An Analysis of the Three Body Problem

by

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(Under the direction of Dr. Malcolm Adams)

Abstract

In this paper we will be presenting my approach to generating two possible solutions to the three-body problem. We will first discuss the theory of reducing the dimension of the dynamical system through the use of symmetry and constants of motion. However, we can reduce no farther than to 5 degrees of freedom, which is still too high to solve and therefore we must change our attack from trying to find a general solution, to simply finding a specific solution.

We will then make use of the program Maple, which will allow me to create a simulation of this system. Through the simulations we will try and find the stable figure eight solution as well as a solution having a light satellite coming in from infinity and being captured in a stable orbit around a tight binary. We can adjust the initial conditions of the three bodies until we have generated a simulation that has the potential to be a stable periodic orbit. Unfortunately Maple will not prove that any solution found is stable, and we are therefore left with only a conjecture as to the stability of the system.

Index words: Celestial Mechanics, n-Body Problem, Three-Body Problem, 3-Body Problem

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Table of Contents

Page

Chapter 1

A Very Brief History of the n-Body Problem

Since mankind first looked to the stars in the night sky, we have wondered about our place in the universe. The first attempts to explain our place held that the Earth was the center of the universe, and that everything else in creation spun about the Earth. Despite the modern knowledge of the incorrectness of this belief, at the time it was understandable. Looking up one could see the stars spin overhead, however, no motion was detectable standing on the Earth, and so the belief lasted.

With the introduction of the Christian religion, and the idea of a divine God who had created all of the matter in the universe, the idea persisted. If God had created the universe, and with it mankind, and since mankind lived on Earth, then clearly God must have created the Earth to be the center of the Universe. This belief was championed by Ptolemy who argued that space was a sphere which rotated around the Earth, which was unmoving. The Catholic church accepted this belief, and vigorously defended it for centuries.

It wasn't until the 1500's that the idea that the universe revolved around the Earth began to change. In the year 1530 a.d. the great Polish astronomer, and cleric, Nicolas Copernicus published the *De Revolutionibus*. In this work Copernicus lays the foundation for modern Astronomy. In the book he claims that, despite the beliefs of the church, the Earth in fact spins on its axis, completing a rotation once a day, and beyond that, the Sun is actually the center of the Solar System, and the Earth revolves around the Sun. Though Copernicus did not suffer the retribution of the Catholic church, many other astronomers of this time that tried to build off the work that he did came into conflict with the church over this idea. It wasn't until the early 1600's that the work done by Copernicus was expanded mathematically.

In 1571 Johannes Kepler was born in what at the time was the Holy Roman Empire. Kepler was a bright, though sickly, child who eventually attended the University of Tübingen to become a Lutheran minister. However, this did not work out for Kepler, and instead he became a mathematics professor in Graz. It was while he was working there in 1596 that he wrote his defense of Copernicus's work. When the Counter-Reformation began, Kepler was forced to leave Graz, and he then moved to Prague, where he teamed up to work with Tycho Brahe. It was here that Kepler's biggest contribution to astronomy occurred.

When Tycho Brahe died in 1601 Kepler replaced him as the Imperial Mathematician. Tycho Brahe had collected very precise data about the orbital path of the planet Mars, and Kepler used this data to determine that Mars' orbit was, in fact, an ellipse. This lead him to publish *Astronomia Nova* in 1609. In this work Kepler laid down what have become known as Kepler's first and second laws of planetary motion. Beyond the two laws of motion, this work is important for another reason. This is the first known published account where a scientist has taken the imperfect data of the real world and converted it into a scientific theorem of exceptional accuracy. After yet another forced move, this time to Linz in 1612, Kepler published *Harmonices Mundi* in which he laid out his third law of planetary motion.

Kepler's first law lays out the orbital patterns of the planets and states that

> *The planets move in ellipses, with the sun at one focus.*

With his second law Kepler discusses the area swept out by a line segment running from the sun to the planet described. It is stated as

> *Equal areas are swept out by the radius vector in equal times.*

More precisely this statement means,

The area swept out in time t is proportional to t.

Finally if a is the length of the major axis of a planet's elliptical orbit, and T is the period of the planet's orbit, then Kepler's third law states

The ratio of a^3/T^2 is the same for all planets.

It would be almost 60 years before the advances made by Kepler could be furthered, and it would take one of the greatest mathematical minds of all time to do it. In 1666 Sir Isaac Newton had begun to develop the basis for his three laws of motion, but it was also at this time that Newton came across the work done by Kepler. Newton's first major accomplishment was to demonstrate that a force directed towards the Sun, now known to be the gravitational pull of the Sun, perfectly explains Kepler's laws. This force directed toward the Sun, he concluded, was proportional to the mass of the object, and inversely proportional to the square of the distance between the Sun and the planet. This explanation leads to the second order differential equation in $\mathbf{R}^3$ which is known as Kepler's problem:

$$\frac{d^2\vec{q}}{dt^2} = \frac{-k\vec{q}}{|\vec{q}|^3}, \tag{1.1}$$

where $\vec{q}$ is the position vector in $\mathbf{R}^3$. Newton proved this equation has as its solutions paths that are conic sections, ie. circles, ellipses, or hyperbolae.

Unfortunately, despite its apparent success in predicting the motion of the planets in our solar system, the equation given by Kepler ignores many influences on the paths of the planets. One of the obvious problems of this equation is that it does not take into account the gravitational pull of any of the other planets in our solar system. In this sense the equation does not give us a true sense of the paths of the planets, but instead it generates an ideal orbit. In order to solve the true equations for our solar system, one would have to generate equations that account for all of the masses in the solar system, and then to try and solve these equations. This leads us to the n-body problem.

The n-body problem is a system of equations that would account for the gravitational pull of n bodies. Unfortunately for us, even with modern problem solving techniques, and relatively fast computers it is impossible to solve the n-body problem for just 3 planets[1]. However, despite not being able to write down a general solution to the 3-body problem, much work has been done on trying to find specific solutions to the three body problem.

In volume 81 of *Contemporary Mathematics* published in 1988 Richard Moeckel lays out a solid basis of facts about the 3-body problem in a paper entitled **Some Qualitative Features of the Three-Body Problem**[6]. Thirteen years later Richard Montgomery published a paper [7] in the *Notices of the AMS* in which he lays out a proof for the existence of a specific solution of the 3-body problem in which three planets of equal mass chase each other around a figure 8. Each planet is exactly one third of a period of revolution behind the planet in front of it. This solution is proven to be stable, and is in fact KAM-stable. It was this article which got me started at looking at the 3-body problem.

The 3-body problem is a system of equations used to determine the path of three celestial objects, as they are attracted to each other by gravity. The equations can all be derived similarly to the one proposed by Kepler, but now instead of just one equation there will be three, one for each planet. Also instead of there just being one term in each equation there will now be two terms, one for the pull from each planet. If we let m_1 be the mass of the first planet, m_2 be the mass of the second planet, and m_3 be the mass of the third planet, and if we let $\vec{q_1}$ be the position vector of the first planet in 3-space, $\vec{q_2}$ be the position vector of the second planet, and $\vec{q_3}$ be the

[1]This fact was proven by Poincaré in his paper [10]

position vector of the third planet, we can begin to write down the equations for the motion of the planets. As in the two body system, the gravitational attraction of each planet will generate a force directed in towards the center of that planet. This is just as in the 2-body case. Thus we see that the force acting on planet 1 is given by

$$m_1 \frac{d^2\vec{q_1}}{dt^2} = -k\left(\frac{m_1 m_2(\vec{q_1}-\vec{q_2})}{|\vec{q_1}-\vec{q_2}|^3} + \frac{m_1 m_3(\vec{q_1}-\vec{q_3})}{|\vec{q_1}-\vec{q_3}|^3}\right) \tag{1.2}$$

where k is a constant. In a similar fashion we can construct the force equations for the other two planets and they are

$$m_2 \frac{d^2\vec{q_2}}{dt^2} = -k\left(\frac{m_2 m_3(\vec{q_2}-\vec{q_3})}{|\vec{q_2}-\vec{q_3}|^3} + \frac{m_2 m_1(\vec{q_2}-\vec{q_1})}{|\vec{q_2}-\vec{q_1}|^3}\right) \tag{1.3}$$

and

$$m_3 \frac{d^2\vec{q_3}}{dt^2} = -k\left(\frac{m_3 m_1(\vec{q_3}-\vec{q_1})}{|\vec{q_3}-\vec{q_1}|^3} + \frac{m_3 m_2(\vec{q_3}-\vec{q_2})}{|\vec{q_3}-\vec{q_2}|^3}\right) \tag{1.4}$$

Where again k is simply a proportionality constant.

At this point we now have the three equations that we would like to solve. However, each one of these equations is, in fact, an equation of 3 further variables. Thus instead of only having to solve a system of 3 second order differential equations, in reality we have to solve a system of 9 second order differential equations. Since these equations are second order, this system is equivalent to a system of eighteen first order differential equations. Because of this, the difficulty factor in solving for the paths of just three planet's orbits has increased dramatically. Instead of finding a general solution to this problem all that can be done is to find specific solutions to the three body problem. Imagine, now, the difficulty in solving the n-body problem that would explain the paths of our solar system.

Chapter 2

Known Results of the Three Body Problem

I will make an attempt in this report to demonstrate some basic solutions of the three body problem. To do this I will make heavy use of the program *Maple* to both set up, and to solve the differential equations defining the problem. I will also discuss the difficulties inherent in trying to prove the existence of these solution.

In the previous chapter I gave a brief discussion of how hard it is to solve the three-body problem. Instead what we can do is to find specific solutions to the three-body problem, with certain initial assumptions. The main assumption that we will make in order to proceed is that the motion of the three bodies is planar.

