MODULI OF WEIGHTED STABLE MAPS AND THEIR GRAVITATIONAL DESCENDANTS

by

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(Under the direction of Valery Alexeev)

Abstract

We study the intersection theory on the moduli spaces of maps of \( n \)-pointed curves \( f : (C, s_1, \ldots, s_n) \to V \) which are stable with respect to the weight data \((a_1, \ldots, a_n)\), \(0 \leq a_i \leq 1\). After describing the structure of these moduli spaces, we define an analog of the gravitational descendants from Gromov-Witten theory using them. We state and prove the equality of some of these descendants to previous numerical invariants of Miller, Morita and Mumford as well as those of Graber, Kock, and Pandharipande. We then prove a formula describing the way each descendant changes as we vary the weights. Finally, we state and prove several nice relationships among these descendants including generalizations of the string, dilaton, and divisor equation.

Index words: Gromov-Witten, moduli, stable map, stable curve, gravitational descendant
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DEDICATION

To those who believed I could, even when I was sure I couldn’t.
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Chapter 1

Introduction

In this introduction, we will give a less technical and rigorous survey of the contents of the remainder of the dissertation. The definitions and theorems stated here will be restated with all necessary rigor and detail in the appropriate sections which follow.

The main object of study is a moduli space. Roughly speaking, a moduli space is a set such that distinct elements represent different objects of a particular type (up to some defined equivalence relation), and every possibility for these objects is represented in this set. In addition to this description as a set, moduli spaces have additional geometric structure themselves. We often exploit this additional structure in order to gain additional insight into the objects which it classifies.

The moduli spaces $\overline{M}_{g,A}(V, \beta)$ of weighted stable maps are the central moduli spaces we will explore. To begin, we fix a smooth projective variety $V$ and a homology class $\beta \in H_2(V, \mathbb{Z})$. Roughly speaking, $\beta$ is the class of a one complex (two real) dimensional compact subvariety in $V$. In addition to this, we fix an ordered $n$-tuple $A = (a_1, \cdots, a_n)$ of real numbers with $0 \leq a_i \leq 1$ and a nonnegative integer $g$.

The case with each $a_i = 1$, usually notated $\overline{M}_{g,n}(V, \beta)$, has been the primary focus of most prior research. In this case, it was discovered that these spaces naturally arise in the study of string theory from physics. Since this time, these spaces have been a key ingredient in the study of many intriguing problems, both old and new. Besides being simply a set, the moduli spaces have the additional geometric structure of a stack, which is a generalization of an algebraic variety. As such, one can hope to connect the geometry of the moduli space itself to the properties of the underlying objects which it classifies. An exciting successful
example of this is found in Gromov-Witten theory. In Gromov-Witten theory, one idea is to interpret the intersection of carefully chosen subspaces of $\overline{M}_{g,n}(V, \beta)$ as counting distinct curves in the variety $V$ with desired properties. Using these methods, one can prove many well known facts such as the existence and uniqueness of a line connecting two distinct points in the plane. The true innovation and strength of these strategies are evident in the predictions one can make and prove in situations in which no previous theories were well equipped to produce.

The generalization of the previous theory of $\overline{M}_{g,n}(V, \beta)$ to arbitrary $A$ was started by Losev and Manin in [LM00] and then by Hassett in [Has03]. Both of these specialized to the case of $V$ being a single point. This theory was extended to any $V$ and $A$ in joint work with Valery Alexeev in [AG06] and simultaneously by Bayer and Manin in [BM06]. Each point of $\overline{M}_{g,A}(V, \beta)$ represents an $n$-pointed $A$-stable map of a genus $g$ curve (topologically a $g$-holed torus) to the variety $V$. These have the following definition.

**Definition 1.1.** An $n$-pointed, genus $g$, $A$-stable map $f : C \to V$ of class $\beta$ has the following properties:

1. $C$ is a genus $g$ connected curve with at most nodes as singularities. Topologically this says that you may pinch the torus so that it has multiple components $C_i$; each component is itself a torus with possibly fewer holes; and each component may intersect the other components locally in a single point called a node.

2. There are $n$ (not necessarily distinct) marked points $x_1, \ldots, x_n$ of $C$. If the point $x_i$ is a node, then we require that $a_i = 0$. If the point $x_i$ is not a node, then we sum the $a_j$ which correspond to the same $x_i$ and have the inequality:

$$\sum a_j \leq 1$$

The upshot of this is that we may repeat a smooth point only whenever the total of its weights is no more than 1.
3. The components $C_i$ which map to a single point in $V$ satisfy the inequality:

$$2(\text{genus of } C_i) - 2 + \#(\text{nodes of } C_i) + \sum_{x_j \in C_i} a_j > 0$$ (1.1)

4. The image of $C$ under $f$ has homology class $\beta$.

For two different weights $A$ and $B$, we note that the inequalities dictate that different subsets of the marked points $x_1, \ldots, x_n$ are allowed to coincide on $C$. For example, in the previous case of $\overline{M}_{g,n}(V, \beta)$ (each $a_i = 1$), no marked points may coincide. Whenever we change the objects which a moduli space classifies, the spaces themselves may change. These changes and their effects on some applications will be studied in greater detail in later sections.

If we have two weights $A, B$ such that $a_i \geq b_i$, we have a reduction morphism between the spaces $\rho : \overline{M}_{g,A}(V, \beta) \to \overline{M}_{g,B}(V, \beta)$. A careful examination of the properties listed in Definition 1.1 reveals that the only property which may be violated whenever we reduce the weights $a_i$ to $b_i$ is that the inequality in Equation (1.1) may no longer be true for components $C_i$ which are mapped to a single point of $V$. This inequality can only be violated whenever $C_i$ has genus zero (meaning it is topologically a sphere), and if it is only connected to one other component of $C$. We say that these components become unstable after this reduction. Should this inequality remain true, however, there is a unique point in $\overline{M}_{g,B}(V, \beta)$ which represents this map once we reduce the weights to $B$, and $\rho$ sends the point in $\overline{M}_{g,A}(V, \beta)$ to this unique point in $\overline{M}_{g,B}(V, \beta)$. From a set theoretic point of view, we can see that $\rho$ is a bijection as a map of sets between these types of points. In fact, this map is actually a bijection as a morphism of moduli spaces on this set. Moreover, these points form an open dense subset of $\overline{M}_{g,A}(V, \beta)$. This means that every map represented in $\overline{M}_{g,A}(V, \beta)$ either remains stable after reduction or is at least “very close” to a map which remains stable.

We contrast this with the situation in which the inequality becomes false for some component $C_i$ of $C$, an example of which is depicted in Figure 1.1. For the genus zero
component, there is only one node, and let us assume that $A$ assigns each of the points $x_1, x_2, x_3$ the weight $a_1 = a_2 = a_3 = 2/3$ and that the map $f_A$ maps this entire component to a single point of $V$.

Figure 1.1: Reduction morphism

![Reduction morphism diagram]

We see that using the weight $A$, the inequality in Equation (1.1) states that $1 > 0$. However, if $B$ assigns the value $b_1 = b_2 = b_3 = 1/3$, we see that this inequality then states $0 > 0$, which is no longer true. The reduction $\rho$ then sends the point representing the map $f_A$ to the point representing the map $\tilde{f}_B$ which replaces the entire genus zero component by a single point on $\tilde{C}$ which is then labeled $x_1, x_2, x_3$. This means that the three previously distinct points of $C$ now correspond to a single point of $\tilde{C}$, and $\tilde{f}_B$ is now stable with respect to $B$. In this case, there are usually many different points of $\overline{M}_{g,A}(V,\beta)$ which all give the same point in $\overline{M}_{g,B}(V,\beta)$.

Whenever an inequality such as this becomes untrue, we say that $\rho$ crosses a wall between $A$ and $B$. We label walls by the set of indices $\sigma$ such that $a_{i_1} + \cdots + a_{i_r} > 1$ but $b_{i_1} + \cdots + b_{i_r} \leq 1$. In the example illustrated above $\sigma = \{1, 2, 3\}$. In the special case whenever there is only one wall crossed between $A$ and $B$, which we call a simple-wall crossing, $\rho$ is a type of map referred to as a blowup of $\overline{M}_{g,B}(V,\beta)$.

Alongside studying how these spaces change, we will explore a generalization of gravitational descendants in this context. The gravitational descendants are numbers which are defined in terms of intersections on $\overline{M}_{g,A}(V,\beta)$ of three central types of classes. The first classes are the evaluation classes. For the definition of these, we start with a class $\gamma_i \in H^*(V,\mathbb{Q})$. The evaluation class $\nu_i^*(\gamma_i)$ is roughly defined to be the locus of points in $\overline{M}_{g,A}(V,\beta)$ representing maps which send $x_i$ into the subvariety of $V$ represented by $\gamma_i$. 
Next, we have the psi classes. These classes are a bit more technical, but geometrically simply give some information about the tangent spaces of the curves. The class $\psi_i$ can be defined to be the first Chern class of the line bundle on $\overline{M}_{g,A}(V,\beta)$ whose fiber at each point is the cotangent space of $C$ at the marked point $x_i$. Equivalently, one may define these by restricting the $i$-th section of the universal curve $\pi : C_{g,A}(V,\beta) \to \overline{M}_{g,A}(V,\beta)$ to the relative dualizing bundle of $\pi$ and taking the first Chern class. We will study these classes and their changes beginning in Section 3.1.

While the spaces $\overline{M}_{g,A}(V,\beta)$ have many nice properties, they may also have some very undesirable pathologies. To compensate for one such pathology (namely that there is an obstruction in the deformation theory of these maps), one defines the virtual fundamental class $[\overline{M}_{g,A}(V,\beta)]^{\text{virt}}$. Its main purpose is to ensure that gravitational descendants, as we will define them below, are indeed a number. These classes are studied in the unweighted case in [BF97], [Beh97], [BM96], and in the weighted case in [BM06]. We will revisit these classes in Section 3.2.1.

**Definition 1.2.** For nonnegative integers $k_i$ and $\gamma_i \in H^{\ast}(V,\mathbb{Q})$, the gravitational descendant is given by the integral:

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{V,\beta}^{V,\beta} := \int \left( \prod_i \psi_i^{k_i} \cup \nu_i^{\ast}(\gamma_i) \right) \cap [\overline{M}_{g,A}(V,\beta)]^{\text{virt}}$$

These are defined to be zero whenever any $k_i$ is negative or:

$$\sum_{i=1}^{n} (k_i + \deg \gamma_i) \neq (1 - g) \dim V + c_1(V) \cdot \beta + (3g - 3 + n)$$

The central result of this dissertation is an unexpectedly nice description of how the descendants change as these spaces cross a wall. The theorem states that the difference between the descendants is again a single descendant, but this time calculated on the “smaller” space $\overline{M}_{g,A_{\sigma}}(V,\beta)$ which will be defined in Example 2.22.7. In order to state this result, we first state the following notational definition.

**Definition 1.3.** Let $\sigma \subset \{1, \ldots, n\}$. We define $\dim \sigma := \# \sigma - 1$, $k_{\sigma} := \sum_{i \in \sigma} k_i - \dim \sigma$ and $\gamma_{\sigma} := \prod_{i \in \sigma} \gamma_i$. 

Theorem 1.4 (Simple-wall Crossing). For two weights \( A \geq B \) which cross exactly one wall, we have the following wall crossing formula:

\[
\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,B}^{V,\beta} + (-1)^{\dim \sigma + 1} \left\langle \tau_{k_\sigma}(\gamma_\sigma) \prod_{j \not\in \sigma} \tau_{k_j}(\gamma_j) \right\rangle_{g,A_\sigma}^{V,\beta}
\]

One nice application of this is that it easily implies that the ordinary unweighted Gromov-Witten invariants, which is whenever each \( k_i = 0 \), coincide with the weighted ones. This being because \( k_\sigma < 0 \) and the correction term is thus zero. In addition, since every set of weights admits a reduction from the space \( \overline{M}_{g,n}(V,\beta) \), we may calculate the weighted descendants in terms of the ones which have been previously studied. Before stating this result, we make one last definition.

Definition 1.5. For a weight \( A = (a_1, \ldots, a_n) \), the set \( \Sigma = \{\sigma_j\} \) is an element of the set \( \Sigma(A) \) iff

1. The \( \sigma_j \) are disjoint subsets of \( \{1, \ldots, n\} \),
2. \( \bigcup \sigma_j = \{1, \ldots, n\} \),
3. For each \( \sigma \in \Sigma \), we have \( \sum_{i \in \sigma} a_i \leq 1 \).

Additionally, define \( \dim \Sigma := \sum_{\sigma \in \Sigma} \dim \sigma \) and denote the number of sets in \( \Sigma \) as \( |\Sigma| \).

Theorem 1.6 (Reduction to Unweighted Descendants). For any admissible weight data \( A \), we have:

\[
\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = \sum_{\Sigma \in \Sigma(A)} (-1)^{\dim \Sigma} \left\langle \prod_{\sigma \in \Sigma} \tau_{k_\sigma}(\gamma_\sigma) \right\rangle_{g,|\Sigma|}^{V,\beta}
\]

An interesting application of Theorem 1.6 is whenever one chooses each \( a_i = 0 \). We will prove in Proposition 3.18 that whenever \( V \) is a point, these gravitational descendants are equal to to the kappa numbers of Miller, Morita and Mumford, which have been studied for quite some time. We will also show in Proposition 3.19 that these descendants are equal to the invariants defined and studied for any \( V \) by Graber, Kock and Pandharipande in
In [GKP02], it is shown that these invariants have numerical significance in application to the characteristic number problem and have many nice relations. Thus, Theorem 1.6 provides an alternate derivation and interpretation of these known results.

In Chapter 4, we will also prove generalizations of other relations in the weighted situation including analogues of the basic pullback relation for the psi classes via the forgetful morphism, as well as the string, dilaton and divisor equations.

Most of the content of this dissertation also appears in the joint paper [AG06]. Some proofs given here are different and are primarily those of the current author. The proofs which are not given here can be found in [AG06] and are primarily due to Alexeev.
2.1 MODULI SPACES AND MODULI PROBLEMS IN Algebraic Geometry

Moduli spaces have taken the center stage in many problems in algebraic geometry since the last century. Here we will give a brief overview of their definitions and some of their central themes. While there are many sources for this information, the explanations and motivations given in [HM98] are closely followed here.

Moduli spaces are constructed in order to solve a moduli problem. A moduli problem typically consists of two things which we specify. The first thing we specify is a class of objects, such as curves or maps with certain properties, together with the notion of what it means to have a family of these objects over a base scheme $B$. Secondly, we specify an equivalence relation $\sim$ on the set $S(B)$ of all families over each $B$.

