assuming the insured will be either healthy or dead at the end of the policy. Applying equation (4.12), we will have that

$$\begin{aligned} H(t,1,2) &= \int_0^t Ce^{-\delta u}(q_{12}(u) + q_{14}(u))du \\ &= \int_0^t Ce^{-\delta u}q_{12}(u)du + \int_0^t Ce^{-\delta u}q_{14}(u)du. \end{aligned} \tag{4.14}$$

The first part of the equation (4.14), $\int_0^t Ce^{-\delta u}q_{12}(u)du$, has been solved in equation (4.13); In addition, we already have $q_{14}(t)$ solved in equation (3.41). Then, for $\int_0^t Ce^{-\delta u}q_{14}(u)du$, we have that

$$
\begin{aligned}
\int_0^t Ce^{-\delta u} q_{14}(u)du &= \int_0^t Ce^{-\delta u}(\frac{\lambda_{11}}{A})^2\frac{\lambda_2}{B}[\frac{ABue^{-Au}}{A-B} - \frac{A^2e^{-Bu} - 2ABe^{-Au} + B^2e^{-Au}}{(A-B)^2} \\
&+\frac{A^2Bue^{-Bu} + AB^2ue^{-Au}}{(A-B)^2} \\
&+\frac{A^3e^{-Bu} - 3A^2Be^{-Bu} + 3AB^2e^{-Au} - B^3e^{-Au}}{(A-B)^3}]du \\
&= C\frac{{\lambda_{11}}^2\lambda_2A^2B}{A^2B(A-B)^3(A+\delta)^2(B+\delta)^2}[(B-A)^3 + 2A^2Be^{-(A+\delta)t}\delta t \\
&+A^2Be^{-(B+\delta)t}\delta t - AB^2e^{-(A+\delta)t}\delta t - 2AB^2e^{-(B+\delta)t}\delta t \\
&+ABe^{-(A+\delta)t}\delta^2 t - ABe^{-(B+\delta)t}\delta^2 t + A^3Be^{-(B+\delta)t}t \\
&+A^3e^{-(B+\delta)t}\delta t + A^2B^2e^{-(A+\delta)t}t - A^2B^2e^{-(B+\delta)t}t \\
&+A^2e^{-(A+\delta)t}\delta^2 t + 2A^2e^{-(B+\delta)t}\delta^2 t - AB^3e^{-(A+\delta)t}t \\
&+Ae^{-(A+\delta)t}\delta^3 t + Ae^{-(B+\delta)t}\delta^3 t - B^3e^{-(A+\delta)t}\delta t \\
&-2B^2e^{-(A+\delta)t}\delta^2 t - B^2e^{-(B+\delta)t}\delta^2 t - Be^{-(A+\delta)t}\delta^3 t \\
&-Be^{-(B+\delta)t}\delta^3 t + 2ABe^{-(A+\delta)t}\delta - 4ABe^{-(B+\delta)t}\delta \\
&+4ABe^{-(A+\delta)t}\delta - 2ABe^{-(B+\delta)t}\delta - 2A^2Be^{-(B+\delta)t} \\
&-A^2Be^{-(B+\delta)t} - 2A^2e^{-(B+\delta)t}\delta + 2A^2e^{-(B+\delta)t}\delta + AB^2e^{-(A+\delta)t} \\
&+2AB^2e^{-(A+\delta)t} + Ae^{-(A+\delta)t}\delta^2 - 4Ae^{-(B+\delta)t}\delta^2 + 2Ae^{-(A+\delta)t}\delta^2 \\
&+Ae^{-(B+\delta)t}\delta^2 - 2B^2e^{-(A+\delta)t}\delta + 2B^2e^{-(A+\delta)t}\delta - Be^{-(A+\delta)t}\delta^2
\end{aligned}
$$

$$
\begin{aligned}
&-2Be^{-(B+\delta)t}\delta^2 + 4Be^{-(A+\delta)t}\delta^2 - Be^{-(B+\delta)t}\delta^2 \\
&+A^3e^{-(B+\delta)t} - B^3e^{-(A+\delta)t} - 2e^{-(B+\delta)t}\delta^3 + 2e^{-(A+\delta)t}\delta^3] \\
= \;& C\frac{{\lambda_{11}}^2\lambda_2A^2B}{A^2B(A-B)^3(A+\delta)^2(B+\delta)^2}[(B-A)^3 + e^{-(B+\delta)t}(A^2B\delta t \\
&-2AB^2\delta t - AB\delta^2 t + A^3Bt + A^3\delta t - A^2B^2t + 2A^2\delta^2 t + A\delta^3 t \\
&-B^2\delta^2 t - B\delta^3 t - 6AB\delta - 3A^2B - 3A\delta^2 - 3B\delta^2 + A^3 - 2\delta^3) \\
&+e^{-(A+\delta)t}(2A^2B\delta t - AB^2\delta t + AB\delta^2 t + A^2B^2t + A^2\delta^2 t \\
&-AB^3t + A\delta^3 t - B^3\delta t - 2B^2\delta^2 t - B\delta^3 t + 6AB\delta + 3AB^2 \\
&+3A\delta^2 + 3B\delta^2 - B^3 + 2\delta^3)]. \qquad (4.15)
\end{aligned}
$$

Figures 4.2 and 4.3 test the sensitivity of expected payout $H(t,1,2)$ to the parameters λ_{11} and λ_2. The two figures show that $H(t,1,2)$ is sensitive to both λ_{11} and λ_2. In Figure 4.2, we take the weekly interest rate δ as 0.000377, the recovery rate λ_2 as 0.25, corresponding to average waiting time to recover from a disease as 4 weeks, the healthy death rate as 0.0026, and λ_{11} as 0.05, 0.033, 0.05, and 0.1, corresponding to the average waiting time to get sick as 40, 30, 20 and 10 weeks. As λ_{11} increases, i.e., the average waiting time for a sickness occurrence decreases, the chance to get sick increases, the expected payout would increase. On the other hand, as time passes by, the expected payout would increase as well. For Figure 4.3, we take λ_{11} as 0.05, corresponding to the average waiting time to get sick as 20 weeks, weekly interest rate δ as 0.000377, healthy death rate $\alpha(z)$ as 0.0026, and the recovery rate, λ_2, as 0.0833, 0.125, 0.25 and 0.5, corresponding to the average waiting time to get recovered 12, 8, 4 and 2 weeks. As λ_2 increases, i.e., as the average waiting time for recovery decreases, as the chance to get sick decreases, the expected payout would decrease as well.

The case where m=3

When $m = 3$, we know that states 2, 4 and 6 would be the three disabled states that the insured would go through assuming the insured will be either healthy or dead in the end of the policy. Applying equation (4.12), we will have that

$$\begin{aligned} H(t,1,3) &= \int_0^t Ce^{-\delta u}(q_{12}(u) + q_{14}(u) + q_{16}(u))du \\ &= \int_0^t Ce^{-\delta u}q_{12}(u)du + \int_0^t Ce^{-\delta u}q_{14}(u)du \\ &\quad + \int_0^t Ce^{-\delta u}q_{16}(u)du. \end{aligned} \tag{4.16}$$

In this equation (4.16), the first and second parts, i.e., $\int_0^t Ce^{-\delta u}q_{12}(u)du$ and $\int_0^t Ce^{-\delta u}q_{14}(u)du$ have been solved in equations (4.14) and (4.15), respectively. We will focus on the calculation of the third part $\int_0^t Ce^{-\delta u}q_{16}(u)du$. Applying (3.47) to the third term of equation (4.16), we will have that

$$\begin{aligned}
\int_0^t Ce^{-\delta u}q_{16}(u)du &= \int_0^t Ce^{-\delta u}(\frac{\lambda_{11}}{A})^3(\frac{\lambda_2}{B})^2\{\frac{A^2B^2u^2(Be^{-Au}-Ae^{-Bu})}{2(A-B)^3} \\
&+\frac{A^3Be^{-Bu}(A-4B)u}{(A-B)^4}+\frac{AB^3e^{-Au}(B-4A)u}{(A-B)^4} \\
&+\frac{A^3e^{-Bu}(A^2-5AB+10B^2)}{(A-B)^5}-\frac{B^3e^{-Au}(10A^2-5AB+B^2)}{(A-B)^5} \\
&-\frac{A^2B^2}{2(A-B)^2}e^{-Au}u^2-\frac{A^3Be^{-Bu}u}{(A-B)^3}-\frac{AB^2e^{-Au}u(3A-B)}{(A-B)^3} \\
&-\frac{1}{(A-B)^4}(A^4e^{-Bu}-4A^3Be^{-Bu}+6A^2B^2e^{-Au} \\
&-4AB^3e^{-Au}+B^4e^{-Au})\}du \\
&= -\frac{C{\lambda_{11}}^3{\lambda_2}^2}{2(A-B)^5(B+\delta)^3(A+\delta)^3A^3B^2}[-10A^4Be^{-(B+\delta)t} \\
&+8A^3B^2e^{-(B+\delta)t}+12A^3B^2e^{-(B+\delta)t}-12A^3e^{-(B+\delta)t}\delta^2 \\
&+12A^3e^{-(B+\delta)t}\delta^2-8A^2B^3e^{-(A+\delta)t}-12A^2B^3e^{-(A+\delta)t} \\
&-8A^2e^{-(A+\delta)t}\delta^3-16A^2e^{-(B+\delta)t}\delta^3+36A^2e^{-(B+\delta)t}\delta^3 \\
&-12A^2e^{-(A+\delta)t}\delta^3+10AB^4e^{-(A+\delta)t}-6Ae^{-(A+\delta)t}\delta^4 \\
&-6Ae^{-(B+\delta)t}\delta^4+36Ae^{-(B+\delta)t}\delta^4-24Ae^{-(A+\delta)t}\delta^4 \\
&+12B^3e^{-(A+\delta)t}\delta^2-12B^3e^{-(A+\delta)t}\delta^2+16B^2e^{-(A+\delta)t}\delta^3 \\
&+8B^2e^{-(B+\delta)t}\delta^3+12B^2e^{-(B+\delta)t}\delta^3-36B^2e^{-(A+\delta)t}\delta^3 \\
&+6Be^{-(A+\delta)t}\delta^4+6Be^{-(B+\delta)t}\delta^4+24Be^{-(B+\delta)t}\delta^4 \\
&-36Be^{-(A+\delta)t}\delta^4-20A^3B^2+10A^4B+20A^2B^3 \\
&-10AB^4-2A^5+2B^5+2A^5Be^{-(B+\delta)t}\delta t^2 \\
&-3A^4B^2e^{-(A+\delta)t}\delta t^2-A^4B^2e^{-(B+\delta)t}\delta t^2
\end{aligned}$$

