

PRESERVICE ELEMENTARY TEACHERS' USE OF MATHEMATICAL DISCUSSION IN  
THE IMPLEMENTATION OF PROBLEM-SOLVING TASKS

by

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Under the Direction of Denise A. Spangler

ABSTRACT

Engaging in mathematical communication holds promise for student learning, but focusing teaching practice on mathematical discourse is challenging. Teacher education has been shown to have weak impact on practice, and the conservative nature of K-12 schools serves only to perpetuate, rather than reform, existing practice. This study describes preservice elementary teachers' initial efforts at facilitating mathematics discussions on problem solving activities.

Participants were enrolled in a mathematics teaching methods course that included a field experience working with pairs of elementary pupils. Data were collected during cycles of planning, enactment, and reflection and included activity plans, construction of hypothetical student-teacher conversations (task dialogues), video of problem enactments, and reflections. Analysis utilized Stein and Smith's (1998) construct of cognitive demand and Hufferd-Ackles, Fuson, and Sherin's (2004) math-talk framework.

Findings indicate repeated enactments, collaboration with peers, and analytic reflections helped participants improve the quality of mathematical discussions with pupils. Participants' visions of leading mathematics discussions generally aligned with enactment, though all participants struggled in hypothesizing pupils' potential strategies. A modified version of the

math-talk framework alongside the cognitive demand framework were useful in making a fine-grained analysis of interactions between participants and pupils. Important links were found between cognitive demand and particular pedagogical moves: Participants who consistently elevated cognitive demand relied on questioning and exploring the pupil's thinking, whereas participants who lowered cognitive demand replaced pupils' thinking with their own. Some moves had varied effects on cognitive demand: eliciting explanations and coordinating collaboration between pupils. Repeated enactments of problems provided evidence that 6 of the 8 participants developed varying degrees of pedagogical content knowledge related to implementing their problems and facilitating mathematical discussions.

Findings point to areas of focus for preservice teacher education. More concentrated practice is needed on enacting pedagogical moves that elicit high-quality mathematical explanations and coordinate meaningful collaboration among pupils. Targeted work on developing pupil solution strategies and responding to unanticipated solutions may help preservice teachers enact instruction that is more aligned with pupil thinking. Task dialogues may serve as predictive tools for teacher educators and may be useful in helping preservice teachers think about and respond to pupil solutions.

INDEX WORDS: Preservice elementary mathematics teachers, Field experiences, Mathematical discussions, Discourse, Math-talk, Cognitive demand, Problem solving

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A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial  
Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2011

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December 2011

DEDICATION

For Jacob and Chris.

## ACKNOWLEDGEMENTS

Had Dr. Jim Wilson not been my first contact with The University of Georgia, I may have decided to go elsewhere, or even decided that graduate school was not for me. I am indebted to him for his early encouragement that brought me to Athens and, later, for his tough critiques that have made my time here worthwhile. I have been fortunate to have Dr. Jeremy Kilpatrick on my committee as well. He has never failed to pose insightful questions, patiently listen to my less-than-insightful answers, and hold me accountable for coming up with better ones. I cannot extend enough thanks to my major professor, Dr. Denise Spangler, for putting aside football rivalries (sometimes) and agreeing to take me on as student. Without her unwavering support, encouragement, guidance, and feedback in the face of what I thought were surely insurmountable obstacles I would not have completed this work. I will always appreciate how she has helped me have confidence in my ideas and pushed my thinking in new directions. She has changed my understanding of what mathematics teaching should be and I am a better teacher for having been her student.

I also owe a debt of gratitude to my fellow graduate students for friendship, reassurance, and advice; my time at Georgia was far more pleasant having you there. Brian and Sharon, my most excellent office mates, Aderhold has not been the same without you. Katy, my fearless co-editor, we are finally caught up! Kyle, Rachael, and (especially) Ginger, your encouragement and feedback on my ideas has been invaluable. Eric, thank you for always listening to my frustrations and never holding it against me later. Colleen and Kelly, your support and hysterical

company made all the late nights studying fun. And Jenny, the best teaching assistant ever, your calm during data collection was (almost) contagious.

My family is awesome and I could not have accomplished this without their help and support. Mom and Dad, I cannot thank you enough for the hours spent driving across multiple states to come cook, clean, babysit, grocery shop, paint, remodel (almost a whole house!), and generally do anything that I've had to ignore so that I could finish this dissertation. Mom, you get bonus points for proofreading this entire document. Dad, you get them for keeping me financially afloat. I have only just begun to understand how much you both have done for me. Joe and Kayce, your friendship and support, even when it has been from afar, has meant as much to me this last year as did in chemistry (oh-so-many years ago). Chris, without your editing and technical support this project would have been a disaster; without your moral support my sanity would be a disaster, and without all your help at home, our family and house would be disaster (a complete list of three). Jacob and I are incredibly lucky to have you. Jacob, thank you for giving me a good reason to take a break; I promise I can take more of them now.

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## CHAPTER ONE

### INTRODUCTION

Implementing ambitious mathematics teaching, in which advanced learning goals are achieved by *all* students, places heavy demands on teachers, in terms of both the content they are expected to know and the pedagogical practices they need to implement. Of all teachers, elementary teachers are often the least prepared to enact such ambitious practice. This study focused on a particular piece of instructional practice, mathematical discourse, and the supports that can be provided in teacher education programs to better prepare elementary teachers for engaging their students in meaningful, effective mathematics discussions.

Since the 1960s, international comparisons of student achievement have shown that American students lag behind those of other industrialized nations in mathematics (U.S. Department of Education, 1992), and over the last 3 decades, low student achievement has garnered increasing national attention (National Commission on Excellence in Education, 1983; National Council of Teachers of Mathematics [NCTM], 1989, 1991, 2000, 2006; National Research Council [NRC], 1989; No Child Left Behind Act of 2001). Though the United States has made gains in student achievement in mathematics since the 1964 First International Mathematics Study, only 40% of fourth graders and 35% of eighth-grade students scored proficient or better on the 2011 administration of the National Assessment of Educational Progress (National Center for Education Statistics, 2011). In 1995, the Third International Mathematics and Science Study (TIMSS) confirmed the trend of low student achievement, and

the 1995 TIMSS Video Study pointed to a possible cause: the nature of mathematics teaching in American classrooms. A comparison of mathematics and science teaching in the United States, and Japan, showed that American teachers state rather than develop concepts and that reasoning takes a back seat to practicing procedures, trends that stood in marked contrast to mathematics teaching in high-achieving Japan (Hiebert & Stigler, 2000). A 1999 follow-up study with a larger sample of countries revealed a fundamental difference within U.S. mathematics classrooms: “Virtually none of the making-connections problems in the U.S. were discussed in a way that made the mathematical connections or relationships visible for students” (TIMSS Video Mathematics Research Group, 2003, p. 773). Though the sample of the teaching episodes is not such that sweeping generalizations about any particular nation’s teaching can be made, it does point to areas in need of increased attention in U.S. mathematics classrooms.

Though TIMSS did not focus on elementary mathematics classrooms, other research supports the notion that K-8 mathematics instruction centered on adept performance of routine procedures and efficient execution of computational algorithms for basic operations is inadequate for an increasingly technology-dependent world (National Commission on Excellence in Education, 1983; NCTM, 1980, 1989; NRC, 1989). At the K-8 level, which is the focus of this study, mathematics goals for students must include, in addition to computational fluency, solving challenging nonroutine problems, using multiple strategies, and explaining and justifying their *own* strategies. Students should explain their ideas to the teacher and fellow students, construct arguments, and question the validity of others’ arguments. Teachers should make connections among mathematical ideas explicit for students, provide them opportunities to think about mathematics conceptually, require them to explain and communicate about mathematical ideas, and focus their work reasoning and problem solving.

### **Why Discourse Is Important**

This study and the teacher education program in which it was set are predicated on the premise that children construct their own knowledge of mathematics based on prior experience and that “individual learning is dependent on social interaction” (van Oers, 1996, p. 93). These ideas originate from Vygotsky’s sociocultural theory that student learning occurs through social interaction in the form of collaboration with others. In the setting of this study, preservice teachers used social interaction in the form of mathematical discussions to help pupils become independent problem-solvers, and they needed to give pupils the opportunity to construct their own mathematics rather than imposing the teachers’ mathematics. NCTM (2000) supports this view as well:

It is important to encourage students to represent their ideas in ways that make sense to them, even if their first representations are not conventional ones. It is also important that they learn conventional forms of representation to facilitate both their learning of mathematics and their communication with others about mathematical ideas. (p. 67)

In order to guide the students in building an understanding of mathematics, a teacher must know the students and how to raise their current activities to a more sophisticated level. The teacher must carefully select and organize students’ mathematical activity in order to make critical aspects of mathematical concepts readily apparent. With this approach to learning underpinning instructional practices, the need for organizing and monitoring social interaction also takes on new importance.

Since the 1980s many mathematics curriculum standards have called for attention to mathematical communication (Blanton, Berenson, & Norwood, 2001; NCTM, 1989, 1991, 2000, 2006; NRC, 2001, 2008; Peressini and & Knuth, 1998). NCTM (2000) considered

communication so important that they devoted an entire process standard to it, setting forth the following goals for students:

organize and consolidate their mathematical thinking through communication; communicate their mathematical thinking coherently and clearly to peers, teachers, and others; analyze and evaluate the mathematical thinking and strategies of others; use the language of mathematics to express mathematical ideas precisely. (p. 194)

The broad description offered is open to a variety of interpretations and, accordingly, it is possible to use mathematical communication in ways that do not support ambitious learning goals for all students. For example, classroom mathematics talk could still focus on executing procedures, and explanations could go unchallenged, thus limiting students' opportunities to build understanding of important mathematics concepts.

The type of communication needed to support mathematical learning entails a significant change from more typical classroom interactions where the teacher initiates interaction with a question, usually with a specific response in mind, students respond, and the teacher evaluates that response (Franke, Kazemi, & Battey, 2007). Instead, the teacher should guide the students' work on challenging mathematics by facilitating and establishing norms for mathematical discourse that requires students to discuss, represent, analyze, evaluate and justify their thinking about mathematical *ideas*. A mathematics discussion is one medium through which discourse can occur, and throughout this paper I use the terms *discourse* and *discussion* interchangeably. Further, I set out to study the development of effective discourse, that which fosters children's understanding of significant mathematical concepts and encourages them to become independent problem-solvers.

Discourse analysis has focused on discourse as a form of collective argumentation (Wood, 1999), discourse mediated by tools (Sfard, 2000), teacher moves to facilitate and maintain a discourse-rich environment (Chapin, O'Conner, & Anderson, 2003), and the qualities

of student-to-student discourse (Kieran, 2001). However, few studies have directly tied participation in a discourse community to student learning. Krummheuer (1995, 2007) made a case for a particular format of discourse, argumentation, as having an influence on student learning. He found that “successful conceptual mathematical learning is reflexively based on the active participation in formatting the core of an argument that is acceptable to all participants” (1995, p. 264). Sfard (2001) also agreed with this idea of the reflexive relationship between learning and discourse, arguing that “communication should not be viewed as a mere aid to thinking, but almost as tantamount to the thinking itself” (p. 1).

With so few studies producing direct evidence that show the value of discourse in promoting learning, it could be argued that discourse is an inappropriate focus for instructional efforts. Yet, a focus on discourse is justified by an understanding of how students learn and the opportunities for learning that discourse can afford. Creating a discourse-oriented mathematics community provides an environment that can support and encourage learning. Mathematical discussions provide teachers the opportunity to assess students, explore their ideas, and extend their thinking (Chapin et al., 2003; O’Connor, 2001; Pirie & Schwarzenberger, 1988). Blanton et al. (2001) claims that “mathematics teachers’ ability to cultivate serious mathematical thinking in students rests on the nature of classroom discourse” (p. 241). By making their thinking public, students have the opportunity to wrestle with significant mathematical ideas.

### **Difficulties in Discourse-Oriented Teaching**

Enacting discourse-based teaching practice has several significant potential roadblocks. Mathematics is to be taught in ways that are unfamiliar to many teachers and students. The changes constitute a major shift in teacher and student roles that can be discomfiting. Students should actively generate knowledge rather than passively listening to teacher lectures. Students

develop authority to judge the validity of their own and each other's arguments; the teacher is not the sole authority of mathematical truth. To facilitate meaningful mathematical communication necessary to achieve the kind of learning NCTM (1989, 1991, 2000, 2006) advocates, the teacher must carefully develop classroom culture and norms for behavior, which represents a dramatic shift from the traditional teacher-centered classroom. Moreover, teachers' views on teaching and learning mathematics may be different from the philosophy underpinning the curricular changes, further compounding the difficulty of enacting these changes to pedagogy.

Focusing teaching practice on discourse requires changing a central feature of instruction: student and teacher interactions. Incorporating reasoning, problem solving, and mathematical communication in mathematics teaching with is no small task, particularly for novice teachers who learned mathematics by mimicking procedures demonstrated by a teacher. Mathematics teachers may believe they are implementing reforms, but researchers have found that the core of their instructional practice remains unchanged (Cohen, 1990; Cohen & Ball, 2001; Elmore, 2008; Spillane, 2004; Tyack & Cuban, 1995). Teachers adopt reforms in a piecemeal fashion. They focus on changing the form of their instruction but not the content. For example, teachers may incorporate group work, manipulative use, or mathematical conversations so that to the casual observer their instruction looks different. But closer examination of their teaching often shows that the nature of their interactions with students has not changed (Cohen, 1990; Spillane, 2004). Instruction still focuses on imitating the teacher, interactions among group members are shallow, and manipulatives are not used to develop mathematical concepts. Schoenfeld (2004) describes the effects of this phenomenon, "When superficial aspects of reform are implemented without the underlying substance, students may not learn much at all" (p. 272). The present study

focused on the intended substance of reforms, the core of teaching and learning: how students and teachers interact.

In addition to challenging pedagogical demands of discourse-based instruction, the increased rigor of mathematics content is likely problematic for many teachers. Justifying why a procedure works, or helping children do so, requires a deeper understanding of content than does only modeling and describing a procedure. Elementary school teachers are of particular interest, as they frequently have the least exposure to rigorous mathematics content (as compared with middle or secondary teachers). They are known to have unconnected, procedure-oriented content knowledge (Ball, 1990; Ma, 1999), and this fragmented understanding is likely to hinder their ability to teach mathematics in ways that make the connections among concepts explicit to students. Teachers lacking a depth of content knowledge will struggle to incorporate higher-order processes of reasoning, problem solving, and mathematical communication into their teaching. Maintaining the cognitive demand of mathematical tasks (e.g., requiring students develop their own strategies for problem solving rather than imitating the teacher's strategy) is essential to ensure that students grapple with challenging mathematics and that their participation in discourse does not become another rote classroom activity. Teachers who do not fully understand the mathematics they are teaching may inadvertently lessen the challenge and mismanage classroom discussions.

### **Rationale**

Difficulties of developing meaningful classroom interactions may be exacerbated by two common problems: Teacher education does not prepare teachers to successfully enact discourse-based instruction (Nathan & Knuth, 2003), and K–12 schools do not have a system in place to support teachers' sustained learning about teaching (Feiman-Nemser, 2001). Questions of when

to intervene in students' work and how to navigate the teacher's role in discourse need to be explored in teacher education before teachers find themselves floundering in their first teaching jobs.

Yet, little research has shown direct links between pedagogical preparation and student learning (Wilson, Floden, & Ferrini-Mundy, 2002). Research has shown the weak impact of teacher education and professional development on teaching practice, and teachers' knowledge of mathematics is often considered the primary factor in students' mathematical deficiencies (Ball, Lubienski, & Mewborn, 2001). Moreover, field experiences do not always provide opportunities for preservice teachers to practice pedagogical strategies touted by their teacher education programs. When presented an opportunity to practice student-centered teaching, preservice teachers struggle to anticipate the direction a lesson may take (Inoue & Buczynski, 2011).

Expected to be experts in all content areas and given minimal preparation in each, elementary teachers may not be equipped to teach mathematics where mathematical discourse, along with the sophisticated reasoning and problem-solving needed to support it, take center stage. Research has shown that preservice and inservice elementary teachers struggle to facilitate mathematical discourse (Baxter & Williams, 1996; Nicol 1999), implement problem-solving instruction (Peterson, Carpenter, & Fennema, 1989), teach using student-centered instruction and new curricula (Cohen, 1990), and explain mathematical concepts underlying basic arithmetic procedures (Ball, 1990). With reasoning, problem solving, and mathematical discourse taking a more central role, and preservice teachers ill-equipped to structure their practice around these processes, a study of their learning about enacting mathematics discussions is warranted. Such a

study can offer insight into the difficulties they face and how preservice teacher education programs might respond.

Additionally, studies of teachers conducting discourse-based instruction often focus on experts (Ball, 1993; Lampert 1990). Though useful for providing exemplars of best practices, they do not provide insight on how typical teachers might develop such practices. More work is needed examining how “ordinary” teachers learn to change instruction (Nathan & Knuth, 2003). Mewborn (2001) agrees that more studies of change over time are required. Thus, this study examined change in ability to facilitate effective mathematics discussions over time and went a step further by looking at how prospective teachers, those with the least experience, might develop their interactions with children.

### **Purpose and Research Questions**

In seeking to understand how preservice elementary teachers learn to teach in ways consistent with NCTM’s (1989, 1991, 2000, 2006) vision for learning and teaching mathematics, I chose to zoom in on a small piece of teaching practice. Because effectively facilitating mathematical discussions about problem solving provides opportunities for students’ mathematical learning, and teacher education programs provide a rich setting for learning to facilitate discussions, I studied preservice elementary teachers leading discussions between two students on problem-solving activities. By focusing on only two students, I provided novice practitioners a group large enough to have student-to-student dialogue but small enough that supervising student contributions and behavior was manageable. In particular, I asked:

- 1) How does preservice elementary teachers’ use of mathematical discussion support the cognitive demand of problem-solving activities?

- 2) How does a cycle of planning, enacting, and reflecting influence preservice elementary teachers' implementation of problem-solving activities over a 6-week field experience?
- 3) How do preservice elementary teachers envision leading mathematical discussions on problem-solving activities and how do these visions align with their enactment?

## CHAPTER TWO

### LITERATURE REVIEW

This chapter provides an overview of literature that underpinned the design of this study and discusses the conceptual framework used in the data analysis. I reviewed several bodies of literature: preservice teacher education, problem solving, cognitive demand, and mathematical discourse. I begin by giving a synopsis of literature on mathematical discourse, which was the focus of this study, and mathematical problem solving which provided the context for participants' work. I consider challenges of implementing instruction focused on mathematical discourse and problem solving. Of those challenges establishing and maintaining high cognitive demand is of particular interest. I contend that in order to improve teachers' ability to establish and sustain instruction focused on discourse, preservice teacher education must play a significant role. However, teacher education has its own hurdles yet to overcome. Thus, I examine two prevailing ideas about teacher education that hold promise for responding to the difficulties of implementing instruction based on mathematical discourse: mathematical knowledge for teaching (MKT) and practice-based teacher education. Drawing on these two constructs, I make a case for the practice-based teacher education that works to develop pedagogical content knowledge, which formed the basis for this study.

## Mathematics Discourse

### Defining Discourse

In general terms, mathematical discourse is simply defined as communication about mathematics. Within this broad definition of mathematical discourse, a teacher who consistently poses questions and elicits correct responses from students would be considered to be including mathematical discourse as a centerpiece of instruction. But such a picture of discourse is incomplete. Students merely answering teacher questions is insufficient. Communication of the form initiation (by teacher), response (from student), and evaluation (by teacher) presents an authoritarian approach that is not representative of the ways that mathematicians approach a study of mathematics. This type of method acculturates the students to a narrow view of mathematics in which all questions have clearly defined correct answers and all problems are solved through a single algorithm that produces one correct solution.

The kind of talk intended by mathematical discourse includes students who “initiate lines of inquiry and challenge the ideas presented by peers, the teacher, and textbooks” (Nathan & Knuth, 2003, p. 176). Ghouseini (2008) draws a distinction between the *discourse of formal mathematics* and the *mathematics discourse in the school classroom*. She defines the former as “reasoning about abstract entities; a process built essentially around the development, justification, and use of mathematical generalizations ... [and] figuring out what is true once the members of the discourse community agree on their definitions and assumptions” and the latter as “ways in which mathematics knowledge is acquired and exchanged in classrooms in the interactions between students and teachers around content” (p. 3). That is, formal mathematical discourse relies on the definitions, assumptions, and norms for argumentation developed by a community of experts that are used to prove general theorems. Translated to the K–12 mathematics classroom, mathematics discourse develops less rigid norms as its norms are determined by non-experts. Thus classroom

mathematics discourse looks like an activity where students share their reasoning with the class and use talk that relies on evidence and mathematical reasoning to justify those ideas (Lampert, 1990) and this is the image of discourse that formed the focus of this study. I elaborate on this description in the next section.

### **Description of a Discourse-Based Mathematics Classroom**

When classroom instruction is based on discourse, both the teachers and students have significant contributions to classroom activity. The teacher does not deliver information and procedures; the teacher instead elicits student thinking, poses challenging questions, positions students to address one another's ideas, connects student ideas, structures presentation of solution methods and representations to highlight key mathematical concepts, and knows when and how to intervene in student discussions. Teachers model the behavior and disposition that they hope to see in their students, and they understand how their pupils think and the possible directions that thinking could take (Wood, Cobb, & Yackel, 1993). The teacher acts as a highlighter, directing student attention to specific solution strategies and the connections among them. The focus of discussion is not restricted to attaining a correct answer; it extends to examining and justifying the process through which a solution is obtained.

Wrong answers often provide the springboard for fruitful discussions. A contradiction between a student solution and a given parameter of a problem can lead to productive discourse. Both correct and incorrect solutions are addressed so that students have the opportunity to understand for themselves why solutions are invalid (Staples & Colonis, 2007). The teacher maintains a balance between "tolerating a child's error, recognizing that it may represent an initial understanding of a new construction, and identifying when it is nonproductive and there is a necessity to constrain it" (Wood et al., 1993, p. 65).

Students too have the substantial task of thinking and communicating about mathematics. They actively generate knowledge through discussion and reasoning with one another; the teacher works together with students, posing problems and questions, to encourage their engagement with mathematics. Throughout a single class, a student's roles may shift from speaker to listener to questioner. Students' classroom activity includes "making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions, and discussing and questioning their own thinking and the thinking of others" (Lampert, 1990, pp. 32–33). The discourse in which students engage includes purposeful talk that is framed in terms of mathematical content or process and that is based upon goals set by the group or teacher (Pirie & Schwarzenberger, 1988). Contributions from students extend beyond factual answers and should move the conversation forward; they offer solutions and justification for their thinking, and challenge and justify the thinking of their classmates. Interaction among students, not just between student and teacher, is a hallmark of discourse that also constitutes a substantial change to the students' role. Student-to-student interaction includes evidence of critical listening, voicing agreement or opposition, and the questioning of conjectures (Pirie & Schwarzenberger, 1988).

### **Frameworks for Analyzing Discourse**

A discourse-driven mathematics class creates a complex classroom dynamic worthy of detailed analysis. Several frameworks have been developed for studying discourse-based classrooms, but their usefulness for looking at teachers learning to facilitate discourse is limited.

Knuth and Peressini (2001) categorize discourse as univocal or dialogic. Any discourse that conveys meaning is univocal, whereas discourse that generates meaning can be considered dialogic. Univocal discourse is the process by which a listener receives the exact message

intended by the speaker. Dialogic discourse begins where univocal discourse ends. After the listener receives the speaker's message, they together generate meaning through shared dialogue, what Lakatos (1976) calls the "zig-zag of discovery" (p. 42). This dialogic discourse is consistent with Lampert's (1990) and Ghouseini's (2008) conceptions of classroom mathematics discourse, in which students create their own meaning of mathematics through communicating with others, thereby generating a shared understanding. In dichotomizing discourse, this framework is overly simplistic, giving the impression that univocal discourse is always bad and dialogic discourse is always good. And, though the univocal-dialogic framework can be used to study the teacher, because it has only two categories using it to assess a teacher's growth over time is difficult. Truxaw (2004) gives a far more detailed picture of dialogic and univocal discourse by using categories of inert and generative verbal assessment. Though her work is more fine-grained than Knuth and Peressini's, it still provides only broad characterizations. The univocal-dialogic framework is more useful for providing snapshots of teachers' practice and general descriptions of their interactions with students than for giving a picture of the small changes over time that were the focus of the current study. I wanted a more robust picture of mathematics teaching that avoided "stereotyping mathematics class discussions as either good or bad" (Crespo, Oslund, & Parks, 2011, p. 130).

The creation of discourse as part of the instructional responsibility of the teacher requires that he or she attend to many social and mathematical norms to establish for the class how and what it means to do mathematics. In the process, a teacher must possess both subject matter knowledge and knowledge of the students' mathematics. Cobb and Yackel (1996) discussed the definition and development of sociomathematical norms in the classrooms of elementary teachers as they implemented an inquiry-based teaching practice. *Mathematical difference*,

determining what constitutes difference among student solution paths, is negotiated together by teacher and students. *Mathematical sophistication*, refers to the value placed on different solutions by the teacher and ultimately the students. Together the evaluation of difference and sophistication inform what is considered an *acceptable justification*, one of the central tasks in a discourse-based mathematics class. This framework is more helpful than Knuth and Peressini's (2001) because it does attend to specific elements of discourse, how the class defines and values different solutions and the justification of those solutions, providing a more multi-faceted way of assessing a teachers' facilitation of discourse. Though a teacher's abilities to recognize difference, sophistication, and adequate explanation are necessary to facilitate mathematics discourse, negotiating difference and sophistication with students requires a detailed understanding of mathematics and pre-existing classroom norms for how to communicate, two things not typically available to novice teachers. Attending to these issues of difference, sophistication, and adequate explanation needs far more sophisticated skills, both mathematically and pedagogically, than are those likely to be found in novice teachers' practice. Because of the difficulty in studying novice teachers using sociomathematical norms to facilitate discourse, this framework was not an appropriate choice for the current study.

However, Hufferd-Ackles, Fuson, and Sherin (2004) developed a framework that addresses some of these problematic issues, thus making it a good fit for this study. Their framework came out of a year-long observation of one teacher as she worked to build what they called a math-talk community in a whole class setting. In a math-talk community, all participants, both teacher and students, work together using discourse to extend everyone's thinking. The framework examines four aspects of classroom discussions: questioning, explaining mathematical thinking, source of ideas, and responsibility for learning. Because of its

attention to these four components, the framework is useful for highlighting specific teacher moves and correlating them to other features of classroom interactions. This framework also takes into account the actions of the teacher and how the student responds to those actions, thus allowing me to evaluate the effect of particular teacher moves. Most importantly, the multi-leveled feature of the framework, which ranges from a strictly teacher-directed classroom to a full-blown description of dialogic discourse, allows for a study of changes over time and can capture the early stages of a novice teacher's work. What was missing in this framework was the ability to associate discourse with opportunities for student learning. Thus, for data analysis in this study, I supplemented the math-talk framework with a framework for cognitive demand (Stein, & Smith, 1998), which is discussed in a later section.

### **Challenges to Discourse-Based Classrooms**

Successfully orchestrating mathematics discourse in the classroom is no easy task. The mathematics education research literature has provided insight on the difficulties teachers face. Before a teacher even enters the classroom, his or her beliefs about mathematics and children may restrict what he or she can accomplish in class. Warfield, Wood, and Lehman (2005) found that a teacher's beliefs about children's ability to construct their own mathematics interfered with the quality of discussion they were able to facilitate. The teacher who is not restricted by her beliefs must nonetheless tackle existing school norms (Blanton et al., 2001) and confront years of students' own school experiences, which is particularly intimidating for novice teachers who have less established reputations and expertise on which to draw.

In achieving classroom mathematics discourse, Lampert (1990) described the roles of the teacher and student undergoing a discomfiting change that contradicts some basic tenets of school mathematics: The teacher is the bearer of mathematical truth, doing mathematics is

performing rules and procedures, and knowing mathematics is the correct application of those rules and procedures. Adler (1999) noted that this type of discourse-oriented teaching practice has the power to give all students access to mathematics, thereby promoting equity, but it is contingent on developing the students' mathematical authority equitably as well. Yet, Peressini and Knuth (1998) found relinquishing mathematical authority to the students posed an insurmountable obstacle for the teacher they studied. Cobb and Yackel's (1996) response to the potential unequal distribution of authority among class members is for each student to develop intellectual autonomy, the ability to rely on his or her own thinking and judgment as sources for mathematical reasoning. In a study of effective mathematical discourse that generated taken-as-shared understanding, Cobb and Yackel found that attending to sociomathematical norms helped the students develop intellectual autonomy. However, as discussed earlier, attention to sociomathematical norms poses a significant challenge to novice teachers.

Mathematical discourse changes the dynamics of the classroom so that students learn from each other. However, the complexities of student-to-student classroom interactions also stand in the way of developing rich discourse communities. Kieran (2001) found, "Making one's emergent thinking available to one's partner in such a way that the interaction be highly mathematically productive for both may be more of a challenge to learners than is suggested by the current mathematics education research literature" (p. 220). Ge and Land (2004) stated that not all types of peer interactions supported students' problem solving. Managing student-to-student talk may require more skill than prospective teachers possess; thus, in this study, I structured an experience to help preservice teachers develop and practice this skill.

Developing discourse in the mathematics classroom that focuses on mathematical content and still maintains social norms for classroom communication is difficult. Baxter and Williams

(1996) examined the social and analytical scaffolding a teacher provided to develop discourse in the classroom. At times, although the students' contributions did not express understanding of content, they were of a particular form they knew the teacher valued. At other times, students viewed discourse practices as a mindless requirement and were not able to appreciate the discourse as a tool to solve problems and further their understanding. The classroom discourse did not always lead to the production of common knowledge as the teacher had expected. Krummheuer (1995) agreed that the form and content of discourse may both constrain and support students' participation and their subsequent learning. Similarly, Nathan and Knuth (2003) also found that their teacher participants "seemed to have in their minds what they expected of their students in terms of dialogue and a solution" (p. 121) and that when students failed to meet these expectations, teachers resorted to univocal discourse to redirect the students to the teachers' way of thinking. Truxaw and DeFranco (2008) showed that teachers tended more towards univocal discourse as well.

Tensions exist between these discourse-oriented goals and the reality of classroom teaching. In trying to be "intellectually honest to both mathematics and the child," Ball (1993, p. 377) explains some dilemmas of practice: respecting children's thinking and managing students' confusion while helping them view themselves and not the teacher as the mathematical authority. Encouraging students to think mathematically implies that the teacher does not explicitly show students how to solve problems. However, allowing students to struggle with mathematics "may create frustration and surrender rather than confidence and competence" (p. 377). The teacher must both welcome students' invented strategies and help them acquire traditional mathematics algorithms and procedures (Lampert, 1990). Conventional mathematics can take precedence over hearing children's mathematics, particularly when children's sense-making efforts are not always

easy to decipher. The teacher must maintain control of the class while also relinquishing mathematical authority to her students. It is too easy for instruction to focus on adept performance of procedures rather than understanding of mathematics. Form and accuracy of answers and proofs can take precedence over meaning, and subsequently understanding is sacrificed (Schoenfeld, 1988). Obtaining the answer outweighs explaining how the answer was obtained. Choosing which pupil ideas are worth following requires knowing mathematics well, while at the same time maintaining equitable participation.

The challenges enumerated here all point to a particular problem of implementing discourse: maintaining high cognitive demand (Stein, Grover, & Henningsen, 1996). Cognitive demand refers to the “the cognitive processes in which students actually engage as they go about working on the task” (p. 461). Maintaining the cognitive demand of mathematical tasks (e.g., requiring students explain their ideas rather than memorize and apply algorithms) goes hand-in-hand with discourse. Teachers can lower the cognitive demand of tasks by suggesting strategies, using leading questions, giving directive hints, giving step-by-step instructions on how to solve problems, offering confusing explanations, or not pressing students for clear and complete explanations. Teachers can support or raise the cognitive demand of the task by following the pupils’ way of thinking, asking pupils to describe their thinking and explain their solutions, and helping pupils generalize their strategies. Stein and colleagues provide a framework for studying influences on task implementation. They found that high demand tasks can devolve into routine performance of procedures. At two points the intended task may change: as the teachers sets up the task and as the teacher implements it with students.

Of interest in this study were how teacher dispositions influence implementation of a task (e.g., how a teacher responds to a struggling student). Henningsen and Stein (1997) state, “The

teacher must proactively and consistently support the students' cognitive activity without reducing the complexity and cognitive demands of the task" (p. 542). Factors that support high cognitive demand implementations include appropriate time, opportunities to see high performance modeled, and consistently pressing students for explanations. Other research in discourse points to specific ways to help teachers facilitate richer mathematical discourse.

### **Ways to Help Teachers Facilitate Discourse**

Ball and Chazan (1999) claim that advice given to teachers about discussion-based instruction is wholly inadequate. Directing teachers to not tell students information gives no hint as to what teachers *should* be doing: "The teacher must have a repertoire of ways to add, stir, slow, redirect the class's work" (p. 9). At present four main areas of teachers' work in leading a discussion have been identified: initiating, taking up, and coordinating participation; making contributions; recording and representing mathematics; and planning for and appraising a discussion (Boerst, Ball, Blunk, & Dray, 2010). Assessments for evaluating these aspects of teachers' work are not yet developed. Chapin et al. (2003) offer five talk moves to help teachers build a discourse community and facilitate effective mathematical talk: teacher use of wait time, teacher revoicing, student revoicing, asking students to apply their reasoning to another's strategy, and prompting students to add on to one another's ideas. Kazemi and Stipek (2001) found that the particular sociomathematical norms for pressing students for explanations that required justification was associated with high-level discourse that helped students develop conceptual understanding. Nathan and Knuth (2003) describe how a teacher learned to shift mathematical authority to her students, thus allowing for increased student contributions to discussions. Parsing leading discussions into these components and providing instruction and

practice in carrying out these specific strategies for facilitating mathematics talk may help to make this challenging pedagogical skill more accessible to preservice teachers.

### **A Context for Discourse: Problem Solving**

Teachers cannot facilitate the kind of mathematical talk I have described in this chapter with conventional mathematics problems found in textbooks. Instead, activities that require students to use reasoning and problem-solving skills are in order. Problem solving gained increased attention in the 1980s as a potential strategy for raising students' low mathematics achievement. A multitude of definitions of problem solving have hindered meaningful discussions of problem solving in teaching (Schoenfeld, 1992). The term is easily mistaken for typical textbook word problems, but word problems do not necessarily involve problem solving. In fact, problem solving is often presented in an isolated section of textbooks that require students to practice particular problem-solving techniques to achieve mastery. Polya's (1957) problem-solving phases (understand the problem, devise a plan, carry out the plan, look back) are an often-cited method for helping students, but teachers can easily misuse them as a step-by-step guide and attempt to turn problem solving into a procedure for children to follow. Problem solving is not solving routine word problems with memorized algorithms; it, like discourse, is a vehicle for learning mathematics. I rely on NCTM's (2000) definition of *problem solving* as "engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings" (p. 52).

The problems used in this study were designed to provide children the opportunity learn mathematics concepts. That is, like NCTM (2000), I wanted pupils to develop new understanding from the process of solving the problem rather than use the problems to show their

existing understanding. Thus problems were grouped according to the mathematical concepts on which their solutions were based: generalizing and explaining patterns, reasoning deductively, reasoning algebraically, working backwards, and making an organized list (see [Figure 1](#) in Chapter 3 for the problem list). Further, the problems in this study needed to provide opportunities for preservice teachers and students to discuss mathematical ideas. Several problems required the students to find multiple solutions, thereby providing opportunities to compare solutions. All problems could be solved in multiple ways, thereby providing a context for students to look for connections among strategies and concepts.

Much of the research in problem solving examines implementations of problem-solving curricula (Charles & Lester, 1984; Lubenski, 2000; Vilansenor & Kepner, 1993), students' problem-solving ability (Carlson & Boom, 2005; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Webb, 1979), or beliefs about problem solving (Cooney, 1985). Yet, some research has illuminated the difficulties teachers encounter in making problem solving central to their teaching practice. Jacobs and Philipp (2010) found that teachers struggled in using the child's thinking in responding to their efforts at problem solving. Buschman (2004) noted that teachers face several challenges in basing their instruction on problem solving: taking up radically different ways of teaching and assessing, developing deeper content knowledge to model problem solving for students, and changing their views of what it means to be good at mathematics. In a self-study, Schettino (2003) also found her willingness to take risks to be a major factor in her ability to implement a problem-solving curriculum in her classroom. Ge and Land (2004) found teachers needed to provide plan problem-specific prompts to effectively scaffold children's problem-solving efforts. All of these difficulties in using problem solving can and should be addressed in teacher education programs.

## **Teacher Education**

Basing instruction on either mathematical communication or problem solving is challenging for teachers, and often the result of poor implementation is lowered cognitive demand. A teacher may be able to help a student solve a problem but in doing so focuses on obtaining an answer, not exploring mathematical concepts. Similarly explanations become descriptions of routine procedures instead of justification for ideas and strategies. It is uncertain if K–12 schools have the capacity to support teachers in overcoming the multitude of challenges to implementing instruction based on discourse. By preparing teachers to facilitate discourse during their preservice education, we may be able to give them some tools to navigate the complexities of learning on-the-job. Nathan and Knuth (2003) claim that mathematics teacher education does not prepare teachers to successfully enact discourse-based instruction. Questions of when to intervene on students' work and how to navigate the teacher's role in discourse need to be explored in teacher education. Yet, the majority of teacher education programs are not structured for nurturing these teaching practices. In the next sections, I highlight dilemmas of teacher education and discuss two responses to these dilemmas.

### **Dilemmas in Teacher Education**

Though teacher education has a myriad of issues needing attention, I review three here that hinder teachers' ability to implement discourse-based instruction. There is no research indicating what the influence of teacher education is on teachers' longterm practice (and, hence, their ability to lead mathematical discussions). A disconnection between theory and practice means that even those who develop some skill at facilitating discourse cannot transfer it to their teaching practice. Weak content knowledge necessarily limits the quality of mathematical discussion a teacher can lead.

**Conflicting Evidence on What Works.** The primary mode of teacher education is that the university provides the theory and field work provides the context in which the theory is practiced. Research has provided contradictory findings about what aspects of teacher education actually lead to effective teaching practices (Wilson et al., 2002). Studies have shown that teachers with education coursework appear to be more effective in the classroom than those who attain certification with preparation in their content area only; Brown and Borko (1992) claimed pedagogical knowledge is a key construct in the differences in novice and experienced teachers and is key component in what makes a good teacher. Others have claimed that those with more teacher education have more successful teaching experiences (Darling-Hammond, 2000). Still others claim that teacher education programs have little influence on preservice teachers' practice (Wideen, Mayer-Smith, & Moon, 1998; Zeichner & Tabachnick, 1981). Feiman-Nemser (2001) called teacher education programs "a weak intervention compared with the influence of teachers' own schooling and their on-the-job experience" (p. 1014). Ball and Cohen (1999) concur that weak teacher education coupled with the "conservative traditions" of schools and their limited capacity for professional development impede the development of effective teaching practices.