We then consider the moduli functor $F : \text{Sch} \to \text{Set}$ from the category $\text{Sch}$ of schemes over an algebraically closed field $k$ to the category $\text{Set}$ of sets. This contravariant functor is defined by sending $F(B)$ to the set $S(B)/\sim$. After constructing this functor, we consider to what extent the functor $F$ is representable by a scheme $\mathcal{M}$. We recall the following definition.

**Definition 2.1.** A contravariant functor $F : \mathcal{C} \to \text{Set}$ from a category $\mathcal{C}$ is **representable** by an object $C$ of $\mathcal{C}$ if there is an isomorphism of functors $\Psi : \text{Hom}(\bullet, C) \to F$. 

Using this definition, we then divide our moduli spaces into two main types.

**Definition 2.2 (Moduli Space).**

1. A scheme $\mathcal{M}$ is a **fine moduli space** for our moduli problem if the moduli functor $F$ is representable by a scheme $\mathcal{M}$.

2. A scheme $\mathcal{M}$ and a natural transformation $\Psi_\mathcal{M} : F \to \text{Hom}_\mathcal{M}(\bullet, \mathcal{M})$ is a **coarse moduli space** for our moduli problem if

   (a) The map $\Psi_{\text{Spec}(k)} : F(\text{Spec}(k)) \to \text{Hom}_\mathcal{M}(\text{Spec}(k), \mathcal{M})$ is a set bijection.

   (b) Given another scheme $\mathcal{M}'$ and natural transformation $\Psi_{\mathcal{M}'}$, there is a unique morphism $\pi : \mathcal{M} \to \mathcal{M}'$ such that the associated natural transformation $\Pi : \text{Hom}_\mathcal{M}(\bullet, \mathcal{M}) \to \text{Hom}_{\mathcal{M}'}(\bullet, \mathcal{M}')$ satisfies $\Psi_{\mathcal{M}'} = \Pi \circ \Psi_\mathcal{M}$.

Representability plays a key role in translating the geometry of families in our moduli problem to the geometry of the moduli space. If we start with a family $\phi : D \to B$, which is an element of $S(B)$, then $\chi := \Psi(\phi)$ is a morphism from $B$ to $\mathcal{M}$. Intuitively then, we can see that closed points of $\mathcal{M}$ classify the objects of our moduli problem, and the map $\chi$ sends a closed point $b$ of $B$ to the moduli point in $\mathcal{M}$ determined by the fiber $D_b$ of $D$ over $b$.

To translate from a moduli point of $\mathcal{M}$ to its family, we consider its universal family.

**Definition 2.3.** The **universal family** $I : \mathcal{U} \to B$ associated to a fine moduli space $\mathcal{M}$ is the pullback of the identity map $\text{id}_{\mathcal{M}, \mathcal{M}}$ by the map $\Psi$ appearing in Definition 2.1.

The universal family has the nice property that, given any morphism $\chi : B \to \mathcal{M}$ as defined above, we get a commutative fiber diagram

$$
\begin{array}{ccc}
D & \longrightarrow & \mathcal{U} \\
\phi \downarrow & & \downarrow I \\
B & \longrightarrow & \mathcal{M}
\end{array}
$$
where $\phi : D \to B$ is in $S(B)$ and $\Psi(\phi) = \chi$. This gives that every family over $B$ is the pullback of $\mathcal{U}$ via a unique map of $B$ to $\mathcal{M}$. Thus whenever we have a fine moduli space, we may use the representability to translate between the geometry of the moduli spaces and that of the objects we are classifying.

The additional natural transformation in the coarse moduli space, while possessing a universal property, still lacks the “clean” answer which a fine moduli space provides. Unfortunately, many interesting moduli problems, including the main ones studied here, will not possess a fine moduli space.

One of the main reasons many moduli problems do not have a fine moduli space is due to the presence of nontrivial automorphisms. A family of objects which has a nontrivial automorphism group is not distinguishable based only on the fibers of the morphism. Thus there cannot be a universal family in the sense we previously discussed.

There are several avenues one can pursue to attain a more desirable answer to the moduli problem than settling for a coarse moduli space. One path often taken, and the one which is employed in this work, is to require representability of our moduli functor by an object in a larger category than that of schemes. This idea leads naturally to the definition and study of a stack. One of the most notable first examples of this is found in [DM69] where Deligne and Mumford employed these techniques in the study of the moduli of curves of a given genus.

The development of the necessary details in order to rigorously define a stack is well beyond the scope of this paper. Instead, we ask the reader to consider a stack to be an object in a category “larger” than that of schemes where the moduli functors are indeed representable. When we enlarge the category of varieties to that of schemes, we add the extra data of a sheaf of rings. In an analogous way, when we enlarge the category of schemes to that of stacks, we add extra data which keeps track of automorphism groups. By adding the data of the automorphism groups, we are able to eliminate the non-uniqueness problem which
we previously discussed. As a result, whenever the automorphism group is finite we are able to attain a fine moduli stack for our moduli functors.

An interesting consequence of this extra data is that the points of a stack are in some sense only “fractional” points. The point is “counted” with value inverse to the size of its automorphism group. This idea is analogous to thinking of a scheme as a variety in which some points are “fat points” and contain extra data about the order of vanishing of defining equations.

In what is to come, the reader will not likely be mislead by substituting the concept of a scheme, or even a variety, whenever stacks are mentioned as long as one keeps in mind the comments we have just made. In fact, the main results of this work were first formulated in a case where the moduli spaces actually are varieties. The generalization of these results to the larger class of moduli problems was almost completely an exercise in formalism. The necessary formalism from intersection theory of which we will soon make use are found most notably in [Vis89].

2.2 First Properties of the Moduli Space of Weighted Stable Maps

For the definitions of individual varieties and pairs, we work over an algebraically closed field. For the definitions of families and moduli functors, we work over a Noetherian base scheme $B$, and all products are fibered products over $B$ and $V$ will denote a flat projective scheme over $B$.

**Definition 2.4.** A **weight** is a real number $0 \leq a_i \leq 1$. A **weight data** is an ordered $n$-tuple $A = (a_1, \ldots, a_n)$ of weights.

In order to simplify notations, we will often use standard abbreviations for the weights. For example, we will write $(1^n)$ instead of $(1, \ldots, 1)$, and $(1^2, \epsilon^{n-2})$ instead of $(1, 1, \epsilon, \ldots, \epsilon)$. 
Definition 2.5. A stable map for the weight data $A$, or an $A$-stable map, is a proper morphism $f : C \to V$ from a connected reduced curve $C$ to a variety $V$, together with $n$ ordered marked points $s_1, \ldots, s_n \in C$ which satisfies the following two conditions:

1. (on singularities) $C$ has at most nodes; for every smooth point $P \in C$, the multiplicity:
   $$\text{mult}_P := \sum_{s_i = P} a_i \leq 1$$
   and for a node $P \in C$, one has $\text{mult}_P = 0$;

2. (numerical) the $\mathbb{R}$-line bundle $\omega_C(\sum a_i s_i)$ is $f$-ample; i.e., for every irreducible component $E$ of $C$ collapsed by $f$ to a point, one has:
   $$\deg \omega_C \left( \sum a_i s_i \right) \big|_E = 2p_a(E) - 2 + |E \cap (C - E)| + \sum_{s_i \in E} a_i > 0$$

A stable curve is a stable map to a point.

Thus, the marked points with $a_i = 0$ may coincide with the nodes, but points with positive weights may not.

As usual, the numerical condition is only a restriction on the collapsed components $E$ which are $\mathbb{P}^1$, elliptic curves, or rational curves with a single node. This being apparent since the only way this inequality could be negative is if $2p_a(E) - 2 < 0$, which is only possible for genus zero and one.

Definition 2.6. Let $V$ be a flat scheme over $B$; if we work over a field, let $V$ be simply a variety and $S/B$ a scheme. A family of $A$-stable maps over $S$ is a morphism of schemes $f : C \to V \times S$ together with sections $g_i : S \to C$ of $\pi = p_2 \circ f : C \to S$, such that

1. $\pi : C \to S$ is flat,

2. every geometric fiber $\left( C, s_i = g_i(S) \right)_s \to V_s$ is an $A$-stable map.
Two families of maps over $S$,

$$(f : C \to V \times S, \{g_i\}), \quad (f' : C' \to V \times S, \{g'_i\})$$

are **isomorphic** if there exists a scheme isomorphism $\tau : C \to C'$ satisfying $\pi = \pi' \circ \tau$ and $g'_i = \tau \circ g_i$.

**Definition 2.7.** The moduli stack $\overline{M}_{g,A}(V)$ associates to every scheme $S/\mathcal{B}$ a category whose objects are families of $A$-stable maps over $S$ such that every curve $C_s$ has arithmetic genus $g$ and whose arrows are isomorphisms of families over $S$.

The moduli functor is defined by associating to $S$ the set of such families modulo isomorphisms.

Stable weighted curves were defined by Hassett [Has03] who constructed their moduli spaces and gave a detailed description.

The statements which we now give are proven, in greater generality than given here, in [AG06]. We will now fix a complex variety $V$, an integer $g \geq 0$ and the weight data $A$. As in the unweighted case, we will further subdivide $\overline{M}_{g,A}(V)$ into a disjoint union

$$\overline{M}_{g,A}(V) = \bigsqcup_{\beta \in H_2(V,\mathbb{Z})} \overline{M}_{g,A}(V,\beta)$$

with pieces $\overline{M}_{g,A}(V,\beta)$ of finite type.

**Remark 2.8.** Whenever the omission of any of $g, A, V, \beta$ is unlikely to lead to confusion, we may do so.

**Theorem 2.9.** The moduli stack $\overline{M}_{g,A}(V,\beta)$ is a proper algebraic Artin stack with finite stabilizer. For each $(g,n,V,\beta)$, there exists $N \in \mathbb{N}$ such that $\overline{M}_{g,A}(V,\beta) \times \mathbb{Z}[1/N]$ (i.e., outside of the finitely many positive characteristics dividing $N$) is a Deligne-Mumford stack.

**Proof.** See Theorem 1.9 in [AG06].
Corollary 2.10. $\overline{M}_{g,A}(V, \beta)$ has a coarse moduli space, a proper algebraic space.

Lemma 2.11. Let $A = (a_1, \ldots, a_n)$ be the weight data and let $A \cup 0^m = (a_1, \ldots, a_n, 0, \ldots, 0)$ be the weight data obtained by adding $m$ zeros. Then

$$\overline{M}_{g,A \cup 0^m}(V, \beta) = C^m_{\overline{M}_{g,A}(V, \beta)}$$

is the $m$-th fibered power of the universal family $C$ over $\overline{M}_{g,A}(V, \beta)$.

Proof. Indeed, the only difference between families in $\overline{M}_{g,A}(V, \beta)$ and in $\overline{M}_{g,A \cup 0^m}(V, \beta)$ is $m$ arbitrary sections. \qed

2.3 Chambers, Walls, and Simplicial Complexes

We now proceed to define some combinatorial objects of which which we will make use in our studies. In this section, we will fix $g, n, V$ and $\beta$.

Definition 2.12. We will call the weight data $A \in [0, 1]^n$ admissible if the stack $\overline{M}_{g,A}(V, \beta)$ is nonempty.

The weight data of the same length have a natural partial order: $A = (a_i) \geq B = (b_i)$ if $a_i \geq b_i$ for all $i$.

Following [Has03], we define the set $D_{g,n,\beta}$, where the admissible weights can theoretically live, and two decompositions of it into finitely many chambers.

Definition 2.13. If $(g, \beta) \neq (0, 0)$ or $(1, 0)$, we set $D_{g,n,\beta}$ to be the cube $[0, 1]^n$.

$D_{1,n,0} = [0, 1]^n$ minus the point $(0^n)$, and $D_{0,n,0}$ is the subset of $[0, 1]^n$ defined by the inequality $\sum_{i=1}^n a_i > 2$.

We define two chamber decompositions of $D_{g,n,\beta}$ obtained by cutting it by finitely many hyperplanes. For the coarse decomposition, the hyperplanes are given by:

$$W_c = \left\{ \sum_{i \in I} a_i = 1 : I \subset \{1, \ldots, n\}, 3 \leq |I| \leq n \right\}$$
For the fine decomposition, they are given by:

\[ W_f = \left\{ \sum_{i \in I} a_i = 1 : I \subset \{1, \ldots, n\}, 2 \leq |I| \leq n \right\} \]

**Remark 2.14.** We corrected an obvious typo in [Has03] which states $|I| \leq n - 2$ in place of $|I| \leq n$ (which is correct for $(g, \beta) = (0, 0)$).

**Definition 2.15.** A chamber is a nonempty locally closed subset of $\mathcal{D}_{g,n,\beta}$ obtained by choosing for each $I \subset \{1, \ldots, n\}$ either the inequality $\sum_{i \in I} a_i > 1$ or the inequality $\sum_{i \in I} a_i \leq 1$.

**Proposition 2.16** (cf. [Has03], Prop.5.1).

1. If $A$ and $B$ belong to the same chamber in the coarse decomposition and have the same zero weights, then $\overline{M}_{g,A}(V, \beta)$ and $\overline{M}_{g,B}(V, \beta)$ naturally coincide.

2. If $A$ and $B$ belong to the same chamber in the fine decomposition and have the same zero weights, then the universal families over $\overline{M}_{g,A}(V, \beta)$ and $\overline{M}_{g,B}(V, \beta)$ naturally coincide.

**Proof.** (1) We have to compare the two conditions of Definition 2.5 for $A$ and $B$. The condition on singularities $\sum_{s_i=1}^P a_i \leq 1$ is obviously the same for $A$ and $B$. The numerical condition says that for any irreducible component $E$ of $C$ collapsed by $f$ to a point, one should have:

\[ \deg \omega_C \left( \sum_{i} a_i s_i \right) \bigg|_E = 2p_a(E) - 2 + |E \cap (C - E)| + \sum_{s_i \in E} a_i > 0 \]

If $(g, \beta) \neq (1, 0)$ or $(0, 0)$, and $E$ is not a $\mathbb{P}^1$ with $|E \cap (C - E)| = 2$, then this condition is either vacuous or says

\[ -1 + \sum_{s_i \in E} a_i > 0 \]

Hence, it is the same for $A$ and $B$ since we have assumed that $A$ and $B$ are in the same chamber. If $(g, \beta) = (1, 0)$ or $(0, 0)$, the additional cases say simply that $A$ and $B$ should be
in \( \mathcal{D}_{g,n,\beta} \). If \( E \) is a \( \mathbb{P}^1 \) with \( |E \cap (C - E)| = 2 \), then the condition says \( \sum_{a_i \in E} a_i > 0 \). Since \( A \) and \( B \) have the same zero weights, this condition is equivalent for \( A \) and \( B \).