In order to proceed mathematically we will need to define the notation we will use throughout the rest of this paper. The first thing we will define is the position vector. This will be defined by letting $\vec{q_j}$ be the position of the j^{th} point mass in planar space. We will next define $\vec{p_j}$ to be the momentum vector for the point mass, again in a planar space. And finally we will define $m_j \epsilon\ \mathbf{R}^+$ to be the mass of the j^{th} body.

If we now define $\vec{q} = (\vec{q}_1, \vec{q}_2, \vec{q}_3)$, let $\vec{p} = (\vec{p}_1, \vec{p}_2, \vec{p}_3)$ where both $\vec{p}, \vec{q}\ \epsilon \mathbf{R}^6$ and finally define M = diag$(m_1, m_1, m_2, m_2, m_3, m_3)$, which is a six by six matrix with zeroes in every entry except through the main diagonal whose entries are given. We can now define the energy function of our dynamical system:

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\vec{p}^{\ T} M^{-1} \vec{p} - U(\vec{q}). \tag{2.1}$$

In the above equation we have defined the potential energy of the system to be $-U(\vec{q})$. This function $U(\vec{q})$ is defined by

$$U(\vec{q}) = \frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2|} + \frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|} + \frac{m_2 m_3}{|\vec{q}_2 - \vec{q}_3|}. \tag{2.2}$$

This energy function $H(\vec{q}, \vec{p})$ is the Hamiltonian, meaning that we can take the partial derivatives of the function in order to generate the system of dynamical equations that we will use to try and find a solution to the three-body problem.

Since the equation is Hamiltonian we know that

$$\frac{\partial H}{\partial p} = \frac{dq}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial q} = \frac{dp}{dt} \tag{2.3}$$

This leads to the following two differential equations.

$$\begin{aligned} \dot{q} &= M^{-1}\vec{p} \\ \dot{p} &= \nabla U(\vec{q}) \end{aligned} \tag{2.4}$$

Unfortunately the two differential equations above generate a dynamical system existing in $\mathbf{R}^{12}$. In order to study this dynamical system it is helpful to reduce the dimension of the problem. To do this we will make use of certain key integrals of motion. I will prove these claims later in chapter 3.

The first integral of motion is the sum of the momenta of the three objects. Since this quantity remains constant, we can, without any loss of generality, take the total momentum of the system to be zero. This means that

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0.$$

From this assumption we can see that the center of mass of the system will be constant, and again we can simply take that to be placed at the origin. Giving us

$$m_1\vec{q}_1 + m_2\vec{q}_2 + m_3\vec{q}_3 = 0$$

Since these two equations are vector equations in $\mathbf{R}^2$ we have now reduced both the momentum vector $\vec{p}$, and the position vector $\vec{q}$ by two degrees, thus reducing the overall degree of the problem from 12 degrees of freedom down to 8 degrees.

Now since equations (2.7) do not change if we simultaneously rotate the position vectors and momentum vectors in $\mathbf{R}^2$, we have that the total angular momentum must be constant, and therefore we get that

$$\vec{p}_1 \times \vec{q}_1 + \vec{p}_2 \times \vec{q}_2 + \vec{p}_3 \times \vec{q}_3 = \omega$$

where ω is a constant. We have now reduced our system by one more degree of freedom. We should also note that this system is symmetric under all rotations so we can remove one more degree of freedom from the system by quotienting our system by this group action.

Finally since the energy function for our system must be conserved we know that

$$H(\vec{q}, \vec{p}) = \tfrac{1}{2}\vec{p}^{\,T} M^{-1}\vec{p} - U(\vec{q}) = h$$

again where h is a constant. This will eliminate one more degree of freedom. We have now reduced our system from 12 degrees of freedom, where we started, to only 5 degrees of freedom. Despite this success in reducing the degree of the system, 5 degrees of freedom is still too many for us to be able to solve this problem. We will define this new five dimensional manifold as $M(h, \omega)$.

At this point it is probably a good idea to shift our attention from an analysis of the momentum of the point masses to an analysis of the shape and sizes of the triangles formed by the planets. In order to do this we will introduce a change of variables, and a new coordinate system[1].

[1] This new coordinate system was discovered by R. McGehee and presented in [5].

Now since the center of mass is a constant, the moment of inertia about the origin will be pivotal. Thus we will define the moment of inertia to be:

$$\Im = \vec{q}^{\,T} M \vec{q} = m_1|\vec{q}_1|^2 + m_2|\vec{q}_2|^2 + m_3|\vec{q}_3|^2.$$

Next we define $r = \sqrt{\Im}$ which measures the size of the triangle that is formed by the three point masses. It also defines the radial length of a polar coordinate system in $\mathbf{R}^6$. We should note at this point that $r = 0$ represents a triple collision of our three point masses (which must occur at the origin). We now define $\vec{s}$ to be the normalized position vector by letting $\vec{s} = \frac{\vec{q}}{r}$. In this way s will measure the shape and angular position of the triangle formed by the bodies. Finally now we will normalize the momentum vector in a slightly different way. By Defining $\vec{z} = \sqrt{r}\vec{p}$.

With our new variables we now have new energy and angular momentum equations. These equations are given by:

$$H(\vec{s}, \vec{z}) = \frac{1}{2}\vec{z}^{\,T} M^{-1} \vec{z} - U(\vec{s}) = hr \tag{2.5}$$

$$\vec{z}_1 \times \vec{s}_1 + \vec{z}_2 \times \vec{s}_2 + \vec{z}_3 \times \vec{s}_3 = \omega\sqrt{r}. \tag{2.6}$$

We will now also have to express our differential equations using these new coordinates, but first we define $v = \vec{s} \cdot \vec{z}$. Thus instead of two equations defining our system we now have three and they are:

$$\begin{aligned} r' &= vr \\ \vec{s}\,' &= \vec{z} - \frac{1}{2}v\vec{s} \\ \vec{z}\,' &= \nabla U(\vec{s}) + \frac{1}{2}v\vec{z} \end{aligned} \tag{2.7}$$

where we have factored out an $r^{\frac{3}{2}}$ from all of the equations, leaving the last two equations free from of any dependence on r. We can now more easily talk about the shape space defined by these equations. Define

$$C = \{(r, \vec{s}) : r \geq 0, \vec{s}^T M \vec{s} = 1, m_1\vec{s}_1 + m_2\vec{s}_2 + m_3\vec{s}_3 = 0\}/S^1$$

to be the space of allowable configurations of the three planets where we quotient out the rotational symmetry. It can be shown that this space C is homeomorphic to $\mathbf{R}^+ \times S^2$, (see [6]).

G. W. Hill [2] realized that it is useful to constrain the configurations through the energy and angular momentum integrals. The subsets of configuration space, called Hill's regions are defined by

$$C(h, \omega) = \{(r, \vec{s}) \epsilon C : \text{for some } \vec{z} \epsilon \mathbf{R}^6, (r, \vec{s}, \vec{z}) \epsilon M(h, w)\}$$

where $M(h, \omega)$ is the five dimensional quotiented energy and angular momentum manifold that we got when we reduced the dimension of our original dynamical system. We see now that $C(h, \omega)$ is simply the projection of $M(h, \omega)$ onto the configuration space.

We can now restrict these Hill's regions even further. We do not need to look at all possible energy's h. If $h \geq 0$ then we must have that the kinetic energy of the system is equal to or higher than the potential energy. This means that the restoring force, gravity, is insufficient to pull the planets back towards each other, and we will simply have the planets moving away from each other after a sufficient amount of time passes. Thus we will restrict $h < 0$. We can also see that once we have chosen the masses for our three bodies that the dynamics of the system only depends on the value of $h\omega^2$ and so we define $\lambda = -h\omega^2$. Now to see exactly what happens to our planets we can fix an $h < 0$ and then let ω vary between $[0, \infty)$.

From this restriction we can now derive some inequalities based on the energy function. Since we know that the kinetic energy term given by $\frac{1}{2}\vec{z}^T M^{-1}\vec{z}$ is non-negative then we know that the potential energy term, given by $U(\vec{s})$ is greater than or equal to $|h|r$. Thus if we fix the shape of the triangle formed by the planets we see that the size of the triangle is restrained to $0 \leq r \leq \frac{U(s_0)}{|h|}$. This constraint rules out certain size triangles in our solution space. The potential energy becomes infinite at a double collision point, and it is minimum at the equilateral triangle solution, thus if we have a double collision we can have any size degenerate triangle, while for any other shape triangle, especially equilateral triangle, shapes of a sufficiently large size are excluded.

Our assumption that the kinetic energy has to be positive can be improved upon. It turns out that one can show that if we fix the angular momentum ω then we can limit the kinetic energy by

$$\frac{1}{2}\vec{z}^T M^{-1}\vec{z} \geq \frac{1}{2}\frac{\omega^2}{r}.$$

Plugging this estimate into the energy equation shows us that

$$U(\vec{s}) \geq |h|r + \frac{\omega^2}{2r}. \tag{2.8}$$

This gives us a nice characterization of $C(h,\omega)$. We know that $C(h,\omega)$ is a solid region in $S^2 \times \mathbf{R}^+$, and that its boundary is given by the equality in (2.8). Since equation (2.8) is quadratic in r, we see that in fact $C(h,\omega)$ must be contained between two sheets which lie over the shape sphere. Since the projection of $C(h,\omega)$ onto the shape sphere is the set of all s for which equation (2.8) holds, with $r \geq 0$, then we can minimize the right hand side of equation (2.8) and we see that

$$U(\vec{s}) \geq \sqrt{2|h|\omega^2} = \sqrt{2\lambda}.$$

This defines the projection of $C(h,\omega)$, and tells us that the Hill's regions lie above a region of the shape sphere that is bounded by an equipotential curve.

For the work that I am doing on this project I will be looking at solutions that have large angular momentum ω. This means that any solutions that I find will be located in a small region around one of the double collision points on our shape sphere. This is because of equation (2.8) that says if we have high angular momentum, we must also have very large potential energy, which we get from the double collision solutions. Since double collisions do not give us periodic orbits, we will instead look at a group of solutions with what is called a tight binary solution. A tight binary is simply two objects orbiting about their center of mass, with the distance between the object held relatively small.