$$-3A^4Be^{-(A+\delta)t}\delta^2t^2 + 4A^4Be^{-(B+\delta)t}\delta^2t^2$$
$$+4A^3B^3e^{-(A+\delta)t}\delta t^2 - 4A^3B^3e^{-(B+\delta)t}\delta t^2$$
$$-8A^3B^2e^{-(B+\delta)t}\delta^2t^2 - 4A^3Be^{-(A+\delta)t}\delta^3t^2$$
$$+A^2B^4e^{-(A+\delta)t}\delta t^2 + 3A^2B^4e^{-(B+\delta)t}\delta t^2$$
$$+8A^2B^3e^{-(A+\delta)t}\delta^2t^2 + 8A^2B^2e^{-(A+\delta)t}\delta^3t^2$$
$$-8A^2B^2e^{-(B+\delta)t}\delta^3t^2 + A^2Be^{-(A+\delta)t}\delta^4t^2$$
$$-4A^2Be^{-(B+\delta)t}\delta^4t^2 - 2AB^5e^{-(A+\delta)t}\delta t^2$$
$$-4AB^4e^{-(A+\delta)t}\delta^2t^2 + 3AB^4e^{-(B+\delta)t}\delta^2t^2$$
$$+4AB^3e^{-(B+\delta)t}\delta^3t^2 + 4AB^2e^{-(A+\delta)t}\delta^4t^2$$
$$-AB^2e^{-(B+\delta)t}\delta^4t^2 + 2ABe^{-(A+\delta)t}\delta^5t^2$$
$$-2ABe^{-(B+\delta)t}\delta^5t^2 - 10A^4Be^{-(B+\delta)t}\delta t$$
$$-24A^3B^2e^{-(A+\delta)t}\delta t - 16A^3B^2e^{-(B+\delta)t}\delta t$$
$$-24A^3Be^{-(A+\delta)t}\delta^2t - 36A^3Be^{-(B+\delta)t}\delta^2t$$
$$+16A^2B^3e^{-(A+\delta)t}\delta t + 24A^2B^3e^{-(B+\delta)t}\delta t$$
$$-12A^2B^2e^{-(A+\delta)t}\delta^2t + 12A^2B^2e^{-(B+\delta)t}\delta^2t$$
$$-32A^2Be^{-(A+\delta)t}\delta^3t - 28A^2Be^{-(B+\delta)t}\delta^3t$$
$$+10AB^4e^{-(A+\delta)t}\delta t + 36AB^3e^{-(A+\delta)t}\delta^2t$$
$$+24AB^3e^{-(B+\delta)t}\delta^2t + 28AB^2e^{-(A+\delta)t}\delta^3t$$
$$+32AB^2e^{-(B+\delta)t}\delta^3t - 2ABe^{-(A+\delta)t}\delta^4t$$
$$+2ABe^{-(B+\delta)t}\delta^4t + 2A^5e^{-(B+\delta)t}$$
$$-2B^5e^{-(A+\delta)t} + 12e^{-(B+\delta)t}\delta^5 - 12e^{-(A+\delta)t}\delta^5$$

$$+36A^2B^2e^{-(B+\delta)t}\delta - 36A^2B^2e^{-(A+\delta)t}\delta - 24A^2Be^{-(A+\delta)t}\delta^2$$

$$-12A^2Be^{-(B+\delta)t}\delta^2 + 72A^2Be^{-(B+\delta)t}\delta^2 - 36A^2Be^{-(A+\delta)t}\delta^2$$

$$+24AB^3e^{-(A+\delta)t}\delta - 24AB^3e^{-(A+\delta)t}\delta + 12AB^2e^{-(A+\delta)t}\delta^2$$

$$+24AB^2e^{-(B+\delta)t}\delta^2 + 36AB^2e^{-(B+\delta)t}\delta^2 - 72AB^2e^{-(A+\delta)t}\delta^2$$

$$-8ABe^{-(A+\delta)t}\delta^3 + 8ABe^{-(B+\delta)t}\delta^3 + 72ABe^{-(B+\delta)t}\delta^3$$

$$-72ABe^{-(A+\delta)t}\delta^3 + A^5B^2e^{-(B+\delta)t}t^2A^5e^{-(B+\delta)t}\delta^2t^2$$

$$-A^4B^3e^{-(A+\delta)t}t^2 - 2A^4B^3e^{-(B+\delta)t}t^2 - A^4e^{-(A+\delta)t}\delta^3t^2$$

$$+3A^4e^{-(B+\delta)t}\delta^3t^2 + 2A^3B^4e^{-(A+\delta)t}t^2 + A^3B^4e^{-(B+\delta)t}t^2$$

$$-2A^3e^{-(A+\delta)t}\delta^4t^2 + 3A^3e^{-(B+\delta)t}\delta^4t^2 - A^2B^5e^{-(A+\delta)t}t^2$$

$$-A^2e^{-(A+\delta)t}\delta^5t^2 + A^2e^{-(B+\delta)t}\delta^5t^2 - B^5e^{-(A+\delta)t}\delta^2t^2$$

$$-3B^4e^{-(A+\delta)t}\delta^3t^2 + B^4e^{-(B+\delta)t}\delta^3t^2 - 3B^3e^{-(A+\delta)t}\delta^4t^2$$

$$+2B^3e^{-(B+\delta)t}\delta^4t^2 - B^2e^{-(A+\delta)t}\delta^5t^2 + B^2e^{-(B+\delta)t}\delta^5t^2$$

$$+2A^5Be^{-(B+\delta)t}t + 2A^5e^{-(B+\delta)t}\delta t - 10A^4B^2e^{-(B+\delta)t}t$$

$$-8A^3B^3e^{-(A+\delta)t}t + 8A^3B^3e^{-(B+\delta)t}t - 8A^3e^{-(A+\delta)t}\delta^3t$$

$$-12A^3e^{-(B+\delta)t}\delta^3t + 10A^2B^4e^{-(A+\delta)t}t - 14A^2e^{-(A+\delta)t}\delta^4t$$

$$-16A^2e^{-(B+\delta)t}\delta^4t - 2AB^5e^{-(A+\delta)t}t - 6Ae^{-(A+\delta)t}\delta^5t$$

$$-6Ae^{-(B+\delta)t}\delta^5t - 2B^5e^{-(A+\delta)t}\delta t + 12B^3e^{-(A+\delta)t}\delta^3t$$

$$+8B^3e^{-(B+\delta)t}\delta^3t + 16B^2e^{-(A+\delta)t}\delta^4t + 14B^2e^{-(B+\delta)t}\delta^4t$$

$$
\begin{aligned}
& +6Be^{-(A+\delta)t}\delta^5 t + 6Be^{-(B+\delta)t}\delta^5 t - 24A^3Be^{-(B+\delta)t}\delta \\
& +24A^3Be^{-(B+\delta)t}\delta - 24A^2B^2e^{-(A+\delta)t}\delta + 24A^2B^2e^{-(B+\delta)t}t] \\
= \quad & -\frac{C{\lambda_{11}}^3{\lambda_2}^2}{2(A-B)^5(B+\delta)^3(A+\delta)^3A^3B^2}[2(B-A)^5 \\
& +e^{-(A+\delta)t}(-20A^2B^3 - 20A^2\delta^3 + 10AB^4 - 30A\delta^4 \\
& -20B^2\delta^3 - 30B\delta^4 - 3A^4B^2\delta t^2 - 3A^4B\delta^2t^2 + 4A^3B^3\delta t^2 \\
& -4A^3B\delta^3t^2 + A^2B^4\delta t^2 + 8A^2B^3\delta^2t^2 + 8A^2B^2\delta^3t^2 \\
& +A^2B\delta^4t^2 - 2AB^5\delta t^2 - 4AB^4\delta^2t^2 + 4AB^2\delta^4t^2 \\
& +2AB\delta^5t^2 - 24A^3B^2\delta t - 24A^3B\delta^2t + 16A^2B^3\delta t \\
& -12A^2B^2\delta^2t - 32A^2B\delta^3t + 10AB^4\delta t + 36AB^3\delta^2t \\
& +28AB^2\delta^3t - 2AB\delta^4t - 2B^5 - 12\delta^5 \\
& -36A^2B^2\delta - 60A^2B\delta^2 - 60AB^2\delta^2 - 80AB\delta^3 - A^4B^3t^2 \\
& -A^4\delta^3t^2 + 2A^3B^4t^2 - 2A^3\delta^4t^2 - A^2B^5t^2 \\
& -A^2\delta^5t^2 - B^5\delta^2t^2 - 3B^4\delta^3t^2 - 3B^3\delta^4t^2 \\
& -B^2\delta^5t^2 - 8A^3B^3t - 8A^3\delta^3t + 10A^2B^4t - 14A^2\delta^4t \\
& -2AB^5t - 6A\delta^5t - 2B^5\delta t + 12B^3\delta^3t \\
& +16B^2\delta^4t + 6B\delta^5t - 24A^2B^2\delta) + e^{-(B+\delta)t}(-10A^4B + 20A^3B^2 \\
& +20A^2\delta^3 + 30A\delta^4 + 20B^2\delta^3 + 30B\delta^4 \\
& +2A^5B\delta t^2 - A^4B^2\delta t^2 + 4A^4B\delta^2t^2 - 4A^3B^3\delta t^2 \\
& -8A^3B^2\delta^2t^2 + 3A^2B^4\delta t^2 - 8A^2B^2\delta^3t^2 - 4A^2B\delta^4t^2 \\
& +3AB^4\delta^2t^2 + 4AB^3\delta^3t^2 - AB^2\delta^4t^2 - 2AB\delta^5t^2
\end{aligned}
$$