(2) By Lemma 2.11, the universal family \( \mathcal{C}_{g,A}(V,\beta) \) is the moduli space for the weight data \( A \cup 0 \); now apply (1).

**Lemma 2.17.** Let \( (a_i) \) be the positive weight data and suppose that for all \( 0 < \epsilon \ll 1 \), the weight data \( (a_i, \epsilon^m) \) is in the interior of the same chamber as \((a_i,0^m)\), and both are in \( \mathcal{D}_{g,n,\beta} \). Then \((a_i,\epsilon^m)\) is admissible iff \((a_i,0^m)\) is admissible.

**Proof.** See Lemma 2.6 in [AG06].

**Corollary 2.18.** For each \((g,n,V,\beta)\) the set of admissible weight data is a union of several chambers.

We now introduce a convenient way to label the chambers.

**Definition 2.19.** Let us identify every subset \( I \subset \{1, \ldots, n\} \) with a simplex \( \sigma(I) \) with vertices in \( I \); we have \( \dim \sigma(I) = |I| - 1 \). The **simplicial complex** \( \Delta_A \) **associated to the weight data** \( A = (a_1, \ldots, a_n) \) consists of simplices \( \sigma(I) \) such that \( \sum_{i \in I} a_i \leq 1 \). We will often identify \( \sigma \) with \( \sigma(I) \) and denote the simplex as simply \( \sigma \).

If \( \sigma' \) is a face of \( \sigma \), i.e., \( \sigma' \subset \sigma \), then \( \sum_{i \in \sigma} a_i \leq 1 \) implies \( \sum_{i \in \sigma'} a_i \leq 1 \). Therefore, \( \Delta_A \) is indeed a complex in the usual sense.

**Remark 2.20.** It is a strong condition for a complex \( \Delta \) to be associated with a chamber.

For example, the complex \( \{12, 34, 1, 2, 3, 4\} \) is not associated to any chamber: the system of inequalities \( a_1 + a_2 \leq 1, a_3 + a_4 \leq 1, a_1 + a_3 > 1, a_2 + a_4 > 1 \) has no solution.

**Corollary 2.21.** With the zero weights fixed:

1. The moduli space \( \overline{M}_{g,\Delta}(V,\beta) \) and universal family \( \mathcal{C}_{g,\Delta}(V,\beta) \) are well defined using any \( A \) such that \( \Delta = \Delta_A \).

2. For the universal family, one has \( \mathcal{C}_{g,\Delta}(V,\beta) = \overline{M}_{g,\text{Cone}_\Delta}(V,\beta) \), where \( \text{Cone}\Delta \) is the simplicial complex on \( n + 1 \) vertices consisting of \( \sigma \) and \( \sigma \cup \{n + 1\} \) for all \( \sigma \in \Delta \).
Proof. The first statement follows from Proposition 2.16. The second follows from Lemma 2.11 combined with Proposition 2.16.

In the following example, we fix notation for several complexes of which we will make frequent use throughout the remainder of this work.

Example 2.22.

1. For the weight data \((1^n)\), \(\Delta\) is the disjoint union of \(n\) vertices.

2. For the weight data \((\varepsilon^n)\), \(\Delta\) contains all simplices; i.e., the support \(|\Delta|\) is homeomorphic to an \((n-1)\)-ball.

3. For the weight data \((1/(r+1), \ldots, 1/(r+1))\) with \(n\) vertices, the complex is given by the \(r\)-skeleton of an \(n\)-simplex; i.e., every subset of vertices with \(r+1\) or fewer vertices is a face of the complex, and we will denote this complex \(\Delta_{n,r}\). Included in this are the previous two examples for \(r = 0\) and \(r = n-1\), respectively. We will reference this for any combination of \(g, V, \beta\) for which the corresponding space is nonempty.

4. For the weight data \((1^2, \varepsilon^{n-2})\) with \(\varepsilon < 1/(n-2)\), the complex \(\Delta_{\varepsilon_{n-2}}\) has the first two vertices isolated, and the remaining vertices form a complete simplex. Whenever we reference this complex, we will assume that \(g = 0, V = \{pt\}, \beta = 0\). See Lemma 3.11 and Example 3.22.2.

5. For the weight data \((1, a_1, \ldots, a_r)\) such that \(\sum a_i > 1\), but for any proper subset one has \(\sum a_i \leq 1\), the complex \(\Delta_{pr^{r-2}}\) is a single isolated vertex plus the boundary of an \((r-1)\)-simplex. Whenever we reference this complex, we will assume that \(g = 0, V = \{pt\}, \beta = 0\). See Lemmas 2.29, 2.35, 3.2 and Example 3.22.3.

6. Corollary 2.21 gives that the complex associated to the universal family of \(\overline{M}_{g,A}(V, \beta)\) is the cone over \(\Delta_A\) with vertex given by the additional section. We denote this as \(\text{Cone}(\Delta_A)\).
7. For any weight data $A$ and $\sigma$ a collection of vertices in $\Delta_A$, we define $\Delta_{A,\sigma}$ to be the complex attained by replacing the vertices of $\sigma$ with a single vertex, which we shall label $\sigma$, and assigning the weight equal to the sum of the weights in $\sigma$. We note that $\gamma \subset \Delta_A$ corresponds to a face of $\Delta_{A,\sigma}$ iff $\gamma$ is a face of $\Delta_A$ and either:

(a) $\gamma \cap \sigma = \emptyset$.

(b) $\sigma \subset \gamma$. In this case, $\gamma$ will correspond to the face of $\Delta_{A,\sigma}$ containing the vertices in $\gamma \setminus \sigma$ as well as the vertex which we label as $\sigma$.

This complex will appear naturally in the wall crossing formula in Theorem 3.24.

### 2.4 Reduction and Forgetful Morphisms

We now begin the study of maps between the moduli spaces with different weights. The properties of these maps will be crucial to our results.

**Theorem 2.23** (Reduction morphism). Fix $g, V, \beta$ and let $A, B \in D_{g,n,\beta}$ be two admissible weight data such that $A \geq B$. Then there exists a natural reduction morphism:

$$\rho_{B,A} : \overline{M}_{g,A}(V, \beta) \to \overline{M}_{g,B}(V, \beta)$$

Given a stable map $(C, \{s_i\}, f)$ for the weight data $A$, its image $\rho_{B,A}(C, \{s_i\}, f)$ is obtained by successively collapsing components of $C$ along which $K_x + \sum a_i s_i$ fails to be $f$-ample.

**Proof.** See Theorem 3.1 in [AG06].

In the next section, we will study the reduction morphism in greater detail and describe its structure as a birational map more carefully.

In addition to the reduction morphism, there is a weighted version of the forgetful morphism which forgets some of the marked points and stabilizes.

**Theorem 2.24** (Forgetful morphism). Let $A = (a_i) \in D_{g,n,\beta}$ and $A' = (a'_i) \in D_{g,m,\beta}$ be two admissible weight data such that $A'$ is obtained from $A$ by dropping the last $n - m$ weights.
Then there exists a natural morphism $\phi_{A',A} : \overline{M}_{g,A}(V,\beta) \to \overline{M}_{g,A'}(V,\beta)$. Moreover, $\phi_{A',A}$ is flat and Gorenstein.

Proof. See Theorem 3.2 in [AG06].

The forgetful morphism’s most important appearance is in the following lemma.

**Lemma 2.25 (Universal family).** Let $A = (a_i)$ be an admissible positive weight data. Then for all $0 < \epsilon \ll \delta \ll 1$, the universal family over $\overline{M}_{g,A}(V,\beta)$ is the moduli for the weight data $(a_i - \delta, \epsilon)$. Moreover, the morphism of the universal family is given by forgetting the section with weight $\epsilon$.

Proof. See Lemma 3.4 in [AG06].

As a result of this lemma, we see that the forgetful morphism corresponds to the universal family whenever we are forgetting a single point with weight sufficiently small. The reader should take care to note that a forgetful morphism which forgets any single section is not necessarily a map of the universal family as is true for the unweighted case. This distinction will play a key role in many formulas which we derive in Chapter 4.

### 2.5 Crossing a Single Wall by Adding a Simplex

We now begin to study a factorization of a general reduction morphisms into many maps, each of which will be easy to describe.

**Definition 2.26.** If $A^+ \geq A$ and $\Delta_A$ is obtained from $\Delta_{A^+}$ by adding a single simplex $\sigma$, i.e., by changing the sign in the single inequality from $\sum_{i \in \sigma} a_i > 1$ to $\leq 1$, then the change from $A^+$ to $A$ will be called a **simple wall crossing**.

In this section, we consider the simple wall crossing given by adding $\sigma$ to $\Delta_{A^+}$ to form $\Delta_A$. We define $r := \dim \sigma + 1$. We denote the reduction morphism $\rho_{A,A^+}$ by $\rho$. Additionally, we suppress the $g, V, \beta$ in much of our notation since these are all fixed.
We will soon see that for simple wall crossings, \( \rho \) is simply a blowup of a substack of \( \overline{M}_A \) which is isomorphic to \( \overline{M}_{A_\sigma} \). We will call the exceptional divisor of this blowup \( D_\sigma \) and will describe its structure as a product of moduli spaces in this section as well. We now state and prove a result which allows us to factor any reduction as a sequence of simple wall crossings.

**Lemma 2.27.** For any two positive weight data \( A \geq B \), there exist weight data \( A' \geq B' \) such that \( \Delta_A = \Delta_{A'} \), \( \Delta_B = \Delta_{B'} \), and the the straight line from \( A' \) to \( B' \) goes through a sequence of simple wall crossings.

**Proof.** Indeed, we may change \((a_i)\) by \((a_i - \epsilon_i)\) for small generic \( \epsilon_i > 0 \), and similarly for \( b_i \).

The condition that the line passes through a non-generic intersection of the walls \( \sum_{i \in I} a_i = 1 \) is a union of hyperplanes. Thus for the generic \( \epsilon_i \) this does not happen.\( \square \)

**Theorem 2.28.** There exists a natural closed embedding \( \iota : \overline{M}_{g,A_\sigma}(V,\beta) \to \overline{M}_{g,A}(V,\beta) \).

The image is a global complete intersection of codimension \( r - 1 \).

**Proof.** (Sketch) The inclusion map \( \iota \) is given by identifying sections of \( \overline{M}_{A_\sigma} \) as follows. The sections in \( \overline{M}_{A_\sigma} \) are by definition labeled by \( i \not\in \sigma \) and by \( \sigma \). The sections with \( i \not\in \sigma \) are identified with the corresponding section in \( \overline{M}_A \). We identify the section labeled by \( \sigma \) with each section of \( \overline{M}_A \) for every \( i \in \sigma \). The subspace is a complete intersection because it is given by the condition that each section for \( i \in \sigma \) are equal.\( \square \)

In the following lemma, we give an explicit description of a genus 0 moduli space which will appear frequently.

**Lemma 2.29.** Consider the complex \( \Delta_{pr-2} \) as defined in Example 2.22.5. We have that \( \overline{M}_{0,\Delta_{pr-2}} \simeq \mathbb{P}^{r-2} \).

**Proof.** If \( C_\delta \) is not \( \mathbb{P}^1 \), then it is a tree of \( \mathbb{P}^1 \)'s with at least two endpoints. For the corresponding irreducible components \( E_j \) of \( C \), one must have \( \sum_{s_i \in E_j} a_i > 1 \), which is not possible by the conditions.
Let us denote the section corresponding to the isolated vertex by $\infty$, and a section corresponding to $a_1$ by 0. Then $\overline{M}_{0,\Delta_{pr-2}}$ is the moduli of $r$ points on $\mathbb{A}^1$, not all of which are equal to 0. Taking into account that $\text{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{G}_m$, we get

$$\overline{M}_{0,\Delta_{pr-2}} = (\mathbb{A}^1)^{r-1}/\mathbb{G}_m = \mathbb{P}^{r-2}. \quad \square$$

**Theorem 2.30** (cf. Prop 4.5 [Has03],) For a simple wall crossing indexed by $\sigma$, $\overline{M}_{g,A^+}(V, \beta)$ is the blowup of $\overline{M}_{g,A}(V, \beta)$ along a subspace isomorphic to $\overline{M}_{g,A^\sigma}(V, \beta)$ which has codim $= \dim \sigma$. The exceptional Cartier divisor is

$$D_\sigma := \overline{M}_{0,\Delta_{pr-2}} \times \overline{M}_{g,A_\sigma}(V, \beta)$$

with the first product having marked points indexed by the indices in $\sigma$ and the second product having marked points indexed by the indices not in $\sigma$. The fiber product of these spaces is taken along an additional marked point of each.

**Proof.** See Theorem 4.8 in [AG06]. \qed

**Corollary 2.31.** The reduction morphism $\rho_{B,A}$ is an isomorphism iff $\Delta_A$ and $\Delta_B$ differ only in several edges, i.e., simplices of dimension 1.

**Proof.** This being because the blowup of a codimension 1 subspace is an isomorphism. \qed

Before stating an additional corollary we make another definition.

**Definition 2.32.** Let $A$ and $B$ be two admissible weights of the same length with $A \geq B$. Define the set

$$\mathcal{F}(A, B) := \{ \sigma \subset \Delta_B : \sigma \not\subset \Delta_A \}$$

**Remark 2.33.** We will endow the set $\mathcal{F}(A, B)$ with two different orderings. The first will be simply the partial order of containment which we will make use in Corollary 2.34. The second will be a total ordering which will be described and used in Chapter 4.

We note that the set $\mathcal{F}(A, B)$ is in obvious bijection with the set of divisors which are contracted by the reduction morphism $\rho_{B,A}$. This observation gives us the following corollary.
Corollary 2.34. Let $A \geq B$ with zero weights fixed. The general reduction morphism 
$\rho_{B,A} : \overline{M}_{g,A}(V,\beta) \rightarrow \overline{M}_{g,B}(V,\beta)$ is given by an arrangement (not necessarily with smooth, reduced nor connected loci) of blowups for subspaces given by the set $\mathcal{F}(A,B)$ with containment as the partial order.