**University/School Disconnection.** Feiman-Nemser (2001) notes two key challenges facing preservice teacher education: the culture of universities that does not reward faculty-teacher collaboration and the culture of schools that does not value teacher collaboration. Both of these hinder teachers' development of nontraditional practice, and that hindrance, in turn, minimizes preservice teachers' opportunities to observe, experience, and practice discourse-based teaching. Several things account for the disconnection between university and school: Cooperating teachers often are not well informed about the activities of the methods courses,

preservice teachers have few opportunities to practice with real students and receive feedback on the methods learned in their coursework, and time in schools is not as carefully planned as it might be in their university courses (Ziechner, 2010). The conservative nature of schools makes going against the grain with pedagogies learned in university courses difficult to sustain, and the incongruity between university preparation and the real work of teaching may contribute to this disconnection. Student teaching, typically viewed as the most significant experience of teacher education programs, has varying impact on education (Eishenhart, Borko, Underhill, Brown, Jones, & Agard, 1993). Student teaching and other field experiences are often viewed as highly disconnected from teacher education programs (Zeichner, 2010; Feiman-Nemser, 2001). Student teachers have limited opportunities to observe or attempt teaching practice focused on discourse. The teachers often feel they are inadequately prepared for student teaching, and teacher educators rarely report their expectations for student teaching being met (Eisenhart et al., 1993).

**Weak Content Knowledge.** Many recommendations for teacher education call for more preparation in mathematics content (Conference Board of the Mathematical Sciences, 2001; National Commission on Excellence in Education, 1983; U.S. Department of Education, 2008). Preservice and inservice elementary teachers' weak content knowledge of mathematics has been confirmed (Ball, 1990; Borko et al, 1992; Ma 1999). In a study of 252 prospective elementary and secondary teachers, Ball used questionnaires and interviews to describe the components of knowing mathematics. The vast majority of her participants were unable to explain the reasoning behind simple arithmetic algorithms and instead merely restated a procedure for solving a problem. In comparing U.S. and Chinese elementary teachers, Ma found that American teachers held their mathematical knowledge as discrete unconnected packets, whereas their Chinese counterparts possessed a deep and connected understanding of concepts. Borko et al. provided a

picture of an elementary teacher unable to construct appropriate representations for dividing fractions or to provide students a clear justification for the traditional algorithm. Yet, teachers' subject-matter preparation improves student learning but only up to a point (Begle, 1972; Monk, 1994). It remains to be determined how to provide teachers the type of content they need (Ball, Hill, & Bass, 2005).

### **Strategies for Improving Teacher Education**

Though much of preservice teacher education focuses on content and pedagogical knowledge and changing preservice teachers' beliefs about teaching and learning mathematics, newly minted teachers do not use this knowledge in teaching (Ebby, 2000). In response to the weak uptake of teacher education, reform is needed. Teacher educators are charged with the task of creating a stronger connection between university programs and the practice of teaching that allow prospective teachers to engage in a deep conceptual examination of mathematics content (Heaton & Lampert, 1993). Two theories about teacher preparation hold promise for attending to these dilemmas.

**Mathematical knowledge for teaching.** Knowledge needed for teaching is central to the question of how to educate and prepare preservice teachers. Although thorough content knowledge is essential, teaching mathematics is more than just knowing content thoroughly. Teaching mathematics effectively means extending beyond just knowing and doing mathematics well; it must also involve representing content in multiple ways and making that content accessible to students. In tracing the shift of focus from content to pedagogy, Shulman (1986) claims that content knowledge is missing from conceptions of effective teaching. Rather than solely evaluating teachers' knowledge in terms of content or pedagogy, it is the interplay between these two, what he termed *pedagogical content knowledge* (PCK), that forms a critical

component of teacher knowledge. He defined PCK as knowledge that relates most directly to the ways of making content understood by others, noting its close connection to the real work of teaching.

Ball (2003) elaborated Shulman's (1986) construct of pedagogical content knowledge to create a practice-based model for mathematics teaching. Ball refuted the claim that more mathematics is the answer to improving teacher, and ultimately student, understanding; success in doing mathematics will not translate to success in teaching mathematics. The mathematical knowledge needed for teaching (MKT) is different from the mathematical knowledge needed for other professions. MKT is composed of both subject-matter knowledge with subcategories of common content knowledge (CCK), specialized content knowledge (SCK), and knowledge of the mathematical horizon (how mathematical ideas connect and develop over the K–12 curriculum), and Shulman's (1986) pedagogical content knowledge with subcategories of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (Ball, Thames, & Phelps, 2008). Whereas CCK refers to the ability to do mathematics accurately and efficiently, SCK involves the mathematical skills that are unique to teaching. For example, assessing the validity and generalizability of students' nonroutine strategies would fall in the domain of SCK. KCS includes the ability to anticipate student thinking and misconceptions, and KCT includes the development, selection, and sequence of content to achieve a particular goal.

In teaching practice, MKT includes how to unpack complex mathematical ideas in seemingly simple problems, represent mathematical concepts in multiple ways, see connections between ideas, pose mathematically rich problems, ask good questions, make mathematics accessible to students, reason mathematically, and judge the validity of solutions and proofs.

These teacher activities do not map back uniquely to one category of MKT; teachers likely draw on several different kinds of knowledge to make and implement decisions. Leading a mathematical discussion likely relies on many aspects of MKT. The present study in particular was designed to help teachers develop problem-specific PCK. It requires teachers to have knowledge of the mathematics behind the problems, knowledge of students' thinking about those mathematical ideas, and an understanding of how to build on that thinking. Ball and Bass (2000) claim such specific preparation cannot train a teacher for all teaching situations he or she might encounter:

Being prepared for these regularities of practice is enabled by what we think of as pedagogical content knowledge, clusters that embed knowledge of mathematics, of students, and of pedagogy. However, no amount of pedagogical content knowledge can prepare a teacher for all of practice, for a significant proportion of teaching is uncertain. (p. 89)

However, by gaining practice in anticipating and responding to students' thinking about particular problems (the regularities of practice), prospective teachers may be able to better respond to uncertainties of practice.

**Practice-based teacher education.** Another piece of the solution to the weak uptake of teacher education is to make it more practice based (Ball, 1990; Lampert, 1990; Feiman-Nemser 2000; Darling-Hammond, 2006). Ball and Cohen (1999) claim that situating teacher learning within the instructional practices they should adopt may be one road to changing instruction and to giving teachers the capacity to sustain such changes in the classroom. Ball and Forzani (2009) describe how teaching is “unnatural,” different from everyday interactions, and “intricate,” involving many actions unnoticeable to the untrained eye (Lampert, 1985). Ball and Bass (2000) advocate rather than examining curriculum or talking to teachers to determine what knowledge is needed for teaching, teacher education should focus on what teachers actually do (practices such

as facilitating a productive mathematical discussion) and then work backwards from those activities to determine what teachers need to know to be able to execute them well.

Practice-based teacher education does not mean that prospective teachers work in schools more often and for longer periods. It does mean that teacher education would “not settle for developing teachers’ beliefs and commitments; instead, it would emphasize repeated opportunities for novices to practice carrying out the interactive work of teaching and not just to talk about that work” (Ball & Bass, 2000, p. 503). More than just completing fieldwork, practice-based teacher education should include the creation and analysis of representations of practice. Ball and Bass suggest instead focusing instruction in teacher education coursework on high leverage practices, “tasks and activities that are essential for skillful beginning teachers to understand, take responsibility for, and be prepared to carry out in order to enact their core instructional responsibilities” (p. 504).

**Images of practice-based teacher education.** Feiman-Nemser (2000) identifies “purposeful, integrated field experience” as one feature of good teacher education programs. Darling-Hammond (2006) adds that teacher education programs must help teachers learn how to continue inquiring about and developing their practice after their formal teacher preparation has ended. Like Feiman-Nemser, she argues for the importance of coherence and integration among courses and clinical work; in fact, she claims that “student teachers see and understand both theory and practice differently if they are taking course work concurrently with fieldwork” (p. 307). Ebby (2000) found that by integrating field work and methods courses that prospective teachers were able to “learn from fieldwork in ways that are generative instead of imitative and in ways that foster future growth” (pp. 94–95) and that the consistency between field placements

and university coursework made it easier for preservice teachers to connect learning in one venue to practice in the other.

Crespo et al. (2011) offer another possibility for practice-based teacher education, using prospective teachers' representations of "imagined performances of mathematics teaching" (p. 120). When prompted with a hypothetical student's incorrect solution to a mathematics problem, preservice teachers were asked to create a hypothetical student-teacher dialogue that would follow. This task serves many purposes: It helps teachers consider mathematical discussions that do not focus on procedures, and it makes their thinking about such discussions evident to teacher educators. Rather than a lesson plan which might specify "I will help students do  $x$ ," this type of assignment requires a teacher to consider the specifics of how he or she might accomplish a goal. Crespo et al. found that prospective teachers' imagined dialogues fell into one of three categories: working with the students' ideas, redirecting the students' ideas, or a hybrid of the two. Creating representations of student-teacher dialogues requires prospective teachers to consider both mathematical and pedagogical concerns, and it does so in the context of students' thinking. This approach is squarely situated in the realm of the work of teaching and holds potential for contributing to practice-based teacher education.

Markovits and Even (1999) used mathematical classroom situations in which preservice elementary teachers responded to real or imagined student questions or ideas. Though they asked their preservice teachers to describe a response rather than propose an actual conversation, they found using this nontraditional representation of teaching revealed insights about both the preservice teachers' mathematics understanding and their pedagogical concerns. In particular, Markovits and Even's analysis of preservice teachers' responses showed a shallow

understanding of mathematics and a belief that questioning should only be used to assess students' performance on procedures explicitly taught by the teacher.

Lampert and Grazani (2009) provide another example of what practice-based teacher education might look like in their study of instructional activities in an Italian teacher education program. Similar to high leverage practices, instructional activities “serve as containers that carry principles, practice, and knowledge (of students, content, curriculum) into practice and support both student learning and teacher learning” (Kazemi, Ghouseini, Beasley, Lampert, & Franke, 2010, p. 10). Instructional activities provide discrete reliable routines that are accessible to novices (Ghouseini, 2008; Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010), and leading a discussion can act as one instructional activity. Situating instructional activities within repeated cycles of planning, teaching, and reflecting allows novices to gain practice at enacting these routines. The essential features of the cycles are deliberate, collective practice combined with strong feedback from peers and teacher educators.

In this study, I was able to take advantage of an existing practice-based mathematics teaching methods course. It was composed of an integrated course and field experience where what was practiced in the methods course was later practiced with children in a school setting. Course activities were designed to support the development of PCK. I also made use of representations of practice (Crespo et al., 2011) and employed a cycle of planning, enactment, and reflection similar to the process used by Kazemi et al. (2010). In the next chapter I describe the methods for data collection and analysis that are built on the frameworks I have outlined here.

## CHAPTER THREE

### METHODS

I conducted this study over a 6-week period in conjunction with the field experience associated with a mathematics teaching methods course for prospective elementary teachers. The field experience described in this report resulted from the university's partnership with a local elementary school, Springfield Elementary.<sup>1</sup> Though I designed the structure and format of this experience, having a field experience focused on a child's mathematical thinking was already an established component of this methods course and was arranged prior to the present study.

The design of this field experience attended to many of the potential problems inherent in practice-based teacher education. I structured the field experience to provide several opportunities for different kinds of assisted performance: planning, co-teaching, analysis of children's mathematical thinking, and co-reflection. Because the teacher education program, and not the elementary school, established interactions between preservice teachers and elementary pupils, it allowed the preservice teachers the opportunity to focus on practices that were coherent with their course work. In this chapter, I provide an overview of the context of the study; specify participant selection procedures; describe the structure of the field experience, data collection process, and instruments used for data collection; and explain how I carried out analysis.

#### **Context**

This study was situated within an early childhood education program at a large university in the southeastern United States. Admission to the program is highly competitive with only 120

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<sup>1</sup> All participant and school names are pseudonyms.

students admitted annually. After completing two years of general coursework, students apply for admittance to the early childhood education program, which includes two years of content courses, content-specific methods courses, general education courses, field experiences, and a semester of full-time student teaching. Completion of the program results in a degree in early childhood education and certification to teach grades PreK–5.

Study participants were enrolled in my section of a mathematics teaching methods course for preservice elementary teachers. This course, entitled Children’s Mathematical Learning, is the first in a required 2-course sequence for the university’s early childhood education program. It focuses on counting, number sense, basic operations with whole and rational numbers, and making sense of standard arithmetic algorithms. A main course goal described in the 2011 syllabus is “becoming aware of children’s mathematical thinking, how it differs from adult thinking, and how it might impact your teaching.” Weekly assignments in which preservice teachers analyze children’s thinking and hypothesize how children will solve upcoming problems complement the in-class discussions and work with children in the field.

Supporting course goals, the focal point of the course is an 8-week supervised field experience in which the class, together with the instructors, makes weekly visits to a local elementary school. The preservice teachers work one-on-one with elementary school pupils on problem-solving tasks. The purpose is to allow the pupils to approach problems in their own way so that the preservice teachers can gain insight into how they think about mathematics. The course instructors do not want the preservice teachers to provide direct instruction on mathematics topics. Instead, the preservice teachers should use techniques discussed and practiced in class to help pupils articulate their thinking and solve problems and use their understanding of the pupils’ thinking to connect the pupils’ work to the underlying mathematical

concepts of each problem. To accomplish this aim, the preservice teachers must facilitate mathematical discourse that reveals and attends to pupils' thinking without advancing preservice teachers' own problem-solving strategies.

### **Participant Selection**

Study participants were in their first semester of coursework after being admitted to the early childhood education program and enrolled in my mathematics teaching methods course for preservice elementary teachers. Though this course was the preservice teachers' first mathematics teaching methods course, they had completed one mathematics content course that focused on arithmetic and problem solving.

To be selected for participation in this study, preservice teachers needed to have (1) good verbal and written communication skills so that they could communicate mathematically with children and analyze mathematics problems, (2) an openness to attempt teaching mathematics in ways that were different from their elementary school experiences, (3) a strong record of attendance, involvement in class, and history of timeliness in completing assignments to ensure they would be present for and actively involved in all field experience sessions and would complete on time the field experience assignments that were part of data collection, and (4) good problem-solving skills to ensure they could successfully complete the types of problems they would be implementing with children. I used several initial class assignments together with observations of class discussion to inform my participant selection.

Two class assignments and observations of class discussions helped me determine which preservice teachers were able to express themselves clearly and were open to exploring, rather than correcting, children's ways of thinking. In their mathematics autobiographies, preservice teachers described their experiences with and feelings about learning and teaching mathematics.

The purpose of this assignment was two-fold: (a) to establish background information for the potential participants and (b) to help them assess how they felt about mathematics, why they felt that way, and how those feelings might influence them as mathematics teachers. A written response to Vivian Paley's (1986) article titled "On Listening to What Children Say" required preservice teachers to express their thoughts, opinions, emerging ideas, or tentative hypotheses. Paley included transcripts of conversations with children in which she failed to correct a child's misconception, and in responding to these instances, preservice teachers revealed their views and opinions on teaching and learning mathematics. Preservice teachers who were adamantly opposed to Paley's techniques or expressed an extreme dislike of mathematics would make poor candidates for a study that asked them to listen to and make sense of children's mathematical thinking. In contrast, those considered for participation in the study had positive experiences with mathematics, responded favorably to the Paley article, or expressed an interest in doing mathematics differently from lecturing at a chalkboard with students diligently completing worksheets. I used observations of preservice teachers' contributions in whole-class or small-group discussions to confirm or disconfirm my initial hypotheses that had been based on their written work. Preservice teachers who responded negatively to learning mathematics or the Paley article would sometimes show a more open perspective when talking in class. Likewise, preservice teachers who responded positively to the Paley article or wholeheartedly agreed with teaching mathematics in new ways might sometimes show that this agreement was superficial when considered in the context of their comments in class. I also used class observations to determine who was comfortable speaking in class and could articulate ideas, opinions, and questions clearly.

Preservice teachers' problem-solving skills were assessed three ways. First, at the beginning of the semester, they completed mathematics problems in class similar to those used in their upcoming field experience; they worked in small groups sharing and discussing solutions. Observations of preservice teachers in the whole-class and small-group settings provided me an opportunity to see how they worked with classmates, approached problems, responded to a strategy different from their own, and explained ideas to others. These observations helped me zero in on which preservice teachers were confident in their abilities, had unusual approaches to problems, and enjoyed participating in class. However, because not each of the 29 preservice teachers in my class was able to discuss his or her solutions in this venue, I supplemented the information in two ways: individual written work and one-on-one interviews. In the problem-solving assignment, preservice teachers completed five problems of the same type they would later implement with elementary children. Preservice teachers had to explain what the problem asked, solve the problems in multiple ways, justify those solutions, explain the underlying mathematics concepts, and propose a follow-up problem (Appendix A). I evaluated this assignment using a rubric (Appendix A) to determine who could not only solve the problems but also explain his or her solutions clearly and identify the mathematics concepts used to solve the problems. Preservice teachers who scored less than 30 out of 40 were not considered for participation in the study. Preservice teachers also participated in a videorecorded problem-solving interview (Appendix B) in which they chose two problems from a provided list and solved them while the course teaching assistant or I observed and asked questions. Whereas the problem-solving assignment showed the preservice teachers' polished solutions and explanations, the interview allowed me to gain insight into how they initially approached problems, corrected mistakes, responded to failed strategies, developed strategies, and explained

their thinking without the advantage of having time to revise and present an ideal response. The interview allowed me to see firsthand the struggles an individual had that he or she might have omitted from a written assignment or that were masked by working in a group.

Based on these assignments and observations, I identified 12 preservice teachers as potential participants. Though some who were invited to participate expressed anxiety about mathematics or a lack of confidence in their ability, all those invited showed good mathematical reasoning and problem-solving skills and were able to express their ideas about mathematics clearly. Their writing was above average for the class. When prompted by me or another class member, they all expressed ideas clearly in class discussion; however, the frequency with which they volunteered to contribute in class varied. All those invited professed to value problem solving, hands-on activities, and mathematical communication as tools for learning. Of the 12 invited, all agreed to participate and completed appropriate consent forms. One participant (Heather), however, was dropped from the study after missing part of the data collection. I assigned the 12 participants to four groups based on their strengths relative to other participants (Table 1). I first assigned to each group one of the four participants I identified as having the best problem-solving skills and then added at least one member who easily communicated with others and at least one member who had unusual approaches to solving problems to each group.

Table 1

*Preservice Teacher Participant Groups*

Participant Groups			
Group G	Group H	Group I	Group J
Beth	Kate	Heather*	Megan
Susan	Nadia	Alice	Rene
Tara	Casey	Dana	Erica

\*Dropped as a participant.

For the field experience, we worked with fifth-grade pupils. Each of the three fifth-grade teachers in the school selected pupils to participate. As in semesters past, teachers were asked to choose pupils they considered academically in the middle, not their strongest or weakest mathematics pupils. The teachers identified 20 pupils and information about the study and consent forms were sent home with all 20 pupils. Of those, 13 agreed to participate. I put these pupils in pairs, and because I intended for the pupil pairs to be the same throughout the study, I did not include in the study one pupil who agreed to participate. Thus, I began the study with six pupil pairs who rotated through the four participant groups. For the first 3-week cycle all six pairs remained the same. However, in the second cycle I adjusted the pairs to account for personality conflicts between pupils and to ensure that no pupil repeated the same set of problems. Because there were 6 pupil pairs and 4 preservice teacher groups participating, for any given week only four of the pupil pairs participated in the study.

### **Structure of Field Experience**

The first 2 weeks of the 8-week field experience were not included in this study. Omitting those weeks provided time for the pupils to understand what the participants expected of them and for the participants to become comfortable both working with the children and being observed by the teaching assistant and me. The remaining 6 weeks of the experience that were the focus of this study were divided into two 3-week cycles. I grouped the problem-solving activities used in the field experience into five sets according to the instructional goal: generalizing and explaining patterns, reasoning deductively, reasoning algebraically, working backwards, and making an organized list. Within each problem set, I chose two problems to focus on for this study (see Figure 1).

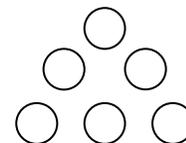
### Generalizing and Explaining Patterns

#### **Problem 1: Phone Club**

Your class made telephones out of strings and juice cans. Each group of students has to work together to make a phone club that connects every person to every other person. If a group had four people, how many strings would be needed to connect every member of the group to every other member of the group? What if you used 28 strings, how many people would be in a group?

#### **Problem 2: 6 Numbers**

Can you put the numbers 1-6 in the triangle shown so that each side adds up to the same amount?



### Making Organized Lists

#### **Problem 1: 12 Pennies**

Place 12 pennies in 3 piles with no two piles having the same number of pennies.

#### **Problem 2: Clock 6**

How many times in a 12-hour period does the sum of the digits on a digital clock equal 6?

### Working Backwards

#### **Problem 1: Crayons**

Mary has some crayons. Doug had 3 times as many as Mary. But Doug gave 4 to the teacher and now John has 2 more crayons than Doug. John has 7 crayons, how many does Mary have?

#### **Problem 2: Puppies**

The pet store advertised that they had lots of new puppies on Monday. The owner took 1 puppy for his son. Then on Tuesday he sold half of the rest of the puppies to a farmer with lots of land. On Wednesday a mom took a half of the puppies that were left for her children. When you got to the pet store on Thursday there were only 4 puppies left to choose from. How many puppies were there on Monday?

### Reasoning Algebraically

#### **Problem 1: Cupcakes**

A baker makes chocolate and vanilla cupcakes. He packages the vanilla ones in boxes of 4 and the chocolate ones in boxes of 6. He made 38 cupcakes and used 8 boxes. How many boxes of vanilla and how many boxes of chocolate did he make? (alternate version: 58 cupcakes and 12 boxes)

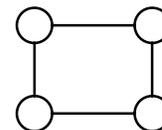
#### **Problem 2: Tickets**

Amy and Judy sold 19 play tickets altogether. Amy sold 5 more tickets than Judy. How many tickets did each girl sell?

### Reasoning Deductively

#### **Problem 1: Castle**

Twenty men need to guard the castle below. For the castle to be safe there should be 7 men guarding each side. The men on the towers count as guards for both walls that connect to the tower. How would you place the 20 men?



#### **Problem 2: Women at the Table**

Five women are seated around a circular table. Mrs. Osborne is sitting between Mrs. Lewis and Mrs. Martin. Ellen is sitting between Cathy and Mrs. Norris. Mrs. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Mrs. Parks on her left and Mrs. Martin on her right. Match the ladies' first names and last names.

Figure 1. Focus problems for each problem set.

For each 3-week cycle, the participant groups were assigned one set of problem-solving activities to implement with a different elementary pupil pair each week (Table 2). Each week, the participants worked on the two focus problems of their set and then, if time allowed, worked on other problems of their choosing in their assigned problem set. On some weeks, the participants were able to implement both focus problems, whereas on other weeks, they were able to do only one. Repeating the same problems three times provided the participants the opportunity to gain familiarity with pupils' approaches to particular problems and to refine their responses to pupil strategies and questions.

Table 2

*Field Experience Organization of Preservice Teacher Groups and Problems*

Cycle	Group J	Group H	Group I	Group G
1: Weeks 3–5	Generalizing & explaining patterns	Making organized lists	Reasoning algebraically	Reasoning deductively
2: Weeks 6–8	Reasoning algebraically	Generalizing & explaining patterns	Working backwards	Making organized lists

In each group, two of the participants led the discussion on problem-solving activities, while the third made a video record of the activities. Within each group, the participants decided how to share leading the problem discussions, and this role changed from week to week depending on the needs and personalities of their pupils and the participants' expertise on particular problems. In some groups, each pupil worked independently with one participant while the third participant alternated videorecording between the two pupil-preservice teacher pairs. In other groups, each pupil worked independently with one participant and then both pupils were prompted by one or more participants to discuss and compare their solutions and strategies. In

this situation, the recording alternated between the individual pairs but also captured the pupils working or explaining their ideas together. In both of these scenarios, the participant doing the recording made decisions about when to record each pupil-preservice teacher pair. However, while one pair was on camera being recorded, the second pair was audiorecorded off camera. As a third possibility, both the pupils worked in concert throughout the problems, usually with some time provided to work alone first, and one participant was responsible for leading each problem or each part of a problem. The most collaborative division of work involved both participants who were not recording working together to lead the pupils, who also worked together, in discussion about the problems. In these two situations, the participant doing the recording was able to capture on camera all of the interactions that occurred. In all four of these cases, the recorder typically interjected comments or suggestions as well, both as asides to her fellow group members and directly to the pupils while they worked. Throughout their time working, the teaching assistant and I circulated, observing and occasionally interjecting or answering the preservice teachers' questions.

### **Data Collection**

Data collection revolved around the field experience associated with participants' mathematics teaching methods course; hence, most of the collected data were part of participants' assigned work for the course. The only data collected that were not a part of required work for the course were the video records participants made of their work with pupils during the field experience.

Data were collected in two 3-week cycles based on cycles of planning, enactment, and reflection (Kazemi et al., 2010). Each participant group was assigned a different set of problems for each cycle. The first element of data collection involved the participants' preparing for their

weekly field experience and receiving feedback on that preparation. In teaching this course in previous semesters, I had observed that writing lesson plans did not adequately prepare preservice teachers to attend to pupils' thinking, anticipate their solution strategies, or facilitate mathematical discussions in the field. Therefore, in this study, before each 3-week cycle of the field experience, the participants first explored pupils' potential solution strategies and considered how to facilitate discussion about those strategies through hypothetical task dialogues (Crespo et al., 2011). In these assignments, the participants created a student-teacher conversation that might follow each of several possible solutions to each of their two focus problems (Figure 2; for complete Task Dialogue assignments see Appendix C). I instructed them to consider what hints and questions could get pupils started and how they might help pupils see flaws in their reasoning, expand their thinking, find a more sophisticated strategy, or generalize their strategy to other problems. They also provided a rationale for their pedagogical decisions in their hypothetical dialogues. The participants used feedback I provided on the task dialogue to complete an activity plan (see Appendix D) for the problems in their assigned set. In that plan, they had to suggest correct and incorrect potential student responses and include how to respond to a student who (1) could not start the problem, (2) was incorrect, (3) was on the right track to a correct solution, or (4) had obtained a correct solution but not connected the process to the underlying mathematical concept. The participants received feedback on their plans before they implemented the problems with pupils. For each subsequent week of that cycle, the participants revised their plans based on their work with pupils and advice from me and the teaching assistant. Thus for the first cycle, planning data included Task Dialogue 2 and activity plans for Weeks 3, 4, and 5. However, in the second cycle, planning data included only Task Dialogue 3 (which also served as their plan for Week 6) and activity plans for Weeks 7 and 8.

<b>Instructional Goal: Make an organized list</b>	
<i>Problem 1: Place 12 pennies in 3 piles with no two piles having the same number of pennies.</i>	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Student randomly puts 12 pennies into 3 piles until she finds 6, 4, 2. Then she moves a penny from the 4 pile to the 2 pile and has 6, 3, 3. "That's not right." She moved the pennies back to 6, 4, 2 and moves one from the 6 pile to the 4 pile. "I keep making them with the same number."	
<b>T:</b>	
<b>Dialogue 2</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Student finds 6, 5, 1, and 6, 4, 2 and 5, 4, 3 by rearranging one penny at a time, then gets stuck.	
<b>T:</b>	
<b>Dialogue 3</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Student has found 5 of the solutions (651, 642, 543, 741, 732) and after several more attempts claims these are the only solutions.	
<b>T:</b>	

*Figure 2.* Sample task dialogue.

The second element of data collection was the video records of the participants' enactments of their problems with their elementary pupils. The video record was essential to have an accurate representation of problem enactments that captured the gestures, work, and use of manipulatives by participants and pupils, as well as their discussions. The teaching assistant and I circulated among all the groups each week. Each week, at least one of us observed each group. We sometimes interjected comments into groups' conversations or answered the participants' questions, which were captured on the recordings. I also reviewed each group's weekly recording and provided feedback on their interactions with pupils. Because the

participants made these recordings themselves, they decided what was recorded. When a pair of pupils was working separately, the recording participant decided when to focus on each pupil. Because the participants and the pupils worked at tables placed in hallways outside of classrooms, there was ambient noise of classes being conducted or students walking through hallways. At times, the participant in charge of recording paused the recording to avoid excessive background noise, to have conversations with pupils about behavior, or to have conversations with her fellow group members about the direction of a problem they were working on or about to start working on.

The third element of data collection was the participants' weekly reflections on their enactment of the problems. Within 10 hours of each weekly session, I posted video recordings on a website available only to me and each groups' participants. The three participants in each group reviewed the recordings, composed individual activity reflections analyzing their own and their pupils' successes and struggles with the activities, and shared the reflections with their fellow group members. Each participant then wrote a brief response to her group members' reflections (see Appendix E for guidelines). The teaching assistant and I made comments on these reflections each week to help the participants modify their plans for the following week. Instead of completing a reflection for the final week, the participants wrote a paper reflecting on their growth in implementing problems and leading mathematical discussions over the entire experience (see Appendix F for final paper guidelines). Table 3 summarizes the class assignments related to the field experience and data collection.

Table 3

*Class Assignments Related to Field Experience*

	Week	Data Collected		
		Planning	Enactment	Reflection
Cycle 0*	1	Task Dialogue 1, Activity Plan 1	Notes	Activity Reflection 1
	2	Activity Plan 2	Notes	None
Cycle 1	3	Task Dialogue 2, Activity Plan 3	Video	Activity Reflection 3
	4	Activity Plan 4	Video	Activity Reflection 4
	5	Activity Plan 5	Video	Activity Reflection 5
Cycle 2	6	Task Dialogue 3	Video	Activity Reflection 6
	7	Activity Plan 7	Video	Activity Reflection 7
	8	Activity Plan 8	Video	Final Paper

\*Assignments for these weeks were not a part of data collection

### Data Analysis

I only analyzed data for those problems that were implemented for all weeks of any given cycle by a group and that were being attempted for the first time by pupils (Table 4). Thus, I eliminated data on the 6 Numbers Problem for Group H and data on the Tickets and the Phone Club Problems for Group J because these groups did not implement these problems all weeks of the cycle. Because Group G failed to do their assigned focus problems all 3 weeks of the first cycle and because their pupils were repeating problems in the second cycle, I eliminated all of their data from this analysis. I chose to eliminate Group I's data on the Crayons Problem because they spent the vast majority of their time each week focusing on the Puppies Problem.

Table 4

*Focus Problems Completed by Preservice Teacher Groups and Their Pupils*

		Groups			
	Week	Group G	Group H	Group I	Group J
Cycle 1	3	<i>Women at the Table</i>	<i>12 pennies, Clock 6s</i>	<i>Cupcakes (58 in 12), Tickets</i>	<i>6 Numbers</i>
		Sara, John	Michelle, Anna	Harold, Rachel	Eli, Oliver
	4	<i>None</i>	<i>12 pennies, Clock 6s</i>	<i>Cupcakes (58 in 12), Tickets</i>	<i>6 Numbers</i>
		Michelle, Anna	Sara, John	Eli, Oliver	Harold, Rachel
	5	<i>Women at the Table</i>	<i>12 pennies, Clock 6s</i>	<i>Cupcakes (58 in 12), Tickets</i>	<i>6 Numbers, Phone Club</i>
	Eli, Oliver	Harold, Rachel	Sara, John	Michelle, Anna	
Cycle 2	6	<i>Clock 6s</i>	<i>Phone Club, 6 Numbers</i>	<i>Crayons, Puppies</i>	<i>Cupcakes (38 in 8), Tickets</i>
		Eli, Oliver	Sara, John	Rachel, Anna	Debbie, Zara
	7	<i>15 pennies, Clock 6s</i>	<i>Phone Club</i>	<i>Crayons, Puppies</i>	<i>Cupcakes (38 in 8), Tickets</i>
		Michelle, Leon	Debbie, Zara	Oliver, John	Anna, Ben
	8	<i>15 pennies</i>	<i>Phone Club</i>	<i>Crayons, Puppies</i>	<i>Cupcakes (48 in 10)</i>
	Michelle, Anna	Leon, Rachel	Debbie, Zara	Ben, John	

Some pupil participants worked with the same participant groups during each of the two cycles, but because they worked on different problems in each cycle, this kind of repetition was not problematic. In one instance, a pupil, Ben, worked with the same group two consecutive weeks in one cycle, and a second pupil, John, worked on the same problem set in Cycle 1 and Cycle 2. Thus, in Week 7, both boys were working together on the Cupcakes Problem for the second time. However, rather than have them repeat the same problem, Group J extended the problem by using different numbers and asked the boys to look for patterns between the two

versions of the problem they had solved. Because the boys were then dealing with unfamiliar aspects of this question, I determined that this kind of repetition in the study was acceptable and included these episodes in the analysis.

### **Preliminary Work**

During data collection, I completed some preliminary analysis as I watched each participant group's weekly video; read task dialogues, activity plans, and activity reflections; and provided them feedback. At this point in the analysis, I got a sense of who was developing detailed plans that focused on students' strategies and mathematical ideas and writing good critical reflections. In watching the videos, I could not consistently assign any characteristics or growth in teaching to an individual participant because of the collaborative nature of the participants' work. However, I was able to determine some general areas of strengths and weaknesses for each participant group and particular sessions that included interesting or unusual pupil work. Based on these preliminary findings, I chose a group that included the widest range of growth and pupil responses to begin my analysis. To examine how the participants' use of mathematical discussions maintained, lowered, or elevated the cognitive demand of the tasks, I used the Hufferd-Ackles et al. (2004) math-talk framework and the Stein et al. (2000) framework for cognitive demand. Before explaining details of the data analysis, I provide a brief overview of these two frameworks.

**Math-talk.** Hufferd-Ackles et al. (2004), through close investigation of one teacher's practice over a 2-year period, developed a framework for examining the development of a math-talk learning community where "the teacher and students use discourse to support the mathematical learning of all participants" with a goal "to understand and extend one's own thinking as well as the thinking of others in the classroom" (p. 82). Organized hierarchically

along four developmental levels, their framework includes four components of classroom talk: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning, hereafter referred to as *questioning*, *explaining*, *ideas*, and *responsibility*, respectively.

**Cognitive demand.** What students learn from classroom experiences is necessarily dependent on the opportunities they have to wrestle with mathematically rich tasks. Stein et al. (1996) analyzed the cognitive demand of tasks as set up and implemented by teachers. By *cognitive demand*, they refer to “the kind and level of thinking required of students in order to successfully engage with and solve the task” (Stein et al., 2000, p. 1). Low-level tasks fall into one of two categories: *memorization or procedures without connections* to understanding, meanings, or concepts. High-level tasks are classified as *procedures with connections* to understanding, meanings, or concepts or *doing mathematics*. Stein et al. found that maintaining the cognitive demand of a task posed a challenge to teachers. Teachers’ tendency to eliminate student struggles by reducing a challenging problem to routine procedures, to select inappropriate tasks for their goals, and to shift the focus of instruction to merely finding a correct answer all contribute to the lowering of a task’s cognitive demand.

### **Enactment Data Analysis**

After all data collection was completed, I transcribed all videos. Based on my preliminary work, I began analysis with Group H’s video. For the video data, I intended to assign a numeric (0–3) level in the categories of cognitive demand and, from the math-talk framework, *questioning*, *explaining*, *ideas*, and *responsibility*. In coding the levels of cognitive demand, I used the guidelines as described in Table 5.

Table 5

*Guidelines for Coding Levels of Cognitive Demand*

Cognitive Demand Level	Indicators	Example
0 Memorization	Teacher reduces task to answering a series of basic skill questions; Students answer questions using only memorized facts and rule	T: Make a list of numbers that add to 6. S: 1 and 5, 2 and 4, and 3 and 3.
1 Procedures without connections <i>to understanding, meanings, or concepts</i>	Teacher and students focuses on students finding solutions to the problem, Teacher asks how student solved a problem, T may suggest strategies for student to carry out	T: You could think of all the options that started with 9. [S nods.] T: Then what would you do? S: Keep adding one to each, like it would be 8 plus that plus that.... T: You want to try that? So you'd make a list so you could keep it organized.
2 Procedures with connections <i>to understanding, meanings, or concepts</i>	Students focus on finding solutions to the problem, Teachers focus on having students explain what patterns were observed, how the solutions were obtained, and why students' strategies worked	T: On the last problem we saw if there's 4 numbers you can arrange them 6 different ways, remember? Why can't we arrange these 6 different ways? S: I started with 2 and 4, but I can't start with a 0 cause there's no time like 0:40 or...
3 Doing mathematics	Teacher encourages students to make and test conjectures, apply reasoning to situations, generalize strategies across multiple similar problems, justify conclusions	S1: ....I'm predicting there will be 12 strings. T: Why are you predicting 12? S1: Each string has to go to 3 people. S2: [draws his solution] I think 6. T: Can you explain yours to S1? S1: I think since you have to do 3 strings to each person you spin it 4 times, so 4 times 3 is 12. T: Why do you think there are 6 strings? S2: There are four people and they all have to connect to each other. So that would mean this person would need to connect to this person and that person. Since they're already connected you only have to draw two lines.

Note: Examples were edited for clarity and correct grammar.

From my preliminary work, I hypothesized that a teaching episode could be assigned different levels in each component of math-talk framework (e.g., a Level 2 in *questioning*, Level 1 in *explaining*, Level 1 in *ideas*, and Level 0 in *responsibility*). This use of the math-talk framework differs from Hufferd-Ackles and colleagues' (2004) findings that the four components of their framework generally developed together (e.g., a Level 2 teacher was a Level 2 in all four components). I initially planned to assign numeric levels in the five categories to each task in each session of each participant group. However, as I coded Group H's video, I encountered several problems. First, the levels of these categories were not consistent across any one task. Second, each participant in the group made different contributions to the task enactment, so even if I were able to assign one level from each category to a task as a whole, I was not able to attribute the pedagogical decisions and moves to one participant. Therefore, I decided to parse each task enactment into segments that allowed me to cleanly apply one level from each category to the pedagogical moves of each participant. To be consistent with activity plans which required preservice teachers to respond to no solution, a correct solution, an incorrect solution, and a partially correct solution, I used these events to act as markers to define segments of the problem enactments in the video data, which could then be coded with levels for each of the five categories.

This four-item list did not sufficiently cover all pupil activities to which the participants responded (Table 6). I added to the list correct solutions with minor errors such as miscounts or small mistakes in adding or subtracting numbers. The participants not only responded to completed pupil solutions; they also intervened as pupils were executing or discussing strategies, or sharing ideas about the problem. To distinguish between these, I considered a response to a strategy as a response to a pupil's approach that the pupil had not finished executing, a response

to a solution as a response to the final answer a pupil presented (which may have been done with or without explanation), and a response to an idea as a response to a pupil's observation or question about the problem. I further subdivided response to ideas and response to strategies to consider the level of correctness (correct, partially correct, and incorrect). The participants responses to pupils' lack of a solution occurred in several different ways: they responded to no solution scenarios before a pupil had started a problem or after a pupil had started and got stuck on reaching a solution.