$$
\begin{aligned}
&-10A^4B\delta t - 16A^3B^2\delta t - 36A^3B\delta^2 t + 24A^2B^3\delta t \\
&+12A^2B^2\delta^2 t - 28A^2B\delta^3 t + 24AB^3\delta^2 t + 32AB^2\delta^3 t \\
&+2AB\delta^4 t + 2A^5 + 12\delta^5 + 36A^2B^2\delta - 12A^2B\delta^2 \\
&+72A^2B\delta^2 + 60AB^2\delta^2 + 80AB\delta^3 + A^5B^2t^2 \\
&+A^5\delta^2t^2 - 2A^4B^3t^2 + 3A^4\delta^3t^2 + A^3B^4t^2 \\
&+3A^3\delta^4t^2 + A^2\delta^5t^2 + B^4\delta^3t^2 + 2B^3\delta^4t^2 \\
&+B^2\delta^5t^2 + 2A^5Bt + 2A^5\delta t - 10A^4B^2t \\
&+8A^3B^3t - 12A^3\delta^3t - 16A^2\delta^4t - 6A\delta^5t \\
&+8B^3\delta^3t + 14B^2\delta^4t + 6B\delta^5t + 24A^2B^2t). \qquad (4.17)
\end{aligned}
$$

Sensitivity of the expected payout

Figures 4.4 and 4.5 test the sensitivity of expected cost $H(t, 1, 3)$ to the parameters λ_{11} and λ_2. The two figures show that $H(t, 1, 3)$ is sensitive to both λ_{11} and λ_2, and has the similar trend with the case where $m = 2$. In Figure 4.4, we take the weekly interest rate δ as 0.000377, which corresponds to an annual interest rate of 2 percent, the recovery rate λ_2 as 0.25 corresponding to average waiting time to recover from a disease as 4 weeks, the healthy death rate as 0.0026 corresponding to a 30-year old person, and λ_{11} as 0.05, 0.1, 0.15, and 0.2, corresponding to the average waiting time to get sick as 20, 10, 7 and 5 weeks. As λ_{11} increases, i.e., as the average waiting time for a sickness occurrence decreases, the chance to get sick increases, so that the expected payout would increases. On the other hand, as time passes by, the expected payout would increase as well. In Figure 4.5, we take the weekly interest rate δ as

0.000377, the recovery rate λ_2 as 0.0833, 0.125, 0.25 and 0.5 corresponding to average waiting time to recover from a disease as 12, 8, 4 and 2 weeks, the healthy death rate $\alpha(z)$ as 0.0026, and λ_{11} as 0.05, corresponding to the average waiting time to get sick as 20 weeks. As λ_2 increases, i.e., as the average waiting time for recovery decreases, the chance to get sick decreases, so that the expected payout would decrease as well.

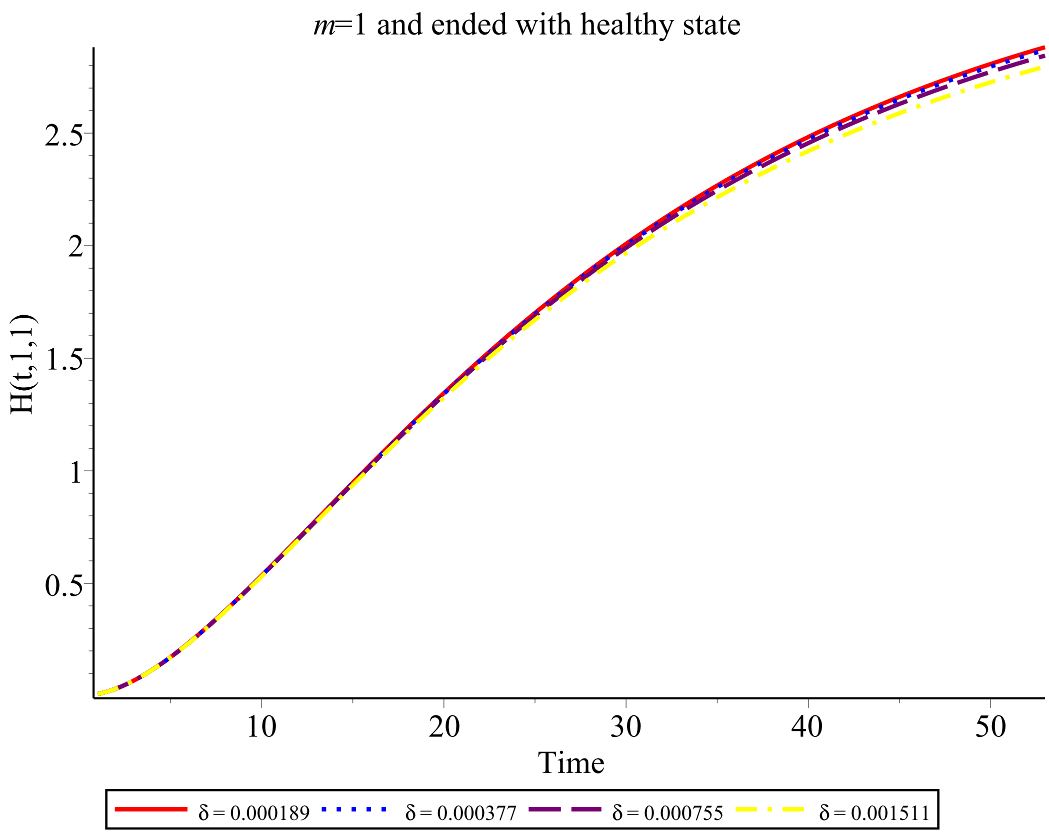

Figure 4.1: Expected payout H(t,1,1) for $\delta = (0.0001, 0.000377, 0.000755, 0.001511)$ where *m=1*

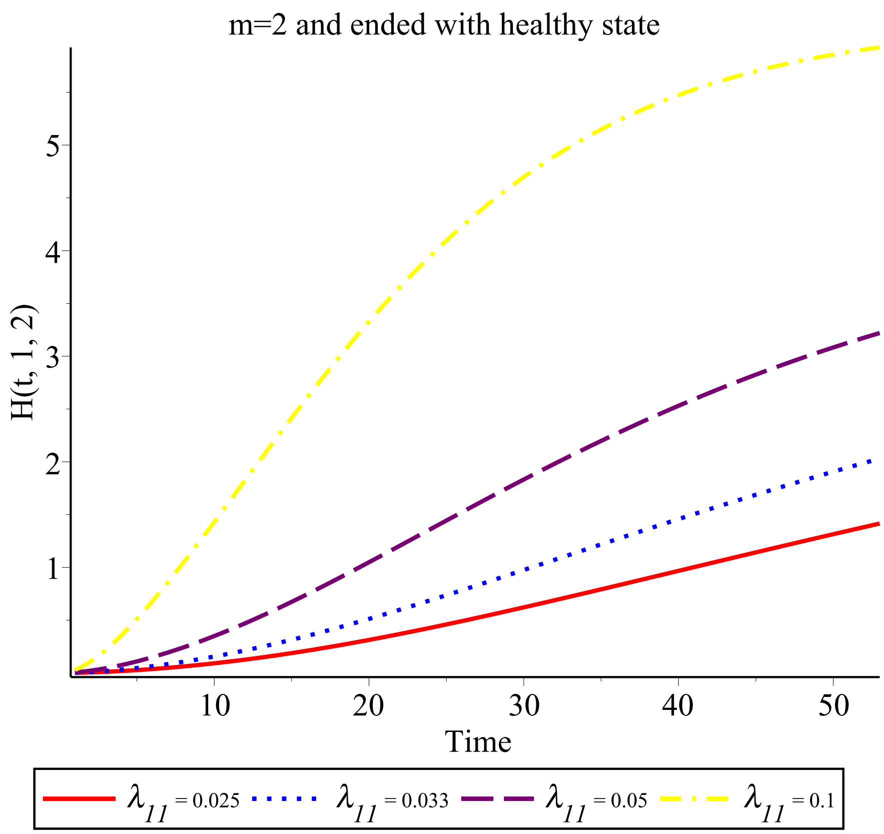

Figure 4.2: Expected payout H(t,1,2) for $\lambda_{11} = (0.025, 0.033, 0.05, 0.10)$ where m=2.