Proof. Clearly we may remove the faces of $\Delta_B$ one at a time to form $\Delta_A$. These are precisely given by $\mathcal{F}(A,B)$. Moreover, we must remove the faces in an order which respects containment, which is the partial order on $\mathcal{F}(A,B)$. Each time we remove a face, Theorem 2.30 gives that we are performing an ordinary blowup. \qed

Lemma 2.35. Consider the complex $\Delta_{pr-2}$ as defined in Example 2.22.5. The universal family to $\overline{M}_{0,\Delta_{pr-2}}$ is the blowup of $\mathbb{P}^{r-1}$ at a point.

Proof. We saw in Example 2.22.6 that the universal family is given by $\text{Cone}(\Delta_{pr-2})$. Recall that $\Delta_{pr-2}$ is given by a single isolated vertex and the boundary of an $(r-1)$ simplex. Hence the moduli space for a complex given by the boundary of an $r$ simplex along with an isolated vertex is isomorphic to $\mathbb{P}^{r-1}$. The cone over $\Delta_{pr-2}$ is clearly the boundary of an $r$ simplex with the face of the $r$ simplex not containing the vertex of the cone removed, along with a 1 simplex connecting the vertex of the cone to the isolated vertex. We may ignore the 1 simplex (up to isomorphism) by Corollary 2.31. To remove this last face of the boundary of the $r$ simplex from the complex for $\mathbb{P}^{r-1}$, we must remove 1 face with $r$ vertices. By Theorem 2.30, this is given by a blowup of a space isomorphic to $\overline{M}_{0,3}$, which is a point. \qed

Theorem 2.30 gives that the divisor $D_{\sigma}$ is a product of two weighted moduli spaces. The first product $\overline{M}_{0,\Delta_{pr-2}} \simeq \mathbb{P}^{r-2}$ has already been described very explicitly above. This part of the product accounts for the curves, all of genus 0, which are contracted to a point by the map such that after the wall crossing, the collection of marked points on $\overline{M}_{0,\Delta_{pr-2}}$ are no longer sufficient to maintain the stability of the map. Thus, this entire component of the
curve is replaced by marking the node which we label repeatedly for each index in the wall crossing. This component of the curve is now stable with respect to the new weights.

We give a brief example of this now. Suppose that we have a complex \( \Delta_{A^+} \) with 5 vertices and contains all 1 cells. We can then form the complex \( \Delta_A \) by adding the additional 2 cell \( \sigma = \{1, 2, 3\} \). Then the maps which contain only the marked points indexed by \( \sigma \) on a contracted genus zero component will now become unstable. As a result, these maps must be contracted by the reduction morphism \( \rho \). An illustration of this example is given in Figure 2.1.

\[ \text{Figure 2.1: Reduction morphism} \]

From this description, we see that the maps which are being contracted can be parameterized by first picking a genus 0 curve \( C_1 \) with marked points indexed by \( \sigma \) and one additional gluing point of \( C_1 \). This is equivalent to giving a curve with a map \( f_1 : C_1 \to V \) mapping \( C_1 \) to a single point of \( V \). This gives the moduli interpretation of the first product since it gives all possible choices for \( C_1 \) with these requirements.

The second choice we must make is a map \( f_2 : C_2 \to V \) which is stable with respect to the indices not in \( \sigma \) along with an additional distinct point which we will use to glue. Additionally, we have that \( p_a(C_2) = g \) and \( (f_2)_*(C_2) = \beta \). This gives the moduli interpretation of the second product. Finally, we create a new map \( f : C \to V \) which is given by a curve \( C \) which is \( C_1 \) and \( C_2 \) joined together to form a node of \( C \). We get a well defined map this way as long as \( f_1 \) and \( f_2 \) both map the gluing points to the same point of \( V \). This last fact accounts for the fiber product.

This locus of maps are all contracted to the subspace of maps where all indices in \( \sigma \) correspond to a single point of our curve. Additionally, the only indices which may coincide
with this new marked point are those which came from $\sigma$ (or those with zero weights which do not change going from $A$ to $A^+$). This gives the moduli description for the base of the blowup. We see that this is given precisely by the complex $\Delta_{A_\sigma}$. In our example above, the complex $\Delta_{A_\sigma}$ is now given by 1 isolated vertex labeled by $\sigma$ and an edge connecting the vertices labeled by 4, 5. The effect on the complexes is thus that the reduction morphism creates $\Delta_{A_\sigma}$ from $\Delta_{A^+}$ (or also $\Delta_A$), by contracting the vertices in $\sigma$ to a new single vertex labeled by $\sigma$. 

Chapter 3

Main Results

3.1 Psi Classes

Let $\pi_A : C_{g,A}(V, \beta) \to \overline{M}_{g,A}(V, \beta)$ be the map of the universal family with sections $s_{i,A}$ and relative dualizing bundle $\omega_{\pi_A}$, and let $N_{s_i}$ be the normal bundle of $s_i$ in the universal family. We refer the interested reader to [Har66, III] for the definition of the relative dualizing bundle.

**Definition 3.1.** For a positive weight data $A$, we define the psi classes to be

$$\psi_{i,g,A} := c_1(s_{i,A}^*(\omega_{\pi_A})) = -c_1(N_{s_i})$$

Note that if $a_i > 0$, then by Definition 2.5, $s_i$ is contained in the locus of $\pi : C \to S$ where $\pi$ is smooth. Hence, $\psi_{i,g,A}$ are the first Chern classes of invertible sheaves. If some of the weights $a_i = 0$, we must adjust this definition.

Let $A = (a_i)$ be the positive weight data. Then by Lemma 2.25, the universal family $C$ over $\overline{M}_{g,A}(V, \beta)$ is $\overline{M}_{g,(a_i-\delta, \epsilon)}(V, \beta)$, and so for the weight-$\epsilon$ section there is a well-defined psi class using the above definition, and let us denote it simply $\psi$.

For the weight data $A \cup 0^n$, we define the psi class for the section $s_j$ with $a_j = 0$ to be the pullback from $C^n$, the $m$-th fibered power over $\mathcal{M}_{g,A}(V, \beta)$, of $\psi$ under the $j$-th projection $p_j$ of:

$$\mathcal{M}_{g,A \cup 0^n}(V, \beta) = C^n \to C : \quad \psi_j := p_j^*(\psi)$$

We may refer to the psi class of a vertex of $\Delta$ with the obvious meaning. We also omit any subscripts of the notation whenever it is unlikely to lead to confusion.
**Lemma 3.2.** Consider the complex $\Delta_{\mathbb{P}^{r-2}}$ as defined in Example 2.22.5. The psi classes of the nonisolated vertices are $-h$, for $h$ the hyperplane section of $\mathbb{P}^{r-2}$, and the isolated vertex has psi class $h$.

**Proof.** Consider the map $\pi$ of the universal curve, which is described in Lemmas 2.29 and 2.35. For the preimages of hyperplanes $H_i$, resp. for the exceptional divisor $E$ of the blowup, we get:

$$N_{s_i} = \mathcal{O}_{H_i}(H_i) = \mathcal{O}_{H_i}(1), \quad \text{resp.} \quad N_{s_i} = \mathcal{O}_E(E) = \mathcal{O}_E(-1)$$

Hence, $\psi_i = -c_1(N_{s_i})$ are as claimed.

3.1.1 Pull-back via reduction

Recall that $D_{\sigma}$ is the divisor parameterizing maps of curves with sections corresponding to $\sigma$ on a contracted genus zero component and the remaining sections on the genus $g$ component as in Corollary 2.30. The blowup of $\overline{M}_A$ induces two blowups in the universal family. We state this in the following lemma.

**Lemma 3.3.** Let $\pi_A : \mathcal{C}_A \to \overline{M}_A$ be the universal family given by forgetting the extra section $\bullet$, and similarly for $A^+$. The reduction morphism $\rho : \overline{M}_{A^+} \to \overline{M}_A$ induces a map $\rho' : \mathcal{C}_{A^+} \to \mathcal{C}_A$ which we may factor as $\rho' = \rho_2 \circ \rho_1$ of two simple blowups in the universal family. They form a diagram

The map $\rho_1 : \overline{M}_{A'} \to \mathcal{C}_A$ is a blowup of codimension $\dim \sigma$ with exceptional divisor $D'_{\sigma}$ parameterized by $\sigma$. The second map $\rho_2 : \mathcal{C}_{A^+} \to \overline{M}_{A'}$ is a blowup of codimension $\dim \sigma + 1$ with exceptional divisor $D'_{\sigma^*}$ parameterized by $\sigma^* := \sigma \cup \{\bullet\}$. 
Proof. Recall that $\mathcal{C}_A = \overline{M}_{\text{Cone}(\Delta_A)}$ with vertex $\bullet$ and similarly for $\mathcal{C}_A^+$. We compare complexes to see that $\text{Cone}(\Delta_A^+) \subset \text{Cone}(\Delta_A)$. We also see that they differ in two faces: the first of which being $\sigma$ and the second being $\sigma \cup \{\bullet\}$. This gives the sequence of two blowups as described. 

Before completing a computation of how the psi classes change under a simple wall crossing, we we review a basic fact from intersection theory which will prove to be key to our calculations.

**Proposition 3.4 (Adjunction Formula).** Let $X$ be Gorenstein over $Y$ with relative dualizing sheaf $\omega_{X/Y}$. Let $\rho : \tilde{X} \to X$ be a blowup along a codimension $r$ complete intersection $Z \subset X$ with exceptional divisor $E$. Then we have that:

$$\omega_{\tilde{X}/Y} = \rho^*(\omega_{X/Y}) \otimes \mathcal{O}_\tilde{X}(E)^{\otimes r-1}$$

**Proof.** Compare to [Ful98], Example 15.4.3. 

We are now ready to state the main result of this section.

**Theorem 3.5.** For any simple wall crossing with positive weight data $A$, we have

$$\psi_{i,A^+} = \begin{cases} 
\rho^*(\psi_{i,A}) + D_\sigma & \text{for } i \in \sigma \\
\rho^*(\psi_{i,A}) & \text{for } i \notin \sigma
\end{cases}$$

**Proof.** By Theorem 2.24, the universal family is locally Gorenstein over $\overline{M}_{\mathcal{A}_\sigma}$, and hence, all relative dualizing sheaves in the following computations are invertible. Applying the results of Lemma 3.3 and Proposition 3.4 we get

$$\omega_{\mathcal{C}_A^{+}/\overline{M}_{\mathcal{A}_\sigma}} = (\rho')^*(\omega_{\mathcal{C}_A/\overline{M}_{\mathcal{A}_\sigma}}) \otimes \mathcal{O}(D')^{\otimes \dim \sigma} \otimes \mathcal{O}(D'_\bullet)^{\otimes \dim \sigma+1}$$

$$\omega_{\mathcal{M}_A^{+}/\overline{M}_{\mathcal{A}_\sigma}} = \rho^*(\omega_{\mathcal{M}_A/\overline{M}_{\mathcal{A}_\sigma}}) \otimes \mathcal{O}(D_\sigma)^{\otimes \dim \sigma}$$

Moreover, since $\pi^*_A(D_\sigma) = D'_\sigma + D'_{\sigma\bullet}$, in combination with the above we have that

$$\omega_{\mathcal{C}_A^{+}/\overline{M}_{\mathcal{A}_\sigma}} \otimes \pi^*_A(\omega^{-1}_{\mathcal{M}_A^{+}/\overline{M}_{\mathcal{A}_\sigma}}) = (\rho')^* \left( \omega_{\mathcal{C}_A/\overline{M}_{\mathcal{A}_\sigma}} \otimes \pi_A^*(\omega^{-1}_{\mathcal{M}_A/\overline{M}_{\mathcal{A}_\sigma}}) \right) \otimes \mathcal{O}(D'_\bullet)$$
We rewrite this to see that
\[ \omega_{\pi_{A^+}} = (\rho')^*(\omega_{\pi_A}) \otimes O(D_{\sigma^*}) \]

We note that the restriction of a section \( s_i \) to \( D_{\sigma^*} \) is \( D_\sigma \) for \( i \in \sigma \) and 0 otherwise. After taking first Chern classes, this gives the result. \( \square \)

Using \( \mathcal{F}(A, B) \) of Definition 2.32 we give the following corollary.

**Corollary 3.6.** For any positive weights \( A \geq B \), we have
\[ \psi_{i,A} = \rho_{B,A}^*(\psi_{i,B}) + \sum_{\sigma \in \mathcal{F}(A, B)} D_{\sigma} \quad i \in \sigma \]

### 3.1.2 Pullback via decreasing from \( \epsilon \) to 0

The following theorem is a technical result needed to ensure that reductions from \( \epsilon \) to 0 do not cause any additional changes to the psi classes. This is needed since we do not claim that this reduction is a blowup. In fact, in cases whenever \( \overline{M}_{g,A}(V, \beta) \) is nonsingular with all positive weights (such as whenever \( V = \{pt\}, \beta = 0 \)), the space with zeros can be quite singular. See [Has03] §2.1.1 for a more detailed discussion of this case.

**Theorem 3.7.** Let \( (a_i) \) be the positive weight data and suppose that for all \( 0 < \epsilon \ll 1 \), the weight data \( A_\epsilon = (a_i, \epsilon^m) \) is in the interior of the same chamber as \( A_0 = (a_i, 0^m) \), and the latter is admissible. Let \( \rho : \overline{M}_{g,A_\epsilon}(V, \beta) \to \overline{M}_{g,A_0}(V, \beta) \) be the reduction morphism. Then we have that:
\[ \psi_{i,A_\epsilon} = \rho^* \psi_{i,A_0} \quad \text{for all } i. \]

**Proof.** See Theorem 5.6 in [AG06]. \( \square \)
3.1.3 Pull-back via the forgetful morphism

We start by considering the pullback of psi classes whenever the forgetful map corresponds to that of the universal curve as given in Lemma 2.11 and Corollary 2.21.

Lemma 3.8. Let $A$ be any admissible weight data of length $n$, and let \( \phi : \overline{M}_{g,A;\emptyset}(V,\beta) \to \overline{M}_{g,A}(V,\beta) \) be the forgetful morphism. Then

\[
\psi_{i,A;\emptyset} = \phi^* \psi_{i,A} \quad \text{for all } 1 \leq i \leq n.
\]

Proof. Indeed, the universal family over \( \overline{M}_{g,A;\emptyset}(V,\beta) \) is the cartesian product

\[
\mathcal{C}_{g,A}(V,\beta) \times \overline{M}_{g,A}(V,\beta) \xrightarrow{\pi} \overline{M}_{g,A;\emptyset}(V,\beta),
\]

and so \( \omega_{\pi;\emptyset} \) is the pullback of \( \omega_\pi \). \qed

We start with a notational definition.