Table 6

*Markers Used to Indicated Segments of Problem Enactments*

Preservice teacher responds to	Types	Description
Solution	Correct	Pupil's completed final answer to a problem or part of a problem, may be presented with or without explanation of strategy or justification
	Correct with minor error	
	Partially correct	
	Incorrect	
Strategy	Correct	Approach pupil described or approach in progress noticed by preservice teacher but not articulated by pupil
	Partially correct	
	Incorrect	
Idea	Correct	Pupil observation about the problem, may be presented as a question
	Partially correct	
	Incorrect	
No Solution	Pupil had not yet started the problem	
	Pupil had started the problem but then got stuck	

The third problematic aspect of coding that arose was accurately capturing each segment with the math-talk framework. As I coded segments with levels for the four components of math-

talk framework, I noticed that the participants' changes to the second and third enactment of a problem were not captured by the levels as presented by Hufferd-Ackles et al. (2004). I read through all the transcripts for Group H to select an example of each level (0–3) of the four components that closely aligned with the examples Hufferd-Ackles et al. provided. Within the data, I found examples for each of the levels for each component except for a Level 3 *responsibility*. As I found these examples, I added more specific detail to the descriptions of component levels to help me more easily identify the levels in my data. For example, to Level 0 *questioning* I added, "Teacher asks many short direct questions and/or yes/no questions in a series." Details were added to descriptions of several levels: Levels 2 and 3 of *responsibility*, Level 2 of *ideas*, and Levels 0, 2, and 3 of *questioning*. I also revised the descriptions of the component levels to better fit the context of this study of two teachers working with a pair of pupils instead of one teacher working with a whole class of pupils as is the context of the original math-talk framework. For example, I modified Level 0 *ideas* from, "Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math," to read, "Teacher shows how to solve or tells correct answers or appropriate strategies." I made revisions of this type for *ideas* at Level 0 and *responsibility* at Level 0. Because Hufferd-Ackles et al. claim that at Level 2 student-to-student talk begins, I also revised the Level 2 descriptions for each component so that they all had the common thread of the teacher prompting student-to-student interactions. In particular, I revised *explaining* Level 2 to include that the teacher prompted students to explain to one another and *ideas* Level 2 to include that the teacher asked students to make sense of one another's ideas. Much of Hufferd-Ackles et al.'s Level 3 descriptions included activities the teacher "expected," and my participants' intentions and expectations were

not something that could be discerned from the video data. Therefore, I rewrote Level 3 descriptions to focus on observable data.

I then composed a general description of each level so that a feature common to all components within a level distinguished it from previous and subsequent levels. Thus, Level 0 is characterized by being entirely teacher-directed; Level 1 is characterized by the teacher's inability to meaningfully attend to student thinking once she has drawn it out; Level 2 is characterized by low-level student-to-student talk occurring only with teacher prompting, and Level 3 is characterized by a high level of student-to-student talk occurring with minimal teacher prompting. With these more explicit guidelines and examples for coding the video data, I then recoded the Group H video data. Still I found segments of the problem enactments that did not fit neatly into one level of each component of the math-talk framework. I concluded mid-levels were needed and proceeded to search for examples of these in the data and, then, to compose descriptions for these. The addition of these levels to the framework occurred in cycles, similar to the process used to develop the framework (Hufferd-Ackles et al., 2004). When I identified several similar segments that did not fit into the revised 0 to 3 levels of a component in the framework, I coded those segments with the appropriate mid-level (0.5, 1.5, 2.5) and composed a description for that cell of the revised framework based on the newly coded segments. I then recoded any earlier video segments to ensure the assignment of levels for that component was consistent across the data. As I found more examples of mid-levels for a math-talk component, I refined the descriptions for that cell. This process continued until I coded all of Group H's video data and all segments fit in exactly one cell for each component. The revised framework with a 7-level scheme is included in Table 7. With the revised framework finalized, I segmented and coded the remaining groups' video data.

Table 7

## Revised 7-Level Math-Talk Framework

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<p>0</p> <p><i>All activity is teacher-directed. Students passively receive information and their only input is to provide answers to yes/no or basic fact questions.</i></p>	<p>T asks many short direct questions and/or yes/no questions in a series; S give short answers</p> <p><i>T: What is 7 and 4? Can you use 10 in one pile? Can we change that pile?</i></p>	<p>T does not elicit S thinking, but does elicit answers; T may answer own questions</p> <p><i>T: What should we try next, 7? That doesn't work, right? How many do we have?</i></p>	<p>T shows how to solve or tells correct answers or appropriate strategies</p> <p><i>T: Make number sentences that add to 12. Count out 5 pennies and we'll trade this in for a nickel.</i></p>	<p>T verifies and shows correct solutions; T shows out to carry out most effective strategies</p> <p><i>T: That's too many. That's 8. You could do 2 plus 4. Let's write that next to this one.</i></p>
<p>0.5</p> <p><i>All activity is still teacher-directed, but there is some opportunity for student input, mostly in the form of factual recall. When teacher poses more complex questions, students are not prepared to answer, often because teacher did not provide adequate time for student work.</i></p>	<p>T asks about strategies before S has applied any strategies</p> <p><i>T: Do you see a pattern or how can we organize our answers (when child has only generated 1 or 2 solutions)?</i></p>	<p>T elicits answers about applying a T-suggested strategy</p> <p><i>T: Let's make a list of numbers that add to 6; what plus what adds to 6?</i></p>	<p>T suggests a strategy via questions for S to apply, T elicits her own strategy, no re-voicing</p> <p><i>T: Why don't we write numbers that add to 6? Why don't you organize your answers?</i></p>	<p>T verifies correctness of S's answers as S carries out T's strategies or T draws conclusions for S's based on S's work</p> <p><i>T: No you can't do 0 o'clock. You have 5:10 and 5:01, how can you rearrange 1:50?</i></p>

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<p><b>1</b></p> <p><i>Teacher shifts her attention to student thinking, but once student ideas are elicited she is unsure how to incorporate them in the discussion. Discussion still includes more teacher talk than student talk.</i></p>	<p>T asks about S's strategies or methods, but does not follow up</p> <p>T: How did you find this? What did you do to get this solution?</p> <p>S: I guessed and checked (though S had a systematic way of guessing and checking that T does not explore).</p>	<p>T elicits strategies and Ss give brief descriptions of how to find solutions; T does not push for more details; T may fill in explanation</p> <p>T: Explain how you got this.</p> <p>S: I just rearranged them into different times.</p> <p>T: And you think you got them all because when you rearrange them you keep getting the same ones.</p>	<p>T elicits S ideas but does not explore them; T may revoice correct and more sophisticated explanation than is given by S</p> <p>S: We knew 10 couldn't be without getting an even number, so we did 9, then we added a number to each pile.</p> <p>T: We started at 12. Then we went to 11, but that was too many. 10 would be 10, 1, 1 and that wouldn't work, so we started at 9 and went down from there.</p>	<p>T gives feedback, but does not verify correctness; T encourages S to execute S's ideas that will lead to correct solutions</p> <p>T: That's a good start, but we're still missing a few more. What if we go back to this solution and look for more with a 3?</p>

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<p><b>1.5</b></p> <p><i>Teacher elicits student thinking and tries to follow students' ways of reasoning, though she may not always be successful in these attempts. Talk is more equally shared between teacher and student.</i></p>	<p>T asks for S's strategies and follows up with open-ended questions; T asks Ss to justify ideas and strategies</p>	<p>T elicits explanation about why (part of) strategy works; T begins pushing for clarification though T may still accept confusing or unclear explanation; T may seek multiple strategies</p>	<p>T elicits and explore S ideas; T helps S articulate idea or reasoning behind a particular strategy S employs</p>	<p>T encourages S to execute S's ideas; T sets up for Ss to evaluate their own errors, though T does not always explicitly follow up on this; S's draw their own conclusions</p>
	<p>T: Why did you use a nickel? What do your answers have in common? Why can't it be 10?</p>	<p>T: Why do you think there's more? S: I think there's more adding, more numbers that equal 10. So I'm thinking of numbers that equal 11. (It is not made clear how 11 relates to the problem.)</p>	<p>T: You did 9, 8, 7, 6; why did you erase the one with a 5? S: There was already one. T: I don't see a 5 anywhere. S: The 6 plus 5 plus 1 was the same as the one I had. T: The 5, 6, 1 was the same as 6, 5, 1? What about the others?</p>	<p>T: You think it's going to be a multiple of 12? Let's keep going and see what happens. (T lets student keep working but does not require to student to articulate why this idea is abandoned in favor of another.)</p>

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<p>2</p> <p><i>Teacher follows student thinking and encourages students to listen to one another. She begins to prompt students to interact, though students may still direct dialogue to the teacher. Teacher may model interaction by speaking to one student on behalf of another student.</i></p>	<p>T prompts Ss to ask questions of each other, T often supplies the exact question to be asked or asks it for the student; Ss ask questions of T or asks T about another S's work</p>	<p>T does not accept poor explanations; T probes Ss for elaboration and looks for multiple strategies; T prompts Ss to explain to each other though Ss still filter explanation through teacher; T may repeat one student's explanation for another</p>	<p>T describes one S's strategies or ideas for another; When asked to compare/contrast solutions or strategies, S's check answers against each other</p>	<p>T holds Ss accountable for listening to others; T asks Ss to make sense of other's ideas, though Ss may struggle with this; T is solely responsible for ensuring ideas are expressed clearly</p>
	<p><i>T: You should ask if she can help you. (to other student)</i> <i>Do you think you can help him?</i></p>	<p><i>T: Our friend said he started with 12, 11, 10 but they didn't work. So he did 9. (to S1) Do you want to explain to her what you did after that? (to S2) You want to watch?</i> <i>S1: I have 5:10.</i> <i>S2: Got it. 1:23?</i> <i>S1: I erased these because I already had them just in a different order.</i> <i>T: Are you explaining to her?</i></p>	<p><i>T: Can you guys compare answers? Why don't you look at his examples to see if that helps you?</i> <i>S1: I have 5:10.</i> <i>S2: Got it. 1:23?</i> <i>S1: Yes. (looking at S1's work) I don't have 4:11.</i></p>	<p><i>T: Do you see what she's saying? Did you pay attention? Can you explain her taking the numbers and adding them to 6? (student does not explain other's idea and requires more prompting from teacher to make sums of 6)</i></p>

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<b>2.5</b> <i>Student-to-student talk occurs, and is maintained beyond initial teacher prompting. Students respond to one another, no longer using teacher as intermediary.</i>	After initial T prompting, Ss ask each other low-level questions without T prompting. T still guides discussion with questions requiring explanation and justification	T probes for complete explanation and looks for multiple strategies; Ss give detailed descriptions of how solved and when prompted by teacher will describe how they solved to another student	When prompted by T, Ss can discuss similarities or differences in their strategies and solutions; Ss confer with one another about how to find solutions	T asks Ss about each other's work, asks Ss to evaluate each other's work (do they agree with another's ideas); Ss hold each other accountable for expressing ideas clearly
	<p><i>S1: What is your answer? Did you get this?</i></p> <p><i>S2: I got that one. Did you get one with 5 in a pile?</i></p> <p><i>S1: No.</i></p> <p><i>S2: Yes you did. There.</i></p> <p><i>T: How are you coming up with these answers? What are you thinking?</i></p>	<p><i>S1: It could be 5.</i></p> <p><i>T: 5 what? What is 5?</i></p> <p><i>S1: These two pairs make 5 so we put them on opposite sides. Then the sides are 10.</i></p> <p><i>T: So you just switched around the numbers that were the same? Tell S2 how you came up with that.</i></p> <p><i>S1: I thought of two numbers that add to the same thing and put them on opposite sides. 1, 4 and 2, 3. The 5 on top number keeps them equal 10 and 10. Then just switch around the bottom.</i></p>	<p><i>S1: 8, 4, 1. Did you do that?</i></p> <p><i>S2: No, it doesn't work.</i></p> <p><i>S1: Wait it works....8, 2, 3 works.</i></p> <p><i>S2: Are you sure?</i></p> <p><i>S1: No it doesn't. 8, 4, 1? No. 8, 2, 2?</i></p> <p><i>S2: Yeah. That's 12. Let's start picking out the numbers with 5.</i></p> <p><i>S1: 5, 4, 2? No 5, 4, 3.</i></p>	<p><i>T: S1 did you see how S2 got that?</i></p> <p><i>S1: She switched these around. The 6 was here last and if she did them in different places they might still be the same. Like the 1 changed the side to a 10.</i></p> <p><i>T: You think that she switched the 1? What about how S2 was picking out the numbers?</i></p> <p><i>S1: They both equal 5. But if she changed these two, that won't be the same.</i></p>

	Questioning	Explaining Thinking	Source of Ideas	Responsibility for Learning
<p><b>3</b></p> <p><i>High level student interaction with minimal teacher prompting. Discussion is guided by student ideas and reasoning.</i></p>	<p>Ss ask each other questions without T prompting; Ss ask each other to explain and justify work; T monitors Ss' questions and may help clarify Ss' questions</p>	<p>T does not have to probe for complete explanation; S's probe one another; When asked by T, Ss can explain one another's ideas; Ss defend and justify to each other and T</p>	<p>Students can generate ideas together; Ss take up apply one another's ideas; T uses S ideas as basis for extension problem</p>	<p>Ss take initiative for their learning; Ss ask for clarification/explanation of another's ideas; Ss judge correctness of others' ideas using their own reasoning; T is not evaluator of idea; T coordinates equitable S participation</p>
	<p>S1: Why did you do that? Why are you erasing all the answers? S2: I'm not. I already have these. I'm just erasing the repeats. Like you had 9, 2, 1 and I had 1, 2, 9. Those are the same. T: She's just making sure you don't have the same ones listed a bunch of times.</p>	<p>S2: First I did 4 strings going out from one person; then I did 3 strings going out from the next. And I saw it was getting one lower each time. T: Why did you do 3 strings from the next person? S2: Because it was already connected to one person. And the next person was already connected to two people that only left 2 connections. S1: I thought there since 4 people had 6, that 28 meant there were 26 people. But that didn't work. Then we figured out 5 people needed 10, so I thought 6 people need 14. Adding 4 each time. But 6 people had 15. So that is when I tried his idea. He said that it goes up by one each time. And when I tried that it worked. 5 was 10, so the next one gains 6 more So after 15 is going to be 21. The next time it grows by 7.</p>	<p>S1: That's 9 also. S2: Yeah but it doesn't work. One side is 9 and they're two 10s. S1: We could start with 2 and 6. That's 8. Probably 8 or 9 would be good. S2: That would be 14 [4 2 6 and 6 3 5] This is 12. S1: I'm going to start with 8. S2: What if we switch these? S1: Well we switched, it still adds to the same. S2: What about these two? That would be 10. This is S1: 11. S2: Not the same. T: Can we have these beside each other? S2: They're the biggest; they should go on different sides. S1: You should have big numbers with really, really small numbers.</p>	

### **Planning and Reflection Data Analysis**

I put all written data for each participant, two task dialogues (TDs), five activity plans (APs) and activity reflections (ARs), and the final paper, in chronological order into one word-processing document and assigned pseudonyms to each participant. I coded each dialogue of each TD, and each teacher response to the four types of solutions (none, incorrect, partial, and correct) in the APs, with levels of each of the five categories of the revised math-talk framework and cognitive demand framework using the guidelines outlined in Tables 5 and 6. In my initial reading of each participant's ARs and final paper, I underlined interesting phrases or passages. In a second reading, I took notes on recurring issues mentioned by the participants for each problem of each cycle. Then for each participant I composed a document that summarized findings from this analysis.

### **Connecting Analysis to Research Questions**

To determine how participants' use of mathematical discussion supported the cognitive demand of the problem-solving activities (Research Question 1), I examined the coded video data two ways. For each participant, I considered how the four components of the math-talk framework were associated with the cognitive demand for a problem enactment. Plotting in chronological order each participant's segments of all three enactments of each problem along the  $x$ -axis and the levels for each category along the  $y$ -axis, I determined which components of math-talk tracked most closely to cognitive demand. For example, Figure 3a shows Kate's cognitive demand and math-talk component levels for all of her segments for the 12 Pennies Problem. It appears that for this problem, the levels of *explaining* most closely matched cognitive demand. Construction of similar figures allowed me to examine how the *questioning*, *explaining*,

*ideas*, and *responsibility* for the segment following a particular kind of pupil solution (none, incorrect, partially correct, correct), strategy or idea was associated with cognitive demand. For example, Figure 3b shows what happened to the cognitive demand and ideas as Nadia responded to incorrect, partially correct, and correct student responses throughout Cycle 1. Together the two types of charts helped me determine how the participant's pedagogical moves in leading mathematics discussions influenced cognitive demand.

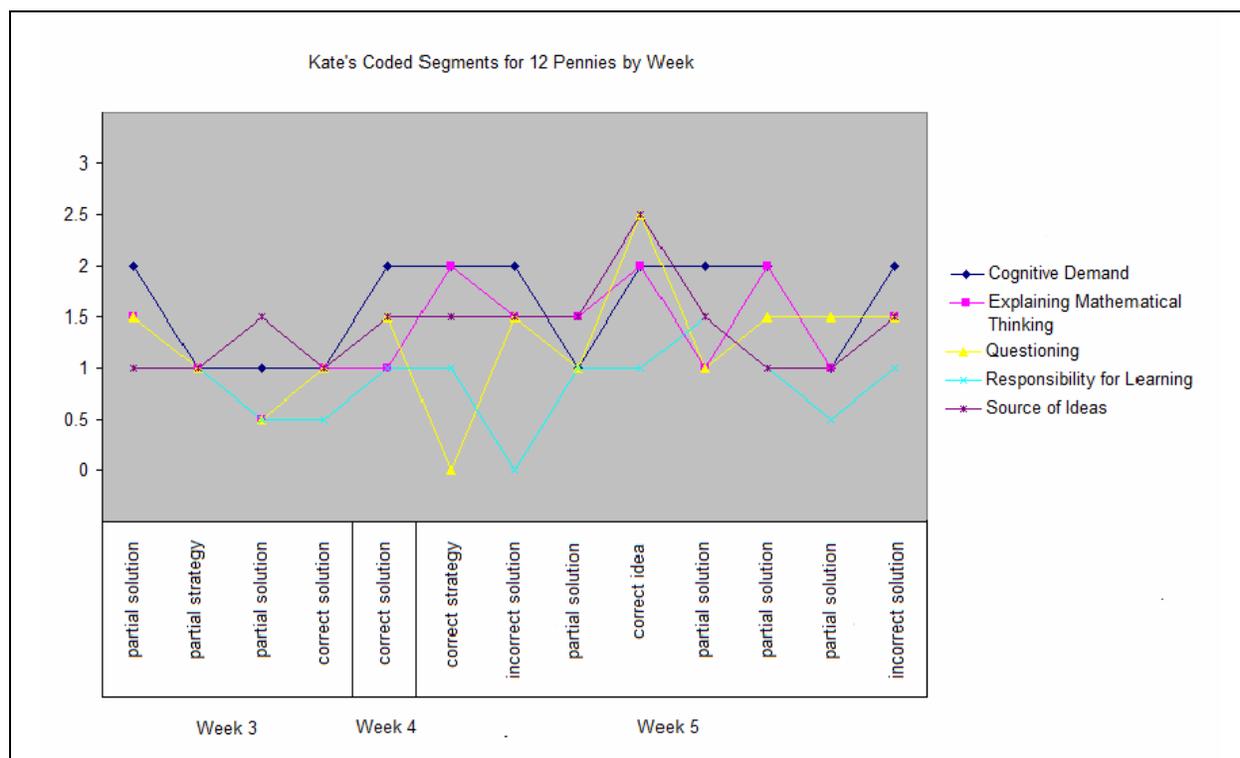


Figure 3a. Kate's segments category levels for 12 pennies.

For each participant, I examined instances of responding to incorrect, nearly correct, and correct pupil thinking and made a brief description of the pedagogical move she used to respond (e.g., corrected error, proposed contradiction, asked for explanation). I also noted the cognitive demand and what components of math-talk the move relied on. Looking across all 8 participants, I was able to determine typical responses to different types of pupil thinking, what components of math-talk they drew on, and how those responses were associated with cognitive demand. In

some instances, I was able to hypothesize what conditions may have influenced the participants' reasons for making a particular pedagogical move.

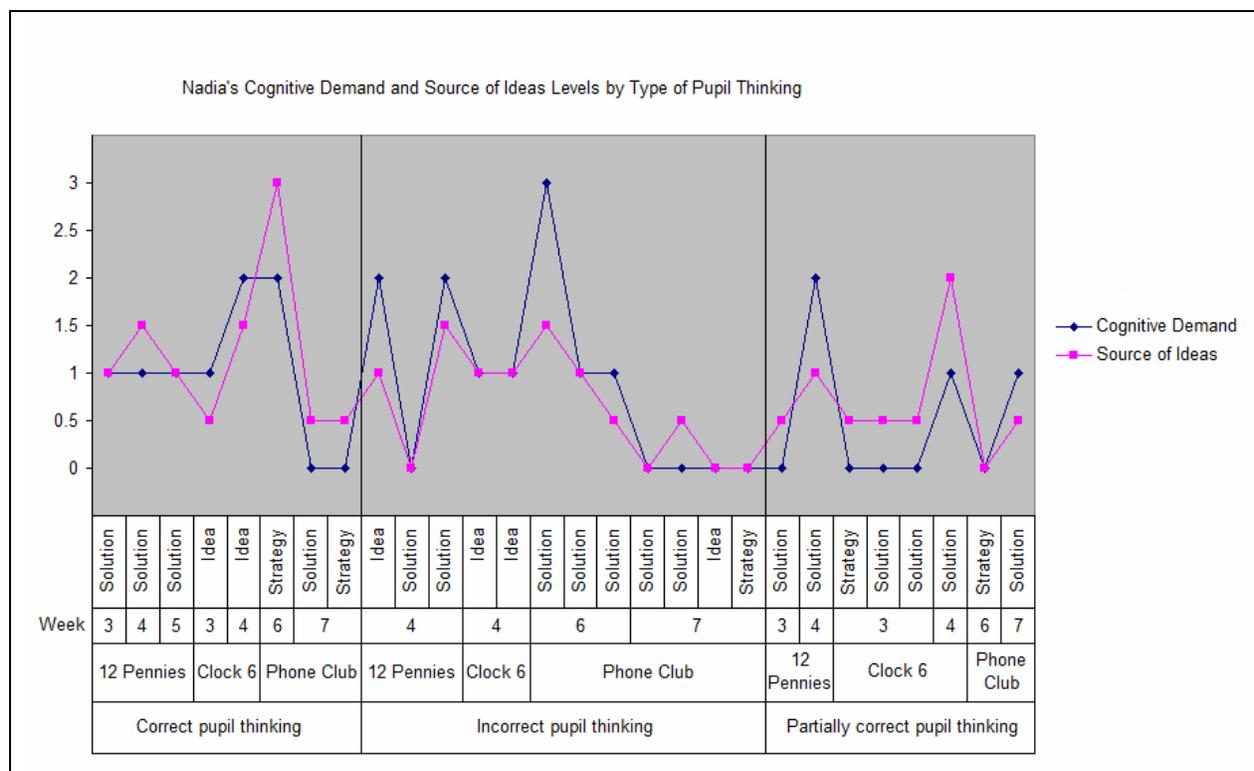


Figure 3b. Nadia's category levels in response to types of student responses in Cycle 1.

To determine how the 3-week cycles influenced the participants' implementation of problem-solving activities over a 6-week field experience (Research Question 2), for each problem of each participant, I looked for trends in their cycles of planning (task dialogues and activity plans), enactment (video data), and reflection (activity reflections and final paper) and the development of their pedagogical content knowledge regarding these problems. For each of the problems for each participant, I took the cycle's task dialogue for each problem as a starting point. By comparing their enactment to their reflection, I decided whether the participants were able to accurately identify their and their pupils' successes and struggles. By comparing their reflection and subsequent activity plan, I decided whether they were making appropriate follow-

up plans to address the identified struggles or to incorporate my or her fellow classmate's suggestions. By comparing the activity plan to the next problem enactment, I could determine whether the reflection and revised plan were translating into changes their practice.

Plotting in chronological order each participant's segments of all enactments of each problem along the  $x$ -axis and the levels for cognitive demand along the  $y$ -axis, I was able to see trends in participants' problem enactment over the course of the study (Figure 4). By comparing the levels of cognitive demand of the coded video data across the enactments of each problem, I determined how each preservice teacher's ability to implement the problem changed over the 3-week cycle. Comparing levels of cognitive demand across all 6 weeks of the study helped me to determine whether the participants were able to transfer their learning from one cycle to the next. I grouped the participants with similar trajectories of cognitive demand and examined their planning and reflection data to determine how they might have influenced the participants' implementations.

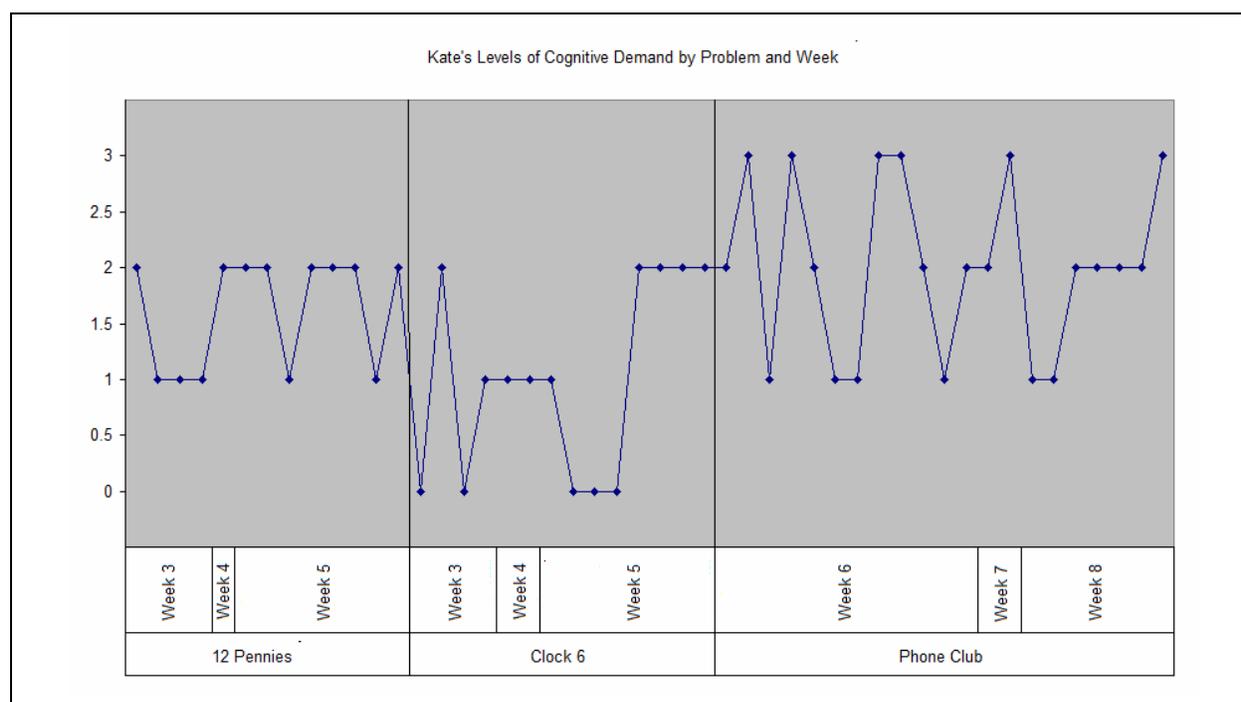


Figure 4. Kate's cognitive demand for all problems.

To determine how participants envisioned leading mathematical discussions on problem-solving activities and how their plans aligned with problem enactment (Research Question 3), I compared each participant's task dialogues and activity plans with her actual enactment of problems. Did the participants with directive task dialogues and activity plans that lowered the cognitive demand also implement these tasks in a directive manner? More interestingly, did those whose task dialogues and activity plans maintain cognitive demand faithfully implement these tasks as planned? Were changes in implementations reflected in plans? Describing the participants' activity plans and task dialogues as directive and nondirective or student-centered and teacher-centered did not adequately capture the key variations in their planning. Further complicating the issue, many participants' plans and enactments changed over time, and not necessarily in parallel ways. To simplify matters, after the participants had been categorized by the trajectory of their mathematics discussions over the 6 weeks, I gave each participant's task dialogues a brief descriptor, noted whether their plans improved in each cycle and how they did or did not, and then determined whether the participants actually followed their task dialogues and plans.

### **Limitations**

This study had two main limitations: the use of groups of participants and the time spent with each pupil pair. Analyzing the implementation of problems by groups of participants made attributing levels of cognitive demand to individual teachers difficult. When I parsed an individual participant's contribution to a segment of collaborative teaching, I could not account for the influence of other group members: Some participants may have been buoyed or restrained by their fellow group members. However, this limitation is balanced by the many benefits to preservice teacher learning that occurred only because they were working in groups. The math-

talk framework used to analyze data in this study focused on building a math-talk community. Each participant group worked with one pupil pair at most twice during the 6 weeks and more frequently only once. Changing pupils weekly certainly affected the sense of community that could be developed and, hence, the levels of math-talk components and cognitive demand that could be achieved.

## CHAPTER FOUR

### RESULTS

In this chapter, I present the results organized by research question. The first question explores how typical pedagogical moves across the group of participants related to cognitive demand. The results for this question are organized by the type of pupil thinking to which the participants were responding. The second question examines changes in each participant's enactments over time; results for this question are presented as four categories of teachers. The final question explores how those four categories of teachers' planning activities aligned with their instruction.

#### **Influence of Mathematical Discussion on Cognitive Demand**

Each participant had idiosyncratic responses to particular types of pupil solutions and strategies. However, typical features of these responses emerged across participants and over different tasks, and these common responses were associated with different levels of cognitive demand. Below I outline some pedagogical moves that commonly appeared in response to four particular kinds of student thinking: the pupil is stuck; the pupil has an incorrect solution or is using an incorrect strategy; the pupil has a correct solution or is using a correct strategy; and the pupil has a nearly correct solution, has some but not all solutions, or is using a strategy that will result in one of these. Within each category of pupil thinking, I describe the common participant pedagogical responses, discuss how these responses corresponded with the cognitive demand, and hypothesize what conditions led the participants to employ particular pedagogical moves.

## Pupil Is Stuck

Pupils were rarely completely at a loss as to how to approach a problem; they were typically willing to execute their ideas. However, problems with a lot of conditions (Clock and Cupcakes Problems) did seem to cause pupils difficulty in starting. Pupils also stalled out during their work on problems that had multiple solutions (12 Pennies, 6 Numbers, and Clock 6 Problems) and problems in which they employed a lengthy guess-and-check strategy (Cupcakes Problem). Participants relied on two pedagogical moves to help a pupil who was uncertain about how to proceed: They *suggested a strategy* for the pupil to try, or they *posed questions* to help the pupil generate ideas about the problem. Both of these moves were associated with different levels of cognitive demand. The versions of these moves that were associated with higher levels of cognitive demand (*procedures with connections*) focused on exploring the pupil's thinking. There was no consistent pattern to the conditions surrounding a participant's choice between *suggesting a strategy* or *posing questions*, though in later weeks they were more likely to maintain higher cognitive demand while using these two pedagogical moves. For example, some questioning led the pupils to a particular approach: In Week 5, when the pupils were stuck on how to start the [Clock 6 Problem](#), Kate asked if they knew what a digital clock was. When the pupils said yes, she did not explore their understanding of the digital clock but asked them to practice summing the digits on an example. When the pupils were asked to explain what the question was asking, they just repeated the question. Though Kate's instinct, to establish that the pupils understood all elements of the question, was a good one, her approach did not make the pupils the *source of ideas*. However, in Week 6, working on the [Phone Club Problem](#), she attuned her broad question to the pupil's thinking instead of her own. A pupil staring at her blank work space mumbled, "What does that mean—every person to every other person?" Kate

responded, “What do you think it means?” (Group H, Week 6, p. 54). This response provided an opportunity for the pupil to describe her understanding through a picture, consult her partner, and explain for herself what the wording meant. Kate shifted the *source of ideas* to the pupils, hence maintaining or raising the cognitive demand.

Similarly, the move of *suggesting a strategy* had several variations. The participants often directly told a pupil how to start a problem. For example, when her pupil confessed that she did not know what to do to begin the [Clock 6 Problem](#), Nadia suggested, “Why don’t we start with writing down numbers that add up to 6?” (Group H Video transcript, p. 7). Nadia zeroed in on an easy mathematical activity of the problem to get the pupil started, but to a confused pupil, it may not have been clear how this suggestion was connected to the conditions of the problem. This issue became apparent later when the pupil identified six and zero as numbers that sum to six but needed further direction from Nadia “to make that into a time” (Group H Video transcript, p. 7). In this series of directive suggestions, the *source of ideas* was Nadia, not her pupil, which lowered cognitive demand to *procedures without connections*. However, suggesting a strategy could also raise the cognitive demand. In Week 8, Kate used a pupil’s previous strategy to offer a suggestion on an extension of the [Phone Club Problem](#) involving 9 people:

Kate: Would you draw out nine people and draw all the strings? Or would you use a different strategy?

C7: Uh. [pause]

Kate: What did you learn from doing this [indicating diagram in Figure 5]?”

C7: I think I’d start drawing strings. I’d stop at the third person and see if I have a pattern. And then...

....

Kate: How about you draw it? Since you didn’t draw this many, I won’t make you draw more than 8. Why don’t you draw a line of 6 people and see how many strings 6

people would have and just start out by drawing the first three like you thought.  
(Group H Video transcript, p. 88)

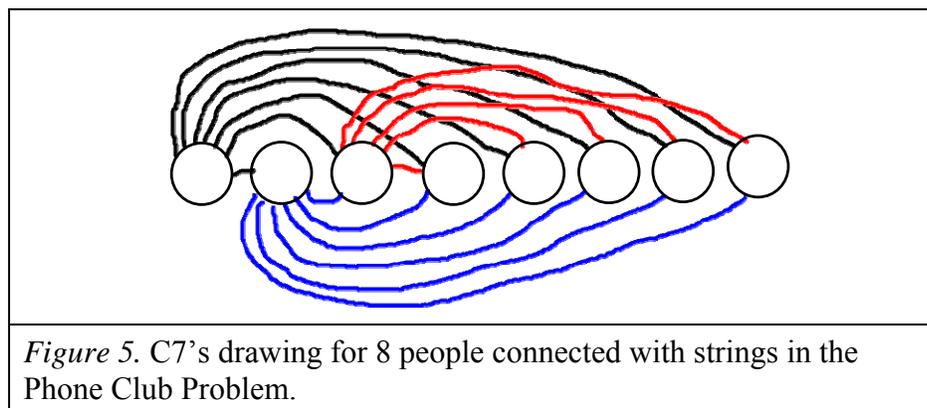


Figure 5. C7's drawing for 8 people connected with strings in the Phone Club Problem.

By connecting to the pupil's earlier work, Kate was able to make the pupil the *source of ideas*, and this action was associated with a higher cognitive demand of *doing mathematics*, as the pupil, not Kate, made and tested a hypothesis.

A third participant reaction was to *repeat the question*, and though this action maintained cognitive demand, it did not result in pupils making any progress. When a pupil paused in her work on the 12 Pennies Problem, unsure of what pennies to put in what pile, Casey repeated the key elements of the question, putting emphasis on words most relevant to the pupil's difficulty: "The question says put ... 12 pennies can be put into 3 piles so that *each pile* has a different number" (Group H Video transcript, p. 16). The participants were aware that they wanted to minimize the help they gave pupil, and some translated this principle as not saying anything more than what was in the original question. *Repeating the question* was most often used when pupils did not know how to begin a problem. The participants may have also used this move when they did not understand the nature of the pupil's confusion or did not know what mathematical ideas to make explicit to help the pupil.

## Pupil Is Incorrect

Each preservice teacher had particular moves that she used when faced with off-the-wall pupil strategies or ideas. One of the most common moves used in response to a drastically incorrect solution was to pose a series of direct short-answer questions to *walk the pupil through the preservice teacher's strategy* for solving the problem. In this scenario, a preservice teacher may even have answered her own questions or posed the questions in such a way that the answer she wanted was obvious. This move generally was associated with a cognitive demand of *memorization*, wherein the task was reduced to recalling previously learned material or answering easy straightforward questions whose connection to problem was not made clear. Nadia, in particular, relied on this technique throughout the 6 weeks. An example follows in which Nadia responded to the pupils' representations to the [Phone Club Problem](#) shown in Figure 6.

In responding this way, Nadia took the *responsibility for learning* from the pupil and made herself, not the pupil, the *source of ideas*. This type of response most often occurred when the incorrect solution was not one the participant had seen before. The participants seemed to panic when presented a solution that deviated from their task dialogues and activity plans and worked to efficiently get the pupil back to more familiar territory. Perhaps they thought that because they were using questions rather than directive instructions, they were maintaining cognitive demand.

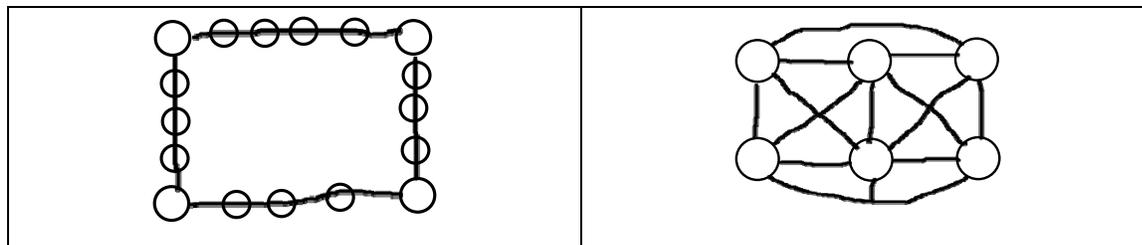


Figure 6a. C10's picture for 28 strings.

Figure 6b. C11's picture for 28 strings.

Nadia: Let's just look at the 4 of us, say this is you, so how many strings do you need to talk to all of us?

C10: Um 4 ... we need 6.

Nadia: Ok, this is you, right [pointing to the gray circle in Figure 7]? So you need a string to talk to me, a string to talk to her, and a string to talk to her, right [pointing to each person in turn]? [to C11] And how many strings do you need?

C11: Uh 4. No, 3.

Nadia: Ok, what you're saying that you're starting, so she's got. ... I'm doing too much [C10 traces over the 3 strings that connect her to the others.]

C10: 3 strings.

Nadia: Ok, so now we're going to go to you [C11, represented by the pink circle in Figure 7]. Now you already have a string connecting you and her [C10, the gray circle in Figure 7], so how many more do you need now? To connect?

C11: 2.

Nadia: 2, right.

Casey: Can you trace it with your blue marker, the two that you need to talk? [C11 traces her two strings in blue]

Nadia: Now Casey, Casey [represented by the brown circle in Figure 7] has a string attached to [her] and a string attached to you [pointing to C10 and C11], how many [more strings] does she need?

C10: One.

C11: One.

Nadia: One. [C10 traces Casey's string in red.] All right, and see when it gets to me [the white circle], I'm already connected to everyone else. I don't need any [more strings]. So do you see ... ? (Group H Video transcript, pp. 73–74)

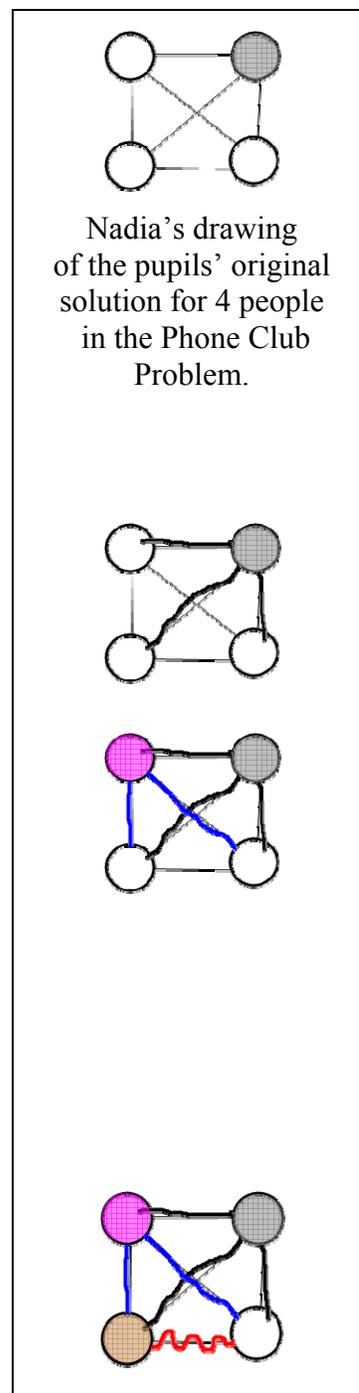


Figure 7. Group's construction of representation of solution to Phone Club problem.

