UNDERSTANDING FRACTIONS

by

SAYONITA GHOSH HAJRA

(Under the Direction of Sybilla Beckmann)

ABSTRACT

I report on the comparison of the syllabi of grades 1 through 7 of the Indian National Council of Educational Research and Training (NCERT) and the Common Core State Standards (CCSS) on Fractions. My study involves interviewing nine students (four from USA and five from India) and analyzing the interviews, which reveals some of the possible problems in the NCERT's syllabi and textbooks on fractions. I present some suggestions to NCERT to improve the standards for the learning of fractions.

INDEX WORDS: Common Core State Standards, Fractions, Mental Actions, National Council of Educational Research and Training

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A Thesis Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment

of the Requirements for the Degree

MASTER OF ARTS

ATHENS, GEORGIA

2014

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DEDICATION

To my parents: Anjushree Ghosh Hajra and Hiranmay Ghosh Hajra.

ACKNOWLEDGEMENTS

I would like to thank Professor Sybilla Beckmann for her continuous support in writing the thesis and her constant guidance and encouragement throughout my teaching career at UGA. I would like to thank Professor Gordana Matic for being so much supportive in my decision to pursue a MA in Mathematics Education. I would like to thank Professor Andrew Izsak for being so flexible in choosing the mathematics education courses and for being in my committee. I would like to thank Professor Jeremy Kilpatrick for being in my committee and for providing feedback on my thesis. I would like to thank Professor Les Steffe and Professor Jim Wilson to guide me throughout my studies in Mathematics Education. I would like to thank my husband, Santosh Kandel, for his love, kindness and support he has shown in every decision I have made. I would like to thank my parents for their endless love and support. I would like to thank my sister, Ipsita, brother-in-law, Diparanjan, and nephew, Diptayan, for their continuous love and support. I would like to thank my friendship partner, Susan Weldon, for her continuous love, support, and valuable friendship. I would like to thank everyone who has influenced me directly or indirectly in my life.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Fractions are complicated constructs in the elementary school mathematics curriculum. Numerous research studies on fractions have revealed this complex construct (Hackenberg, 2007; 2010; Lamon, 1996; 1999; Steffe & Olive, 2010). There are varied levels of understanding among students on fractions even on the simplest of all fractions: one-half (Watanabe, 1996). That being said, how are current education systems helping students in their understanding of fractions? How do the approaches to understanding fractions differ in different countries? How are textbooks written to facilitate learning? I was thinking about these questions for a long time. More specifically, I focus on the Indian Education system and I attempt to study the Indian syllabi on fractions. While studying the syllabi, I raised some questions. This thesis answers the following questions:

1. How do the Indian syllabi stand in comparison with some standards from the United States on fractions?
2. When and what are students learning in these two countries?
3. What are the possible issues with the Indian syllabi or textbooks on fractions?
4. What are some suggestions for improving the Indian syllabi and the textbooks on fractions?

My study "Understanding Fractions" focuses on finding the answers to the above-mentioned questions. I arrange the thesis into six chapters. The second chapter is "Overview on Indian School System". It briefly discusses the ancient Indian education to the most current Indian education system. The Government of India established an autonomous organization called the National Council of Educational Research and Training (NCERT) in the year 1961 to assist and advice both the Central Government and the State Governments on curricula, textbooks and other education related matters. Here I briefly discuss the role of NCERT in the education system in India.

The third chapter is “Comparisons between the two countries’ approach on fractions”, which addresses research questions 1 and 2. Research question 1 deals with the comparison of standards on fractions from the USA and India. Research question 2 is about understanding when and what students are learning in these two countries. There are 1.03 million government-run schools in India. NCERT writes syllabi and textbooks for the entire Central government-run schools and gives advice to the State run-schools. In this chapter, I present a comparison between the NCERT’s syllabi and the Common Core State Standards for Mathematics (CCSSM) standards on fractions, studying grade 1 through grade 7 syllabi and NCERT’s textbooks on fractions. The major difference in the approaches is in the use of the definition of fractions. NCERT uses the “part-out-of whole” definition of fractions. In addition, CCSSM defines unit fraction in terms of “parts-out-of-whole” and all other fractions (proper and improper) is defined in terms of iteration (a fraction a/b defined as a parts of size 1/b) that is missing from NCERT. The meaning of improper fractions is not explicitly defined in NCERT, improper fractions are explained in terms of mixed fractions. NCERT focuses on congruent parts of a whole, whereas CCSSM focuses on recognizing equal parts of the same whole, which might not be congruent. In this chapter, I discuss these differences on two countries’ approaches on fractions.

The fourth chapter is “Analysis of Students’ Interviews”. I present my findings based on students’ interviews on fraction tasks. Those tasks were about explaining fractions, explaining improper fractions, understanding fractions as operator, understanding fractions on number line, explaining equivalent fractions, explaining fraction addition, understanding units in word problems. I found that even though all the participants in the study were high performing students in their respective classes, they have different levels of understanding and they use different mental actions to solve a particular task.

The fifth chapter is “Possible Problems with NCERT’s Syllabi and Textbooks on Fractions”, which addresses research question 3. Research question 3 is about understanding the possible issues in the Indian syllabi and textbooks. In this chapter, I present the possible discrepancies in the NCERT’s textbooks and the syllabi. I connect my findings from the analysis of the Indian students’ interviews and the analysis of the NCERT’s syllabi and Textbooks on fractions. My study reveals some loopholes in NCERT’s syllabi and the textbooks on fractions. The first loophole is the use of the part-whole definition of fraction only to define fraction from grade 4 to grade 7. This puts restriction on students’ understanding of improper fractions. The second loophole is the overemphasis on developing procedural skills rather than in understanding and developing concepts. The third loophole is the use of notations before they are formally introduced in the respective grades. The fourth loophole is the absence of story-based problems for fraction operations emphasizing different units. In this chapter, I discuss how these could be the sources for the misconceptions of the Indian students.

The sixth chapter is “Suggestions to the National Council of Educational Research and Training”, which addresses research question 4. Research question 4 is on providing suggestions for improving the Indian syllabi and textbooks on fractions. In this chapter, I present some suggestions to improve the current issues in NCERT’s syllabi and textbooks, which would help in learning of fractions. These suggestions are based on my findings from the students’ interviews and from my study of the NCERT’s syllabi and textbooks. I use various research literatures to suggest ways to improve the current NCERT syllabi and textbooks.

CHAPTER 2

OVERVIEW OF THE INDIAN SCHOOL SYSTEM

India is a South Asian democratic country governed under the parliamentary system. There are 28 States and 7 Union Territories. There are 22 official languages spoken in India; the official language of the Central Government is Hindi and English is the secondary official language (“Languages”, n.d.). According to the provisional results of the 2011 census, the literacy rate in India is about 74.04%, 82.14% for males and 65.46% for females (“Languages”, n.d.). The 1961 Indian census recognized 1652 classified mother tongues that portray the vast diversity among people (Mallikarjun, 2002).

In spite of struggling with major issues like poverty, illiteracy, and terrorism, India has been able to display a strong mathematical tradition. India has produced mathematicians like Aryabhata, Brahmagupta, Ramanujan, Harish-Chandra and many more (“Mathematics education in India- status and outlook”, 2012). The root of mathematics education practice in India is very ancient. The first known schools are the Gurukuls. Gurukul is a Sanskrit word, *guru* means teacher and *kul* means family. Usually Gurukuls were in isolated places inside the forests far from the cities where the male teacher lived with his family. Students were supposed to stay with the teacher’s family, work for the household and learn from the teacher through formal and informal interactions (Agarkar & Pradhan, 2002; Rai, 1981). Gurukuls were like the present day residential schools. The existence of Gurukuls dates back to Vedic age (1500 BC-600

BC) (Sharma & Sharma, 2004). The Gurukuls were not under the control and influence of the government. Learning in Gurukuls was based on day-to-day experiences. More educational institutions flourished in the Buddhist period (Sharma & Sharma, 2004). These institutions were very orderly and study topics included arithmetic, astronomy, literature, logic, etc. (Sharma & Sharma, 2004).

Arithmetic and astronomy were the core areas of study. Astronomy and Geometry were considered essential for determining auspicious times for performing religious rituals and for the construction of sacrificial altars respectively ("National Focus Group on Teaching of Mathematics", 2006). This is a glimpse of the ancient India.

The system of education in India experienced a major change upon the arrival of the British as they introduced a Western system of education for the proper functioning of the British Empire ("National Focus Group on Teaching of Mathematics", 2006). After the independence of India in 1947, there have been many educational reforms and implementations of various educational policies and acts. One such example is the Right to Education RTE Act 2009, which provides free and compulsory education to children between ages of 6-14 years ("Right to Education", n.d.). And another such example is the adoption of Mid-Day Meal Scheme. Dropouts from schools are a major issue for this poverty-stricken country. The Indian Government has adopted this scheme to retain and enroll children in the schools. This scheme is the world's largest school feeding program, feeding about 120 million children in about 1.2 million schools across India ("Mid-day Meal", n.d.).

Current Indian school education system

The Indian higher education system is among one of the largest education system in the world. It is framed from pre-primary to the post-graduate level ("Mathematics education in India - status and outlook", 2012). The primary level is grades 1 to 5; upper primary is grades 6 and 7; secondary is grades 8 to 10 and the higher secondary is grades 11 and 12 (Figure 2.1). Students learn from a curriculum that is mostly common, which makes Indian education system a centralized education system (Kumar, n.d.). Students mainly learn three languages in school namely English, Hindi and their respective mother tongues (Kumar, n.d.).

LEVELS OF EDUCATION
Post-Graduate (for 2 years)
Undergraduate (for 3-5 years)
Higher Secondary (Grades 11 & 12)
Secondary (Grades 8, 9 & 10)
Upper Primary (Grades 6 & 7)
Primary (Grades 1-5)
Pre- Primary (Kindergarten)

Figure 2.1. Different levels of education in India.

Source: "Mathematics education in India- status and outlook", 2012, p. 168. (Modified)

The education system is mainly managed by: The central government in Delhi, the state governments, and the local private sources (Figure 2.2) ("Mathematics education in India- status and outlook", 2012, p. 168). This makes the Indian education system a top-down education system.

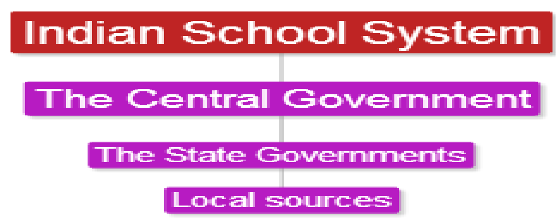


Figure 2.2. A top-down Indian education system.

The central government mainly supervises and helps with the funding, whereas the states are responsible for the following functions ("Mathematics education in India- status and outlook", 2012):

- Design and implement curricula,
- Develop syllabi,
- Make policies for hiring and training teachers,
- Provide various certifications,
- Monitor schools and set standards.

There are further divisions of educational districts within these states but curricula are not created locally ("Mathematics education in India- status and outlook", 2012).

The Ministry of Human Resource Development regulates the overall Indian education system. Each state government has its own Education Ministry and a Central

Advisory Board (CAB) on Education. CAB acts as a bridge between the central government and the states and in between states (Figure 2.3). There are **43** Boards of School Education in India.

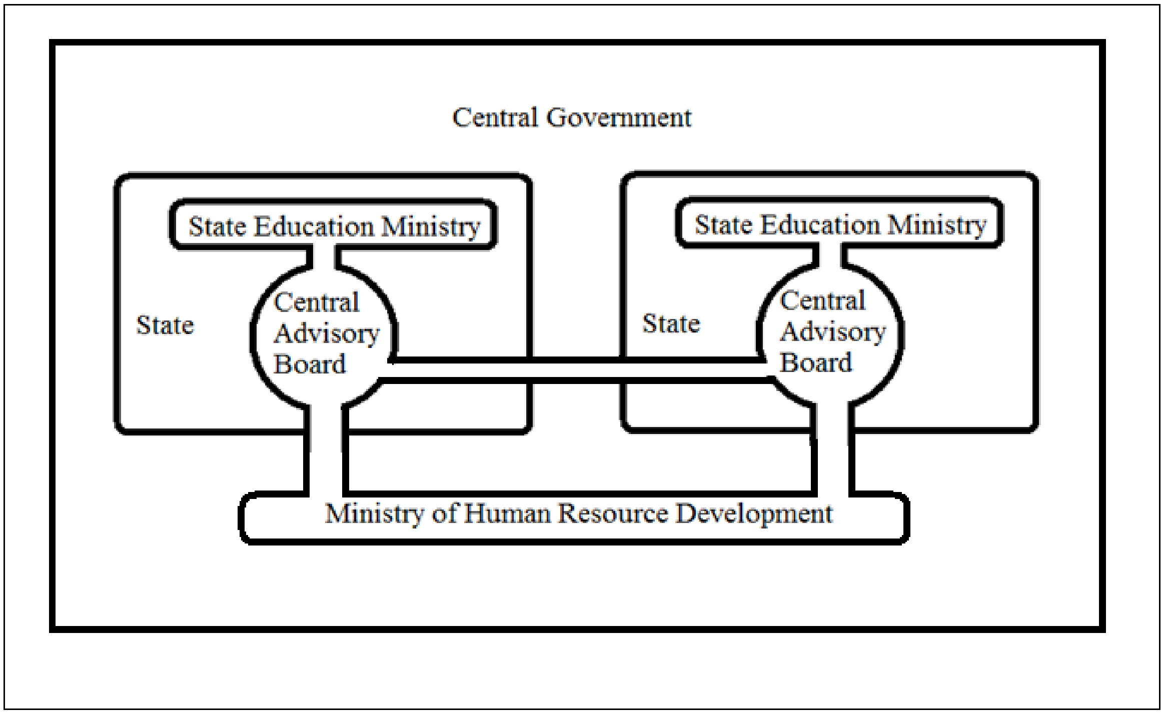


Figure 2.3. CAB connecting Central and State education system.

The oldest education board was the Uttar Pradesh Board of High School and Intermediate Education established in 1921 (Kumar, n.d.). As Kumar (n.d.) wrote:

> The Uttar Pradesh (a state in India) Board of High School and Intermediate Education was the first Board set up in India in the year 1921 with jurisdiction over Rajputana, Central India and Gwalior. In 1929, the Board of High School and Intermediate Education, Rajputana, was established. Later, boards were established in some of the states. But eventually, in 1952, the constitution of the board was amended and it was renamed Central Board of Secondary Education (CBSE). All schools in Delhi and some other regions came under the Board. It was the function of the Board to decide on things like curriculum, textbooks and examination system for all schools affiliated to it. Today there are thousands of schools affiliated to the Board, both within India and in many other countries from Afghanistan to Zimbabwe. (para. 3)

In 1961, Government of India established an autonomous organization called the National Council of Educational Research and Training (NCERT), to assist and advise central and state governments for qualitative improvement in school education. NCERT designs curricula for the Central Board of Secondary Education, which is a Board of Education for public and private schools, under the Government of India. It gives advice to the other Boards of Education too. The following are the major objectives of NCERT ("NCERT", n.d.):

- Funding and promoting research in fields related to school education;
- Writing and publishing model textbooks, supplementary materials, journals and other teaching related literatures;
- Organizing trainings for teachers;
- Developing and publicizing innovative educational techniques and practices;
- Cooperating with state educational departments, universities, non-governmental organizations (NGOs) and other educational institutions;
- Undertaking goals for universalization of elementary education.

In addition, NCERT collaborates with other countries in bilateral cultural exchange programs in the fields of school education, and trains educational personnel from other developing countries ("NCERT", n.d.). NCERT acts as an advisor to different states' education boards.

Each state with its department of education operates its own school system. Each state has its own council called the State Council for Educational Research and Training (SCERT) which helps the state's education department to develop curricula and pedagogical strategies and to propose evaluation approaches that follow the guidelines of

NCERT (Kumar, n.d.). There are three different types of schools under the state government (Kumar, n.d.): The first kinds are the government schools run by the state government with low tuition fees. The second kinds are the private schools with high tuition fees. The third kinds are the private schools getting grant-aids from the government with moderate tuition fees.

Apart from state level education boards, there are two national level education boards in India (Kumar, n.d.). One is the Central Board of Secondary Education (CBSE) and the other one is the Council of Indian Certificate of Secondary Education (ICSE) (Kumar, n.d.). There are a number of schools called “Central Schools” run by the central government under CBSE located in all the main urban areas in India, which follow a common schedule (Kumar, n.d.). These schools follow NCERT’s syllabi and textbooks; all the subjects are taught in English except social studies, which are taught in Hindi (Kumar, n.d.). There are many private schools that follow CBSE syllabus written by NCERT but follow different textbooks and have different teaching schedules (Kumar, n.d.). According to Kumar (n.d.), there are 141 schools in twenty-one different countries affiliated to CBSE. The second board ICSE was formed in 1958 and was established as a body for conducting public examinations in 1973 (Kumar, n.d.). A large number of private schools all over India are affiliated to this board (Kumar, n.d.). Both the boards conduct an all India school leaving examinations for all the schools affiliated to them after the tenth and twelfth year of schooling (at the end of grade 10 and grade 12) (Kumar, n.d.). Admissions in grade 11 and in colleges are strictly based on the performances on these two examinations (Kumar, n.d.). Beside these schools, there are small numbers of private expensive residential schools that follow foreign curricula having good infrastructure and with low student-teacher ratio (Kumar, n.d.).

The following table (Table 2.1) is a final report of Sarva Shiksha Abhiyan taken from ("Mathematics education in India- status and outlook", 2012, p. 168) which depicts a large primary education system in India.

Table 2.1. Final report of Sarva Shiksha Abhiyan 2008-2009.

	Total in millions	Number in rural areas (in millions)
Number of Children (ages 6-11)	134 (male: 69, female: 65)	108
Number of schools	1.28	0.8
Number of teachers	5.8	4.5

Source: "Mathematics education in India- status and outlook", 2012, p. 168.

Sarva Shiksha Abhiyan is an Indian Government education program for universalization of elementary school education ("Sarva Shiksha Abhiyan", n.d.). As stated in the mission statement of SSA ("Sarva Shiksha Abhiyan", n.d.):

> SSA is being implemented in partnership with State Governments to cover the entire country and address the needs of 192 million children in 1.1 million habitations. The program seeks to open new schools in those habitations which do not have schooling facilities and strengthen existing school infrastructure through provision of additional class rooms, toilets, drinking water, maintenance grant and school improvement grants. Existing schools with inadequate teacher strength are provided with additional teachers, while the capacity of existing teachers is being strengthened by extensive training, grants for developing teaching-learning materials and strengthening of the academic support structure at a cluster, block and district level. SSA seeks to provide quality elementary education including life skills. SSA has a special focus on girl's education and children with special needs. SSA also seeks to provide computer education to bridge the digital divide. (para. 1)

Out of 1.28 million schools, 1.03 million schools are government-run ("Mathematics education in India- status and outlook", 2012). The government runs more than eighty percent of the schools and NCERT is involved directly or indirectly in the development of the curriculum for these schools (Figure 2.4).

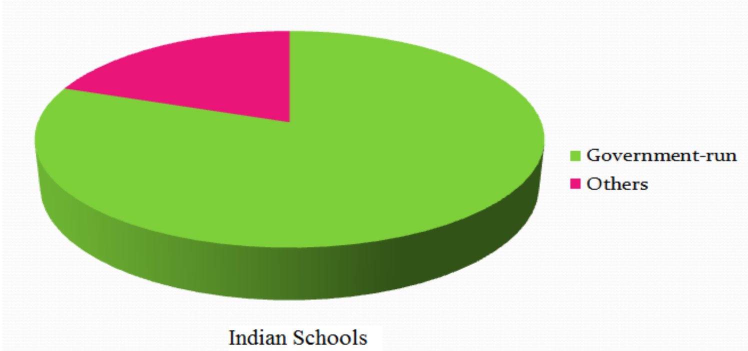


Figure 2.4. Government-run verses other schools in India.

This large number portrays how important it is for NCERT to develop a balanced and appropriate curriculum for all of these schools. This brings my focus to study the NCERT's syllabi and textbooks. In the coming chapters, I will give a detailed analysis of my study.

CHAPTER 3

COMPARISONS BETWEEN THE TWO COUNTRIES APPROACHES TO FRACTIONS

This chapter presents a detailed survey of the syllabi and textbooks of India's National Council of Educational Research and Training (NCERT) ("NCERT", n.d.) from grade 1 through grade 7 along with a comparison with the USA's Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative, 2010) on fractions. This chapter consists of four sections. The first section gives an outline of the Common Core State Standards; the second section explains the need of the comparison of the Indian syllabi with the US standards; the third section brings the grade by grade comparisons of the two countries standards and syllabi respectively; and the fourth section presents a summary of the comparisons between the NCERT's syllabi and the CCSSM's standards on fractions.

Common Core State Standards

The U.S. education system is a bottom-up education system. Education is generally controlled at three levels: local, state, and federal (Howson, Keitel, & Kilpatrick, 1981). Usually the states are responsible for the educational standards. Even though there is population mobility and there are large national textbook publishers, the U.S. education system is diverse to some extent (Schmidt, 2012). The Common Core State Standards Initiative is a state-led effort, launched in 2009 by state leaders, including governors and state commissioners of education from 48 states, 2 territories and the District of Columbia, through their membership in the National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO) to bring together the diverse state curricula (Common Core State Standards Initiative, 2010). The final standards were released in June 2010 (Common Core State Standards Initiative, 2010). The goal of the Mathematics Standards is to move towards greater focus and coherence in the mathematics curriculum (Common Core State Standards Initiative, 2010). Out of 50 states, forty-five states, the District of Columbia, four territories and the Department of Defense Education Activity have adopted the Common Core State Standards (Figure 3.1) (Common Core State Standards Initiative, 2010).

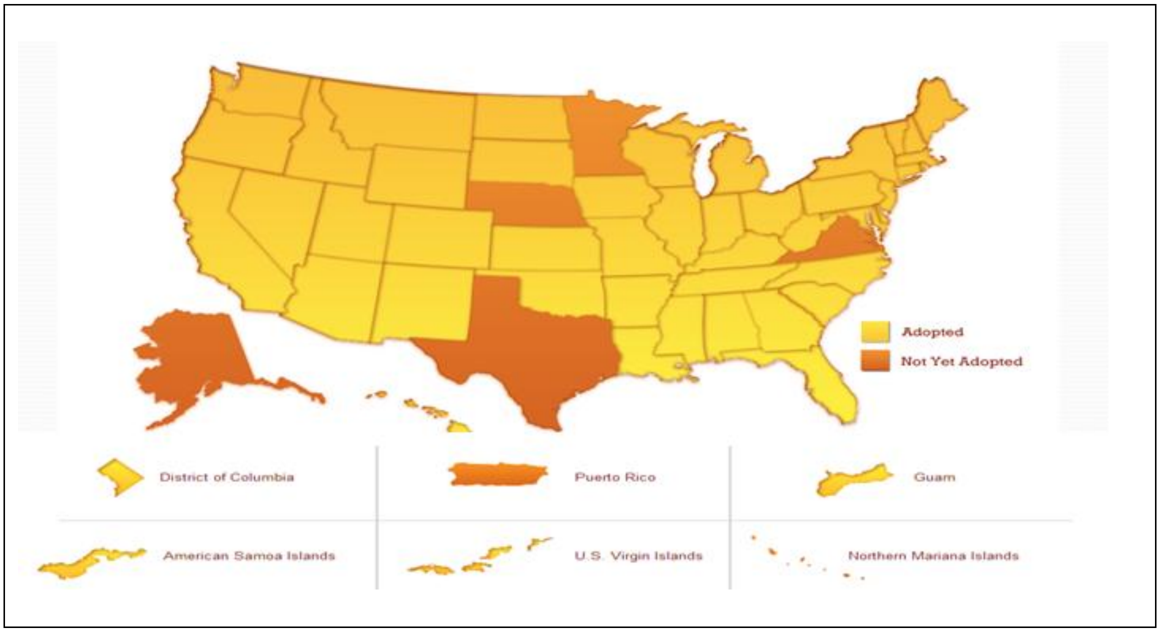


Figure 3.1. Adoption of Common Core State Standards by states.

Source: Common Core State Standards Initiative, 2010.

These standards specify what students should understand and be able to do in their course of study of mathematics. Although these standards are grade-specific standards, they do not specify any specific pedagogy or teaching materials necessary to support students who are well below or well above specific grade level expectations. These standards are written in a way to give a clear picture of what is being expected from students entering colleges (Common Core State Standards Initiative, 2010).

Need for comparison

Being the second most populous country in the world with 1.2 billion people; with growing work force and India's aspirations for generating engineers, there is a need for mathematically sound work force in India. Whatever career path one chooses, one has to deal with numbers, including fractions. One uses fractions in day-to-day activities, like in cooking, grocery, banking etc. Therefore, everyone needs to understand fractions in a meaningful way. However, the current scenario is not very pleasant for India. According to the National Focus Group under NCERT, fractions and decimals constitute a major problem area in primary and upper primary education ("National Focus Group on Teaching of Mathematics", 2006). As stated in ("National Focus Group on Teaching of Mathematics", 2006, p. 15):

> There is some evidence that the introduction of operations on fractions coincides with the beginnings of fear of mathematics. The content in these areas needs careful reconsideration. Everyday contexts in which fractions appear, and in which arithmetical operations need to be done on them, have largely disappeared with the introduction of metric units and decimal currency. At present, the child is presented with a number of contrived situations in which operations have to be performed on fractions. Moreover, these operations have to be done using a set of rules which appear arbitrary (often even to the teacher), and have to be memorized - this at a time when the child is still grappling with the rules for operating on whole numbers. While the importance of fractions in the conceptual

> structure of mathematics is undeniable, the above considerations seem to suggest that less emphasis on operations with fractions at the primary level is called for.

This suggests that NCERT itself recognized fraction as a problem area in the Indian education system. Also in (“Preparing for Fractions”, 2011, p. 1), it is mentioned that international comparisons like Trends in International Mathematics and Science Study (TIMSS, n.d.) reveal that too many U.S. students have trouble making the transition from whole number arithmetic to fractions. India is not a participant in TIMSS, and there are no such comparisons of Indian standards or curriculum available to date. This made me interested in comparing US standards with the NCERT’s syllabi on fractions.

I decided to compare the NCERT’s syllabi with CCSSM on fractions as CCSSM standards have been adopted by most states and it functions as a common set of standards. As mentioned in chapter 2, the Indian schools run by the government affiliated with the Central Board of Secondary Education (CBSE) use NCERT’s syllabi and textbooks. In addition, other education boards seek guidance from NCERT. Hence, the comparison seems reasonable.

In the next section, I present comparisons between the NCERT’s syllabi and CCSSM on fractions in an attempt to find the strengths and weaknesses of the NCERT syllabi on fractions, which is discussed in length in the subsequent chapters.

NCERT’s syllabi and CCSSM standards on fractions

In this section, I discuss grades 1 through 7 of NCERT’s syllabi in comparison to the CCSSM on fractions. I compare the NCERT’s syllabi that were developed in 2006 aligned with the philosophy of Yashpal report (Yaspal Committee Report, n.d.) and the

National Focus Group for Teaching Learning Mathematics (National Focus Group on Teaching of Mathematics, 2006). My comparison is based on NCERT's textbooks for grades 1 through 7 because NCERT's syllabi were written for the textbook writers and NCERT's textbooks are used in the government schools affiliated to CBSE across India. The standards are on fractions as well as on length measurement. The rationale for involving the length measurement standards is that the early standards on fractions in the CCSSM involve fractions on number lines, which requires attention to length. The standards from CCSSM that are stated below are directly quoted from Common Core State Standards Initiative (2010) and presented in bullet form; the NCERT's syllabi are directly quoted from ("Syllabi", 2006) and the figures are directly taken from NCERT's textbooks ("Textbook1", n.d.; "Textbook2", n.d.; "Textbook3", n.d.; "Textbook4", n.d.; "Textbook5", n.d.; "Textbook6", n.d.; "Textbook7", n.d.).

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 1

1.MD Measurement and Data

- Measure lengths indirectly and by iterating length units:
  - 1.MD.A.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.
  - 1.MD.A.2 Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

1.G Geometry

- Reason with shapes and their attributes:
  - 1.G.A.3 Partition[1] circles and rectangles into two and four equal shares. Describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 1

Length Measurement:

- Distinguishes between near, far, thin, thick, longer/ taller, shorter, high, low.
- Seriates objects by comparing their length.
- Measures short lengths in terms of non-uniform units (in the context of games e.g. 'Gilli Danda' and 'marble games').
- Estimates distance and length, and verifies using non-uniform units (e.g. hand span etc.).