Definition 3.9. For a weight data $A$ and index $k$, we define $A_{k,\epsilon}$ to be a weight data associated to \( \mathrm{Cone}_k(\Delta_A \setminus \{a_k\}) \). That is, the weight data $A_{k,\epsilon}$ will give the universal family over $A \setminus \{a_k\}$ whenever we forget the $k$-th section.

We note that, just as in the standard unweighted case, the faces in $\mathcal{F}(A, A_{k,\epsilon})$ are in bijection with divisors $D_{\sigma}$ of $\overline{M}_A$ whose maps become unstable after forgetting the $k$-th section.

We recall that in the unweighted case, we have the well known basic pullback relationship which states that:

\[
\psi_{i,n+1} = \phi^*(\psi_{i,n}) + D_{i,n+1}
\]

where $D_{i,n+1}$ is the divisor with only the marked points $i, n+1$ on a genus zero contracted component, and $\phi$ is the morphism which forgets the $n+1$-st point and stabilizes. Using these observations, we state and prove the analogue of the basic pullback relationship.
Theorem 3.10 (Basic Pullback Relation). For any admissible weight data $A$, if $A' := A \setminus \{a_k\}$ is also admissible, then

$$\psi_{i,A} = \phi_{A',A}^*(\psi_{i,A'}) + \sum_{\sigma \in \mathcal{F}(A,A_k,\epsilon)} D_\sigma$$

Proof. We factor the forgetful morphism first by a reduction, then by the map of the universal family. By Theorem 3.7, we may assume that $A$ is positive. We decrease the weights $a_j$ a little to get into the interior of a chamber. Then we decrease the weight $a_k$ we are about to forget to $\epsilon$, and then to 0. The psi classes will change as claimed by Corollary 3.6 and Lemma 3.8. Then we apply Theorem 3.7 one more time if $A$ was not originally positive.

The next lemma pertains to one of the first examples of weighted moduli spaces which were studied by A. Losev and Yu. I. Manin in [LM00]. It is, interestingly enough, the toric variety associated to the permutohedron, the convex hull of the $S_n$–orbit of $(1, 2, \ldots, n)$. In fact, this moduli space can be interpreted as the moduli space of stable $(n-2)$-pointed genus zero curves $(\mathbb{G}_m \curvearrowright C, s_2, \ldots, s_{n-1})$ with torus action.

Lemma 3.11. Consider the complex $\Delta_{\mathcal{L}_{n-2}}$ for the weight data $(1^2, \epsilon^{n-2})$, as in Example 2.22.4. Then the psi classes of the nonisolated vertices are zero.

Proof. By Theorem 3.10, the psi classes of the nonisolated vertices pull back from $\Delta_{\mathcal{L}_1}$. $\overline{M}_{\Delta_{\mathcal{L}_1}}$ is zero dimensional and isomorphic to $\overline{M}_{0,3}$. Whence the pullback is zero.

3.2 Gravitational Descendants

3.2.1 Virtual Fundamental Class

A crucial ingredient in the theory of stable maps is the notion of the virtual fundamental class of $\overline{M}_{g,A}(V, \beta)$. The virtual fundamental classes are defined as a substitute for the
usual fundamental class of a space. This is necessary because the deformation theory of these spaces is often obstructed. These obstructions result in the actual dimension of the moduli spaces to differ from the expected dimension.

When the deformation theory is unobstructed, the virtual fundamental class corresponds to the usual fundamental class of the space. Two well studied unobstructed cases are the case for any genus and $V$ being a point and also for $g = 0$ and $V$ a convex variety; i.e., for every morphism $\mu : \mathbb{P}^1 \to V$, we have $H^1(\mathbb{P}^1, \mu^*(T_V)) = 0$. The latter class of varieties includes projective spaces, grassmannians, and $G/B$.

In the unweighted case, these classes are treated in [BF97] and [Beh97], among others. The weighted context was treated in A. Bayer’s dissertation whose results are in [BM06].

3.2.2 Properties of the Virtual Fundamental Class

We repeat from [BM06] the the facts of which we will make use in our calculations. These statements are originally found in §4 of [BM06] and are proven to hold later in later sections of their work. We make some notational changes to their statements to adhere to our conventions.

**Theorem 3.12** (Theorem 6.3.2, [BM06]). *There exist a theory of virtual fundamental classes for weighted stable maps with the following properties:*

1. **Mapping to a point.** If $\beta = 0$, then

   $$[\overline{M}_{g,A}(V,0)]^\text{virt} = c_{g \dim V}(R^1\pi_*f^*TV)$$

2. **Forgetting a tail.** Assume that $\phi_{A,A\cup\{e\}}$ is the universal curve over $\overline{M}_{g,A}(V,\beta)$. In particular, this implies that $\phi_{A,A\cup\{e\}}$ is flat, and thus defines a pull-back in intersection theory. We require:

   $$\phi_{A,A\cup\{e\}}^*[\overline{M}_{g,A}(V,\beta)]^\text{virt} = [\overline{M}_{g,A\cup\{e\}}(V,\beta)]^\text{virt}$$
3. Combining tails. Consider the closed embedding \( i : \overline{M}_{g,A} (V, \beta) \to \overline{M}_{g,A} (V, \beta) \) of Theorem 2.28. We require that:

\[
i! [\overline{M}_{g,A} (V, \beta)]^\text{virt} = [\overline{M}_{g,A} (V, \beta)]^\text{virt}
\]

4. Gluing. We fix \( g_1, A_1, g_2, A_2 \) and some \( \beta \in H^+_2 (V, \mathbb{Z}) \). Set \( g = g_1 + g_2 \) and \( A = A_1 \cup A_2 \). Consider the gluing morphisms

\[
\mu_{\beta_1, \beta_2} : \overline{M}_{g_1,A_1 \cup \{1\}} (V, \beta_1) \times \overline{M}_{g_2,A_2 \cup \{1\}} (V, \beta_2) \times V \times V \to \overline{M}_{g,A} (V, \beta)
\]

for all \( \beta_1, \beta_2 \) with \( \beta_1 + \beta_2 = \beta \). The union of their images is the boundary component in \( \overline{M}_{g,A} (V, \beta) \) given as the pull-back

\[
\begin{array}{ccc}
\overline{M}_{g_1,A_1 \cup \{1\}} (V, \beta) & \longrightarrow & \overline{M}_{g,A} (V, \beta) \\
\downarrow & & \downarrow \\
\overline{M}_{g_1,A_1 \cup \{1\}} \times \overline{M}_{g_2,A_2 \cup \{1\}} & \mu & \longrightarrow \overline{M}_{g,A}
\end{array}
\]

Since the moduli spaces of weighted stable curves are smooth, \( \mu \) is a local complete intersection morphism and defines a pull-back \( \mu^! [\overline{M}_{g,A} (V, \beta)]^\text{virt} \). On the other hand, via the diagonal \( \Delta : V \to V \times V \), we may pull-back the virtual fundamental class on the product \( \overline{M}_{g_1,A_1 \cup \{1\}} (V, \beta_1) \times \overline{M}_{g_2,A_2 \cup \{1\}} (V, \beta_2) \) to the fibre product that is the source of \( \mu_{\beta_1, \beta_2} \). We require that:

\[
\mu^! [\overline{M}_{g,A} (V, \beta)]^\text{virt} = \sum_{\beta_1 + \beta_2 = \beta} \mu_{\beta_1, \beta_2}^! \left( [\overline{M}_{g_1,A_1 \cup \{1\}} (V, \beta_1)]^\text{virt} \times [\overline{M}_{g_2,A_2 \cup \{1\}} (V, \beta_2)]^\text{virt} \right)
\]

5. Kontsevich-stable maps. If all weights are 1, then \( [\overline{M}_{g,A} (V, \beta)]^\text{virt} \) agrees with the definition of virtual fundamental classes of [BF97, Beh97].

6. Reducing weights. Given two set of weights \( A \geq B \), we require compatibility with the reduction morphism \( \rho_{B,A} \):

\[
\rho_{B,A}^* [\overline{M}_{g,A} (V, \beta)]^\text{virt} = [\overline{M}_{g,B} (V, \beta)]^\text{virt}
\]
Remark 3.13. It should be noted that in [BM06] as well as in [AG06], we actually use properties (5),(6) as the definition of the virtual fundamental class. In [BM06], they show that the additional properties listed above then hold.

3.2.3 Evaluation Morphisms

The moduli spaces $\overline{M}_{g,A}(V,\beta)$ are equipped with $n$ evaluation morphisms $\nu_{i,A} : \overline{M}_{g,A}(V,\beta) \to V$ defined by $\nu_{i,A}([C,\{s_i\},f]) = f(s_i)$.

Definition 3.14. For each index $i$, fix $\gamma_i \in A^*(V,Q)$. The evaluation classes are the pullback classes $\nu_{i,A}^*(\gamma_i)$.

Essentially, the $i$-th evaluation morphism pulls back a class $\gamma_i$ of $V$ to the locus of $A$-stable maps such that the image of the $i$-th marked point is contained in the subvariety represented by $\gamma_i$.

3.2.4 Definition of Gravitational Descendants

Definition 3.15. We define an analogue of the usual notion of the gravitational descendants of Gromov-Witten theory which we denote as:

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,A}^{V,\beta} := \int \left( \prod \psi_{i,A}^{k_i} \cup \nu_{i,A}^*(\gamma_i) \right) \cap \left[ \overline{M}_{g,A}(V,\beta) \right]^{\text{virt}}$$

where $\gamma_i \in A^*(V,Q)$ ($\gamma_i \in H^*(V,Q)$ when working over $\mathbb{C}$), and each $k_i$ is a nonnegative integer.

As usual, these are defined to be zero unless:

$$\sum_{i=1}^n (k_i + \deg \gamma_i) = (1 - g) \dim V - K_V \cdot \beta + (3g - 3 + n)$$

Whenever any $k_i$ is negative, we define this to be zero as well.

We note that whenever $k_i = 0$, these are simply the Gromov-Witten invariants of $V$. We warn the reader to treat the $\tau$’s as noncommuting variables and to not shift indices without discretion as the symmetry of these descendants is very often broken. One can
describe the commuting properties of the $\tau$’s in terms of the symmetries of the complex $\Delta_A$, but we will make no use of this and leave it to the reader. When the weight is $(1^n)$, we omit the weight and note the number of marked sections, the genus $g$, $V$ and $\beta$.

The first property of the weighted descendants is this:

**Lemma 3.16.** For $g, n, V, \beta$ fixed, each descendant $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,A}^{V,\beta}$ is constant as $A$ varies in a chamber.

**Proof.** Indeed, when $A$ varies so that the zero weights remain the same, the moduli space and the universal family stay constant by Proposition 2.16. When some coefficients decrease from $\epsilon$ to 0, the intersection stays constant by Theorem 3.7 and the projection formula.

3.3 Kappa Classes of Miller, Morita and Mumford and the Numerical Invariants of Graber, Kock and Pandharipande

Some weighted descendants are actually very familiar. We repeat here a short discussion of the kappa classes of Miller, Morita and Mumford as found in [Wit91]. For this, we set $V = \{pt\}$ and $\beta = 0$ and fix $n$ and $g$.

Let $C^n_g$ be the $n$-fold fiber product of the universal curve over $\overline{M}_{g,0}$, which is isomorphic to $\overline{M}_{g,1}$, and $\pi_i$ be the $i$-th projection. Let $K_{\mathcal{C}_i/M}$ be the cotangent bundle to the fibers of the $i$-th factor of $C^n_g$ and define $\hat{\mathcal{L}}_i := \pi_i^*(K_{\mathcal{C}_i/M})$. Then we have the following definition.

**Definition 3.17.** The **kappa numbers** of Miller, Morita and Mumford are given by the intersection

$$\langle \kappa_{k_1-1} \cdots \kappa_{k_n-1} \rangle_{g,n} := \int_{C^n_g} \prod_{i=1}^n c_1(\hat{\mathcal{L}}_i)^{k_i}$$

With this definition in mind, we see that the following lemma is almost immediate.

**Proposition 3.18.** The descendant for the weight data $(0^n)$ are the intersections of the Miller, Morita and Mumford kappa classes

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,(\epsilon^n)} = \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,(0^n)} = \langle \kappa_{k_1-1} \cdots \kappa_{k_n-1} \rangle$$
Proof. The first identity is by Theorem 3.7. The second identity is simply by the definition of the kappa numbers and by our definition of the psi classes for the weight data \((0^n)\).

A second familiar example of a previously studied theory which we reinterpret in this context is by Graber, Kock and Pandharipande in [GKP02]. This example is for any \(V, \beta, g, n\). The motivation for their studies was in application to the enumerative connection they proved to the characteristic number problem. In their work, the psi classes themselves are modified, but the moduli spaces are not changed. We give a brief overview of the connection to their work now. A modification of the psi classes is defined as follows.

Suppose \(\beta > 0\) or \(g > 0\). For each marked point \(s_i\), let

\[
\hat{\pi}_i : \overline{M}_{g,n}(V, \beta) \to \overline{M}_{g,\{s_i\}}(V, \beta)
\]

be the morphism which forgets all marked points but \(s_i\). The modified psi class \(\overline{\psi}_i\) on \(\overline{M}_{g,n}(V, \beta)\) is by definition:

\[
\overline{\psi}_i := \hat{\pi}_i^*(\psi_i)
\]

Therefore, these modified psi classes are simply the pullbacks of the psi classes for the weight data \((0^n)\) via the reduction morphism \(\rho : \overline{M}_{g,(1^n)}(V, \beta) \to \overline{M}_{g,(0^n)}(V, \beta)\). We have thus proven the following proposition.

**Proposition 3.19.** The modified gravitational descendants using psi classes \(\psi_1, \ldots, \psi_m\) and modified psi classes \(\overline{\psi}_m+1, \ldots, \overline{\psi}_n\) are the gravitational descendants for the weight data \((1^m, 0^{n-m})\).