In responding this way, Nadia took the *responsibility for learning* from the pupil and made herself, not the pupil, the *source of ideas*. This type of response most often occurred when the incorrect solution was not one the participant had seen before. The participants seemed to panic when presented a solution that deviated from their task dialogues and activity plans and worked to efficiently get the pupil back to more familiar territory. Perhaps they thought that because they were using questions rather than directive instructions, they were maintaining cognitive demand.

Instead of using *questioning* to respond to an incorrect solution, another potential preservice teacher response that also lowered cognitive demand was *to correct* what the preservice teacher perceived as the pupil's misunderstanding of the problem. In the following example, Casey's pupil composed a solution to the 12 Pennies Problem using a nickel and 7 pennies. Casey instructed her pupil how to change his solution to fit the parameters of the [12 Pennies Problem](#), using only 12 pennies distributed among 3 piles:

Casey: Why are you using a nickel?

C6: 5 with, yeah, 5 plus 7.

Casey: You're showing me, you're using 7 pennies and a nickel? How many pennies are in a nickel?

C6: Um, 5.

Casey: Ok, can you count out 5 pennies, and we'll trade this in.

C6: Well, isn't that still the same?

Casey: It's the same, but we're supposed to be using pennies in the problem. So if you want to show us you're using 5 pennies, you would just put these 5 pennies in a pile. You understand?

C6: Oh.

Casey: So make sure you count out just 12 pennies. So you have 7; how many more do you need? (Group H Video transcript, pp. 16–17)

Though Casey took the time to clarify what the pupil was doing, she did not show that she understood the pupil's reasoning behind this solution. It seemed the pupil interpreted the question as a typical question children might be asked about change: In what ways you can make 12 cents using any coins? By not connecting her instructions to the pupil's way of thinking about the problem, she eliminated an opportunity for him to understand the problem constraints on his own, thus lowering the cognitive demand to *memorization*. When the participants corrected pupils in this way, it often happened early in the problem or as the pupil was beginning a new strategy; the participants were unwilling to let pupils make major errors as they were just getting started on a problem.

A third response to incorrect solutions that placed more *responsibility for learning* on the pupil was to explore the pupil's solution and allow the pupil to *continue an incorrect way of thinking* until it played itself out. In the following example, Nadia has asked why a pupil thought the solution to the [12 Pennies Problem](#) was 36:

C1: Because if there's 3 groups, and there's 12...

Nadia: 12 pennies.

C1: Yeah, so there's 12 pennies in each one. ... It would equal 36. But that's just a guess.

Nadia: A guess? Well, let's keep going and see how many you can find.

C1: At first I thought it was going to be 4 because 3 divided by 12. Then I thought it would be 6. It's going to be a multiple of 12.

Nadia: You think it's going to be a multiple of 12?

C1: Or a multiple of 3. Yeah, I have no idea.

Nadia: Ok, let's just keep going and see what happens. (Group H Video transcript, p. 24)

Ultimately the pupil drops this line of thinking when she was unable to confirm any of these predictions. Though this is an improvement over the previous two participant responses to

incorrect solutions, it still does not elevate the cognitive demand to *doing mathematics*. Nadia did not ask the pupil to explain why this solution did not work, nor did she ask why the pupil abandoned this line of thinking in favor of another. The decision to make this type of response over a more directive one may have been influenced by the quality of the pupil's solution. When a participant thought that the pupil had some reasoning that led to the solution, the participant may have believed that less direct guidance was needed.

Another move in response to a severely flawed solution was to ask a few specific questions that were intended to get the *pupil to identify his or her own mistake*. Though the success in getting the pupil to identify the error or misconception varied, this move typically was associated with an elevated cognitive demand of *procedures with connections*. A notable example of this response occurred with Alice and Dana helping one pupil with the [Puppies Problem](#). The pupil wrote in a column from bottom to top 4, 4, 16, 1, which looked to be a promising start. When Alice asked how he obtained these numbers, he was able to describe a working backward strategy but then concluded the pet store started with 25 puppies (calculated by summing his list of 4, 4, 16, and 1). Alice then tried several approaches to help him identify the mistake: "You want to see if 25 puppies works by working it forwards?" He persisted in using his current backward strategy. She tried giving him two options to help clarify his thinking: "Do you think you are supposed to add, or do you think that each [number in the list] represents the total number at that time?" When he launched into another explanation, she picked up on a sensible part of it saying, "I hear you say he took this [the one puppy taken at the beginning of the problem] first. You are saying there is 24 puppies. Let's work it out." After she started him on the forward strategy to check 25, he claimed that 18 would be left and, reworking it again, that 21 would be the answer if you add his newly generated list: 12, 6, 3. At this point, Dana,

who was recording, stepped in: “What do those numbers represent? Which one is Monday? What is this?” Alice picked up on this cue from Dana, and they both worked to have him explain each number in his list. Trying to connect what the numbers represented to his idea about summing so that he could see the contradiction between what he was saying and the operations he was performing, Alice then asked him, “Explain where you are getting 25. Can you tell me again what each number represents?” (Group I Video transcript, pp. 70–84). Though they continued to provide him opportunities to identify the flaw in his solution, the pupil never moved past the idea of summing the list.

Some participants tried the strategy of *getting the pupil to identify his or her mistake* because they knew they were not supposed to correct pupils or guide them through adult ways of reasoning. Of participants who tried this strategy, some consistently did so throughout the 6 weeks and also used it in their task dialogues, whereas others only attempted this strategy in later weeks of the study after getting feedback from instructors or peers that more directive responses were not effective in helping pupils be autonomous problem-solvers. Other attempts at getting the pupil to self-correct were the result of the participants’ assessment that the incorrect response could be easily caught and corrected by the pupil. Attempting to get the pupils to identify their errors was most successful when the error was not based on a fundamental misconception. The lack of success Dana and Alice experienced can be attributed to their initial assessment that error was a simple mistake in applying a working-backward heuristic and their failure to bring up in their discussion the particular issue of what the whole was at each step of the problem.

### **Pupil Is Correct**

At the other end of the spectrum were cases in which the pupils were able to find correct solutions or could use a correct strategy with minimal or no teacher intervention. Although one

might think that finding a correct solution with little guidance from adults is indicative of high cognitive demand, that was not always the case. One common response by the participants to correct solutions and strategies that lowered cognitive demand to *procedures without connections* was to do no more than praise pupils for their correct solutions and *move to the next problem*. This response was most often employed when the pupil had taken a long time to arrive at a correct solution, and the participant thought the pupil was burned out on that problem. In particular, this response was common with problems having multiple solutions: the Clock 6, 12 Pennies, and 6 Numbers Problems. It also occurred when a participant was unaware or unsure of the mathematics underlying the problem's solution and thus was unable or unwilling to explore those ideas with the pupils. For example, in their first week, completing the [12 Pennies Problem](#) with pupils, Kate, Casey, and Nadia (Group H) were unsure of the solution. When Nadia's pupil, who had 7 solutions, asked if she needed to find the additional solutions that she saw on her partner's board, Nadia responded, "No, there's not— ... let's let her [the other pupil] work through it before we explain ours." In her reflection, Nadia explained, "Our group was a little confused as to whether or not we should count these 'differences' in the piles like 921 and 192. At the end of the problem, we summarized by saying that there are 7 completely different combinations and you can arrange one combination 6 different ways." Nadia did not elicit this explanation from either pupil, nor did she make explicit that her pupil could be certain she had accounted for all possibilities by constructing her list of options systematically. Kate explained:

This got kind of confusing for us all, but [the pupils] were able to grasp that if you had 3 numbers that worked [added to 12 with no two piles having the same number], then there were 6 different ways they could be reorganized into different piles. After they claimed they understood that, we moved on to the next problem though because they were tired of it. (Kate, Activity Reflection 3, p. 10)

In the first weeks of the study, *moving to the next problem* without eliciting pupils' explanations was a common response to correct solutions. However, as the study progressed and the participants reflected on their work with pupils and received feedback on that work from the course instructor and peers, this response became less common. For example, in the first week of Cycle 1, when a pupil correctly found a solution to the [6 Numbers Problem](#) in which all sides summed to 11, Erica confirmed the pupil's solution and, rather than exploring that solution, prompted him to keep looking for others, "Yay! So what's our magic number? So far we got—? [Pupil replies "11."] Great job. So, you think there is any more?" (Group J Video transcript, p. 5). This response is in contrast to a more pedagogically sophisticated response to a correct solution. In Week 3, a pupil found the correct solution to the [Cupcakes Problem](#): 5 chocolate boxes and 7 vanilla boxes. Rather than accept this correct information, Dana followed up with questions to help her pupil consider why this was the only solution:

Dana: Yeah. So you don't think any others would work?

C7: Well, ... I don't know, um. ... I don't think so ...

Dana: Why did this one work? 7 vanilla [boxes] and 5 chocolate [boxes]. ... I know you know. I just want to know, why did this work, having 7 vanilla [boxes] and 5 chocolates [boxes]?

C7: Because if I ... if I had, like ... if I had, like, too many, too many chocolate cupcakes, it will go over ... it will go over, like, 58 [the total number of cupcakes] ... of how many cupcakes. Like ... 4 ... 4 is like ...

Dana: So, do you think there always needs to be less chocolate cupcakes than vanilla?

C7: Um, I don't think so. Well, I don't think so, because, like, if you ... you don't, like ... if you did ... you can have ...

Dana: Like, could I have 8 chocolate and 4 vanilla?

C7: Oh, I don't think ... I don't think that would work.

Dana: That wouldn't work? Why wouldn't that work?

C7: Because if you ... if you ... if you have, if you do 8 chocolate and 4 vanilla, so there's ... Well, we would have 12 boxes. We have 12 boxes of cupcakes, but I don't think we would have 58 cupcakes.

Dana: Why not?

C7: Because I already ... I tried ...

Dana: Well, you did try.

C7: But it ... it's a bigger number.

Dana: Right.

C7: If I try it with an even bigger number, it will probably be ... even bigger.

Dana: Right, and that's because why? (Group I Video transcript, pp. 6–8)

Though this is an atypical response for Dana early in the study, it is an example of another pedagogical move made in response to a correct solution; she required her pupil to *provide some supporting explanation*. Those participants who responded to correct solutions by asking pupils to explain how they solved the problem or why their strategy worked fell into two categories: They consistently required pupils to explain over the 6-week study, or they initially did not elicit explanations but began to do so over the course of the study. For three participants (Kate, Rene, and Megan), asking pupils to explain their solution was their standard response to correct solutions during the entire study. These participants seemed to understand that the goal of their work with pupils was not finding answers but explaining those answers. However, the quality of the explanations they elicited varied. This variation was also true for the participants who did not employ such a response initially but began to do so as the study progressed. Most participants' first attempts at eliciting explanations (regardless of when in the study their first attempts occurred) initially accepted any explanation given by a pupil. In those instances, the participants did not probe the pupils for complete explanations. They were satisfied that the pupil could offer an explanation of how he or she had arrived at a solution and did not pose follow-up

questions or require the pupil to explain why that process worked. This response was typically associated with a *procedures without connections* level of cognitive demand.

Responding to pupils' explanations proved challenging for some participants. Confusing and unclear explanations were particularly difficult to manage. One response to a pupil's confusing explanation was simply to *ignore* it. Nadia described how she became aware of this problem in her teaching:

Another area I struggled with at first was genuinely listening to and understanding the child's explanation and strategy. When problems took a long time and we were running short on time, I had a tendency to just nod and say yes while the student was explaining what they did. I had watched them work through the entire problem so I thought I knew their strategy pretty well, but it is still so important to understand what they are saying. I noticed that I was doing this one week with C6 and C1. I was watching the video to write my reflection, and I noticed that when C6 was explaining an answer I said "right" on the tape, but I had to rewind the video a couple of times to understand what he meant. (Nadia, final paper, p. 52)

Over the course of the study, all of the participants eventually attempted to push for clarification of pupils' explanations; however, their success varied. They encountered four difficulties: continuing to accept unclear or confusing explanations, revocicing their own ideas instead of the pupils' ideas, accepting incomplete explanations by asking only for how a strategy was carried out and not why that strategy yielded a correct solution, and obtaining clear and complete explanations but doing so through excessive teacher guidance. These difficulties resulted in cognitive demand varying between *procedures with connections* and *procedures without connections*. For example, Casey attempted to push for a better explanation when she asked why a pupil started his list for the [12 Pennies Problem](#) with 9, but ultimately *let an unclear explanation go unchallenged*:

C6: [writes  $9 + 2 + 1 = 12$ ]

Casey: Why did you start with 9?

C6: I just thought the first time to do 11 plus 1 [referring to an earlier attempt where he wrote:  $11 + 1 = 12$ ]. Then I just I thought I can subtract 2 and try to make 3 piles out of it.

Casey: So you knew that 11 didn't work, because 11 plus 1 is 12, but you only had 2 numbers or 2 piles. Why didn't you do  $10 + 1 + 1$ ?

C6: Because one cannot have two.

Casey: Good, so you just went down to 9. So what would your next step be? (Group H Video transcript, p. 18)

The pupil explained how he based his choice of 9 on an incorrect initial choice of 11. Casey revoiced his idea and then, pushing for a more complete explanation, asked why he did not use 10 instead of 9. The pupil's explanation for this step, however, was unclear. A logical follow-up might have been, "Are you saying that one pile cannot have two pennies?" Instead, Casey accepted the explanation, assuming that the pupil meant two piles could not have one penny each. This unclear explanation, followed by Casey's prompt for the "next step," established a cognitive demand of *procedures without connections*.

Another response to a confusing explanation that lowered cognitive demand was for the *teacher to revoice her own idea instead of the pupil's*. Both this move and *continuing to accept an unclear explanation* denied the pupil the opportunity to gain practice explaining his ideas and denied the participant the opportunity to see the mathematics from the pupil's point of view.

When a pupil gave a confusing explanation for a solution to the [12 Pennies Problem](#), rather than push for clarification, Nadia revoiced the strategy that she had tried to walk her pupil through.

Kate, on the other hand, ignored the first pupil's strategy and asked the other pupil to explain:

C3: I did 3 circles and put pennies in them and guessed and checked.

Kate: Ok.

C3 gave incomplete explanation.

Kate accepted the explanation.

<p>Nadia: Ok. But at some point that stopped working, right? We kept getting the same thing? So what'd we do?</p> <p>C3: So we, well we knew 10 could be that without getting an even number, so we put in 9, and then we added each number to each circle.</p> <p>Nadia: So, we started at 10, and we knew ... well, we started at 12; we knew we couldn't do that, and we went to 11. We knew we couldn't have that many in one pile. We went to 10, and it would have been 10, 1, 1, which wouldn't have worked, so then we started with 9. And we just worked our way down from there.</p> <p>Kate: Ok. [to C2] Do you want to explain kind of what you did? (Group H Video, p. 5)</p>	<p>Nadia pushed for more detail.</p> <p>C3 gave confusing explanation.</p> <p>Nadia revoiced her own strategy in lieu of the pupil's and Kate ignored the confusing explanation in favor of eliciting the other pupil's ideas.</p>
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The participants made improvements in pushing pupils to better articulate their explanations, but they still *did not ask the pupils to consider why a particular strategy worked*. In the following example from Week 8, Casey helped her pupil identify and clearly describe a pattern in the [Phone Club Problem](#):

- Casey: So, do you notice anything about that [the number of strings coming from each person, see her drawing in [Figure 5](#)]?
- C7: It's counting down from 7. I mean, like, it's going down by one.
- Kate: Do you want to see if it keeps doing that?
- C7: Yeah. [C7 keeps working]
- Kate: So why don't you draw lines for the forth person?
- C7: Ok.
- ....
- Casey: Talk to me about it. To make sure I understand what you did, tell me what you did first.
- C7: Well, first I did 4, like, for the first question. Then I tried 6. I put 6 people on there. And I thought about and took it off. Then I did 7 people, and I was drawing strings from each person. For my first person, it went to, the line went to 7 people,

for the second it went to 6, and for the third it went to 5, for the fourth it went to 4, for the fifth it went to 3, for the sixth it went to 2, for the 1 it was 1.

Casey: How many did the first one have again?

C7: 7.

Casey: It had 7?

C7: Yeah, yeah, no, I mean, like, for the first one it connected to 7 people.

Casey: So 7, it had 7. And then how many?

C7: Then 6.

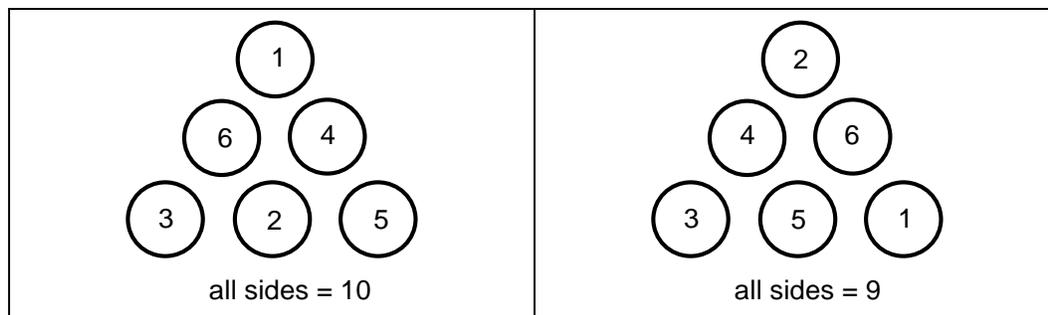
Casey: Then how many?

C7: 5 ... 4 ... 3 ... 2 ... 1. (Group H Video transcript, pp. 84–85)

Though this explanation was associated with a *procedures with connections* level of cognitive demand, it falls short in attending to why the pattern exists. In this particular example, the pupil was drawing the pattern for 8 people, but Casey did not address why the pattern starts counting down from 7 instead of 8. This failure caused a difficulty later when the pupil saw that starting with 8 strings and “counting down” ( $8 + 7 + \dots + 1$ ) gave her the correct 28 total strings. However, she then erroneously concluded that because she had started with 8 strings, it meant she had started with 8 people when, in fact, starting with 8 strings meant she must have had 9 people.

The participants who appreciated the need for clear and complete explanations still at times struggled to elicit explanations from pupils without being *overly directive in helping the pupil articulate his or her thinking*, and the task still had varying levels of cognitive demand. In the example that follows, the pupil had obtained one correct solution to the [6 Numbers Problem](#) where the smallest numbers are in the corner (see Figure 8a). Rene was trying to help the pupil explain why her solution worked (large numbers were each on different sides) and use that idea to help her find other solutions:

Rene: So, do you notice anything about where your numbers are, which numbers are where? [pause] Let's look at the numbers and see. Look at the first one, the second one. Can you see any similarities between your first answer [Figure 8a] and your second [Figure 8b]?



*Figure 8a.* C2's first answer to the 6 Numbers Problem.

*Figure 8b.* C2's second answer to the 6 Numbers Problem.

C2: No.

Rene: None?

C2: Except for, they all same numbers, like 5 and 3.

Rene: Oh, they do [have] all the same [numbers], that's right. Um ... Let's see where the numbers are. On here [indicating Figure 8b]. Where is? ... Where is your number 6?

C2: Side.

Rene: What about your number 5?

C2: It's on the side.

Rene: What about your number 4?

C2: Side.

Rene: What about 3?

[C2 points to 3 in bottom corner.]

Rene: In the corner? What about 2?

C2: Top.

Rene: In the top corner? And what about number 1?

....

[C2 tries rotating her current solution.]

Rene: Well, let's think about it. So, what numbers were on your corners? What numbers did you use to make the corners of your triangle?

C2: Lower numbers.

Rene: Yeah, you did. That's right. You had the lower numbers, 1, 2, and 3. Well, what about on the insides, the middle?

C2: Larger.

Rene: Larger numbers. ... Why do you think that is? Why would the tiny numbers be on the corners and the larger numbers be on the inside?

C2: Maybe because ... only you can get the larger numbers together. So maybe ... (Group J Video transcript, pp. 37–38)

The pupil did not connect her answers to these questions to the larger issue of solving the problem. Rather than notice this pattern, she noticed that rotating the triangle would yield the same sum for each side but different positions for the sides. Rene eventually steered her back to the original pattern and continued posing directive questions to help the pupil articulate the pattern. Megan, on the other hand, reached a similar roadblock with her pupil on the [6 Numbers Problem](#) (Figure 9), but she is an example of a participant who was more successful at eliciting good explanations without being overly directive:

Megan: Do you notice anything about how you position the numbers?

C5: No.

Megan: No. Ok. Well, keep trying and see what else you can find.

C5: Yeah. It goes 1, 2, 3 [pointing to corners], and then it skips and 4, 5, 6.

Megan: Ok. So the smaller numbers ...

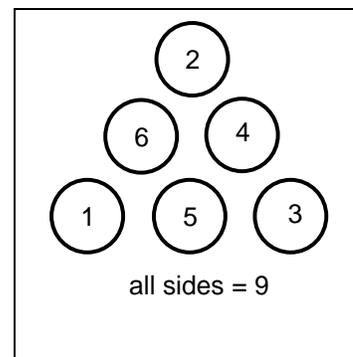


Figure 9a. C5's solution to 6 Numbers Problem.

C5: Around the outside. And the bigger numbers are on the inside.

Megan: Why does that make the solution work?

C5: Because ... I don't know.

Megan: You don't know?

[C5 switches numbers around]

Megan: What are you trying to do with that one side? [163 side]

C5: Make it 8.

Megan: Why do you want to make it 8?

C5: I don't know. I'm just...

Megan: You're just going to try it?

C5: Yeah.

Megan: Ok.

[C5 continues switching numbers around.]

C5: I'm trying to make [sums of] 6 [on each side].

Megan: You want to try and make 6 now? Ok.

C5: Can't make 6.

Megan: Ok, what do you think you should try now? 8 didn't work. 6 didn't work. But 9 did.

C5: 7.

Megan: Ok. Try that.

C5: 7 doesn't work either I don't think.

Megan: So, 6 didn't work. 7 didn't work. 8 didn't work. 9 did. What could you try next?

C5: Anything below 6 cannot work.

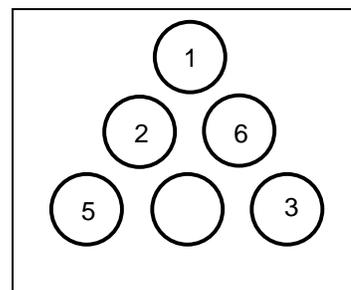


Figure 9b. C5's work on 6 Numbers Problem to rule out a solution where sides sum to 8.

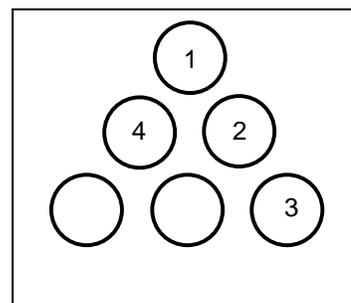


Figure 9c. C5's work on 6 Numbers Problem to rule out a solution where side sum to 6.

Megan: Ok. Why?

C5: 6 and under cannot work.

Megan: Why?

C5: Because that's the highest card.

...

[Later in the session Megan returns to this explanation.]

C5: It can't be lower than 9.

Megan: Try to explain it to her [C7] again. Why?

C5: Because they're too high.

Megan: What do you mean by that?

C5: The numbers are too high.

Megan: What do you mean by "they're too high"? (Group J Video transcript, pp. 12–13, 26–27 )

Though Megan's pupil did not see the pattern, rather than force that approach by resorting to a series of directive questions, she let the pupil keep working and tried to follow his thinking. As he made new observations about the problem, she continued to require him to justify those observations and pushed him when his reasoning was inadequate. In her work with this pupil, Megan elevated the cognitive demand to *doing mathematics*.

A variation on the pedagogical move of requiring pupils to provide an explanation was to ask them to *direct that explanation to a peer*. This move had wide variations in how it was implemented and, again, depending on the quality of the explanation elicited, it established a cognitive demand of *procedures without connections* or *procedures with connections*. In most instances of this move, the preservice teacher had each of the two pupils take turns explaining their solutions to one another after both had finished solving the problem. As the study

progressed, the participants began orchestrating these exchanges during problem solving so that the pupil's ideas might influence each other. Most often, this move was employed by *asking one pupil to give another a hint*. In an instance that was associated with a cognitive demand of *procedures without connections*, Casey urged, "So our friend here has found four of the answers. Do you think maybe you could help him a little bit?" (Group H Video transcript, p. 20). When the pupil offered a hint for the [12 Pennies Problem](#) of making sums of 10, her partner ignored her contribution and continued his strategy. Rarely did a participant ask one pupil questions about another's explanations, but when that did happen, it was associated with a higher cognitive demand, usually *doing mathematics*, as the pupils had to understand two approaches to a problem and compare them. In the excerpt below, two pupils described two approaches to solving the [Cupcakes Problem](#); one started with the correct number of boxes and added cupcakes until he reached the correct number of cupcakes, whereas the other started with the correct number of cupcakes and grouped them into boxes until he had the correct number of boxes:

Rene: How did you solve it?

C6: Well, I just drew out like dots, like guess and checking. And so that's mostly how I got it. Just guess and checking.

Megan: So, you didn't draw out the dots first. You just sort of added them as you went—right?

C6: Yeah.

Rene: So, how is that different? How is the way you solved it different from the way C12 solved it?

C6: Well, I guess he just started out, like he just said, that he made all boxes, but I just did it in my head.

Rene: What was C6 saying? How is the way that you solved it different from the way he solved it?

C12: Because he ... I drew mine out, and he just did each of them in his head. Guessed and checked. (Group J Video transcript, pp. 90–91)

Though the boys may not have appreciated the nuances of their two strategies, the participants held them responsible for both listening to and understanding another's explanation, which served to elevate the cognitive demand.

The response that most consistently elevated the cognitive demand to *doing mathematics* was to *extend the question* so that the pupil had to generalize a strategy, apply his or her reasoning in a new situation, or solve the same problem in a different way. A popular way of extending the question was to alter the numbers or conditions of the problem slightly (e.g., 8 boxes of cupcakes to 10, or that piles can have the same number of pennies). However, in later weeks, as the participants got more comfortable with their problems, they tried more challenging ideas. Kate suggested this question as a follow-up to the Phone Club Problem: "Looking at the four of us, I had to have a line connected to me so I could talk to you, but I also had a line connected to you so I could hear you? What about that? How many strings would that be?" (Group H Video transcript, p. 89). Their initial attempts at getting pupils to solve a problem in a different way were often met with resistance or indifference (as one pupil said, "I think I could do it like this [with a picture] but I think I could because I already know the answer" (Group I Video transcript, p. 71)). However, the participants began having success when they modeled this practice and made specific suggestions. After comparing two strategies for the [Cupcakes Problem](#), Rene and Megan then prompted their pupils to show how they would solve it using different colored connecting cubes. Their pupils modeled their solution by putting the blocks in stacks of 4 and 6 to represent the different sizes of boxes. They then manipulated the blocks into different combinations of 4- and 6-stacks to confirm that no other combination of boxes yielded the correct number of cupcakes. Over the course of the study, as the participants gained more

experience extending the question, they extended the questions in novel ways that were more mathematically sophisticated.

### **Pupil Is Almost Correct**

Another potential and common pupil response was to be nearly correct. Such responses included missing part of an explanation, not attending to some restriction of the problem, missing one or two solutions of a multiple solution problem, or making a minor computation error. The participants employed two moves in response to nearly correct solutions that lowered the cognitive demand to *procedures without connections* or *memorization*: They *corrected the pupil's error*, or they *drew the conclusion* for the pupil. In both of these moves, the participant took the responsibility for learning away from the pupil.

In the case where a small error had been made, participants often *corrected the error*. For example, the participants frequently pointed out when a pupil's digits did not add to 6 in the [Clock 6 Problem](#). Similarly in the [6 Numbers Problem](#), Erica pointed out why a solution did not work, "That one works and that one works, but this isn't working because we need another big number on that side." The participants opted to *draw a conclusion* for the pupil when other methods of getting the pupil to see an idea failed or when they thought the pupil had done the majority of the work but just needed a little push to see the big idea of the problem. In Week 3, Kate was working on helping a pupil see a connection among several of her solutions to the [Clock 6 Problem](#), but when she was unable to elicit a good description of the pupil's strategy, she described a strategy what she saw in the pupil's work:

[After C3 suggested 2:31 as a solution C2 generated several additional solutions as seen Figure 10a.]

2:31	3:21	<b>2</b> : <b>31</b>	3: <i>21</i>
3:12	1:23	<b>3</b> : <i>12</i>	1: <u>23</u>
1:32		<b>1</b> : <u>32</u>	
2:13		2: <b>13</b>	

Figure 10a. C2's solutions to Clock 6 Problem using 2, 3, and 1.

Figure 10b. Map to show how Kate discussed C2's solutions.

Kate: How did you get those so quick? What did you do?

C2: I just ... rearranged them quickly.

Kate: I think you rearranged them. But look. You did 2, 3, and 1 [points to the boxed numbers in Figure 10b]. You put each of the numbers first, and then you put those other two second, right? But then look what you did—you immediately went back to 2 and switched these two numbers [the bold, red 1 and 3 in Figure 10b giving her 2:31 and 2:13]. And then you went back to 3 and switched these two numbers [the italicized, blue 1 and 2 in Figure 10b giving her 3:12 and 3:21]. Then you went back to 1 and switched these [the underlined, green 1 and 3 giving in Figure 10b giving her 2:31 and 2:13]. Did you realize you did that? Yeah?

C2: No. (Group H Video transcript, p. 12)

A pedagogical move that was associated with varying levels of cognitive demand was to ask pupils to *compare an idea with their partner*. When pupils were missing a few solutions of a multi-solution problem, the participants employed this strategy to help complete this list of solutions by seeing what solutions a partner had found. Though in this type of response the participant kept the *responsibility for learning* and *source of ideas* on the pupils, the *questioning* and *explaining* were not maintained at high levels. Typically once pupils found the answer they were missing, the participants did not require the pupil to explore why he or she had missed that solution or how the other pupil had found it. This move was far more successful when the participants scaffolded the pupils' interaction by asking them to attend to specific features of each other's solutions. This move, in turn, elevated the *questioning* and *explaining* and

established a higher cognitive demand. In the following example, Megan and Rene were trying to get one pupil to help another see why a side sum of 8 is not possible in the [6 Numbers](#)

Problem:

Megan: [to C5] Give her [C7] an example [of how to place the numbers so that the sum of the numbers on one side of the triangle will be 8].

....

C5: Yes. 5 plus 1 ... 5 plus 3 would be able to equal 8. Ok, sure, you can do 4 plus ... Ok. Look. Two rows [sides] equal it, but then the bottom row [side] couldn't (see Figure 11).

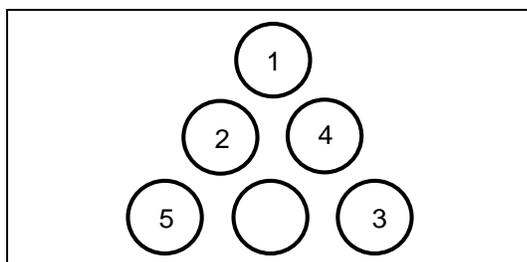


Figure 11. C5's construction of 8 on two sides.

Rene: Oh. Why's that?

C5: Because ... some of the numbers, some of the numbers, some of the [remaining] numbers are too low.

Megan: [to C5] Look. She [C7] just made 8. [Pointing to the gray 125 side in Figure 12.] What happened next? ... You've got to help [C7] figure out why is it [the other two sides are] not 8.

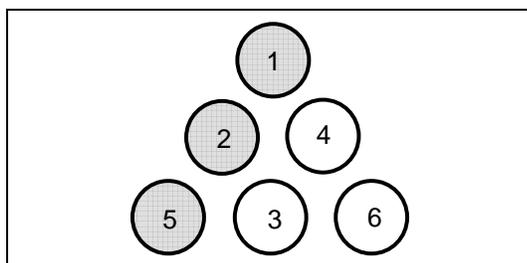


Figure 12. C7's construction of one side with 8.

....

Rene: C5 help us out. [to C7] You chose 1 and 2 [to put with 5 to make 8 in Figure 12]. Why did you choose that? (Group J Video transcript, pp. 28–29)

A more sophisticated response to nearly correct solutions that raised the cognitive demand was to *question the pupil's response* or *propose a contradiction to his or her reasoning*. This move was typically effective in helping pupils correct their own thinking and kept the *responsibility for learning* on the pupils instead of the teacher. After working with her pupil on rearranging numbers to make new times for the Clock 6 Problem, Kate wanted her pupil to notice a pattern:

On the last problem we saw if there's 4 [she actually meant 3] numbers you can arrange them 6 different ways, remember? Why can't we arrange these [the numbers 0, 2, and 4] 6 different ways? You started with the 2 and you started with the 4, why can't you start with the 0 for this problem? (Group H Video transcript, p. 11)

By directing her pupil's attention to specific elements of earlier work, Kate was able to keep the pupil the source of ideas and encourage her to explain her thinking.

By directing her pupil's attention to specific elements of earlier work, Kate was able to keep the pupil the source of ideas and encourage her to explain her thinking.

Another move that maintained the cognitive demand by making the pupil the *source of ideas* was to *use questioning to help the pupil articulate thinking or a strategy*. This response was more sophisticated than *drawing the conclusion* for the pupil. In the example below, the participant used questioning to help the pupil identify some salient aspects of his work, look for patterns in his approach, and justify his ideas. Megan was working to help a pupil identify an upper lower bound to the side sums in the [6 Numbers Problem](#) (see his work in Figure 13):

C5: So, I tried turning ... I'm just putting the bigger numbers on the outside [see Figure 13a].

Megan: You are? Why do you do that?

C5: Because it has to.

Megan: It has to?

C5: Never mind ...

Megan: No, that's good. Why do you think you have to do that?

C5: Because it equals a big number right here [653 side]. And it's complicated.

Megan: It's complicated? Let's see, because. ... What do you think would happen if we put the 6 and the 5 together [pointing the gray circles in Figure 13b]?

C5: It would be ... It's 11.

Megan: Ok. And what makes, what happens to the other side when you do that?

[C5 moves 1 to make 641 and notices he already has 542, the gray circles and dashed red circles, respectively, in Figure 13c.]

C5: Maybe it can be 11 Megan: 11? Ok. Try 11.

C5: 11 ... Oh, never mind.

Megan: Keep trying. It's a good idea.

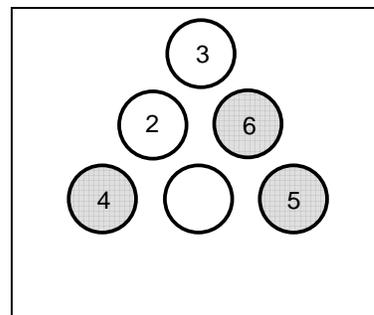


Figure 13a. C5's initial attempt.

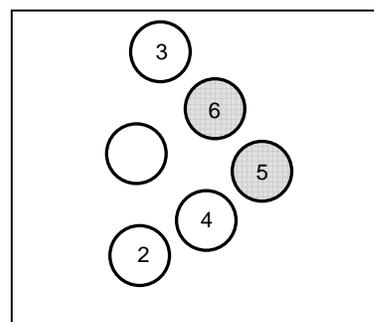


Figure 13b. C5's considers 5 and 6 on the same side.

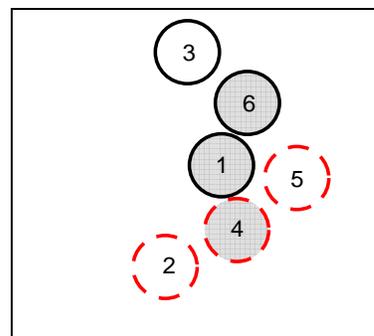


Figure 13c. C5 makes 11.

C5: Yeah, 11 works [see Figure 13d].

Megan: Ok. Let's see. 5, 11. 5, 11, 9, 10, 11. Ok. Good job. That's really good. So, now you've got 11, too. Let's see. Let's look at how the numbers are distributed. You're right that the 6 and the 5 are separated, and the bigger numbers are pretty much on the outside, right? Ok. What do you think you could try now?

C5: 13. 12 maybe. [C5 works on finding more solutions.]

Megan: Let's see. So what do you have here?

C5: ... 12, and this is 12 [see Figure 13e].

Megan: Good job. Ok, how did you figure out this one so fast?

C5: I don't know.

Megan: You don't know? What strategy do you think you used? Did you just move them around and check it? Guess and check, you think?

C5: Yeah.

Megan: Was there anything that you had looked at from your other solutions that made you know where to put these numbers?

C5: Yeah. Because I know to put the bigger numbers on the outside.

Megan: Ok.

C5: Ok ... 13. [C5 can get only two sides that sum to 13; see Figure 13f]. ... The last one is 10.

Megan: Ok, what does that mean? Do you think you found all the solutions now? ... Why? (Group J Video transcript, pp. 14–15)

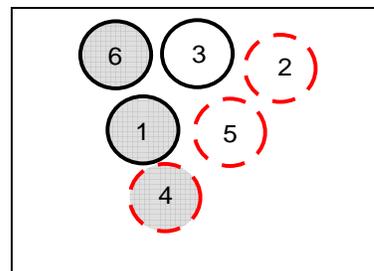


Figure 13d. C5 makes 11 on all sides.

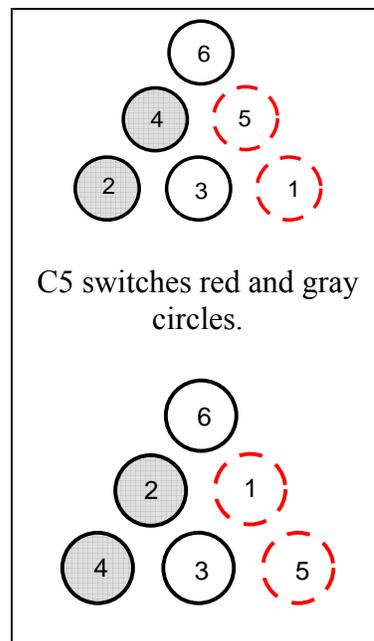


Figure 13e. C5 makes 12 on all sides.

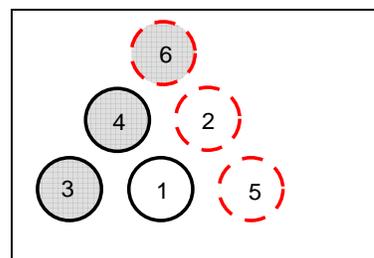


Figure 13f. C5 makes 13 on two sides.

Part of how Megan kept the pupil the *source of ideas* was to verbalize conclusions only after the pupil had made them first, give him time to test his ideas, and then ask him to justify those ideas. Pushing him to articulate his thinking also elevated the level of *questioning* and *explaining*. By attending to all three of these components, she was able to achieve high levels of cognitive demand.

### **Summary**

The participants employed pedagogical moves in their discussions that both raised and lowered the cognitive demand (Table 8). Those moves that lowered cognitive demand either ignored the pupil's reasoning or supplanted it with that of the participant. All but two of the moves that elevated cognitive demand relied on *questioning* to elicit and explore the pupil's thinking. *Proposing a contradiction*, though not relying on *questioning*, also elicited the pupil's thinking by requiring him or her to defend, and hopefully identify the flaw in, his or her thinking. *Extending the problem*, which consistently achieved the highest level of cognitive demand, though not focused on exploring the pupil's thinking, did require the pupil to apply his or her reasoning in a new way. All three of the pedagogical moves that had no impact on cognitive demand merely perpetuated what the pupil was already doing and were characterized by minimal teacher intervention. The teachers often used moves that maintained demand when they were trying to avoid being directive but did not know how to use moves that raised the cognitive demand.