NCERT'S TEXT BOOK FOR GRADE 1

Chapter 7: "Measurement", p. 93

This chapter begins with general comparison. For example, distinguishing near, far, thin, thick, taller, shorter, heavy, light. Then length measurement is discussed using non-standard units such as spanning hands, pencils, human feet etc. The syllabus uses the

[1] Partition means dividing a region.

word "non-uniform units" for grade 1. However, throughout this chapter, only non-standard units are discussed. Length measurement of a table, a mat, a book, etc. using hands, human feet, pencil and rod has been discussed (Figure 3.2). Some other activities in this chapter include measuring the distance between two trees using one's feet.

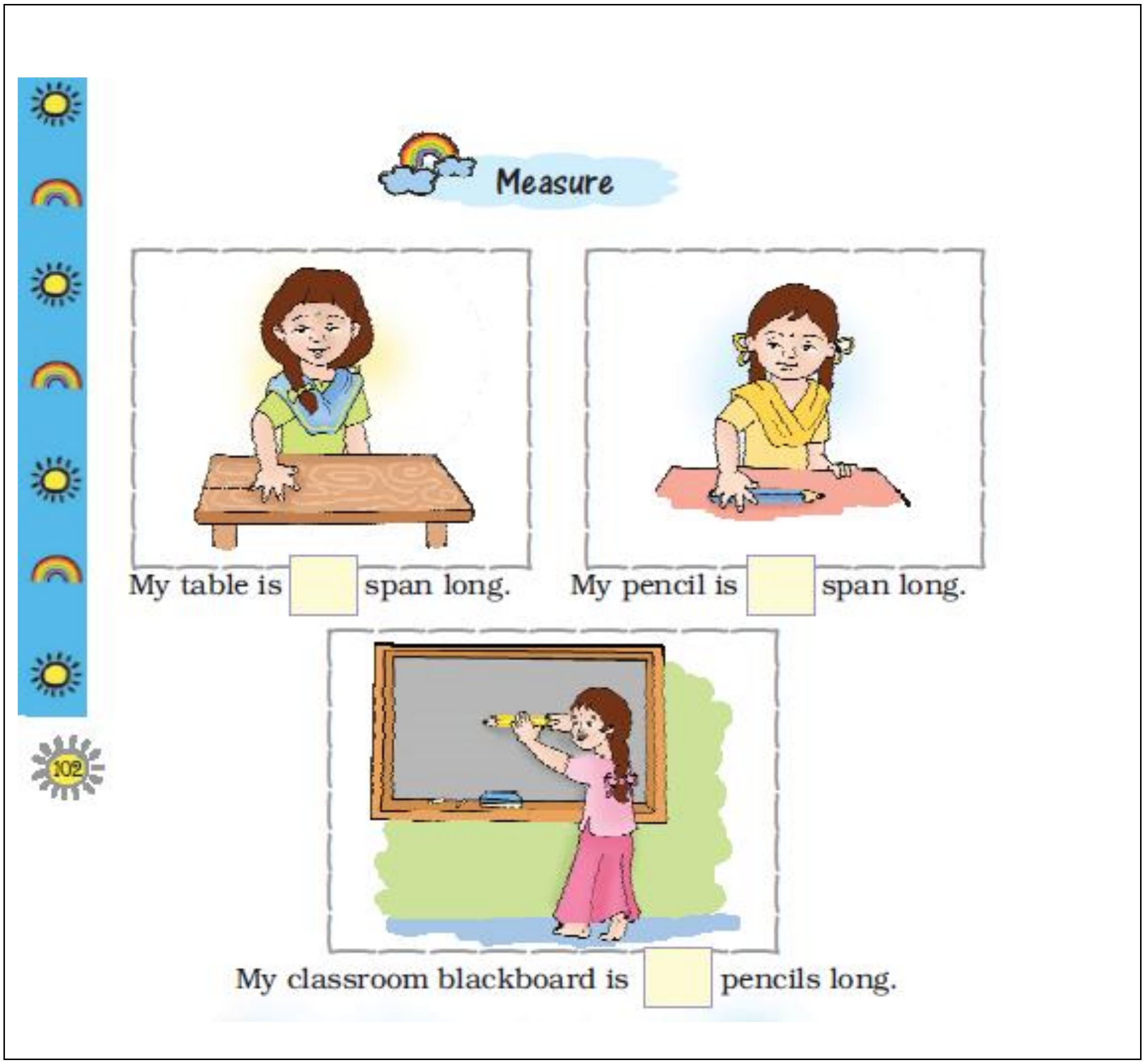


Figure 3.2. Length measurements using non-standard units.

Source: "Textbook1", n.d., p. 102.

COMPARISON IN GRADE 1

Both standards have focused on the development of the concept of length units. Attention to units is essential for understanding fractions; lengths and length units are important for understanding fractions on number lines. Ordering objects by length is seen in both the standards. Both of them deal with non-standard length units in this grade. A common feature in both is arranging objects by comparing lengths. Both the U.S. and Indian grade 1 standards and syllabi respectively focus on understanding the length measurement of an object as a whole number of equal length units that spans it without any gaps or overlaps. In both the countries' syllabi and standards respectively, the length of a given quantity is obtained by iterating a smaller unit. In CCSSM grade 1, the words halves, fourths and quarters using circle patterns have been described. Also the phrases *half of*, *fourth of*, and *quarter of* are used in this grade of CCSSM. In NCERT's grade 1 syllabus, the words halves; fourths and quarters are introduced much later (in grade 4). I found that in NCERT's grade 1 syllabus, the term "non-uniform" unit is used for "non-standard" unit.

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 2

2.MD Measurement and Data

- Measure and estimate lengths in standard units.
    - 2.MD.A.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
- Relate addition and subtraction to length.

- 2.MD.B.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
- 2.MD.B.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

2.G Geometry

- Reason with shapes and their attributes:
- 2.G.A.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 2

Measurement

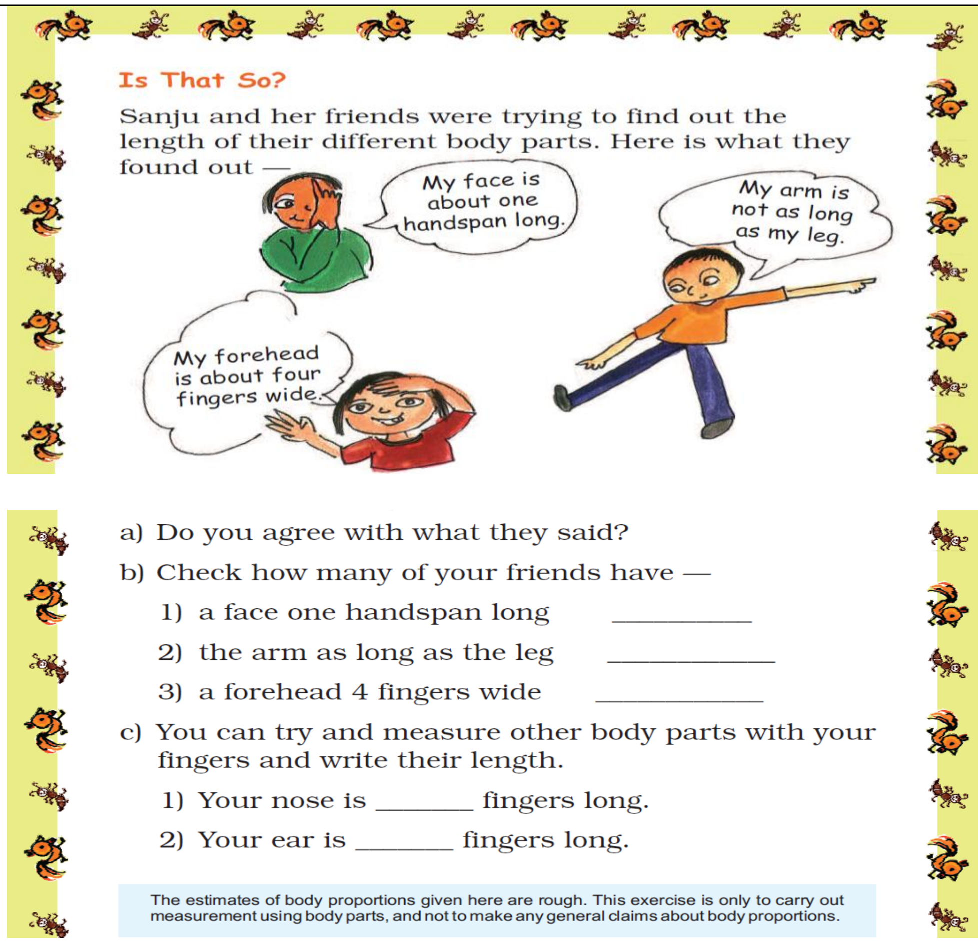
Length:

- Measures lengths & distances along short and long paths using uniform (non-standard) units, extends to longer lengths.

NCERT'S TEXT BOOK FOR GRADE 2

Chapter 13: "The Longest Step", p. 104

This chapter is about measurements using non-standard units of length measurement. Here using hand spans, fingers, matchsticks lengths of different objects are measured. Figure 3.3 depicts the use of non-standard units in length measurement. The distance between two objects is measured in this chapter using non-standard units. This chapter is a recapitulation of chapter 7: "Measurement" of grade 1.



Is That So?

Sanju and her friends were trying to find out the length of their different body parts. Here is what they found out —

My face is about one handspan long.

My arm is not as long as my leg.

My forehead is about four fingers wide.

a) Do you agree with what they said?

b) Check how many of your friends have —

1) a face one handspan long ____________

2) the arm as long as the leg ____________

3) a forehead 4 fingers wide ____________

c) You can try and measure other body parts with your fingers and write their length.

1) Your nose is ________ fingers long.

2) Your ear is ________ fingers long.

The estimates of body proportions given here are rough. This exercise is only to carry out measurement using body parts, and not to make any general claims about body proportions.

Figure 3.3. Non-standard length measurements.

Source: "Textbook2", n.d., p. 110.

COMPARISON IN GRADE 2

Standard units seen in Grade 2 of CCSSM standards are not introduced in grade 2 of NCERT's syllabus. Concepts of partitioning a whole (rectangle and circle), use of the words "halves", "fourths", "thirds" are not seen in the NCERT's syllabus for grade 2. Halves, fourths, thirds are introduced in grade 4 of NCERT's syllabus. Number lines for whole numbers are introduced in CCSSM for grade 2, which is missing from the grade 2 of NCERT's syllabus. Number lines are introduced in grade 6 of NCERT's syllabus. Fingers are used as a uniform (non-standard) unit in grade 2 of NCERT's syllabus. I consider fingers, a unit of measurement, as a non-uniform unit since different fingers in a hand are of different lengths. Spanning a finger one after another finger to measure length suggests the use of non-uniform units in grade 2 of NCERT's syllabus.

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 3

3.NF Number and Operations- Fractions

- Develop understanding of fractions as numbers.
  - 3.NF.A.1 (Part 1)[2] Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; (Part 2)[2] understand a/b as the quantity formed by a parts of size 1/b.
  - 3.NF.A.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.
  - 3.NF.A.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

[2] CCSSM does not use (Part 1) and (Part 2) separately, I considered them separately for the convenience of my study.

        - 3.NF.A.3a (Part 1)[2] Understand two fractions as equivalent (equal) if they are the same size, or (Part 2)[2] the same point on the number line.
        - 3.NF.A.3b Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.
        - 3.NF.A.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram*.
        - 3.NF.A.3d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

3.G Geometry

- Reason with shapes and their attributes.
    - 3.G.A.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape*.

NCERT’S SYLLABUS FOR MATHEMATICS GRADE 3

Measurement

Length:

- Appreciates the need for a standard unit.
- Measures length using appropriate standard units of length by choosing between centimeters, and meters.
- Estimates the length of given object in standard units and verifies by measuring.
- Uses a ruler.
- Relates centimeter, and meter.

Patterns:

- Identifies simple symmetrical shapes and patterns.
- Makes patterns and designs from straight lines and other geometrical shapes.

Geometry:

Shapes & Spatial Understanding

- Creates shapes through paper folding, paper cutting.

NCERT’S TEXT BOOK FOR GRADE 3

Chapter 1: “Where to look from”, p. 1

In this chapter, halves are introduced as mirror images of an object. Halves are described using folding a paper so that the two parts look exactly the same. Halves are also discussed by cutting a shape in a plane in two parts such that two parts look alike. Figure 3.4 shows two tasks, one is to identify if the dotted lines marked in the given shapes produces two identical mirror halves and the other task is to draw dotted lines on the given shapes to divide the shape into two similar halves.

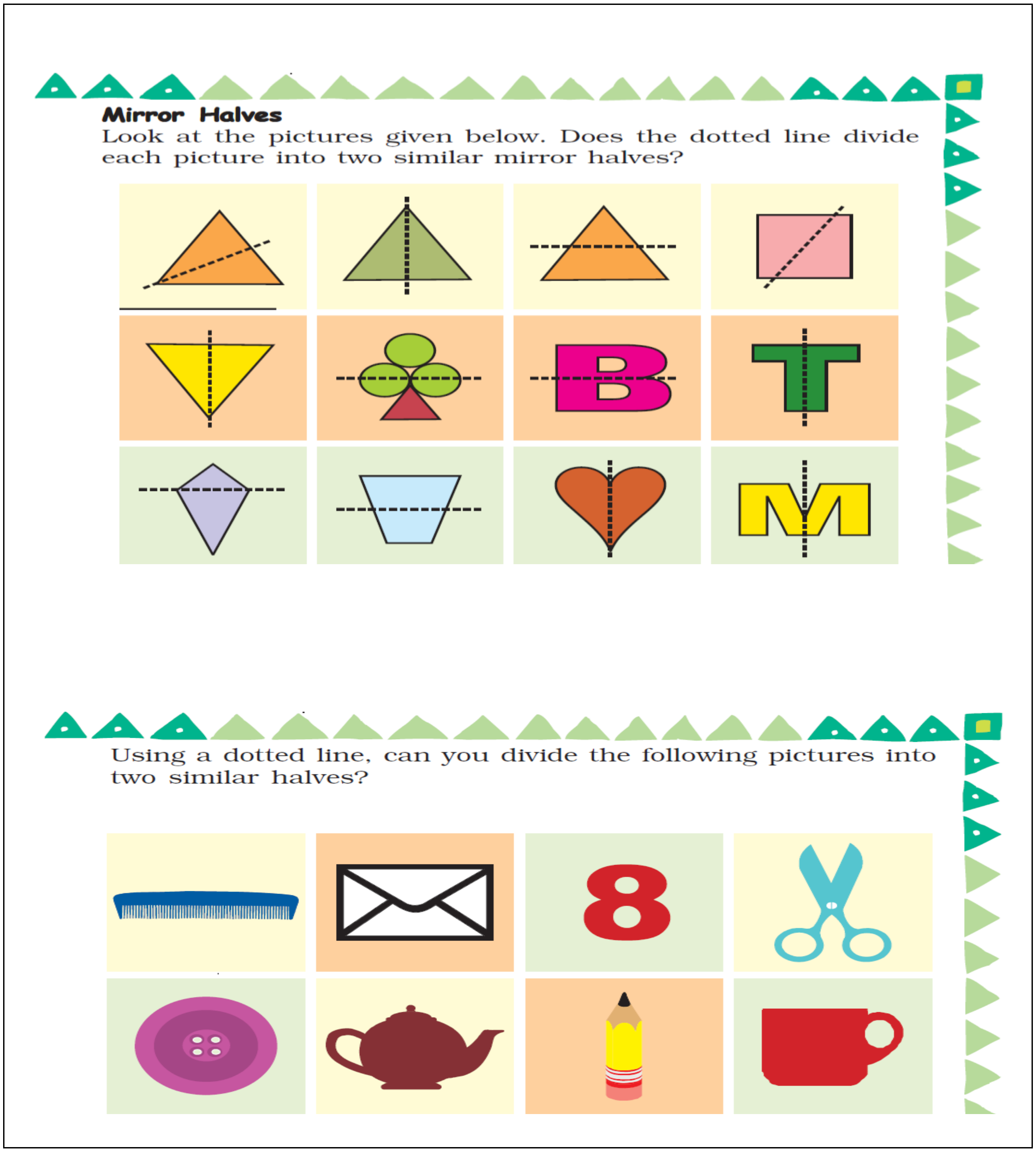


Figure 3.4. Mirror halves.

Source: “Textbook3”, n.d., pp. 9-10.

Chapter 4: “Long and Short”, p. 46

In this chapter, the inconsistency involving non-standard units has been explained through an example (Figure 3.5). The example discusses how the length measured in non-standard units varies from person to person. The example demonstrates how 7 units in terms of “arms of rope” give different lengths of rope. The importance of standard units for length measurement is shown through this example. Later in this chapter standard units for length measurements are introduced.

Centimeter and meter scales are introduced in this chapter. Tasks involve understanding which scale (centimeter/ meter) is to be used to measure length of different objects. Figure 3.6 demonstrates a task that focuses on understanding which units (centimeters or meters) are appropriate for measuring the given quantities.

COMPARISON IN GRADE 3

The concepts of fraction 1/b (one part when the whole is partitioned into b equal parts) and a/b (a parts of size 1/b) have been developed in grade 3 of CCSSM, which is not yet developed in the NCERT’s grade 3 syllabus. Number lines and visual models are used in explaining fractions, in developing equivalent fractions and in comparing fractions in the CCSSM’s grade 3. Mirror halves are discussed in NCERT’s grade 3, while mirror symmetry is discussed in grade 4 of CCSSM after the introduction of fraction in grade 3. NCERT’s grade 3 syllabus on standard units for measuring lengths and distances exactly fits with CCSSM’s 2.MD standards except for number lines. Number lines are introduced in grade 6 of NCERT’s syllabus. In grade 3 of CCSSM, area, set and length models of fractions are discussed. When comparing fractions, CCSSM emphasize recognizing that comparisons are valid only when the two fractions refer to the same whole.

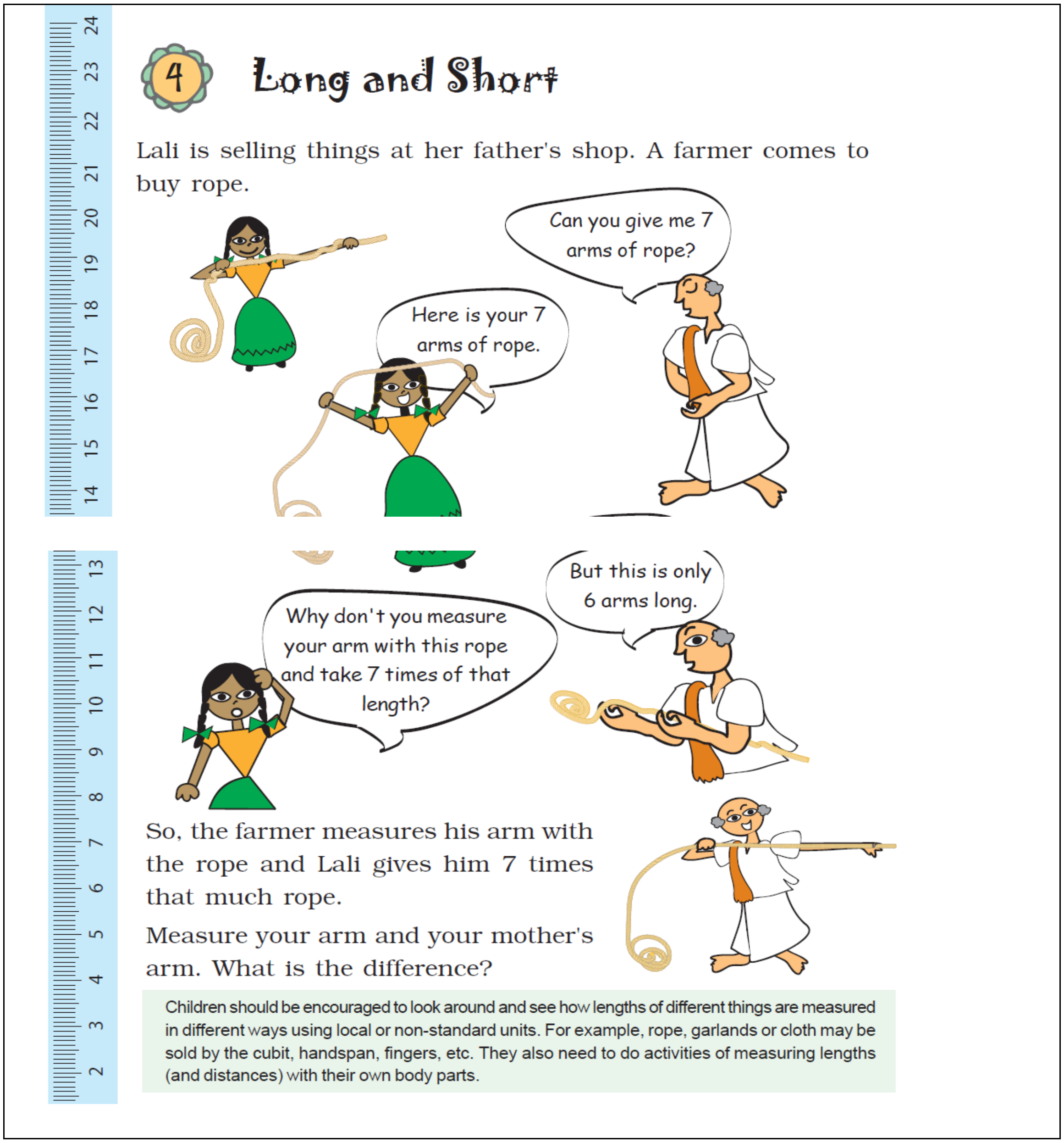
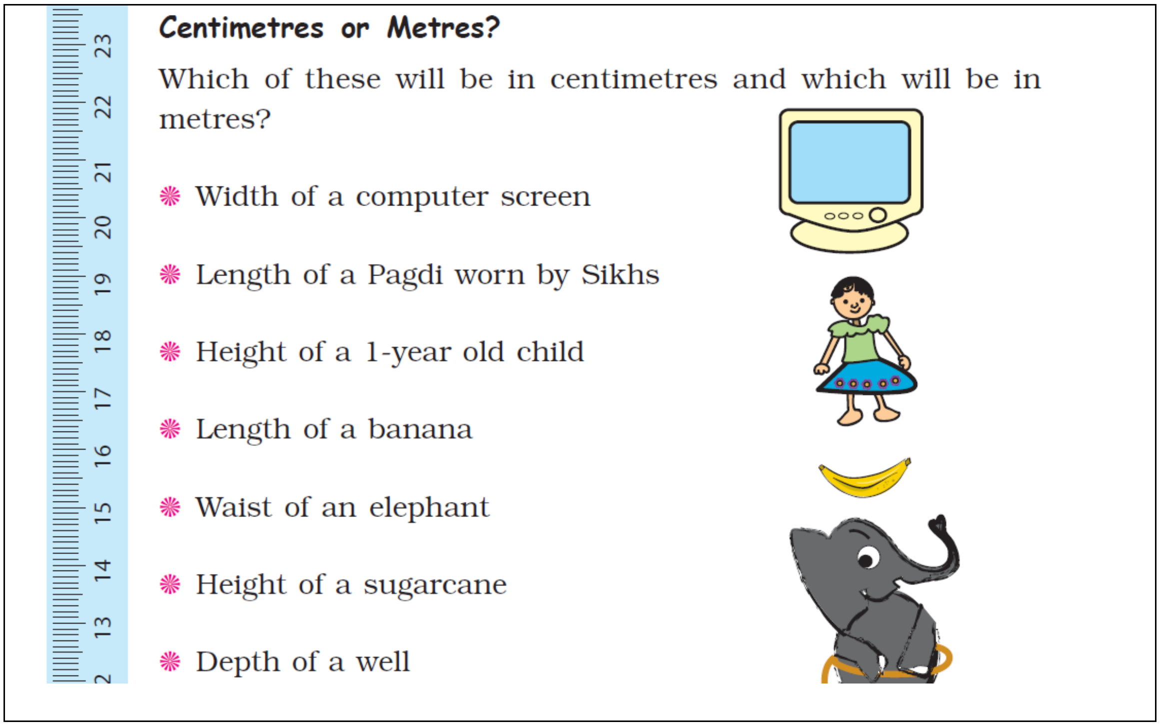


Figure 3.5. Example showing non-standard units in grade 3.

Source: “Textbook3”, n.d., p. 46.



Centimetres or Metres?

Which of these will be in centimetres and which will be in metres?

- Width of a computer screen
- Length of a Pagdi worn by Sikhs
- Height of a 1-year old child
- Length of a banana
- Waist of an elephant
- Height of a sugarcane
- Depth of a well

Figure 3.6. A task involving standard units in grade 3.

Source: “Textbook3”, n.d., p. 56.

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 4

4.NF Number and Operations- Fractions

- Extend understanding of fraction equivalence and ordering.
    - 4.NF.A.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
    - 4.NF.A.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or

by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

- Build fractions from unit fractions
  - 4.NF.B.3 Understand a fraction a/b with a>1 as a sum of fractions 1/b.
    - 4.NF.B.3a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.
    - 4.NF.B.3b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.
    - 4.NF.B.3c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.
    - 4.NF.B.3d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.
  - 4.NF.B.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

- 4.NF.B.4a Understand a fraction *a*/*b* as a multiple of 1/*b*. *For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).*
- 4.NF.B.4b Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number.
- 4.NF.B.4c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

- Understand decimal notation for fractions, and compare decimal fractions.
  - 4.NF.C.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.
  - 4.NF.C.6 Use decimal notation for fractions with denominators 10 or 100.
  - 4.NF.C.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

4.MD Measurement & Data

- Solve problems involving measurement and conversion of measurements.
  - 4.MD.A.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money,

including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

4.G Geometry

- 4.G.A.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 4

Geometry

Shapes & Spatial Understanding:

- Explores intuitively the reflections through inkblots, paper cutting and paper folding.

Fractional Numbers:

- Identifies half, one fourth and three-fourths of a whole.
- Identifies the symbols 1/2, 1/4, 3/4.
- Explains the meaning of 1/2, 1/4 and 3/4.
- Appreciates equivalence of 2/4 and 1/2; and of 2/2, 3/3, 4/4 and 1.

Patterns:

- Identifies geometrical patterns based on symmetry.

NCERT'S TEXT BOOK FOR GRADE 4

Chapter 9: "Halves and Quarters", p. 94

In this chapter fraction is formally introduced for the first time. The following are discussed in this chapter:

- Visual models of rectangles and circles are used to understand what halves; quarters and three-fourths of a whole would look like.
- Tasks involve multiple ways to cut a rectangle and circles into halves and also in fourths maintaining equal parts.
- Notations for half, quarter and three-fourth of a whole are given as 1/2, 1/4, 3/4 respectively.
- 1/2 is explained as "1 part out of 2" (Figure 3.7).

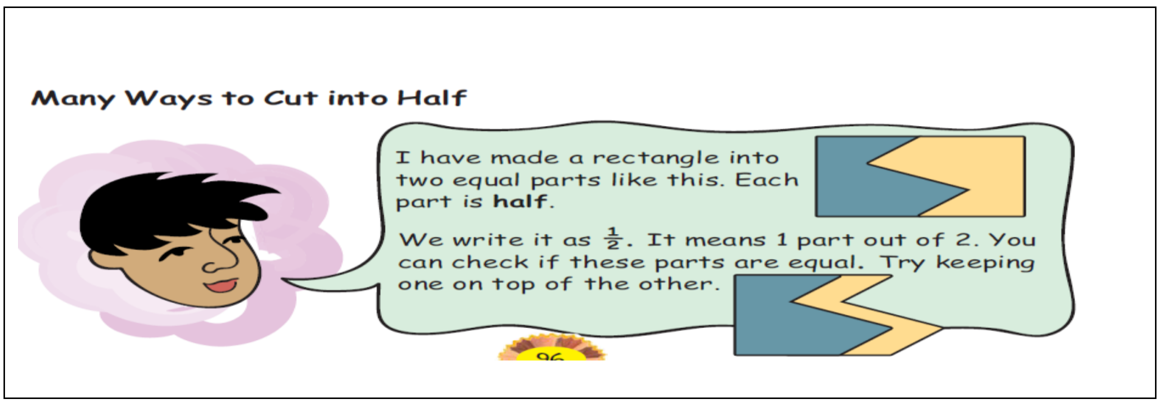


Figure 3.7. Meaning and notation of *Half*.