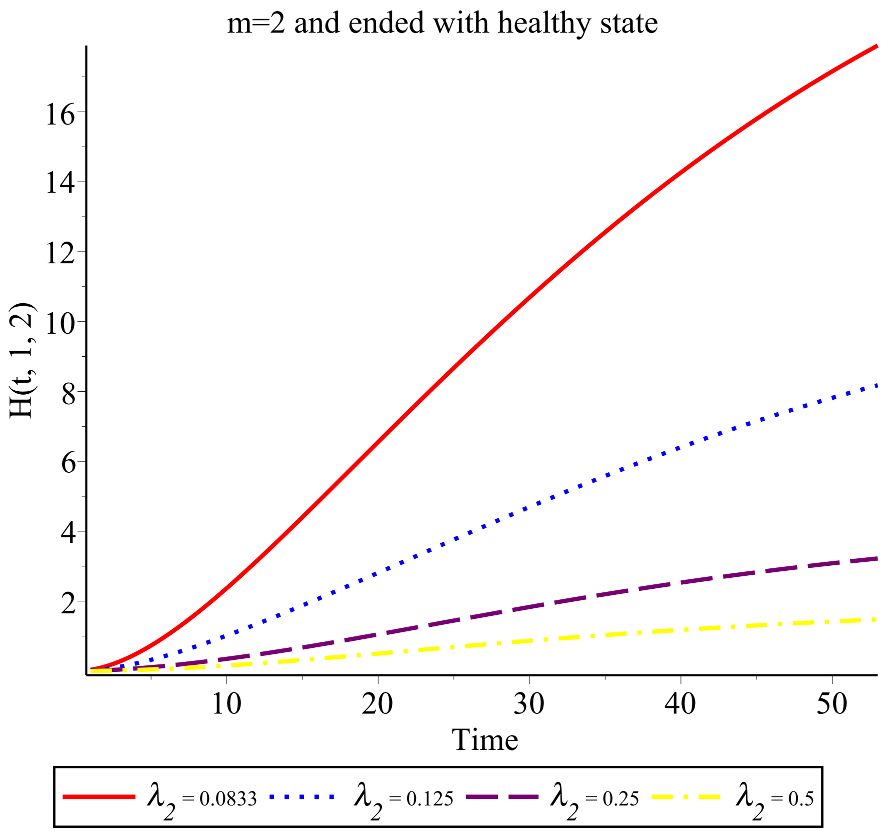

Figure 4.3: Expected payout H(t,1,2) for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$ where m=2.

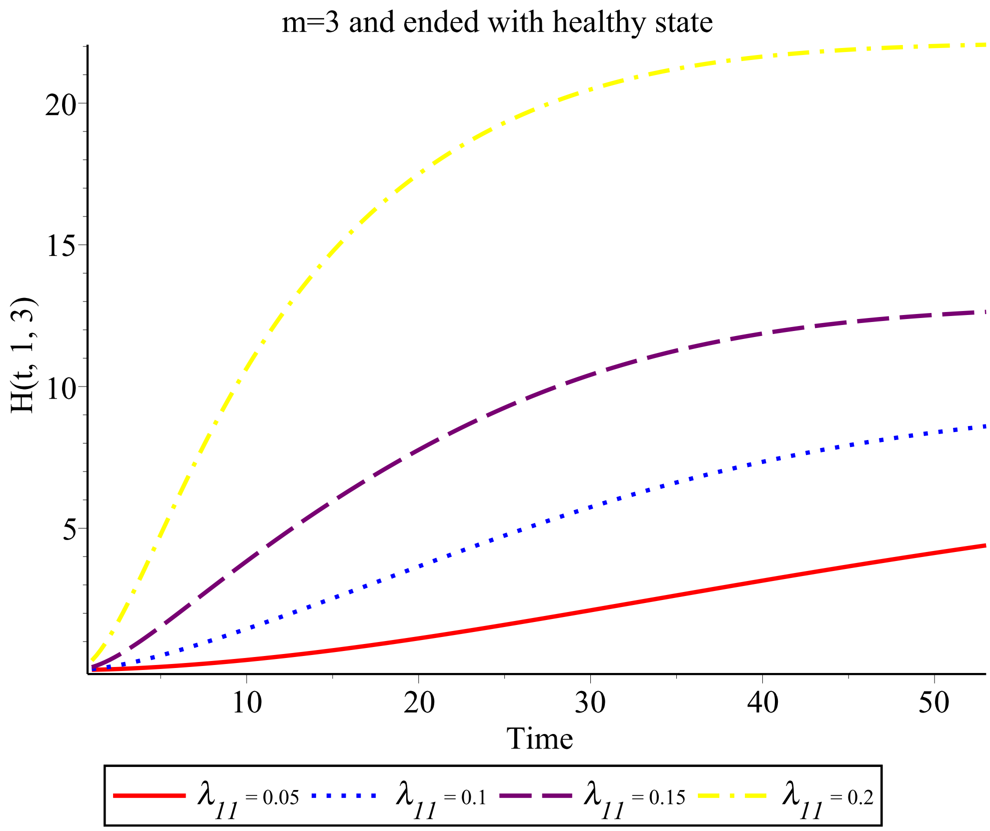

Figure 4.4: Expected payout H(t,1,3) for $\lambda_{11} = (0.05, 0.1, 0.15, 0.2)$, where *m=2*

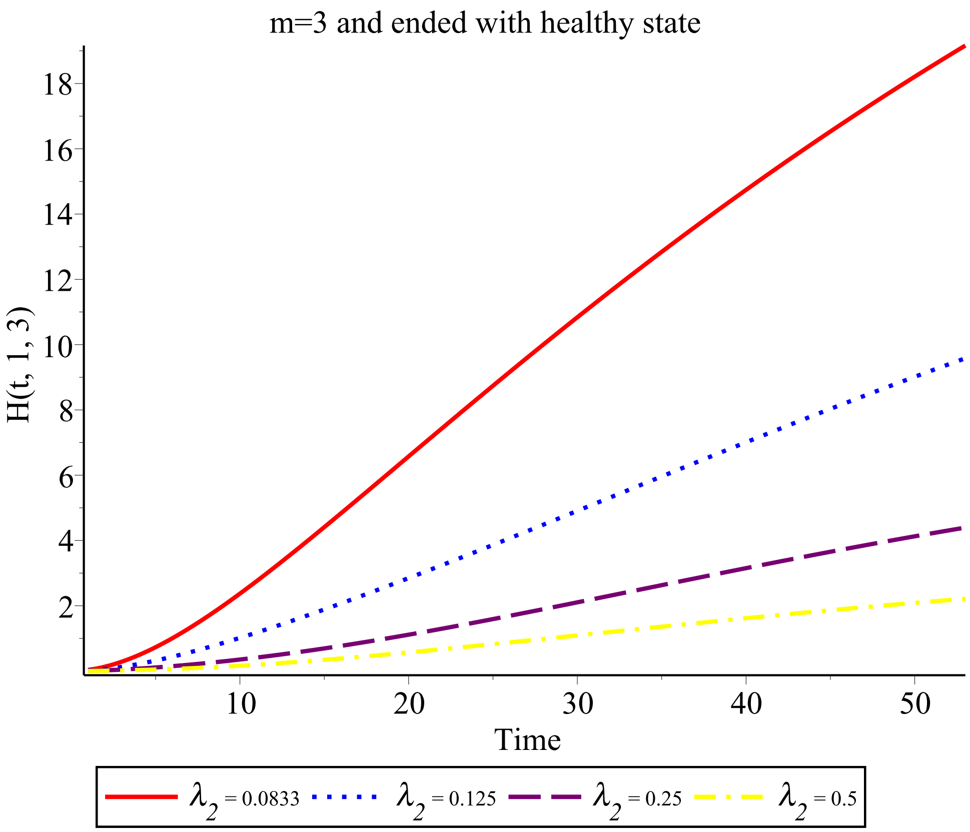

Figure 4.5: Expected payout H(t,1,3) for $\lambda_2 = (0.0833, 0.125, 0.25, 0.5)$, where *m=2*

References

Akkouchi, M. (2008). On the Convolution of Exponential Distributions. *Journal of the Chungcheong Mathematical Society*, 21, 501-510.

Billard, L. and Dayananda, P. W. A. (2014a). A Multi-Stage Compartmental Model for HIV-infected individuals:I - Waiting time approach. *Mathematical Biosciences*, 249, 92-101.

Billard, L. and Dayananda, P. W. A. (2014b). A Multi-Stage Compartmental Model for HIV-Infected Individuals: II - Application to Insurance Functions and Health-Care Costs Approach. *Mathematical Biosciences*, 249, 102-109.

Pitacco, E. (1995). Actuarial Models for Pricing Disability Benefit: Towards a Unifying Approach. *Insurance: Mathematics and Economics*, 16, 39-62.

Chapter 5

Future Work

In this chapter, we discuss some ideas about future work.