Table 8

*Preservice Teacher Moves Made in Response to Pupil Thinking*

Pupil Thinking	Effect of PST move			
	Lower Cognitive Demand	Maintain Cognitive Demand	Raise Cognitive Demand	Variable Cognitive Demand
Stuck		Repeat the question		Suggest a strategy Hint from a peer Pose questions to help the pupil generate ideas
Incorrect	Walk the pupil through the PST's strategy  Correct pupil's error/misconception	Continue an incorrect way of thinking	Question to help pupil to identify his or her own mistake	
Nearly Correct	Correct pupil's error/misconception  Draw conclusion for the pupil		Question to help pupil articulate thinking/strategy  Question pupil's response or propose a contradiction	Compare an idea with their partner
Correct	Move to the next problem  Ignore confusing explanation  Push for clarity but continue to accept unclear/confusing explanations  Revoice PST idea instead of the pupil's idea  Give excessive guidance to obtain clear/complete explanation	Ask how a strategy was carried out not why that strategy yielded a correct solution	Extend the question	Ask for supporting explanation  Push pupil to clarify confusing/incomplete explanation  Direct explanation to a peer

The moves that are of particular interest are those that had varying effects on the cognitive demand, because they potentially could have a great impact on cognitive demand. They included moves that focused on eliciting clear and complete explanations and all the moves that required pupils to attend to one another's thinking. Merely getting pupils to talk to one another was not enough to raise the cognitive demand; the quality of what was said mattered. Also, asking for explanation and pushing for clarification and completeness of explanations raised the demand only as far as the quality of the explanation that was elicited.

Elements of the math-talk framework that proved most difficult for participants to reach at high levels were *questioning* and *responsibility*, whereas *explaining* and *ideas* were more often at the highest levels. The highest levels of *questioning* and *responsibility* required that the pupils, not the teacher, be responsible for asking questions and evaluating mathematical arguments, duties traditionally relegated to the teacher. On the other hand, the highest levels of *explaining* and *ideas* required the pupils to be the source of ideas and express those ideas clearly. The relative ease of achieving the highest level in these areas could be due to the fact that explaining their ideas is a more common pupil activity in traditional classrooms than is questioning or evaluating mathematics.

### **Cycle of Planning, Enacting, and Reflecting Influences Implementation**

For the second research question, I explored how the repeated process of planning for a mathematical discussion, enacting a discussion, and reflecting on that enactment influenced subsequent implementations of the same task and mathematical discussions. Tracing each participant's problem implementations over 6 weeks (two 3-week cycles) allowed me to see changes in the cognitive demand and components of math-talk. Looking across the eight participants, I categorized them into four groups: inconsistent discussion leaders, proficient but

still improving discussion leaders, inadequate but improving discussion leaders, and teacher-directed discussion leaders. I describe the participants in each group and suggest what unique features of each participant's experience might have influenced the trajectory of her discussions over the 6 weeks.

### **Inconsistent Discussion Leaders**

Two participants showed inconsistent performance over the 6 weeks; that is, they did not show either a gradual improvement each week in their ability to maintain high cognitive demand, nor did they show improvement in how they responded to particular types of pupil thinking. The nature of their inconsistency was different: The cognitive demand and quality of Nadia's mathematical discussions varied widely during the first cycle and appeared to decline in the second cycle, whereas Megan's performance fluctuated throughout the study but with less dramatic high and low points. There also was no trend across Nadia's ability to respond to different kinds of student thinking (see [Figure 3b](#) in Chapter 3). Megan, on the other hand, showed little change across the 6 weeks in how she responded to varying levels of sophistication in pupil solutions. Further, at the outset of the study, Megan was a moderately strong discussion leader, able to maintain a *procedures without connections* or *procedures with connections* level of cognitive demand, only once dipping to the lowest level of cognitive demand, whereas Nadia's cognitive demand frequently dropped to a level of *memorization*. In this section, I describe the trajectory of these two participants and offer explanations for the changes in their implementation of tasks over the course of the study.

The quality of Nadia's mathematical discussion decreased over the 6 weeks, and in general, cognitive demand was lower during the second 3-week cycle. She struggled with responding to incorrect solutions in ways that maintained the cognitive demand of the task. In the

Week 3 implementation of the [Clock 6 Problem](#), she reduced the pupil's efforts to following her directions and responding to basic fact questions while she verified the pupil's work as they progressed.

C3: 4:10.

Nadia: 4:10. 4 plus 1 plus 0?

C3: 4:11.

Nadia: Mmhmm.

C3: 5 plus ... 5 plus 1 plus 0.

Nadia: Mmhmm.

C3: 2 plus 3 plus, no ... 2 plus ... 2 plus 4 plus 2.

Nadia: No. That's too many. That's 8.

C3: Oh. 2 plus 4.

Nadia: You could do 2 plus 4.

C3: 2 plus 4? Equals 6. Oh.

Nadia: Mmhmm.

C3: 1 plus 5.

Nadia: Ok. So, let's write that one  $[1 + 5]$  next to this one  $[5:10]$ .

C3: This one?

Nadia: Like next to the 1. Like write 1.50.

C3: This one  $[5:10]$ ?

Nadia: Yeah. See how it's the same number rearranged. Is there another way you could rearrange that? [C3, just added 1:50 beside 5:10 at Nadia's direction, then adds 5:01.]

Nadia: Mmhmm. There's one more.

C3: Could I do 0?

Nadia: No. Ok, so we have 5:10 and this 5:01, now we have 1:50 what [times are here] that could we rearrange [the digits of]? (Group H Video transcript, p. 8)

However, there were moments of high cognitive demand in Nadia's second cycle. The only instance in which she achieved a cognitive demand of *doing mathematics*, she was working in concert with her group members to encourage their two pupils to make and test hypotheses. Nadia picked up on the direction of both her pupils and the other members of her group and pushed the pupils to justify their thinking and clearly articulate their ideas in the following example from the [Phone Club Problem](#).

Kate: Ok. ... All right. The next part of this question says what if you used 28 strings, how many people would be in a group? What do you think would be a good way to going about to getting to that answer would be? Do you want to just immediately draw 28 strings, or do you want to work up to getting there or ...

C6: Since 6 strings took 4 people, I think 12 strings, it would probably take 8 people. And it's 6 ... [C6 is doubling 6 to get 12 and 4 to get 8].

C1: I think it might be 2 less each time.

C6: Yeah.

Nadia: Why do you think that?

C1: Because it was 4 and 6. So maybe it could be ...

Casey: What is there 2 less of?

C1: 10 and then 12. And you keep on working up.

Nadia: 10 and 12 what?

C1: 10 people and 12 strings, and keep on working up like that.

C6: Wait, so, 6 people ... 6 times ...  $6 \times 5$ .

Casey: You can try it.

Nadia: You want to try it?

....

- C1: So 26 people and 28 people ... that's going to be a lot of people. [C1 has written 10, 12 and 26, 28. She starts making a circle of tick marks, erases, draws 16 circles, writes 26 and 28 again, does 26 circles and starts drawing in lines.] It's 26 ... [counts up to 26].
- C6: So, um, 6 times 5 ... 4 times 5 ... and that'd be 20 people ... and since 20 people ... it must be more 'cause 30 people, and I need 28. ... I need one third more. ...
- Kate: You guys want to pause for a second and talk about to each other what you're doing? 'Cause y'all are both doing different strategies. So how about you explain first, C6? (Group H Video transcript, p. 56)

Unfortunately, Nadia was not able to continue or repeat these results in her next implementation of the Phone Club Problem. In Week 7, after a pupil had made several incorrect attempts at drawing a picture of people connected with 28 strings, Nadia walked both pupils step-by-step through constructing the picture for 4 people and then for 5 people (see [Figure 6](#) and [Figure 7](#)). Nadia also did not show any growth over the 6 weeks in how she responded to different types of pupil thinking; she began lowering cognitive demand in response to correct and nearly correct pupil thinking and only sporadically raised cognitive demand in response to incorrect pupil thinking.

Nadia's cognitive demand in Cycle 2 was typically lower than in Cycle 1 (see Figure 14). There was no pattern to the rise and fall of Nadia's cognitive demand across the three problems. It declined for all implementations of the 12 Pennies Problem and declined even more sharply over her enactments of the Phone Club Problem but showed some improvement across implementations of the Clock 6s Problem. In Cycle 1, Nadia's *questioning* was most closely associated with cognitive demand. *Source of ideas* and *responsibility for learning* were also associated with cognitive demand, though less closely. *Explaining mathematical thinking* was not closely aligned with cognitive demand or the other components of math-talk, which does account for some of her struggles. However, in Cycle 2, *questioning* and *explaining* were both closely aligned with cognitive demand, whereas *source of ideas* and *responsibility* were less well

aligned with cognitive demand, which fits with her directive approach to helping pupils work through incorrect solutions on the Phone Club Problem. Across both cycles, the more “off” a pupil solution was, the more directive Nadia became.

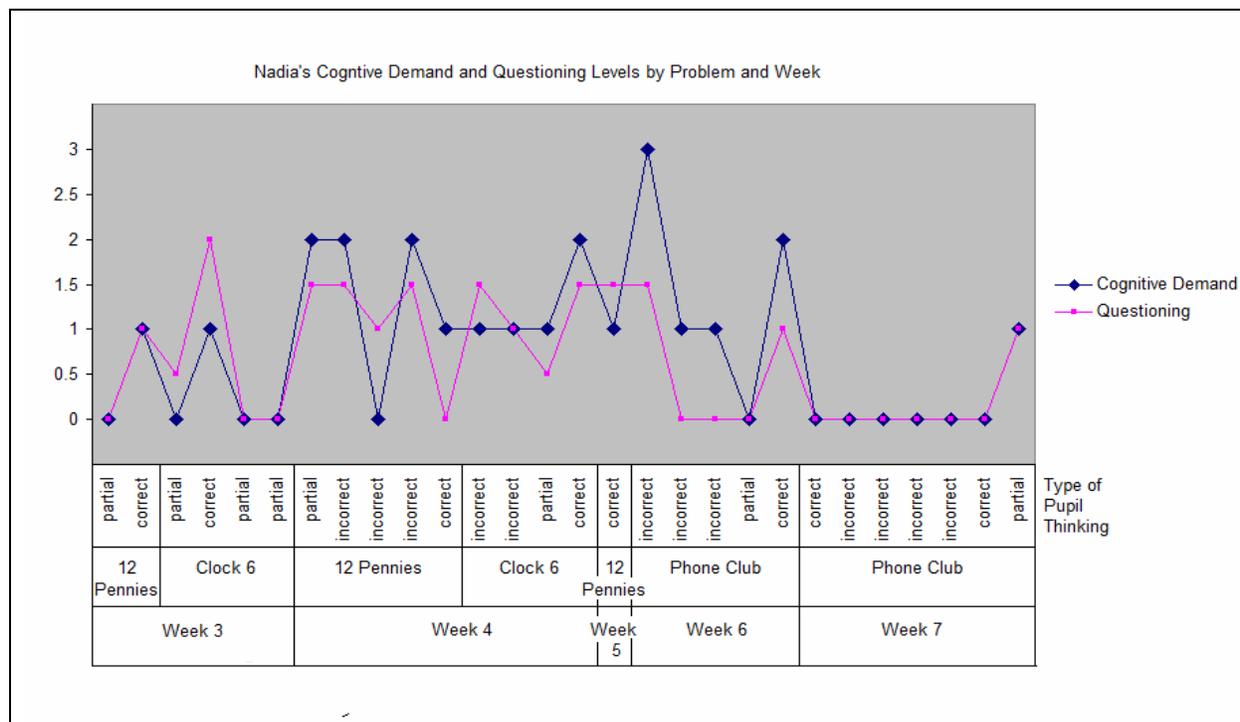


Figure 14. Nadia's cognitive demand across all problem enactments.

However, there were positive results that stemmed from Nadia's seemingly downward spiral, and these results may in part account for her apparent decline. First, Nadia was one of the few participants who incorporated actual pupil work she observed in one week into her activity plan for the subsequent week. In Week 6, when a student erroneously predicted that 12 strings were needed to create a phone club for 4 people, she included it as a potential student solution in her Week 7 activity plan and addressed how she would respond if she encountered it again.

Second, though her initial attempts at analyzing her teaching were shallow, Nadia made dramatic improvements in her self-analysis. She could accurately identify instances in which she railroaded the mathematical discussion; in her Week 5 reflection, she pointed out, “We ask

questions and then immediately answer them for the student and do not give them a chance to answer it. Or we will ask questions that are more rhetorical and do not really need an answer” (Nadia, Activity Reflection 5, p. 25). In Week 6, she noticed that she had accepted unclear or confusing solutions without pushing pupils to clarify ideas she did not understand. In her Week 7 enactment, in the midst of providing especially directive assistance to her pupils, she even stated, “I’m doing too much” (Group H Video transcript, p. 74).

Third, though Nadia was aware of her missteps, even as they were happening, and was not able to self-correct in the moment of teaching, she was able to make corrections in subsequent weeks after reflecting on what happened. For example, after directing a pupil through the [12 Pennies Problem](#) in Week 3, she consciously made the decision to react differently in Week 4:

Even though I knew this was wrong I still let her work out the problem because I thought that maybe in the end it might help her. When she makes another incorrect answer with 10-1-1 I stop her and talk about those two answers. I told her the problem again so that maybe she could figure out for herself why it was wrong. She still did not understand why they were wrong so I got out the manipulatives and asked her to make the piles 10-1-1. (Nadia, Activity Reflection 4, p. 16)

Because Nadia needed the opportunity to reflect on her practice and could not make in-the-moment adjustments to her teaching, she struggled when she was faced with pupil solutions that were far outside the constraints of any for which she had prepared. In the second cycle, she encountered more solutions that she had not anticipated and more solutions that were more seriously flawed than in previous weeks. So by chance, the pupil responses that were most difficult for her to manage happened to occur more frequently during her second cycle, giving a potentially false appearance of the quality of her mathematical discussions decreasing over the course of the study.

Megan, on the other hand, had fewer dramatic swings in cognitive demand than Nadia (see Figure 15). This difference is in part because she generally maintained cognitive demand at a higher level than Nadia, only dropping to *memorization* once when correcting a pupil's miscount. However, like Nadia, the only two instances she reached a *doing mathematics* level of cognitive demand were in concert with one of her group members. In Week 5, Megan intervened in Rene's conversation with one pupil to ask what patterns she noticed and to suggest that another pupil offer a hint to the 6 Numbers Problem. Rene then directed the pupils to consider whether all the solutions had been found and, to help them eliminate potential solutions, Megan revoiced other solutions they ruled out. Then she and Rene worked together to direct their two pupils to aspects of each other's reasoning. In Week 8, Megan joined in when Rene was trying to get two pupils to justify that the Cupcakes Problem had only one solution. Again, Megan interjected to encourage one pupil to help the other and to pose questions to clarify the pupils' strategies. Because her cognitive demand remained fairly consistent between *procedures without connections* and *procedures with connections*, one might claim that Megan is an example of a participant who did not change. However, the difference between these two levels of cognitive demand is quite significant; it is the difference between focusing on getting correct answers and understanding ideas.

Megan showed improvement across her enactments of the 6 Numbers Problem, moving from a cognitive demand of *procedures without connections* to *procedures with connections*, but in her enactments of the Cupcakes Problem, she oscillated between these two levels. She remained roughly the same in response to incorrect pupil thinking, lowered the cognitive demand over time in response to correct pupil thinking, but did improve in how she addressed partially correct pupil thinking. The following example shows how Megan used *questioning* and

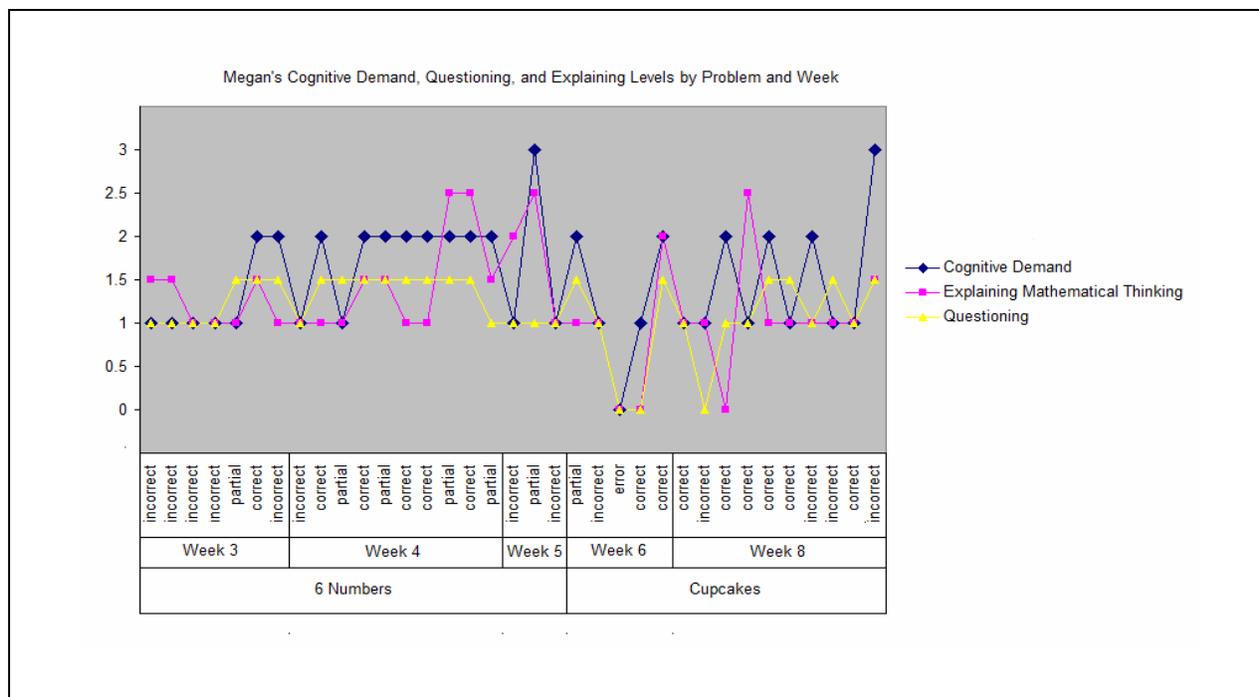


Figure 15. Megan's cognitive demand across all enactments.

revoicing without correcting the pupil's work to explore a partial strategy of grouping blocks in the [Cupcake Problem](#):

Megan: What do your yellow blocks represent?

C11: Vanilla. These [pulling out orange] are chocolate.

Megan: Ok. How do you decide how many pull out?

C11: Well, I'm still kind of seeing if it's going to balance. I'm still trying to figure it ...

Megan: What do you mean *balance*?

C11: Like, uh, for them, well, for them to equal 38, because the question said make 38 cupcakes, and he used 8 boxes. So I have to put them in 8 groups of cupcakes.

[C11 makes groups of 6.]

C11: I don't have enough.

Megan: How are you grouping them?

C11: I actually don't know.

Megan: It looks like you have a reason for grouping them like you are.

C11: So these are the vanilla ...

Megan: Ok.

C11: Packed up into 4.

Megan: Ok. ... So what's the problem?

C11: They're equal.

Megan: Ok. How many cupcakes in total do you have between the two groups?

C11: Well, this has 19 [the yellow vanilla], and that has 19 [the orange chocolate]. 19 and 19 is 38.

Megan: Ok. So you have the right amount of cupcakes, but they're not grouped, right? So what do you think you need to do? [C11 has 3 groups of 6 red and 1 extra red for 19, and 4 groups of 4 yellow and one group of 3 extra yellows for 19.] (Group J Video transcript, p. 53)

Megan kept *responsibility for learning* on the pupil by making her the *source of ideas* in this exchange. Though she elicited mostly brief explanations, she continued to push the pupil to articulate her work. This effort was particularly improved over her response in Week 3, where after a pupil found one correct solution to the 6 Numbers Problem and ignored Megan's attempt to help him see a pattern, Megan could only repeatedly prompt him to look for more solutions. For Megan, none of the components of math-talk tracked particularly closely to cognitive demand, and this separation may account for her inconsistent performance. Though she often asked good questions requiring justification of pupil ideas and follow-up questions to help pupils articulate their thinking, she did not always elicit well-developed responses from pupils. In the example that follows, Megan worked working to elicit an explanation from her pupil for why particular side sums for the [6 Numbers Problem](#) could not be constructed. However, she was repeatedly shut down by brief pupil responses throughout the session. An adequate explanation is not reached until the near the end of the problem:

Megan: Do you think there are [more solutions]?

C5: I don't think that [a side sum of 15] would work though.

Megan: Why not?

C5: I don't know. Because it's too high.

Megan: It's too high? Ok. (Group J Video transcript, p. 14)

....

Megan: C5. C5 help her [C7] out.

C5: I don't know.

Megan: Help her out.

C5: They're [the numbers] too high [to make all 3 sides sum to 8]. (Group J Video transcript, p. 27)

....

Megan: Why couldn't 8 be a solution?

C5: Because it is a number that's too high.

Megan: [to C7] Try doing 8. Try doing 8. (Group J Video transcript, p. 28)

....

Megan: Give her [C7] an example

C5: Like 5 plus 1 would equal like 6. (Group J Video transcript, p. 28)

....

C5: Look. 2 rows equal it, but then the bottom row couldn't.

[C7 constructs sides that sum to 8 shown in Figure 16.]

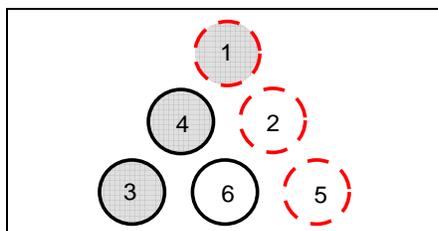


Figure 16. C7's construction of 2 sides that sum to 8.

C5: Equals 8....[Indicating the sides C7 is pointing to in Figure 16]. ...And that does not equal 8 [Pointing to bottom side of the triangle in Figure 16]. (Group J Video transcript, pp. 28–29)

Similarly, some of Megan’s lower level strategy-focused questions actually elicited powerful pupil ideas that rated high on the source-of-ideas scale. For example, later in the session excerpted previously she directed one pupil who was struggling to clarify an idea to a counterexample in another pupil’s work, saying, “Look, she just made 8. What happened next?” (Group J Video transcript, p. 28). The simple question forced her pupil to refute another’s pupil’s claim. These conflicting relationships (high-level *questioning* and low-level *explaining* and low-level *questioning* and high-level *source of ideas*) may have contributed to the variations in the cognitive demand in her discussions.

Megan did not make meaningful changes to her activity plans each week, but, like Nadia, seemed to need the time to reflect after her sessions to notice her strengths and weaknesses and to consider how to improve subsequent enactments. In the first week of each cycle, Megan’s commentary and analysis were fairly shallow. However, with each week of additional work on a problem, she noticed more about pupils’ approaches and her own responses. In Week 4, she realized that she needed to probe pupils’ thinking more: “Again, my questioning is effective on the surface, but to make sure students fully understand a concept, I need to continue to ask questions” (Megan, Activity Reflection 4, p. 9). She also realized upon watching the video that what her pupil called a guess-and-check strategy actually looked more purposeful. She commented, “Next time, I need to be more aware during ‘wait time’ and try to figure out the students’ reasoning so I can ask them questions about it right after they do something” (Megan data, p. 10). This self-critique was evident in with her future enactments; in Week 6, while watching a pupil seemingly randomly group cubes, Megan noticed a strategy and tried to help the child articulate it: “It looks like you have a reason for grouping them like you are” (Megan,

Activity Reflection 6, p. 53). In the second week of the Cupcakes Problem, she was able to identify and explain a pupil's difficulty in solving it: "Because C2 had organized the blocks into chocolate and vanilla by color, she had a little difficulty understanding that the color did not actually matter, and that it is only the number of cupcakes in the box that determines if it is chocolate or vanilla" (Megan, Activity Reflection 7, p. 26). In the final week of the 6 Numbers Problem, a week during which Megan observed as video recorder, she finally seemed to make some connections across the enactments:

After working on this problem with students and then observing it being worked on this week, I have noticed several main concepts that help the student find the 4 solutions. Typically, the students recognize the pattern of the placement of big and small numbers first. Once they are able to do that, they can more easily find additional solutions. The next 2 main patterns are found interchangeably depending on the student. They realize that anything below 8 can not be a solution and anything above 12 can not be a solution either. They also see that the solutions are probably 9, 10, 11, and 12. In other words, if the students find 9 and 11 as their first 2 solutions, they will guess that 10 might be a solution also. Knowing that these 3 concepts are important in finding the full solution set is helpful because we are more prepared to ask the right questions to prompt the students to see the patterns. (Megan, Activity Reflection 5, p. 16)

Multiple implementations of this problem helped her develop a roadmap for how pupils' approached the problem and gave her a better sense of how to respond to their work. However, she did not develop this understanding quickly enough for it to have improved her implementations of the 6 Numbers Problem with pupils.

Though she could identify a specific difficulty and correct in later sessions, Megan was unable to maintain her improvements in all areas simultaneously. Though she learned to explore and explain correct solutions in the first cycle, in the second cycle she reverted to verifying pupils' solutions. Likewise, her ability to elicit explanations improved in the first cycle, but showed several low points in the second cycle, with only two instances of eliciting complete explanations. Though she rarely revoiced her idea instead of her pupil's, she struggled in helping

a pupil articulate his or her ideas. Her only area of consistent, dramatic growth was in coordinating pupils' collaboration; however this may be because the other group members were also trying to improve in this area. Multiple enactments helped Megan develop a rich understanding of pupils' mathematics, but she typically was not able to use that information to improve her problem enactments. Perhaps, given another opportunity to implement these problems, there would be evidence of Megan's growth and improvement.

### **Proficient but Still Improving Discussion Leaders**

I classified two participants, Kate and Rene, as proficient and improving at leading mathematical discussions (see [Figure 4](#) in Chapter 3 and Figure 17a). At the outset of the study, they showed some strengths in leading mathematical discussions. In their first enactment, though both had instances of low cognitive demand, they generally maintained *procedures without connections* or *procedures with connections* levels of cognitive demand. By the end of the study, they could consistently maintain cognitive demand at levels of *procedures with connections* or *doing mathematics* rather than just have occasional spikes of high cognitive demand.

Additionally, both Rene and Kate were able to achieve the highest level of cognitive demand in response to correct, incorrect, and nearly correct pupil thinking, whereas the other participants did not make gains in all three of these areas.

Rene employed talk moves that elevated cognitive demand as her pupils were closer to obtaining correct solutions (Figure 17b). In every implementation, she elevated the cognitive demand to *doing mathematics* when responding to pupils' correct solutions. Interestingly, she also elevated cognitive demand when pupils had stalled out in the problem-solving process by asking one pupil to give another a hint or explain an idea and coordinating the pupils work together. In her second implementation of the 6 Numbers Problem, she had more instances of *doing mathematics*

than in her first, and this also represented a greater proportion of her total interactions with pupils. However, her improvements did not carry over into the second cycle. With the Cupcakes Problem, she again improved over the course of its implementations. Over the course of the 6 weeks, she improved in elevating the cognitive demand in response to correct and incorrect pupil thinking. She generally maintained high cognitive demand for nearly correct pupil thinking as well. It seems that the levels of all four components of math-talk stayed fairly close to those of cognitive demand. Her attention to all components of math-talk could account for success in reaching the highest level of cognitive demand. In the bold statements in the following excerpt, Rene used questions that required pupils to justify, directed the pupils to explain each other's ideas, pushed them to clarify their thinking, and held them responsible for evaluating one another's ideas.

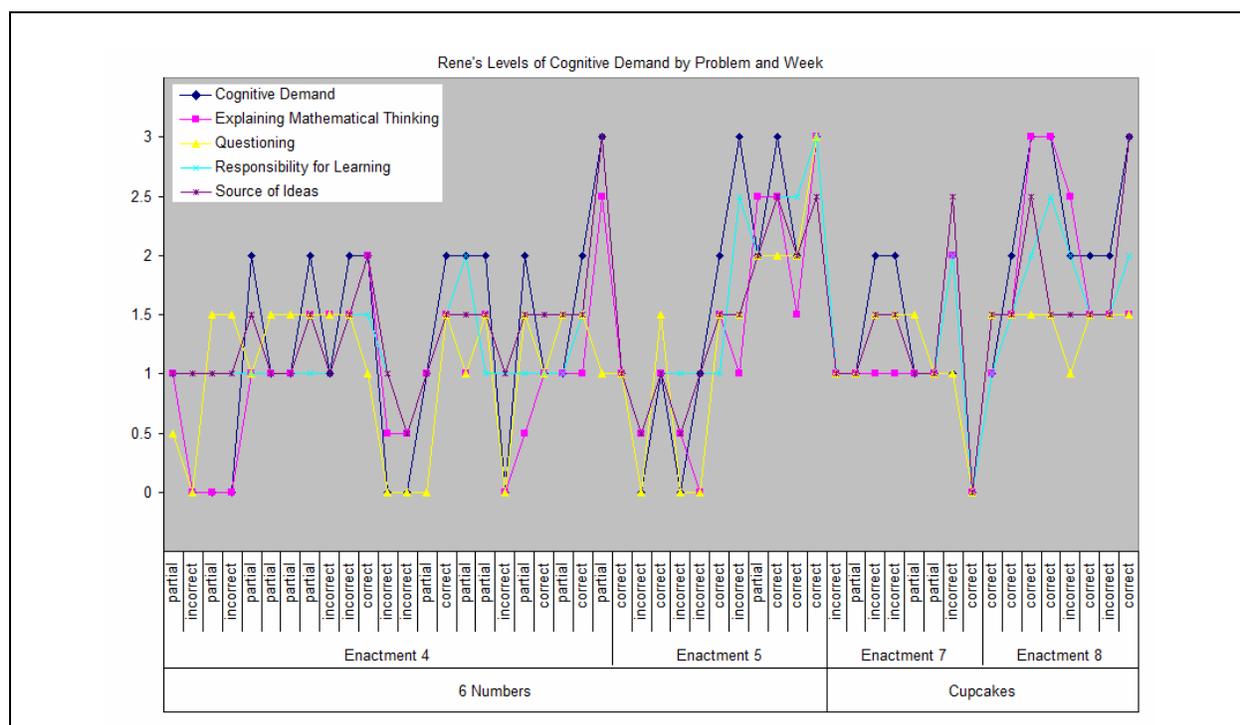


Figure 17a. Rene's cognitive demand across all problem enactments.

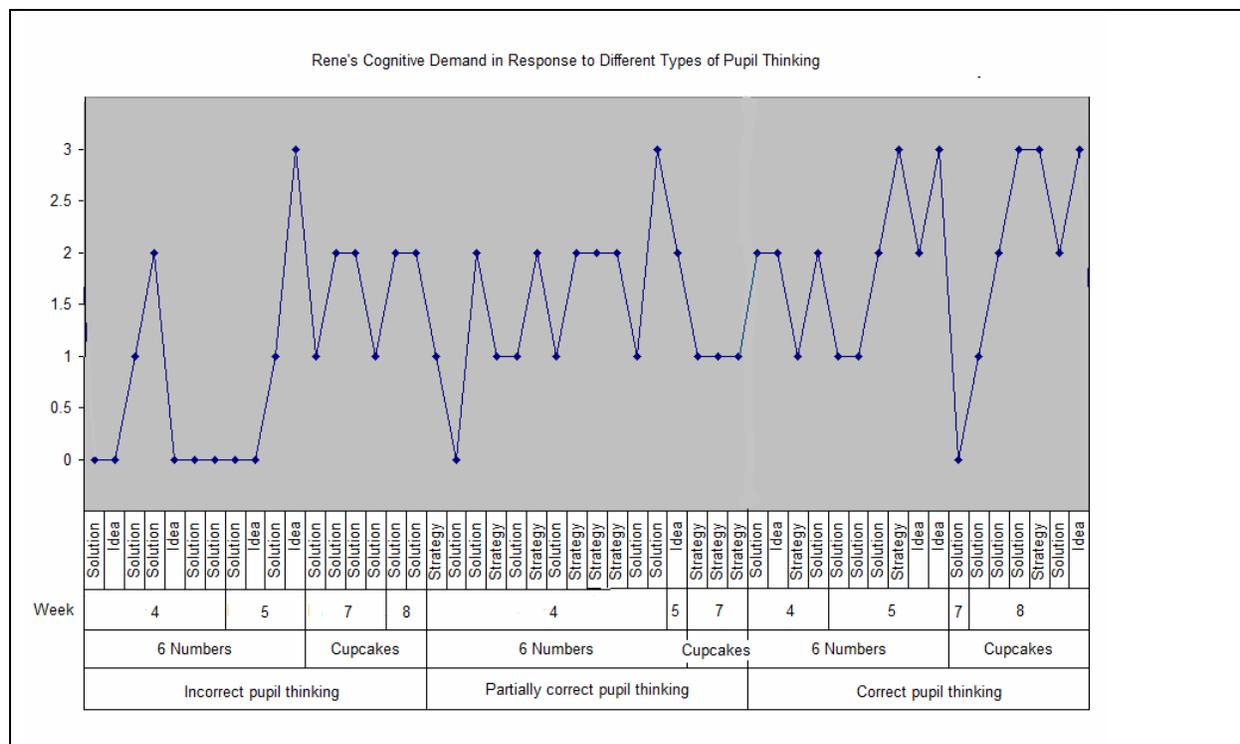


Figure 17b. Rene's cognitive demand across in response to different types of pupil solutions.

In the teaching episode from which this excerpt of the [6 Numbers Problem](#) was taken, Rene was able to maintain high cognitive demand for over 9 minutes through several conversational turns:

C3: Like say that you used all the big numbers [4 5 6] on this side and just say that equals like 13. And like you'd have to use all these [1 2 3] and you wouldn't have enough.

Rene: Oh Ok. [to C2] **So what's she saying ....do you know what she's saying? No? Well she... You can ask me, you can ask C3.**

C2: Say it again.

C3: Ok. Well, say that, I'm just saying that *I* couldn't get 13 basically with all the numbers. But if you had more numbers you probably could.

Rene: **Are you saying if you had more than just the 1 through 6?**

C3: Yeah. Like if you had 7, 8, 9, 10.

Rene: That's a good point. So C2 do you think we could get 13 with the numbers we have?

C2: Probably not.

Rene: **Why not?**

C2: Well, because maybe you could, but like...

Rene: **C3's saying no, we definitely can't. So you agree with that?**

C2: You might be able to. I don't know.

...[C2 decides you cannot get 13 on a side]

C2: I don't know. I just can't [explain it].

Rene: **C3, how did you explain it?**

C3: Just that you can't get that number. I mean it's too high for all the numbers that you have right now basically.

Rene: Ok. [To C2] **So what was she saying? Let's build off what she was saying.**

C2: All these numbers [1-6] aren't high enough.

Rene: They're not big enough. That's right. Well, some of them are, 'cause you did have a side of 13. I saw you create that a couple of times. [Makes a side of 13 with 652]

C2: But it's only one side.

Rene: But it's only one side. Ah. So, do you think you can make any number bigger than 12?

C2: Probably not.

Rene: Probably not, Not with these numbers? Well, what about smaller than our smallest set we found? We found 9. Do you think you can make a number smaller than 9? Like 8?

C2: I don't think you can make 8, because ... like, if you were using 6 on one side, it would be like 6 plus 2 [picks up 6 and 2] equals 8, but ... you divided 2, then it would be 1, and you don't have more 2 numbers [you need 6 plus 2 to get 8, so dividing 2 you get 1, so it would be 6 plus 1 plus 1 to make a side of 8, but you don't have two 1s].

Rene: That's a good point. **What was she saying C3? I know it's kind of hard to hear. It's getting loud. Well, ask C2. Seriously, I think she made an excellent point.**

C3: C2, what did you say?

- C2: Like you can't do that because if you were doing like 6 plus 2, that's only 2 numbers. And if you do 2 in half it would be, like, 1, and you don't two 1's.
- C3: What?
- C2: This is the last time I'll explain this. [Laughs.] Ok. So, if you were to have like 6 plus 2 equals 8, you would ... that's just like an example. Then you would, that would only be 2 numbers. (Group J Video transcript, pp. 45–47)

In the course of this conversation, Rene maintains high cognitive demand and does so by attending to all four components of math-talk.

Kate's data tell a slightly different story (see [Figure 4](#) in Chapter 3). She improved her ability to maintain and elevate cognitive demand across implementations of her three problems, and she showed that she was able to carry over that improvement to the second cycle, rather than start from scratch as Rene did in the second cycle. Over the three enactments of the 12 Pennies Problem in the first cycle, she had more instances of interacting with pupils and more consistently maintained the cognitive demand at *procedures with connections*. Though her discussions of the Clock 6 Problem in the first two enactments tended to establish low cognitive demand, in the third enactment of it she achieved and maintained *procedures with connections*, regardless of the correctness of the pupils' thinking. In her second cycle, enacting the Phone Club Problem, she never fell below a cognitive demand of *procedures without connections*, and she frequently achieved the highest level of *doing mathematics*. In her final implementation of the Phone Club Problem she consistently maintained cognitive demand at *procedures with connections* or better. Like Rene, she managed to have all four components track fairly closely to cognitive demand, lending further support to the hypothesis that to achieve and sustain high cognitive demand even in response to different types of pupil thinking requires attending simultaneously to *questioning, explaining, ideas, and responsibility*.

Kate and Rene also showed they had the ability to assist their fellow group members as they were leading discussions; Kate intervened to head off her fellow group members' directive instruction, and Rene helped hers elevate cognitive demand to *doing mathematics*. When Nadia and Casey walked their pupils through drawing a representation for 4 people in the Phone Club Problem, but the pupils seemed uncertain about how to use this information, Kate suggested they try constructing a representation for 5 people. When Nadia was directionally guiding their pupils to methodically draw each person's strings in the [Phone Club Problem](#), Kate stepped in to curtail Nadia's excessive guidance by asking, "How about you guys make a prediction as to how many strings this person is going to need?" (Group H Video transcript, p. 78). In nearly every implementation, Rene was able to work in concert with her colleagues to develop collaboration among their pupils.

The differences in these two types of improvement made by Rene and Kate, steady over the 6 weeks as compared with steady over each cycle, may be due to differences in their reflections. Each week, Kate perceptively analyzed herself, whereas Rene's reflections focused on providing rich descriptions of pupil work and critiquing her group members rather than herself. Though Rene was able to identify strengths and weaknesses in others' work, she was less apt to make suggestions for her own practice. On the other hand, Kate addressed several recurring themes each week and evaluated her and her colleagues' progress in these areas. She also perceptively analyzed the pupils' work to make claims about what they understood. By measuring herself with the same yardstick each week, Kate was able to make consistent progress, whereas Rene had to start anew in the second cycle. Rene's growth in leading discussions was tied to specific problems, but Kate's analytical reflections allowed her to improve across problems.