Source: "Textbook4", n.d., p. 96.

- The text describes two parts being equal if placing one on top of the other exactly fits each other.
- In addition, quarter is described as 1 part out of 4 and three-fourths is explained as 3 parts out of 4.
- Some tasks involved producing the whole when the fractional part (1/2, 1/4) of the whole is given.
- Visual models are used to show the equivalence of 4/4 and the whole; 2/4 and 1/2.
- Tasks involved reasoning questions, for example, why some shaded region is not half of the whole.
- Fractions are combined with the money related tasks, for example, if 1 Kg of potato cost Rs. 8, then how much does 1/2 Kg, 1/4 Kg and 3/4 Kg of potatoes cost? (Figure 3.8). I view this task as multiplication of a whole number with a fraction. In addition, this task requires using fractions as operators[3], because the task requires finding ½ of Rs. 8.
- There are some other tasks in this chapter. One such task is to match the fraction notation with their corresponding English name and picture (Figure 3.9).
- There are conversion tasks using fractions in this chapter, for example, conversion of 1/2 of a meter to centimeter, 1/4 of a litre to milliliters (Figure 3.10).

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[3] Behr, Lesh, Post & Silver (1983) said fraction as operator imposes an algebraic interpretation of a fraction a/b. Their interpretation was a/b could be thought of as a function transforming geometric figures to similar geometric figures a/b times as big or it could be viewed as a function transforming a set into another with a/b times as many elements. Behr et al mentioned that when a/b operates on a continuous object, e.g. length, a/b acts as a stretcher-shrinker. When a length L is operated on by a/b, the length is stretched to a times its length and then it is shrunk by a factor of b. They mentioned that when a/b operates on a discrete set, a/b acts as a multiplier-divider. When a discrete set with n elements is operated on by a/b, it transforms the set into a set with na elements and then the number is reduced to na/b.

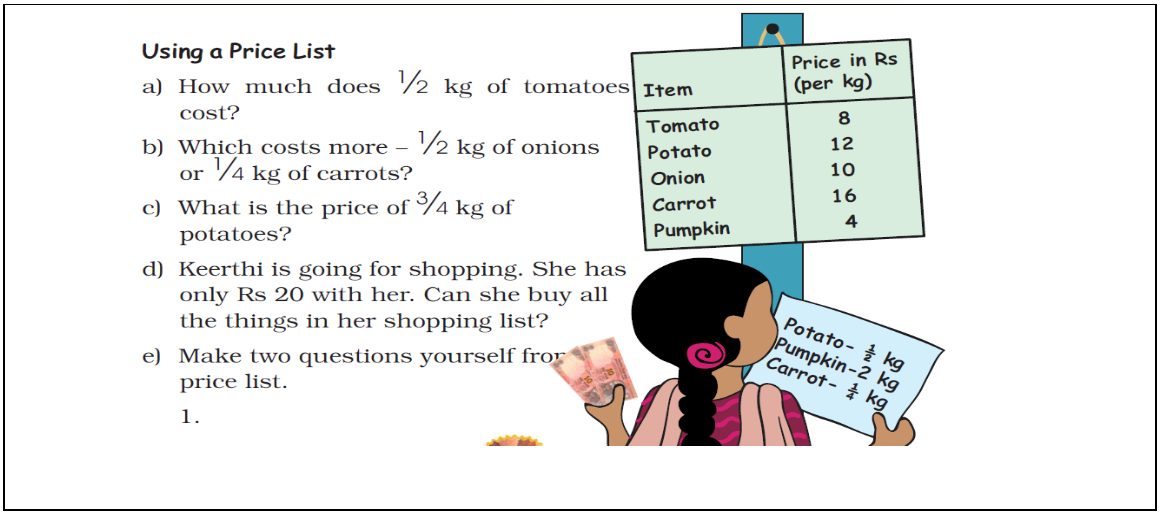


Figure 3.8. Fraction and money related task.

Source: "Textbook4", n.d., p. 100.

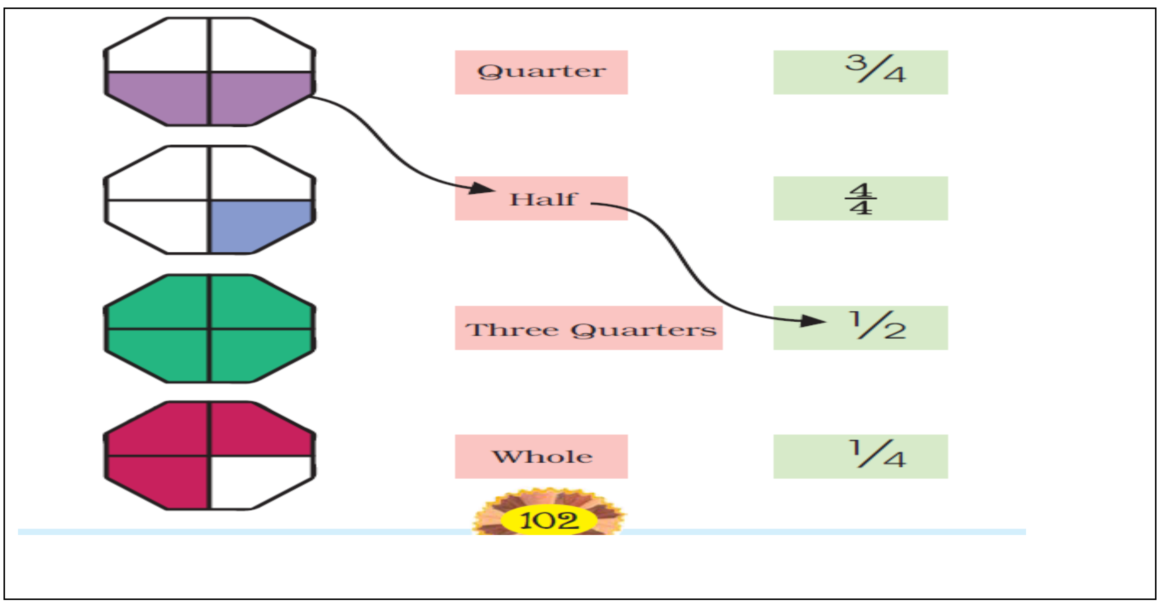
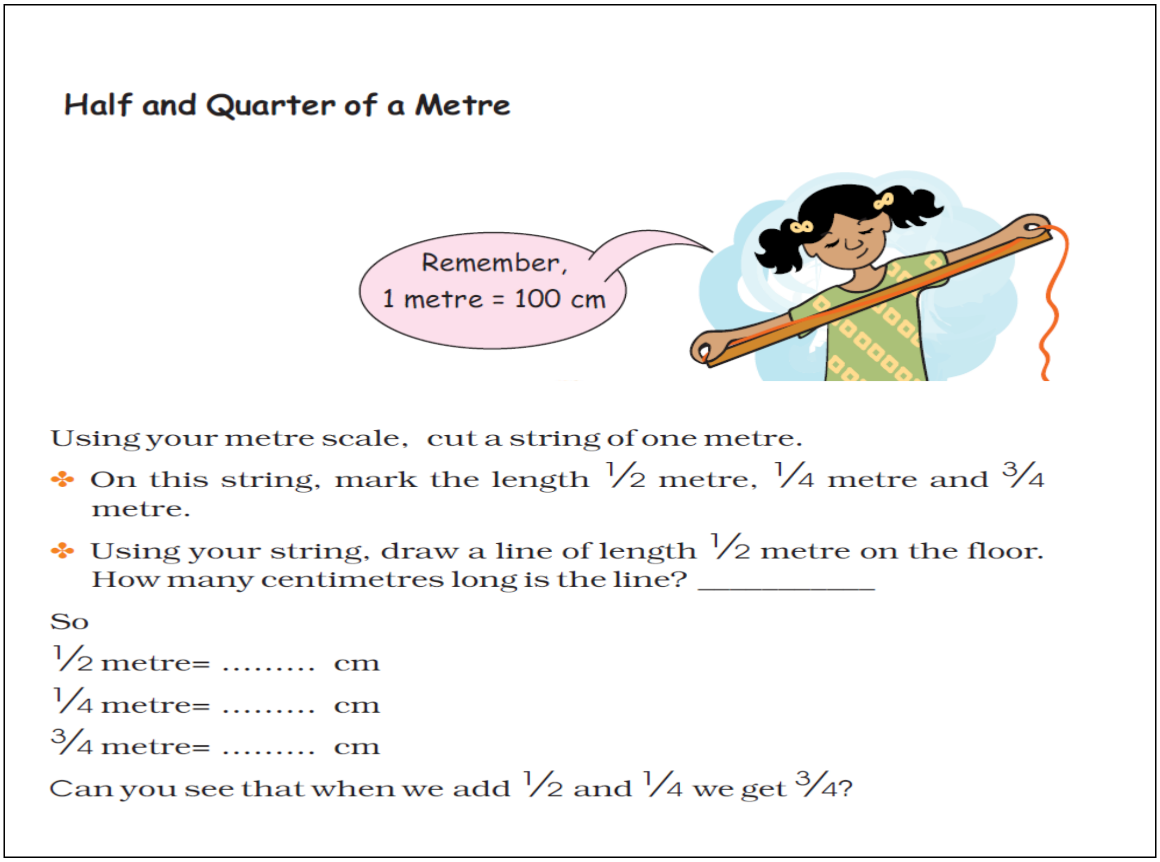


Figure 3.9. Identify and match the appropriate fraction.

Source: "Textbook4", n.d., p. 102.



Half and Quarter of a Metre

Using your metre scale, cut a string of one metre.

- On this string, mark the length $\frac{1}{2}$ metre, $\frac{1}{4}$ metre and $\frac{3}{4}$ metre.
- Using your string, draw a line of length $\frac{1}{2}$ metre on the floor. How many centimetres long is the line? ___________

So

$\frac{1}{2}$ metre= .... cm

$\frac{1}{4}$ metre= .... cm

$\frac{3}{4}$ metre= .... cm

Can you see that when we add $\frac{1}{2}$ and $\frac{1}{4}$ we get $\frac{3}{4}$?

Figure 3.10. Conversion of meters to centimeters.

Source: “Textbook4”, n.d., p. 104.

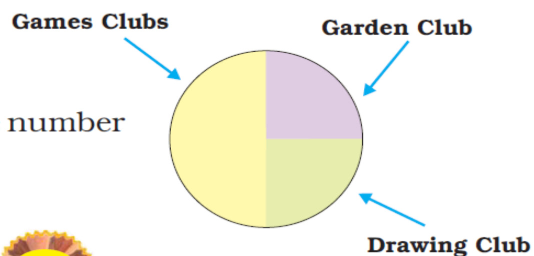
Chapter 14: “Smart Charts”, p. 162

In this chapter, Chapati charts (Figure 3.11) are used to understand, explore, and solve word problems on proper fractions. Chapati charts are pie charts. The circular disc in a Chapati chart is divided into parts and some parts of it are shaded to represent the fraction. Figure 3.12 shows a task involving Chapati chart and fractions. Both figures 3.11 and 3.12 involve tasks that I view as fraction multiplication with whole numbers.

Chapati Chart

All children of a school take part in different clubs:

The *Chapati* Chart shows the number of children in different clubs.



168

From the picture we can see that:

a) Half the children in the class take part in the Games Club.

b) One fourth of the children are members of the Garden Club.

c) The Drawing Club has one fourth of the children of the class.

If there are 200 students in the school, look at the above *Chapati* Chart and tell the number of members in each club:

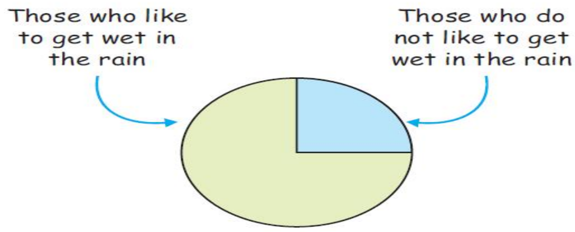
- The Games Club has ______ members.
- The Garden Club has ______ members.
- There are _____ members in the Drawing Club.

Figure 3.11. Chapati chart.

Source: "Textbook4", n.d., pp. 168-169.

Getting Wet in the Rain

Who likes to get wet in the rain? A child made this *Chapati* Chart after asking his friends.



See the *Chapati* Chart and tell:

1) How many children like to get wet in the rain?

a) half b) one-fourth c) three-fourth

2) How many children do not like to get wet in the rain?

a) half b) one-fourth c) three-fourth

If the number of children in the class is 28, then tell the number of children

- **who like to get wet in the rain __________**
- **who do not like to get wet in the rain __________**

Figure 3.12. Task with Chapati chart and fraction.

Source: “Textbook4”, n.d., pp. 169-170.

COMPARISON IN GRADE 4

I found similarities in grades 1, 2, and 3 of CCSSM standards with NCERT’s syllabus for grade 4 on fractions namely circles and rectangles are used in both for partitioning into equal parts. The striking difference is the use of “part-whole” definition of fraction in the NCERT’s syllabus, which is only used in CCSSM 3.NF.A.1 (Part 1) for unit fractions. In NCERT’s textbook, proper fraction is explained as “part out of …” i.e. part-whole definition of fraction. Only proper fractions are introduced in grade 4 of NCERT’s textbook. Another difference is that CCSSM has a standard in grade 2, which emphasize that equal shares of identical wholes need not have the same shape (Common Core State Standards Initiative, 2010). In this grade, NCERT focuses on parts of same shape. Figure 3.7 demonstrates how half is explained in the textbook where two parts are compared by placing one on top of the other. Visual models are used to show equivalence of 4/4 with whole, 2/4 with 1/2 in NCERT's syllabus, which is seen in CCSSM grade 3 on fractions. CCSSM's definition of fraction a/b as a parts of size 1/b is not used in the NCERT's syllabi. I also found that only continuous units are discussed in grade 4 of the NCERT's syllabus. The focus is on the area model of fractions. In NCERT's grade 4, reflection is considered through paper folding, paper cutting and inkblots, which is consistent with the CCSSM's geometry standards. I found tasks involving generation of the whole from given unit fractional (one-half, one-fourth) part of the whole in this grade of NCERT's textbook.

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 5

5.NBT Number and Operations in Base Ten

- Understand the place value system.
  - 5.NBT.A.1 Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
  - 5.NBT.A.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the

placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

    - 5.NBT.A.3 Read, write, and compare decimals to thousandths.
    - 5.NBT.A.4 Use place value understanding to round decimals to any place.

- Perform operations with multi-digit whole numbers and with decimals to hundredths.
    - 5.NBT.B.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

5.NF Number and Operations- Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
    - 5.NF.A.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.
    - 5.NF.A.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness to answers.

For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.

- Apply and extend previous understandings of multiplication and division.
  - 5.NF.B.3 Interpret a fraction as division of the numerator by the denominator (a/b= a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.
  - 5.NF.B.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
    - 5.NF.B.4a Interpret the product (a/b) x q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a x q ÷ b.
    - 5.NF.B.4b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.
  - 5.NF.B.5 Interpret multiplication as scaling (resizing), by:
    - 5.NF.B.5a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

        - 5.NF.B.5b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence a/b = (n x a)/ (n x b) to the effect of multiplying a/b by 1.
    - 5.NF.B.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.
    - 5.NF.B.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.
        - 5.NF.B.7a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.
        - 5.NF.B.7b Interpret division of a whole number by a unit fraction, and compute such quotients.
        - 5.NF.B.7c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

5.MD Measurement and Data

- Convert like measurement units within a given measurement system.
- Represent and interpret data.

- 5.MD.B.2 Make line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 5

Measurement

Length:

- Applies simple fractions to quantities.
- Converts fractional larger unit into complete smaller units.

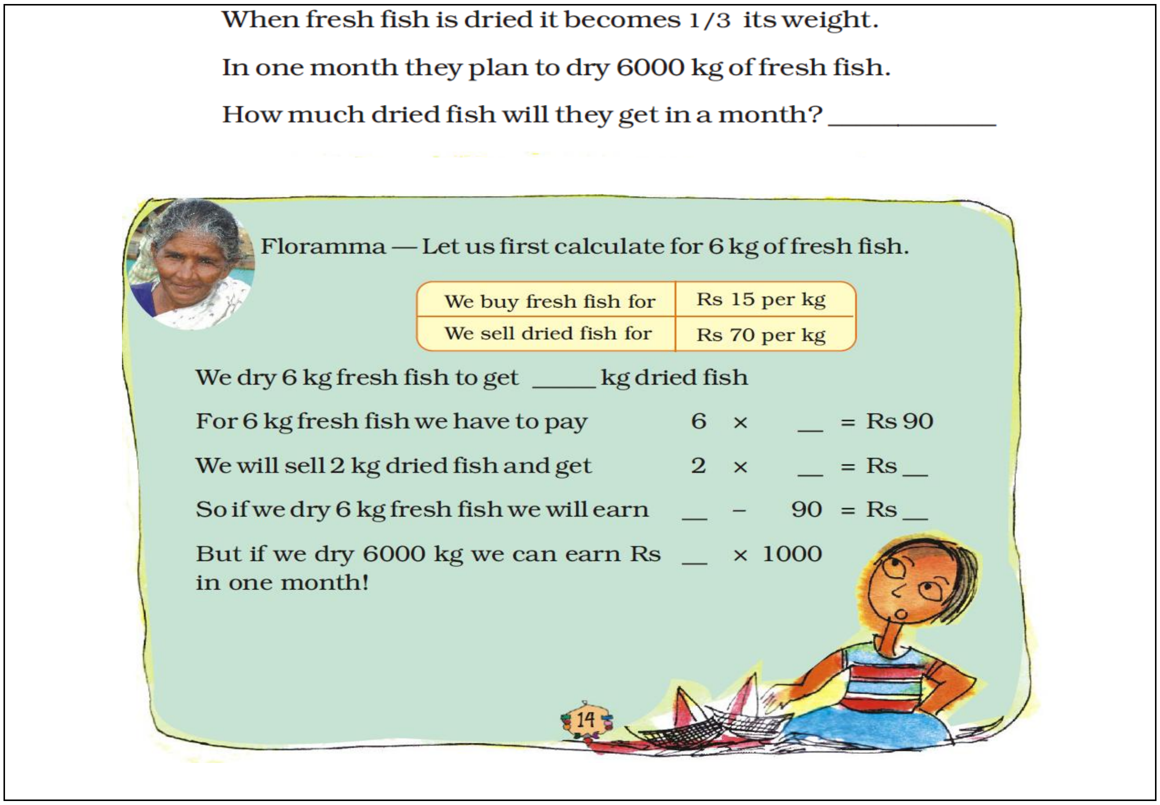
Fractional Numbers:

- Finds the fractional part of a collection.
- Compares fractions.
- Identifies equivalent fractions.
- Estimates the degree of closeness of a fraction to known fractions (1/2, 1/4, 3/4 etc.).
- Uses decimal fractions in the context of units of length and money.
- Expresses a given fraction in decimal notation and vice versa.

NCERT'S TEXT BOOK FOR GRADE 5

Chapter 1: "The Fish Tale", p. 1

In this chapter, word problem involving fraction multiplication with whole number is discussed. Figure 3.13 shows a task involving fraction multiplication with whole number.



When fresh fish is dried it becomes 1/3 its weight.

In one month they plan to dry 6000 kg of fresh fish.

How much dried fish will they get in a month? ____________

Floramma — Let us first calculate for 6 kg of fresh fish.

We buy fresh fish for	Rs 15 per kg
We sell dried fish for	Rs 70 per kg

We dry 6 kg fresh fish to get _____ kg dried fish

For 6 kg fresh fish we have to pay 6 × __ = Rs 90

We will sell 2 kg dried fish and get 2 × __ = Rs __

So if we dry 6 kg fresh fish we will earn __ – 90 = Rs __

But if we dry 6000 kg we can earn Rs __ × 1000 in one month!

14

Figure 3.13. Multiplication of a fraction with whole number.

Source: "Textbook5", n.d., p. 14.

Chapter 4: "Parts and Wholes", p.50

In this chapter, the following are discussed:

- Using flags from different countries, parts of strips and patterns, 1/3, 3/4, 1/2 are explored in terms of "parts out of …" definition of fraction.
- Figure 3.14 portrays candy bar problem showing unlike[4] fractions on the same bar. The task involves sharing a chocolate bar among four persons having different fractional parts of the whole.

[4] Unlike fractions are fractions that have different denominators. For example: 2/3 and 4/5 are unlike fractions.

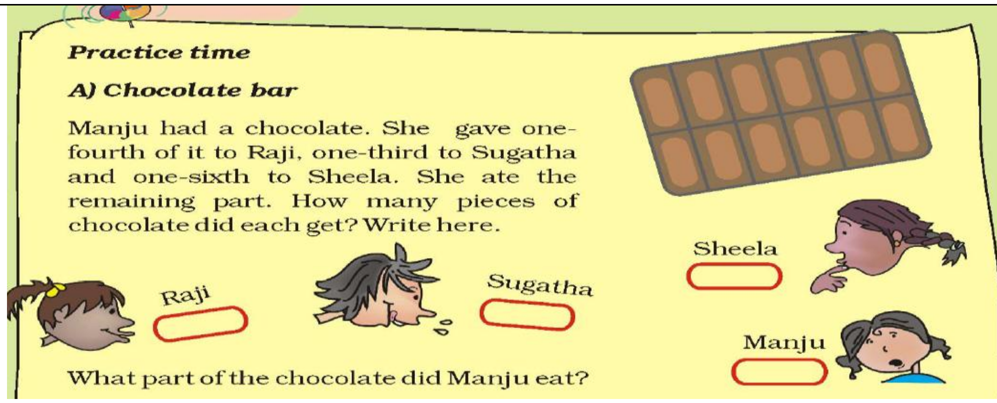
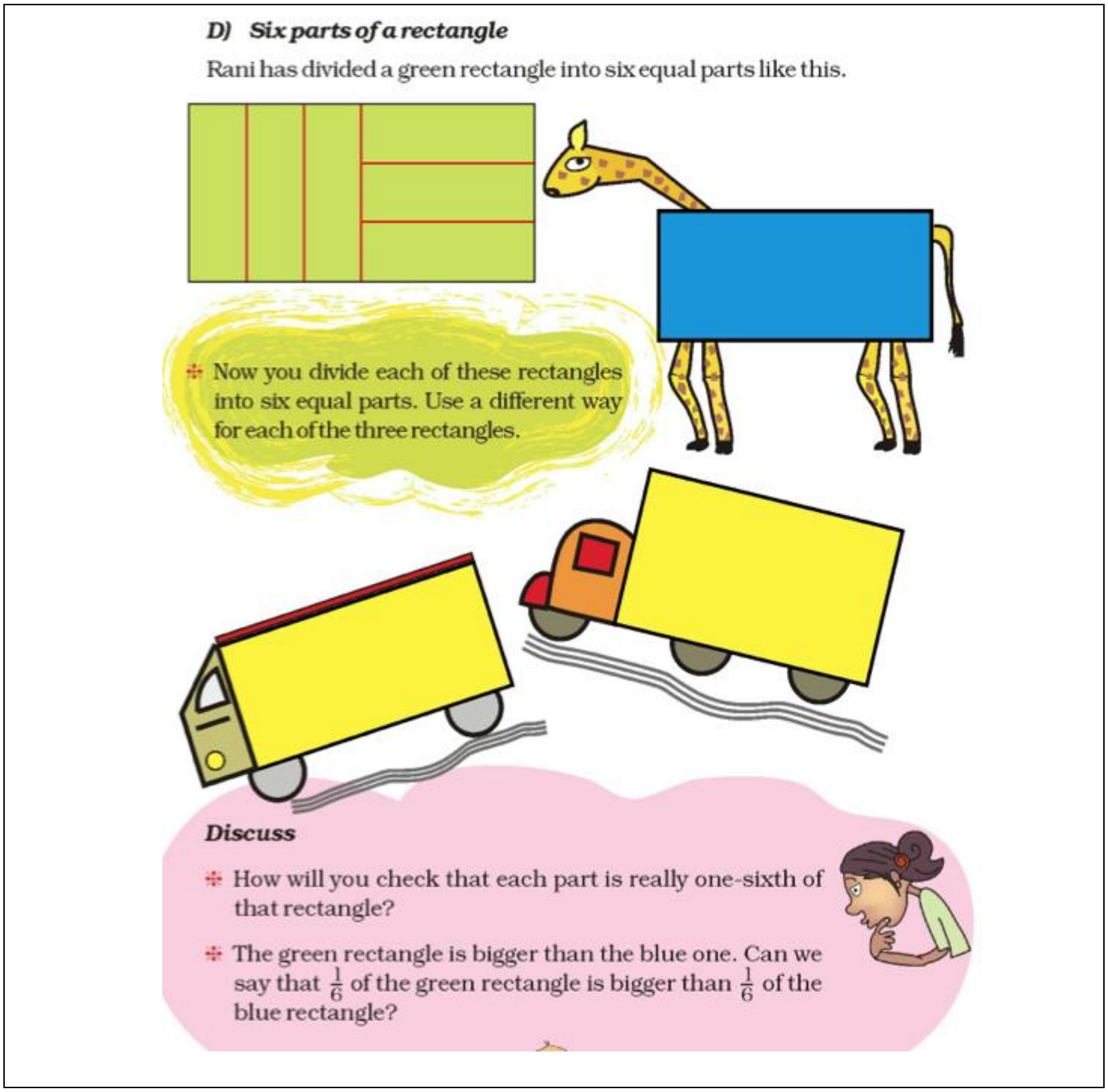


Figure 3.14. Partitioning a chocolate bar.

Source: "Textbook5", n.d., p. 52.

- This chapter focuses on understanding 1/8 and 1/6. I found some reasoning tasks in this chapter. Figure 3.15 shows two of such tasks. One of the tasks requires understanding why each fractional part of the whole is really one-sixth of the rectangle. The other task requires understanding of why the fractional part of a larger whole is always bigger than the same fractional part of a smaller whole.
- This chapter requires understanding proper fractions by coloring different patterns: A square grid is given and students are asked to color some of the squares in the grid. Then they are asked what fraction of the grid is colored and what fraction of the grid is not colored. Figure 3.16 is one of the tasks taken from the textbook.
- Understanding equivalent fractions: 1/2 = 2/4 = 3/6 = 4/8 by coloring circle patterns. Figure 3.17 demonstrates one such task.



D) Six parts of a rectangle

Rani has divided a green rectangle into six equal parts like this.

- Now you divide each of these rectangles into six equal parts. Use a different way for each of the three rectangles.

Discuss

- How will you check that each part is really one-sixth of that rectangle?
- The green rectangle is bigger than the blue one. Can we say that $\frac{1}{6}$ of the green rectangle is bigger than $\frac{1}{6}$ of the blue rectangle?

Figure 3.15. Reasoning based task.

Source: “Textbook5”, n.d., p. 54.

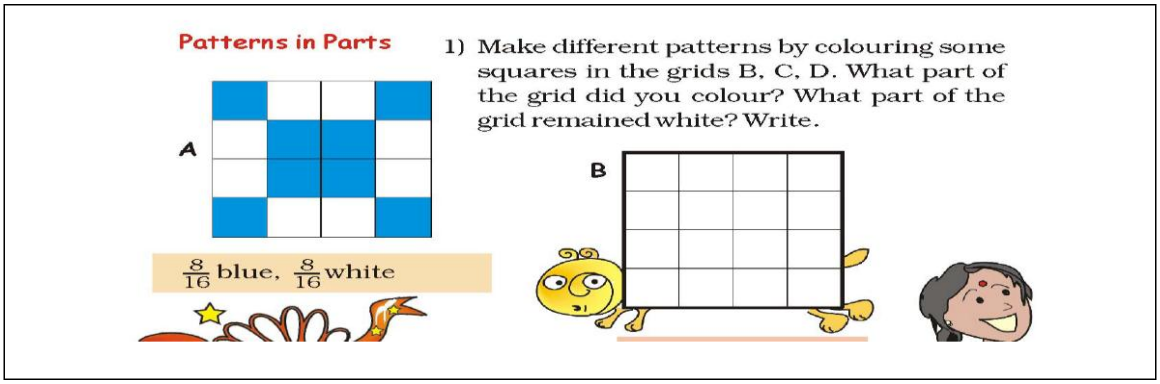


Figure 3.16. Part-whole problem.