5.1 Discussion of the Death Distribution

In Chapter 3 and 4, we focused on the health insurance, in which the insured will receive a payout paid when he/she becomes sick. Next, we plan to focus on the probability of transferring to death in the long term. In this section, we plan to change the distribution of the waiting time to death. In chapter 3 and 4, we assume that the waiting time to death for the first time, D_1, is exponentially distributed with death rate $\alpha(z)$. In future work, we plan to compare the probability of transferring to death with four distributions for D_1, i.e., the death rate for individual in a healthy status. The four distributions are exponential, uniform, Weibull and Gompertz distribution based on their hazard function. First De Moivre's Law assumes that the death happens uniformly over the interval of deaths, i.e., the waiting time for the next death to happen is uniformly distributed with density $f_z(z)$ as

$$f_z(z) \;=\; 1/w, \quad for \quad 0<z<w. \tag{5.1}$$

where w is the extreme age, and z here is the age of the insured. The hazard function, $h_z(z)$, will be

$$h_z(z) \;=\; 1/(w-z), \quad for \quad 0<z<w. \tag{5.2}$$

The hazard function shows that the risk to death will increase with age. Second we assume that D_1 has a Weibull distribution with pdf as follows

$$f_z(z) \;=\; kz^n e^{-\frac{kz^{n+1}}{n+1}}, \quad for \quad z>0. \tag{5.3}$$

Similarly, z denotes age here. The hazard function, $h_z(z)$, will be

$$h_z(z) \;=\; kx^n, \quad for \quad z>0. \tag{5.4}$$

The hazard function shows that the risk to death will be linearly related to x^n. Next, we assume that D_1 is Gompertz distributed with pdf as follows

$$f_z(z) \;=\; ae^{bz}e^{-\frac{a}{b}(e^{bz}-1)}, \quad for \quad z>0. \tag{5.5}$$

The hazard function, $h_z(z)$, will be

$$h_z(z) = ae^{bz} \tag{5.6}$$

We know from the hazard function that the mortality risk is exponential increasing with the age function bx.

We assumed in Chapters 3 and 4 that D_1 is exponentially distributed with healthy death rate $\alpha(z)$. The associated hazard function is $\alpha(z)$. Based on the results from chapter 3 and 4, we plan to run the simulation about the transition probability to death based on the four different distributions.

5.2 Discussion of Different Distributions

In chapter 3 and 4, we assume that all the waiting times are exponentially distributed for calculation simplicity. Once we limit the insurance period to one year, it is reasonable to assume that the number of iterations between healthy and disabled states m is equal to 2 based on the results of chapter 3 and 4. Then, we plan to try different distributions for the waiting times.

Appendices

Appendix A

Some Appendix

The following code is for figure 3.4

```
 f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

print(`output redirected...`); # input placeholder

lambda__11 := 0.384e-1;
zlist := [0.1e-2, 0.26e-2, 0.42e-2, 0.57e-2];
A1 := lambda__11+zlist[1];
A2 := lambda__11+zlist[2];
A3 := lambda__11+zlist[3];
A4 := lambda__11+zlist[4];
functionlist := [exp(-A1*t), exp(-A2*t), exp(-A3*t), exp(-A4*t)];
colorlist := [red, blue, purple, yellow];
legendlist := [alpha(z) = 0.1e-2, alpha(z) = 0.26e-2,
alpha(z) = 0.42e-2, alpha(z) = 0.57e-2];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
```

```
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], thickness = 2,
labels = [Time, q__11(t)],
labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.5

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);


P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
 h[1] = 0 .. t);
P(W[1] < 0) := lambda[11]/A;
q12t := P(W[1] < 0)*(P(H__1 < t)-P(H__1+H__2 < t));
g := proc (A, B) options operator, arrow; 0.384e-1*(1-exp(-A*t)
+(A*exp(-B*t)-exp(-A*t)*B-A+B)/(A-B))/A end proc;
g(0.410e-1, 0.677e-1);
g(0.410e-1, .1302);
g(0.410e-1, .2552);
g(0.410e-1, .5052);
functionlist := [g(0.410e-1, 0.677e-1), g(0.410e-1, .1302),
g(0.410e-1, .2552), g(0.410e-1, .5052)];
NULL;
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.625e-1, lambda__2 = .1250,
 lambda__2 = .2500, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], thickness = 2,
labels = [Time, q__12(t)], labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.6

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);


P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
h[1] = 0 .. t);
P(W[1] < 0) := lambda[11]/A;
P(W[2] < 0) := lambda[2]/B;

q13t := `&x`(`&x`(P(W[1] < 0), P(W[2] < 0)), P(H__1+H__2 < t));

A = 0.384e-1+0.26e-2;
B = [0.625e-1+0.52e-2, .125+0.52e-2, .25+0.52e-2, .5+0.52e-2];
g := proc (lambda__2, B) options operator, arrow;
(-1)*0.384e-1*lambda__2*(0.436e-1*exp(-B*t)-exp((-1)*0.436e-1*t)*B
-0.436e-1+B)/(0.436e-1*B*(0.436e-1-B)) end proc;
NULL;
g(0.625e-1, 0.677e-1);
g(.125, .1302);
g(.25, .2552);
g(.5, .5052);
functionlist := [g(0.625e-1, 0.677e-1), g(.125, .1302), g(.25, .2552),
 g(.5, .5052)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.625e-1, lambda__2 = .1250, lambda__2 = .2500,
lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];


plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8], 'location' = bottom],
thickness = 2,
labels = [Time, q__13(t)], labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.8

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
 h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]), h[13] = 0 .. t-h[2]),
 h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
 h[24] = 0 .. t-h[13]), h[13] = 0 .. t);

P(W[1] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

q13t := `&x`(`&x`(P(W[1] > 0), P(W[2] > 0)), [P(H__1+H__2 < t)-P(H__1+H__2+H__3 < t)]);
NULL;
f := proc (A, B) options operator, arrow; -(A*exp(-B*t)
-exp(-A*t)*B-A+B)/(A-B)-(A^2*B*t*exp(-A*t)
-A*B^2*t*exp(-A*t)-A^2*exp(-B*t)
+2*A*B*exp(-A*t)-B^2*exp(-A*t)+A^2-2*A*B+B^2)/(A-B)^2 end proc;
f(0.276e-1, .2552);
f(0.3593e-1, .2552);
f(0.526e-1, .2552);

f(.1026, .2552);
lambda__11 = [1/40.0, 1/30.0, 1/20.0, 1/10.0];
alpha__z = 0.26e-2;
A = [1/40+0.26e-2, 1/30+0.26e-2, 1/20+0.26e-2, 1/10+0.26e-2];
B = .25+0.52e-2;
lambda__2 = .25;
(0.25e-1*.25)/(0.276e-1*.2552);
(0.333e-1*.25)/(0.3593e-1*.2552);
(0.5e-1*.25)/(0.526e-1*.2552);
(.1*.25)/(.1026*.2552);
functionlist := [.8873404207*f(0.276e-1, .2552),
.9079174326*f(0.3593e-1, .2552),
.9312013540*f(0.526e-1, .2552), .9547990492*f(.1026, .2552)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__11 = 0.25e-1, lambda__11 = 0.333e-1,
```

```
lambda__11 = 0.5e-1, lambda__11 = .1];
linestylelist := [solid, dot, dash, dashdot];

plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom],
labels = [Time, q__13(t)], labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.9

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

q13t := `&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
[P(H__1+H__2 < t)-P(H__1+H__2+H__3 < t)]);

f := proc (A, B) options operator, arrow; -(A*exp(-B*t)
-exp(-A*t)*B-A+B)/(A-B)-(A^2*B*t*exp(-A*t)
-A*B^2*t*exp(-A*t)-A^2*exp(-B*t)+2*B*A*exp(-A*t)
-B^2*exp(-A*t)+A^2-2*B*A+B^2)/(A-B)^2 end proc;
NULL;
lambda__11 = 0.5e-1;
alpha__z = 0.26e-2;
A = 0.526e-1;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2, .5+0.52e-2];
alpha__2 = 0.52e-2;
```

```
f(0.526e-1, 0.8853e-1);

f(0.526e-1, .1302);
f(0.526e-1, .2552);
f(0.526e-1, .5052);

(0.5e-1*0.833e-1)/(0.526e-1*0.885e-1);
(0.5e-1*.125)/(0.526e-1*.1302);
(0.5e-1*.25)/(0.526e-1*.2552);
(0.5e-1*.5)/(0.526e-1*.5052);
functionlist := [.8947176215*f(0.526e-1, 0.8853e-1),
.9126059353*f(0.526e-1, .1302),
.9312013540*f(0.526e-1, .2552), .9407861661*f(0.526e-1, .5052)];

colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.83e-1, lambda__2 = .125, lambda__2 = .25,
 lambda__2 = .5];

linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])],
t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], labels = [Time, q__13(t)],
labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
 resolution = 8000000);
```

The following code is for figure 3.10

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
 h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
```

```
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

q14t := `&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
 P(W[3] > 0)), [P(H__1+H__2+H__3 < t)
 -P(H__1+H__2+H__3+H__4 < t)]);
f := proc (A, B) options operator, arrow; (A^2*B*t*exp(-A*t)
-A*B^2*t*exp(-A*t)-A^2*exp(-B*t)+2*B*A*exp(-A*t)
-B^2*exp(-A*t)+A^2-2*B*A+B^2)/(A-B)^2+(A^3*B*exp(-B*t)*t
-A^2*B^2*exp(-B*t)*t+A^2*B^2*exp(-A*t)*t-A*B^3*exp(-A*t)*t
+A^3*exp(-B*t)-3*A^2*B*exp(-B*t)+3*A*B^2*exp(-A*t)-B^3*exp(-A*t)
-A^3+3*A^2*B-3*A*B^2+B^3)/(A-B)^3 end proc;
NULL;
lambda__11 = [0.25e-1, 0.33e-1, 0.5e-1, .1];
alpha__z = 0.26e-2;
A = [0.25e-1+0.26e-2, 0.3333e-1+0.26e-2, 0.5e-1+0.26e-2,
 .1+0.26e-2];
lambda__2 = .25;
B = .25+0.52e-2;
alpha__2 = 0.52e-2;

f(0.276e-1, .2552);

f(0.3593e-1, .2552);
f(0.526e-1, .2552);
f(.1026, .2552);