Sundman [15] gives us a nice theorem about the configurations of the three body problem assuming a tight binary construct with two of the planets.

Theorem 1 *Let the angular momentum be non-zero. Then any orbit passing sufficiently close to triple collision is of the following type: the configuration is a tight binary for all time and the short side of the triangle remains bounded while the other two sides tend to infinity in both forward and backward time.*

Sundman's theorem leads to some interesting open question about tight binary systems. One such question is can we find a tight binary solution of the three body problem in which the short side of the triangle bounded for all time, while the other two sides tend to infinity in one time direction, yet remain bounded in the other time direction? In chapter we will use Maple to try and find some experimental evidence about this question.

Chapter 3

Analysis

In this chapter we will provide the proofs of the claims that were made in chapter 2. Recall that the energy of the system is given by

$$E = \frac{1}{2}\vec{p}^T M^{-1}\vec{p} - U(\vec{q}).$$

Where $U(\vec{q})$ is the potential energy given by:

$$U(\vec{q}) = \frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2|} + \frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|} + \frac{m_2 m_3}{|\vec{q}_2 - \vec{q}_3|}$$

Now since this is a closed system we know that there can be no other energy added to the system. Thus the total energy of the system must be constant and so we get

$$E = \frac{1}{2}\vec{p}^T M^{-1}\vec{p} - U(\vec{q}) = h,$$

where h is a constant. Since this equation is a constant, we know it is Hamiltonian and therefore we can change notation to

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\vec{p}^T M^{-1}\vec{p} - U(\vec{q}) = h. \tag{3.1}$$

This tells us that our system is subject to Hamilton's equation. These give us a system of equations that we can use to model our planet system by taking the partial derivatives of the energy function. Hamilton's equations are given by:

$$\begin{aligned} \frac{d\vec{q}}{dt} &= \frac{\partial H}{\partial \vec{p}} \\ \frac{d\vec{p}}{dt} &= -\frac{\partial H}{\partial \vec{q}}. \end{aligned} \tag{3.2}$$

Now by applying equation (3.2) to our energy equation we can generate a system of dynamical equations, shown below, with which we can try and find a particular solution to the three body problem.

$$\begin{aligned} \frac{d\vec{q}}{dt} &= M^{-1}\vec{p} \\ \frac{d\vec{p}}{dt} &= \nabla U(\vec{q}). \end{aligned} \tag{3.3}$$

We can make a slight notation change now to make things a little easier. We will from now on denote the time derivative by $\dot{\vec{p}}$ instead of the more cumbersome $\frac{d\vec{p}}{dt}$. Both $\dot{\vec{p}}$ and $\dot{\vec{q}}$ define differential equations in $\mathbf{R}^6$, so that equation (3.3) defines a dynamical system in $\mathbf{R}^{12}$. We have no chance of solving a system with this many degrees of freedom so we will need to lower the overall dimension of the system. We can reduce these equations to a five-dimensional system by making use of specific integrals of motion.

The first integral of motion that we will make use of tells us that the total momentum of the system is a constant and therefore we can assume it to be zero without a loss of generality.

Claim 1 *The total momentum of the dynamical system does not change in time, i.e.* $\frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = 0$ *and so we can choose* $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$

Proof: Since our dynamical system is Hamiltonian, we know that

$$\dot{\vec{p}}_1 = \frac{\partial}{\partial \vec{q}_1} U(\vec{q}) \text{ and that } \dot{\vec{p}}_2 = \frac{\partial}{\partial \vec{q}_2} U(\vec{q}) \text{ finally that } \dot{\vec{p}}_3 = \frac{\partial}{\partial \vec{q}_3} U(\vec{q}).$$

By $\frac{\partial}{\partial q_i}$ we mean to take the gradient in the q_i direction. This means that

$$\dot{\vec{p}}_1 = \frac{\partial}{\partial \vec{q}_1} \left(\frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2|} + \frac{m_2 m_3}{|\vec{q}_2 - \vec{q}_3|} + \frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|} \right).$$

But since the middle term in the above expression has no $\vec{q}_1$ term we see that its derivative with respect to $\vec{q}_1$ is zero and therefore we can ignore it. Similarly the third term has no $\vec{q}_2$ dependance and can be dropped for $\dot{\vec{p}_2}$ and the first term has no $\vec{q}_3$ term it is dropped for $\dot{\vec{p}_3}$. Now we can write down the simplified equation for each term and we get:

$$\begin{aligned}
\dot{\vec{p}_1} &= \frac{\partial}{\partial \vec{q}_1}\left(\frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2|} + \frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|}\right) \\
\dot{\vec{p}_2} &= \frac{\partial}{\partial \vec{q}_2}\left(\frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2} + \frac{m_2 m_2}{|\vec{q}_2 - \vec{q}_3|}\right) \\
\dot{\vec{p}_3} &= \frac{\partial}{\partial \vec{q}_3}\left(\frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|} + \frac{m_2 m_3}{|\vec{q}_2 - \vec{q}_3|}\right).
\end{aligned} \tag{3.4}$$

Before we continue we will need to note two important facts. The first is that we can look at the partial derivatives of each term with respect to the x and y coordinates independently. The second thing that we should note is that

$$|\vec{q}_1 - \vec{q}_2|$$

can be rewritten as

$$\sqrt{(q_{1,x} - q_{2,x})^2 + (q_{1,y} + q_{2,y})^2}.$$

Where $q_{i,x}$ and $q_{i,y}$ is the notation used to denote the x-component and the y-component for the vector $\vec{q}_i$. Combining these two facts together allows us to rewrite (3.4) as

$$\begin{aligned}
\dot{\vec{p}_1} &= \left(\frac{\partial}{\partial q_{1,x}}, \frac{\partial}{\partial q_{1,y}}\right)\left(\frac{m_1 m_2}{\sqrt{(q_{1,x} - q_{2,x})^2 + (q_{1,y} - q_{2,y})^2}} + \frac{m_1 m_3}{\sqrt{(q_{1,x} - q_{3,x})^2 + (q_{1,y} - q_{3,y})^2}}\right) \\
\dot{\vec{p}_2} &= \left(\frac{\partial}{\partial q_{2,x}}, \frac{\partial}{\partial q_{2,y}}\right)\left(\frac{m_1 m_2}{\sqrt{(q_{1,x} - q_{2,x})^2 + (q_{1,y} - q_{2,y})^2}} + \frac{m_2 m_3}{\sqrt{(q_{2,x} - q_{3,x})^2 + (q_{2,y} - q_{3,y})^2}}\right) \\
\dot{\vec{p}_3} &= \left(\frac{\partial}{\partial q_{3,x}}, \frac{\partial}{\partial q_{3,y}}\right)\left(\frac{m_2 m_3}{\sqrt{(q_{2,x} - q_{3,x})^2 + (q_{2,y} - q_{3,y})^2}} + \frac{m_1 m_3}{\sqrt{(q_{1,x} - q_{3,x})^2 + (q_{1,y} - q_{3,y})^2}}\right).
\end{aligned}$$

Now we can differentiate each expression which leaves us with:

$$\dot{\vec{p}_1} = -\frac{1}{2}\frac{2m_1 m_2((q_{1,x} - q_{2,x}), (q_{1,y} - q_{2,y}))}{((q_{1,x} - q_{2,x})^2 + (q_{1,y} - q_{2,y})^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{2m_1 m_3((q_{1,x} - q_{3,x}), (q_{1,y} - q_{3,y}))}{((q_{1,x} - q_{3,x})^2 + (q_{1,y} - q_{3,y})^2)^{\frac{3}{2}}}$$

$$\dot{\vec{p_2}} = \frac{1}{2}\frac{2m_1m_2((q_{1,x}-q_{2,x}),(q_{1,y}-q_{2,y}))}{((q_{1,x}-q_{2,x})^2+(q_{1,y}-q_{2,y})^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{2m_2m_3((q_{2,x}-q_{3,x}),(q_{2,y}-q_{3,y}))}{((q_{2,x}-q_{3,x})^2+(q_{2,y}-q_{3,y})^2)^{\frac{3}{2}}}$$
$$\dot{\vec{p_3}} = \frac{1}{2}\frac{2m_1m_3((q_{1,x}-q_{3,x}),(q_{1,y}-q_{3,y}))}{((q_{1,x}-q_{3,x})^2+(q_{1,y}-q_{3,y})^2)^{\frac{3}{2}}} + \frac{1}{2}\frac{2m_2m_3((q_{2,x}-q_{3,x}),(q_{2,y}-q_{3,y}))}{((q_{2,x}-q_{3,x})^2+(q_{2,y}-q_{3,y})^2)^{\frac{3}{2}}}.$$

Now we can reduce our fractions, and condense our terms where possible. This leaves us with:

$$\dot{\vec{p_1}} = \frac{-m_1m_2(\vec{q_1}-\vec{q_2})}{|\vec{q_1}-\vec{q_2}|^3} + \frac{-m_1m_3(\vec{q_1}-\vec{q_3})}{|\vec{q_1}-\vec{q_3}|^3} \tag{3.5}$$
$$\dot{\vec{p_2}} = \frac{m_1m_2(\vec{q_1}-\vec{q_2})}{|\vec{q_1}-\vec{q_2}|^3} + \frac{-m_2m_3(\vec{q_2}-\vec{q_3})}{|\vec{q_2}-\vec{q_3}|^3} \tag{3.6}$$
$$\dot{\vec{p_3}} = \frac{m_1m_3(\vec{q_1}-\vec{q_3})}{|\vec{q_1}-\vec{q_3}|^3} + \frac{m_2m_3(\vec{q_2}-\vec{q_3})}{|\vec{q_2}-\vec{q_3}|^3}. \tag{3.7}$$

It is clear from the above equations that when we add $\dot{\vec{p_1}} + \dot{\vec{p_2}} + \dot{\vec{p_3}}$ together we will get zero. Thus $\vec{p_1} + \vec{p_2} + \vec{p_3} = k$ where k is a constant that we take to be zero, and we have proven our claim.