Source: "Textbook5", n.d., p. 57.

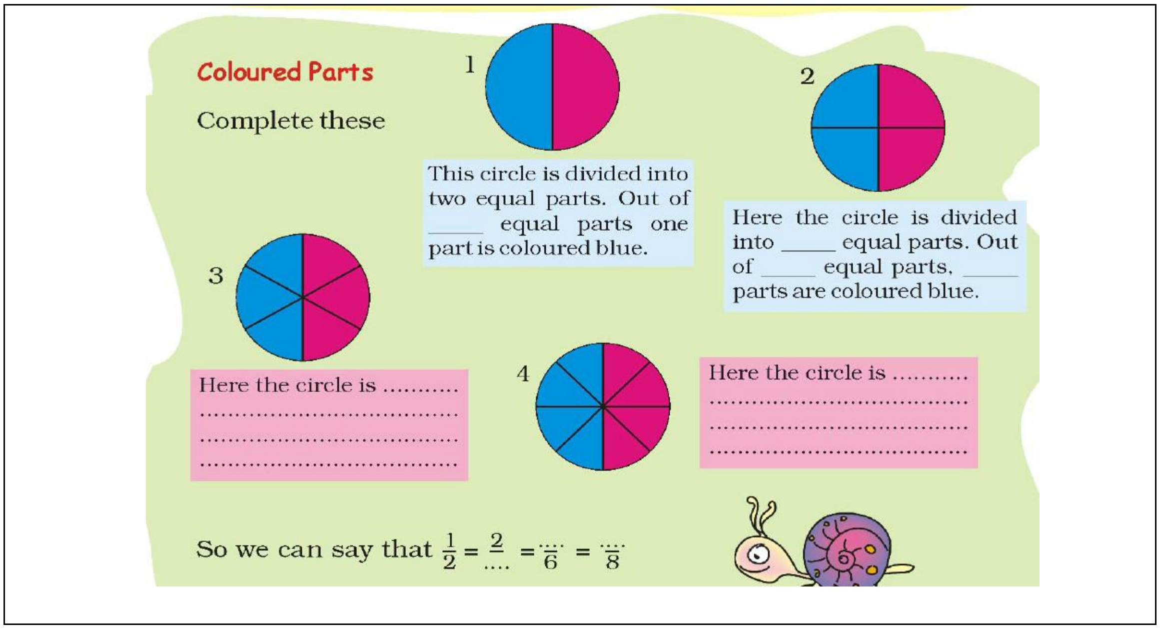


Figure 3.17. Equivalence of fractions.

Source: "Textbook5", n.d., p. 62.

- I found money related fraction tasks in this chapter. The tasks require understanding paise as fractional part of a rupee. Students are asked about the fractional relationship between 10, 20, 25, 50 paise with one rupee respectively (Figure 3.18).

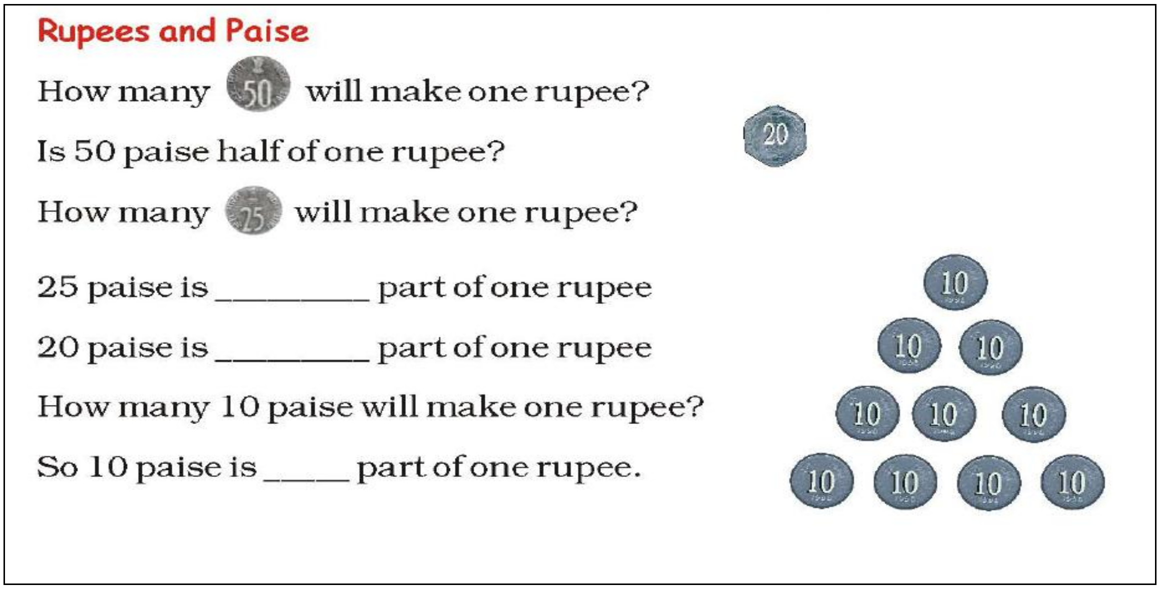
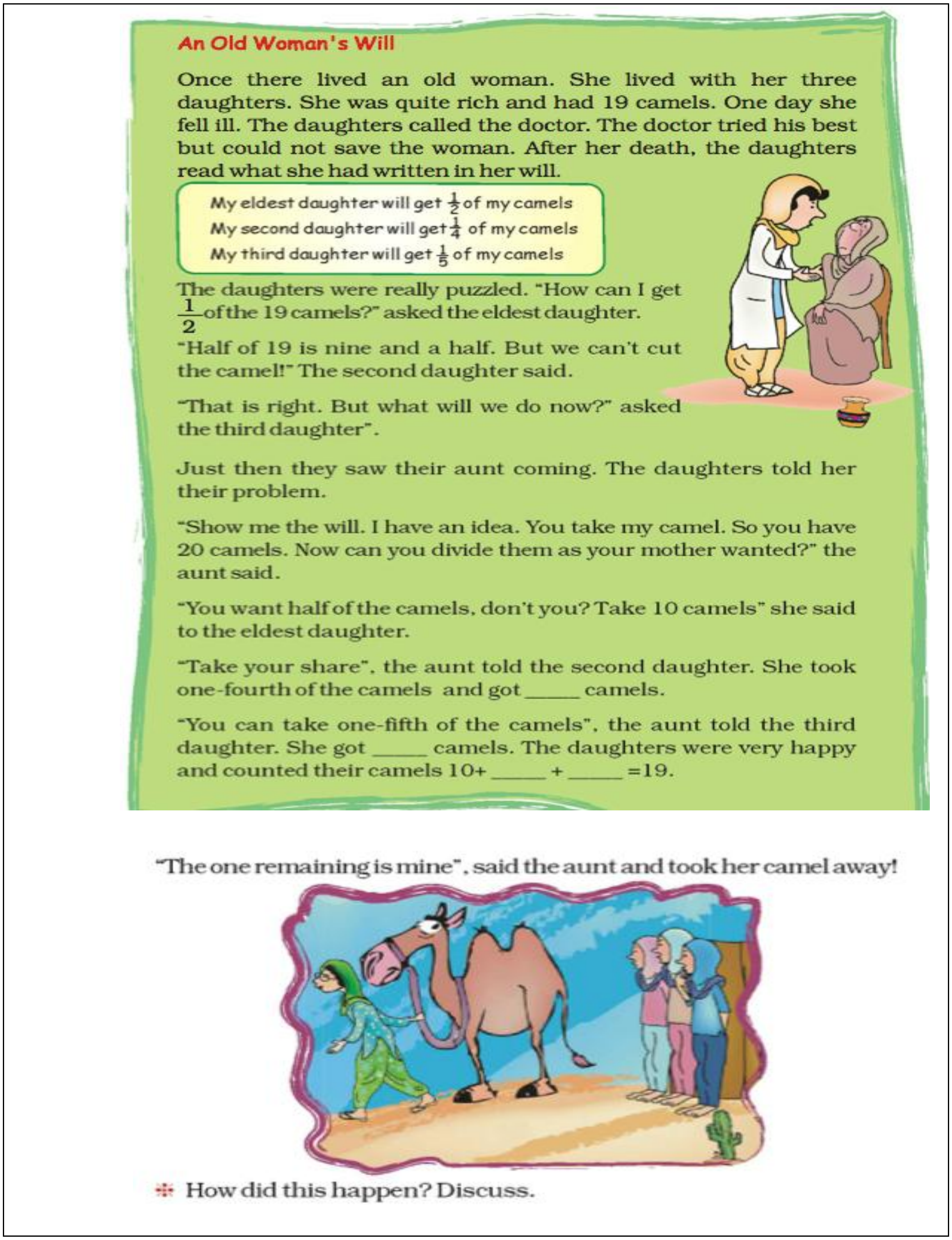


Figure 3.18. Money related fraction task.

Source: “Textbook5”, n.d., p. 65.

- I found many story problems in this chapter. Figure 3.19 shows one of the story problems. The focus of this chapter is on understanding and reasoning mathematically.
- I found tasks on understanding and reproducing the whole from a given fractional part of the whole. Figure 3.20 shows a task in constructing a whole from a given unit fractional part of the whole. I infer that students solving the task would use iteration, i.e., repeating a part of the whole to reproduce the whole.



An Old Woman's Will

Once there lived an old woman. She lived with her three daughters. She was quite rich and had 19 camels. One day she fell ill. The daughters called the doctor. The doctor tried his best but could not save the woman. After her death, the daughters read what she had written in her will.

My eldest daughter will get $\frac{1}{2}$ of my camels
My second daughter will get $\frac{1}{4}$ of my camels
My third daughter will get $\frac{1}{5}$ of my camels

The daughters were really puzzled. "How can I get $\frac{1}{2}$ of the 19 camels?" asked the eldest daughter.

"Half of 19 is nine and a half. But we can't cut the camel!" The second daughter said.

"That is right. But what will we do now?" asked the third daughter".

Just then they saw their aunt coming. The daughters told her their problem.

"Show me the will. I have an idea. You take my camel. So you have 20 camels. Now can you divide them as your mother wanted?" the aunt said.

"You want half of the camels, don't you? Take 10 camels" she said to the eldest daughter.

"Take your share", the aunt told the second daughter. She took one-fourth of the camels and got _____ camels.

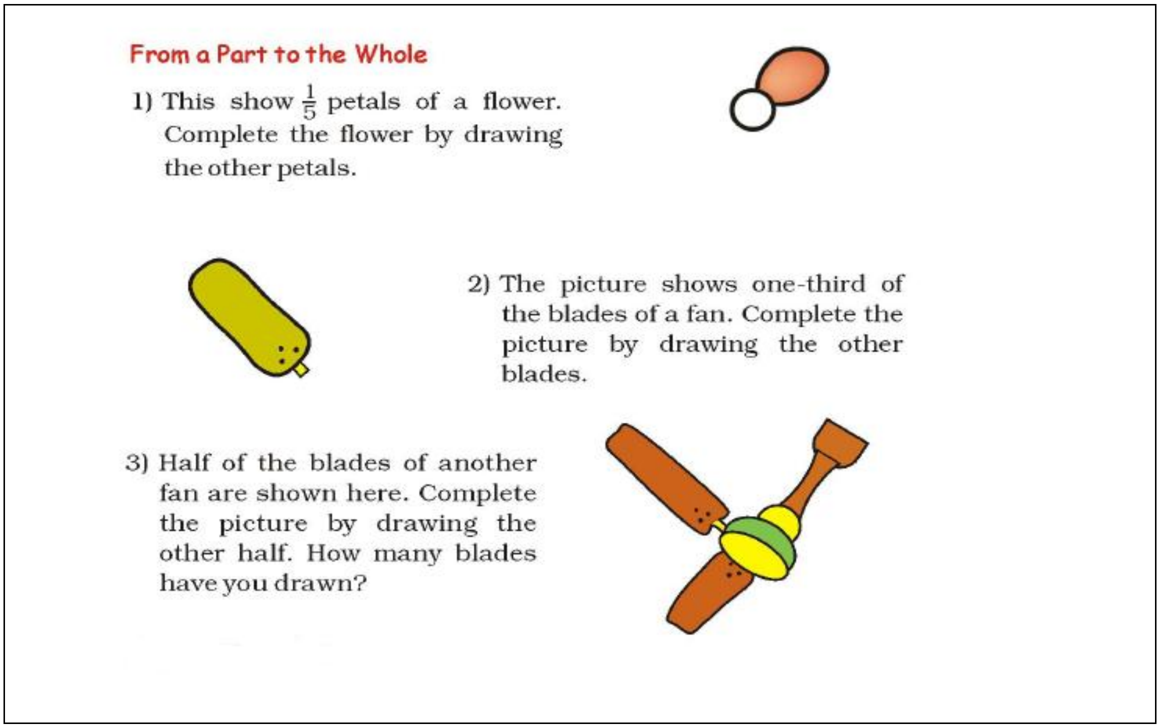
"You can take one-fifth of the camels", the aunt told the third daughter. She got _____ camels. The daughters were very happy and counted their camels 10+ _____ + _____ =19.

"The one remaining is mine", said the aunt and took her camel away!

How did this happen? Discuss.

Figure 3.19. Story problem.

Source: "Textbook5", n.d., p. 66.



From a Part to the Whole

1) This show $\frac{1}{5}$ petals of a flower. Complete the flower by drawing the other petals.

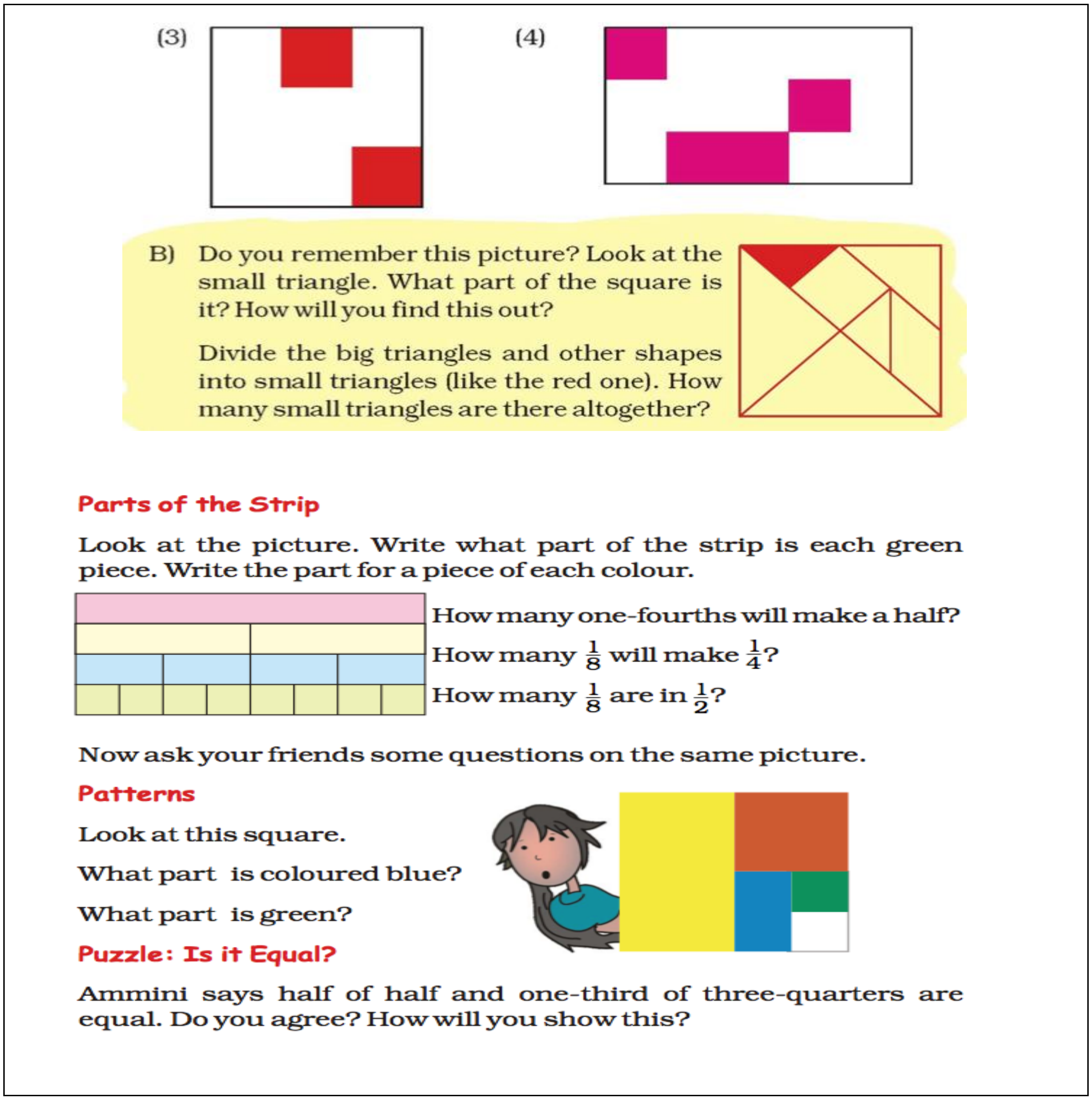
2) The picture shows one-third of the blades of a fan. Complete the picture by drawing the other blades.

3) Half of the blades of another fan are shown here. Complete the picture by drawing the other half. How many blades have you drawn?

Figure 3.20. Whole from the given fractional part.

Source: “Textbook5”, n.d., p. 65.

- I found reasoning based tasks in this grade. Figure 3.21 shows some of the tasks. The tasks involve a picture of unpartitioned whole and a given shaded part of the whole. The tasks require students to partition the whole and figure out the fractional part of the whole.



(3) (4)

B) Do you remember this picture? Look at the small triangle. What part of the square is it? How will you find this out?

Divide the big triangles and other shapes into small triangles (like the red one). How many small triangles are there altogether?

Parts of the Strip

Look at the picture. Write what part of the strip is each green piece. Write the part for a piece of each colour.

How many one-fourths will make a half?

How many $\frac{1}{8}$ will make $\frac{1}{4}$?

How many $\frac{1}{8}$ are in $\frac{1}{2}$?

Now ask your friends some questions on the same picture.

Patterns

Look at this square.

What part is coloured blue?

What part is green?

Puzzle: Is it Equal?

Ammini says half of half and one-third of three-quarters are equal. Do you agree? How will you show this?

Figure 3.21. Fractional part of a whole.

Source: "Textbook5", n.d., pp. 62, 64.

Chapter 5: "Does it Look the Same?", p. 71

In this chapter mirror halves is revisited, reflections through inkblots are seen and various turns are discussed.

The turns discussed in this chapter are:

- Half a turn and one-fourth turn (Figure 3.22).
- One-third turns (Figure 3.23).
- One-sixth turns (Figure 3.24).

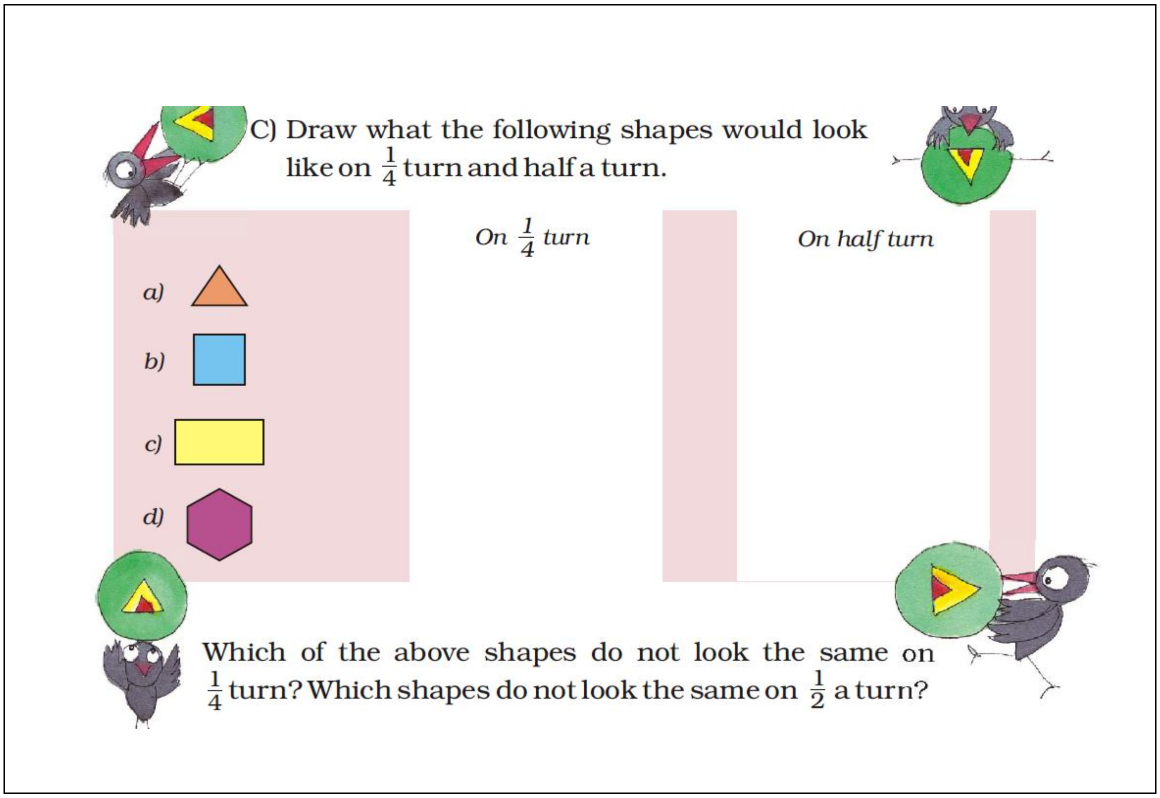


Figure 3.22. One-fourth turn and half a turn.

Source: “Textbook5”, n.d., p. 84.

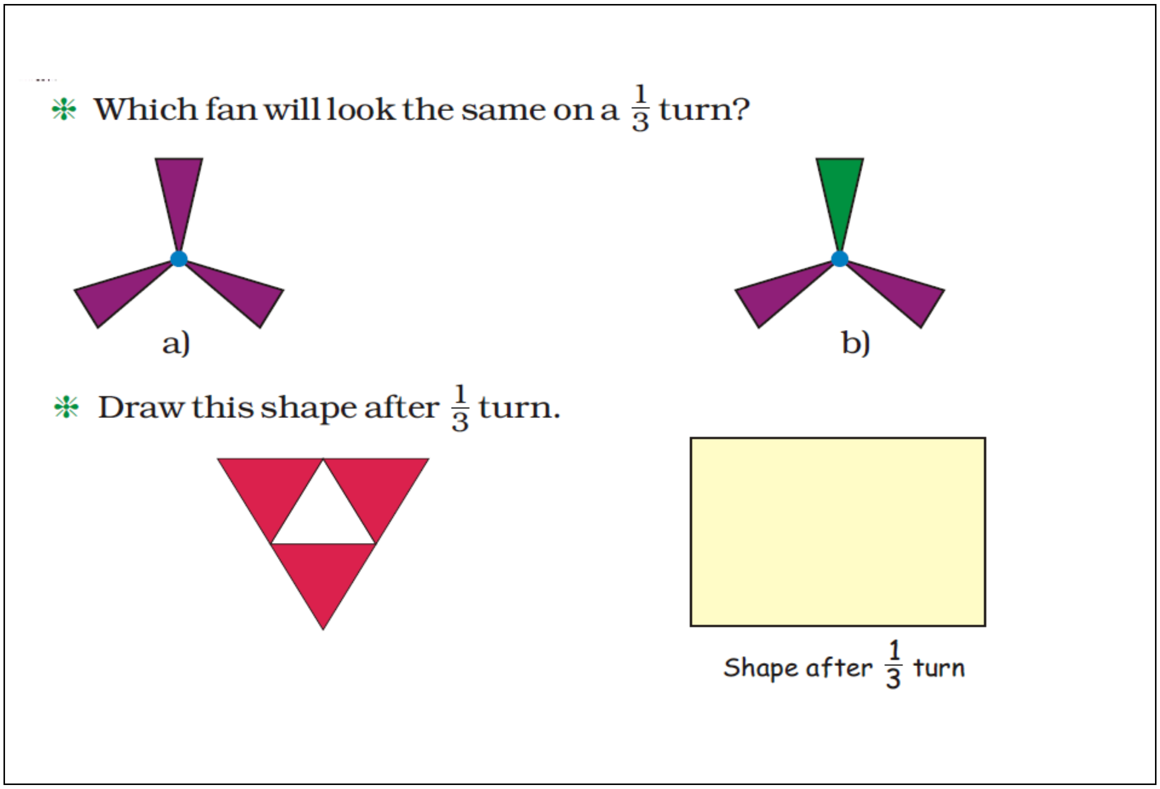
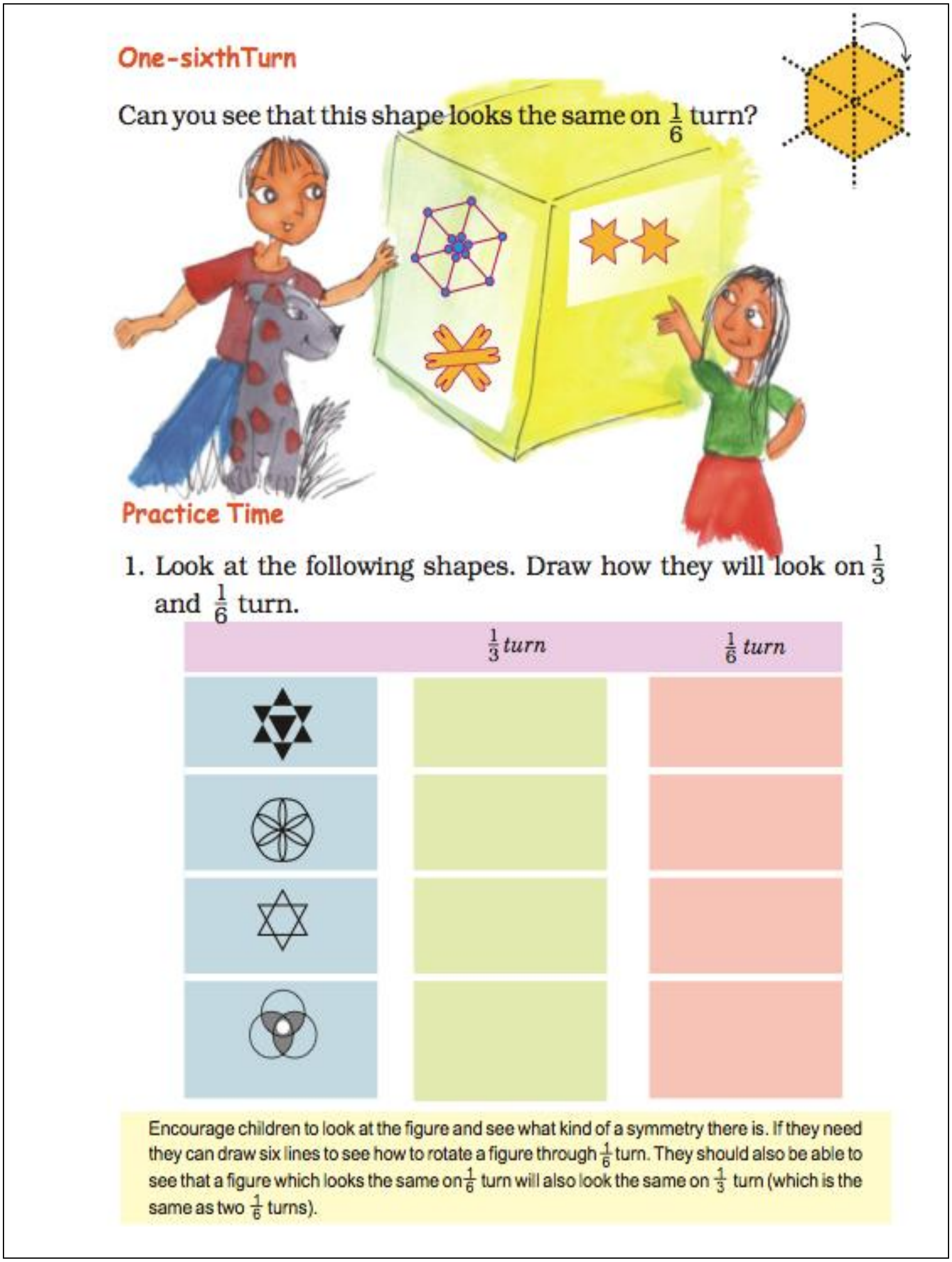


Figure 3.23. One-third turns.