.25*0.25e-1^2/(.2552*0.276e-1^2);
.25*0.33e-1^2/(.2552*0.3593e-1^2);
.25*0.5e-1^2/(.2552*0.526e-1^2);
.25*.1^2/(.2552*.1026^2);
functionlist := [.8037503811*f(0.276e-1, .2552),
.8263666539*f(0.3593e-1, .2552),
.8851723898*f(0.526e-1, .2552), .9306033618*f(.1026, .2552)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__11 = 0.25e-1, lambda__2 = 0.33e-1,
lambda__2 = 0.5e-1, lambda__2 = .1];
linestylelist := [solid, dot, dash, dashdot];

plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom],
```

```
labels = [Time, q__14(t)], labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
resolution = 8000000);
```

The following code is for figure 3.11

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

q14t := `&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
 P(W[3] > 0)), [P(H__1+H__2+H__3 < t)-P(H__1
 +H__2+H__3+H__4 < t)]);
NULL;
f := proc (A, B) options operator, arrow; (A^2*B*t*exp(-A*t)
-A*B^2*t*exp(-A*t)-A^2*exp(-B*t)
+2*B*A*exp(-A*t)-B^2*exp(-A*t)+A^2-2*B*A+B^2)/(A-B)^2
+(A^3*B*exp(-B*t)*t-A^2*B^2*exp(-B*t)*t+A^2*B^2*exp(-A*t)*t-A*B^3*exp(-A*t)*t
+A^3*exp(-B*t)-3*A^2*B*exp(-B*t)+3*A*B^2*exp(-A*t)-B^3*exp(-A*t)-A^3
+3*A^2*B-3*A*B^2+B^3)/(A-B)^3 end proc;
lambda__11 = 0.5e-1;
alpha__z = 0.26e-2;
A = 0.526e-1;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2,
 1/2.0+0.52e-2];
alpha__2 = 0.52e-2;

f(0.526e-1, 0.8853e-1);
```

```
f(0.526e-1, .1302);
f(0.526e-1, .2552);
f(0.526e-1, .5052);

0.8333e-1*0.5e-1^2/(0.8853e-1*0.526e-1^2);
.125*0.5e-1^2/(.1302*0.526e-1^2);
.25*0.5e-1^2/(.2552*0.526e-1^2);
.5*0.5e-1^2/(.5052*0.526e-1^2);
functionlist := [.8505100268*f(0.526e-1, 0.8853e-1),
.8674961362*f(0.526e-1, .1302), .8851723898*f(0.526e-1, .2552), .8942834278*f(0.526e-1, .5052)];

colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.833e-1, lambda__2 = .125,
lambda__2 = .25, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];

plot([seq(functionlist[k], k = [1, 2, 3, 4])],
 t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k],
k = [1, 2, 3, 4])], 'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], labels = [Time, q__14(t)],
 labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
 resolution = 8000000);
```

The following code is for figure 3.12

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
 h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
```

```
P(W[2] > 0) := lambda[2]/B;

P(W[4] > 0) := lambda[2]/B;
q15t := `&x`(`&x`(`&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
 P(W[3] > 0)), P(W[4] > 0)),
[P(H__1+H__2+H__3+H__4 < t)]);

f := proc (A, B) options operator, arrow; -(A^3*B*exp(-B*t)*t
-A^2*B^2*exp(-B*t)*t
+A^2*B^2*exp(-A*t)*t-A*B^3*exp(-A*t)*t+A^3*exp(-B*t)
-3*A^2*B*exp(-B*t)+3*A*B^2*exp(-A*t)
-B^3*exp(-A*t)-A^3+3*A^2*B-3*A*B^2+B^3)/(A-B)^3 end proc;
NULL;
lambda__11 = [0.25e-1, 0.3333e-1, 0.5e-1, .1];
alpha__z = 0.26e-2;
A = [0.25e-1+0.26e-2, 0.3333e-1+0.26e-2, 0.5e-1+0.26e-2,
.1+0.26e-2];
lambda__2 = .25;
B = .2552;
f(0.276e-1, .2552);
f(0.3593e-1, .2552);
f(0.526e-1, .2552);
f(.1026, .2552);
0.25e-1^2*.25^2/(0.276e-1^2*.2552^2);
0.3333e-1^2*.25^2/(0.3593e-1^2*.2552^2);
0.5e-1^2*.25^2/(0.526e-1^2*.2552^2);
.1^2*.25^2/(.1026^2*.2552^2);
functionlist := [.7873730222*f(0.276e-1, .2552),
.8257999841*f(0.3593e-1, .2552), .8671359617*f(0.526e-1, .2552),
.9116412243*f(.1026, .2552)];

colorlist := [red, blue, purple, yellow];
legendlist := [lambda__11 = 0.25e-1, lambda__11 = 0.333e-1,
 lambda__11 = 0.5e-1, lambda__11 = .1];

linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k],
 k = [1, 2, 3, 4])], 'legendstyle' = ['font' = [TIMES, ROMAN, 8],
 'location' = bottom], labels = [Time, q__15(t)],
  labeldirections = ["horizontal", "vertical"],
  'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
  resolution = 8000000);
```

The following code is for figure 3.13

```
f(h[1]) := A*exp(-A*h[1]);
```

```
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
 h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

P(W[4] > 0) := lambda[2]/B;
q15t := `&x`(`&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
P(W[3] > 0)), P(W[4] > 0)), [P(H__1+H__2+H__3+H__4 < t)]);

NULL;
f := proc (A, B) options operator, arrow; -(A^3*B*exp(-B*t)*t
-A^2*B^2*exp(-B*t)*t+A^2*B^2*exp(-A*t)*t-A*B^3*exp(-A*t)*t
+A^3*exp(-B*t)-3*A^2*B*exp(-B*t)+3*A*B^2*exp(-A*t)-B^3*exp(-A*t)
-A^3+3*A^2*B-3*A*B^2+B^3)/(A-B)^3 end proc;
lambda__11 = 0.5e-1;
alpha__z = 0.26e-2;
A = 0.526e-1;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2,
1/2.0+0.52e-2];
f(0.526e-1, 0.8853e-1);
f(0.526e-1, .1302);
f(0.526e-1, .2552);
f(0.526e-1, .5052);
0.5e-1^2*0.833e-1^2/(0.526e-1^2*0.8853e-1^2);
0.5e-1^2*.125^2/(0.526e-1^2*.1302^2);
0.5e-1^2*.25^2/(0.526e-1^2*.2552^2);
0.5e-1^2*.5^2/(0.526e-1^2*.5052^2);
functionlist := [.7999771728*f(0.526e-1, 0.8853e-1),
.8328495931*f(0.526e-1, .1302),
.8671359617*f(0.526e-1, .2552), .8850786103*f(0.526e-1, .5052)];
```

```
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.833e-1, lambda__2 = .125,
lambda__2 = .25, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k],
k = [1, 2, 3, 4])], 'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], labels = [Time, q__15(t)],
labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
 resolution = 8000000);
```

The following code is for figure 3.15

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
 h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
 h[24] = 0 .. t-h[13]), h[13] = 0 .. t);

P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
 h[24] = 0 .. t-h[135]), h[135] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
h[246] = 0 .. t-h[135]), h[135] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

P(W[4] > 0) := lambda[2]/B;
P(W[6] > 0) := lambda[2]/B;
P(W[5] > 0) := lambda[11]/A;
q15t := `&x`(`&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
P(W[3] > 0)), P(W[4] > 0)), [P(H__1+H__2+H__3+H__4 < t)
```

```
-P(H__1+H__2+H__3+H__4+H__5 < t)]);
f := proc (A, B) options operator, arrow; -(A^3*B*exp(-B*t)*t
-A^2*B^2*exp(-B*t)*t+A^2*B^2*exp(-A*t)*t-A*B^3*exp(-A*t)*t
+A^3*exp(-B*t)-3*A^2*B*exp(-B*t)+3*A*B^2*exp(-A*t)
-B^3*exp(-A*t)-A^3+3*A^2*B-3*A*B^2+B^3)/(A-B)^3
+(1/2)*(A^4*B^2*exp(-A*t)*t^2-2*A^3*B^3*exp(-A*t)*t^2
+A^2*B^4*exp(-A*t)*t^2+2*A^4*B*exp(-B*t)*t
-2*A^3*B^2*exp(-B*t)*t
+6*A^3*B^2*exp(-A*t)*t-8*A^2*B^3*exp(-A*t)*t
+2*A*B^4*exp(-A*t)*t+2*A^4*exp(-B*t)-8*A^3*B*exp(-B*t)
+12*A^2*B^2*exp(-A*t)
-8*A*B^3*exp(-A*t)+2*B^4*exp(-A*t)-2*A^4+8*A^3*B-12*A^2*B^2
+8*A*B^3-2*B^4)/(A-B)^4 end proc;
lambda__11 = [1/20.0, 1/10.0, 1/5.0, 1/2.0];
alpha__z = 0.26e-2;
A = [1/20.0+0.26e-2, 1/10.0+0.26e-2, 1/5.0+0.26e-2, 1/2.0+0.26e-2];

NULL;
lambda__2 = .25;

B = .25+0.52e-2;
f(0.526e-1, .2552);
f(.1026, .2552);
f(.2026, .2552);
f(.5026, .2552);
0.5e-1^2*.25^2/(0.526e-1^2*.2552^2);
.1^2*.25^2/(.1026^2*.2552^2);
.2^2*.25^2/(.2026^2*.2552^2);
.5^2*.25^2/(.5026^2*.2552^2);
functionlist := [.8671359617*f(0.526e-1, .2552),
.9116412243*f(.1026, .2552),
.9351898543*f(.2026, .2552), .9497596557*f(.5026, .2552)];
colorlist := [red, blue, purple, yellow];

legendlist := [lambda__11 = 0.5e-1, lambda__11 = .1,
lambda__11 = .2, lambda__11 = .5];
linestylelist := [solid, dot, dash, dashdot];



plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k], k = [1, 2, 3, 4])],
'legend' = [seq(legendlist[k],
k = [1, 2, 3, 4])], 'legendstyle' = ['font' = [TIMES, ROMAN, 8],
'location' = bottom], labels = [Time, q__15(t)],
labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
```

```
 resolution = 8000000);
```