Since the forgetful morphism used in the definition above does not exist for the case whenever \(g = 0, \beta = 0\), a different construction, with the same application, was necessary. This construction is extended to this remaining case by Kock in [Koc04]. Here the modified psi classes are constructed with the following twist: Start with \(\overline{M}_{0,n+3}(V, 0)\) with three additional distinguished marked points \(q_1, q_2, q_3\). For each of the other marked points
$p_i, i \leq n$, define

$$\hat{\pi}_i : \overline{M}_{0,n+3}(V,0) \to \overline{M}_{0,(q_1,q_2,q_3,p_i)} \simeq \mathbb{P}^1$$

to be the map which forgets the sections not in $\{q_1,q_2,q_3,p_i\}$ as well as the map to $V$. For $i \leq n$, the definition is extended to this case to be:

$$\overline{\psi}_i := \hat{\pi}_i^*(-2h)$$

That is, it is the pull-back of the class of degree $-2$ on $\mathbb{P}^1$. A short calculation is needed to see the connection, and we make it in the following lemma.

**Lemma 3.20.** Consider the four pointed space $\overline{M}_{0,A}$ with labeled points $\{q_1,q_2,q_3,p_i\}$ and the weight data $A = ((1-\varepsilon)^3,\varepsilon)$ with $\varepsilon < 1/n$. Then $\overline{M}_{0,A} \simeq \mathbb{P}^1$ and $\psi_{p_i,A}$ has degree $-2$.

**Proof.** We clearly have that $\overline{M}_{0,A} \simeq \mathbb{P}^1$ for any admissible $A$. So we need only compute the degree of each psi class. The unweighted basic pullback relation recalled in §3.1.3 easily gives that each psi class $\psi_{p_i,(1^4)}$ on the unweighted space $\overline{M}_{0,4}$ has degree 1. We note that $\Delta_A$ contains precisely the faces for each vertex as well as the edges connecting the vertex $p_i$ to each of the vertices $q_j$. Moreover, each divisor $D_{q_j,p_i}$ is simply a point of $\mathbb{P}^1$ and has degree 1. Additionally, the reduction morphism $\rho$ from $(1^4)$ to $A$ is an isomorphism. So application of Corollary 3.6 gives that:

$$\psi_{p_i,(1^4)} = \rho^*(\psi_{p_i,A}) + D_{q_1,p_i} + D_{q_2,p_i} + D_{q_3,p_i}$$

which we may easily solve to see $\psi_{p_i,A}$ has degree $-2$. ∎

### 3.3.1 Generating functions

A useful way of packaging these invariants is the use of generating functions. We define these below and use them to compactly give several examples of gravitational descendants.

**Definition 3.21.** For fixed $n, V, \beta, \gamma_i$, we define the generating polynomial for the descendants to be:

$$e_{g,A}(t) := \sum_{k_1,\ldots,k_n} \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,A}^V,\beta t^{k}$$
and the exponential generating polynomial to be:

\[ E_{g,A}(t) := \sum_{k_1, \ldots, k_n} \frac{1}{k!} \langle \tau_{k_1} (\gamma_1) \cdots \tau_{k_n} (\gamma_n) \rangle_{V, \beta} V, \beta_{g,A} t^k \]

and use the customary multi-index conventions that \( t := (t_1, \ldots, t_n) \), \( k! := k_1! \cdots k_n! \) and \( t^k := t_1^{k_1} \cdots t_n^{k_n} \).

These are polynomials because \( \sum k_i \) is a constant based on the choice of \( n, V, \beta, \gamma_i \) by the dimension constraint on these descendants, and hence there are only finitely many monomials in these sums.

**Example 3.22.**

1. In the case of genus zero and weight \((1^n)\), it is well known that:

\[ e_{0,n} = (t_1 + \cdots + t_n)^{n-3} \]

2. It follows immediately from Lemma 3.11 and the above example, that:

\[ e_{\Delta_{\mathcal{L}^n}} = (t_1 + t_2)^{n-3} \]

3. It also follows immediately from Lemma 3.2 that:

\[ E_{\Delta_{\mathcal{L}^{n-2}}} = \frac{(t_1 - t_2 - \cdots - t_{r+1})^{r-2}}{(r-2)!} \]

### 3.4 Wall Crossing Formula

In this section, we consider the simple wall crossing given by adding \( \sigma \) to \( \Delta_{\mathcal{A}^+} \) to form \( \Delta_{\mathcal{A}} \). As before, denote the divisor corresponding to the crossing as \( D_{\sigma} \), and the reduction morphism \( \rho_{\mathcal{A}, \mathcal{A}^+} \) by \( \rho \). The image of the divisor \( D_{\sigma} \) is given by the complex \( \Delta_{\mathcal{A}_\sigma} \) which is attained from that of \( \Delta_{\mathcal{A}^+} \) (or also of \( \Delta_{\mathcal{A}} \)) by contracting the vertices in \( \sigma \) to a disconnected vertex which we label by \( \sigma \). We also make the following notational definition which we will use often.
**Definition 3.23.** Let $\sigma$ be a collection of vertices of $\Delta_A$ and assume we are given a class $\tau_{k_i}(\gamma_i)$ for each vertex. Then we define $k_\sigma := \sum_{i \in \sigma} k_i - \dim \sigma$ and $\gamma_\sigma := \prod_{i \in \sigma} \gamma_i$.

**Theorem 3.24** (Simple-wall Crossing). We have the following wall crossing formula:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{V,\beta, g,A} = \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A} +$$

$$+ (-1)^{\dim \sigma + 1} \left\langle \tau_{k_\sigma}(\gamma_\sigma) \prod_{j \not\in \sigma} \tau_{k_j}(\gamma_j) \right\rangle_{g,A_\sigma}$$

**Remark 3.25.** Theorem 3.24 gives an answer to a question posed by Alexeev which can be found as [Ale06, Question 7.3].

Before giving the proof of this result, we state some corollaries and prove some preliminary results.

**Corollary 3.26.** The Gromov-Witten Invariants of $\overline{M}_{g,A}(V, \beta)$ are equal to the unweighted invariants for any weight $A$.

**Proof.** This follows from setting $k_i = 0$ and noting that $k_\sigma$ is thus always negative.

**Lemma 3.27.** The normal bundle of $\overline{M}_{g,A_\sigma}(V, \beta)$ in $\overline{M}_{g,A}(V, \beta)$ is given by

$$N = O(-\psi_\sigma)^{\oplus \dim \sigma}.$$ 

**Proof.** Let $\overline{M}$ be the complete intersection of two sections $s_1$ and $s_2$ in the universal family $C_{\overline{M}}$. Then $\overline{M}$ is also the complete intersection of $s_1$ and $C_{\overline{M}}$. This implies that the normal bundle of $\overline{M}$ in $s_2$ is isomorphic to the normal bundle of $\overline{M}$ in $C_{\overline{M}}$, which is $O_{\overline{M}}(-\psi_1)$ by the definition of psi classes. On the locus where all the points $s_i$ coincide, we have

$$O(-\psi_1) = O(-\psi_\sigma).$$

Since $\overline{M}_{g,A_\sigma}(V, \beta)$ is a complete intersection of $\dim \sigma$ Cartier divisors, its normal bundle is a direct sum of $\dim \sigma$ of these line bundles.

$\blacksquare$
Lemma 3.28 (Splitting Lemma). For any $A$, we have:

$$
\mu^! \left[ \overline{M}_{g,A}(V, \beta) \right]^{\text{virt}} = \left[ \overline{M}_{g,A_1} \right] \times \left[ \overline{M}_{g,A_\sigma}(V, \beta) \right]^{\text{virt}}
$$

where $\mu$ is the embedding $\mu : \overline{M}_{g,A_1}(V, \beta) \times \overline{M}_{g,A_\sigma}(V, \beta) \to \overline{M}_{g,A}(V, \beta)$.

Proof. See Lemma 7.6 in [AG06].

Lemma 3.29. One has:

$$
\sum_{p>0} (-1)^{p-1} \rho_* \left( D^p_\sigma \cap \left[ \overline{M}_{g,A^+}(V, \beta) \right]^{\text{virt}} \right) = s(N) \cap \left[ \overline{M}_{g,A_\sigma}(V, \beta) \right]^{\text{virt}}
$$

where $s(N)$ is the Segre class of the normal bundle.

Proof. The latter equality is by the previous lemma:

$$
s(N) = c(\mathcal{O}(-\psi_\sigma)^{\oplus \dim \sigma})^{-1} = (1 - \psi_\sigma)^{-\dim \sigma}
$$

For the first, recall that

$$
D_\sigma = \overline{M}_{0,\Delta_{pr-2}} \times \overline{M}_{g,A_\sigma}(V, \beta) = \mathbb{P}^{r-2} \times \overline{M}_{g,A_\sigma}(V, \beta)
$$

is the exceptional divisor of the blowup. Let $\mu : D_\sigma \to \overline{M}_{g,A^+}(V, \beta)$ be the embedding and $\eta : D_\sigma \to \overline{M}_{g,A_\sigma}(V, \beta)$ be the projection. Then

$$
\sum_{p>0} (-1)^{p-1} \rho_* \left( D^p_\sigma \cap \left[ \overline{M}_{g,A^+}(V, \beta) \right]^{\text{virt}} \right) = \sum_{p>0} (-1)^{p-1} \eta_* \left( D^p_\sigma \cap \mu^! \left[ \overline{M}_{g,A^+}(V, \beta) \right]^{\text{virt}} \right)
$$

because $\mathcal{O}_{D_\sigma}(-D_\sigma) = \mathcal{O}_{\mathbb{P}(N)}(1)$

$$
= \sum_{q \geq 0} \eta_* \left( c_1(\mathcal{O}_{\mathbb{P}(N)}(1))^q \cap \left[ \mathbb{P}^{r-2} \times \overline{M}_{g,A_\sigma}(V, \beta) \right]^{\text{virt}} \right) \quad \text{by Lemma 3.28}
$$

$$
= s(N) \cap \left[ \overline{M}_{g,A_\sigma}(V, \beta) \right]^{\text{virt}} \quad \text{by the definition of the Segre classes.}
$$
Corollary 3.30. We have the following identity

\[
\rho_*(D^p) = \begin{cases} 
(-1)^{p+1} \binom{p-1}{p-\dim \sigma} \psi^p \dim \sigma & \text{for } p \geq \dim \sigma \\
0 & \text{for } p < \dim \sigma
\end{cases}
\]

Proof. We compute the right hand side in the previous lemma and equate dimensions to get the claimed results. \qed

An essential tool in our calculation will be the projection formula which we recall in the following proposition. For a proof, see Theorem 3.2.c of [Ful98].

Proposition 3.31 (Projection Formula). Let \( E \) be a vector bundle on \( X \), \( f : X' \to X \) a proper morphism. Then

\[
f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)
\]

for all cycles \( \alpha \) on \( X' \), and for all \( i \).

We have now gathered the necessary facts in order to give a proof of Theorem 3.24. We give two proofs here. The first proof is an elegant and concise proof which is due to Alexeev and is found as Theorem 7.2 in [AG06]. The second proof is due to the current author.

First Proof of Theorem 3.24. We recall a few facts in preparing to apply the projection formula. Theorem 2.30 gives that the reduction morphism \( \rho \) is given by a blowup along \( \overline{M}_{g,A^+}(V,\beta) \). It is evident that the evaluation morphisms commute with reductions, and so the projection formula allows us to push them forward unchanged. In addition, Theorem 3.5 dictates that for \( i \in \sigma \), \( \psi_{i,A^+} = \rho^*(\psi_{i,A}) + D_\sigma \), but the remaining classes are pullbacks.

We also note that for \( i \in \sigma \), whenever we restrict \( \psi_{i,A} \) to \( \overline{M}_{g,A^+}(V,\beta) \) we get \( \psi_\sigma \). We are now ready to compute:

\[
\left< \prod_{i \in \sigma} \tau_{k_i}(\gamma_i) \prod_{j \notin \sigma} \tau_{k_j}(\gamma_j) \right>_{g,A^+} = \prod_{i \in \sigma} (\rho^*\psi_{i,A} + D_\sigma)^{k_i} \nu_{i,A^+}^{\ast} (\gamma_i) \prod_{j \notin \sigma} (\rho^*\psi_{j,A})^{k_j} \nu_{i,A^+}^{\ast} (\gamma_i) \cap [\overline{M}_{g,A^+}(V,\beta)]^\text{virt}
\]
Let us expand each \((\rho^* \psi_{i,A} + D_\sigma)^{k_i}\) and multiply them out. The only term that does not contain a positive power of \(D_\sigma\) is, by the projection formula and Theorem 3.12.6, the descendant computed on \(\overline{M}_{g,A}(V, \beta)\). Let us deal with the rest.

Note that all the terms with the evaluation classes and with \(\rho^* \psi_{j,A}\) are pullbacks from \(\overline{M}_{g,A}(V, \beta)\). Let us call this part \(\rho^* \tau_J(\gamma)\).

Now look at the remaining part, with \(D_\sigma\) and \(\rho^* \psi_{i,A}\). We now observe that, up to the sign \((-1)^{\sum k_i+1}\), it is the homogeneous degree \(\sum k_i\) part of:

\[
\prod_{i \in \sigma} \rho^*(1 - \psi_{i,A})^{k_i} \times \sum_{p > 0} (-1)^p D_\sigma^p \quad \text{applied to } [\overline{M}_{g,A^+}(V, \beta)]^\text{virt}
\]

By the projection formula \(\rho_*(\rho^* \alpha \cap \beta) = \alpha \cap \rho_* \beta\) and Lemma 3.29, we are reduced to computing, up to the sign \((-1)^{\sum k_i+1}\), the homogeneous degree \(\sum k_i - \dim \sigma = k_\sigma\) part of:

\[
\prod_{i \in \sigma} (1 - \psi_{i,A})^{k_i} \times (1 - \psi_\sigma)^{-\dim \sigma} \quad \text{applied to } \tau_J(\gamma) \cap [\overline{M}_{g,A_\sigma}(V, \beta)]^\text{virt}
\]

But each \(\psi_{i,A}\) restrict to \(\psi_\sigma\) on \(\overline{M}_{g,A_\sigma}(V, \beta)\), so we need to compute the degree \(k_\sigma\) part of:

\[
\prod_{i \in \sigma} (1 - \psi_{i,A})^{k_i} \times (1 - \psi_\sigma)^{-\dim \sigma} = (1 - \psi_\sigma)^{k_\sigma}
\]

which is \((-1)^{k_\sigma} \psi^{k_\sigma}\). This gives the formula. Note also that when \(k_\sigma < 0\), we get zero; all monomials have nonnegative degree.

\[\square\]

**Second Proof of Theorem 3.24.** We recall a few facts in preparing to apply the projection formula. Theorem 2.30 gives that the reduction morphism \(\rho\) is given by a blowup along \(\overline{M}_{g,A_\sigma}(V, \beta)\). It is evident that the evaluation morphisms commute with reductions, and so the projection formula allows us to push them forward unchanged. In addition, Theorem 3.5 dictates that for \(i \in \sigma\), \(\psi_{i,A^+} = \rho^*(\psi_{i,A}) + D_\sigma\), but the remaining classes are pullbacks.