Kate focused on knowing when to step in, managing the tension between helping and telling pupils how to solve, and encouraging and coordinating pupils' collaboration. Her initial concern "was that it seemed kind of hard to get them to put their thinking into organized lists without feeling like I was straight out telling them" (Kate, Activity Reflection 3, p. 12). She continued to work at this issue throughout the study. In Week 5 she discussed her increasing awareness of this issue:

Last week I was just recording the other two girls, and I applauded them on their ability to wait out the student's answers; and I said I was going to try to focus on it this week for my turn of teaching. However, there were so many instances that I caught myself talking or presenting questions to the students while they were clearly in the middle of a thought! What makes me annoyed with myself is that I could even see that they were thinking... I just jumped in to try to help them when they really didn't need it. (Kate, Activity Reflection 5, p. 26)

In her final paper, Kate referred to this issue again, noting her efforts to improve, "I realize I need to work on not trying to lead them on with obvious hints so much. ... In the following weeks (especially weeks 7 and 8) though, I made a point to remember to back off and let them be more in control" (Kate, final paper, p. 60). Each week, she evaluated the group's success in getting pupils to work together and made suggestions for how to improve in subsequent weeks. Her consistent awareness that she "re-voiced their [the pupils'] thoughts too close to our own ways of thinking" (Kate, Activity Reflection 5, p. 29) shows her attentiveness to the impact of minute details of her actions on the quality of discussion she facilitated.

Rene's reflections lacked Kate's intent self-analysis. However, Rene's reflections do provide some insight on her reasons for particular pedagogical moves and show moments where she experienced a change in thinking. For example, in Week 4, Rene's interaction with her pupil on the [6 Numbers Problem](#) was directive:

Rene: You're just switching them around?

C7: Yeah. 7, 8, 9. That's 11. 4, 5, 6, 7 ... uh ... 5 ... 10, 11, 12. That's 12. 7, 8, 9, 11. Wait, wait, wait. Well, yeah. That's 11. It was close.

Rene: It was. It's getting closer. Why do you think that is? Why do you think that was closer? Let's move it back [see Figure 18]. Let's look at the numbers that we're using. They're close together.

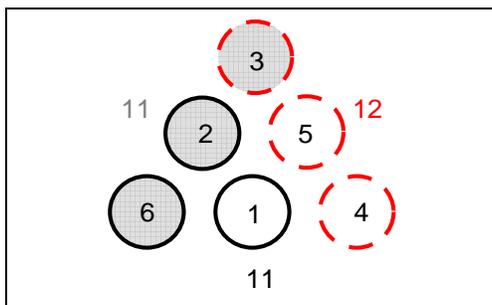


Figure 18. C7's incorrect solution to the 6 Numbers Problem.

C7: Yeah.

Rene: So, what did this side [the 354 side] equal again?

C7: 12.

Rene: Mmhmm. And what about this one [the 236 side]?

C7: 11.

Rene: 11.

C7: And this [the 614 side] has got 11, too.

Rene: Oh, that one does, too. Let's look at this. So, why do you think this side equaled almost the same number as this side [pointing the gray 11 side and red 12 side]? You said they were close. Why do you think that is?

C7: Um ...

Rene: Let's look at the numbers we use.

C7: Uh ... I don't know ... I just know that ...

Rene: Well, let's see. Let's say the numbers that we use then. Which numbers did we do? (Group J Video transcript, p. 7)

In her reflection for that week, she justified her decision of providing so little time for her pupil to work independently before she intervened:

C7 relied on her fingers to find the sum of each side. She had a positive attitude, but was quick to suggest that she didn't know why she made a certain move followed by a giggle. For this reason, I decided to focus on pushing her to be articulate and for her to form a strategy which she could clearly explain. Although she did a great job of saying what was on her mind as she worked, I encouraged her to think about the consequences of each move before she did it. (Rene, Activity Reflection 4, p. 19)

This information is helpful because it provides context for her rather directive interactions with her pupil with mostly low-level cognitive demand. Because her pupil did not initially have a strategy (Rene was not willing to accept a guess-and-check strategy), Rene felt it was her job to question her pupil until she could articulate a strategy. Her reflection shows that her directive interactions with the pupil were a purposeful decision on her part; she had a rationale for the pedagogical decisions that lowered cognitive demand. Further, because these low-level directive questions ultimately resulted in more sophisticated mathematical reasoning from the pupil, she was able to raise cognitive demand and set the stage for both pupils to compare ideas and work together:

Rene: Do you have a certain strategy that you're using?

C5: Bigger numbers on the outside.

Rene: What did we just find out about ours?

C7: Oh, ok. That like if we have like 2 big numbers on the same row it made like equal even higher number [the sum of that row]. And then if we have like lower numbers, we get an even lower number or, like. ... Like, if you switch a big number and, um ... like, we have this right here [switches a 2 and 6]. And then we switch it with a 6. (Group J Video transcript, p. 16)

In Week 6, she was surprised to discover that just asking the pupils to solve the problem using a different method led to productive mathematics: "I felt as though this wouldn't work since they both know the answer, why wouldn't [they] automatically group the cubes accordingly? However, they do participate correctly. ..." (Rene, Activity Reflection 6, p. 37). She went on to describe 10 more minutes of work where the girls solved the problem using cubes

and organized charts and discussed why they thought there was only one solution. However, she seemed not to register that this was a worthwhile exploration when, the following week, she explained her difficulties in extending the problems: “I really don’t know how much further I can take these problems since there are no patterns to find” (Rene, Activity Reflection 7, p. 47). The following week, she was challenged to extend the Cupcakes Problem for two pupils who had done it in an earlier session. She and her fellow group member, Megan, spent the entire session working on extending the problem; they changed the numbers, had the pupils compare and explain each other’s strategies, try different strategies, and explain why there could be only one solution. The experience gave Rene the opportunity to re-examine the mathematics inherent in the Cupcakes Problem.

Both Rene’s and Kate’s plans were very detailed and thoughtful. Rene seemed to use her plans as guides to be followed. In her case, using her plans in this way may have helped her implementation because her plans gave great attention to having pupils explain and justify their ideas. The following example from her 6 Numbers Problem task dialogue shows Rene’s focus on generating explanations:

T: And how did you think to come up with this solution?

S: I wanted to spread out the larger numbers.

T: Why? How would that help?

S: So that it would be even when you add each side up.

T: Even? What do you mean by even?

S: So you don’t have two big numbers on one side. This way, you put a really big number with a really small number and another big number with another small number.

T: Why would you want to do that?

S: Because I knew that if you added 6 and 5, the two largest numbers, to the same side, the other sides couldn't add up to a number as big as 11 using three of any of the numbers that are left. (Rene, Task Dialogue 2, pp. 1–2)

She probed for complete and clear explanations, even questioning her hypothetical pupil's particular terminology. Kate's first plans for each cycle of the study were fairly strong; she included many questions to elicit explanation and justification. She revised her plans each week to target particular difficulties pupils encountered. For example, with the Clock 6 Problem, she carefully modified her plan each week to address pupils' confusion about what the question was asking. By the end of the first cycle, she included details about exploring her pupils' understanding of digital clocks before launching the problem.

Both Kate and Rene had detailed plans. Though Kate's plans were useful, her self-critical reflections seemed to play a bigger role in her week-to-week improvements. However, Rene's plans helped her improve over each cycle, although her mostly descriptive reflections seemed to have little impact on her enactments.

### **Inadequate but Improving Discussion Leaders**

Two participants, Casey and Dana, struggled to achieve high cognitive demand in their discussions. At the start of the study, their problem enactments were mostly low cognitive demand, with occasional instances of high cognitive demand. They are classified as improving because in subsequent implementations of most of their problems, they achieved higher levels of cognitive demand than they did in their first implementations. Dana showed consistent improvement across her implementations of the Cupcakes and Puppies Problems, moving from *memorization* to *procedures with connections* (see Figure 19). Casey showed improvements across the 12 Pennies and Phone Club Problems, moving from sporadic cognitive demand to more consistent cognitive demand of *procedures without connections* in the 12 Pennies Problem and *procedures with connections* in the Phone Club Problem. In her final enactment of the Phone



solutions if we had more pennies?” and concluded for them that “if we change the rules of the problem, it would come up with a different answer” (Group H Video transcript, p. 23). But in Weeks 7 and 8 of the [Phone Club Problem](#), she maintained cognitive demand at *procedures with connections* by asking pupils specific questions about their work to help them notice and articulate a pattern such as, “How did you know to draw your strings that way?” (Group H Video transcript, p. 70) and asking questions about that explanation. In this instance, she tried to help pupils recognize patterns in their work rather than point out these patterns for them. In Week 5 when her pupils were stumped applying their reasoning from the [12 Pennies Problem](#) to a scenario with 15, she asked “What is the highest number you could start with if we had 15?” (Group H Video transcript, p. 41). This question gave the pupils a starting point, helped connect them to the previous problem with 12 pennies, and sparked a debate between the pupils about the highest number (13 or 12) that could be put in pile.

Casey was merely complying with class assignment requirements rather than critically reflecting on her work with pupils and making thoughtful changes to her plans each week. Many of the issues she cited as problematic did not relate directly to her interactions with pupils or their understanding of mathematics. Her reflections in the first cycle generally focused on keeping the pupils on-task, coordinating collaboration, and pacing of pupils’ work. In Week 3, Casey’s primary worry was not having the answers to the questions. She did not trust that because she could do similar problems on her own that she could successfully do these problems and support pupils in doing them. In the first cycle, both her reflections and plans showed that she lacked an understanding of the mathematical concepts underlying her problems. For example, though she understood the lists in the [12 Pennies](#) and [Clock 6](#) Problems needed to be organized, it was not clear that she understood that by organizing the list you could ensure that

you had checked all possible combinations. She seemed to think that the pupils were supposed to organize the list just for the sake of being organized. In her implementations, though she looked at patterns in her pupils' lists for the 12 Pennies and Clock 6 Problems, she failed to address the mathematical concept of making a systematic list to be certain all solutions were found. Though Casey claimed to value having the pupils explain their thinking, other comments, such as the following, contradicted this claim: "They got the answers and that's really the most important part" (Casey, Activity Reflection 5, data, p. 19).

Her reflections did, however, address more pedagogical issues when she responded to her fellow group members' questions and concerns. Nadia's Week 4 reflection helped Casey be more critical of her work with the pupils: "I didn't think about asking better question[s] until Nadia pointed it out. ... We need to give the students plenty of time to answer the question, not just answer it ourselves" (Casey data, p. 16). Reacting to her fellow group members' comments helped Casey to become more aware of areas in which she needed to improve. In Week 6, she showed a burgeoning awareness of her focus on answers and not explanations when she responded to Nadia's commentary:

Nadia's concern about not fully understanding a student's reasoning is valid. I think we assume they understand how they solved the problem and that we do too. But, sometimes we don't know why they solved a problem a certain way or how they even got to the answer! I am guilty of hearing an explanation and just nodding my head or saying "good job!" when I don't even know what is going on. I didn't notice that I did that until Jordan pointed it out. (Casey, Activity Reflection 5, p. 25)

By Week 7, Casey was finally picking up on the same issues as her group members and reflections were improving. After discussing her own struggles with how to help their pupils, she related her struggles to those Kate experienced:

I totally understand the need to jump in! I felt like I wasn't helping the girls solve the problem at all. I just stared at their white boards while they attempted to solve the problem wrong in a variety of time consuming ways. I didn't know if it was

beneficial or detrimental to jump in and tell them they're doing it. (Casey, Activity Reflection 7, p. 30)

The improvement in Casey's reflections parallels the improvements in her enactments in the second cycle.

Like Casey, Dana also showed improvement across her implementations. In her first implementation of the [Cupcakes Problem](#), she interjected early in the problem to correct her pupil's misconception that there were 6 boxes of chocolate cupcakes and 4 boxes of vanilla cupcakes instead of 6 chocolate cupcakes to a box and 4 vanilla cupcakes to a box. Rather than giving her pupil time to attempt a guess-and-check strategy, she tried to guide her to see that her initial guess would not work:

C7: Ok. ...And I'm going to start with a random number here.

Dana: Ok.

C7: Ok, I'm going to do a ... like ... 5 [5 boxes of 6].

Dana: Ok, why did you pick 5?

C7: I did ... I tried like ... I just decided to maybe do ... one ... one less box ... I decided to do one less box as that, one more box here [she has 4 boxes of 4s and 5 boxes of 6 for a total of 46 cupcakes].

Dana: So how many boxes of vanilla do you have here?

C7: I have 5.

Dana: And you have how many chocolate boxes?

C7: Oh, ...5 [Adds another box of 4, for a total of 50 cupcakes].

Dana: So do you know how many cupcakes that give you the total?

C7: Ok. This is 20 [5 boxes of 4]. That is ... ok ... add together [with the 30 from 5 boxes of 6]... 50 ... um ... that same

Dana: What could you do to ...

C7: ... I still have 10 boxes. But I have ...

Dana: You still have how many, how many boxes have you used?

C7: I have 10 boxes here, and I have 50 cupcakes. Right there. I'm close to 58 cupcakes.

Dana: You are?

C7: And so, I think I'm going to go ... I think I might go up, maybe try to going up. I don't know. I just want to try it. [Adds another box of 4 and another box of 6 for 6 boxes of 4 and 6 boxes of 6 and 60 cupcakes total.]

Dana: So now you have too many [C7 has 60 cupcakes].

C7: Yeah.

Dana: So, you know that you're close. How many boxes do you have total between these two right now ... before you erase that?

C7: I have 60 [cupcakes]. 12 [boxes]. (Group I Video transcript, pp. 3–4)

Many of Dana's questions in this segment of conversation reduce the problem to counting the number of cupcakes and boxes with Dana drawing the appropriate conclusions for the pupil. This was typical of her initial implementation of problems: establish cognitive demand at the *memorization* level with her only push for high cognitive demand at the end of the problem when the pupil had obtained a correct answer. In her second implementation of problems, she maintained the cognitive demand at *procedures without connections*. In her group's second enactment of the [Puppies Problem](#) she saw Alice faltering; as the pupil became more entrenched in his incorrect solution, Alice became more directive in trying to get him to identify his own mistakes. Dana intervened to skillfully explore their pupil's thinking, asking what different parts of his representation meant and how they connected to the original question. In doing so, she elevated the cognitive demand to *procedures without connections* and *procedures with connections*, and Alice was able to follow her lead and also raised cognitive demand from *memorization* to *procedures without connections*. Unfortunately, Alice could not sustain the higher cognitive demand without Dana's continued intervention. Dana continued to interject with

questions to explore the pupil's thinking such as: "How did you get from 33 to 16?" (Group I Video transcript, p. 80), "Which places in the problem did you take half?" (p. 81), "What do you know about Monday and Tuesday? What does the problem say?" (p. 78). When she saw Alice try to force a work-backward strategy, she advised, "He's working backwards. I think you should go back" (p. 84).

Dana showed real improvement here as she asked a few careful questions, backed off to let Alice take the lead, and stepped in again as she saw Alice struggling. Dana not only had to carefully consider when to intervene on the pupil's thinking, she also had to understand what Alice was trying to accomplish and when to intervene in Alice's work with the pupil. Dana described this experience:

We ended up spending about 20 minutes trying to get to the root of John's thinking and encourage him to see his mistake.... We kept hitting a roadblock because John's mind was fixed on his idea of what the problem was asking and it was hard for us to dissect his thoughts. I found myself wanting to just explain and clarify; it was so hard attempting to get him to realize his own mistake." (Dana, Activity Reflection 7, p. 39)

Although their pupil was never able to correct his misunderstanding, neither Dana nor Alice corrected his thinking; instead, they were comfortable ending the session with the pupil not having resolved the issue.

I have found I tend to ask yes or no questions when my students are not able to elaborate, as an easy way out. I also noticed I could be pretty repetitive with my questions when I am not getting the answers I want. However, when I was able to ask more open-ended questions and students responded, I found asking questions without an answer already in my mind left room for student ideas rather than my own. (Dana, final paper, p. 57)

In the cases of Dana and Casey, their improvement may be attributed at least in part to the actions of their fellow group members. The cognitive demand of Casey's implementations was pulled up by her colleagues, as was the level of analysis of her reflections. Part of her improvement can likely be attributed to her trying to imitate the practices modeled by her

colleagues. Dana, on the other hand, made improvements as she was trying to assist a struggling colleague. The added complexity of trying to understand her fellow preservice teacher's thinking, in addition to the pupil's thinking, may have influenced her better questioning and willingness to explore the pupil's thinking, which in turn improved the cognitive demand of that enactment.

### **Teacher-Directed Discussion Leaders**

Two participants, Erica and Alice, showed no change in the quality and structure of their mathematical discussions over 6 weeks. Both of these participants led discussions that were highly teacher-directed and remained so during the 6 weeks. Whereas other participants had changes in the cognitive demand over the course of any single problem implementation, the cognitive demand of these two participants' implementations was nearly constant for the entire implementation, and their interventions consistently maintained low cognitive demand.

Both Erica's and Alice's problem implementations remained mostly at a *procedures without connections* level of cognitive demand, occasionally dropping to the *memorization* level. Erica's performance across all implementations of the 6 Numbers Problem remained mostly the same, with only a one-time uptick at the end of her final implementation (see Figure 20). Her implementations of the Cupcakes Problem hovered between the two lowest levels of cognitive demand. Alice's implementations of the Cupcakes and Tickets Problems in the first cycle also remained constant at *procedures without connections*, with occasional drops to *memorization*. She attained *procedures with connections* only twice, once in each implementation of the Puppies Problem.

Though both Erica and Alice did achieve instances of high cognitive demand, *procedures with connections*, these instances do not show improvement in their ability to orchestrate

discussion. Erica's only instance of high cognitive demand was achieved in concert with her fellow group members, who had a specific goal of encouraging pupil-to-pupil interaction. Of Alice's two instances of high cognitive demand, only one was achieved without the intervention of a fellow group member. That instance of solo high cognitive demand was only a small proportion of her total interactions and the only instance in which she attempted to explore a pupil's correct solution. Thus, these instances of high cognitive demand were not typical for either participant.

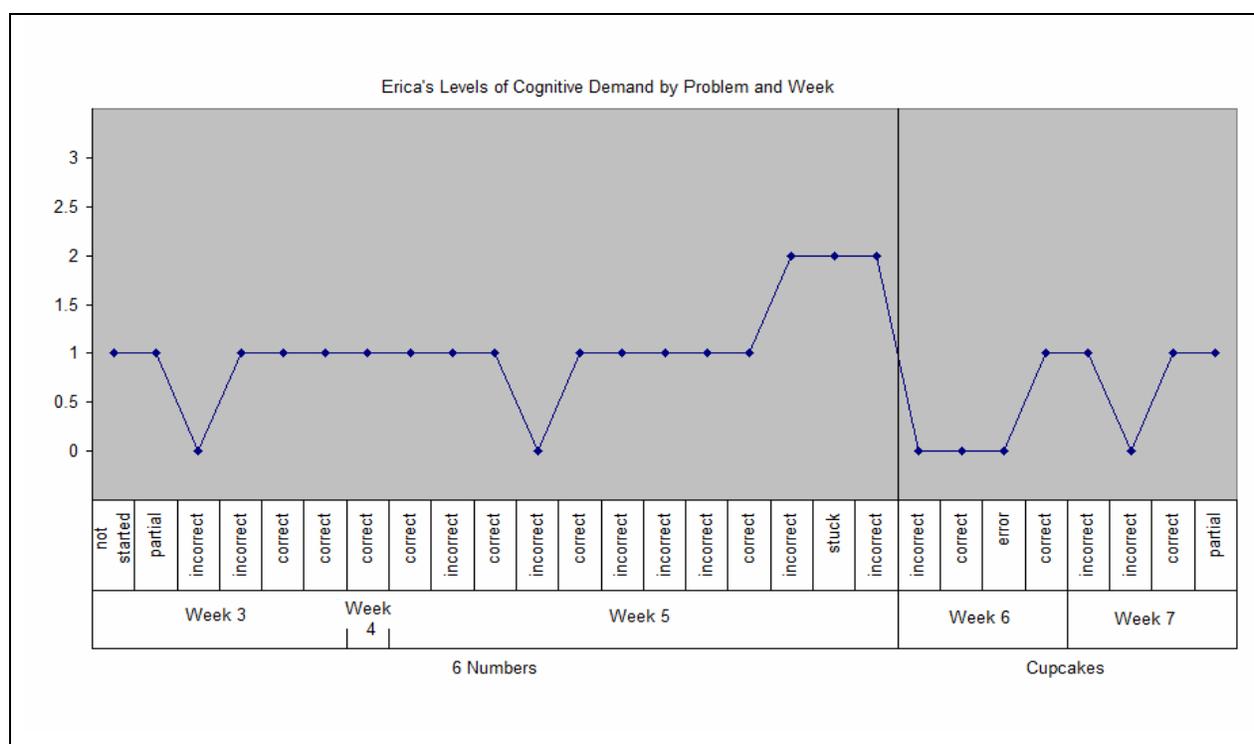


Figure 20. Erica's levels of cognitive demand across all problems.

Erica's responses to pupil work typically involved correcting errors, drawing conclusions for her pupils, or congratulating them on correct solutions without requiring much explanation. The following exchange on the 6 Numbers Problem in Week 5 is typical of her directive work with pupils:

Erica: Ok, we have a big number here and a big number here [Pointing to the 6 and 4 of the 164 side in Figure 21], do you think that's why? ... That one works and that one works [Pointing to the 164 and 425 sides which both add to 11], but this isn't working because we need another big number on that side. [Pointing to the 531 side which only adds to 9.]

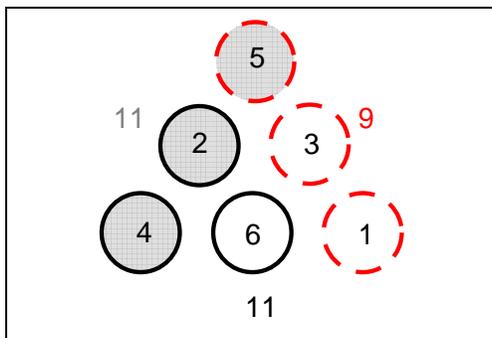


Figure 21. C8's first incorrect solution to the 6 Numbers Problem.

C8: 11, 11, 9.

Erica: You keep getting 11, 11, and 9 ...

C8: 10 ...

Erica: Ok, so all the patterns we've done so far, the 6 and 4 have been in the same row. [Points to them, C8 erases, and writes a different arrangement of the numbers]

C8: Ok. Darn, I always get ...

Erica: You got 11 again 'cause you have the 6 and 4 together. (Group J Video transcript, p. 3)

Here Erica drew conclusions for her pupil and repeatedly directed him to the same strategy, separating the 4 and 6. Like her group members, her typical response to correct solutions was to verify that all conditions of the problem were met. However, she generally failed to follow up by eliciting any explanation as in this example with the [Cupcakes Problem](#):

Erica: So we have all of them. So, what are our 4 [boxes of 4]?

C12: Chocolate is 6 [cupcakes to a box].

Erica: Ok. Chocolate is 6. So, how many boxes of 6 do we have?

C12: 1, 2, 3.

Erica: Ok. And how many boxes of 4?

C12: 1, 2, 3, 4. [There are 5 and he has written 5 on his board]

Erica: So, let's make sure we have all our conditions met. Do we have 38 cupcakes? In these boxes? Do you want to count up, make sure we have 38 in here?

C12: [He counts] 38.

Erica: That's great. ... How many boxes of vanilla? [3] How many boxes of chocolate? [5]... 8 boxes total. That's right. Good job. (Group J Video transcript, p. 75)

Alice had similar responses to correct solutions; she typically verified all the conditions of the problem were met and congratulated the pupil for his work. Only once did she explore a child's correct solution and ask her to consider why the problem did not have other solutions.

Also similar to Erica, Alice struggled when her pupil obtained incorrect answers. In the following example, she repeatedly pushed her correction to the pupil's thinking in the [Puppies](#)

Problem:

C6: Yeah, and since there was 4 left [in the statement of the problem], but ... but there was 4 left and 18 [from adding 12 and 6 of the list he created: 12, 6, 3].

Alice: So, do you think you're supposed to add these numbers together, or do you think each one of these numbers [represents] the total number of puppies at that time?

C6: The total number of puppies at that time.

Alice: So, do you think you would add them [the number in C6's list of 4, 4, 16, 1 or his list 12, 6, 3], or do you think they just represent the total?

C6: The total.

Alice: Ok. So if they just represent the total, how many puppies would be left after she took half of that [referring to the 6 in the 12, 6, 3 list]?

C6: 3?

Alice: So would that number [21, the sum of the 12, 6, 3 list] work? Would 25 [the sum of the 4, 4, 16, 1 list] work?

....

Alice: [pause] So, I thought you told me that that each of these numbers [represents] the total number of puppies at that time, right? So, if ... so, would we add them, or would we not add them?

C6: Well, the number that was on Monday.

Alice: Ok. Which one is Monday? Point to ... What is that [points to 12]? (Group I Video transcript, p.75)

Alice also had some responses that were unique to her. She often tried to ask pupils about strategies before they had actually developed any. In same session focused on the [Puppies Problem](#), she began questioning her pupil early in the problem-solving process and began asking her pupil to make deductions about the solution that he was not ready to answer.

Alice: So, C6, can you tell me what you did?

C6: Well, I'm just writing all my information down.

Alice: Ok.

C6: Ok, so one plus one, so that's one. So that's one puppy he already had and how many puppies were there on Monday? Ok. And Tuesday, he sold half of them.

Alice: How many puppies did you say were there on Monday?

C6: Here were ... well, how many ... I didn't answer.

Alice: Ok. I thought you said it, I'm sorry.

C6: And so, Tuesday, he sold half. On Wednesday, she took half of them. ... And there was 5 ... so that's 5 [adding the one that was taken to the 4 left].

Alice: By looking at that problem, it doesn't tell us how many total number of puppies we have, but what can you tell me about the numbers? We do it, like ... It would have to be odd number of puppies or even number puppies. What do you think? (Group I Video transcript, p. 72)

At times when pupils *did* have strategies, Alice allowed them to get by with fairly weak explanations of how they solved problems. Whereas participants from other categories rejected guessing as a valid strategy, Alice and Erica frequently accepted it as an explanation when the pupil had employed a more sophisticated strategy but was unable or unwilling to articulate it. In

following excerpt of the [Cupcakes Problem](#), the pupil constructed a list systematically. He began with a list of six 4's to represent the boxes of 4 vanilla cupcakes and six 6's to represent the boxes of chocolate cupcakes, for a total of 60 cupcakes. Then, maintaining the total number of 12 boxes constant, he added a 4 and erased a 6, to give him 7 boxes of 4 vanilla cupcakes each, 5 boxes of chocolate cupcakes and 58 cupcakes total.

Alice: You have 12 boxes here?

C8: No.

Alice: Sure?

C8: Oh, wait, yes I do.

Alice: So, you have 12 boxes with 58 ... So why did you start off making a list?

C8: 'Cause ... I guess ... [shrugs] it just seemed easier

Alice: It just seemed easier. When you started you just used a random number to go in each list, is that right?

C8: Yeah.

Alice: You just used a random number of vanilla and a random number of chocolate. ...And then from there what did you do?

C8: Saw how much I had and I thought about what I could do to change them up and get the answer.

Alice: So, guess and check. Did you do a lot of guessing and checking?

C8: Yeah.

Alice: Great job. (Group I Video transcript, p. 18)

Though Erica asked follow-up questions in response to pupils' explanations more often than Alice did, she still did not elicit complete explanations. In only three instances did Erica's questions requiring justification actually elicit complete explanations.

Interestingly, both participants showed different associations between the components of math-talk and cognitive demand. For Alice, all four components were closely aligned with

cognitive demand; all stayed at low levels. For Erica, only *explaining* aligned with cognitive demand, so this component of math-talk, over the other three, may have had a more significant influence on the cognitive demand she was able to achieve.

Erica's and Alice's planning activities were similar to their enactments. Alice provided directive guidance when faced with errors, and this was also true for her task dialogues:

The part where you add 4 to 8 and get 12 would you get the right answer if you worked that part backwards? Is 8 half of 12 equal to 8? The student would reply no. From here, I would ask the student what he or she would do to find the number when halved is equal to 8? The student should reply 16. I would then ask the student how he or she got this and why would 16 work. Hopefully they would see that they just had to double the numbers in the problem since half of the numbers were taken. (Alice, Task Dialogue 3, p. 28)

Erica consistently maintained a low level of cognitive demand of *memorization*, as she frequently directly guided pupils to the solution (Figure 22). In only a few instances, and only in response to correct solutions, did Erica elevate the cognitive demand to *procedures without connections*. In both of the planning examples below, Erica's tendency to explain for the pupil and elicit only short answers is apparent. Alice, on the other hand, did make more careful plans for responding to pupil's correct solutions. For the Puppies Problem, she planned to ask the pupil to find the solution a different way, for the Tickets Problem, she included a different question from her problem set that had a similar set up, and for the Cupcakes Problem, she included questions to get the pupil to justify his strategy. Unfortunately, she never implemented any of these plans.

Neither Erica nor Alice took good advantage of the opportunity to revise their plans each week. At best, Erica highlighted a concept she needed to emphasize more. For example, in her Week 5 activity plan, she wrote, "Help the student recognize that one side cannot have all 3 of the biggest numbers" (Erica, Activity Plan 5, p. 16), but she gave no detail on what she would do or say to help the pupil with this realization. Alice updated her plans to indicate what type of

solution her pupil obtained and how effective her planned response to that solution was, but she did not consider possible changes based on this update. The planning these participants engaged in likely accounted for some of the low cognitive demand of their enactments. Erica even noted this possibility herself in her reflection for Week 5:

I asked if she [the pupil] saw a pattern and could tell that I had worded it incorrectly ‘cause she said she did not understand what I was asking. I asked her if she saw a pattern in where the high and low numbers were placed in the triangle. In reflection I wish I would have planned better because I really felt like I gave the answer away instead rewording the question better. (p. 18)

Task Dialogue for Cupcakes	Activity Plan for 6 Numbers
<p>Prompt: Student says he’s got too many cupcakes.</p>	<p>Prompt: Student has [125-546-631]</p> <p>Student: I don’t get how you can have each side with the same number when all of the numbers of different. I need numbers that are the same. Like two 2’s.</p>
<p>S: We would have 4 boxes of vanilla with 4 in each boxes, 3 boxes of chocolate with 6 in each box. And 4 left over from the one we broke up. T: Since we only have 4 left over and cannot make a box of chocolate because we need 6 that 4 must go with the box of what? S: Vanilla T: Correct, and if the 4 is vanilla what do we have altogether? S: We have 5 boxes of vanilla with 4 cupcakes in each box and 3 boxes of chocolate with 6 cupcakes in each box. 38 cupcakes 8 boxes total. T: Ok so we have 2 cupcakes to many. Let’s say we broke up a box of cupcakes, the box of 6 and took the 2 cupcakes out of there. How many would we have?</p>	<p>T: Can we add different numbers together to get the same numbers? For example, to get the number 6 can we add <math>3+1+2</math>? S: Yes so we need to add the number together to get the same number. T: Close we need to add the numbers together to get a new number. See if we were trying to get 6 then we would have 3, 2, 1 on one side but we would not be able to get just “6” on any other side. So our common number must be greater than 6.</p>

*Figure 22.* Examples of Erica’s planning.

Because Erica had no high-level planned questions to rely on, when she had to develop questions in-the-moment of teaching, she made several mistakes: (a) having not considered pupil’s thinking or their need for wait time, she assumed her initial question was inadequate, and

(b) she reverted to a more directive question to guide the pupil's thinking to the line of reasoning she wanted. Alice simply relied on the directive questions she planned, ignored her more demanding questions, and failed to make meaningful changes to her plans each week. Part of the reason Alice failed to make such changes is likely because she provided virtually no analysis of her teaching and only brief superficial commentary to her fellow group members. This indicates that she did not appreciate the impact of a teacher's interventions on a pupil's mathematical work or understanding. Her first critique of her own teaching did not occur until Week 6: "I feel that I may have guided C2 more than I should have. I don't feel as if I gave her enough time to think about what she was doing and to fully articulate her thought process" (Alice, Activity Reflection 6, p. 31). Though she realized she was being too directive in Week 6, she made no changes to alleviate this problem, and it continued into Week 7: "Some of my questioning was definitely too direct but I was not sure as to how else to approach the situation" (Alice, Activity Reflection 7, p. 37).

Erica's reflections, however, offer some insight into Erica's pedagogical decisions. She was generally unaware of her own missteps, even though these were frequently pointed out to her by her group members. Each week Rene gave a detailed critique of Erica's work that included specific suggestions for how to improve:

For example, at 2:15, Erica tells the student what his problem is and does not question him first as to why he stopped writing or gives him a chance to verbalize that two sides added up to 11, but not the third. Rather, she acknowledges the issue for the child, assuming he understood the problem. While this was probably true, I think it's important to have the student tell you this in his own words so that he can become more accustomed to talking through his reasoning. (Rene, Activity Reflection 3, p. 10)

My only concern is that Erica might be telling [her pupil] important information rather than having [her pupil] deduce it on her own. I would just suggest getting the student to verbalize such discoveries, although it may take longer. (Rene, Activity Reflection 5, p. 29)

Because Rene gave so much critical feedback, it could be that Erica was overwhelmed and she simply tuned it out. It could also have been a conscious decision: Erica may have felt Rene was intervening too much, and Erica over compensated in the other direction. She noticed:

Rene has a different approach than I and I wanted to know if I should use it. I have seen on a number of occasions that Rene asks a lot of questions. ... I start with this approach but I eventually just give in and began giving hints to the answer. I think this is because I know I did not like when my teachers continued with the '21 questions' as I called it. (Erica, Activity Reflection 5, p. 19)

Further compounding Erica's difficulties in improving, she was not aware of the effect of her own or other's interventions. In Week 3, she described Rene's pupil's work on the [6 Numbers Problem](#) as looking for a pattern before she had found any solutions and compared this to what she considered a more logical approach (guess and check to generate solutions, then look for a pattern) that her own pupil had taken. Erica expressed her confusion,

I did not really understand at first why [Rene's pupil] was going the 'hard' way around it. After watching the recording I get that she [the pupil] was trying to understand the problem because as soon as she did she starting getting the solutions" (Erica, Activity Reflection 3, p. 13).

What Erica seemed to miss was that the pupil's search for patterns was prompted by Rene's *questioning*. She also revealed confusion about pushing the pupils to articulate their explanations:

Should we have the students explain their solutions until we understand, is that our job? It is one thing to have the student explain his work, but it is a shorter process when I understand what they did. When I don't understand what they did I feel like they are repeating themselves because, I am not understanding. (p. 30)

This information may help explain Erica's consistent and low-level performance over the course of the study. She both failed to plan effectively (and to make detailed revisions to those plans), and was not fully aware of her difficulties or the impact and purpose of her interventions with pupils.

## Summary

Findings from this study of repeated enactments provide evidence that 6 of the 8 teachers developed pedagogical content knowledge related to implementing these problems and facilitating mathematical discussions. Rene, Kate, Casey, and Dana all showed some degree of improvement in their ability to establish and maintain higher cognitive demand during their mathematical discussions with pupils. Rene and Kate achieved the highest levels of cognitive demand and were able to sustain it over several turns in the conversation. However, because Rene did not carry improvements from the first cycle to the second, her pedagogical content knowledge may be specific to these problems. Kate, on the other hand, showed consistent improvement across all 6 weeks, without showing any backsliding as she started the second cycle, and, because of this improvement, she may have developed pedagogical content knowledge about leading mathematics discussions that was not specific to one set of problems. It could be that Kate will be better equipped than Rene to transfer the skills she developed to future work with pupils. Though Dana and Casey showed less significant improvement, the fact that they were able to achieve higher cognitive demand by the end of study than they did at the beginning of the study indicated that they too developed pedagogical content knowledge. Dana developed some generic strategies and questions for exploring a child's thinking. Though Casey may only have been imitating effective techniques she observed in her fellow group members, because she learned to recognize effective techniques and use them herself shows that she also developed some pedagogical content knowledge. However, because her development was so dependent on her involvement with other participants, she may not be able to transfer this knowledge to other settings of working with pupils.

Megan's and Nadia's performance, though inconsistent, gave evidence of their potential to elevate cognitive demand; both achieved *procedures with connections* levels of cognitive demand. Elements of each of their reflections also indicated that they developed pedagogical content knowledge. Nadia was able to identify her missteps but unable to correct them in the moment of teaching. However, because she learned to critically analyze her practice, there is a possibility that she will be able to use this skill to continue refining her teaching practice. Similarly, Megan was able to develop trajectories of pupil work on particular problems. This ability to analyze several pupils' work and draw general conclusions about ways that typical pupils approach problems is a skill that, if continued, can inform her future mathematical work with pupils.

Unfortunately, Alice and Erica did not show evidence of development of pedagogical content knowledge. However, they were exposed to mathematical discussions that established high cognitive demand and even took part in assisting in these discussions. Furthermore, they were able to serve as nonexamples for their fellow group members; Alice and Erica's struggles may have helped their group members lead better mathematical discussions.

### **Alignment of Planning and Enactment**

The third question explores how the participants' vision of mathematical discussions aligned with their enactment of tasks. To answer this question I compared participants' task dialogues and activity plans against their enactment of tasks. Some participants' task dialogues and activity plans foreshadowed the quality of the discussions they would lead with pupils. In other instances, there was a disconnection between what a participant envisioned and the quality of discussion they were actually able to achieve with pupils (Table 9). Additionally, in some cases, there were consistent findings among participants who had growth in their ability to

facilitate mathematics discussions: In particular, those who made the most significant improvement to their mathematical discussions, Rene and Kate, had the most well-conceptualized vision of what leading mathematical discussions should be, and those with no development beyond their teacher-directed discussions, Erica and Alice, had an equally directive vision for leading discussions. The other two categories of inconsistent discussions (Nadia and Megan) and inadequate but improving discussions (Dana and Casey) did not have consistent relationships between planning and enactment across group members. While Casey's problem enactments improved over the 6 weeks, her vision of mathematical discussions as seen in her task dialogues and activity plans did not change. Dana, on the other hand, whose problem enactments also improved, developed a richer vision of mathematical discussions that included more attention to pupil thinking. Nadia's planning activity, much like her enactments, showed no consistent trends over the course of the study. However, Megan, who also had inconsistent performance in her discussions, had weak plans and, thus, at times her enactments were better than what she planned.

Table 9

*A Comparison of Participants' Task Dialogue, Plans, and Enactments*

Participant	Enactment Improved?	Task Dialogue Description	Plan Improved?	Plans and Task Dialogue Followed?
Kate	yes	concept-focused	yes	yes
Rene	yes	explaining-focused	yes	yes
Casey	yes	answer-focused	no	yes/ no*
Dana	yes	Incomplete/exploring pupil reasoning*	yes	no/yes*
Alice	no	highly directive	no	yes
Erica	no	highly directive	no	yes
Megan	inconsistent	incomplete/directive*	no	inconsistent
Nadia	inconsistent	incomplete/directive*	yes	inconsistent

\*Indicates a change from Cycle 1 and 2.

## Directive Planning and Enactment

Not surprisingly, the two participants who showed no change over the 6 weeks in their teacher-directed discussions also had the most directive task dialogues. Alice's and Erica's task dialogues were highly directive for both cycles, and their plans consistently followed the same trend, with few helpful revisions made each week. They both used specific short-answer questions to guide pupils to solutions in their plans *and* their enactments, as seen in Figure 23. Neither included meaningful follow-up questions to correct solutions, typically only asking the pupil to explain a strategy again. Only once in her enactments did Alice explore a pupil's correct solution, and Erica only engaged in this when it was instigated by one of her fellow group members.