Our next claim will also reduce the degree of the system by two. This integral of motion tells us that the center of mass of our system is a constant and therefore as above we can take it to be the origin.

Claim 2 *Assuming that* $\vec{p_1} + \vec{p_2} + \vec{p_3} = 0$, *then the center of mass of the dynamical system is a constant, i.e.*

$$m_1\vec{q_1} + m_2\vec{q_2} + m_3\vec{q_3} = C \tag{3.8}$$

where C *is a constant.*

Proof: Once again if we can show that the derivative of $m_1\vec{q_1} + m_2\vec{q_2} + m_3\vec{q_3}$ is equal to zero than we will have proven our claim. We know that

$$\dot{\vec{q}} = M^{-1}\vec{p}.$$

But $\vec{p}$ can be written as a 1×6 matrix. M on the other hand is a 6×6 matrix, but with every entry zero except down the main diagonal. Thus $M^{-1}\vec{p}$ becomes

$$\left(\frac{1}{m_1}p_{1,x}, \frac{1}{m_1}p_{1,y}, \frac{1}{m_2}p_{2,x}, \frac{1}{m_2}p_{2,y}, \frac{1}{m_3}p_{3,x}, \frac{1}{m_3}p_{3,y}\right).$$

Now when we compare the terms we see that

$$\dot{\vec{q_1}} = \frac{1}{m_1}\vec{p_1}$$

$$\dot{\vec{q_2}} = \frac{1}{m_2}\vec{p_2}$$

$$\dot{\vec{q_3}} = \frac{1}{m_3}\vec{p_3}$$

and so finally we see that

$$\begin{aligned} m_1\dot{\vec{q_1}} + m_2\dot{\vec{q_2}} + m_3\dot{\vec{q_3}} &= \frac{m_1}{m_1}\vec{p_1} + \frac{m_2}{m_2}\vec{p_2} + \frac{m_3}{m_3}\vec{p_3} = \\ &= \vec{p_1} + \vec{p_2} + \vec{p_3}. \end{aligned}$$

But we have chosen to take :

$$\vec{p_1} + \vec{p_2} + \vec{p_3} = 0,$$

and so we have proven claim 2.

Next I will prove that the angular momentum of our system, ω is a constant. This will allow us to remove one more degree of freedom from the dynamical system.

Claim 3 *The angular momentum of our system is constant i.e.*

$$\vec{p_1} \times \vec{q_1} + \vec{p_2} \times \vec{q_2} + \vec{p_3} \times \vec{q_3} = \omega, \tag{3.9}$$

where ω is a constant.

Proof: Once again if we can show that the derivative of (3.9) with respect to time is zero then we will have proven the claim. So we would like to show that

$$\frac{d}{dt}(\vec{p_1} \times \vec{q_1}) + \frac{d}{dt}(\vec{p_2} \times \vec{q_2}) + \frac{d}{dt}(\vec{p_3} \times \vec{q_3}) = 0.$$

Now we should note that

$$\frac{d}{dt}(\vec{p} \times \vec{q}) = \dot{\vec{p}} \times \vec{q} + \vec{p} \times \dot{\vec{q}},$$

but $\dot{\vec{q}} = M^{-1}\vec{p}$ so that

$$\vec{p} \times \dot{\vec{q}} = \vec{p} \times M^{-1}\vec{p} = M^{-1}(\vec{p} \times \vec{p}) = 0,$$

since the cross product of a vector with itself is zero, and thus our problem is now to show that

$$\dot{\vec{p}}_1 \times \vec{q}_1 + \dot{\vec{p}}_2 \times \vec{q}_2 + \dot{\vec{p}}_3 \times \vec{q}_3 = 0. \tag{3.10}$$

We can ease the notation by referring back to equations (3.5)-(3.7) and by making the following substitution:

$$\phi_1 = \frac{m_1 m_2}{|\vec{q}_1 - \vec{q}_2|^3},\ \phi_2 = \frac{m_2 m_3}{|\vec{q}_2 - \vec{q}_3|^3},\ \text{and}\ \phi_3 = \frac{m_1 m_3}{|\vec{q}_1 - \vec{q}_3|^3}.$$

So we can now take (3.9) and try and prove our claim. We can now substitute in what $\dot{\vec{p}}_i$ equals from (3.5)-(3.7) and we get

$$= (-\phi_1(\vec{q}_1 - \vec{q}_2) - \phi_3(\vec{q}_1 - \vec{q}_3)) \times \vec{q}_1 + (\phi_1(\vec{q}_1 - \vec{q}_2) - \ \phi_2(\vec{q}_2 - \vec{q}_3)) \times \vec{q}_2(\phi_3(\vec{q}_1 - \vec{q}_3) + \phi_2(\vec{q}_2 - \vec{q}_3)) \times \vec{q}_3.$$

Next we will distribute the cross product through the parenthesis

$$= -\phi_1(\vec{q}_1 - \vec{q}_2) \times \vec{q}_1 - \phi_3(\vec{q}_1 - \vec{q}_3) \times \vec{q}_1 + \phi_1(\vec{q}_1 - \vec{q}_2) \times \vec{q}_2 - \phi_2(\vec{q}_2 - \vec{q}_3) \times \vec{q}_2 + \phi_3(\vec{q}_1 - \vec{q}_3) \times \vec{q}_3 + \phi_2(\vec{q}_2 - \vec{q}_3) \times \vec{q}_3$$

and through the next level of parenthesis

$$= \phi_1(\vec{q}_2 \times \vec{q}_1) + \phi_3(\vec{q}_3 \times \vec{q}_1) + \phi_1(\vec{q}_1 \times \vec{q}_2) + \phi_2(\vec{q}_3 \times \vec{q}_2) + \phi_3(\vec{q}_1 \times \vec{q}_3) + \phi_2(\vec{q}_2 \times \vec{q}_3)$$

and finally we will reorganize the cross product by making use of the fact that $\vec{q}_i \times \vec{q}_j$ is the same as $-(\vec{q}_j \times \vec{q}_i)$

$$= -\phi_1(\vec{q}_1 \times \vec{q}_2) + \phi_1(\vec{q}_1 \times \vec{q}_2) - \phi_3(\vec{q}_1 \times \vec{q}_3) + \phi_3(\vec{q}_1 \times \vec{q}_3) - \phi_2(\vec{q}_2 \times \vec{q}_3) + \phi_2(\vec{q}_2 \times \vec{q}_3)$$

$$= 0.$$

This has lowered the degrees of freedom by one more. So we have succeeded in reducing the degrees of freedom for the dynamical down from 12 to 7.

We can eliminate two more degrees of freedom in the system by making the following observations. We can identify all of the vectors $(\vec{q}, \vec{p})$ that differ by only a simultaneous rotation of each vector through the same angle since the system is symmetric under rotations. Our final reduction in degree comes from looking at the total energy of the system. Since we know that the total energy of the system is conserved we can further reduce the degree of our system by one. Thus we are left with a total of five degrees of freedom. Despite having reduced our dynamical system by seven degrees of freedom, five degrees of freedom is still to many to solve explicitly.

Since we will be talking about the three-body problem in a more geometrical method it will help to change the coordinate system to one that was discovered by McGehee [5]. We define:

$$\Im = \vec{q}^{\,T} M \vec{q} = m_1|\vec{q}_1|^2 + m_2|\vec{q}_2|^2 + m_3|\vec{q}_3|^2,$$

and define the variable $r = \sqrt{\Im}$ to be the radial variable in a 6 dimensional polar coordinate system. The size of the triangle formed by the three planets is represented by the variable r. We measure the shape of the triangle by the normalized position vector given by $\vec{s} = \frac{\vec{q}}{r}$. We normalize the momentum vector in a different way, by defining $\vec{z} = \sqrt{r}\vec{p}$.

Now when we substitute these coordinates into our energy function we get:

$$H(\vec{s}r, \frac{\vec{z}}{\sqrt{r}}) = \frac{1}{2}\frac{\vec{z}^{\,T}}{\sqrt{r}} M^{-1} \frac{\vec{z}}{\sqrt{r}} - \frac{m_1 m_2}{|\vec{s}_1 r - \vec{s}_2 r|} - \frac{m_2 m_3}{|r\vec{s}_2 - r\vec{s}_3|} - \frac{m_1 m_3}{|r\vec{s}_1 - r\vec{s}_3|}.$$

factoring out the r's we get:

$$h = H(\vec{s}r, \frac{\vec{z}}{\sqrt{r}}) = \frac{1}{2r}\vec{z}^{\,T} M^{-1} \vec{z} - \frac{U(\vec{s})}{r} = \frac{1}{r}H(\vec{s}, \vec{z}).$$

So that we see in our new coordinate system $H(\vec{s}, \vec{z}) = hr$. We also get a new equation for the angular momentum. Substituting our new coordinates into equation (3.5) yields our new equation:

$$\frac{\vec{z}_1}{\sqrt{r}} \times \vec{s}_1 r + \frac{\vec{z}_2}{\sqrt{r}} \times \vec{s}_2 r + \frac{\vec{z}_3}{\sqrt{r}} \times \vec{s}_3 r =$$
$$= \sqrt{(r)}(\vec{z}_1 \times \vec{s}_1) + \sqrt{r}(\vec{z}_2 \times \vec{s}_2) + \sqrt{r}(\vec{z}_3 \times \vec{s}_3) =$$
$$\sqrt{r}(\vec{z}_1 \times \vec{s}_1 + \vec{z}_2 \times \vec{z}_2 + \vec{z}_3 \times \vec{s}_3).$$

Which shows that:

$$\vec{z}_1 \times \vec{s}_1 + \vec{z}_2 \times \vec{s}_2 + \vec{z}_3 \times \vec{s}_3 = \frac{\omega}{\sqrt{r}}.$$