By definition, we have that:

\[
\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{A^+} = \int_{\overline{M}_{A^+}} \prod_{i=1}^n \psi_{i,A^+}^{k_i} \nu_{A^+}(\gamma_i) \quad \text{which we may rewrite as}
\]
We note that for \( i \in \sigma \), whenever we restrict \( \psi_{i,A} \) to \( \overline{M}_{g,A,\sigma}(V, \beta) \) we get \( \psi_{\sigma} \). Since we will apply the projection formula shortly, we treat all \( \psi_{i,A} \) in terms containing a positive power of \( D_\sigma \) as \( \psi_{\sigma} \) now in order to simplify the expansion. Recall also that \( k_\sigma := \sum_{i \in \sigma} k_i - \dim \sigma \).

For convenience let us define \( m := k_\sigma + \dim \sigma \). Thus we expand to get:

\[
\prod_{i \in \sigma} (\rho^*(\psi_{i,A}) + D_\sigma)^{k_i} = \prod_{i \in \sigma} \rho^*(\psi_{i,A})^{k_i} + \sum_{p=1}^{m} \binom{m}{p} D_\sigma^p (\rho^*(\psi_\sigma))^{m-p}
\]

We now substitute this expression into our previous calculation and apply the projection formula as well as Corollary 3.30 and Theorem 3.12.6 to get:

\[
\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{A^+} = \int_{\overline{M}_{g,A,\sigma}} \rho^* \left( \prod_{i=1}^{n} \psi_{i,A}^{k_i} \nu_A^*(\gamma_i) \right) + \int_{\overline{M}_{g,A,\sigma}} \rho^* \left( \prod_{i \notin \sigma} \psi_{i,A}^{k_i} \cdot \prod_{i=1}^{n} \nu_A^*(\gamma_i) \right) \sum_{p=1}^{m} \binom{m}{p} D_\sigma^p (\rho^*(\psi_\sigma))^{m-p}
\]

by simply substituting and regathering the terms with no \( D_\sigma \)

\[
= \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,\sigma} + \int_{\overline{M}_{g,\sigma}} \left( \nu_A^*(\gamma_\sigma) \prod_{i \notin \sigma} \psi_{i,A,\sigma}^{k_i} \nu_A^*(\gamma_i) \right) \cdot \sum_{p=\dim \sigma}^{m} \binom{m}{p} \left( -1 \right)^{p+1} \binom{p-1}{p-\dim \sigma} \psi_\sigma^{p-\dim \sigma} \psi_\sigma^{m-p}
\]

after applying the projection formula and Corollary 3.30

\[
= \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,\sigma} + \left\langle \tau_{k_\sigma}(\gamma_\sigma) \prod_{j \notin \sigma} \tau_{k_j}(\gamma_j) \right\rangle_{g,\sigma} \cdot C
\]

by realizing the power of \( \psi_\sigma \) no longer depends on \( p \).

where we have

\[
C := \sum_{p=\dim \sigma}^{m} (-1)^{p+1} \binom{m}{p} \binom{p-1}{p-\dim \sigma}
\]
To calculate the constant, make the change of variables by letting $p' = p - \dim \sigma$. Then this sum is easily seen to be given in terms of the Gauss Hypergeometric Function (see Remark 3.32 below) as:

$$\textstyle 2F_1 \left( \begin{array}{c} -k_\sigma, \dim \sigma \\ \dim \sigma + 1 \end{array} \right) \cdot (-1)^{\dim \sigma + 1} \left( \frac{m}{\dim \sigma} \right) = (-1)^{\dim \sigma + 1}$$

**Remark 3.32.** Recall that a hypergeometric series is a series $\sum_{k=0}^{\infty} t_k z^k$ such that the ratio $t_{k+1}/t_k$ is a rational function which is written without loss of generality as

$$\frac{(k + a_1)(k + a_2) \cdots (k + a_p)}{(k + 1)(k + b_1)(k + b_2) \cdots (k + b_q)}$$

The notation used for these functions is

$$\textstyle pF_q \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right) z$$

It is standard for $t_0 = 1$. The special case whenever $p = 2, q = 1$ is called the Gauss Hypergeometric Function. For simplification one often lets $a_1 = a, a_2 = b$ and $b_1 = c$. It is then a well known fact that

$$\textstyle 2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

**Definition 3.33.** Let $\Sigma = \{\sigma\}$ be a partition of $\{1, \ldots, n\}$ into a disjoint union of subsets. We say that $\Sigma$ is $\Delta_A$-admissible if each $\sigma$ is in $\Delta_A$. For each $\sigma \in \Sigma$, we define $\dim \Sigma := \sum_{\sigma \in \Sigma} \dim \sigma$ and denote the number of sets in the partition as $|\Sigma|$. We denote the set of $\Delta_A$-admissible partitions by $\Sigma(A)$. In addition, we notate $\Sigma(A, B)$ to be the set of partitions which are $\Delta_B$-admissible, but not $\Delta_A$-admissible.

We are now ready to state a cousin of the string equation which reduces the calculation of weighted descendants to that of the standard unweighted descendants.
Theorem 3.34 (Reduction to Unweighted Descendants). For any admissible weight data $A$, 
\[
\left< \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,A}^{V,\beta} = \sum_{\Sigma \in \Sigma(A)} (-1)^{\dim \Sigma} \left< \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}}(\gamma_{\sigma}) \right>_{g,|\Sigma|}^{V,\beta}
\]

Proof. Pick a simplex $\sigma$ in $\Delta_A$ and apply Theorem 3.24 to get two complexes: one without $\sigma$, and one with $\sigma$ collapsed: it has one vertex instead of $\sigma$, disjoint from the rest. Now continue this inductively. The end result is the alternating sum over partitions of descendants on complexes which are disjoint unions of vertices, i.e., the unweighted descendants. \hfill \Box

Corollary 3.35. The products of Miller, Morita and Mumford classes are expressed in the following way through the products of psi classes:
\[
\left< \kappa_{k_1} \cdots \kappa_{k_n-1} \right>_{g,n} = \sum_{\text{all partitions } \Sigma} (-1)^{\dim \Sigma} \left< \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}} \right>_{g,|\Sigma|}^{V,\beta}
\]

The inverse of this relation, i.e., expressing the psi numbers in terms of the kappa numbers, is due to C. Faber and can be found in [AC96, 1.13].

Corollary 3.36. For any $A \geq B$, we have:
\[
\left< \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,B}^{V,\beta} = \left< \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,A}^{V,\beta} + \\
+ \sum_{\Sigma \in \Sigma(A,B)} (-1)^{\dim \Sigma} \left< \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}}(\gamma_{\sigma}) \right>_{g,|\Sigma|}^{V,\beta}
\]
Chapter 4

General Reductions and Additional Relationships

4.1 General Reduction

We now consider any two complexes $\Delta_A$ and $\Delta_B$ such that $\Delta_A \subset \Delta_B$. We combine the results of the previous sections to give a formula for the descendants on $\overline{M}_{g,A}(V,\beta)$ in terms of those on $\overline{M}_{g,B}(V,\beta)$ and its subspaces. After deriving these results in general, we will be able to state and prove analogs of the well known string, dilaton and divisor equations of Gromov-Witten theory.

We make several more definitions of which we will make frequent use in the remainder of this work. The reader may also like to recall Definition 2.32 for the set $\mathcal{F}(A,B)$ which we shall now revisit. After giving these definitions, we give Example 4.8 to illustrate these ideas.

**Definition 4.1.** Define $\Sigma_{st}(A,B) \subset \Sigma(A,B)$ with $\Sigma \in \Sigma_{st}(A,B)$ if and only if every positive dimensional $\sigma \in \Sigma$ is also in $\mathcal{F}(A,B)$. Here we use $st$ since all positive dimensional sets are strictly in $\Delta_B$, but not $\Delta_A$.

We define one additional relevant complex which generalizes $\Delta_{A_{st}}$ of Example 2.22.7.

**Definition 4.2.** For any weight data $A$ and $\Sigma \in \Sigma(A)$, we define the complex $\Delta_{A\Sigma}$ as follows:

1. For every $\sigma \in \Sigma$, we have a vertex which we shall also label $\sigma$.

2. For every subset $\Gamma \subset \Sigma$ there is a face of $\Delta_{A\Sigma}$, which we shall label $\Gamma$, if $\bigcup_{\sigma \in \Gamma} \sigma$ is a face of $\Delta_A$.
Remark 4.3. The above complex can be seen to be a generalization of $\Delta_{A\sigma}$ where we view $\Delta_{A\sigma}$ as corresponding to a partition with only one positive dimensional set whereas the complex $\Delta_{A\Sigma}$ may have more than one positive dimensional set.

Definition 4.4. Fix a factorization of $\rho = \rho_{B,A}$ into simple wall crossings. This gives an order in which the faces in $\mathcal{F}(A, B)$ are added to $\Delta_A$ in order to form $\Delta_B$. For $\sigma_1, \sigma_2 \in \mathcal{F}(A, B)$, we say that $\sigma_1 \leq_\rho \sigma_2$ if and only if $\sigma_1$ is added to $\Delta_A$ before $\sigma_2$.

Remark 4.5. The ordering given in Definition 4.4 refines the natural partial order of containment among the faces so that we obtain a total ordering. Moreover, a different choice of ordering will not have a tangible impact on the results we derive from it. In particular, the descendants which appear in the remaining formulas of this work would be exactly the same using any choice of factorization. See Example 4.8, where we discuss this in an example.

We now define a graph which will organize all the data we need for the later results.

Definition 4.6 (Reduction Graph). For any two comparable weights $A \geq B$, we fix an ordering given by $\leq_\rho$ on $\mathcal{F}(A, B)$ and define a labeled directed graph $G_{A,B,\leq_\rho}$ (or simply $G_{A,B}$ whenever $\leq_\rho$ is clear) with the following description:

1. Vertices are labeled by $\Sigma$ for every $\Sigma \in \Sigma^{ud}(A, B)$.

2. A directed edge from $\Sigma_1$ to $\Sigma_2$ iff

   (a) There exists a positive dimensional face $\Gamma$ of $\Delta_{B\Sigma_1}$ such that $\Delta_{B\Sigma_2}$ is attained from $\Delta_{B\Sigma_1}$ by collapsing the vertices of $\Gamma$ to form a vertex of $\Delta_{B\Sigma_2}$ which we may label $\gamma$. On the level of partitions this says that we combine $|\Sigma_1| - |\Sigma_2| + 1$ of the sets in the partition $\Sigma_1$ to form the new partition $\Sigma_2$.

   (b) For every positive dimensional $\sigma \in \Sigma_1$, we have that $\sigma \leq_\rho \gamma$.

We assign these edges the value:

$$(-1)^{|\Sigma_1|+|\Sigma_2|+1} = (-1)^{\dim \Gamma + 1}$$
3. One special vertex which we label by $\Delta_B$.

4. Additionally, there are edges from the vertex labeled $\Delta_B$ to the vertices labeled $\Sigma$ such that $\Sigma$ contains precisely 1 positive dimensional set $\sigma \in \mathcal{F}(A, B)$. We label these edges by $(-1)^{\dim \sigma + 1}$.

We form its directed adjacency matrix $W_{\leq \rho}(A, B)$ whose entries $w_{v_i, v_j}$ has the value assigned to the edge connecting $v_i$ to $v_j$. Finally define

$$W_{\leq \rho}^\infty(A, B) := \sum_{\ell=1}^{\infty} W_{\leq \rho}(A, B)^\ell$$

**Lemma 4.7.** The matrix $W(A, B)$ is nilpotent. Hence $W^\infty(A, B)$ is well defined.

**Proof.** This is clear since the conditions we have imposed on the directions of the edges insure that there are no cycles in this graph. By general graph theory facts, the $k$-th power of this matrix counts (with weights) the paths of length $k$ (e.g. see [Sta99, §4.7]). Having only finitely many vertices and no cycles eliminates the possibility of having a path of arbitrarily large length. \qed

We note that the edges are one of two types. The type given in 4.6.4 connects to “first order” correction descendants. That is, the spaces $\overline{M}_{B\Sigma}$ will correspond to loci which we blowup directly in our factorization. The types of edges given in 4.6.2 connect to subspaces of $\overline{M}_B$ which were blownup as a result of pulling back spaces under other blowups. That is, they correspond to subspaces of $\overline{M}_B$ which contain loci of a blowup we performed. A vertex for a partition with only 1 set of positive dimension may have edges of both types.

**Example 4.8.** Consider the reduction morphism $\rho_{B,A}$ for the complexes $\Delta_A, \Delta_B$ in Figure 4.1.

The set $\Sigma^{st}(A, B)$ consists of the partitions:

$$\Sigma_1 = \{\{1, 3\}, \{2\}, \{4\}\}$$

$$\Sigma_2 = \{\{2, 3\}, \{1\}, \{4\}\}$$
We give the set $\mathcal{F}(A, B)$ the ordering $\leq_{\rho}$:

$$\{1, 3\} \leq_{\rho} \{2, 4\} \leq_{\rho} \{2, 3\} \leq_{\rho} \{1, 2, 3\} \leq_{\rho} \{1, 2, 4\}$$

This gives us the first reduction graph in Figure 4.2.

Note that if we had instead used the ordering given by switching the first two in the sequence to get $\leq_{\rho'}$:

$$\{2, 4\} \leq_{\rho'} \{1, 3\} \leq_{\rho'} \{2, 3\} \leq_{\rho'} \{1, 2, 3\} \leq_{\rho'} \{1, 2, 4\}$$

Then the resulting graph would simply have an edge from from $\Sigma_2$ to $\Sigma_6$ instead of from $\Sigma_1$ to $\Sigma_6$. This is the second graph depicted in Figure 4.2.
Figure 4.2: Reduction graph for $\rho_{B,A}$ using two different orderings

We have adjacency matrices given by:

$W_{\leq \rho}(A,B) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$

$W_{\leq \rho'}(A,B) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$

After summing powers, we have:

$W_{\leq \rho}(A,B) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$

$W_{\leq \rho'}(A,B) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$
The important point to note concerning using different orders is that the first column of both of these matrices are equal. This is the only column which we will make use in our statements to come.