Source: “Textbook5”, n.d., p. 84.



One-sixthTurn

Can you see that this shape looks the same on $\frac{1}{6}$ turn?

Practice Time

1. Look at the following shapes. Draw how they will look on $\frac{1}{3}$ and $\frac{1}{6}$ turn.

	$\frac{1}{3}$ *turn*	$\frac{1}{6}$ *turn*

Encourage children to look at the figure and see what kind of a symmetry there is. If they need they can draw six lines to see how to rotate a figure through $\frac{1}{6}$ turn. They should also be able to see that a figure which looks the same on $\frac{1}{6}$ turn will also look the same on $\frac{1}{3}$ turn (which is the same as two $\frac{1}{6}$ turns).

Figure 3.24. One-sixth turns.

Source: “Textbook5”, n.d., p. 85.

Chapter 7: “Can you see the Pattern?”, p. 99

In this chapter, various problems on turns and patterns are discussed. Figure 3.25 shows one such example.

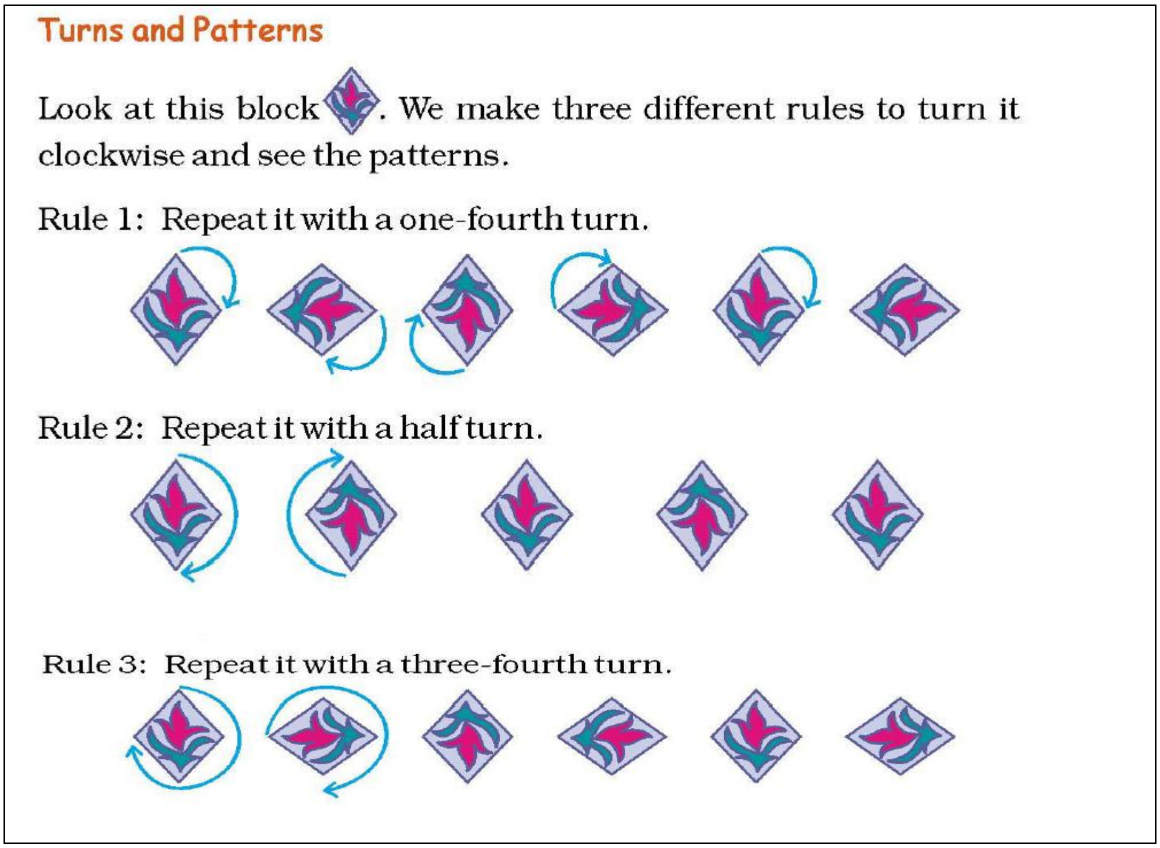
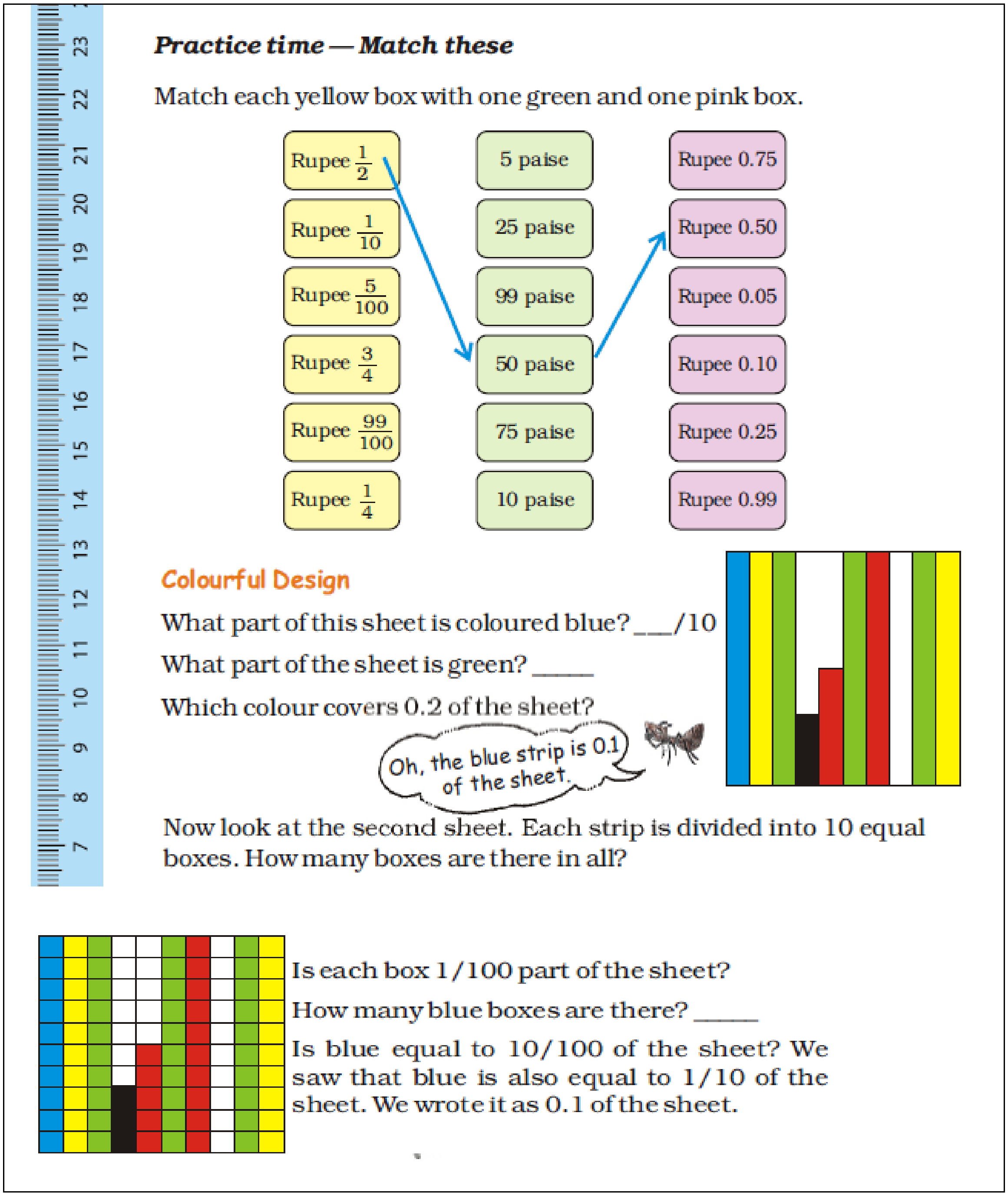


Figure 3.25. Turns and patterns.

Source: “Textbook5”, n.d., p. 100.

Chapter 10: “Tenths and Hundredths”, p. 134

In this chapter millimeter and centimeter is discussed. It is shown that one-millimeter equals 0.1 centimeters. The relationship between 1/10 and 0.1 is discussed (Figure 3.26).



Practice time — Match these

Match each yellow box with one green and one pink box.

Rupee $\frac{1}{2}$	5 paise	Rupee 0.75
Rupee $\frac{1}{10}$	25 paise	Rupee 0.50
Rupee $\frac{5}{100}$	99 paise	Rupee 0.05
Rupee $\frac{3}{4}$	50 paise	Rupee 0.10
Rupee $\frac{99}{100}$	75 paise	Rupee 0.25
Rupee $\frac{1}{4}$	10 paise	Rupee 0.99

Colourful Design

What part of this sheet is coloured blue? ___/10

What part of the sheet is green? _____

Which colour covers 0.2 of the sheet?

Oh, the blue strip is 0.1 of the sheet.

Now look at the second sheet. Each strip is divided into 10 equal boxes. How many boxes are there in all?

Is each box 1/100 part of the sheet?

How many blue boxes are there? _____

Is blue equal to 10/100 of the sheet? We saw that blue is also equal to 1/10 of the sheet. We wrote it as 0.1 of the sheet.

Figure 3.26. Tenths and hundredths.

Source: “Textbook5”, n.d., p. 140.

COMPARISON IN GRADE 5

I found similarities with grade 3 and 4 standards of the CCSSM with NCERT's syllabus for grade 5 on fractions. In the context of length and money, decimal fractions (tenths and hundredths) are introduced in NCERT's syllabus that is seen in grade 4 of CCSSM. NCERT introduces 1/2, 1/3, 1/4, 1/6 turns in this grade, which I consider part of rotational symmetry. Rotational symmetry is introduced in grade 8 of CCSSM. Lines of symmetry are discussed in grade 4 of CCSSM that is seen in grade 5 of NCERT's syllabus while revisiting mirror halves. I found the concept of iteration in the task of reproducing the whole from a given fractional part of the whole in chapter 4 of NCERT's grade 5 textbook which is closely related to CCSSM's 4.NF.B.3a on fraction addition as both use iteration to get the whole. I infer that more focus is given on continuous unit (area model of fraction) rather than discrete units (set model of fraction) in this grade of NCERT as I found only three tasks involving discrete units across chapter 4 of grade 5 of NCERT's textbook. Figure 3.27 demonstrates one such task that involves discrete units.

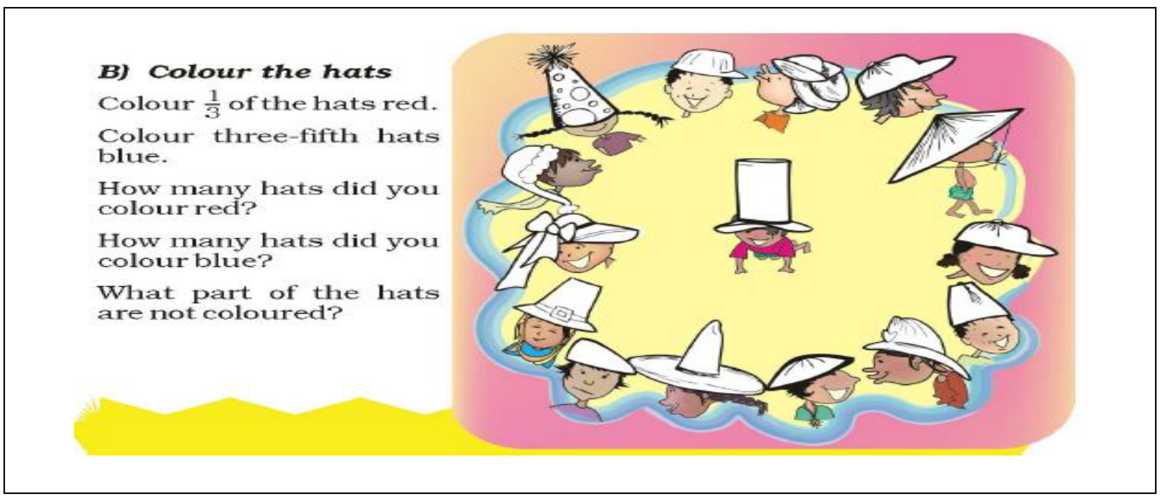


Figure 3.27. Hat coloring task.

Source: "Textbook5", n.d., p. 53.

Comparison grid for grades 1 through 5

I present the comparison between grades 1-5 NCERT's syllabi with CCSSM on fractions as a tree diagram (Figure 3.28). The tree diagram gives the standards for NCERT's syllabi from grades 1-5 with the corresponding CCSSM standards (shown in parenthesis).

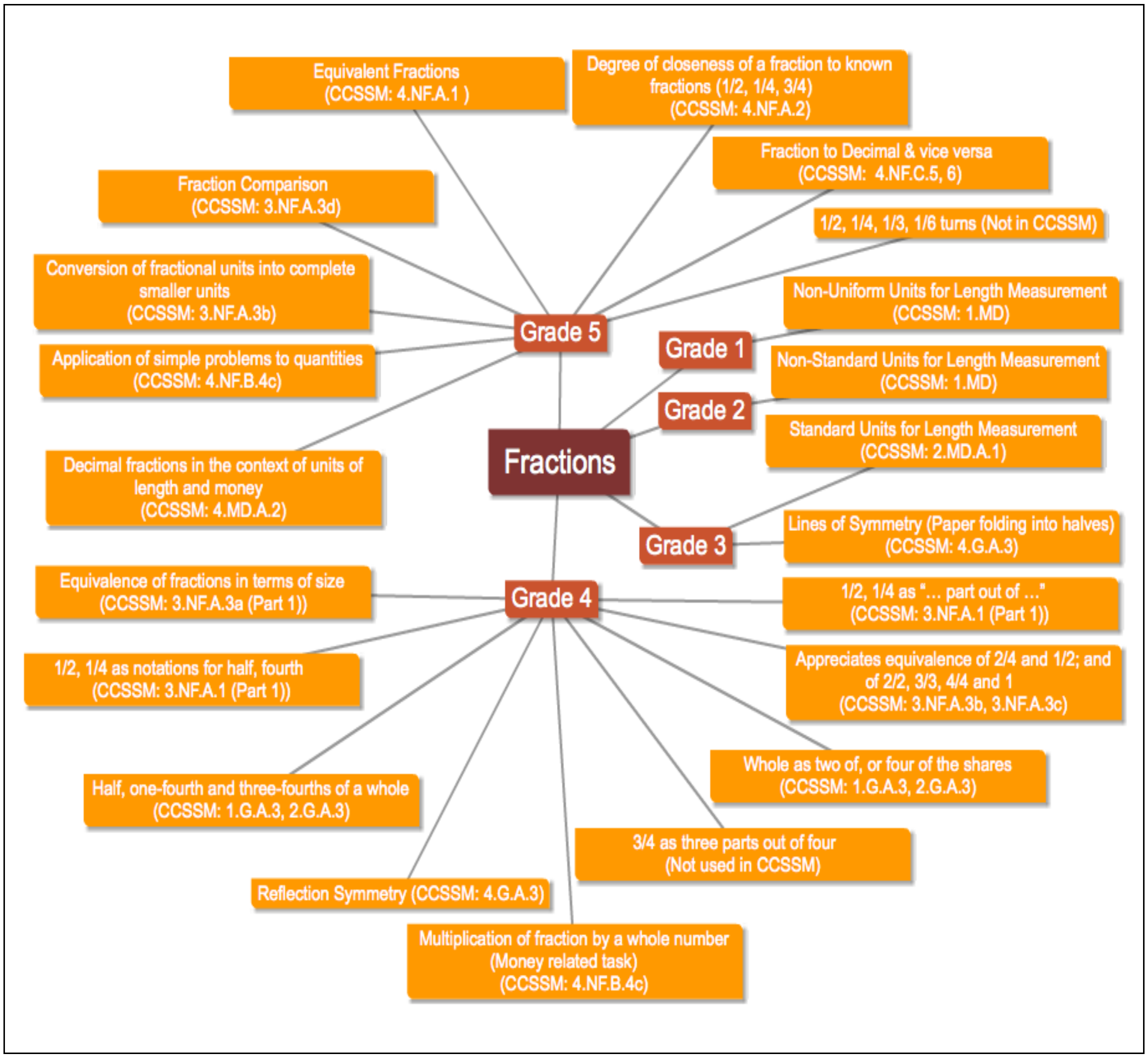


Figure 3.28. Comparison of NCERT's topics with the CCSSM.

Next, I show a grade-by-grade comparison of the NCERT's syllabi with CCSSM standards on fractions. I have organized table 3.1 with the first column being the

NCERT's syllabi on fractions starting from grade 1. The second column representing the CCSSM standards that are closely related to the NCERT's syllabi on fractions. This table gives a quick view of CCSSM's standards corresponding to NCERT's on fractions.

Table 3.1. Comparison between grades 1-5.

NCERT's Syllabi on Fractions	CCSSM on Fractions
Grade 1 1. Non-uniform units for length measurement **Grade 2** 1. Uniform (Non-Standard) units for length measurement	1. Grade 1: 1.MD
Grade 3 1. Standard units for length measurement **2.** Lines of symmetry (Paper folding into halves)	1. Grade 2: 2.MD.A.1 2. Grade 4: 4.G.A.3
Grade 4 1. Half, one-fourth and three-fourths of a whole 2. Symbols 1/2, 1/4 3. 1/2, 1/4 as "… part out of …" 4. 3/4 as three parts out of four 5. Whole as two of, or four of the shares 6. Equivalence of fractions in terms of size 7. Appreciates equivalence of 2/4 and 1/2; and of 2/2, 3/3, 4/4 and 1. 8. Multiplication of fraction by a whole number (Money related task) **9.** Reflections through inkblots, paper cutting and paper folding, geometrical patterns based on symmetry	1. Grade 1, Grade 2: 1.G.A.3, 2.G.A.3 2. Grade 3: 3.NF.A.1 (Part 1) 3. Grade 3: 3.NF.A.1 (Part 1) 4. Not used in CCSSM 5. Grade 1, Grade 2: 1.G.A.3, 2.G.A.3 6. Grade 3: 3.NF.A.3a (Part 1) 7. Grade 3: 3.NF.A.3b, 3.NF.A.3c 8. Grade 4: 4.NF.B.4c 9. Grade 4: 4.G.A.3
Grade 5 1. Application of simple problems to quantities 2. Conversion of fractional units into complete smaller units 3. Finding fractional part of a collection 4. Fraction Comparison 5. Equivalent Fractions 6. Degree of closeness of a fraction to known fractions (1/2, 1/4, 3/4) 7. Decimal fractions in the context of units of length and money 8. Fraction to Decimal & vice versa 9. 1/2, 1/4, 1/3, 1/6 turns **10.** Area models of fractions	1. Grade 4: 4.NF.B.4c 2. Grade 3: 3.NF.A.3b 3. Grade 4: 4.NF.B.3a (Closely related) 4. Grade 3: 3.NF.A.3d 5. Grade 4: 4.NF.A.1 6. Grade 4: 4.NF.A.2 7. Grade 4: 4.MD.A.2 8. Grade 4: 4.NF.C.5, 4.NF.C.6 9. Not in CCSSM (But rotational symmetry is in 8.G.A.3) 10. Grade 3: 3.G.A.2

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 6

6.RP Ratios and Proportional Relationships

- Understand ratio concepts and use ratio reasoning to solve problems.

6.NS The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
    - 6.NS.A.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.
- Compute fluently with multi-digit numbers and find common factors and multiples.
    - 6.NS.B.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.
- Apply and extend previous understandings of numbers to the system of rational numbers.
    - 6.NS.C.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 6

Fractions:

- Revision of what a fraction is
- Fraction as a part of whole
- Representation of fractions (pictorially and on number line)
- Fraction as a division
- Proper, improper and mixed fractions
- Equivalent fractions
- Comparison of fractions
- Addition and subtraction of fractions (Avoid large and complicated unnecessary tasks)
- Review of the idea of a decimal fraction
- Place value in the context of decimal fraction
- Inter conversion of fractions and decimal fractions (avoid recurring decimals at this stage)
- Word problems involving addition and subtraction of decimals (two operations together on money, mass, length, and temperature).

Ratio and Proportions

- Concept of ratio
- Proportion as equality of two ratios
- Unitary method (with only direct variation implied)
- Word problems

Geometry

(iii) Symmetry: (reflection)

- Observation and identification of 2-D symmetrical objects for reflection symmetry.
- Operation of reflection (taking mirror images) of simple 2-D objects.
- Recognizing reflection symmetry (identifying axes).

NCERT'S TEXT BOOK FOR GRADE 6

Chapter 7: "Fractions", p. 133

In this chapter, the following are introduced:

- Number lines for fractions: Number lines are introduced earlier in this grade in chapter 2 on whole numbers. In chapter 7, fractions are shown in the number line. The simplest of all fractions, half is shown on the number line. Half is considered greater than 0 and less than 1, hence it is placed in between 0 and 1. To locate the exact position of half, the space between 0 and 1 is divided into two equal parts. One of the parts is called one-half and is shown on the number line as a tick mark halfway between 0 and 1. Similarly, one-third is explained using partitioning the space in between 0 to 1 into three parts and showing one of the parts as 1/3. While showing 2/3 on the number line, grade 4's part-whole definition of fraction is revisited. It is emphasized that 2/3 means two parts out of three, so there are two equidistant tick marks in between 0 and 1, the second one denoting 2/3. I consider this as a length model of fraction.

- Improper and mixed fractions: Figure 3.29 shows how improper fractions and mixed fractions are introduced in grade 6. The connection between mixed fractions and improper fractions is shown by a sharing situation: some objects are shared equally among some people. The example in Figure 3.29 makes the connection between division and fraction as well. The example shows the amount each person gets in two ways: as a whole and a fractional part of a whole, and as an improper fraction.
- Conversion of improper fractions to mixed fraction and vice versa: The focus is on the algorithm for converting improper fractions to mixed fractions, which is shown in Figure 3.30. The standard division algorithm is used to convert improper fractions to mixed fractions.
- Understanding equivalent fractions: Different tasks are used here to explain the equivalent fractions. The tasks involved visual models with both continuous and some discrete units. Some of the tasks are shown in Figure 3.31.
- HCF: The highest common factor (HCF) is used here to find the equivalent fractions as shown in Figure 3.32. A procedure is explained how one can take the HCF of the numerator and denominator and then can divide both the numerator and the denominator by the HCF. It is not explained why one can do that.
- Comparing fractions: Comparisons between fractions are discussed in this chapter. It is shown how two fractions can be compared using diagrams (Figure 3.33). The emphasis is on learning a systematic procedure to compare fractions. Like fractions (fractions with same denominators) are compared by comparing the numerator (Figure 3.34).

7.5 Improper and Mixed Fractions

Anagha, Ravi, Reshma and John shared their tiffin. Along with their food, they had also, brought 5 apples. After eating the other food, the four friends wanted to eat apples.

How can they share five apples among four of them?

Anagha said, 'Let each of us have one full apple and a quarter of the fifth apple.'

Anagha Ravi Reshma John

Reshma said, 'That is fine, but we can also divide each of the five apples into 4 equal parts and take one-quarter from each apple.'

Anagha Ravi Reshma John

Ravi said, 'In both the ways of sharing each of us would get the same share, i.e., 5 quarters. Since 4 quarters make one whole, we can also say that each of us would get 1 whole and one quarter. The value of each share would be five divided by four. Is it written as $5 \div 4$?' John said, 'Yes the same as $\frac{5}{4}$'. Reshma added that in $\frac{5}{4}$, the numerator is bigger than the denominator. The fractions, where the numerator is bigger than the denominator are called **improper fractions.**

Figure 3.29. Improper and mixed fractions.

Source: "Textbook6", n.d., pp. 138-139.

Example 1 : Express the following as mixed fractions :

(a) $\frac{17}{4}$ (b) $\frac{11}{3}$ (c) $\frac{27}{5}$ (d) $\frac{7}{3}$

Solution : (a) $\frac{17}{4}$

$$\begin{array}{r|l} 4 & \overline{17}\ \ 4 \\ & -16 \\ \hline & \ \ \ 1 \end{array}$$

i.e. 4 whole and $\frac{1}{4}$ more, or $4\frac{1}{4}$

(b) $\frac{11}{3}$

$$\begin{array}{r|l} 3 & \overline{11}\ \ 3 \\ & -\ 9 \\ \hline & \ \ \ 2 \end{array}$$

i.e. 3 whole and $\frac{2}{3}$ more, or $3\frac{2}{3}$

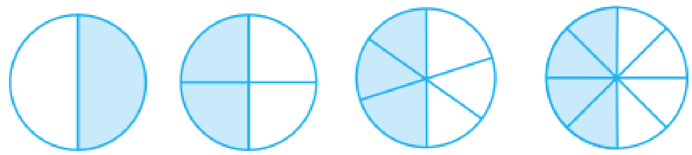
$$\left[\textit{Alternatively}, \frac{11}{3} = \frac{9+2}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3} = 3\frac{2}{3}\right]$$

Figure 3.30. Improper fraction to mixed fraction.

Source: “Textbook6”, n.d., p. 140.

1. Write the fractions. Are all these fractions equivalent?

(a)



2. Write the fractions and pair up the equivalent fractions from each row.

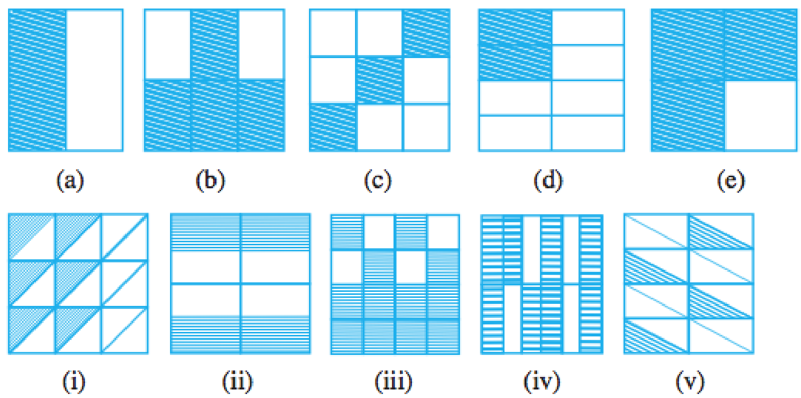


Figure 3.31. Equivalent fractions in grade 6.

Source: “Textbook6”, n.d., p. 146.

The shortest way

The shortest way to find the equivalent fraction in the simplest form is to find the HCF of the numerator and denominator, and then divide both of them by the HCF.

A Game

The equivalent fractions given here are quite interesting. Each one of them uses all the digits from 1 to 9 once!

$$\frac{2}{6} = \frac{3}{9} = \frac{58}{174}$$

$$\frac{2}{4} = \frac{3}{6} = \frac{79}{158}$$

Try to find two more such equivalent fractions.

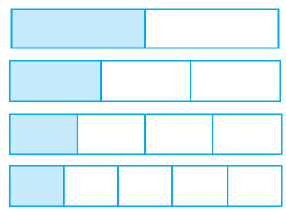
Figure 3.32. HCF.

Source: “Textbook6”, n.d., p. 145.

7.9 Comparing Fractions

Sohni has $3\frac{1}{2}$ rotis in her plate and Rita has $2\frac{3}{4}$ rotis in her plate. Who has more rotis in her plate? Clearly, Sohni has 3 full rotis and more and Rita has less than 3 rotis. So, Sohni has more rotis.

Consider $\frac{1}{2}$ and $\frac{1}{3}$ as shown in Fig. 7.12. The portion of the whole corresponding to $\frac{1}{2}$ is clearly larger than the portion of the same whole corresponding to $\frac{1}{3}$.



$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$

Fig 7.12

So $\frac{1}{2}$ is greater than $\frac{1}{3}$.

But often it is not easy to say which one out of a pair of fractions is larger. For example, which is greater, $\frac{1}{4}$ or $\frac{3}{10}$? For this, we may wish to show the fractions using figures (as in fig. 7.12), but drawing figures may not be easy especially with denominators like 13. We should therefore like to have a systematic procedure to compare fractions. It is particularly easy to compare like fractions.