Now that we have a new coordinate system it has become vital that we generate a new system of differential equations. We will start with the r variable.

$$r = \sqrt{\vec{q}^T M \vec{q}}$$

and so differentiating on the right hand side with respect to time gives us:

$$\dot{r} = \frac{1}{2}\frac{\dot{\vec{q}}^T M\vec{q} + \vec{q}^T M\dot{\vec{q}}}{\sqrt{(\vec{q}^T M\vec{q})}} = \frac{1}{2}\frac{2\vec{q}^T M\dot{\vec{q}}}{\sqrt{\vec{q}^T M\vec{q}}}.$$

But $\dot{\vec{q}} = M^{-1}\vec{p}$ and so the above equation becomes:

$$\dot{r} = \frac{\vec{q}^{\,T}\vec{p}}{\sqrt{\vec{q}^{\,T} M\vec{q}}}.$$

Now if we substitute our new coordinates back in we get:

$$\dot{r} = \frac{r\vec{s} \cdot \frac{\vec{z}}{\sqrt{r}}}{r}$$

and now when we simplify we get

$$\frac{\vec{s} \cdot \vec{z}}{\sqrt{r}}. \tag{3.11}$$

For our second differential equation we start with $\vec{s} = \frac{\vec{q}}{r}$ and again we take the time derivative on the right hand side, this time we get:

$$\dot{\vec{s}} = \frac{\dot{\vec{q}}r - \vec{q}\dot{r}}{r^2}.$$

Which becomes

$$\dot{\vec{s}} = \frac{M^{-1}\vec{p}r - \frac{\vec{q}\vec{s}\cdot\vec{z}}{\sqrt{r}}}{r^2}.$$

Continuing on gives us

$$\dot{\vec{s}} = \frac{\frac{M^{-1}\vec{z}r}{\sqrt{r}} - \frac{\vec{s}(\vec{s}\dot{\vec{z}})r}{\sqrt{r}}}{r^2}.$$

And so finally we see that our second differential equation is:

$$\dot{\vec{s}} = \frac{M^{-1}\vec{z} - \vec{s}^2\vec{z}}{r^{\frac{3}{2}}}. \tag{3.12}$$

Our final differential equation comes from $\vec{z} = \sqrt{r}\vec{p}$, and again differentiating the right hand side with respect to time.

$$\dot{\vec{z}} = \frac{1}{2}\frac{\dot{r}\vec{p}}{\sqrt{r}} + \sqrt{r}\dot{\vec{p}}$$

So we now substitute in for $\dot{r}$ and for $\dot{\vec{p}}$ and $\vec{p}$ we get:

$$\dot{\vec{z}} = \frac{1}{2}\frac{(\vec{s}\dot{\vec{z}})\vec{z}}{r^{\frac{3}{2}}} + \sqrt{r}\nabla U(\vec{q}). \tag{3.13}$$

Now we should analyze $\nabla U(\vec{s})$ to determine its relation to $\nabla U(\vec{q})$. So

$$\nabla U(\vec{s}) = \nabla U(\frac{\vec{q}}{r}).$$

But this is

$$\frac{m_1m_2\frac{(\vec{q}_1-\vec{q}_2)}{r}}{|\frac{\vec{q}_1}{r} - \frac{\vec{q}_2}{r}|^3} + \frac{m_1m_3\frac{(\vec{q}_1-\vec{q}_3)}{r}}{|\frac{\vec{q}_1}{r} - \frac{\vec{q}_3}{r}|^3} + \frac{m_2m_3\frac{(\vec{q}_2-\vec{q}_3)}{r}}{|\frac{\vec{q}_2}{r} - \frac{\vec{q}_3}{r}|^3}.$$

So once again factoring out the r we get:

$$\frac{\frac{m_1m_2(\vec{q}_1-\vec{q}_2)}{r}}{\frac{|\vec{q}_1-\vec{q}_2|^3}{r^3}} + \frac{\frac{m_1m_3(\vec{q}_1-\vec{q}_3)}{r}}{\frac{|\vec{q}_1-\vec{q}_3|^3}{r^3}} + \frac{\frac{m_2m_3(\vec{q}_2-\vec{q}_3)}{r}}{\frac{|\vec{q}_2-\vec{q}_3|^3}{r^3}}.$$

And finally by simplifying we see that

$$\nabla U(\vec{s}) = r^2\frac{m_1m_2(\vec{q}_1 - \vec{q}_2)}{|\vec{q}_1 - \vec{q}_2|^3} + r^2\frac{m_1m_3(\vec{q}_1 - \vec{q}_3)}{|\vec{q}_1 - \vec{q}_3|^3} + r^2\frac{m_2m_3(\vec{q}_2 - \vec{q}_3)}{|\vec{q}_2 - \vec{q}_3|^3} = r^2\nabla U(\vec{q})$$

Now going back to equation (3.12) we end up with:

$$\dot{\vec{z}} = \frac{1}{2}\frac{\vec{s}\vec{z}^2}{r^{\frac{3}{2}}} + \frac{\nabla U(\vec{s})}{r^{\frac{3}{2}}}. \tag{3.14}$$

When we examine the three new differential equations we see that two of the equation have an $r^{\frac{3}{2}}$ term in the denominator. We can now multiply through by a factor of $r^{\frac{3}{2}}$ in all three equations and we will only be changing the parameterizations of the solutions generated. We will now denote the derivative with respect to this new parametrization by $'$ and to help clean up the notation some we will define $v = \vec{s} \cdot \vec{z}$. So finally our three new differential equations become:

$$\begin{aligned} r' &= vr \\ \vec{s}\,' &= M^{-1}\vec{z} - \vec{s}v \\ \vec{z}\,' &= \nabla U(\vec{s}) + \frac{1}{2}v\vec{z}. \end{aligned} \tag{3.15}$$

This change of coordinates helps with understanding the geometrical look of this dynamical system. However, in the attempt that I made in trying to find a particular solution to the three-body problem equations (3.15) are unnecessary, and so I will be using the $\dot{\vec{p}}$ and $\dot{\vec{q}}$ equations that we started with as given in equation (2.4).

Chapter 4

My Work

In the last two chapters we used a change of coordinates in order to view the system in a more geometric light. Unfortunately for the analytical work that I did on this project these new coordinates are not as efficient. Instead I will go back to the equations as defined in (1.2), (1.3), and (1.4). To display these equations as a dynamical system I will need to reduce the order of the system to first degree differential equations by introducing velocity variables. I now have the following equations to solve:

$$\begin{aligned}
\frac{d\vec{q}_1}{dt} &= \frac{1}{m_1 m_1}\vec{p}_1 \\
\frac{d\vec{p}_1}{dt} &= m_1 m_2 \frac{\vec{q}_1 - \vec{q}_2}{|\vec{q}_1 - \vec{q}_2|^3} + m_1 m_3 \frac{\vec{q}_1 - \vec{q}_3}{|\vec{q}_1 - \vec{q}_3|^3} \\
\frac{d\vec{q}_2}{dt} &= \frac{1}{m_2 m_2}\vec{p}_2 \\
\frac{d\vec{p}_2}{dt} &= m_2 m_1 \frac{\vec{q}_2 - \vec{q}_1}{|\vec{q}_2 - \vec{q}_1|^3} + m_2 m_3 \frac{\vec{q}_2 - \vec{q}_3}{|\vec{q}_2 - \vec{q}_3|^3} \\
\frac{d\vec{q}_3}{dt} &= \frac{1}{m_3 m_3}\vec{p}_3 \\
\frac{d\vec{p}_3}{dt} &= m_3 m_1 \frac{\vec{q}_3 - \vec{q}_1}{|\vec{q}_3 - \vec{q}_1|^3} + m_3 m_2 \frac{\vec{q}_3 - \vec{q}_2}{|\vec{q}_3 - \vec{q}_2|^3}
\end{aligned}$$

where $\vec{q}_i \ \epsilon\ \mathbf{R}^3$ is the three dimensional position vector of the i^{th}. From now on we will suppress the vectorial arrows to ease the typography.

We can further ease the analysis of this system by further breaking apart the differential equations in to both an x and y direction. We can ignore the z direction as we are assuming that the planets are moving in a planar sense, and so the rate of change in the z-direction is therefore 0. Now instead of 6 equations we will end up with 12 first order differential equations. To make the entering of these equations in Maple easier I will now switch from the $\vec{q}$ and $\vec{p}$ to an (x, y) and v notation where x_i is the x value of the i^{th} position vector and y_i is the y value of the vector. Also $v_{x,i}$ is the x value of the velocity of the i^{th} planet, and $v_{y,i}$ is the y component of the velocity. It is these equations I will use in the Maple software to try and find approximate solutions to the n-body problem. The twelve equations are:

$$
\begin{aligned}
\frac{dx_1}{dt} &= v_{x,1} \\
\frac{dv_{x,1}}{dt} &= \frac{-m_2(x_1 - x_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} - \frac{m_3(x_1 - x_3)}{((x_1 - x_3)^2 + (y_1 - y_3)^2)^{3/2}} \\
\frac{dy_1}{dt} &= v_{y,1} \\
\frac{dv_{y,1}}{dt} &= \frac{-m_2(y_1 - y_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} - \frac{m_3(y_1 - y_2)}{((x_1 - x_3)^2 + (y_1 - y_3)^2)^{3/2}} \\
\frac{dx_2}{dt} &= v_{x,2} \\
\frac{dv_{x,2}}{dt} &= \frac{-m_1(x_1 - x_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} - \frac{m_3(x_2 - x_3)}{((x_2 - x_3)^2 + (y_2 - y_3)^2)^{3/2}} \\
\frac{dy_2}{dt} &= v_{y,2} \\
\frac{dv_{y,2}}{dt} &= \frac{-m_1(y_1 - y_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} - \frac{m_3(y_2 - y_3)}{((x_2 - x_3)^2 + (y_2 - y_3)^2)^{3/2}} \\
\frac{dx_3}{dt} &= v_{x,3} \\
\frac{dv_{x,3}}{dt} &= \frac{-m_1(x_1 - x_2)}{((x_1 - x_3)^2 + (y_1 - y_3)^2)^{3/2}} - \frac{m_2(x_2 - x_3)}{((x_2 - x_3)^2 + (y_2 - y_3)^2)^{3/2}} \\
\frac{dy_3}{dt} &= v_{y,3} \\
\frac{dv_{y,3}}{dt} &= \frac{-m_1(y_1 - y_3)}{((x_1 - x_3)^2 + (y_1 - y_3)^2)^{3/2}} - \frac{m_2(y_2 - y_3)}{((x_2 - x_3)^2 + (y_2 - y_3)^2)^{3/2}}.
\end{aligned}
$$

It is with these 12 equations that I began my analysis of the three-body problem using Maple as a tool to try and approximate certain orbits.