**Theorem 4.9 (General Reduction).** Let $A \geq B$. Let $w_{\Sigma}$ be the entry of $W^\infty(A,B)$ in the row for the vertex labeled by $\Sigma$ and the column for the vertex labeled by $\Delta_B$. Then we have:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,B}^{V,\beta} + \sum_{\Sigma \in \Sigma^{st}(A,B)} w_{\Sigma} \cdot \left\langle \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}}(\gamma_{\sigma}) \right\rangle_{g,B_{\Sigma}}^{V,\beta}$$

**Proof.** We proceed by induction on the the number of faces we add to our complex. The case whenever we add only 1 face is precisely the wall crossing formula found in Theorem 3.24. Fix a factorization, and hence an ordering on $\mathcal{F}(A,B)$, so that $\rho_{B,A} = \rho_{B,B'} \circ \rho_{B',A}$ such that $\rho_{B',A}$ adds the faces $\sigma_1, \ldots, \sigma_{N-1}$ to $\Delta_A$ to get $\Delta_{B'}$ and $\rho_{B,B'}$ is a simple wall crossing adding $\sigma_N$ to $\Delta_{B'}$ to get $\Delta_B$. Our inductive hypothesis gives that our formula holds for the reduction $\rho_{B',A}$. So we need only study the effects of this additional step on the objects of which we make use to state our formula.

First we note that $\Sigma^{st}(A,B') \subsetneq \Sigma^{st}(A,B)$. Moreover, the conditions giving an edge in the reduction graph $G_{A,B'}$ from $A$ to $B'$ would still be satisfied when we consider the reduction from $A$ to $B$ and hence the edges would still be in the reduction graph $G_{A,B}$ from $A$ to $B$.

We may identify the vertex of $G_{A,B'}$ labeled $\Delta_{B'}$ with the vertex labeled $\Delta_B$ in $G_{A,B}$. The result is that $G_{A,B'}$ may be naturally identified with a subgraph of $G_{A,B}$.

We consider now the vertices in $G_{A,B} \setminus G_{A,B'}$, which are labeled by partitions in $\mathcal{V} := \Sigma^{st}(A,B) \setminus \Sigma^{st}(A,B')$. We clearly have that $\mathcal{F}(A,B) \setminus \mathcal{F}(A,B') = \{\sigma_N\}$. Thus all partitions in $\mathcal{V}$ contain the set $\sigma_N$. In particular, the partition with the only positive dimensional set $\sigma_N$, which we will denote $\Sigma_N$, is also in $\mathcal{V}$. As a result, there is an edge from $\Delta_B$ to $\Sigma_N$ with label given by the wall crossing formula. Finally, we observe that the complexes such that $\Delta_{B_{\Sigma}} \neq \Delta_{B_{\Sigma}'}$ have correction descendants corresponding to a partition
in \( \mathcal{V} \). Each of these complexes differ by exactly 1 face. The union of the labeling sets for this additional face is \( \sigma_N \). The definition of \( G_{A,B} \) dictates that each of these vertices gives an edge to a vertex in \( \mathcal{V} \). Moreover, the label of this edge is consistent with the wall crossing formula. The order requirement in the definition of \( G_{A,B} \) on positive dimensional faces gives that there is a bijection between the edges \( G_{A,B} \setminus G_{A,B'} \) and the new correction descendants. Without the order requirement, there would be edges from \( \Sigma_N \) to partitions containing \( \sigma_N \) and additional positive dimensional sets. But \( \Delta_{B\Sigma_N} \) already contains a face for the other positive dimensional sets, and so there is no correction descendant corresponding to the additional positive dimensional sets.

To summarize the above, we see that new vertices are indexed precisely by the elements of \( \Sigma_{st}(A,B) \setminus \Sigma_{st}(A,B') \). Additionally, all new edges begin at a vertex of \( G_{A,B'} \) and terminate at one of the new vertices. Thus all new paths in \( G_{A,B} \) from \( \Delta_B \) terminate at a new vertex.

Now to the formula. Our induction hypothesis gives us that

\[
\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \rangle_{g,A}^{V,\beta} = \langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \rangle_{g,B'}^{V,\beta} + \sum_{\Sigma \in \Sigma_{st}(A,B')} w'_{\Sigma} \cdot \langle \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{g,B'_{\Sigma}}^{V,\beta}
\]

The wall crossing formula gives us that the first descendant is given by

\[
\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \rangle_{g,B'}^{V,\beta} = \langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \rangle_{g,B}^{V,\beta} + (-1)^{\dim \Sigma_N} + \langle \prod_{j \notin \sigma_N} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{g,B_{\Sigma_N}}^{V,\beta}
\]

Moreover, the properties of the new edges and vertices as explained above give that the descendants which change as we go from \( \Delta_{B'_{\Sigma}} \) to \( \Delta_{B_{\Sigma}} \) are precisely given by changing our index set from \( \Sigma_{st}(A,B') \) to \( \Sigma_{st}(A,B) \). Moreover, we calculate the coefficients using \( G_{A,B} \) instead of \( G_{A,B'} \). Combining all of this gives the result. \( \square \)
Example 4.10. Continuing using the information from Example 4.8, we apply the above result to see that

\[
\langle \prod_{i=1}^{n} \tau_{k_{i}}(\gamma_{i}) \rangle_{V,\beta}^{g,A} = \langle \prod_{i=1}^{n} \tau_{k_{i}}(\gamma_{i}) \rangle_{V,\beta}^{g,B} + \langle \prod_{\sigma \in \Sigma_1} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{V,\beta}^{g,B_{1}} + \\
+ \langle \prod_{\sigma \in \Sigma_2} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{V,\beta}^{g,B_{2}} + \langle \prod_{\sigma \in \Sigma_3} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{V,\beta}^{g,B_{3}} + \\
+ \langle \prod_{\sigma \in \Sigma_6} \tau_{k_{\sigma}}(\gamma_{\sigma}) \rangle_{V,\beta}^{g,B_{6}}.
\]

4.2 Dilaton, String and Divisor Equations

For the remainder of this section, we define \( A \) to be of length \( n+1 \) and \( A' := A \setminus \{a_{n+1}\} \), and assume that both \( A \) and \( A' \) are admissible. Moreover let \( A_{\epsilon} := A' \cup \{\epsilon\} \), where \( \epsilon \) is chosen so that the map \( \phi_{A',A_{\epsilon}} \) corresponds to the universal curve. We will also fix \( g, V \) and \( \beta \). We are now in a position to use our results to derive analogues of the well known dilaton, string and divisor equations. Each of these equations differ from previously stated relationships in that we will be able to express the value of a descendant \textit{strictly} in terms of descendants which are calculated on a space with fewer marked points.

We derive each first in the case when \( \phi_{A',A} \) corresponds to the universal curve, and then in the general case whenever the reduction \( \rho_{A_{\epsilon},A} \) crosses walls.

4.2.1 \( A \)-Dilaton Equation

Recall that the unweighted Dilaton Equation states that

\[
\langle \tau_{1} \prod_{i=1}^{n} \tau_{k_{i}}(\gamma_{i}) \rangle_{V,\beta}^{g,n+1} = (2g - 2 + n) \langle \prod_{i=1}^{n} \tau_{k_{i}}(\gamma_{i}) \rangle_{V,\beta}^{g,n}
\]

The customary proof of this equality is to apply a push-pull type argument with the forgetful map, using the pullback relation, and reduce this to calculating the degree of a fiber. There is no harm in this argument, and we use it here as well.
We first give a version of the dilaton equation for the case whenever the forgetful map corresponds to the map of the universal curve, i.e., $\Delta_A = \text{Cone}(\Delta_{A'})$.

**Theorem 4.11** (Cone Dilaton Equation). Assume $k_{n+1} = 1$ and $\Delta_A = \text{Cone}(\Delta_{A'})$. Then,

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = (2g - 2) \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'}^{V,\beta}.$$

**Proof.** Indeed, the pullback relation of Theorem 3.10 gives that each psi class $1 \leq i \leq n$ is given by the pullback. The result thus follows from the projection formula and the degree of fiber being $2g - 2$. \qed

**Theorem 4.12** (General A-Dilaton Equation). Assume $k_{n+1} = 1$ and $A$ has length $n + 1$. Retain the additional notation of Theorem 4.9. Then we have:

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = (2g - 2) \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'}^{V,\beta} + \sum_{\Sigma \in \Sigma_{\text{st}}(A,A_{\epsilon})} w_{\Sigma} \cdot \left\langle \prod_{\sigma \in \Sigma} \tau_{k_{\sigma}}(\gamma_{\sigma}) \right\rangle_{g,A_{\Sigma}}^{V,\beta}.$$

**Proof.** This formula follows immediately from factoring $\phi_{A',A} = \phi_{A',A_{\epsilon}} \circ \rho_{A_{\epsilon},A}$ and applying Theorem 4.9 followed by Theorem 4.11. \qed

**Remark 4.13.** We see that if we are in the symmetric case of $\Delta_{n+1,r}$ we have that

$$\Delta_{n+1,r} \cup \bigcup_{\sigma \in \Sigma_{\text{st}}(A,A_{\epsilon})} \sigma = \text{Cone}(\Delta_{n,r}).$$

The upshot of this is that the set $\Sigma_{\text{st}}(A,A_{\epsilon})$ is in a natural bijection with $\Sigma_{\text{st}}(A,A_{\epsilon})$. When we combine this observation with Remark 4.3, we can state the somewhat simpler expression in the theorem which follows.

**Theorem 4.14** (Symmetric A-Dilaton Equation). Assume $k_{n+1} = 1$ and $A$ corresponds to $\Delta_{n+1,r}$. Then we have:

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A}^{V,\beta} = (2g - 2) \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'}^{V,\beta} + \sum_{\sigma \in \Sigma_{\text{st}}(A,A_{\epsilon})} (-1)^{\dim \sigma + 1} \left\langle \tau_{k_{\sigma}}(\gamma_{\sigma}) \prod_{j \notin \sigma} \tau_{k_{j}}(\gamma_{j}) \right\rangle_{g,A_{\sigma}}^{V,\beta}.$$
Proof. This follows from our above remark and noting the simple structure of the graph. We have only a vertex labeled $\Delta_A$ to each vertex labeled by $\sigma \in F(A,A)$. \hfill $\square$

### 4.2.2 $A$-String Equation

The unweighted string equation states:

$$
\left< \tau_0 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,n+1}^{V,\beta} = \sum_{\ell=1}^{n} \left< \tau_{k_{\ell}-1}(\gamma_{\ell}) \prod_{i \neq \ell} \tau_{k_i}(\gamma_i) \right>_{g,n}^{V,\beta}
$$

We again give statements in the weighted case.

**Theorem 4.15** (Cone String Equation). Assume $k_{n+1} = 0$ and $\Delta_A = \text{Cone}(\Delta_{A'})$. Then we have:

$$
\left< \tau_0 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,A}^{V,\beta} = 0
$$

**Proof.** Indeed, this product pulls back from a space of a lower dimension and is thus zero. \hfill $\square$

**Theorem 4.16** (General $A$-String Equation). Assume $k_{n+1} = 0$ and $A$ has length $n+1$.

Retain the additional notation of Theorem 4.9. Then we have:

$$
\left< \tau_0 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,A}^{V,\beta} = \sum_{\Sigma \in \Sigma^{\sigma}(A,A)} w_\Sigma \cdot \left< \prod_{\sigma \in \Sigma} \tau_{k_\sigma}(\gamma_\sigma) \right>_{g,A \Sigma}^{V,\beta}
$$

**Proof.** The proof of this is identical to that of Theorem 4.12 after noting the differences in Theorem 4.11 and Theorem 4.15. \hfill $\square$

**Theorem 4.17** (Symmetric $A$-String Equation). Assume $k_{n+1} = 0$ and $A$ corresponds to $\Delta_{n+1,r}$. Then we have:

$$
\left< \tau_0 \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right>_{g,A}^{V,\beta} = \sum_{\sigma \in F(A,A)} (-1)^{\dim \sigma+1} \left< \tau_{k_\sigma}(\gamma_\sigma) \prod_{i \notin \sigma} \tau_{k_i}(\gamma_i) \right>_{g,A_\sigma}^{V,\beta}
$$

**Proof.** Again, the proof of this is virtually identical to that of Theorem 4.14 after noting the differences in Theorem 4.11 and Theorem 4.15. \hfill $\square$
4.2.3  A-Divisor Equation

In the same spirit as the string and dilaton equations, there is the usual unweighted divisor equation which states that for $D \in A^1(V, \mathbb{Q})$:

$$\left\langle \tau_0(D) \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A} = \int D \cdot \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'} +$$

$$+ \sum_{\ell=1}^{n} \left\langle \tau_{k_{\ell-1}}(\gamma_\ell \cup D) \prod_{i \neq \ell} \tau_{k_i}(\gamma_i) \right\rangle_{g,n}$$

We state this now in the same cases as we stated the string and dilaton equations. The proofs of each are very much in the same spirit as previous proofs and are left to the reader.

**Theorem 4.18** (Cone A-Divisor Equation). Assume $k_{n+1} = 0$ and $\Delta_A = \text{Cone}(\Delta_{A'})$. For $\gamma_{n+1} = D \in A^1(V, \mathbb{Q})$, we have:

$$\left\langle \tau_0(D) \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A} = \int D \cdot \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'}$$

**Theorem 4.19** (General A-Divisor Equation). Assume $k_{n+1} = 0$ and $A$ has length $n + 1$. Retain the additional notation of Theorem 4.9. For $\gamma_{n+1} = D \in A^1(V, \mathbb{Q})$, we have:

$$\left\langle \tau_0(D) \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A} = \int D \cdot \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'} +$$

$$+ \sum_{\Sigma \in \Sigma^*(A,A_\epsilon)} w_\Sigma \cdot \left\langle \prod_{\sigma \in \Sigma} \tau_{k_\sigma}(\gamma_{\sigma} \cup D) \prod_{i \not\in \sigma} \tau_{k_i}(\gamma_i) \right\rangle_{g,A_\Sigma}$$

**Theorem 4.20** (Symmetric A-Divisor Equation). Assume $k_{n+1} = 0$ and $A$ corresponds to $\Delta_{n+1,r}$. For $D \in A^1(V, \mathbb{Q})$, we have:

$$\left\langle \tau_0(D) \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A} = \int D \cdot \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,A'} +$$

$$+ \sum_{\sigma \in \mathcal{F}(A,A_\epsilon)} (-1)^{\dim \sigma + 1} \left\langle \tau_{k_\sigma}(\gamma_{\sigma} \cup D) \prod_{i \not\in \sigma} \tau_{k_i}(\gamma_i) \right\rangle_{g,A_\sigma}$$
Bibliography