The following code is for figure 3.16

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);

P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);

P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
h[24] = 0 .. t-h[135]), h[135] = 0 .. t);


P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
 h[246] = 0 .. t-h[135]), h[135] = 0 .. t);

P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

P(W[4] > 0) := lambda[2]/B;
P(W[6] > 0) := lambda[2]/B;
P(W[5] > 0) := lambda[11]/A;
q15t := `&x`(`&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)),
P(W[3] > 0)), P(W[4] > 0)), [P(H__1+H__2+H__3+H__4 < t)
-P(H__1+H__2+H__3+H__4+H__5 < t)]);
f := proc (A, B) options operator, arrow; -(A^3*B*exp(-B*t)*t
-A^2*B^2*exp(-B*t)*t+A^2*B^2*exp(-A*t)*t
-A*B^3*exp(-A*t)*t+A^3*exp(-B*t)-3*A^2*B*exp(-B*t)
+3*A*B^2*exp(-A*t)-B^3*exp(-A*t)-A^3+3*A^2*B-3*A*B^2
+B^3)/(A-B)^3+(1/2)*(A^4*B^2*exp(-A*t)*t^2
-2*A^3*B^3*exp(-A*t)*t^2+A^2*B^4*exp(-A*t)*t^2
+2*A^4*B*exp(-B*t)*t-2*A^3*B^2*exp(-B*t)*t+6*A^3*B^2*exp(-A*t)*t
-8*A^2*B^3*exp(-A*t)*t+2*A*B^4*exp(-A*t)*t
```

```
+2*A^4*exp(-B*t)-8*A^3*B*exp(-B*t)
+12*A^2*B^2*exp(-A*t)-8*A*B^3*exp(-A*t)
+2*B^4*exp(-A*t)-2*A^4+8*A^3*B-12*A^2*B^2
+8*A*B^3-2*B^4)/(A-B)^4 end proc;
NULL;

lambda__11 = .2;
alpha__z = 0.26e-2;
A = .2026;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2,
 1/2.0+0.52e-2];
f(.2026, 0.8853e-1);
f(.2026, .1302);
f(.2026, .2552);
f(.2026, .5052);
.2^2*0.8333333333e-1^2/(.2026^2*0.8853e-1^2);
.2^2*.125^2/(.2026^2*.1302^2);
.2^2*.25^2/(.2026^2*.2552^2);
.2^2*.5^2/(.2026^2*.5052^2);
functionlist := [.8634509842*f(.2026, 0.8853e-1),
.8982126493*f(.2026, .1302),
.9351898543*f(.2026, .2552), .9545406634*f(.2026, .5052)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.833e-1, lambda__2 = .125,
 lambda__2 = .25, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
 'colour' = [seq(colorlist[k],
k = [1, 2, 3, 4])], 'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8], 'location' = bottom],
labels = [Time, q__15(t)], labeldirections = ["horizontal", "vertical"],
linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
resolution = 8000000);
```

The following code is for figure 3.17

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);
```

```
P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);

P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
 h[24] = 0 .. t-h[135]), h[135] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
 h[246] = 0 .. t-h[135]), h[135] = 0 .. t);
P(W[1] > 0) := lambda[11]/A;
P(W[3] > 0) := lambda[11]/A;
P(W[2] > 0) := lambda[2]/B;

P(W[4] > 0) := lambda[2]/B;
P(W[6] > 0) := lambda[2]/B;
P(W[5] > 0) := lambda[11]/A;
q16t := `&x`(`&x`(`&x`(`&x`(`&x`(P(W[1] > 0), P(W[2] > 0)), P(W[3] > 0)),
P(W[4] > 0)), P(W[5] > 0)), [P(H__1+H__2+H__3+H__4+H__5 < t)
-P(H__1+H__2+H__3+H__4+H__5+H__6 < t)]);
f := proc (A, B) options operator, arrow; -(1/2)*(A^4*B^2*exp(-A*t)*t^2
-2*A^3*B^3*exp(-A*t)*t^2+A^2*B^4*exp(-A*t)*t^2
+2*A^4*B*exp(-B*t)*t-2*A^3*B^2*exp(-B*t)*t+6*A^3*B^2*exp(-A*t)*t
-8*A^2*B^3*exp(-A*t)*t+2*A*B^4*exp(-A*t)*t
+2*A^4*exp(-B*t)-8*A^3*B*exp(-B*t)+12*A^2*B^2*exp(-A*t)-8*A*B^3*exp(-A*t)
+2*B^4*exp(-A*t)-2*A^4+8*A^3*B-12*A^2*B^2+8*A*B^3-2*B^4)/(A-B)^4
+(1/2)*(A^5*B^2*exp(-B*t)*t^2-A^4*B^3*exp(-A*t)*t^2-2*A^4*B^3*exp(-B*t)*t^2
+2*A^3*B^4*exp(-A*t)*t^2+A^3*B^4*exp(-B*t)*t^2-A^2*B^5*exp(-A*t)*t^2
+2*A^5*B*exp(-B*t)*t-10*A^4*B^2*exp(-B*t)*t-8*A^3*B^3*exp(-A*t)*t
+8*A^3*B^3*exp(-B*t)*t+10*A^2*B^4*exp(-A*t)*t-2*A*B^5*exp(-A*t)*t
+2*A^5*exp(-B*t)-10*A^4*B*exp(-B*t)+20*A^3*B^2*exp(-B*t)-20*A^2*B^3*exp(-A*t)
+10*A*B^4*exp(-A*t)-2*B^5*exp(-A*t)-2*A^5+10*A^4*B-20*A^3*B^2+20*A^2*B^3-10*A*B^4+2*B^5)/(A-B)^5 end proc;

NULL;

lambda__11 = [0.5e-1, .1, .2, .5];
alpha__z = 0.26e-2;
A = [0.5e-1+0.26e-2, .1+0.26e-2, .2+0.26e-2, .5+0.26e-2];
lambda__2 = .25;
B = .2552;
f(0.526e-1, .2552);
f(.1026, .2552);
f(.2026, .2552);
f(.5026, .2552);
```

```
0.5e-1^3*.25^2/(0.526e-1^3*.2552^2);
.1^3*.25^2/(.1026^3*.2552^2);
.2^3*.25^2/(.2026^3*.2552^2);
.5^3*.25^2/(.5026^3*.2552^2);
functionlist := [.8242737280*f(0.526e-1, .2552), .8885392050*f(.1026, .2552),
.9231884049*f(.2026, .2552), .9448464538*f(.5026, .2552)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__11 = 0.5e-1, lambda__11 = .1, lambda__11 = .2, lambda__11 = .5];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
 'colour' = [seq(colorlist[k],
 k = [1, 2, 3, 4])], 'legend' = [seq(legendlist[k],
 k = [1, 2, 3, 4])],
  'legendstyle' = ['font' = [TIMES, ROMAN, 8],
  'location' = bottom], labels = [Time, q__16(t)],
  labeldirections = ["horizontal", "vertical"], 'linestyle' = [seq(linestylelist[k],
   k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.18

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);


P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]), h[2] = 0 .. t-h[1]),
 h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]), h[13] = 0 .. t-h[2]),
 h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
 h[24] = 0 .. t-h[135]), h[135] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
h[246] = 0 .. t-h[135]), h[135] = 0 .. t);

q16t := P(H__1+H__2+H__3+H__4+H__5 < t)
-P(H__1+H__2+H__3+H__4+H__5+H__6 < t);
f := proc (A, B) options operator, arrow;
-(1/2)*(A^4*B^2*exp(-A*t)*t^2-2*A^3*B^3*exp(-A*t)*t^2
```

```
+A^2*B^4*exp(-A*t)*t^2+2*A^4*B*exp(-B*t)*t-2*A^3*B^2*exp(-B*t)*t
+6*A^3*B^2*exp(-A*t)*t-8*A^2*B^3*exp(-A*t)*t
+2*A*B^4*exp(-A*t)*t+2*A^4*exp(-B*t)-8*A^3*B*exp(-B*t)
+12*A^2*B^2*exp(-A*t)-8*A*B^3*exp(-A*t)
+2*B^4*exp(-A*t)-2*A^4+8*A^3*B-12*A^2*B^2
+8*A*B^3-2*B^4)/(A-B)^4
+(1/2)*(A^5*B^2*exp(-B*t)*t^2-2*A^4*B^3*exp(-B*t)*t^2
-A^4*B^3*exp(-A*t)*t^2+A^3*B^4*exp(-B*t)*t^2
+2*A^3*B^4*exp(-A*t)*t^2-A^2*B^5*exp(-A*t)*t^2
+2*A^5*B*exp(-B*t)*t-10*A^4*B^2*exp(-B*t)*t
+8*A^3*B^3*exp(-B*t)*t
-8*A^3*B^3*exp(-A*t)*t+10*A^2*B^4*exp(-A*t)*t
-2*A*B^5*exp(-A*t)*t+2*A^5*exp(-B*t)-10*A^4*B*exp(-B*t)
+20*A^3*B^2*exp(-B*t)-20*A^2*B^3*exp(-A*t)
+10*A*B^4*exp(-A*t)-2*B^5*exp(-A*t)-2*A^5
+10*A^4*B-20*A^3*B^2
+20*A^2*B^3-10*A*B^4+2*B^5)/(A-B)^5 end proc;
lambda__11 = .2;
alpha__z = 0.26e-2;
A = .2026;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2, 1/2.0+0.52e-2];
f(.2026, 0.8853e-1);
f(.2026, .1302);
f(.2026, .2552);
f(.2026, .5052);
.2^3*0.8333e-1^2/(.2026^3*0.88533e-1^2);
.2^3*.125^2/(.2026^3*.1302^2);
.2^3*.25^2/(.2026^3*.2552^2);
.2^3*.5^2/(.2026^3*.5052^2);
functionlist := [.8522442228*f(.2026, 0.8853e-1),
.8866857348*f(.2026, .1302),
 .9231884049*f(.2026, .2552), .9422908823*f(.2026, .5052)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.833e-1, lambda__2 = .125,
lambda__2 = .25, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];
plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
'colour' = [seq(colorlist[k],
 k = [1, 2, 3, 4])], 'legend' = [seq(legendlist[k],
 k = [1, 2, 3, 4])],
 'legendstyle' = ['font' = [TIMES, ROMAN, 8], 'location' = bottom],
 labels = [Time, q__16(t)], labeldirections = ["horizontal", "vertical"],
 'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```

The following code is for figure 3.19