Task Dialogue	Enactment
<p>T: Ok but was the puppies only halved once?            What does the problem say about the people who bought the puppies.            The student should say that puppies were halved twice.            I would then ask the students how many times he or she halved the puppies.            The student should reply only once.            Here, I may ask the student what would happen if he started working from the four puppies left over and ask if he or she would like to try it. (Alice, Task Dialogue 2, p. 28)</p>	<p>Alice: How many did you have here?            C2: [counts] 25...and then, then he took the pups, he took half of 24 which is 12 so...4, 5, 6, 7, 8, 9, 10, 11, 12 [takes away 12]...and then another woman took. Half 1 2 3 ....Oh! I think I know!            Alice: What?            C2: Well, if 4 more than that... 4 more than that...?            Alice: Wait, repeat that one more time?            C2: Well, like, if, cause...well, wait, let me think...maybe we have to do an even number? Because...like, if on the odd numbers we take one and one...            Alice: let's look at the question...You have 25 puppies and took half of the puppies, you have 24 and you take half of 24 and that gave you 12. Are you taking...and then the women took half of what was left. (Group I Video transcript, pp. 65–66)</p>

Figure 23a. A comparison of Alice's plans and enactments.

Task Dialogue	Enactment
<p>Student counts the dots in the picture and says he's got too many cupcakes.</p> <p>T: Ok so we have 2 cupcakes too many. Let's say we broke up a box of cupcakes, the box of 6 and took the 2 cupcakes out of there. How many would we have?</p> <p>S: We would have 4 boxes of vanilla with 4 in each boxes, 3 boxes of chocolate with 6 in each box. And 4 left over from the one we broke up.</p> <p>T: Since we only have 4 left over and cannot make a box of chocolate because we need 6 that 4 must go with the box of what?</p> <p>S: Vanilla</p> <p>T: Correct, and if the 4 is vanilla what do we have altogether?</p> <p>S: We have 5 boxes of vanilla with 4 cupcakes in each box and 3 boxes of chocolate with 6 cupcakes in each box. 38 cupcakes 8 boxes total. (Erica Task Dialogue 2, p. 21)</p>	<p>Erica: When you do division how many answers do you get?</p> <p>W: 1.</p> <p>Tia: Looking at this problem, how many boxes do we need?</p> <p>W: How many? uh, 2.</p> <p>Tia: Yeah so can we solve it with division?</p> <p>W: No. [he erases it.]</p> <p>Tia: You can use the cubes. You can use pictures.</p> <p>W: 38. [W is drawing circles]</p> <p>Tia: 38 Cupcakes</p> <p>W: In 8 boxes. [draws 3 big boxes]</p> <p>....</p> <p>Tia So 38 cupcakes that cover 8 boxes. So step 1, step 2 is right, ok?</p> <p>W: 38 cupcakes in all.</p> <p>Tia: 38 cupcakes in 8 boxes.</p> <p>W: Ok.... Are there the same amount of vanilla as chocolate?</p> <p>Tia: It doesn't have to be. (Group J Video transcript, p. 74)</p>

*Figure 23b.* A comparison of Erica's plans and enactments.

### **Strong Plans and Enactments**

The participants who were able to achieve and sustain the highest level of cognitive demand had plans that included rich mathematical discussions. Rene's task dialogues and plans focused on gradually raising the cognitive demand and requiring pupils to explain their reasoning. Kate's plans focused on making mathematical concepts underlying the problems evident. All of Rene's enactments had a similar structure that was mirrored in her task dialogue and activity plans: She began by establishing low cognitive demand as she asked pupils many questions requiring them to explain strategies and moving to high cognitive demand once pupils obtained correct solutions that she could ask them to explore and compare. She wrote, "Enforce the need to double check your answers and read the original problem once you think you have

found a solution,” (Rene, Activity Plan 3, p. 8) and in all of her enactments, she consistently stepped her pupils through double-checking their answers. She posed questions from her activity plans in her enactments. To help a pupil who could only get two of the three sides of the triangle in the [6 Numbers Problem](#) to have the same sum, she planned to ask “Why do you think these two sides equal a number so much greater than this side?” (Rene, Activity Plan 4, p. 16). She not only asked this question several times, she asked a modified version of it: “So why do you think this side equaled almost the same number as this side? You said they were close. Why do you think that is?” (Group J Video transcript, p. 8). In planning for correct solutions she consistently sought justification such as, “Why did you need to spread out the larger numbers? Do just the large numbers need to be spread out? Why?” (Rene Activity Plan, p. 8).

However, when pupils did not respond as she anticipated, she did become more directive, trying to get them back to the line of reasoning she prepared for in her activity plan. When one of her planned questions did not elicit the response she wrote in her task dialogue, Rene reverted to a series of direct short answer questions to help her pupil develop an explanation (see [excerpt](#) on p. 84). Rene improved in attending to solutions by using specific pupil language in her plans and noting that she needed to explore what the pupils meant by terms like *even* and *reversing*.

Kate used her planning activities slightly differently than Rene; her task dialogues and plans were venues to test specific approaches for helping pupils overcome difficulties. In subsequent weeks she would modify her plans based on her pupils’ prior struggles or successes. For a pupil who had generated some, but not all, solutions to the [12 Pennies Problem](#) she planned to ask what could be concluded if the first pile was 6. However, in her Week 4 enactment when she asked, “What do you notice has to be true about the other two piles when your first pile has 6 in it?” it degenerated into a directive conversation. She addressed this in her Week 5 plan adding,

“Are there any more solutions to be made with 6 in the first pile?” (Kate, Activity Plan 5, p. 23) as a follow-up question to avoid the previous week’s missteps.

Kate’s goals for her work with pupils can be summed up by the rationale she used to justify a pedagogical move to extend the Phone Club problem: “Have students search for pattern in answers so that they can finish it on a quicker level, as well as understand the mechanics of how and why the problem works this way” (Kate, Task Dialogue 3, p. 34). Both Kate’s planning activities and interactions with pupils were focused on making mathematical concepts explicit. She maintained high demand even when her pupils were stuck. To prepare for responding to correct solutions, she developed questions that would make pupils justify their strategies such as, “Explain why you decided to set it up this way? Why did you not make lists where there were 5 or 6 pennies in the first piles?” (Kate, Activity Plan 5, p. 23–24). She also considered how to maintain high cognitive demand for incorrect pupil thinking as well, as seen in the task dialogue excerpt below. In this example Kate was able to give hints that built on pupil ideas and maintained cognitive demand:

S: Student randomly puts 12 pennies into 3 piles until she finds 6, 4, 2. Then she moves a penny from the 4 pile to the 2 pile and has 6, 3, 3. “That’s not right.” She moves the pennies back to 6, 4, 2 and moves one from the 6 pile to the 4 pile. “I keep making them with the same number.”

T: That’s ok- why don’t you try thinking about what happens when you’re moving them. Why do you ‘keep making them with the same number’?

S: Because this is too hard for me.

T: No it’s not- you’re doing fine! Let’s see. What do you notice has to be true about the other two piles when your first pile has 6 in it?

S: The other two piles add up to 6. I already know they have to because there are 12 pennies and  $6 + 6$  is 12.

T: Ok! That’s a great realization- you’re smarter than you give yourself credit. So you know that the other two piles have to equal 6. You already tried 4 & 2, and then 3 & 3

didn't work because of the rule. Can you make anything else if the first pile keeps just 6?

S: Hmmm.. Oh I know that 1 & 5 makes 6 too! That is another option. So then 642 and 615 are my answers. There are probably more though. How can I find those? I don't know what to do!

T: Well do you remember what we just learned with the last reasoning we did? That when the first pile has 6, the other two had to add to 6. So how can we use that reasoning to find more answers if there are any? (Kate, Task Dialogue 2, pp. 1–2)

Both Rene and Kate seemed to use their plans as guides for their mathematical discussions with pupils. For Kate this meant testing her ideas for how to help pupils and focusing on making mathematical concepts in the problems evident to them. For Rene, when pupils veered away from her plan, she generally moved them back to it. While this meant she initially struggled to build off of pupils' strategies, especially those that she did not anticipate, she was able to maintain cognitive demand by connecting the procedures to underlying mathematics concepts, requiring pupils to explain each other's strategies, and extending the questions to get pupils to apply the strategies in a new context.

### **Improving Plans and Enactments**

In Dana's first task dialogue she failed to include the pupils' role in the conversation. Though she had some good questions, such as "Why did you start with 6 groups of 6 and 6 groups of 4? Why did you decide to take away 2 cupcakes from a group of 6? What did that change?" (Dana, Task Dialogue 2, p. 2), she failed to consider how she would handle it if her pupils did not respond well to her questions. Her failure to consider pupil's thinking may have contributed to her tendency to reduce tasks to answering basic fact questions since she had no planned follow-up questions to rely on.

In her task dialogue for the second cycle she focused more on exploring pupil reasoning and connecting it to the ideas in the problem:

S: 4 is half so  $4+4$  is 8, then half were gone again, so  $8+$  another 4 is 12 and one was taken away is 13.

T: Why did you choose to add 4 to the 8?

S: because another half were gone again and half of 8 is four.

T: another half of what were gone again? What does the problem say?

S: It says half of what were left

T: Ok so what does 8 represent?

S: it represents the number of puppies before half were taken away.

T: How does 8 relate to the first half taken away?

S: OH! I got it! 8 is the half of what they started with on Tuesday. So that would mean its 16 plus one more which is 17. (Dana, Task Dialogue 3, p. 30)

Dana's pattern of trying to understand how a pupil arrived at a solution and directing him to connect that reasoning to the conditions of the problem so that he might identify his own errors held true for her enactments as well. As discussed in an earlier example where she assisted Alice with a struggling pupil, she employed some similar questioning as shown above to explore and understand the pupil's reasoning. Developing some generic questions such as "What does  $x$  represent?" an, "How does  $x$  in your work relate to the problem?" may have been the key to her improved enactments.

### **Divergent Enactments and Plans**

Casey's task dialogues were a small improvement over being directive; both Casey's planning and her enactments were focused on answers. In Cycle 1, Casey's task dialogues and activity plans followed closely with her enactment of tasks. In her first dialogue and plan, her questions focused on eliciting strategy descriptions. Her most frequently planned questions was, "How did you get that answer?" and in her enactments she not only frequently posed *how* questions. However, she often missed the mathematical ideas underlying the problems. She did

not address why strategies worked or how strategies guaranteed that all solutions were found, a key mathematical concept to her problems in the first cycle. In the case of the [12 Pennies Problem](#), she was happy a pupil found all the answers and though she explored the patterns in his list, she did not help him make the connection between those patterns and knowing that he got all the solutions:

Casey: Right exactly. So...I think you found all the answers. What do you think?

C6: Probably.

Casey: Because you are getting the same answer in different ways or you're getting piles that have the same number of pennies....Good. Well you did. You did a good job. [to C1] Did you find all of them too? (Group H Video transcript, p. 22)

Another issue that surfaced in both Casey's plans and enactments was that she frequently drew conclusions for her pupils. Though her pupil had an organized list of number sentences ( $9 + 2 + 1$ ,  $8 + 3 + 1$ ,  $7 + 4 + 1$ ,  $6 + 5 + 1$ ), she did not push him to continue this organization (e.g., all combinations with 2 in one pile:  $7 + 3 + 2$ ,  $6 + 4 + 2$ ) to find other solutions. Instead of building off his work, she encouraged a guess-and-check strategy with manipulatives and, instead of asking him to explain how he knew he had found all solutions, a question that addressed the mathematics behind the problem's solution, she summarized an explanation for him. Her explanation matched the hypothetical pupil explanation she suggested in her task dialogue, "Because I put all the numbers where they could go" (Casey, Task Dialogue 2, p. 2).

This trend continued in her implementations of the [Clock 6 Problem](#). In lieu of asking the pupils to explain, she summarized for them:

You [to C6] were noticing that [pattern] too. Because 6 is the one that she's [C1] talking about that would only have one combination [6:00] and then it would go up from there [5 o'clock would have two combinations, 4 o'clock would have 3 combinations, etc]. So you [C6] were finding the pattern we just kind of ran out of time. But you guys [C1 and C6] understand? (p. 31)

She did not use her pupil's explanation as a way to evaluate his understanding. In addition to drawing conclusions for her pupils and explaining their ideas, she often suggested the strategies they would need to solve problems. Though she suggested a different strategy, switching numbers, in her task dialogue and activity plans, the pedagogical move of directing the pupil is still evident in her implementation of the [12 Pennies Problem](#):

Casey: ... 7 plus what equals 12. Here's a seven [pointing to 7 4 1] what's 4 plus 1?

C7: 6.

Casey: No. 4 plus 1?

C7: 5.

Casey: Yeah. Ok. If you first pile has 7 then your second and third pile have to give you 5 because 7 plus 5 is 12. So right here you have 7, what is 3 plus 2? (Group H Video transcript, p. 40)

Even her early attempts at extending questions resulted in directly guiding her pupils to draw conclusions about the problem or drawing conclusions for them. In her Week 4 activity plan she suggested asking, "How would the problem change if we had a different number of pennies?" She posed this question in the Week 4 enactment but followed up this question with, "Do you think you'd have more solutions if you had more pennies?" (Group H Video transcript, p. 22). This follow-up elicited one-word answers before she explained to the pupils the effect of having more pennies.

Casey mostly tried to meet bare minimum requirements for class assignments rather than use her reflections to inform subsequent activity plans. Casey's vision of instruction aligned with her plans, but for her this meant taking the responsibility from the pupils and replacing their ideas with her own. Her interactions with her pupils focused on attaining answers, not justifying claims or ideas. Though she could elicit explanations of how pupils solved, she rarely explored these and, instead, revoiced her own more sophisticated versions of the pupils' work. She made

few modifications to her plans in the first cycle and none in the second cycle. However, her enacted mathematical discussions did improve as she helped pupils articulate their reasoning rather than draw conclusions for them. Though Casey planned to ask, “Do you notice a pattern? Do you think there is a better way to organize the people and the strings?” (Casey data, p. 21) for the [Phone Club Problem](#), when she actually attempted this with a pupil she worked to help her articulate her reasoning:

Casey: Can I borrow your eraser [to make space so that Casey can draw C7’s picture]... to make sure I understand what you did? Tell me what you did first.

C7: I did 4 people. And then, and then I drew telephone strings from each of them. And I started with the first person, I started with the first person and then I brought that string over to the second. [With each sentence Casey is recreating her picture, as seen in Figure 24 ]. Then I brought it to the third. Then I brought it to the fourth [red lines going over circles in Figure 24]. Then I got the second string and brought it to the first [blue line going under circle in Figure 24]. Oh wait, no, no, no, never mind.

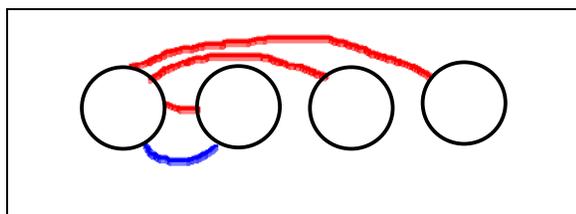


Figure 24. Casey’s reconstruction of C7’s solution to the Phone Club Problem.

Casey: That’s what you did. Right?

C7: Yeah.

Casey: Ok, why did you just say you don’t need to do that?

C7: Because I already have a string connecting them [Points to 1-2 connection)

.....

C7: Ok, then connect the second to the third. And then the fourth. Connect the second to the fourth. And then the third, the third to the fourth.

.....

Casey: You're on the right track. You've drawn lines from the first one to each one, from the second one to each one, and from the third one to each one. So which one do you need to do next?

C7: Well, first I did [the] 4 [person case] like for the first question. Then I tried 6 [people]. I put 6 people on there. And I thought about and took it off. Then I did 7 people and I was drawing strings from each person. For my first person it went to the line went to 7 people, for the second it went to 6, and for the third it went to 5, for the fourth it went to 4, for the fifth it went to 3, for the sixth it went to 2, for the one it was one.

....

Casey: So do you notice anything about that?

C7: It's counting down from 7. I mean like, it's going down by one. (Group H Video transcript, pp. 84–86)

In this example, the pupil tried to ignore a mistake she made with connecting the first person to the second and the second person to the first. But Casey did not let the mistake slide; she used it as an opportunity to explore the problem constraints and, as a result of exploring the error, the pupil had to examine the problem more closely. Casey's improvement in leading discussions may be attributed not to her planning activities but to her interactions with her fellow group members who she had watched effectively employ similar tactics in previous weeks. It could be that Casey was merely imitating practices she observed and recognized as more successful.

Nadia and Megan with their inconsistent enactments presented a different picture of divergent visions and enactments. Nadia's task dialogue for the first cycle was incomplete in that it only addressed how the pupil would follow her directions; she did not take into account the pupil's thinking or reasoning. In her task dialogue for the second cycle, her hypothetical teacher attempted to elicit the pupil's thinking, but, even though she was prompted with nearly correct pupil reasoning, Nadia followed up with weak and incorrect pupil thinking (Figure 25). Nadia had limited conceptions of pupil thinking, and this was evident in her task dialogues. However, her plans improved; each week she made changes based on pupils' prior struggles. After both her

fellow group members and her pupils were confused about the order of solutions in the [12 Pennies Problem](#) (e.g., is a combination of 9 2 1 the same as 2 1 9), she included a question about the order of the piles in her plan for the next week. When she saw that having pupils compare solutions was an effective strategy, she included this activity in a subsequent plan. Nadia's improving plans but difficulties envisioning pupil's reasoning may give some insight into her struggles when she faced incorrect solutions. Unpacking the reasoning behind pupil's incorrect solutions may have required more sophisticated conceptions of pupil thinking than Nadia had developed, and this may have influenced her inconsistent performance.

<b>Nadia's Task Dialogue for Cycle 1</b>	<b>Nadia's Task Dialogue for Cycle 2</b>
<p>Prompt: Student says, "So 6:00, if I put 6 anywhere else, it doesn't make a time. 3:30, 3:03, I can't put 0 in the hour place. 4:20, 4:02 2:40, 2:04. 5:10, 5:01, 1:50, 1:05."</p> <p>You are doing a great job so far! Why don't you write down all the answers you have come up with so far? Do you notice a pattern about your answers?"</p> <p>If the student doesn't I would ask him to write down all the equations that equal 6. He would write: <math>5+1</math>, <math>4+2</math>, <math>3+3</math>, and <math>6+0</math>.</p> <p>Then I would ask if he noticed anything about these equations. Then I would ask if there is any way to "break down" some of these numbers to make the equation longer. (Nadia, Task Dialogue 2, p. 3)</p>	<p>Prompt: Student draws correct representation for 4 and 5 people. After counting the strings in the 5 people case he says that is not enough. He needs 6 people. But gets confused drawing the lines for 6 people.</p> <p>T: What do you mean that is not enough? S: There is not enough strings for all the people T: Why do you say that? S: I just feel like there should be more T: well why don't you start with the first person and then make sure that he can talk to every other person. (Nadia, Task Dialogue 3 p. 28)</p>

*Figure 25.* Excerpts form Nadia's Task Dialogues.

Megan's task dialogues in the first cycle followed a similar pattern to Nadia's, short on pupil thinking, but their task dialogues in the second cycle had dissimilar problems. Whereas Nadia suggested strategies, Megan employed a series of directive questions to guide pupils to

correct solutions, as can be seen in the following excerpt from her task dialogue for the second cycle:

Prompt: Student draws 8 boxes. He draws a line down the middle to make 2 groups of 4 boxes each. He puts four dots in one group of 4 and 6 dots in the other group of 4. Then he says 4 vanilla and 4 chocolate.

T: Why did you divide the boxes into 2 equal groups? How many cupcakes are in both sets of boxes?

S: 40 cupcakes

T: Does that follow the rules set in the problem?

S: No, there are only supposed to be 38 cupcakes

T: How many extra cupcakes do you have?

S: 2 cupcakes

T: What could you do to make sure you only have 38 cupcakes?

S: take away 2 cupcakes from the boxes (Megan, Task Dialogue 3, p. 19)

In all of her plans, Megan elevated cognitive demand as pupil solutions were closer to being correct. Most of her planned questions were too open to be helpful; in response to a pupil who could not start the problem she suggested, “Is there a specific number that the sides need to add up to? Could there be multiple solutions?” (Megan, Activity Plan 4, p. 7). For a pupil with an incorrect solution she offered, “What is the smallest number the sides can add up to?” (p. 7). She acknowledged this weakness herself noting that “my questions were typically not direct enough or posed at the appropriate time” (Megan, final paper, p. 38). Though she did make improvements in preparing for correct and nearly correct solutions in her planning, she only showed improvement in her enactments when responding to incorrect and nearly correct pupil thinking. In both cycles she had extension questions, but she clearly had not attempted to solve them as both were not possible with the information she had provided. This worked out in her favor as she did not attempt these extensions with any of her pupils. Given Megan’s planning

activities, it is surprising that her implementations were not consistently teacher-directed like Alice and Erica. Though some of her success might be due to her ability to work well with Rene, a participant who was a stronger discussion leader, this does not account for all of the variation in her enactments. Though she was not able to achieve a *doing mathematics* level of cognitive demand on her own, at several points Megan achieved procedures with connections while working on her own with a pupil. Using her planning data, I am unable to explain the sporadic successes that Megan did experience.

### **Summary**

For the most part participants' visions of leading mathematical discussions aligned with how those discussions were enacted. Unsurprisingly, the preservice teachers with the best- and worst-conceptualized plans most closely aligned with their enactments. The preservice teachers with the most-well-developed plans that focused on mathematical concepts and pupil reasoning, Kate and Rene, also had the mathematical discussions that achieved the highest cognitive demand; Erica and Alice with the most teacher-directed task dialogues had similarly teacher-directed discussions with pupils.

The more interesting findings came from the four preservice teachers whose task dialogues and plans fell between these two extremes. Both Casey and Dana began the study with limited visions of mathematical discussions. Though the improvement of Dana's task dialogues matched the improvement in her enactments, this pattern did not hold true for Casey. While Casey's interactions with her pupils improved, her task dialogues and plans showed little change. It could be that Casey's improvement resulted from mimicking the work of her stronger group members, rather than as a result of her developing a better understanding of pupils' thinking.

Like Casey, Nadia and Megan's vision of facilitating discussions did not match their enactments. Though their visions of mathematics discussion varied between not attending to pupil thinking and directly guiding pupils, their enactments had points of high cognitive demand. Their superficial visions of mathematical discussions were the result of weak conceptions of pupils' thinking more so than they were the result of weak ability to facilitate discussion. These two participants struggled in anticipating how pupils thought about mathematics, but when faced with actual pupil reasoning in their enactments they were sometimes able to respond in ways that maintained the cognitive demand.

## CHAPTER 5

### CONCLUSION

Though research has not been able to directly tie participation in mathematical discourse to student learning, many claim that communication holds promise for helping students build an understanding of mathematics (Bauersfeld, 1995; Cobb & Yackel, 1996; Krummheuer, 1995, 2007; Lampert 1990; Lampert & Cobb, 2003; Sfard, 2000, 2001). Amid calls to increase communication in mathematics (NCTM 1989, 1991, 2000), teachers struggle to move mathematics conversations beyond a traditional Initiation-Response-Evaluation sequence (Franke, et. al, 2007). For those teachers who do attempt to facilitate conversations based on their pupils' thinking and grounded in mathematical reasoning, additional difficulties arise: developing autonomy in pupils, providing appropriate supports, balancing mathematical challenge with accessibility, respecting pupil ideas, and maintaining integrity to the discipline (Adler, 1999, Ball, 1993; Baxter & Williams, 1996; Knuth & Peressini, 1998). These difficulties are even more pronounced for prospective elementary mathematics teachers, who are likely relatively unfamiliar with pupils' ways of thinking mathematically and have the least rigorous mathematics content preparation.

Overcoming the apprenticeship of observation (Lortie, 1975) is one of many challenges to preservice teacher education. In response to weak carryover of what is taught in preservice teacher education to actual teaching practice, research points to the need for practice-based teacher education (Ball & Forzani, 2009; Darling-Hammond, 2006; Feiman-Nemser, 2001;

Lampert & Ball, 1999). By practice-based, I mean that learning to teach should focus on more than the acquisition of knowledge about teaching practice and should “emphasize repeated opportunities for novices to practice carrying out the interactive work of teaching and not just to talk about that work” (Ball & Forzani, 2009, p. 503). To that end, connecting learning in preservice methods courses to field experience seems a readymade solution. However, field placements may further compound the problem because of the mismatch between the teaching practice advocated in preservice teacher education and what preservice teachers experience in field placements. Feiman-Nemser (2001) suggests that practice-based teacher education should not focus solely on whole-class instruction and that other types of assisted performance activities (e.g., studying children, co-teaching) may be fruitful sites for teacher learning.

Thus I designed the current study to examine how preservice elementary teachers learned to lead mathematical discussions during a 6-week field experience set in a practice-based teacher education program. Through fine-grained analysis of their work with pupils, I sought to determine what pedagogical moves for facilitating mathematics discussions elevated or maintained high cognitive demand and constituted teachable high leverage practices (Ball & Bass, 2000), how a cycle of planning, enactment, and reflection influenced preservice elementary teachers’ development of pedagogical content knowledge of leading discussions, and how preservice teachers’ visions of leading discussion aligned with their enactment.

To explore these questions, I chose as participants preservice elementary teachers from among enrollees in my section of an elementary mathematics teaching methods course that focused on children’s thinking. I collected data during the course’s fieldwork in two 3-week cycles of planning for, enactment of, and reflection on (Kazemi et al., 2010) facilitating discussions with and between pupils on nonroutine mathematics problems. As part of the

planning participants explored pupils' potential solution strategies and considered how to facilitate discussion about those strategies through *task dialogues* (Crespo et al., 2011), assignments in which I asked them to develop a hypothetical pupil-teacher dialogue in response to several possible pupil solution strategies. Participants then used my feedback to complete an activity plan where they had to suggest several potential pupil responses and describe how they would respond. I provided feedback again prior to implementation of their plans. Video records of participant groups' weekly sessions with pupils served as the enactment data. Reflection data were each participant group's collective activity reflections in which they analyzed their own and their pupil's work and responded to each team members' reflections and individual final papers in which they reflected on the entire field experience.

I analyzed data using a cognitive demand framework (Stein, et al., 2000) and a revised version of the math-talk framework (Hufferd-Ackles, et al., 2004). I coded each dialogue of the task dialogues, each type of teacher response in the activity plans, and carefully parsed segments of all the teaching episodes from the videorecords by assigning one of four levels of cognitive demand and one of seven levels of each category of the math-talk framework. In reading each participant's reflections, I looked for themes in the issues mentioned by participants for each problem of each cycle. To determine what pedagogical moves supported cognitive demand I examined which components of math-talk aligned with high cognitive demand. To determine how the cycles influenced participants' implementation of problem solving activities over the field experience, I looked for trends in their cycles of planning, enactment, and reflection and the development of their pedagogical content knowledge for each problem. Levels of cognitive demand of the coded video data across the enactments of each problem indicated how each participant's ability to implement the problem changed over the 6 weeks. Comparing

participants' task dialogues and plans to their enactments allowed me to determine if their visions of leading mathematical discussions aligned with their practice.

Findings from this study highlighted two high leverage practices, coordinating pupil collaboration and eliciting complete justifications for solutions, that were also difficult for participants to enact. Moves that lowered cognitive demand replaced the pupils' reasoning with that of the participants and placed responsibility for learning with the preservice teacher. Moves that raised the cognitive demand relied on questioning that explored pupils' thinking and pressing for complete explanations. Extending the question most consistently elevated the cognitive demand.

This study provided evidence that participants developed pedagogical content knowledge for leading mathematics discussions. Six participants showed improvement in the levels of cognitive demand they were able to achieve with pupils, and this improvement could be traced to their collaborative work with peers, opportunities to refine responses to pupils, and collective reflection on their teaching. Two participants did not show growth in their ability to facilitate discussions that also sustained high levels of cognitive demand.

Participants' visions of leading mathematics discussions generally aligned with their enactments of those discussions, especially for the best and worst discussion leaders. Those participants whose visions of discussions and enactments did not align either put minimal effort into planning and did not use planning activities to inform their work with pupils or had limited conceptions of how pupils thought mathematically. One of the biggest struggles for all participants was anticipating pupil solutions and responding to solutions that they had not anticipated.

Though not all of the preservice teachers who participated showed improvement, this study showed that it is possible for preservice teachers to foster discourse with pupils in a problem-solving context. An analysis using math-talk and cognitive demand frameworks proved fruitful in assessing the quality of mathematics discussions. Task dialogues had a predictive power in establishing how preservice teachers would fare in leading discussions, and they may be useful in helping the preservice teachers anticipate solutions, which was one of their biggest struggles. Repeated enactments, collaboration with group members, and analytic reflection all helped to improve these preservice teachers' discussions with children.

Findings of this study revealed aspects of this teacher education experience that promoted preservice teacher learning, tools that were useful in assessing that learning, and areas that may need greater attention in teacher education. Two elements of the study's design promoted teacher learning: working in groups and repeated enactments. Enacting problems and facilitating mathematical discussion in groups was beneficial for the preservice teachers in several ways. Working in groups acted as a type of assisted performance (Feiman-Nemser, 2001; Mewborn & Stinson, 2007) that provided opportunities for preservice teacher learning that otherwise might have been missed. First, stronger discussion leaders, those who were able to sustain high cognitive demand, were able to model practice for participants who struggled in leading discussions, and weaker discussion leaders were thus able to adopt some of those practices. Second, even those participants who were not able to imitate the stronger discussion leaders were at least able to observe good mathematics discussions, something they would not have been able to do if working alone. Third, in a more direct show of assistance, the stronger discussion leaders were able to intervene in the work of their struggling fellow group members and pull them out of downward spirals. By virtue of being able to decipher pupil thinking, understand their fellow

participants' thinking, and know when and how to step in, participants who intervened to help fellow group members showed they developed sophisticated pedagogical content knowledge. Fourth, because multiple participants within a group had the same goal for pupil learning (e.g., explaining or working collaboratively), they were less likely to get derailed when working with pupils whose thinking they could not always predict.

Repeated enactments of the same problems with different pupils provided opportunities for participants to develop pedagogical content knowledge about the specific problems and, more generally, about leading mathematical discussions. Of the 6 participants who showed varying degrees of improvement, 4 participants' improvements can be traced to the repeated enactments. They were able to hypothesize responses to pupil's thinking, test those ideas, and then make revisions in subsequent enactments. Even the 2 participants who did not show overall improvement provided evidence of having developed pedagogical content knowledge as a result of repeated enactments. They learned to accurately assess their work with pupils and recognize areas in which improvements were needed and, they developed an understanding of the ways pupils approached problems and the connections among those approaches.

Several tools in this study were useful for assessing and supporting participants' learning. The math-talk framework (Hufferd-Ackles et al., 2004) proved a good starting point for assessing preservice teachers' abilities to facilitate mathematics discussions. The four components of the framework were appropriate dimensions along which to evaluate the participants' mathematical discussions. However, descriptions of some of the levels were too generic or included commentary about what the teacher expected, which could not be determined from data in this study. Originally created from observations of one teacher leading whole-class discussion, the framework did not accurately capture the small incremental changes made by

novice teachers or the different dynamic of working with pairs of pupils rather than whole classes, and thus I needed to add mid-levels. The revised framework was more helpful than the original version in seeing week-to-week changes in preservice teachers' discussions. Further, associating math-talk with cognitive demand (Stein et al., 2000) allowed for the identification of particular pedagogical moves that facilitated discussion and raised or maintained high cognitive demand. Combining these two frameworks gave a fine-grained picture of the preservice teachers' interactions with pupils.

Though I cannot claim that the task dialogue assignments led to improvement in teachers' work, there were some associations between improved enactments and the development of more student-centered visions of mathematical discussions. Task dialogues often matched participants' enactments. Because of the high degree of alignment between how participants envisioned discussions and how they actually carried them out, task dialogues may be a useful tool for predicting preservice teachers' skill at facilitating discussions before they interact with pupils. In general, the reflections proved more helpful to the participants' mathematical discussions than did their planning. In their reflections, they addressed broad pedagogical dilemmas or issues tied to specific problems. Some of the preservice teachers with weak plans had rich analytical reflections in which they discussed pedagogical dilemmas that were not always apparent in their plans and task dialogues. Reflections may have been a more fruitful ground than planning for considering teaching practice because group members had to reflect on each other's work and reflections, whereas group members planned individually and were never asked to do any co-planning. Some participants' reflections were more analytical when responding to fellow group members' ideas than when they wrote about their individual work with pupils.

This study serves an existence proof that preservice teachers can learn to lead mathematics discussions that achieve high levels of cognitive demand. However, it also points to areas of preservice teachers' work with children that need more consideration. The participants struggled most in attending to incorrect and correct pupil thinking; whereas they had better responses when with faced with nearly correct ideas or pupils who stalled out in problem solving. Low-cognitive-demand moves were used more often than high-cognitive-demand moves in response to incorrect thinking. At the beginning of the study, most of the participants treated correct solutions as the final product of problem-solving activity; justification or generalization of correct strategies was not sought. The widely varying cognitive demand associated with attending pupils to one another's thinking and eliciting complete explanations signifies that these moves have the potential to develop high cognitive demand, but that implementing them effectively is challenging for preservice teachers. All participants displayed difficulty in hypothesizing pupils' potential solution strategies.

### **Implications for Teacher Education**

This study has several implications for preservice teacher education. First, it gives a glimpse of what pedagogical moves have the best results in terms of sustaining a high cognitive demand (and hence providing the best opportunities for pupil learning) that are also the most difficult for preservice teachers to implement well (those that resulted in varying levels of cognitive demand). Therefore, this study points to some high leverage practices that can create powerful opportunities for pupil learning but that are challenging for preservice teachers to implement: eliciting high-quality mathematical explanations and coordinating meaningful collaboration among pupils. These two practices are in need of increased attention in teacher education. This study highlights types of pupil thinking that preservice teachers struggled to

respond to and gives examples of the problematic aspects of those responses. Just as the preservice teachers were better able to respond to pupil thinking they had anticipated, teacher educators will be better equipped to manage and head off some of the more egregious missteps of preservice teachers in leading discussions. For example, teacher educators can attend to specific moves that lower cognitive demand before preservice teachers work with pupils and provide methods for preservice teachers to assess themselves in carrying out those moves.

Second, working in teams was beneficial for preservice teacher learning. The participants whose discussions maintained high cognitive demand modeled practice that was at times taken up by participants who led teacher-centered discussions, though they were often unable to maintain it independently for long stretches. Focusing their analysis on instances where other group members intervened to elevate cognitive demand might benefit struggling preservice teachers more than reflecting on an entire session of working with children. This focus could help them isolate effective pedagogical moves and make it more likely that they would be able to implement them independently. Observing struggles of their other group members allowed participants to better articulate and implement good pedagogical moves. Because reflecting and teaching together as a group was helpful but individual planning activities were less so, co-planning in addition to co-reflecting and co-teaching should be incorporated in the planning, enacting, and reflecting cycle. Video records were essential in helping the participants accurately assess the quality of interactions with pupils. The ability to capture their own and the pupils' exact words allowed many of the participants to notice instances where they explained their ideas instead of a pupil's, asked directive questions, or failed to give pupils adequate wait time.

Though some participants had more well-developed task dialogues than others, all of the participants struggled in anticipating potential pupil solutions, and they tended to be the most

directive when responding to an unanticipated solution. Thus, targeted work on developing pupils' solution strategies and responding to unanticipated solutions is needed, and task dialogues could be an appropriate tool for forcing preservice teachers to think about children's solutions. Because developing collaboration among pupils and eliciting a complete explanation can have such variable, yet potentially positive, powerful effects on cognitive demand, preservice teachers need deliberate practice in these two areas. Explicit instruction and practice in assessing their own work with pupils using the math-talk and cognitive demand frameworks may better help preservice teachers identify effective and ineffective ways to intervene on and extend pupils' problem-solving activity.

### **Future Research**

Results of this study indicate two primary avenues for further research: What led to these participants' results, and how will they fare in future teaching endeavors? It would be of interest to determine what might account for the different growth of the four categories of participants. Some combination of beliefs, content knowledge, comfort level with pupils, and confidence in mathematics may play into the quality of the mathematical discussions a preservice teacher is able to facilitate. Tracking struggles and successes of preservice teachers from a variety of different backgrounds in these areas could highlight the factors that are most influential in preservice teachers' initial attempts at and growth in facilitating mathematics discussions.

Because it is likely that content knowledge plays a role in leading mathematics discussions and different grade levels of teachers have different levels of content expertise, a comparison of elementary, middle, and secondary preservice teachers' skill in leading mathematics discussions could provide insights into both the influence of content knowledge and the specific supports needed by different levels of preservice teachers. That is, it would be

helpful to determine how the learning experiences provided by teacher education should be differentiated for preservice teachers with strong versus weak content knowledge.

Finally, it would be useful to know how preservice teachers lead mathematics discussions and implement problem-solving tasks in different settings and at different points in their development towards becoming an experienced teacher. Given more time, do inconsistent discussion leaders ultimately show improvement? Do those with only small improvement continue improving, or do they regress to more directive tactics? Are the most successful discussion leaders able to maintain that momentum in a whole-class setting or into student teaching and the beginning of their careers?

### **Concluding Remarks**

Practice-based teacher education can provide settings where the theory of preservice teachers' university training has a greater chance of being translated to teachers' interactions with pupils (Ball & Cohen, 1999; Ball & Forzani, 2009; Darling-Hammond, 2006; Feiman-Nemser, 2001; Lampert & Ball, 1999). This study provides an example of what practice-based teacher education might look like in a mathematics methods course for early childhood education majors. The results of this study show that preservice teachers can, in fact, learn a great deal from thoughtfully designed, authentic forays into the practice of teaching. However, this study also illuminates some of the complexities of supporting preservice teachers in a practice-based setting. The literature makes clear that practice-based teacher education does not consist simply of sending preservice teachers into schools but rather necessitates intensive and interactive involvement of the teacher educator to structure opportunities to interact with pupils in ways that allow for repeated practice of pedagogical strategies and to provide feedback as the preservice teachers are engaged in practice. This study highlights the nature of the challenges faced by

preservice teachers while engaged in the practice of teaching and identifies ways in which teacher educators might provide intervention and support.

## REFERENCES

- Adler, J. (1999). The dilemma of transparency: Seeing and seeing through talk in the mathematics classroom. *Journal for Research in Mathematics Education*, 30, 47–64.
- Ball, D. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90, 449–466.
- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93, 373–397.
- Ball, D. (2003, February). *What mathematical knowledge is needed for teaching mathematics?* Paper presented at the U.S. Department of Education, Secretary's Mathematics Summit, Washington, DC.
- Ball, D. L., & Bass, H. (2000). Interweaving Content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.
- Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco, CA: Jossey Bass.
- Ball, D. L., & Forzani, F.M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, 60, 497–511.
- Ball, D. L., Hill, H. C, & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14–17, 20–22, 43–46.
- Ball, D., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). New York, NY: Macmillan.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389–407.
- Bauersfeld, H. (1995). "Language games" in the mathematics classroom: Their functions and effects. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 271–291). Hillsdale, NJ: Erlbaum.