Try These

1. You get one-fifth of a bottle of juice and your sister gets one-third of a bottle of juice. Who gets more?

Figure 3.33. Comparing fractions.

Source: “Textbook6”, n.d., p. 148.

Let us compare two like fractions: $\frac{3}{8}$ and $\frac{5}{8}$.

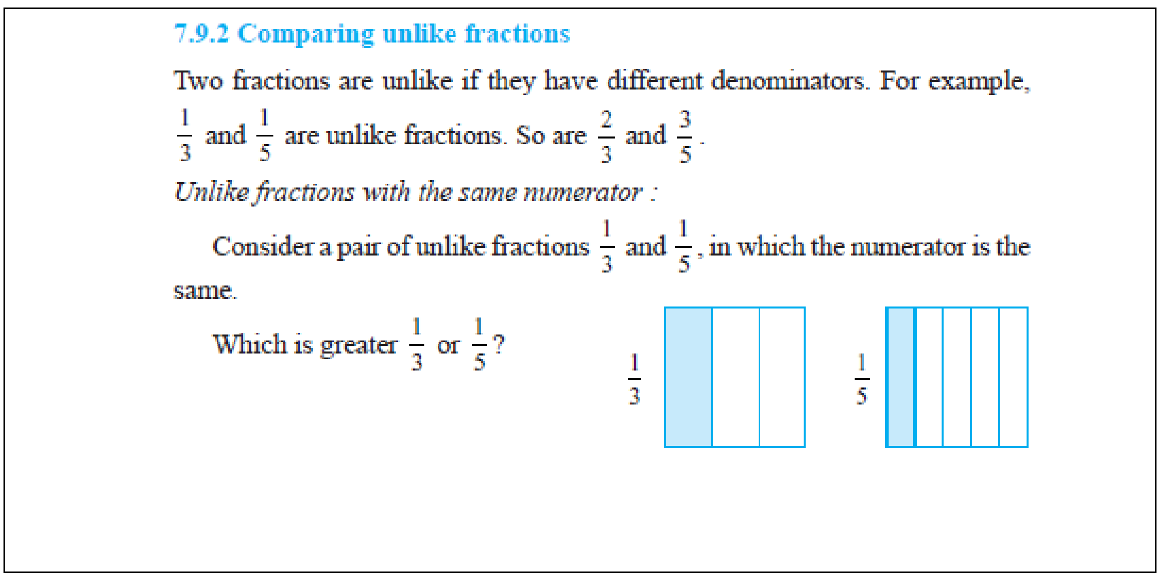
$\frac{3}{8}$ $\frac{5}{8}$

In both the fractions the whole is divided into 8 equal parts. For $\frac{3}{8}$ and $\frac{5}{8}$, we take 3 and 5 parts respectively out of the 8 equal parts. Clearly, out of 8 equal parts, the portion corresponding to 5 parts is larger than the portion corresponding to 3 parts. Hence, $\frac{5}{8} > \frac{3}{8}$. Note the number of the parts taken is given by the numerator. It is, therefore, clear that for two fractions with the same denominator, the fraction with the greater numerator is greater. Between $\frac{4}{5}$ and $\frac{3}{5}$, $\frac{4}{5}$ is greater. Between $\frac{11}{20}$ and $\frac{13}{20}$, $\frac{13}{20}$ is greater and so on.

Figure 3.34. Comparing like fractions.

Source: “Textbook6”, n.d., p. 149.

- Comparison of unlike fractions: Later in this chapter, two unlike fractions are compared. Unlike fractions are fractions with different denominators (Figure 3.35). It is discussed why 1/3 is greater than 1/5. One third is described by dividing the whole into three parts where one of the parts denotes one-third. Similarly dividing the whole into five equal parts and taking one part, one obtains one fifth. The whole is divided into a smaller number of parts in 1/3 than in 1/5; therefore the equal parts we get in 1/3 are larger than the equal parts we get in 1/5, which implies 1/3 > 1/5.



7.9.2 Comparing unlike fractions

Two fractions are unlike if they have different denominators. For example, $\frac{1}{3}$ and $\frac{1}{5}$ are unlike fractions. So are $\frac{2}{3}$ and $\frac{3}{5}$.

Unlike fractions with the same numerator :

Consider a pair of unlike fractions $\frac{1}{3}$ and $\frac{1}{5}$, in which the numerator is the same.

Which is greater $\frac{1}{3}$ or $\frac{1}{5}$?

$\frac{1}{3}$ $\frac{1}{5}$

Figure 3.35. Comparing unlike fractions.

Source: “Textbook6”, n.d., p. 149.

- Adding and subtracting like fractions: Here grid diagrams are used for fraction addition as shown in Figure 3.36. Grid diagrams use the area model of fraction.

7.10.1 Adding or subtracting like fractions

All fractions cannot be added orally. We need to know how they can be added in different situations and learn the procedure for it. We begin by looking at addition of like fractions.

Take a 7 × 4 grid sheet (Fig 7.13). The sheet has seven boxes in each row and four boxes in each column.

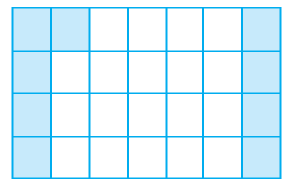


Fig 7.13

How many boxes are there in total?

Colour five of its boxes in green.

What fraction of the whole is the green region?

Now colour another four of its boxes in yellow.

What fraction of the whole is this yellow region?

What fraction of the whole is coloured altogether?

Figure 3.36. Like fraction addition.

Source: “Textbook6”, n.d., p. 155.

- Adding and subtracting unlike fractions: Equivalent fractions are used here to add unlike fractions (Figure 3.37). At first, the unlike fractions are converted into like fractions and then addition of like fractions are applied here.

The focus is on understanding the procedure. There is no pictorial explanation found in the textbook. I found mixed fraction addition and subtraction in this chapter (Figure 3.38). Mixed fraction addition is done by grouping the different wholes together and then operating on the proper fractional parts of the whole.

7.10.2 Adding and subtracting fractions

We have learnt to add and subtract like fractions. It is also not very difficult to add fractions that do not have the same denominator. When we have to add or subtract fractions we first find equivalent fractions with the same denominator and then proceed.

What added to $\frac{1}{5}$ gives $\frac{1}{2}$? This means subtract $\frac{1}{5}$ from $\frac{1}{2}$ to get the required number.

Since $\frac{1}{5}$ and $\frac{1}{2}$ are unlike fractions, in order to subtract them, we first find their equivalent fractions with the same denominator. These are $\frac{2}{10}$ and $\frac{5}{10}$ respectively.

This is because $\frac{1}{2}=\frac{1\times5}{2\times5}=\frac{5}{10}$ and $\frac{1}{5}=\frac{1\times2}{5\times2}=\frac{2}{10}$

Therefore, $\frac{1}{2}-\frac{1}{5}=\frac{5}{10}-\frac{2}{10}=\frac{5-2}{10}=\frac{3}{10}$

Note that 10 is the least common multiple (LCM) of 2 and 5.

Figure 3.37. Fraction addition and subtraction.

Source: "Textbook6", n.d., p. 158.

How do we add or subtract mixed fractions?

Mixed fractions can be written either as a whole part plus a proper fraction or entirely as an improper fraction. One way to add (or subtract) mixed fractions is to do the operation seperately for the whole parts and the other way is to write the mixed fractions as improper fractions and then directly add (or subtract) them.

Example 11 : Add $2\frac{4}{5}$ and $3\frac{5}{6}$

Solution : $2\frac{4}{5}+3\frac{5}{6}=2+\frac{4}{5}+3+\frac{5}{6}=5+\frac{4}{5}+\frac{5}{6}$

Now $\frac{4}{5}+\frac{5}{6}=\frac{4\times6}{5\times6}+\frac{5\times5}{6\times5}$ (Since LCM of 5 and 6 = 30)

$=\frac{24}{30}+\frac{25}{30}=\frac{49}{30}=\frac{30+19}{30}=1+\frac{19}{30}$

Thus, $5+\frac{4}{5}+\frac{5}{6}=5+1+\frac{19}{30}=6+\frac{19}{30}=6\frac{19}{30}$

And, therefore, $2\frac{4}{5}+3\frac{5}{6}=6\frac{19}{30}$

Figure 3.38. Add and subtract mixed fractions.

Source: "Textbook6", n.d., p. 159.

Chapter 8: “Decimals”, p. 164

In this chapter, tenths and hundredths are discussed using blocks and grid diagrams. Decimals are represented on a number line as shown in Figure 3.39.

Representing Decimals on number line

We represented fractions on a number line. Let us now represent decimals too on a number line. Let us represent 0.6 on a number line.

We know that 0.6 is more than zero but less than one. There are 6 tenths in it. Divide the unit length between 0 and 1 into 10 equal parts and take 6 parts as shown below :

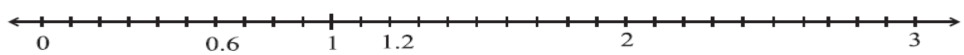


Figure 3.39. Decimals on number line.

Source: “Textbook6”, n.d., p. 166.

The following are discussed in this chapter:

1. Conversion of fractions to decimals and decimals to fractions. Also, two decimals are compared using a grid (Figure 3.40).

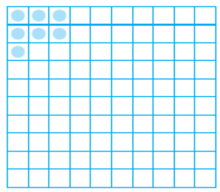
2. Solving tasks about money, length, and weight with decimals. Here addition and subtraction of decimal numbers are discussed (Figure 3.41).

8.4 Comparing Decimals

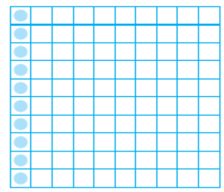
Can you tell which is greater, 0.07 or 0.1?

Take two pieces of square papers of the same size. Divide them into 100 equal parts. For 0.07 we have to shade 7 parts out of 100.

Now, $0.1 = \frac{1}{10} = \frac{10}{100}$, so, for 0.1, shade 10 parts out 100.



$0.07 = \frac{7}{100}$



$0.1 = \frac{1}{10} = \frac{10}{100}$

This means 0.1>0.07

Let us now compare the numbers 32.55 and 32.5. In this case , we first compare the whole part. We see that the whole part for both the nunbers is 32 and, hence, equal.

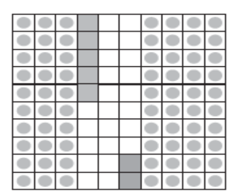
We, however, know that the two numbers are not equal. So, we now compare the tenth part. We find that for 32.55 and 32.5, the tenth part is also equal, then we compare the hundredth part.

Figure 3.40. Comparing decimals.

Source: “Textbook6”, n.d., p. 174.

Mark 0.35 in this square by shading 3 tenths and colouring 5 hundredths.

Mark 0.42 in this square by shading 4 tenths and colouring 2 hundredths.



Now count the total number of tenths in the square and the total number of hundredths in the square.

	Ones	Tenths	Hundredths
	0	3	5
+	0	4	2
	0	7	7

Therefore, 0.35 + 0.42 = 0.77

Figure 3.41. Adding decimals.

Source: “Textbook6”, n.d., p. 178.

COMPARISON IN GRADE 6

NCERT's grade 6 syllabus on fraction is a blend of CCSSM's grades 3, 4, 5 and 6 standards on fraction. Number lines for fractions are introduced in this grade of NCERT's syllabus, which is seen in grade 3 of CCSSM. Improper fractions are introduced here with focus on the part-whole definition of fractions. Here more emphasis is given on understanding improper fractions as a whole and some fractional parts of the whole. CCSSM defines a fraction a/b as a parts of size 1/b, regardless of whether a is greater than b or b is greater than a. In this grade of NCERT's syllabus, mirror halves are revisited in reflection symmetry. The concepts of iteration and discrete units are seen in this grade of NCERT's syllabus. I found tasks in the textbooks on equivalent fractions that do not focus on making the wholes equal. On the other hand, CCSSM strongly emphasizes the importance of the same whole in comparing fractions and in generating equivalent fractions. NCERT introduces fraction addition and subtraction for both like and unlike fractions in grade 6 while CCSSM introduces those in two different grades, one in grade 4 and another in grade 5. I found the use of the highest common factor (HCF) in calculating equivalent fractions in NCERT's textbook of grade 6, which is not explicitly seen in fraction addition and subtraction with unlike denominators in the standards of CCSSM, where fractions with unlike denominators are replaced by equivalent fractions.

Comparison grid for grade 6

The following table 3.2 gives a topic-by-topic comparison for grade 6. I have organized table 3.2 with the first column being the NCERT's Grade 6 syllabus on fractions. The second column presents the CCSSM standards that are closely related to the NCERT's syllabus on fractions. The comparison chart shows when different concepts are introduced in the NCERT's syllabus for grade 6 in comparison to the CCSSM standards.

Table 3.2. Comparison of grade 6.

NCERT'S GRADE 6 SYLLABUS ON FRACTIONS	CCSSM ON FRACTIONS
1. Fraction as a part of whole	1. Grade 3: 3.NF.A.1 (Part 1)
2. Representations of fractions on number line (Length model of fraction)	2. Grade 3: 3.NF.A.2
3. Fraction as division	3. Grade 5: 5.NF.B.3
4. Proper, improper and mixed fractions	4. Grades 3, 4: 3.NF.A.1, 4.NF.B.3c
5. Equivalent fractions	5. Grade 3, 4: 3.NF.A.3a (Part 2), 4.NF.A.1
6. Comparison of fractions	6. Grades 3, 4: 3.NF.A.3a, 3d; 4.NF.A.2
7. Addition and subtraction of fractions	7. Grades 4, 5: 4.NF.B.3a, 4.NF.B.3b, 4.NF.B.3c, 4.NF.B.3d; 5.NF.A.1, 5.NF.A.2
8. Simplest form of a fraction	8. Not in CCSSM
9. Review of the idea of a decimal fraction	9. Grade 4: 4.NF.C.5, 4.NF.C.6
10. Place value in the context of decimal fraction	10. Grade 5: 5.NBT.A.1, 5.NBT.A.2, 5.NBT.A.3, 5.NBT.A.4
11. Tenths and hundredths	11. Grade 4: 4.NF.C.7
12. Inter conversion of fractions and decimal fractions	12. Grade 4: 4.NF.C.5, 4.NF.C.6
13. Addition and subtraction of decimals	13. Grade 5, 6: 5.NBT.B.7, 6.NS.B.3
14. Word problems involving addition and subtraction of decimals (two operations together on money, mass, length, and temperature).	14. Grade 4: 4.MD.A.2
15. Symmetry	15. Grade 4: 4.G.A.3

Figure 3.42 presents a tree diagram depicting how NCERT's grade 6 syllabus on fractions stands in comparison to CCSSM on fractions.

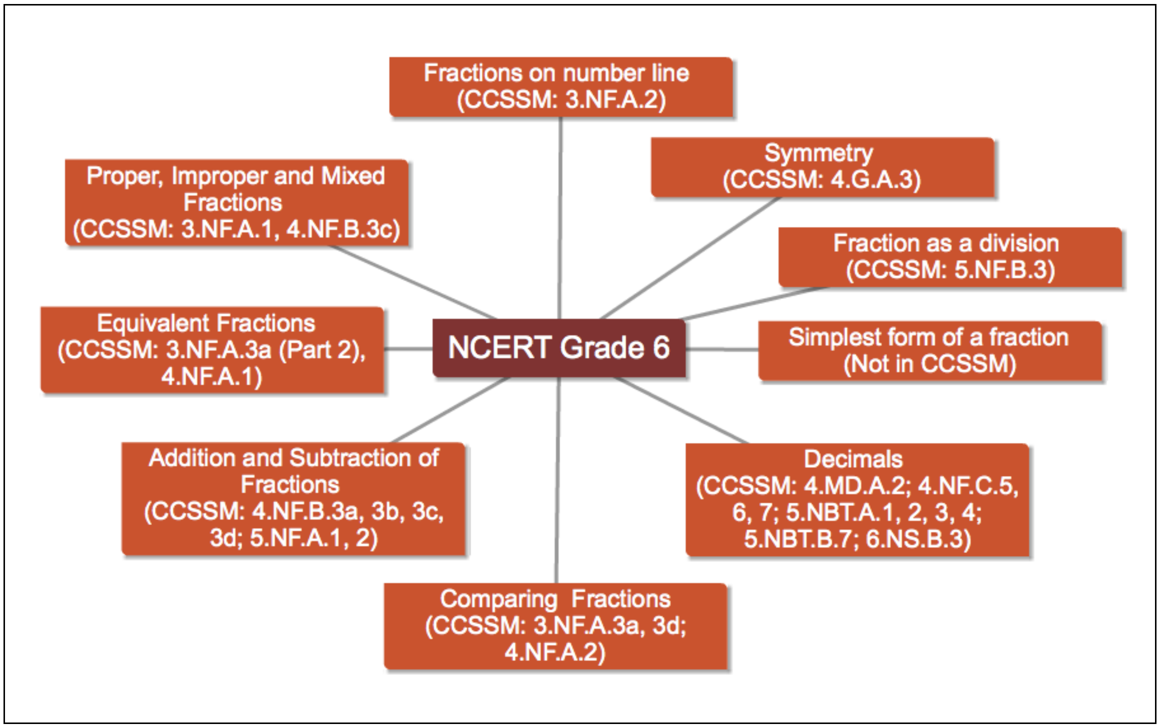


Figure 3.42. Tree diagram comparing grade 6.

COMMON CORE STATE STANDARDS FOR MATHEMATICS GRADE 7

7. RP Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

7. NS The Number System

- Apply and extend previous understandings of operations with fractions.
    - 7.NS.A.1 Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

- 7.NS.A.2 Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
    - 7.NS.A.2d Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
- 7.NS.A.3 Solve real-world and mathematical problems involving the four operations with rational numbers.

NCERT'S SYLLABUS FOR MATHEMATICS GRADE 7

Number System

Fractions and rational numbers:

- Multiplication of fractions
- Fraction as an operator
- Reciprocal of a fraction
- Division of fractions
- Word problems involving mixed fractions
- Introduction to rational numbers (with representation on number line)
- Operations on rational numbers (all operations)
- Representation of rational number as a decimal
- Word problems on rational numbers (all operations)
- Multiplication and division of decimal fractions
- Conversion of units (length & mass)
- Word problems (including all operations)

Ratio and Proportion

- Ratio and proportion (revision)
- Unitary method continued, consolidation, general expression
- Percentage- an introduction
- Understanding percentage as a fraction with denominator 100
- Converting fractions and decimals into percentage and vice-versa
- Application to profit and loss (single transaction only)
- Application to simple interest (time period in complete years)

Geometry

(iii) Symmetry

- Recalling reflection symmetry
- Idea of rotational symmetry, observations of rotational symmetry of 2-D objects. (90 degrees, 120 degrees, 180 degrees).
- Operation of rotation through 90 degrees and 180 degrees of simple figures.
- Examples of figures with both rotation and reflection symmetry (both operations).
- Examples of figures that have reflection and rotation symmetry and vice-versa.

NCERT'S TEXT BOOK FOR GRADE 7

Chapter 2: "Fractions and Decimals", p. 29

In this chapter, fraction is described as an operator "of". Figure 3.43 shows how it is explained in the textbook. Here "of" in English is translated to "multiply" in math.

Fraction as an operator 'of'

Observe these figures (Fig 2.6)

The two squares are exactly similar.

Each shaded portion represents $\frac{1}{2}$ of 1.

So, both the shaded portions together will represent $\frac{1}{2}$ of 2.

Combine the 2 shaded $\frac{1}{2}$ parts. It represents 1.

So, we say $\frac{1}{2}$ of 2 is 1. We can also get it as $\frac{1}{2} \times 2 = 1$.

Thus, $\frac{1}{2}$ of $2 = \frac{1}{2} \times 2 = 1$

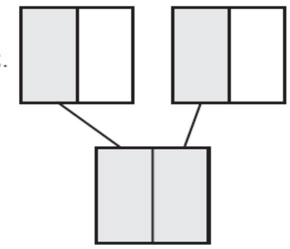


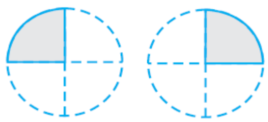
Fig 2.6

Figure 3.43. Fraction as an operator of.

Source: "Textbook7", n.d., p. 34.

In this chapter, fraction multiplication is discussed. Fraction multiplication is divided into three categories: multiplication of a fraction by a whole number (Figure 3.44), multiplication of a fraction by a fraction (Figure 3.45), and multiplication of a mixed number by a whole number or fraction. The first category, multiplication of a fraction by a whole number is discussed as repeated addition. I infer that the use of repeated addition suggests iteration.

Multiplication of a Fraction by a Whole Number



Observe the pictures at the left (Fig 2.1). Each shaded part is $\frac{1}{4}$ part of a circle. How much will the two shaded parts represent together? They will represent $\frac{1}{4}+\frac{1}{4}=2\times\frac{1}{4}$.

Fig 2.1

Combining the two shaded parts, we get Fig 2.2 . What part of a circle does the shaded part in Fig 2.2 represent? It represents $\frac{2}{4}$ part of a circle .

Fig 2.2

The shaded portions in Fig 2.1 taken together are the same as the shaded portion in Fig 2.2, i.e., we get Fig 2.3.

Fig 2.3

or $\quad 2\times\frac{1}{4}=\frac{2}{4}$.

Let us now find $3\times\frac{1}{2}$.

We have $\quad 3\times\frac{1}{2} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

We also have $\quad \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{1+1+1}{2} = \frac{3\times 1}{2} = \frac{3}{2}$

So $\quad 3\times\frac{1}{2} = \frac{3\times 1}{2} = \frac{3}{2}$

Similarly $\quad \frac{2}{3}\times 5 = \frac{2\times 5}{3} = ?$

Can you tell $\quad 3\times\frac{2}{7} = ? \qquad 4\times\frac{3}{5} = ?$

Figure 3.44. Multiply a fraction by a whole number.

Source: “Textbook7”, n.d., pp. 32-33.

The second category is multiplication of a fraction by a fraction. Here unit fractions are considered first (Figure 3.45). The third category is multiplication of a mixed number by a fraction (Figure 3.46). Here a mixed number is converted into fraction and then fraction multiplication is done.

Multiplication of a Fraction by a Fraction

To do this we first learn to find the products like $\frac{1}{2} \times \frac{1}{3}$.

Fig 2.8

(a) How do we find $\frac{1}{3}$ of a whole? We divide the whole in three equal parts. Each of the three parts represents $\frac{1}{3}$ of the whole. Take one part of these three parts, and shade it as shown in Fig 2.8.

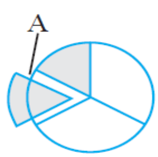


Fig 2.9

(b) How will you find $\frac{1}{2}$ of this shaded part? Divide this one-third ($\frac{1}{3}$) shaded part into two equal parts. Each of these two parts represents $\frac{1}{2}$ of $\frac{1}{3}$ i.e., $\frac{1}{2} \times \frac{1}{3}$ (Fig 2.9).

Take out 1 part of these two and name it 'A'. 'A' represents $\frac{1}{2} \times \frac{1}{3}$.

(c) What fraction is 'A' of the whole? For this, divide each of the remaining $\frac{1}{3}$ parts also in two equal parts. How many such equal parts do you have now?
There are six such equal parts. 'A' is one of these parts.

So, 'A' is $\frac{1}{6}$ of the whole. Thus, $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

How did we decide that 'A' was $\frac{1}{6}$ of the whole? The whole was divided in $6 = 2 \times 3$ parts and $1 = 1 \times 1$ part was taken out of it.

Thus, $$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = \frac{1 \times 1}{2 \times 3}$$

or $$\frac{1}{2} \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3}$$

Figure 3.45. Multiplication of a fraction by a fraction.

Source: "Textbook7", n.d., pp. 37-38.

Example 6 Sushant reads $\frac{1}{3}$ part of a book in 1 hour. How much part of the book will he read in $2\frac{1}{5}$ hours?

Solution The part of the book read by Sushant in 1 hour $= \frac{1}{3}$.

So, the part of the book read by him in $2\frac{1}{5}$ hours $= 2\frac{1}{5} \times \frac{1}{3}$

$$= \frac{11}{5} \times \frac{1}{3} = \frac{11 \times 1}{5 \times 3} = \frac{11}{15}$$

Let us now find $\frac{1}{2} \times \frac{5}{3}$. We know that $\frac{5}{3} = \frac{1}{3} \times 5$.

So, $\frac{1}{2} \times \frac{5}{3} = \frac{1}{2} \times \frac{1}{3} \times 5 = \frac{1}{6} \times 5 = \frac{5}{6}$

Also, $\frac{5}{6} = \frac{1 \times 5}{2 \times 3}$. Thus, $\frac{1}{2} \times \frac{5}{3} = \frac{1 \times 5}{2 \times 3} = \frac{5}{6}$.

This is also shown by the figures drawn below. Each of these five equal shapes (Fig 2.10) are parts of five similar circles. Take one such shape. To obtain this shape we first divide a circle in three equal parts. Further divide each of these three parts in two equal parts. One part out of it is the shape we considered. What will it represent?

It will represent $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. The total of such parts would be $5 \times \frac{1}{6} = \frac{5}{6}$.

Fig 2.10

Try These

Find: $\frac{1}{3} \times \frac{4}{5}$; $\frac{2}{3} \times \frac{1}{5}$

Similarly $\frac{3}{5} \times \frac{1}{7} = \frac{3 \times 1}{5 \times 7} = \frac{3}{35}$.

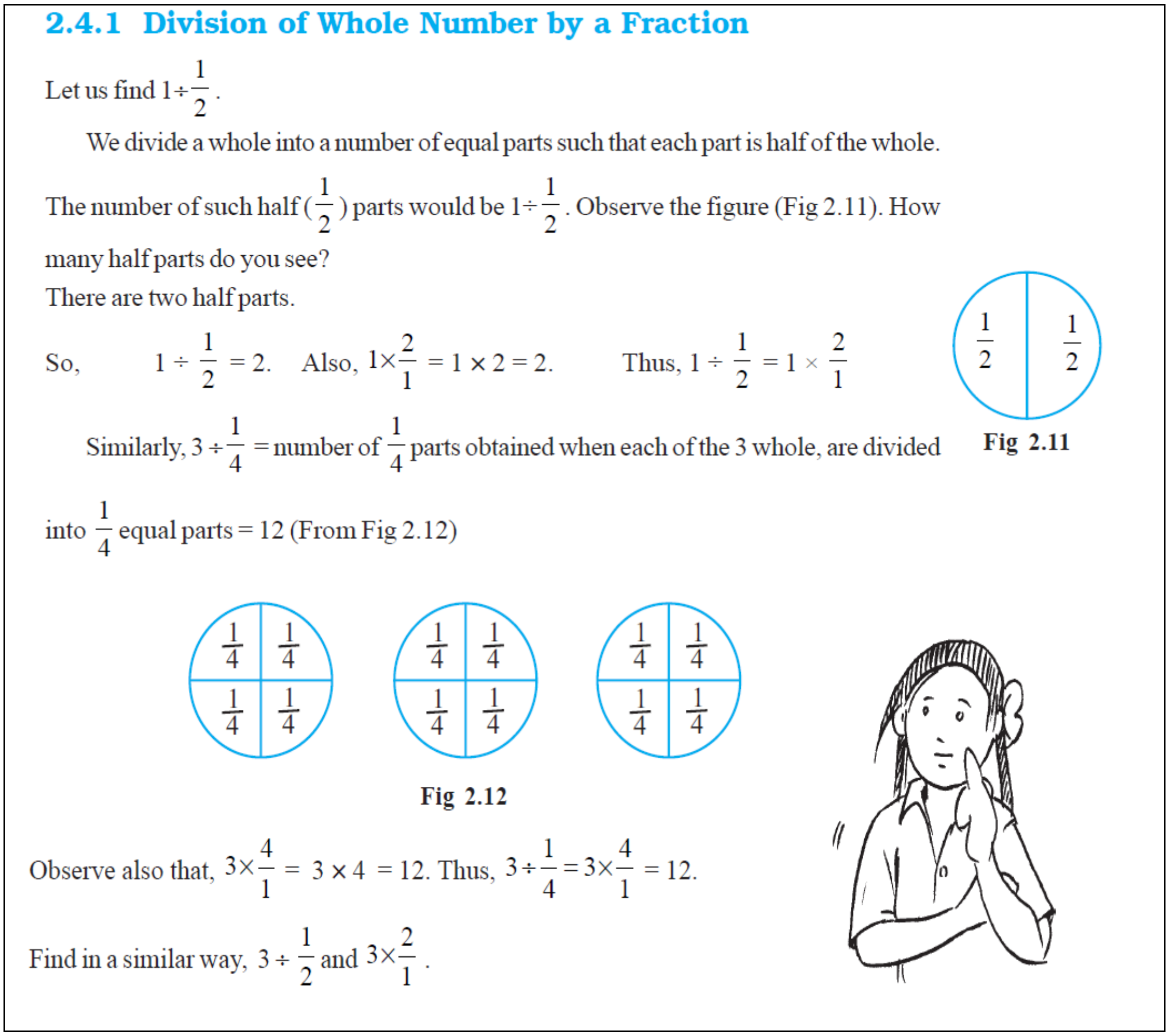
We can thus find $\frac{2}{3} \times \frac{7}{5}$ as $\frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}$.