The first orbit I tried to approximate was the figure eight orbit. This orbit was proven to be stable by Richard Montgomery in [7]. I took the 12 equations as defined above, plugged them in to Maple, and then used Maple to draw and display the figure eight solution, with a particular set of initial conditions. In order to simplify the work that I did, I took $m_1 = m_2 = m_3 = 1$. I then had Maple generate the phase portrait for, as well as a particular solution to, the dynamical system I had defined[1].

The initial conditions that generate the figure 8 orbit were found on Montgomery's website [8]. The initial conditions for the position of each planet were:

$$x_1 = -.97000436 \quad y_1 = .24308753 \tag{4.1}$$

$$x_2 = .97000436 \quad y_2 = -.24308753 \tag{4.2}$$

$$x_3 = 0 \quad y_3 = 0 \tag{4.3}$$

and the initial velocity given to each planet in the system was:

$$v_{x,1} = -.46620369 \quad v_{y,1} = -.43236573 \tag{4.4}$$

$$v_{x,2} = -.46620369 \quad v_{y,2} = -.43236573 \tag{4.5}$$

$$v_{x,3} = .93240737 \quad v_{y,3} = .8647314. \tag{4.6}$$

By using the *phaseportrait* command in Maple as well as the given initial conditions (4.1)-(4.6), and the 12 differential equations given above I was able to generate the following figure:

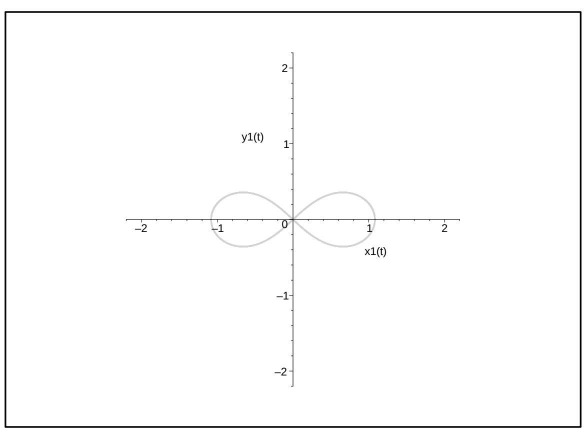

Figure 4.1: Maple's graph of a figure eight solution to the three body problem.

[1]See Appendix A for the Maple code used to generate this solution.

Now that I had a good picture of the figure eight solution I attempted to generate a movie that would show the motion of the three bodies along the figure eight. To do this some serious modifications to the original worksheet needed to be made. To generate the movie I needed to have maple solve the differential equations and output the results as a list procedure. The list procedure form, outputs the results as a function, in my case, the results were output as a function of time. We can then take these results and plug them back into our original equations, and use this to generate the frames of the movie that we watch. When we do this we get an initial frame like: We can then play this movie and watch the three planets chase each other

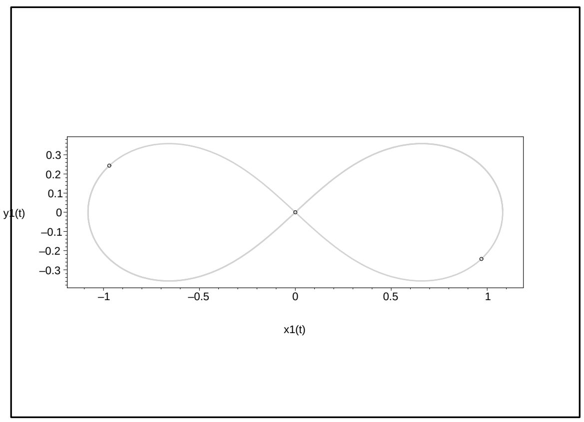

Figure 4.2: Initial frame of the figure eight solution generated in Maple.

around the figure eight[2].

After finding the Maple approximation to this problem I then went on to look at an open question Moeckel poses in his article. Can a planet's position move to infinity in one time direction, but be contained in a stable orbit in the other time direction? One way to think about this problem is to imagine a satellite moving in from very far away and falling into a stable orbit around another planet or a sun.

To find a solution like this we first must realize that the only way a satellite can reach a stable orbit after coming in from infinity is if there is some way for the

[2]The Maple code used to generate this movie can be found in Appendix B

satellite to transfer some of its energy to the objects it is trying to orbit. Since we want stability in the system it is clear that we will need three bodies here and we will need two of the bodies to be orbiting each other in what is called a tight binary system. In this way we will allow the energy from the satellite to be absorbed by the orbital energy of the tight binary. In this way I had hoped to find such a solution.

In this attempt I once again had to define the same twelve equations as before. Though to make the next step easier I temporarily defined the mass of the third planet to be zero. This way I could ignore its effect on the other two planets so that I would be able to easily set up the tight binary system that I would need, also to make things easier I normalized the distance of the binary planets from the center of mass for the tight binary, so that each was 1 unit away from the center of mass, taken to be the origin. I was now ready to determine the initial conditions for the tight binary, which were:

$$x_1 = 1 \quad v_{x,1} = \tfrac{1}{2} \tag{4.7}$$

$$x_2 = 1 \quad v_{x,2} = \tfrac{1}{2}, \tag{4.8}$$

with $y_1 = y_2 = v_{y,1} = v_{y,2} = 0$.

Now that I had the solution for the tight binary system I was ready to reinsert the third planet in to the system by increasing its mass from 0. My idea was that if the third planet was fairly small then it would have the best chance of being captured, and orbiting the other two planets. With that in mind I set the mass of the third planet to be $\frac{1}{25000}$ that of the other masses. Since I still had the masses normalized that gave me the masses $m_1 = m_2 = 1$ and $m_3 = \frac{1}{25000}$. Now it was simply a matter of picking an initial position and velocity for the third planet and seeing if it ever stayed in an orbit around the tight binary[3].

[3]See Appendix C for the Maple code

This solution has proven to be much more difficult to find than the figure eight solution. So far all initial conditions that I have picked either have the third planet moving to infinity in both time directions, or if I manage to find one not going to infinity in one time direction, it turns out to be in orbit around the binary for all time. The following graph will show this first type of example. I have been looking

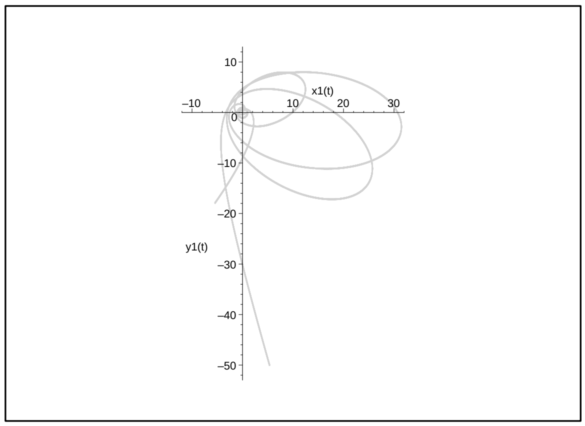

Figure 4.3: Example of failed attempt.

at this problem for months and I have been unable to generate any solution to the problem.

All the work I have done on this system has been through the use of the mathematical software Maple. A software program like this can not be used to prove the existence of a stable solution to the three-body problem. In fact all we can do with a program like this is generate likely candidates for solutions. The program, and the movies it generates can not be used to imply any behavior of the system as the time line goes to infinity. All we can determine from the movie is stability over a certain period of time.

Another problem with using a software program to try and approximate solutions to the three-body problem is that the solutions are in fact approximate. Maple can not generate exact answers and therefore every term used in the movies is truncated or rounded off in some way. This means that the error terms between each frame of the movie, and the ”exact” results could be rather significant. Both of these factors mean that any solution found using a software program like Maple is by no means proven to be an actual solution to the three-body problem. Instead they can just be used to point one in the right direction to a solution.

I would like to continue my experimentation into finding a solution to the problem that I proposed. I had also considered trying to find a solution to the three-body problem that was a variation of the solution found by Montgomery. In my variation to this solution, instead of all three planets chasing each other through the figure eight, I had thought to try and find a solution with two large planets, possibly in a binary configuration, with a third smaller planet orbiting them in a figure eight pattern. Further open questions on this topic can be found in the work done by Moeckel, and by Montgomery.