- Baxter, J., & Williams, S. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. *Elementary School Journal*, 97, 21–38.
- Begle, E. G. (1972) *Teacher knowledge and student achievement in algebra* (SMSG Reports, No. 9). Stanford, CA: Stanford University, School Mathematics Study Group.
- Blanton, M. L., Berenson, S. B., & Norwood, K. S. (2001). Using classroom discourse to understand a prospective mathematics teacher's developing practice. *Teaching and Teacher Education*, 17, 227–242.
- Boerst, T., Ball, D. L., Blunk, M., & Dray, A. (2010). *Toward a description of the mathematical work of leading a discussion*. Paper presented at National Council of Teachers of Mathematics Annual Research Pre-session Conference, San Diego, CA.
- Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, 23, 194–222.
- Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209–239). New York, NY: Macmillan.
- Buschman, L. (2004). Teaching problem solving in mathematics. *Teaching Children Mathematics*, 10, 302–309.
- Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, 58, 45–75.
- Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, 24, 428–441.
- Chapin, S. H., O'Connor, C., & Anderson, N. C. (2003). *Classroom discussions using math talk to help students learn, grades 1–6*. Sausalito, CA: Math Solutions.
- Charles, R. I., & Lester, F. K. (1984). An evaluation of a process-oriented instructional program in mathematical problem solving in Grades 5 and 7. *Journal for Research in Mathematics Education*, 15, 15–34.
- Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics*, 19(2), 2–10.
- Cobb, P., & Yackel, E. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27, 458–477.
- Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis*, 12, 311–329.

- Cohen, D. K., & Ball, D. L. (2001). Making change: Instruction and its improvement. *Phi Delta Kappan*, 83, 73–77.
- National Governors' Association and Council of Chief State School Officers. (2010). *Common core state standards for mathematics*.. Retrieved from [http://www.corestandards.org/assets/CCSSI\\_Math%20Standards.pdf](http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf)
- Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Washington, DC: Author.
- Cooney, T. J. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education*, 16, 324–336.
- Crespo, S., Oslund, J., & Parks, A. (2011). Imagining mathematics teaching practice: Prospective teachers generate representations of a class discussion. *ZDM: The International Journal on Mathematics Education*, 43, 119–131.
- Darling-Hammond, L. (2000). How teacher education matters. *Journal of Teacher Education*, 51, 166–173.
- Darling-Hammond, L. (2006). Constructing 21st-century teacher education. *Journal of Teacher Education*, 57, 300–314.
- Ebby, C. B. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice. *Journal of Mathematics Teacher Education*, 3, 69–97.
- Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24, 8–40.
- Elmore, R. F. (2008). *School reform from the inside out: Policy, practice, and performance*. Cambridge, MA: Harvard Education Press.
- Feiman-Nemser, S. (2001). From preparation to practice: Designing a continuum to strengthen and sustain teaching. *Teachers College Record*, 103, 1013–1055.
- Franke, M., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Reston, VA: National Council of Teachers of Mathematics.
- Ge, X., & Land, S. M. (2004). A conceptual framework for scaffolding ill-structured problem-solving processes using question prompts and peer interactions. *Educational Technology Research and Development*, 52, 5–22.
- Ghousseini, H. N. (2008). *Learning with routines: Preservice teachers learning to lead classroom mathematics discussions*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor.

- Heaton, R. M., & Lampert, M. (1993). Learning to hear new voices: Inventing a new pedagogy of teacher education. In M. McLaughlin, D. K. Cohen, & J. E. Talbert (Eds.), *Teaching for understanding: Challenges for policy and practice* (pp. 43–83). San Francisco, CA: Jossey Bass.
- Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28, 524–549.
- Hiebert, J., & Stigler, J. W. (2000). A proposal for improving classroom teaching: Lessons from the TIMSS Video Study. *Elementary School Journal*, 101, 3–20.
- Hufferd-Ackles, K., Fuson, K., & Sherin, M. G. (2004). Describing levels and components of a math-talk community. *Journal for Research in Mathematics Education*, 35, 81–116.
- Inoue, N., & Buczynski, S. (2011). You asked open-ended questions, and now what?: Understanding the nature of stumbling blocks in teaching inquiry lessons. *Mathematics Educator*, 20, 10–23.
- Jacobs, V. R., & Philipp, R. A. (2010). Supporting children's problem solving. *Teaching Children Mathematics*, 17, 98–105.
- Kazemi, E., Beasley, H., Chan, A. G., Franke, M., Ghouseini, H., & Lampert, M. (April, 2010). *Learning ambitious teaching through cycles of investigation and enactment*. Presented at the National Council of Teachers of Mathematics (NCTM) Annual Research Pre-session Conference, San Diego, CA.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal*, 102, 59–80.
- Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics*, 46, 187–228.
- Knuth, E., & Peressini, D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School*, 6, 320–325.
- Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 229–269). Hillsdale, NJ: Erlbaum.
- Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal for Research in Mathematics Education*, 26, 60–82.
- Lakatos, I. (1976). *Proofs and refutation: The logic of mathematical discovery*. New York, NY: Cambridge University Press.

- Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review*, 55, 178–194.
- Lampert, M. (1988). What can research on teacher education tell us about improving quality in mathematics education? *Teaching and Teacher Education*, 4, 157–170.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27, 29–63.
- Lampert, M., & Ball, D. (1999). Aligning teacher education with contemporary K–12 reform visions. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of teaching and policy*. New York, NY: Jossey-Bass.
- Lampert, M., Beasley, H., Ghouseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein, & L. Kucan (Eds.), *Instructional explanations in the disciplines*, (pp. 129–141). New York, NY: Springer.
- Lampert, M., & Cobb, P. (2003) Communication and learning in the mathematics classroom. In J. Kilpatrick & D. Shifter, (Eds.), *Research companion to Principles and Standards for School Mathematics* (pp. 237–249). Reston, VA: National Council of Teachers of Mathematics.
- Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *Elementary School Journal*, 109, 491–509.
- Lortie, D. C. (1975). *Schoolteacher: A sociological study*. Chicago, IL: University of Chicago Press.
- Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class. *Journal for Research in Mathematics Education*, 31, 454–482.
- Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.
- Markovits, Z., & Even, R. (1999). The decimal point situation: A close look at the use of mathematics-classroom-situations in teacher education. *Teaching and Teacher Education*, 15, 653–665.
- Mewborn, D. (2001). Teachers' content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Education Research Journal*, 3, 28–36.
- Mewborn, D. S., & Stinson, D. W. (2007). Teacher education as assisted performance. *Teachers College Record*, 109, 1457–1487.
- Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review*, 13, 125–145.

- Nathan, M. J., & Knuth, E. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*, 175–207.
- National Center for Education Statistics (2011). *The Nation's Report Card: Mathematics 2011* (NCES 2012–458). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education, Washington, D.C.
- National Commission on Excellence in Education. (1983). *A nation at risk: The imperative for educational reform*. Washington, DC: Government Printing Office. Retrieved from <http://www.ed.gov/pubs/NatAtRisk/risk.html>
- National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.
- National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, DC: National Academy Press.
- National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study Committee. Washington, DC: National Academies Press.
- Nicol, C. (1999). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics, 37*, 45–66.
- No Child Left Behind (NCLB) Act of 2001, 20 U.S.C.A. § 6301 *et seq.* (West 2003).
- Paley, V. G. (1986). On listening to what the children say. *Harvard Educational Review, 56*, 122–131.
- O'Connor, M. C. (2001). “Can any fraction be turned into a decimal?” A case study of a mathematical group discussion. *Educational Studies in Mathematics, 46*, 143–185.
- Peressini, D., & Knuth, E. (1998). Why are you talking when you could be listening? The role of discourse in the professional development of mathematics teachers. *Teaching and Teacher Education, 14*, 107–125.

- Peterson, P. L., Carpenter, T., & Fennema, E. (1989). Teachers' knowledge of students' knowledge in mathematics problem solving: Correlation and case analyses. *Journal of Educational Psychology, 81*, 558–569.
- Pirie, S. E. B., & Schwarzenberger, R. L. E. (1988). Mathematical discussion and mathematical understanding. *Educational Studies in Mathematics, 19*, 459–470.
- Polya, G. (1957) *How to solve it: A new aspect of mathematical problem solving* (2nd ed.). Princeton, NJ: Princeton University Press.
- Schettino, C. (2003). Transitioning to a problem solving curriculum. *Mathematics Teacher, 96*, 534–537.
- Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of “well-taught” mathematics courses. *Educational Psychologist, 23*, 145–166.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.
- Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*, 253–286.
- Sfard, A. (2000). Symbolizing mathematical reality into being—Or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 37–98). Mahwah, NJ: Erlbaum.
- Sfard, A. (2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the Twenty-First Conference of the North American Chapter of the International Group of Psychology for Mathematics Education* (pp. 23–44). Columbus, Ohio: Clearing House for Science, Mathematics, and Environmental Education.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.
- Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*, 344–350.
- Spillane, J. P. (2004). *Standards deviation: How schools misunderstand education policy*. Cambridge, MA: Harvard University Press.
- Staples, M., & Colonis, M. M. (2007). Making the most of mathematical discussions. *Mathematics Teacher, 101*, 257–261.
- Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455–488.

- Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.
- TIMSS Video Mathematics Research Group. (2003). Understanding and improving mathematics teaching: Highlights from the TIMSS 1999 Video Study. *Phi Delta Kappan*, 85, 768–775.
- Truxaw, M. P. (2004). Mediating mathematical meaning through discourse: An investigation of discursive practices of middle grades mathematics teachers. (Doctoral dissertation, University of Connecticut, 2004). *Dissertation Abstracts International*, 65(08), 2888B.
- Truxaw, M. P., & DeFranco, T. C. (2008). Mapping mathematics classroom discourse and its implications for models of teaching. *Journal for Research in Mathematics Education*, 39, 489–525.
- Tyack, D., & Cuban, L. (1995). *Tinkering toward utopia: A century of public school reform*. Cambridge, MA: Harvard University Press.
- U.S. Department of Education. (1992). *International Mathematics and Science Assessments: What Have We Learned? Executive Summary*. Retrieved from <http://nces.ed.gov/pubs92/web/92011.asp>
- U.S. Department of Education. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Retrieved from <http://www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>
- Van Oers, B. (1996). Learning mathematics as meaningful activity. In L. Steffe, P. Nesher, P. Cobb., G. Goldin., & B. Greer (Eds.), *Theories of mathematical learning* (pp. 91-114). Hillsdale, NJ: Erlbaum.
- Villaseñor, A., & Kepner, H. S. (1993). Arithmetic from a problem-solving perspective: An urban implementation. *Journal for Research in Mathematics Education*, 24, 62–69.
- Warfield, J., Wood, T., & Lehman, J. D. (2005). Autonomy, beliefs and the learning of elementary mathematics teachers. *Teaching and Teacher Education*, 21, 439–456.
- Webb, N. L. (1979). Processes, conceptual knowledge, and mathematical problem-solving ability. *Journal for Research in Mathematics Education*, 10, 83–93.
- Wideen, M., Mayer-Smith, J., & Moon, B. (1998). A critical analysis of the research on learning to teach: Making the case for an ecological perspective on inquiry. *Review of Educational Research*, 68, 130–178.
- Wilburne, J. M. (2006). Preparing preservice elementary school teachers to teach problem solving. *Teaching Children Mathematics*, 12, 454–463.

- Wilson, S. M., Floden, R. E., & Ferrini-Mundy, J. (2002). Teacher preparation research: An insider's view from the outside. *Journal of Teacher Education*, 53, 190–204.
- Wood, T., Cobb, P., & Yackel, E. (1993). The nature of whole class discussion. *Journal for Research in Mathematics Education Monographs*, 6, 55–68.
- Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education*, 30, 171–191.
- Zeichner, K., & Tabachnick, B. R. (1981). Are the effects of university teacher education washed out by school experiences? *Journal of Teacher Education*, 32, 7–11.
- Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college- and university-based teacher education, *Journal of Teacher Education*, 61, 89–99.

## APPENDIX A

**Problem-Solving Assignment**

**Guidelines:** Problem-solving responses must include all of the following. Include as much detail as possible. The most important components of the assignment are your description of how you solved the problem and identifying the mathematics underlying the problems. Finding the solution is only part of the problem-solving process. Be creative in your thinking. As a future teacher, you should realize that often more than one way can be used to solve many problems. You will need to encourage this realization whenever possible to promote creative thinking in your classroom.

**A. Understand the problem:** Explain the problem in your own words. Doing so will help you be sure that you know what is being asked and what you need to find. Describe what information is given, what the problem is asking you to find, and what information is needed. Identify hidden or important information.

**B. Solve the problem:** Solve the problem describing the process. Identify your solution, and justify the mathematics. Describe and justify your thought processes while solving the problem. Use multiple strategies to solve the problem where possible.

**C. Explain the mathematics:** What mathematical concepts, skills, or ideas did you use to solve this problem? If you used multiple strategies, what were the advantages of each strategy? Which strategy was the most efficient or sophisticated? How are the strategies connected?

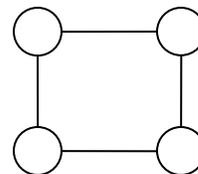
**D. Extend the problem:** This step is important and will be valuable to you as a future teacher. Create one new problem using this problem as the basis. It should require students to use higher-level thinking and/or require them to explore mathematical connections with the original problem. You do not have to solve the problem.

**Problems:**

1. In a soccer championship there are 6 teams. If all teams are going to play each other, how many games will there be in the championship?
2. List all the ways 10 pennies can be put into 3 piles so that each pile has a different number of pennies.
3. There is a jar of cookies on the counter. Allyson skipped lunch so she was hungry and ate half the jar of cookies. Then Melissa came by and took a third of the cookies. Later on, Alicia took a fourth of the cookies to have as snack on her way to class. Then, Jennifer

ate two cookies. At the end of the day, when Megan saw the cookie jar, there was only one cookie left. How many cookies were in the jar to begin with?

4. There are some bicycles and tricycles at the playground. There are 27 wheels and 12 cycles. How many are bicycles and how many are tricycles?
5. 20 men need to guard the castle pictured. For the castle to be safe, there must be 7 men guarding each wall. The men on the towers count as guards for both walls that connect to the tower. How would you place the 20 men?



**Rubric**  
(Wilburn, 2006)

	<b>Emerging (0.5)</b>	<b>Developing (1.0)</b>	<b>Proficient (1.5)</b>	<b>Exemplary (2.0)</b>
<p><b>Understanding the problem</b> Does the student's interpretation of the problem reflect the key issues?</p>	Incorrect interpretation of the problem. Used wrong info to solve the problem.	Interpretation of the problem was mostly correct. Some but not all relevant info was used.	Mathematical interpretation was correct. Appropriate info used.	Interpretation of the problem was correct. Identified hidden or important info not readily apparent.
<p><b>Solving the problem</b> Does the student identify and use multiple strategies and follow a logical and verifiable process toward a solution? Given the approach taken, is the solution performed in an accurate and complete manner?</p>	Strategies were not appropriate. Student did not seem to know where to begin. Reasoning did not support work. Errors in computation were serious enough to flaw solution. Mathematical representations were incorrect or labeled incorrectly. Solution was incorrect. No evidence provided to show how answer was obtained.	Attempted to use an appropriate strategy and/or offered little explanation of your strategy. Minor errors in computation. Representations were essentially correct but not accurately or completely labeled. Inefficient choice of procedures impeded success. The evidence for solution was inconsistent or unclear.	Used an appropriate, efficient strategy to solve the problem and justified each step of work. Representation(s) fit the task and were complete and accurate. Essentially accurate computations. Essentially correct solution. Work clearly supported your solution. Evidence for solution was clear and consistent.	Used insightful strategies for solving the problem and justified all steps. Representation(s) and diagrams were well organized and detailed. Completely accurate computations. Used multiple representations for verifying solution. Showed multiple ways to compute answer. Evidence for solution was extremely clear and consistent.
<p><b>Explaining the mathematics</b> Does the student clearly explain the mathematical concepts behind the problem?</p>	Incorrectly identified mathematical ideas underlying the problem.	Identified correct but insignificant mathematical ideas underlying the problem.	Identified some of the ideas underlying the problem but explained these in a confusing way.	Identified all key mathematical ideas underlying the problem and explained these in a clear articulate way.

<p><b>Extending the problem</b> Does the student create a higher-level thinking problem that uses the same concepts or extend the problem to a generalizable strategy?</p>	<p>Extension changed the problem set up in a way that changed the mathematical concepts addressed or no extension to general case was made.</p>	<p>Extension simply replaced numbers or ideas from the given problem or failed to make extension to general case. Very little creative and critical thinking was evident.</p>	<p>Extension showed some creativity in requiring the use of higher-level thinking requiring connections with previously learned mathematics or attempted to extend problem to a general case.</p>	<p>Extension was creative and required the use of higher-level thinking/connections with previously learned mathematics with more depth. Or successfully extended problem to general case.</p>
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## APPENDIX B

**Problem-Solving Interview****Interviewer copy**

Interviewer: *Look at these problems and pick two of them to solve. They may not form the same category. You probably won't know the answer right away. You may have to think about these for a little while. You can use any of the tools available to help you. You can ask questions to clarify anything you don't understand. You can also ask for hints if you get stuck. While you are solving them try to say out loud what you are thinking. I will ask you questions while you are working about what you are doing or thinking.*

*While student is solving researcher asks probing questions:*

*Why are you trying that strategy?*

*What types of diagrams/pictures/charts can you use?*

*How can you organize your work?*

*What if I changed this number?*

*I saw another student do x, how does that compare to your strategy?*

*What are the important pieces of information in this problem?*

*Can you explain what the problem is asking you to find in your own words?*

*How do you know your solution is correct?*

*How do you know your strategy works?*

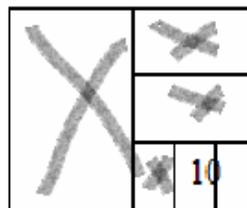
*Specific probing questions for each problem follow in italics.*

1. *Generalizing and Explaining Patterns:* In a soccer championship there are 6 teams. If all teams are going to play each other, how many games will there be in the championship?
  - a. *What if there were 8 teams? How would you find the answer for 10 teams? 100 teams?*
  
2. *Generalizing and Explaining Patterns:* Move three toothpicks from the arrangement below so as to form exactly 3 squares, all the same size.
  - a. *Generalize the pattern.*



3. *Making an Organized List:* How many different ways can I make 25 cents?
  - a. *How do you know you found them all?*

4. *Working Backwards*: Sue baked some cookies. She ate 1 of them. She shared the rest of the cookies evenly between her sister and herself. She ate two more cookies from her half, and had two cookies left. How many cookies did Sue bake?
- Why did you decide to use this strategy? What other strategies might be possible? (picture, algebra/fractions, guess and check)*
  - If student used guess and check, how did you decide which numbers to guess?*
  - How would solve this problem by working it backwards?*
  - If picture not used, can you explain how someone could come up with a picture?*
5. *Working Backwards*: Your school is having a car wash and car washers are in four teams. Team #1 washed one-half of all the cars that came. Team #2 washed two-thirds of the cars that were left. Team #3 washed one-third of the cars that Team #2 left. Finally, Team #4 washed the last 10 cars. How many cars did the four teams wash?
- Why did you decide to use this strategy? What other strategies might be possible? (picture, algebra/fractions, guess and check)*
  - If student used guess and check, how did you decide which numbers to guess?*
  - How would solve this problem by working it backwards?*
  - If picture not used, can you explain how someone could come up with this picture?*

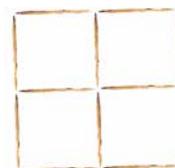


6. *Reasoning Algebraically*: In a field there are some cows and some chickens. I know there are 10 animals. The total number of feet is 32. How many cows and how many chickens are there in the field?
- How does your solution compare to drawing ten bodies with two legs [four legs] each and then adding [taking away] 2 legs to [from] each body until you have 32?*
7. *Reasoning Algebraically*: Nadia has 3 more jelly beans than Brian. All together, Nadia and Brian have 15 jelly beans. How many jelly beans does Brian have? How many jelly beans does Nadia have?
- How do you know there is only one solution?*
  - What makes this problem challenging?*
8. *Reasoning Deductively*: Five women are seated around a circular table. Mrs. Osborne is sitting between Mrs. Lewis and Mrs. Martin. Ellen is sitting between Cathy and Mrs. Norris. Mrs. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Mrs. Parks on her left and Mrs. Martin on her right. Match the ladies' first names and last names.
- What clue was most helpful?*
  - Why did you use clues in that order?*

## Student copy

Pick two problems from different categories.

1. *Generalizing and Explaining Patterns*: In a soccer championship there are 6 teams. If all teams are going to play each other, how many games will there be in the championship?
2. *Generalizing and Explaining Patterns*: Move three toothpicks from the arrangement below so as to form exactly 3 squares, all the same size.



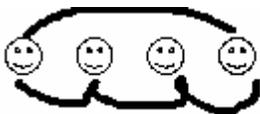
3. *Making an Organized List*: How many different ways can I make 25 cents?
4. *Working Backwards*: Sue baked some cookies. She ate 1 of them. She shared the rest of the cookies evenly between her sister and herself. She ate two more cookies from her half, and had two cookies left. How many cookies did Sue bake?
5. *Working Backwards*: Your school is having a car wash and car washers are in four teams. Team #1 washed one-half of all the cars that came. Team #2 washed two-thirds of the cars that were left. Team #3 washed one-third of the cars that Team #2 left. Finally, Team #4 washed the last 10 cars. How many cars did the four teams wash?
6. *Reasoning Algebraically*: In a field there are some cows and some chickens. I know there are 10 animals. The total number of feet is 32. How many cows and how many chickens are there in the field?
7. *Reasoning Algebraically*: Nadia has 3 more jelly beans than Brian. All together, Nadia and Brian have 15 jelly beans. How many jelly beans does Brian have? How many jelly beans does Nadia have?
8. *Reasoning Deductively*: Five women are seated around a circular table. Mrs. Osborne is sitting between Mrs. Lewis and Mrs. Martin. Ellen is sitting between Cathy and Mrs. Norris. Mrs. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Mrs. Parks on her left and Mrs. Martin on her right. Match the ladies' first names and last names.

## APPENDIX C

## Task Dialogue Assignments

*Task Dialogue Instructions*

Given the student responses to the following problem, suggest a student-teacher conversation that might follow. Consider both the role of the teacher and the student. What hints/questions can get students started, help them see the flaws in their reasoning, and expand their thinking? What questions help them find a more sophisticated strategy or generalize their strategy to other problems? How might students respond, and modify or extend their reasoning?

<b>Problem Set #1: Generalize and explain patterns</b>	
<p><b>Problem 1:</b> Your class made telephones out of strings and juice cans. Each group of students has to work together to make a phone club that connects every person to every other person. If a group had four people, how many strings would be needed to connect every member of the group to every other member of the group? What if you used 28 strings? How many people would be in a group?</p>	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<p><b>S:</b> Student draws and counts 4 strings.</p>  <p><b>T:</b></p>	
<b>Dialogue 2</b>	<b>Rationale For Teacher Moves</b>
<p><b>S:</b> Student draws and counts 6</p>  <p>Then draws and continues...until</p>	



He counts 26 lines and adds one more person:



He says 13 people are needed for 28 strings.

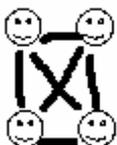
**T:**

### Dialogue 3

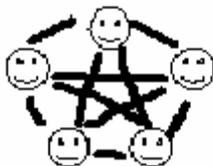
### Rationale For Teacher Moves

**S:**

Student draws and counts 6 strings for 4 people.



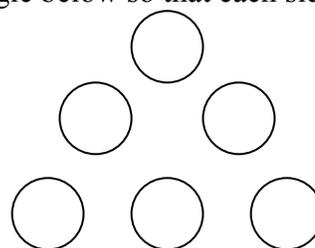
Then draws and after counting the strings says that is not enough. He needs 6 people. But gets confused drawing the lines



for 6 people.

**T:**

**Problem 2:** Can you put the numbers 1-6 in the triangle below so that each side adds up to the same amount?



**Dialogue 1**

**S:** Student is moving around number and is using fingers to add up the sides. After several minutes of trying she has

3  
2 4  
6 1 5

and says that is the closest she can get the two sides.

**T:**

**Rationale For Teacher Moves**

**Dialogue 2**

**S:** Student uses some logical reasoning to “spread out” the larger numbers and gets the solution

1  
6 5  
2 4 3

and is finished.

**T:**

**Rationale For Teacher Moves**

**Dialogue 3**

**S:** Student finds the solution above and then “reverses” the solution to find

6  
1 2  
5 3 4

**T:**

**Rationale For Teacher Moves**

Dialogue 4	Rationale For Teacher Moves
<p><b>S:</b> After finding the two solutions above and prompted to find another, the student says, “There can’t be anymore because all the big numbers would be together.</p> $\begin{array}{c} 6 \\ 1 \quad 2 \\ 5 \quad 3 \quad 4 \end{array}$ <p><b>T:</b></p>	

<b>Problem Set #2: Make an organized list</b>	
<b>Problem 1:</b> Place 12 pennies in 3 piles with no two piles having the same number of pennies.	
Dialogue 1	Rationale for Teacher Moves
<p><b>S:</b> Student randomly puts 12 pennies into 3 piles until she finds 6, 4, 2. Then she moves a penny from the 4 pile to the 2 pile and has 6, 3, 3. “That’s not right.” She move the pennies back to 6, 4, 2 and moves one from the 6 pile to the 4 pile. “I keep making them with the same number.”</p> <p><b>T:</b></p>	
Dialogue 2	Rationale For Teacher Moves
<p><b>S:</b> Student finds 6, 5, 1, and 6, 4, 2 and 5, 4, 3 by rearranging one penny at a time, then gets stuck.</p> <p><b>T:</b></p>	
Dialogue 3	Rationale For Teacher Moves
<p><b>S:</b> Student has found 5 of the solutions (651, 642, 543, 741, 732) and after several more attempts claims these are the only solutions.</p> <p><b>T:</b></p>	

<b>Problem 2:</b> How many times in a 12 hour period does the sum of the digits on a digital clock equal 6?	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Things that add to six are 6 and 0, 1 and 5, 2 and 4, and 3 and 3. So to make times I get 6:00, 3:30, 4:20, 5:10.  <b>T:</b>	
<b>Dialogue 2</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> So 6:00, 3:30, switching the threes doesn't change the time, 4:20, 2:40, I can't have 0 o'clock 5:10, 1:50.  <b>T:</b>	
<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> So 6:00, if I put 6 anywhere else, it doesn't make a time. 3:30, 3:03, I can't put 0 in the hour place. 4:20, 4:02 2:40, 2:04. 5:10, 5:01, 1:50, 1:05.  <b>T:</b>	
<b>Dialogue 4</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> Student has all the times with 0's (6:00, 4:20, 4:02 2:40, 2:04. 5:10, 5:01, 1:50, 1:05). Then 4:11, 3:21, 2:22. "Oh and changing them around. If I rearrange these too that should be all."  <b>T:</b>	

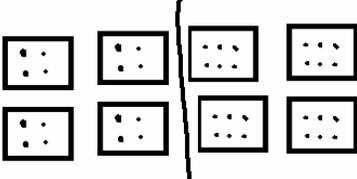
**Problem Set #3: Working Backwards**

**Problem 1:** Mary has some crayons. Doug had 3 times as many as Mary. But Doug but gave 4 to the teacher and now John has 2 more crayons that Doug. John has 7 crayons, how many does Mary have?

<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> How many crayons does Mary have? (doesn't understand what "some	

crayons” means)  <b>T:</b>	
<b>Dialogue 2</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> If Mary has 10, then Doug has 30. He gave away 4 is 26. John has two more that is 28. But he is supposed to 7. That’s way too many.  <b>T:</b>	
<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> John has 7 and that is two more than Doug, so Doug has 9. He gave four to the teacher so that is 5. Mary has 3 times as many so that is 15.  <b>T:</b>	
<b>Problem 2:</b> The pet store advertised that they had lots of new puppies on Monday. The owner took 1 puppy for his son. Then on Tuesday he sold half of the rest of the puppies to a farmer with lots of land. On Wednesday a mom took a half of the puppies that were left for her children. When you got to the pet store on Thursday there were only 4 puppies left to choose from. How many puppies were there on Monday?	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> 100 is lots of puppies.  <b>T:</b>	
<b>Dialogue 2</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> Student uses random guess and check beginning with 11 puppies (take away one is 10, half is 5, half is 2.5) then moving to 21 puppies (take away one is 20, half is 10, half is 5).  <b>T:</b>	
<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> If there were 10 puppies, and the owner took one that is nine. Half of	

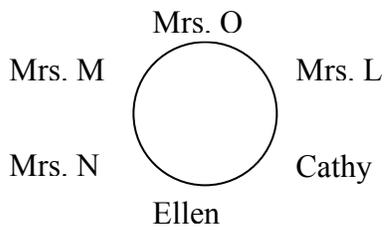
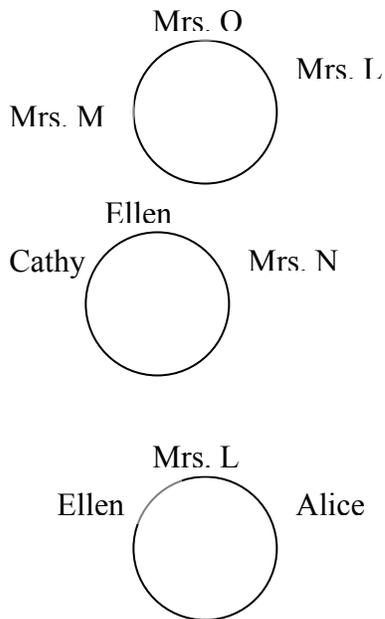
<p>nine is...you can't half 9. It has to be an odd number. 9 puppies take away one for the owner is 8, then half is 4. There were 9.</p> <p><b>T:</b></p>	
<b>Dialogue 4</b>	<b>Rationale For Teacher Moves</b>
<p><b>S:</b> 4 is half so <math>4+4</math> is 8, then half were gone again, so <math>8+</math> another 4 is 12 and one was taken away is 13.</p> <p><b>T:</b></p>	

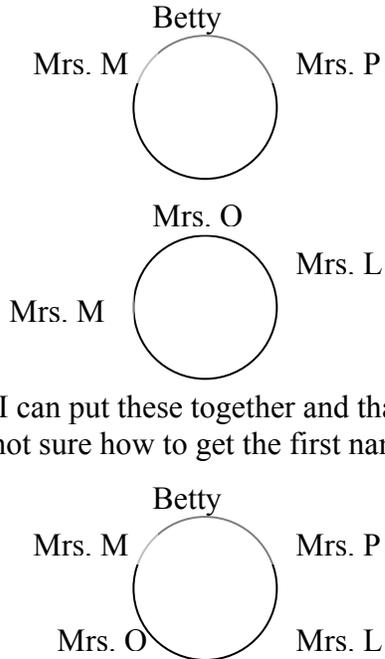
<b>Problem Set #4: Reasoning algebraically</b>	
<p><b>Problem 1:</b> A baker makes chocolate and vanilla cupcakes. He packages the vanilla ones in boxes of 4 and the chocolate ones in boxes of 6. He made 38 cupcakes and used 8 boxes. How many boxes of vanilla and how many boxes of chocolate did he make?</p>	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<p><b>S:</b> Student draws or represents 38 cupcakes. He groups vanilla into groups of 4 and has 8 groups of 4. He makes one group of 6 with remaining cupcakes. He counts the boxes and says 8 vanilla and 1 chocolate.</p> <p><b>T:</b></p>	
<b>Dialogue 2</b>	<b>Rationale For Teacher Moves</b>
<p><b>S:</b> Student draws 8 boxes. He draws a line down the middle to make 2 groups of 4 boxes each. He puts four dots in one group of 4 and 6 dots in the other group of 4. Then he says 4 vanilla and 4 chocolate.</p>  <p><b>T:</b></p>	

<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> Students draws the same picture as above but counts the dots and says he's got too many cupcakes.  <b>T:</b>	
<b>Problem 2:</b> Amy and Judy sold 19 play tickets altogether. Amy sold 5 more tickets than Judy. How many tickets did each girl sell?	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Amy has 5. So Amy has 19, 20, 21, 22, 23, 24.  <b>T:</b>	
<b>Dialogue 2</b>	<b>Rationale for Teacher Moves</b>
<b>S:</b> Amy has 10 and Judy has 9.  <b>T:</b>	
<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> Amy has 10 and Judy has 9. Amy only has one more than Judy. So Amy needs 4 more; that is 14 (taking 4 from Judy and giving them to Amy). Then Judy has 5. But then Amy has 9 more than Judy.  <b>T:</b>	
<b>Dialogue 4</b>	<b>Rationale For Teacher Moves</b>
<b>S:</b> To get 19, we have 10+9, 11+8, 12+7, 10 and 9 are 1 apart, 11 and 8 are 3 apart, 12 and 7 are 5 apart. So Amy sold 12 and Judy sold 7.  <b>T:</b>	

**Problem Set #5: Reasoning deductively**

**Problem 1:** Five women are seated around a circular table. Mrs. Osborne is sitting between

Mrs. Lewis and Mrs. Martin. Ellen is sitting between Cathy and Mrs. Norris. Mrs. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Mrs. Parks on her left and Mrs. Martin on her right. Match the ladies' first names and last names.	
<b>Dialogue 1</b>	<b>Rationale for Teacher Moves</b>
<p><b>S:</b> Students draw this after reading the first two clues. When he tries to add Mrs. Lewis between Ellen and Alice, he says, "Something's wrong, but I don't know what."</p> 	
<b>Dialogue 2</b>	<b>Rationale for Teacher Moves</b>
<p><b>S:</b> Student draws the pictures below. Ellen is in two pictures, so I should combine them. How can I do that?</p>  <p><b>T:</b></p>	
<b>Dialogue 3</b>	<b>Rationale For Teacher Moves</b>
<p><b>S:</b> After drawing above pictures, student says Mrs. Osborne is either Ellen or Alice, but nothing helps me figure out which one. If I make a chart that doesn't help</p>	

<p>either. What do I do with Cathy and Doris are sisters?  <b>T:</b></p>	
<p><b>Dialogue 4</b></p>	<p><b>Rationale For Teacher Moves</b></p>
<p><b>S:</b> Student draws</p>  <p>I can put these together and that gives me all five, but I'm not sure how to get the first names.</p> <p><b>T:</b></p>	

## APPENDIX D

**Activity Plan Assignment Instructions**

For each problem do the following:

**Possible student solution methods and how you might respond:** This should include not just final solutions children might come up with, but how they might find these solutions.

**Can't start:** If your child is totally stumped or makes desperate guesses ("I don't know, why I just guessed" or you suspect they just picked numbers out of the problem) what hints/questions can get them started? Note there is a spot for you to put "possible can't start solutions" but obviously there are not solutions for the child not being able to start.

**Incorrect:** If your child suggests something that is totally wrong, what hints/questions would help them see the flaws in their reasoning and find a good starting point?

**On the right track:** If your child has an answer that is supported by some good reasoning, but missed a few important aspects of the problem, what questions would help them reconsider and rethink their reasoning?

**Correct and extend:** If (and when) your child has a completely correct answer supported by reasoning, what questions help them find a more sophisticated strategy or generalize their strategy to other problems? (This is where you can accomplish the instructional goals.)

\*\*Your plan for the following week should not read: "We will work on the castle problem. I will ask him what strategies will be helpful and guide him to see that there should be 8 men in the towers."

**Problem Planning Matrix**

		<b>Possible Student Solutions</b>	<b>Questions to Ask</b>	<b>Hints to Give</b>
<b>Problem:</b>	Can't Start			

	Incorrect Solution			
	Partially Correct/Incomplete Solution			
	Correct Solution			

Copy and paste the chart above for as many problems as you include.

IF YOU INCLUDED:	You will earn these points:	Points you earn:
List of 2-3 problems appropriate for your child	1	
More than one possible strategy the child might use and potential mistakes/errors/confusion child may have	1	
Hints to help get them started & hints and questions to help child get “unstuck” that don’t give away the answer	1	
Questions that explore their thinking and require them to explain their reasoning	1	
Questions or ways to extend the problem, questions that help the child generalize strategies, connect two strategies, or find a better strategy	1	
TOTAL Points:	5	

## APPENDIX E

**Activity Reflection Assignment Instructions***Analysis of Activity*

Remember that we aren't only interested in finding the answers, but (you and the children) understanding how the answer was determined. Use evidence from your work with the children to determine what mathematics concepts in the problems they understood. Consider the kinds of strategies the children used, how they represented and made connections between those strategies, their use of diagrams, charts, pictures to record their thinking, their use of tools to reach solutions and explain their thinking, explanations children provided, questions they asked, and their interaction with each other.

In analyzing your (or your teammate's) work with the children pay attention to: the how problems were modified, how children's understanding of problems was ensured, how incorrect strategies were explored, how and what other strategies were encouraged, how assistance was timed, how minor errors were managed, how strategies were elicited, and how problems were extended. In analyzing your (or teammate's) ability to lead discussions consider how a problem was set up, how contributions were initiated and coordinated, how representations were recorded and shared. You should also include elements of the math talk framework: the types of questions that were asked, the quality of explanations generated, the source of mathematical ideas, and responsibility for learning.

You should not address each of these elements for every problem. Rather you should choose important moments from the teaching episode and discuss the areas above that you deem relevant.

*Questions/Concerns*

Any questions you want ask about something that happened or issues you want to draw our attention to.

*Response to Teammate's Reflection*

There are many options for how to respond your teammate's reflections. You may give suggestions for ways to resolve any problems or dilemmas that your teammates explain in their reflection. You can respond to their questions/concerns. You provide suggestions for how you led a discussion or helped a child differently. You may offer your interpretation on teammates' teaching and leading of discussion. Please frame your response to be constructive and not overly critical.

Have one person from your team compile each person's part and submit one document for everyone.

IF YOU INCLUDED:	You will earn these points:	Points you earn:
Insightful analysis of child's activity to make claims about what mathematics he/she understands; insightful analysis of your work with the child that includes questions you have or suggestions for improvement	2	
Evidence to support claims with no sweeping generalizations made about the child or the child's classroom teacher	2	
Reaction to Team member's analysis with thoughtful recommendations for next time	1	
*Your report should be well-written and complete. You will lose points for errors in mathematics.*		
TOTAL Points:	5	

## APPENDIX F

**Final Paper Guidelines**

The purpose of this assignment is to give you a chance to reflect on your growth over the semester. Your paper should show evidence of reflection and analysis on the semester. Do not simply create a “scrapbook” in which you tell a chronological story of your semester.

You should include the following:

*Part I: Your understanding of mathematics teaching*

Describe the most important aspects of teaching mathematics. What are the jobs a mathematics teacher needs to be able to do well? Why do you think so? Be specific in identifying the knowledge, skills, and/or beliefs you think are most helpful for teaching mathematics. Draw on course readings, class discussion, and experiences at Barrow.

*Part II: Analysis of your work at Barrow**Children: An analysis of the children’s successes and struggles.*

What did the children learn and how do you know (provide evidence and examples to support your claims)? With what did they struggle? Use the “analyzing student thinking” document (posted on elc) to help organize your analysis and frame your discussion. You can borrow ideas and language from these documents.

*You: An analysis of your successes and struggles with teaching and planning for teaching.*

What did you learn about teaching, planning for teaching, and interacting with children about mathematics and how do you know (provide evidence and examples to support your claims)? How did your ability to plan activities for and interact with your children progress? With what did you struggle? Consider your ability to elicit and follow your children’s thinking, anticipate their solutions, guide their thinking, pose questions and hints that did not give away the answer, and make the underlying mathematical concepts evident to them. For each problem set, how did your ability to anticipate and respond to students’ solutions, provide and elicit good explanations, ask questions and give hints, and lead discussions change over time? Use the “analyzing your teaching,” and the “analyzing your math talk” documents (on elc) to help you organize this section, evaluate yourself, and frame your discussion. You can borrow ideas and language from these documents.

*Part III: Conclusion*

How and why have your views on teaching mathematics changed over the course of the semester? What do you consider your strengths in teaching mathematics? In what areas do you feel you need improvement? How will you address these areas? Looking back at

the skills, knowledge, and/or beliefs you identified in part I, what are your goals for yourself as a mathematics teacher and for your students?