So, we find that we multiply two fractions as $\frac{\text{Product of Numerators}}{\text{Product of Denominators}}$.

Figure 3.46. Fraction multiplication.

Source: “Textbook7”, n.d., pp. 39-40.

In this chapter, fraction division is discussed. There are three categories of fraction division. They are: Division of a whole number by a fraction (Figure 3.47). The second category of fraction division is division of a fraction by a whole number (Figure 3.48) and the third category is division of a fraction by a fraction (Figure 3.48).



2.4.1 Division of Whole Number by a Fraction

Let us find $1 \div \frac{1}{2}$.

We divide a whole into a number of equal parts such that each part is half of the whole. The number of such half ($\frac{1}{2}$) parts would be $1 \div \frac{1}{2}$. Observe the figure (Fig 2.11). How many half parts do you see?

There are two half parts.

So, $1 \div \frac{1}{2} = 2$. Also, $1 \times \frac{2}{1} = 1 \times 2 = 2$. Thus, $1 \div \frac{1}{2} = 1 \times \frac{2}{1}$

Fig 2.11

Similarly, $3 \div \frac{1}{4}$ = number of $\frac{1}{4}$ parts obtained when each of the 3 whole, are divided into $\frac{1}{4}$ equal parts = 12 (From Fig 2.12)

Fig 2.12

Observe also that, $3 \times \frac{4}{1} = 3 \times 4 = 12$. Thus, $3 \div \frac{1}{4} = 3 \times \frac{4}{1} = 12$.

Find in a similar way, $3 \div \frac{1}{2}$ and $3 \times \frac{2}{1}$.

Figure 3.47. Division of whole number by a fraction.

Source: “Textbook7”, n.d., p. 43.

2.4.2 Division of a Fraction by a Whole Number

- What will be $\frac{3}{4} \div 3$?

  Based on our earlier observations we have: $\frac{3}{4} \div 3 = \frac{3}{4} \div \frac{3}{1} = \frac{3}{4} \times \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$

  So, $\frac{2}{3} \div 7 = \frac{2}{3} \times \frac{1}{7} = \quad ?$ What is $\frac{5}{7} \div 6$, $\frac{2}{7} \div 8$?

- While dividing mixed fractions by whole numbers, convert the mixed fractions into improper fractions. That is,

  $2\frac{2}{3} \div 5 = \frac{8}{3} \div 5 =$ ------ ; $4\frac{2}{5} \div 3 =$ ------ = ------; $2\frac{3}{5} \div 2 =$ ------ = ------

2.4.3 Division of a Fraction by Another Fraction

We can now find $\frac{1}{3} \div \frac{5}{6}$.

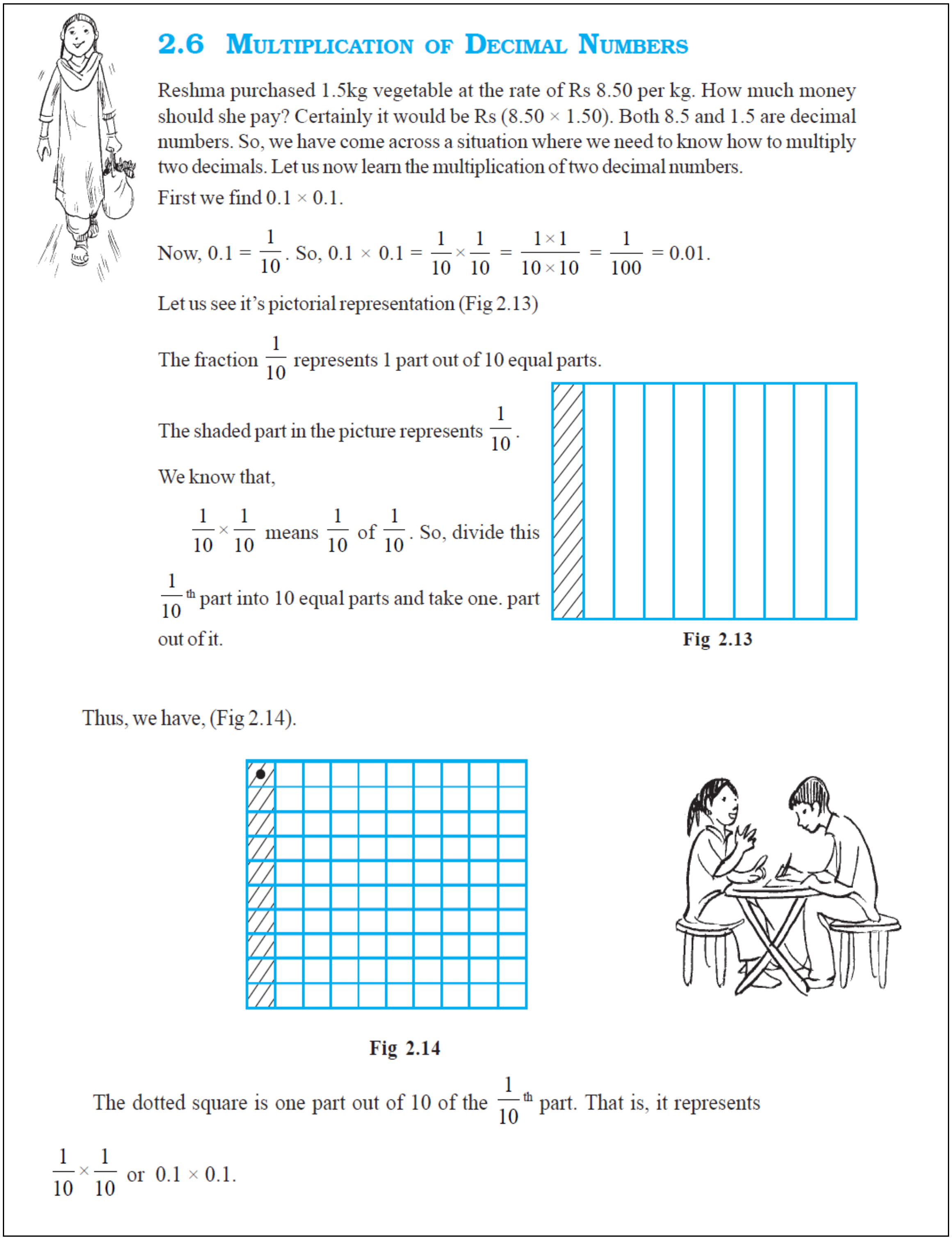
$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \times$ reciprocal of $\frac{5}{6} = \frac{1}{3} \times \frac{5}{6} = \frac{2}{5}$.

Similarly, $\frac{8}{5} \div \frac{2}{3} = \frac{8}{5} \times$ reciprocal of $\frac{2}{3} = ?$ and, $\frac{1}{2} \div \frac{3}{4} = ?$

Figure 3.48. Fraction division.

Source: "Textbook7", n.d., p. 45.

Multiplication of decimal numbers is discussed in this chapter. Here decimal numbers are converted to fractions and then fraction multiplication is done. Money related word problems are solved in this chapter using decimal multiplication (Figure 3.49). Division of decimals is introduced in this chapter. They are divided into three types: Division by 10, 100 and 1000; division of a decimal by a whole number (Figure 3.50) and division of a decimal by a decimal.



2.6 Multiplication of Decimal Numbers

Reshma purchased 1.5kg vegetable at the rate of Rs 8.50 per kg. How much money should she pay? Certainly it would be Rs (8.50 × 1.50). Both 8.5 and 1.5 are decimal numbers. So, we have come across a situation where we need to know how to multiply two decimals. Let us now learn the multiplication of two decimal numbers.

First we find 0.1 × 0.1.

Now, $0.1 = \frac{1}{10}$. So, $0.1 \times 0.1 = \frac{1}{10} \times \frac{1}{10} = \frac{1 \times 1}{10 \times 10} = \frac{1}{100} = 0.01$.

Let us see it's pictorial representation (Fig 2.13)

The fraction $\frac{1}{10}$ represents 1 part out of 10 equal parts.

The shaded part in the picture represents $\frac{1}{10}$.

We know that,

$\frac{1}{10} \times \frac{1}{10}$ means $\frac{1}{10}$ of $\frac{1}{10}$. So, divide this $\frac{1}{10}$th part into 10 equal parts and take one. part out of it.

Fig 2.13

Thus, we have, (Fig 2.14).

Fig 2.14

The dotted square is one part out of 10 of the $\frac{1}{10}$th part. That is, it represents $\frac{1}{10} \times \frac{1}{10}$ or 0.1×0.1.

Figure 3.49. Multiplication of decimal numbers.

Source: "Textbook7", n.d., pp. 48-49.

2.7.2 Division of a Decimal Number by a Whole Number

TRY THESE

(i) $35.7 \div 3 = ?$;

(ii) $25.5 \div 3 = ?$

Let us find $\frac{6.4}{2}$. Remember we also write it as $6.4 \div 2$.

So, $6.4 \div 2 = \frac{64}{10} \div 2 = \frac{64}{10} \times \frac{1}{2}$ as learnt in fractions.

$$= \frac{64 \times 1}{10 \times 2} = \frac{1 \times 64}{10 \times 2} = \frac{1}{10} \times \frac{64}{2} = \frac{1}{10} \times 32 = \frac{32}{10} = 3.2$$

Or, let us first divide 64 by 2. We get 32. There is one digit to the right of the decimal point in 6.4. Place the decimal in 32 such that there would be one digit to its right. We get 3.2 again.

TRY THESE

(i) $43.15 \div 5 = ?$;

(ii) $82.44 \div 6 = ?$

To find $19.5 \div 5$, first find $195 \div 5$. We get 39. There is one digit to the right of the decimal point in 19.5. Place the decimal point in 39 such that there would be one digit to its right. You will get 3.9.

Now, $12.96 \div 4 = \frac{1296}{100} \div 4 = \frac{1296}{100} \times \frac{1}{4} = \frac{1}{100} \times \frac{1296}{4} = \frac{1}{100} \times 324 = 3.24$

Or, divide 1296 by 4. You get 324. There are two digits to the right of the decimal in 12.96. Making similar placement of the decimal in 324, you will get 3.24.

Figure 3.50. Division of a decimal by a whole number.

Source: "Textbook7", n.d., p. 53.

There is a section on the value of a product in this chapter of NCERT's grade 7 textbook. Here it is discussed what happens to the value of the product when two fractions are multiplied (Figure 3.51). It is noted that the product of two whole numbers is always greater than each of the two whole numbers; here it is extended to understand what happens to the product of two fractions. Is it bigger or smaller than each of the two fractions? It is emphasized through multiple tasks that product of two fractions are less than both the fractions.

Value of the Products

TRY THESE

Find: $\frac{8}{3} \times \frac{4}{7}$; $\frac{3}{4} \times \frac{2}{3}$.

You have seen that the product of two whole numbers is bigger than each of the two whole numbers. For example, $3 \times 4 = 12$ and $12 > 4, 12 > 3$. What happens to the value of the product when we multiply two fractions?

Let us first consider the product of two proper fractions.

We have,

$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$	$\frac{8}{15} < \frac{2}{3}, \frac{8}{15} < \frac{4}{5}$	Product is less than both the fractions
$\frac{1}{5} \times \frac{2}{7}$ = ---------	--------,--------	--------------------------------------
$\frac{3}{5} \times \frac{\square}{8} = \frac{21}{40}$	--------,--------	--------------------------------------
$\frac{2}{\square} \times \frac{4}{9} = \frac{8}{45}$	--------,--------	--------------------------------------

Figure 3.51. Value of the products.

Source: “Textbook7”, n.d., p. 40.

COMPARISON IN GRADE 7

NCERT’s grade 7 syllabus on fraction is a blend of CCSSM’s grades 4, 5, 6, 7 and 8 standards on fraction. In this syllabus, fraction is conceived as an operator that is seen formally in CCSSM’s grade 4 standards. Three types of fraction multiplication are discussed in NCERT’s grade 7. First is the multiplication of fraction by a whole number, second is the multiplication of a fraction by a fraction and the third is the multiplication of a mixed number by a whole number or a fraction. Multiplication of fraction or a whole number by a fraction is seen in grade 5 of CCSSM, though their approach focuses more on understanding the visual model and creating story related tasks for fraction multiplication.

CCSSM extends the previous understandings of whole number multiplication to multiply a fraction by a whole number, which involves thinking of fractions as operators as we go from 4*3 as 4 groups of 3 to 4*1/3 as 4 groups of 1/3. NCERT uses fraction as operators to understand fraction multiplication. Another part of this CCSSM's standards focuses on finding the area of a rectangle with fractional side lengths, which is not seen in NCERT's syllabus.

Division of fractions is discussed in this grade of NCERT's syllabus. Divisions are explained by converting it to multiplication and changing the divisor to a unit fraction i.e. as (a/b) ÷ c means (a/b) X (1/c). While fraction division (division of fraction by a whole number) is introduced in CCSSM for grade 5 but in CCSSM the relationship between multiplication and division is explained as (a/b) ÷ c = d because d X c = a/b. Division of a fraction by a fraction discussed in NCERT's grade 7 is introduced in CCSSM's grade 6. Multiplication and division of decimal numbers by powers of 10 is discussed in NCERT's grade 7 which are seen in CCSSM's grade 5. CCSSM extends the previous understanding of division to divide unit fractions by whole numbers and whole numbers by unit fractions. My study of the NCERT's textbook examples shows no such connection between whole number division and fraction division. Rational numbers are introduced in this grade in NCERT's syllabus that is seen in CCSSM for grade 6. Operations on rational numbers are introduced in grade 7 of NCERT's syllabus, which are seen in CCSSM's grade 7. Rotational symmetry discussed in this grade of NCERT's syllabus is seen in CCSSM's grade 8 standards 8.G.A.1 and 8.G.A.3.

Comparison grid for grade 7

The following table 3.3 gives topic-by-topic comparison in grade 7 of NCERT's syllabus. I have organized table 3.3 with the first column being the NCERT's Grade 7 syllabus on fractions. The second columns representing which of the CCSSM standards are closely related to the NCERT's syllabus on fractions.

Table 3.3. Comparison of grade 7.

NCERT'S SYLLABUS FOR FRACTIONS	CCSSM ON FRACTIONS
Grade 7	
1. Fraction as an operator "of"	1. Grade 4: 4.NF.B.4a
2. Multiplication of fraction by a whole number	2. Grade 4: 4.NF.B.4b
3. Multiplication of fraction by a fraction	3. Grade 5: 5.NF.B.4 (more elaborated)
4. Multiplication of mixed numbers by a whole number or a fraction	4. Grade 5: 5.NF.B.6
5. Division of a fraction by a whole number	5. Grade 5: 5.NF.B.7a, 5.NF.B.7c
6. Division of whole number by a fraction	6. Grade 5: 5.NF.B.7b, 5.NF.B.7c
7. Division of fraction by a fraction	7. Grade 6: 6.NS.A.1
8. Multiplication of decimal numbers by 10, 100 and 1000	8. Grade 5: 5.NBT.A.2
9. Division by 10, 100 and 1000	9. Grade 5: 5.NBT.A.2
10. Division of a decimal by a whole number	10. Grade 6: 6.NS.B.3
11. Division of a decimal by a decimal	11. Grade 6: 6.NS.B.3
12. Rational numbers on number line	12. Grade 6: 6.NS.C.6
13. Operations on rational numbers (all operations)	13. Grade 7: 7.NS.A.1, 7.NS.A.2
14. Representation of rational number as a decimal	14. Grade 7: 7.NS.A.2d
15. Word problems on rational numbers (all operations)	15. Grade 7: 7.NS.A.3
16. Rotational symmetry	16. Grade 8: 8.G.A.1, 8.G.A.3
17. Value of the products	17. Grade 5: 5.NF.B.5b

Figure 3.52 presents a tree diagram depicting how NCERT's grade 7 syllabus on fractions stands in comparison to CCSSM's standards on fractions.

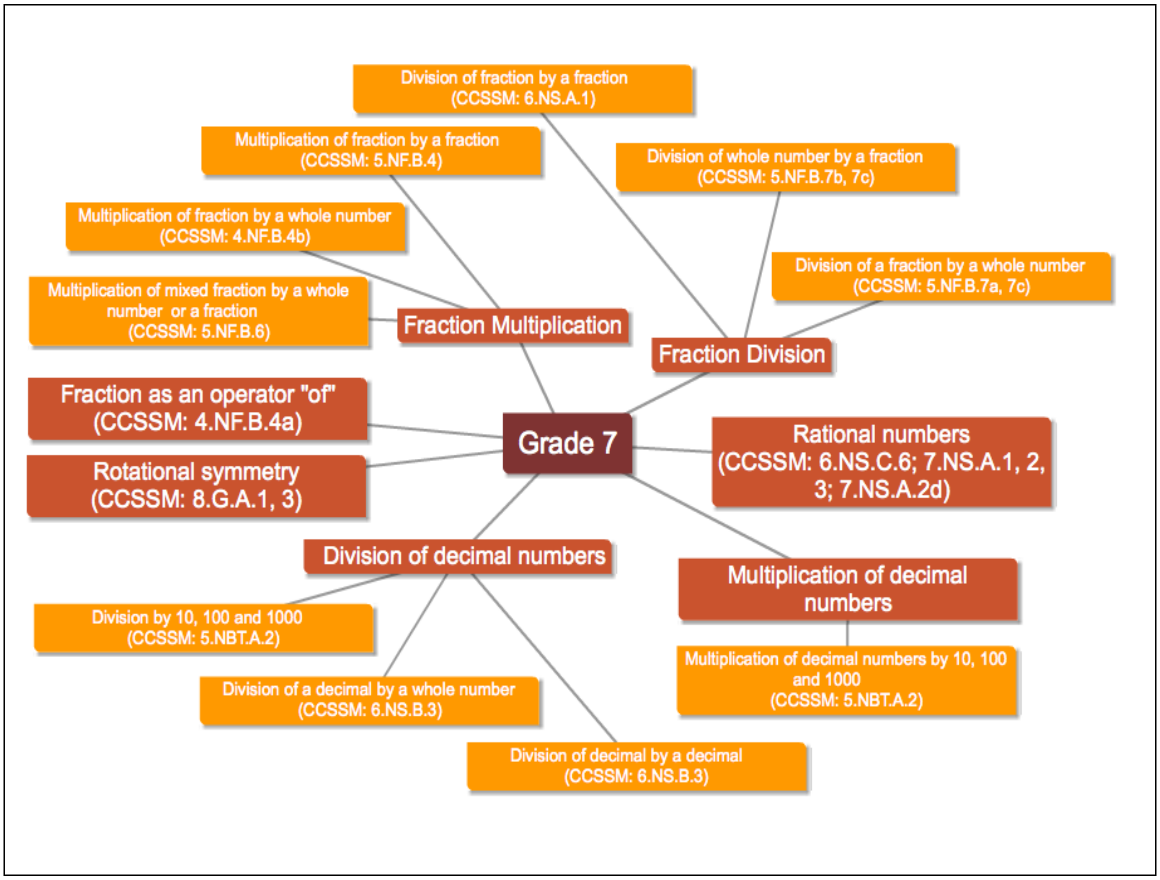


Figure 3.52. Comparison of grade 7.

Comparison matrix

I present the comparison between the NCERT's syllabi between grades 1 through 7 with CCSSM's standards in a matrix form shown in table 3.4. The entries in each cell are of the form (the NCERT's syllabus code, corresponding CCSSM standards). The NCERT's syllabus code is used from table 3.1, 3.2 and 3.3. For example, in NCERT's grade 1 syllabus, no. 1. "Non-uniform units for length measurement" is denoted by 1.1; in grade 5, no. 7. "Fraction to decimal & vice versa" is denoted by 5.7. One thing we see from the upper diagonal nature of the comparison matrix on the next page is that NCERT standards are generally later than the corresponding CCSSM ones.

Table 3.4. Comparison matrix between NCERT's syllabi and CCSSM standards.

NCERT / CCSSM	NCERT Grade 1	NCERT Grade 2	NCERT Grade 3	NCERT Grade 4	NCERT Grade 5	NCERT Grade 6	NCERT Grade 7	Not used in NCERT
CCSSM Grade 1	(1.1, 1.MD)	(2.1, 1.MD)		(4.1, 1.G.A.3) (4.5, 1.G.A.3)				
CCSSM Grade 2			(3.1, 2.MD.A.1)	(4.1, 2.G.A.3) (4.5, 2.G.A.3)				
CCSSM Grade 3				(4.2, 3.NF.A.1 (Part 1)) (4.3, 3.NF.A.1 (Part 1)) (4.6, 3.NF.A.3a (Part 1)) (4.7, 3.NF.A.3b, 3c)	(5.2, 3.NF.A.3b) (5.4, 3.NF.A.3d) (5.10, 3.G.A.2)	(6.1, 3.NF.A.1 (Part 1)) (6.2, 3.NF.A.2) (6.4, 3.NF.A.1) (6.5, 3.NF.A.3a (Part 2)) (6.6, 3.NF.A.3a, 3d)		(-, 3.NF.A.1 (Part 2))
CCSSM Grade 4			(3.2, 4.G.A.3)	(4.8, 4.NF.B.4c) (4.9, 4.G.A.3)	(5.1,4.NF.B.4c) (5.3, 4.NF.B.3a) (5.5, 4.NF.A.1) (5.6, 4.NF.A..2) (5.7, 4.MD.A..2) (5.8, 4.NF.C.5, C.6)	(6.4, 4.NF.B.3c) (6.5, 4.NF.A.1) (6.6, 4.NF.A.2) (6.7, 4.NF.B. 3a, 3b, 3c, 3d) (6.9, 4.NF.C.5, C.6) (6.11, 4.NF.C.7) (6.12, 4.NF.C.5, C.6) (6.14, 4.MD.A.2) (6.15, 4.G.A.3)	(7.1, 4.NF.B.4a) (7.2, 4.NF.B.4b)	
CCSSM Grade 5						(6.3, 5.NF.B.3) (6.4, 5.NF.B.3) (6.7, 5.NF.A.1, A.2) (6.10, 5.NBT.A.1, A.2, A.3, A.4) (6.13, 5.NBT.B.7)	(7.3, 5.NF.B.4) (7.4, 5.NF.B.6) (7.5, 5.NF.B.7a, 7c) (7.6, 5.NF.B.7b, 7c) (7.8, 5.NBT.A.2) (7.9, 5.NBT.A.2) (7.17, 5.NF.B.5b)	(-, 5.MD.B.2) (-, 5.NF.B.4a) (-, 5.NF.B.4b) (-, 5.NF.B.5a)
CCSSM Grade 6						(6.13, 6.NS.B.3)	(7.7, 6.NS.A.1) (7.10, 6.NS.B.3) (7.11, 6.NS.B.3) (7.12, 6.NS.C.6)	
CCSSM Grade 7							(7.13, 7.NS.A.1) (7.13, 7.NS.A.2) (7.14, 7.NS.A.2d) (7.15, 7.NS.A.3)	
CCSSM Grade 8							(7.16, 8.G.A.1) (7.16, 8.G.A.3)	
Not used in CCSSM				(4.4, -)	(5.9, -)	(6.8, -)		

Summary of the comparisons between NCERT's syllabi and CCSSM's standards

The following table 3.5 presents a summary of the comparisons between the NCERT's syllabi from grades 1 through 7 and the CCSSM standards on fractions.

Table 3.5: Comparison between NCERT's syllabi and CCSSM standards.

NCERT'S SYLLABI	CCSSM STANDARDS
GRADE 1 1. Length measurement • Non-uniform length units • Measure lengths of objects using hand spans and other objects • Arrange objects by lengths	**GRADE 1** 1. Length measurement • Non-standard length units • Measuring length indirectly and iterating non-standard length units • Comparing the lengths of two objects by using a third object • Arrange three objects with respect to their lengths 2. Geometry • Partitioning circles and rectangles into two or four equal shares • Describing those shares as half of, fourth of, quarter of • Representing the whole as two of the shares, four of the shares
GRADE 2 1. Length measurement • Uniform (non-standard) units.	**GRADE 2** 1. Length measurement • Standard length units • Represent whole numbers on the number line 2. Geometry • Partitioning circles and rectangles into two, three or four equal shares • Describing those shares as half of, third of, fourth of • Representing whole as two halves, three thirds and four fourths.
GRADE 3 1. Length measurement • Standard length units 2. Symmetrical shapes and patterns • Mirror halves, paper folding and paper cutting • Halves as mirror images	**GRADE 3** 1. Formal definition of fraction • 1/b as part out of whole • a/b as a parts of size 1/b • Fractions on number line 2. Equivalent fractions 3. Compare fractions (same numerators/ same denominators) 4. Understand equivalent fractions: 1/2= 2/4, 4/6= 2/3, 3= 3/1, 6/1=6, 4/4= 1

	5. Geometry • Partition shape into parts with equal areas • Area of each part is described as a unit fraction of the whole
GRADE 4 1. Formal definition of fraction (part-whole) 2. Symbols for fractions 3. Equivalent fractions: 2/4= 1/2, 2/2= 3/3= 4/4= 1 4. Reflections through inkblots, paper cutting, paper folding 5. Fractions in money, length, volume related tasks 6. Chapati charts (pie charts) 7. Part to whole: Reproducing whole from a fractional part of the whole	**GRADE 4** 1. Equivalent fractions: a/b= (nxa)/(nxb) (visual model) 2. Compare fractions (different numerators and different denominators) 3. Fraction addition and subtraction (with same denominator) 4. Add and subtract mixed numbers with like denominator 5. Word problems involving addition and subtraction of fractions referring to the same whole and with like denominators 6. Multiplication of fraction by a whole number 7. Decimal notation for fractions 8. Compare decimal fractions 9. Word problems for addition, subtraction, multiplication and decimals involving money, liquid volumes, time and weight 10. Geometry • Symmetry (Reflections), paper folding
GRADE 5 1. Recapitulation of grade 4's part-whole definition of fraction, only proper fractions 2. Reproducing whole from a fractional part of the whole 3. Comparing fractions 4. Equivalent fractions 5. Estimating closeness of a fraction to known fractions (1/2, 1/4, 3/4) 6. Decimal fractions in context of length units and money 7. Conversion of fraction to decimal and vice versa 8. Multiplication of fraction with whole number 9. Mirror halves revisited 10. Various turns (1/2, 1/3, 1/4, 1/6) (rotational symmetry)	**GRADE 5** 1. Decimals • Compare decimals to thousandths 2. Add and subtract fractions with unlike denominators 3. Word problems involving addition and subtraction of fractions (including unlike denominators) referring to the same whole 4. Fraction as division of the numerator by the denominator 5. Fraction multiplication • (a/b) x q = (a x q) ÷ b • Finding area of a rectangle with unit squares of unit fractional side lengths • Representing fraction products as areas of rectangles with fractional side lengths • If a/b > 1 then explain pa/b > p and if a/b < 1 then explain pa/b< p for any nonzero natural number p • Without performing the multiplication, compare the size of a product to the size of one factor 6. Value of the products 7. Word problems involving multiplication of

	fractions and mixed numbers 8. Division of unit fractions by whole numbers and whole numbers by unit fractions 9. Word problems involving division (as in 7)
GRADE 6 1. Part-whole definition of fraction 2. Fraction on the number line 3. Fraction as division 4. Improper and mixed fractions 5. Equivalent fractions 6. Comparison of fractions (both like and unlike denominators) 7. Add and subtract fractions (both like and unlike denominators) 8. Decimal fractions and place value 9. Conversion of fraction to decimal fraction 10. Word problems involving addition and subtraction of decimals on money, mass, length and temperature related tasks 11. Using HCF for equivalent fractions 12. Ratios and proportions 13. Reflection symmetry- Mirror symmetry revisited	**GRADE 6** 1. Ratios and proportional relationships 2. Division of fraction by fraction, solving word problems 3. Fluently add, subtract, multiply and divide multi digit decimals 4. Finding greatest common factor of two whole numbers 5. Rational number as a point on the number line
GRADE 7 1. Fraction multiplication 2. Value of the products 3. Fractions as operators 4. Reciprocal of a fraction 5. Division of fractions 6. Rational number as a point on the number line 7. Rational numbers as decimals 8. Word problem on rational numbers 9. Multiplication and division of decimal fractions 10. Ratio and proportions revisited 11. Geometry • Reflection symmetry revisited • Rotational symmetry	**GRADE 7** 1. Ratio and proportional relationships 2. Add and subtract rational number 3. Using number line for addition and subtraction 4. Multiplication and division of rational numbers 5. Convert rational numbers to decimals using long division 6. Real world problem involving the four operations with rational numbers

Here I present a quick review of the comparisons between NCERT's syllabi and CCSSM's standards on a grade-by-grade basis.