Appendix A

Figure 8 Phaseportrait

The following Maple code was used to generate the figure eight solution to the three-body problem.

```
> restart:with(plots):with(DEtools):
```

I will define the twelve differential equations below.

```
> eqns1:=t->diff(x1(t),t)=vx1(t):
> eqns2:=t->diff(vx1(t),t)=-(x1(t)-x3(t))/(((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2))^(3/2)-(x1(t)-x2(t))/(((x1(t)-x2(t))^2+
> (y1(t)-y2(t))^2))^(3/2):

> eqns3:=t->diff(y1(t),t)=vy1(t):
> eqns4:=t->diff(vy1(t),t)=-(y1(t)-y3(t))/(((x1(t)-x3(t))^2
> +(y1(t)-y3(t))^2))^(3/2)-(y1(t)-y2(t))/(((x1(t)-x2(t))^2+
> (y1(t)-y2(t))^2))^(3/2):

> eqns5:=t->diff(x2(t),t)=vx2(t):
> eqns6:=t->diff(vx2(t),t)=-(x2(t)-x3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2)-(x2(t)-x1(t))/(((x2(t)-x1(t))^2+
> (y2(t)-y1(t))^2)^(3/2)):

> eqns7:=t->diff(y2(t),t)=vy2(t):
> eqns8:=t->diff(vy2(t),t)=-(y2(t)-y3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2)-(y2(t)-y1(t))/(((x2(t)-x1(t))^2+
> (y2(t)-y1(t))^2)^(3/2)):

> eqns9:=t->diff(x3(t),t)=vx3(t):
```

```
> eqns10:=t->diff(vx3(t),t)=-(x3(t)-x1(t))/(((x3(t)-x1(t))^2
> +(y3(t)-y1(t))^2))^(3/2)-(x3(t)-x2(t))/(((x3(t)-x2(t))^2
> +(y3(t)-y2(t))^2))^(3/2):
```

```
> eqns11:=t->diff(y3(t),t)=vy3(t):
> eqns12:=t->diff(vy3(t),t)=-(y3(t)-y1(t))/(((x3(t)-x1(t))^2
> +(y3(t)-y1(t))^2))^(3/2)-(y3(t)-y2(t))/(((x3(t)-x2(t))^2
> +(y3(t)-y2(t))^2))^(3/2):
```

The following are the initial conditions for the solution.

```
> X1:=-0.97000436:Y1:=0.24308753:
> VX1:=-0.46620369:VY1:=-0.43236573:
> X2:=0.97000436:Y2:=-0.24308753:
> VX2:=-0.46620369:VY2:=-0.43236573:
> X3:=0:Y3:=0:VX3:=0.93240737:VY3:=0.86473146:
```

The below commands generate the solution for planet 1.

```
> pp1:=phaseportrait([eqns1(t),eqns2(t),eqns3(t),
> eqns4(t),eqns5(t),eqns6(t),eqns7(t),eqns8(t),
> eqns9(t),eqns10(t),eqns11(t),eqns12(t)],
> [x1(t),y1(t),vx1(t),vy1(t),x2(t),y2(t),vx2(t),
> vy2(t),x3(t),y3(t),vx3(t),vy3(t)],t=0..40,
> [[x1(0)=X1,y1(0)=Y1,vx1(0)=VX1,vy1(0)=VY1,
> x2(0)=X2,y2(0)=Y2,vx2(0)=VX2,vy2(0)=VY2,
> x3(0)=X3,y3(0)=Y3,vx3(0)=VX3,vy3(0)=VY3]],
> scene=[x1(t),y1(t)],stepsize=.01,x1=-2..2,
> y1=-2..2,color=[red],scaling=constrained):
```

This will generate the solution for the second planet.

```
> pp2:=phaseportrait([eqns1(t),eqns2(t),eqns3(t),
> eqns4(t),eqns5(t),eqns6(t),eqns7(t),eqns8(t),
> eqns9(t),eqns10(t),eqns11(t),eqns12(t)],
> [x1(t),y1(t),vx1(t),vy1(t),x2(t),y2(t),vx2(t),
> vy2(t),x3(t),y3(t),vx3(t),vy3(t)],t=0..40,
> [[x1(0)=X1,y1(0)=Y1,vx1(0)=VX1,vy1(0)=VY1,
> x2(0)=X2,y2(0)=Y2,vx2(0)=VX2,vy2(0)=VY2,
> x3(0)=X3,y3(0)=Y3,vx3(0)=VX3,vy3(0)=VY3]],
> scene=[x2(t),y2(t)],stepsize=.01,x2=-2..2,
> y2=-2..2,color=[red],scaling=constrained):
```

Below is the solution for the third planet.

```
> pp3:=phaseportrait([eqns1(t),eqns2(t),eqns3(t),
> eqns4(t),eqns5(t),eqns6(t),eqns7(t),eqns8(t),
> eqns9(t),eqns10(t),eqns11(t),eqns12(t)],
> [x1(t),y1(t),vx1(t),vy1(t),x2(t),y2(t),
> vx2(t),vy2(t),x3(t),y3(t),vx3(t),vy3(t)],
> t=0..40,[[x1(0)=X1,y1(0)=Y1,vx1(0)=VX1,
> vy1(0)=VY1,x2(0)=X2,y2(0)=Y2,vx2(0)=VX2,
> vy2(0)=VY2,x3(0)=X3,y3(0)=Y3,vx3(0)=VX3,
> vy3(0)=VY3]],scene=[x3(t),y3(t)],
> stepsize=.01,x3=-2..2,y3=-2..2,color=[red],
> scaling=constrained):
```

The command below will display all three solutions on the same graph.

```
> display([pp1,pp2,pp3]);
```

Appendix B

Figure 8 Movie

This is the Maple code that I used to generate the movie demonstrating the figure eight solution to the three-body problem.

```
> restart:with(plots):with(plottools):with(DEtools):Digits:=8:
```

m here is the mass of the planets.

```
> m:=1:
```

I will now define the equations defining the motion of the bodies as eqns1 through eqns12.

```
> eqns1:=t->D(x1)(t)=vx1(t):
> eqns2:=t->D(vx1)(t)=-m*((x1(t)-x3(t))/(((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2))^(3/2)+(x1(t)-x2(t))/(((x1(t)-x2(t))^2+
> (y1(t)-y2(t))^2))^(3/2)):

> eqns3:=t->D(y1)(t)=vy1(t):
> eqns4:=t->D(vy1)(t)=-m*((y1(t)-y3(t))/(((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2))^(3/2)+(y1(t)-y2(t))/(((x1(t)-x2(t))^2+
> (y1(t)-y2(t))^2))^(3/2)):

> eqns5:=t->D(x2)(t)=vx2(t):
> eqns6:=t->D(vx2)(t)=-m*((x2(t)-x3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2)+(x2(t)-x1(t))/(((x2(t)-x1(t))^2+
> (y2(t)-y1(t))^2)^(3/2))):

> eqns7:=t->D(y2)(t)=vy2(t):
```

```
> eqns8:=t->D(vy2)(t)=-m*((y2(t)-y3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2)+(y2(t)-y1(t))/(((x2(t)-x1(t))^2+
> (y2(t)-y1(t))^2)^(3/2))):

> eqns9:=t->D(x3)(t)=vx3(t):
> eqns10:=t->D(vx3)(t)=-m*((x3(t)-x1(t))/(((x3(t)-x1(t))^2+
> (y3(t)-y1(t))^2))^(3/2)+(x3(t)-x2(t))/(((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2))^(3/2)):

> eqns11:=t->D(y3)(t)=vy3(t):
> eqns12:=t->D(vy3)(t)=-m*((y3(t)-y1(t))/(((x3(t)-x1(t))^2+
> (y3(t)-y1(t))^2))^(3/2)+(y3(t)-y2(t))/(((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2))^(3/2)):
```

Below I will define the initial conditions

```
> X1:=-.97000436: Y1:=.24308753:
> VX1:=-.46620369: VY1:=-.43236573:
> X2:=.97000436:  Y2:=-.24308753:
> VX2:=-.46620369: VY2:=-.43236573:
> X3:=0:          Y3:=0:
> VX3:=.93240737:  VY3:=.86473146:
> gg1:=DEplot([eqns1(t),eqns2(t),eqns3(t),eqns4(t),eqns5(t),
> eqns6(t),eqns7(t),eqns8(t),eqns9(t),eqns10(t),eqns11(t),
> eqns12(t)],[x1(t),y1(t),vx1(t),vy1(t),x2(t),y2(t),vx2(t),
> vy2(t),x3(t),y3(t),vx3(t),vy3(t)],t=0..10,
> [[x1(0)=X1,x2(0)=X2,x3(0)=X3,y1(0)=Y1,y2(0)=Y2,y3(0)=Y3,
> vx1(0)=VX1,vy1(0)=VY1,vx2(0)=VX2,vy2(0)=VY2,vx3(0)=VX3,
> vy3(0)=VY3]],scene=[x1(t),y1(t)],
> stepsize=.01,scaling=constrained, arrows=none):
```

The following beginns the program to plot the movie.

```
> g:=dsolve({eqns1(t),eqns2(t),eqns3(t),eqns4(t),eqns5(t),
> eqns6(t),eqns7(t),eqns8(t),eqns9(t),eqns10(t),eqns11(t),
> eqns12(t),x1(0)=X1,y1(0)=Y1,vx1(0)=VX1,vy1(0)=VY1,
> x2(0)=X2,y2(0)=Y2,vx2(0)=VX2,vy2(0)=VY2,x3(0)=X3,
> y3(0)=Y3,vx3(0)=VX3,vy3(0)=VY3},{x1(t),y1(t),vx1(t),
> vy1(t),x2(t),y2(t),vx2(t),vy2(t),x3(t),y3(t),
> vx3(t),vy3(t)},type=numeric,output=listprocedure);
> gx1:=subs(g,x1(t)):gy1:=subs(g,y1(t)):
> gx2:=subs(g,x2(t)):gy2:=subs(g,y2(t)):
> gx3:=subs(g,x3(t)):gy3:=subs(g,y3(t)):
> for i from 1 to 100 do
> px1[i]:=gx1((i-1)/10):py1[i]:=gy1((i-1)/10):
> px2[i]:=gx2((i-1)/10):py2[i]:=gy2((i-1)/10):
> px3[i]:=gx3((i-1)/10):py3[i]:=gy3((i-1)/10):
> pic[i]:=pointplot([[px1[i],py1[i]],[px2[i],py2[i]],
> [px3[i],py3[i]]],symbol=circle,axes=boxed):end do:
> display(seq(display([gg1,pic[i]]),i=1..100),
> insequence=true,scaling=constrained);
```