```
f(h[1]) := A*exp(-A*h[1]);
```

```
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);


P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
 h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
 h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
 h[24] = 0 .. t-h[13]), h[13] = 0 .. t);

P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
 h[24] = 0 .. t-h[135]), h[135] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
 h[246] = 0 .. t-h[135]), h[135] = 0 .. t);
q17t := P(H__1+H__2+H__3+H__4+H__5+H__6 < t);
f := proc (A, B) options operator, arrow;
-(1/2)*(A^5*B^2*exp(-B*t)*t^2-A^4*B^3*exp(-A*t)*t^2
-2*A^4*B^3*exp(-B*t)*t^2+2*A^3*B^4*exp(-A*t)*t^2
+A^3*B^4*exp(-B*t)*t^2-A^2*B^5*exp(-A*t)*t^2
+2*A^5*B*exp(-B*t)*t-10*A^4*B^2*exp(-B*t)*t
-8*A^3*B^3*exp(-A*t)*t+8*A^3*B^3*exp(-B*t)*t
+10*A^2*B^4*exp(-A*t)*t-2*A*B^5*exp(-A*t)*t
+2*A^5*exp(-B*t)-10*A^4*B*exp(-B*t)+20*A^3*B^2*exp(-B*t)
-20*A^2*B^3*exp(-A*t)+10*A*B^4*exp(-A*t)
-2*B^5*exp(-A*t)-2*A^5+10*A^4*B-20*A^3*B^2+20*A^2*B^3
-10*A*B^4+2*B^5)/(A-B)^5 end proc;

lambda__11 = [0.5e-1, .1, .2, .5];
alpha__z = 0.26e-2;
A = [0.5e-1+0.26e-2, .1+0.26e-2, .2+0.26e-2, .5+0.26e-2];
lambda__2 = .25;
B = .2552;
f(0.526e-1, .2552);
f(.1026, .2552);
f(.2026, .2552);
f(.5026, .2552);
0.5e-1^3*.25^3/(0.526e-1^3*.2552^3);
.1^3*.25^3/(.1026^3*.2552^3);
```

```
.2^3*.25^3/(.2026^3*.2552^3);
.5^3*.25^3/(.5026^3*.2552^3);
functionlist := [.80747818170*f(0.526e-1, .2552),
.8704341741*f(.1026, .2552), .9043773555*f(.2026, .2552),
.9255940965*f(.5026, .2552)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__11 = 0.5e-1, lambda__11 = .1,
lambda__11 = .2, lambda__11 = .5];
linestylelist := [solid, dot, dash, dashdot];

plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
 'colour' = [seq(colorlist[k],
k = [1, 2, 3, 4])], 'legend' = [seq(legendlist[k],
k = [1, 2, 3, 4])],
'legendstyle' = ['font' = [TIMES, ROMAN, 8], 'location' = bottom],
labels = [Time, q__17(t)],
labeldirections = ["horizontal", "vertical"],
'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])],
 resolution = 8000000);
```

The following code is for figure 3.20

```
f(h[1]) := A*exp(-A*h[1]);
f(h[2]) := B*exp(-B*h[2]);
f(h[3]) := A*exp(-A*h[3]);
f(h[4]) := B*exp(-B*h[4]);
f(h[5]) := A*exp(-A*h[5]);
f(h[6]) := B*exp(-B*h[6]);
f(h[13]) := A^2*h[13]*exp(-A*h[13]);
f(h[24]) := B^2*h[24]*exp(-B*h[24]);
f(h[135]) := (1/2)*A^3*h[135]^2*exp(-A*h[135]);
f(h[246]) := (1/2)*B^3*h[246]^2*exp(-B*h[246]);
P(H__1 < t) := int(f(h[1]), h[1] = 0 .. t);
P(H__1+H__2 < t) := int(int(f(h[1])*f(h[2]),
h[2] = 0 .. t-h[1]), h[1] = 0 .. t);
P(H__1+H__2+H__3 < t) := int(int(f(h[2])*f(h[13]),
h[13] = 0 .. t-h[2]), h[2] = 0 .. t);
P(H__1+H__2+H__3+H__4 < t) := int(int(f(h[13])*f(h[24]),
h[24] = 0 .. t-h[13]), h[13] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5 < t) := int(int(f(h[135])*f(h[24]),
 h[24] = 0 .. t-h[135]), h[135] = 0 .. t);
P(H__1+H__2+H__3+H__4+H__5+H__6 < t) := int(int(f(h[135])*f(h[246]),
 h[246] = 0 .. t-h[135]), h[135] = 0 .. t);
f := proc (A, B) options operator, arrow;
-(1/2)*(A^5*B^2*exp(-B*t)*t^2-2*A^4*B^3*exp(-B*t)*t^2
-A^4*B^3*exp(-A*t)*t^2+A^3*B^4*exp(-B*t)*t^2
+2*A^3*B^4*exp(-A*t)*t^2-A^2*B^5*exp(-A*t)*t^2
```

```
+2*A^5*B*exp(-B*t)*t-10*A^4*B^2*exp(-B*t)*t
+10*A^2*B^4*exp(-A*t)*t-2*A*B^5*exp(-A*t)*t
+2*A^5*exp(-B*t)-10*A^4*B*exp(-B*t)
+20*A^3*B^2*exp(-B*t)-20*A^2*B^3*exp(-A*t)
+10*A*B^4*exp(-A*t)
-2*B^5*exp(-A*t)-2*A^5+10*A^4*B-20*A^3*B^2
+20*A^2*B^3-10*A*B^4+2*B^5)/(A-B)^5 end proc;

lambda__11 = .2;
alpha__z = 0.26e-2;
A = .2026;
lambda__2 = [1/12.0, 1/8.0, 1/4.0, 1/2.0];
B = [1/12.0+0.52e-2, 1/8.0+0.52e-2, 1/4.0+0.52e-2,
1/2.0+0.52e-2];
f(.2026, 0.8853e-1);
f(.2026, .1302);
f(.2026, .2552);
f(.2026, .5052);
.2^3*0.8333e-1^3/(.2026^3*0.88533e-1^3);
.2^3*.125^3/(.2026^3*.1302^3);
.2^3*.25^3/(.2026^3*.2552^3);
.2^3*.5^3/(.2026^3*.5052^3);
functionlist := [.8021586423*f(.2026, 0.8853e-1),
 .8512727868*f(.2026, .1302),
.9043773555*f(.2026, .2552), .9325919264*f(.2026, .5052)];
colorlist := [red, blue, purple, yellow];
legendlist := [lambda__2 = 0.833e-1, lambda__2 = .125,
lambda__2 = .25, lambda__2 = .5];
linestylelist := [solid, dot, dash, dashdot];

plot([seq(functionlist[k], k = [1, 2, 3, 4])], t = 1 .. 53,
 'colour' = [seq(colorlist[k],
 k = [1, 2, 3, 4])], 'legend' = [seq(legendlist[k], k = [1, 2, 3, 4])],
 'legendstyle' = ['font' = [TIMES, ROMAN, 8], 'location' = bottom],
 labels = [Time, q__17(t)], labeldirections = ["horizontal", "vertical"],
 'linestyle' = [seq(linestylelist[k], k = [1, 2, 3, 4])], resolution = 8000000);
```