Appendix C

My Solution

The following Maple code shows the program used to try and find the solution to three-body problem, with a lighter planet coming in from infinity and being trapped in a stable orbit around a tight binary.

```
> restart:with(plots):with(DEtools):
```

I will now define the masses of the three planets.

```
> m1:=1:
> m2:=1:
> m3:=1/25000:
```

I will now define the 12 differential equations

```
> eq1:=t->diff(x1(t),t)=vx1(t):
> eq2:=t->diff(vx1(t),t)=-m2*((x1(t)-x2(t))/
> (((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)))-
> m3*((x1(t)-x3(t))/((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2)^(3/2)):
> eq3:=t->diff(x2(t),t)=vx2(t):
> eq4:=t->diff(vx2(t),t)=-m1*(x2(t)-x1(t))/
> (((x2(t)-x1(t))^2+(y2(t)-y1(t))^2)^(3/2))-
> m3*(x2(t)-x3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2):
> eq5:=t->diff(y1(t),t)=vy1(t):
> eq6:=t->diff(vy1(t),t)=-m2*(y1(t)-y2(t))/
> ((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)-
> m3*(y1(t)-y3(t))/((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2)^(3/2):
> eq7:=t->diff(y2(t),t)=vy2(t):
> eq8:=t->diff(vy2(t),t)=-m1*(y2(t)-y1(t))/
> ((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)-
> m3*(y2(t)-y3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2):
> eq9:=t->diff(x3(t),t)=vx3(t):
> eq10:=t->diff(vx3(t),t)=-m1*(x3(t)-x1(t))/
> ((x1(t)-x3(t))^2+(y1(t)-y3(t))^2)^(3/2)-
> m2*(x3(t)-x2(t))/((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2)^(3/2):
> eq11:=t->diff(y3(t),t)=vy3(t):
> eq12:=t->diff(vy3(t),t)=-m1*(y3(t)-y1(t))/
> ((x1(t)-x3(t))^2+(y1(t)-y3(t))^2)^(3/2)-
> m2*(y3(t)-y2(t))/((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2)^(3/2):
```

I will now define the initial conditions for this solution.

```
> X1:=1:Y1:=0:
> VX1:=-1/(2*m1)*m3*VX3:VY1:=.5-1/(2*m1)*m3*VY3:
> X2:=-1:Y2:=0:
> VX2:=-1/(2*m2)*m3*VX3:VY2:=-.5-1/(2*m2)*m3*VY3:
> X3:=8:Y3:=8:VX3:=-.3:VY3:=0:
```

I will now generate the solution for the first planet.

```
> pp1:=phaseportrait([eq1(t),eq2(t),eq3(t),eq4(t),
> eq5(t),eq6(t),eq7(t),eq8(t),eq9(t),eq10(t),
> eq11(t),eq12(t)],[x1(t),y1(t),vx1(t),vy1(t),
> x2(t),y2(t),vx2(t),vy2(t),x3(t),y3(t),
> vx3(t),vy3(t)],t=0..40,[[x1(0)=X1,y1(0)=Y1,
> vx1(0)=VX1,vy1(0)=VY1,x2(0)=X2,y2(0)=Y2,
> vx2(0)=VX2,vy2(0)=VY2,x3(0)=X3,y3(0)=Y3,
> vx3(0)=VX3,vy3(0)=VY3]],scene=[x1(t),y1(t)],
> stepsize=.01,x1=-2..2,y1=-2..2,color=[red],
> scaling=constrained):
```

This is the solution for the second planet.

```
> pp2:=phaseportrait([eq1(t),eq2(t),eq3(t),eq4(t),
> eq5(t),eq6(t),eq7(t),eq8(t),eq9(t),eq10(t),
> eq11(t),eq12(t)],[x1(t),y1(t),vx1(t),vy1(t),
> x2(t),y2(t),vx2(t),vy2(t),x3(t),y3(t),
> vx3(t),vy3(t)],t=0..40,[[x1(0)=X1,y1(0)=Y1,
> vx1(0)=VX1,vy1(0)=VY1,x2(0)=X2,y2(0)=Y2,
> vx2(0)=VX2,vy2(0)=VY2,x3(0)=X3,y3(0)=Y3,
> vx3(0)=VX3,vy3(0)=VY3]],scene=[x2(t),y2(t)],
> stepsize=.01,x2=-2..2,y2=-2..2,color=[red],
> scaling=constrained):
```

And finally for the third planet.

```
> pp3:=phaseportrait([eq1(t),eq2(t),eq3(t),eq4(t),
> eq5(t),eq6(t),eq7(t),eq8(t),eq9(t),eq10(t),
> eq11(t),eq12(t)],[x1(t),y1(t),vx1(t),vy1(t),
> x2(t),y2(t),vx2(t),vy2(t),x3(t),y3(t),
> vx3(t),vy3(t)],t=-100..2000,[[x1(0)=X1,y1(0)=Y1,
> vx1(0)=VX1,vy1(0)=VY1,x2(0)=X2,y2(0)=Y2,
> vx2(0)=VX2,vy2(0)=VY2,x3(0)=X3,y3(0)=Y3,
> vx3(0)=VX3,vy3(0)=VY3]],scene=[x3(t),y3(t)],
> stepsize=.01,x3=-10..30,y3=-50..10,color=[red],
> scaling=constrained):
```

I will now finally display the graph of all three solutions
> .

```
> display([pp1,pp2,pp3]);
```

Appendix D

My Solution Movie

The following code will show the program that I used to generate a movie of the captured planet problem I looked at.

```
> restart:with(plots):with(DEtools):with(plottools):
```

Below I will define the masses of the three planets.

```
> m1:=1:
> m2:=1:
> m3:=1/25000:
```

Now I will define the 12 equations of the dynamical system

```
> eq1:=t->diff(x1(t),t)=vx1(t):
> eq2:=t->diff(vx1(t),t)=-m2*((x1(t)-x2(t))/
> (((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)))-
> m3*((x1(t)-x3(t))/((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2)^(3/2)):
> eq3:=t->diff(x2(t),t)=vx2(t):
> eq4:=t->diff(vx2(t),t)=-m1*(x2(t)-x1(t))/
> (((x2(t)-x1(t))^2+(y2(t)-y1(t))^2)^(3/2))-
> m3*(x2(t)-x3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2):
> eq5:=t->diff(y1(t),t)=vy1(t):
> eq6:=t->diff(vy1(t),t)=-m2*(y1(t)-y2(t))/
> ((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)-
> m3*(y1(t)-y3(t))/((x1(t)-x3(t))^2+
> (y1(t)-y3(t))^2)^(3/2):
> eq7:=t->diff(y2(t),t)=vy2(t):
> eq8:=t->diff(vy2(t),t)=-m1*(y2(t)-y1(t))/
> ((x1(t)-x2(t))^2+(y1(t)-y2(t))^2)^(3/2)-
> m3*(y2(t)-y3(t))/((x2(t)-x3(t))^2+
> (y2(t)-y3(t))^2)^(3/2):
> eq9:=t->diff(x3(t),t)=vx3(t):
> eq10:=t->diff(vx3(t),t)=-m1*(x3(t)-x1(t))/
> ((x1(t)-x3(t))^2+(y1(t)-y3(t))^2)^(3/2)-
> m2*(x3(t)-x2(t))/((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2)^(3/2):
> eq11:=t->diff(y3(t),t)=vy3(t):
> eq12:=t->diff(vy3(t),t)=-m1*(y3(t)-y1(t))/
> ((x1(t)-x3(t))^2+(y1(t)-y3(t))^2)^(3/2)-
> m2*(y3(t)-y2(t))/((x3(t)-x2(t))^2+
> (y3(t)-y2(t))^2)^(3/2):
```

Below I will define the initial conditions.

```
> X1:=1:Y1:=0:
> VX1:=-1/(2*m1)*m3*VX3:VY1:=.5-1/(2*m1)*m3*VY3:
> X2:=-1:Y2:=0:
> VX2:=-1/(2*m2)*m3*VX3:VY2:=-.5-1/(2*m2)*m3*VY3:
> X3:=8:Y3:=8:
> VX3:=-.3:VY3:=0:
```

The following commands begin to generate the movie.

```
> g:=dsolve({eq1(t),eq2(t),eq3(t),eq4(t),eq5(t),
> eq6(t),eq7(t),eq8(t),eq9(t),eq10(t),eq11(t),
> eq12(t),x1(0)=X1,x2(0)=X2,vx1(0)=VX1,vx2(0)=VX2,
> y1(0)=Y1,y2(0)=Y2,vy1(0)=VY1,vy2(0)=VY2,
> x3(0)=X3,y3(0)=Y3,vx3(0)=VX3,vy3(0)=VY3},
> {x1(t),x2(t),y1(t),y2(t),vx1(t),vx2(t),
> vy1(t),vy2(t),x3(t),y3(t),vx3(t),vy3(t)},
> type=numeric,output=listprocedure);
```

I will now determine the number of frames for the movie.

```
> p:=2500:
> gx1:=subs(g,x1(t)):gy1:=subs(g,y1(t)):
> gx2:=subs(g,x2(t)):gy2:=subs(g,y2(t)):
> gx3:=subs(g,x3(t)):gy3:=subs(g,y3(t)):
```

Next I will generate the frames for the movie

```
> for i from 0 to p do
> px1[i]:=gx1(-(i-1)):py1[i]:=gy1(-(i-1)):
> px2[i]:=gx2(-(i-1)):py2[i]:=gy2(-(i-1)):
> px3[i]:=gx3(-(i-1)):py3[i]:=gy3(-(i-1)):
> pic1[i]:=pointplot([[px1[i],py1[i]]],symbol=circle):
> pic2[i]:=pointplot([[px2[i],py2[i]]],symbol=circle):
> pic3[i]:=pointplot([[px3[i],py3[i]]],symbol=box):
> end:
```

Finally I will display the movie

```
> display(seq(display([pic1[i],pic2[i],pic3[i]]),i=0..p),
> insequence=true,scaling=constrained);
```

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