Grade 1: Grade 1 of NCERT's syllabus and CCSSM's standards are similar. Although NCERT's syllabus describes the use of non-uniform units in grade 1, I found examples of only non-standard units in the grade 1 of NCERT's textbook. Fraction is introduced in grade 1 of CCSSM in terms of partitioning a whole into halves, and into quarters. Both describe length units by iterating a quantity without any gaps or overlaps.

Grade 2: NCERT's grade 2 syllabus is based on recapitulation of its grade 1 syllabus. Hand spans, fingers are used to measure length of an object. I consider "fingers" as non-uniform unit as different fingers in a hand have different widths (sizes). Standard length units and number lines are introduced in grade 2 of CCSSM which are seen in grade 3 and grade 6 of NCERT's syllabus respectively. In CCSSM, fractions are revisited in this grade in geometry with the introduction of thirds.

Grade 3: Standard length units are introduced in grade 3 of NCERT's syllabus, which are seen in grade 2 of CCSSM. Mirror halves are introduced in NCERT's syllabus that is not seen until now in CCSSM. Formal definition of fraction is introduced in grade 3 of CCSSM. Length model, set model and area model of fractions are discussed in this grade of CCSSM. When comparing fractions, CCSSM emphasize recognizing that comparisons are valid only when the two fractions refer to the same whole. I found that the fraction definition in CCSSM emphasizes understanding the concept of iteration. I infer that a/b as a parts of size 1/b suggests a/b as 1/b + 1/b + … + 1/b (a times) which is seen in grade 4 of CCSSM.

Grade 4: Formal definition of fraction as part out of whole is introduced in grade 4 of NCERT's syllabus. Here only proper fractions and area model of fraction is discussed.

This suggests the focus is on continuous units. CCSSM's definition of fraction includes both proper and improper fractions; the definition works for any fraction in general. Fraction addition and subtraction of like fractions (fractions with same denominators) are discussed in grade 4 of CCSSM. Multiplication of fraction by a whole number (money related task) is seen in both NCERT's textbook and in CCSSM standards for grade 4. The money related task emphasizes treating fractions as operators. The introduction of multiplication of a fraction by a whole number in this grade of CCSSM is a continuous progression from grade 3's definition of fraction as a parts of size 1/b. It suggests viewing a/b as a x (1/b). Mirror halves and reflections are introduced in grade 4 of CCSSM after the formal introduction of fractions in CCSSM's grade 3 which is seen in grades 3 and 4 of NCERT's syllabi. Decimal fractions are introduced in CCSSM's grade 4. I found the concept of iteration in chapter 9 of NCERT's grade 4 textbook while reproducing the whole from a given fractional (one-half, one-fourth) part of the whole. Another difference is that CCSSM has a standard in grade 2 which emphasizes that equal shares of identical wholes need not have the same shape. However, NCERT focuses on parts of same shape where each of the parts of a whole has the same shape.

Grade 5: In grade 5 of NCERT's textbook, focus is on continuous units. For example, considering a rectangular bar or a circular disc as a whole. I also found few set models for fractions. I found similarities with grade 3 and 4 standards of the CCSSM with NCERT's syllabus for grade 5 on fractions. Lines of symmetry are discussed in grade 4 of CCSSM that is seen in grade 5 of NCERT's syllabus while revisiting mirror halves. I found the concept of iteration in chapter 4 of NCERT's grade 5 textbook in the task of reproducing the whole from a given fractional part of the whole which is closely related to fraction addition in grade 4 of CCSSM. Mirror halves are introduced before fractions in grade 3 and revisited in grade 5 of NCERT's syllabus, which is first seen in the line of symmetry in CCSSM's grade 4 standards. Reflection symmetry is introduced later in grade 6 of NCERT's syllabus. Although rotational symmetry is introduced in grade 7 of NCERT's syllabus, various turns (1/2, 1/3, 1/4, 1/6) are discussed in grade 5. Those turns are not discussed in CCSSM; rotational symmetry is seen in grade 8 of CCSSM.

Grade 6: Improper and mixed numbers are introduced in grade 6 of NCERT's syllabus. Here emphasis is given more on understanding improper fraction as a whole, and some fractional parts of the whole but improper fractions are also explained by partitioning and distributing the parts of the whole. NCERT introduces fraction addition and subtraction for both like and unlike fractions in grade 6 while CCSSM introduces those in two different grades, one in grade 4 and another in grade 5. Number line is introduced in grade 2 of CCSSM, while it is introduced in grade 6 of NCERT's syllabus. In CCSSM grade 3, number line is used to plot fractions. Grade 6 of NCERT's syllabus focuses on discrete units along with continuous units. Iterations are used in equivalent fractions task in grade 6 of NCERT's syllabus. CCSSM focus on understanding equivalent fractions with the whole being same, NCERT does not focus on the same whole. In this grade of NCERT's syllabus, area, length and set models of fractions are used. HCF is used in calculating the equivalent fractions in NCERT's textbook of grade 6, which is seen in fraction addition and subtraction with unlike denominators in grade 5 standard of CCSSM where fractions with unlike denominators are replaced by equivalent fractions.

Grade 7: NCERT's grade 7 syllabus on fraction is a blend of CCSSM's grades 4, 5, 6, 7 and 8 standards on fraction. In NCERT's grade 7 syllabus, fraction is considered as an operator that is seen formally in CCSSM's grade 4 standards. Fraction multiplication and division of fraction are discussed in NCERT's grade 7. Divisions are explained by converting it to multiplication and changing the divisor to a unit fraction i.e. as (a/b) ÷ c means (a/b) X (1/c). While fraction division (division of fraction by a whole number) is introduced in CCSSM for grade 5 but in CCSSM, the relationship between multiplication and division is explained as (a/b) ÷ c = d because d X c = a/b. Division of fraction by a fraction discussed in NCERT's grade 7 is introduced in CCSSM's grade 6. NCERT discusses value of the products in grade 7 while CCSSM discusses in grade 5. CCSSM extends the prior understanding of whole number multiplication and whole number division to fraction multiplication and fraction division, but no such connection in seen in NCERT's textbook examples. NCERT uses fraction as operator for both fraction multiplication and fraction division. Rational numbers are introduced in grade 7 of NCERT's syllabus that is seen in CCSSM for grade 6. Operations on rational numbers are introduced in grade 7 of NCERT's syllabus, which are seen in CCSSM's grade 7. In grade 7 of NCERT, partitioning and iteration are used as operations on fractions. Rotational symmetry is discussed in this grade of NCERT that is seen in grade 8 of CCSSM.

In this chapter, I found some significant differences between the two countries approaches to fractions. First, fractions appeared in different grade levels in these two countries. In CCSSM, it is in grade 1, whereas it appeared in NCERT much later in grade

4. Table 3.4 shows that fraction topics generally appear later in NCERT syllabi. The main difference in CCSSM and NCERT is in the definition of fraction. NCERT uses “parts-out-of-whole” definition of fractions whereas CCSSM has two separate definitions of fractions. CCSSM defines unit fraction in terms of “parts-out-of-whole” and all other fractions (proper and improper) is defined in terms of iteration (a fraction a/b defined as a parts of size 1/b). In NCERT, the meaning of improper fractions is not explicitly defined, improper fractions are explained in terms of mixed fractions. This might be the result of not emphasizing unit fractions in NCERT. NCERT focuses on congruent parts of a whole, whereas CCSSM focuses on recognizing equal parts of the same whole, which might not be congruent. NCERT advocates procedural fluency in equivalent fractions, wholes are ignored here; whereas CCSSM emphasizes the importance of the same whole in equivalent fractions. In NCERT, fraction multiplication and division are not developed as extension of whole number multiplication and division; whereas in CCSSM, fraction multiplication and division are extensions of prior understanding of whole number multiplication and whole number division.

CHAPTER 4

ANALYSIS OF STUDENTS' INTERVIEWS

Research has revealed that fractions are the most complicated construct in the elementary school mathematics (Smith, 2002). Students have different levels of understanding on fractions even on the simplest fraction one-half (Watanabe, 1996). In this chapter, I analyze the interviews with nine high performing students (four from the US and five from India) in between grades 5 to 9. The questionnaire for the interview consisted of fraction questions such as explaining a fraction, explaining an improper fraction, drawing fractions on the number line, explaining equivalent fractions using diagrams, and tasks with fraction addition and fraction subtraction (see Appendix A). By posing different fraction tasks, I was able to observe students' ways of operating with fractions. In an effort to understand and describe their thinking, I categorized the nine students' mental operations using Norton & McCloskey's (2008) categories, with a few modifications based on my findings.

This chapter consists of two sections. The first section provides a detailed discussion of the theoretical framework used to analyze the students' interviews. In this section, I discuss a constructivist view of how knowledge is constructed and present different types of mental actions. The second section presents an analysis of the students' interviews. The result shows that even though all the students were high performing students, they used different mental operations to complete a given task. The operations they used often indicated significant constraints in their understanding of fractions.

The theoretical framework

CONSTRUCTING KNOWLEDGE

Mathematical learning is an ongoing process of accommodation of prior experiences (Hackenberg, 2010). When we experience a situation, we try to fit our current experiences with prior experiences. When the experience is familiar, we feel as if we know it as it fit into our previous experiences. According to Piaget, this mental action is called assimilation, i.e., fitting our experiences into existing ones (Van de Walle, 1998). Other times, disequilibrium is created when we cannot fit current ideas into previously developed understanding (Von Glasersfeld, 1995). We need to modify our ideas or generate new ideas to fit the current idea to attain equilibrium. This action of maintaining equilibrium by adjusting and generating ideas is called accommodation. Constructivism is based on these mental activities of assimilation and accommodation (Van de Walle, 1998).

Constructivists view knowledge as constructed by the learner through assimilation and accommodation. According to von Glasersfeld, "knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the (knowing) subject" (Von Glasersfeld, 1990, p.22). According to Van de Walle (1998), for constructing and understanding a new idea, we need to make connections between old ideas and new ones. New ideas depend on the network of existing ideas called schemas:

> Networks of ideas that presently exist in the learner's mind determine how an idea might be constructed. These integrated networks, frequently referred to as cognitive schemas, are both the product of constructing knowledge and the tools with which new knowledge is constructed. The more connections with the existing network of ideas, the better the new ideas are understood. As learning occurs, the networks are rearranged, added to, or otherwise modified. (p. 25)

OPERATIONS

According to Norton & McCloskey (2008), "operations are mental actions that have been abstracted from experience to become available for use in various situations" (p. 49). For example, rotation is an operation that we experience when we twist. A child using rotation as mental action can recognize a square even if it rotated at an angle, i.e. one of the sides is not horizontal but is slanted (Norton & McCloskey, 2008).

In my study, I have used the term "object" and "quantity" simultaneously. "A quantity is schematic: It is composed of an object, a quality of the object, an appropriate unit or dimension and a process by which to assign a numerical value to the quality" (Thompson, 1994, pp. 7-8).

In this section, I describe several key operations that students use to understand fractions (Norton & McCloskey, 2008; Steffe & Olive, 2010). I have subdivided some of the mental actions and introduced further categories based on my findings from the student interviews. A summary of the mental actions is given in table 4.1. A detailed description of the mental actions follows:

- ***Unitizing (U)*** is a foundational mental operation. Unitizing is the operation of producing a single object as a unit or a collection of objects to represent a unit or a whole. According to Norton & McCloskey (2008), "anything can be taken as a unit by establishing it as a separate entity, such as extracting foreground from background" (p. 49). I have further categorized unitizing as:
    - ***Unitizing a Whole (UW)***: Here a single object is considered as a unit or a whole. This produces a continuous unit. For example, recognizing a rectangular candy bar as a unit or a whole.

- *Unitizing a Collection (UC)*: Here a collection of objects represents a unit or a whole. Unitizing a collection produces a discrete composite unit. For example, a child takes a collection of five marbles as a unit. After a student unitizes a collection of objects, he or she views the object as a measurable entity. There is some numerical value assigned to each object after unitizing. For example, each marble in the collection of five marbles will be one-fifth of the collection.

- ***Partitioning (P)*** is the most important operation for producing fractional knowledge (Hackenberg, 2010; Lamon 1996; Mack, 2001; Steffe & Olive 2010). Partitioning involves dividing a whole into parts, but not necessarily equal parts. I subdivided partitioning further into two categories:
  - ***Continuous Partitioning (CP)*** *is* associated with partitioning a continuous unit. This mental action is associated with the mental action of unitizing a whole. When a child conceives of a continuous whole as a unit, the mental operation involved in making a fraction is partitioning the continuous unit. Hence, I called this operation continuous partitioning. "Partitioning a continuous unit (e.g., a candy bar), a continuous composite unit (e.g., three-fifths of a candy bar) produces equal parts by marking separations within the unit" (Norton & McCloskey, 2008, pp. 49-50). Norton & McCloskey (2008) discussed a confusing aspect of partitioning for students:

    > One aspect of partitioning that can confuse students is the idea that partitioning a unit into n parts requires only n-1 marks, leading

> some students to claim they have produced fifths, for instance, instead of sixths. (p. 50)

  - ***Discrete Partitioning (DP)*** is associated with partitioning a discrete composite unit. Here partitioning is associated with the mental operation of unitizing a collection. If a collection of objects is seen as unitized, a selection of objects from the given collection of objects may be chosen to represent a fraction. I call such an operation discrete partitioning. Discrete partitioning involves partitioning into equal parts. For example, selecting four marbles out of five marbles to demonstrate four-fifths of the set of marbles.

- ***Disembedding (D)*** is the mental action of imaginatively pulling out a part from a whole (continuous or discrete) without mentally destroying the whole. In other words, a student pulls out copies of some number of parts within the whole, while leaving the whole intact (Norton & McCloskey, 2008; Steffe & Olive, 2010). For example, a student might produce three-fourths by partitioning the whole into four parts and then pulling out only three of those parts to show three-fourths while maintaining the four parts as a unit (Norton & McCloskey, 2008). A student pulling out three parts out of the four parts would consider those three parts in comparison to the original four parts. According to Steffe and Olive (1996), "Disembedding is the fundamental mental operation on which part-whole comparisons are based" (p. 118).

Discrete partitioning and disembedding are two different mental operations. In discrete partitioning, a child might not have the understanding of the relationship of each part with respect to the whole. A child using discrete partitioning could select a number of items from the given perceptual items to represent a fraction. For example, a child choosing 4 marbles out of 7 marbles to denote 4/7. A child using only discrete partitioning will see three marbles are remaining instead of three out of seven marbles. A child using disembedding as mental operation will see the remaining three marbles as three out of seven marbles and the four chosen marbles will mean four out of the seven marbles. The child will be able to keep the relationship of the parts from the whole even after the operation.

- ***Identifying (Id)*** is another mental action. Here students rely on key words to solve problems. For example, while solving word problems like "Tom pours 1/2 cup of water into an empty bowl. Then Tom pours in another 1/3. How many cups of water are in the bowl?" (Beckmann, 2011), the student focuses on key words such as "pours in" and identifies it with "*adding*".

- ***Retrieving (R)*** is another mental action that students use while solving problems. Students' prior experiences with similar problems experienced in the past trigger this mental action. Here students recall specific formulas or rules that they have seen before in their math classes. Students might not have abstracted the meaning of the math rules they use. For example, a child converts improper fractions to mixed fractions using division whenever he encounters improper fractions; a child uses fraction addition (subtraction) rules to add (subtract) two fractions, etc.

Identifying and retrieving operations are two different mental operations. Identifying lets the child identify closely related experiences with the current situation. Retrieving lets a child perform similarly to a situation which has been experienced earlier.

- ***Iterating (I)*** is the mental action of repeating a part of a unit in order to make a larger amount (Hackenberg, 2010; McCloskey & Norton, 2009; Steffe & Olive, 2010). According to Norton & McCloskey (2008), "iterating involves repeating a part to produce identical copies of the original" (p. 50). A student might use iteration to form a continuous unit or to form a discrete composite unit (Norton & McCloskey, 2008). Students using the iterating operation understand that any part of the whole can be iterated to get the whole (Norton & McCloskey, 2008). For example, given one-fourth of a stick, a child repeats the one-fourth stick four times to draw a whole.

- ***Splitting (Sp)*** is the simultaneous composition of partitioning and iterating operations (McCloskey & Norton, 2009; Steffe & Olive, 2010). "Beyond partitioning and iterating sequentially, students who are "splitters" can exploit the inverse nature of these two operations to solve problems and re-create the whole from any fractional piece" (McCloskey & Norton, 2009, p. 46). A student can partition a bar to find one-third of the bar and may be able to find a bar five times as long as a given bar by iterating (McCloskey & Norton, 2009). According to McCloskey & Norton (2009), “the splitting operation would enable the student to solve the task: “This bar is five times as long as another bar. Draw the bar.” (pp. 46-47). A splitter will recognize that the given piece was made by iterating the unknown whole five times and hence need to partition the whole into five equal parts. This task uses the language suggestive of iteration rather than partitioning (McCloskey & Norton, 2009). A child using splitting operation can posit a hypothetical unit outside the given whole. The child can imagine a part of the whole out of the whole and repeat it simultaneously to generate the whole. Therefore splitting involves disembedding.

- ***Estimating (E)*** is another mental action. A child can estimate if a fraction is close to 0, 1/2 or 1 by looking into visual models of fraction. Some students use a number line to represent a fraction. Even though the student might not have the exact sense of the position of the fraction on the number line, they can view the fraction somewhere in between certain benchmark numbers, such as whole numbers. I call such an operation estimating. For example, showing four-thirds on a number line in between 1 and 2 without specifying its exact position.

Table 4.1: Mental operations used by students.

Operation	Description	Example of Using the Operation
Unitizing (U)	Producing a single object as a unit or a collection of objects as a unit.	Drawing a single rectangular bar to begin with.
• Unitizing a Whole (UW)	Treating an object as a unit or whole.	
• Unitizing a Collection (UC)	Treating a collection of objects as a unit or whole.	Treating two circles as a whole.
Partitioning (P)	Dividing a whole into parts not necessarily equal parts.	Sharing a candy bar among three people.
• Continuous Partitioning (CP)	Partitioning a continuous unit.	
• Discrete Partitioning (DP)	Choosing a number of objects from the collection of objects.	Showing one-third by choosing one marble from 3 marbles.
Disembedding (D)	Mentally pulling out a part out of a whole (continuous or discrete) without mentally destroying the whole.	After slicing the pizza in fifths, showing what three-fifths of the pizza would look like and being able to recognize the pieces left as two-fifths of the pizza, not just two pieces left.
Identifying (ID)	Searching for key words in a problem.	Adding 1/2 + 1/3 thinking "pours in" as addition to solve the problem: Tom pours 1/2 cup of water into an empty bowl. Then Tom pours in another 1/3. How many cups of water are in the bowl?
Retrieving (R)	Applying fraction operation rules.	• Converting an improper fraction to a mixed fraction by division. • Adding 1/2 + 2/3 using a fraction addition rule.
Iterating (I)	Repeating a part to produce a larger amount and repeating to produce identical copies of the part.	Repeating a one-fourth piece three times to generate a three-fourths piece.
Splitting (SP)	Simultaneous composition of partitioning and iterating operations.	"This bar is six times as long as your bar. Draw your bar". A student partitions the bar into six equal parts, takes out one part, and iterates five more times to see if it produces the given bar.
Estimating (E)	Estimating the fraction by comparing it with 0, 1/2 or 1, or other numbers.	Showing 4/3 on a number line in between 1 and 2 without exactly locating the position.

FIRST ORDER AND SECOND ORDER MATHEMATICAL KNOWLEDGE

According to Steffe in Steffe & Olive (2010), first order mathematical knowledge is one's own mathematical knowledge. It is "the models an individual construct to organize, comprehend, and control his or her experience" (p. 16). Second order mathematical knowledge is "the models observers may construct of the observed person's knowledge" (p. 16). Observers use their first order mathematical knowledge to construct the second order mathematical knowledge of the observed person. Steffe called second order models, social models as they are constructed through social processes (Steffe & Olive, 2010). It is critical to distinguish between first- and second- order mathematical knowledge to avoid mixing children's mathematical concepts and operations with conventional school mathematics (Steffe & Olive, 2010).

UNITS COORDINATION

There are three levels of units coordination (Steffe and Olive, 2010).

One level of units coordination: In this level, the individual has only the sense for single units. For example, a student seeing 7 marbles considers each one to be one unit but does not count them all to see 7 units together.

Two levels of units coordination: In this level, individuals can form composite units. For example, 7 units is 7 one units. An individual using second level of units coordination understands 7 units is obtained by combining 7 one units together.

Three levels of units coordination: It consists of three different units construction (Figure 4.1). For instance, to understand 4/3, one can pick three marbles to denote the whole (second level) and then uses another marble to denote 4/3. Here the individual is treating each marble as one unit (First Level).

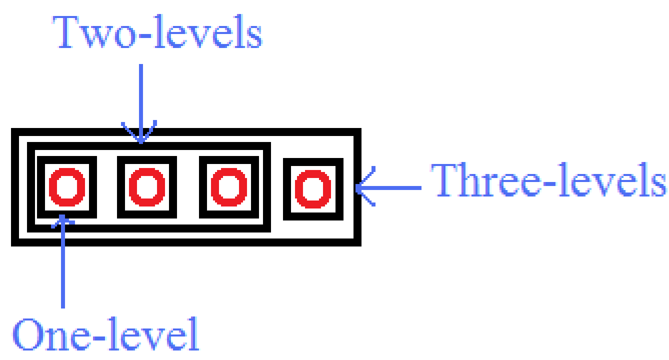


Figure 4.1. Three-levels of units coordination.

Three levels of units coordination is critical for understanding of fractions conceptually (Hackenberg, 2007; Steffe & Olive, 2010). In the mental actions, disembedding uses two levels of units coordination, whereas splitting uses three levels of units coordination.

The research study

PARTICIPANTS

I have a convenience sample. I approached friends whose children were high performing students. Nine students between grades 5 to 9 who had worked on fraction addition and fraction subtraction agreed to participate in this study. Out of nine, four students were studying in 3 different states (Georgia, Indiana and Virginia) in the United States and the other five students were from different schools in Kolkata, India. All these students were high achievers and A students in their classes. Students from the USA were from grades 5, 6, 7 and 9. The Indian students were from grades 6, 7, 8 and 9. Out of the five Indian students, two students were studying under the Central Board of Secondary Education CBSE, two were studying under the Indian Certificate of Secondary Education ICSE and one student was from West Bengal Board of Secondary Education WBBSE. NCERT is directly involved with CBSE and indirectly to the other boards of education. I have used the following names for the participants:

Student Name (U: U.S., I: India; Corresponding Grade): Grade; Gender; State, Country (Curriculum followed) (Interview Type)

Student 1 (U.5): Grade 5; M; Indiana, USA (Standards not known) (Face-to-face interview)

Student 2 (U.6): Grade 6; F; Georgia, USA (Common Core Georgia Performance Standards CCGPS) (Face-to-face interview)

Student 3 (U.7): Grade 7; F; Virginia, USA (Fairfax County Public Schools FCPS Standards) (Skype interview)

Student 4 (U.9): Grade 9; M; Georgia, USA (Common Core Georgia Performance Standards CCGPS) (Face-to-face interview)

Student 5 (I.6): Grade 6; F; Kolkata, India (ICSE) (Skype interview)

Student 6 (I.7): Grade 7; F; Kolkata, India (CBSE) (Skype interview)

Student 7 (I.8-1): Grade 8; M; Kolkata, India (ICSE) (Skype interview)

Student 8 (I.8-2): Grade 8; F; Kolkata, India (CBSE) (Skype interview)

Student 9 (I.9): Grade 9; F; Kolkata, India (WBBSE) (Skype interview)

PROCEDURE

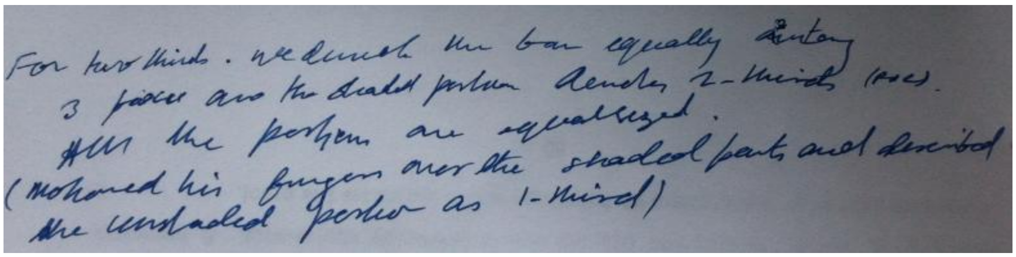
All of these 9 students individually participated in a 60-minute clinical task-based interview. Except for three face-to-face interviews, six interviews were taken over Skype.

The interview questions involved explaining examples of fractions, finding a fraction on the number line, adding two fractions with same and different denominators, and deciding whether a given answer to the word problems related to fraction addition and subtraction were valid (see Appendix A). Tasks were presented to participants one at a time and were asked to explain their work. The students recorded their answers on papers. Researcher took detailed notes for each student.

DATA ANALYSIS FOR INTERVIEWS

I use the different types of mental actions to analyze children's fractional knowledge. The main goal of the analysis of the student interviews is to understand the ways and means of operation of these nine high-performing students. I aim to find the ways of operating of these nine students that were to be consistent with the models from prior research (Norton & McCloskey, 2008; Steffe & Olive, 2010). Students' ways of operation and the tasks might suggest some measures teachers can take to improve fractional learning. In this regard, I took detailed notes for each student and summaries of each student's work on every problem. An example of my notes and corresponding approximate transcript looks like Figure 4.2.

After I wrote the summary of the second-order model of each student, I shared the models with a colleague. The colleague was a mathematics graduate student. First, I explained the modified categories of mental actions and gave him a set of interviews. He was asked to identify the mental actions used by the students in some of the problems. We discussed my model with his on some problems. Whenever we did not have an agreement, we discussed and refined our interpretations. Final decisions were recorded after a general agreement between the researcher and the graduate student.



Approximate Transcript: *For two-thirds. We divide the bar equally into three pieces and the shaded portion denotes two-thirds (picture). All the portions are equal-sized. (Student motioned his fingers over the shaded parts and described the unshaded portion as one-third.)*

Figure 4.2. An example of my notes and corresponding approximate transcript.

Analysis of students' interviews

In this section, I present analysis of the students' interviews. In the analysis, I focus on how students make sense of fractions, both proper and improper fractions, plotting fractions on number line, equivalent fractions, and units in word problems. Tasks are shown in Appendix A. I categorize the tasks into seven categories. Category 1 includes questions 1, 2, 3 and 4. I call this category *Explaining Fractions.* Category 2 includes questions 5, 6 and 7. I call this category *Explaining Improper Fractions.* Category 3 includes questions 8 and 9, I call this *Fractions as operator*. Category 4 is *Fractions on number line*, it includes questions 10 and 11. Category 5 is *Equivalent Fractions;* includes questions 12 and 13. Category 6 is *Fraction Addition;* it includes questions 14, 15, 16, 17 and 18. The last category is units in word problems; it includes questions 20 and 21. I used these categories to make sense of students' understanding of fractions.

This study documents differences in students' ways of operating within and across different grade levels across the two countries (see table 4.2). Even though all the students were A students in their respective classes, they used different mental operations to solve a particular task. This study also shows the current understanding of fractions and fraction operations among these high performing students. The study supports unitizing and partitioning as critical components in building fractional knowledge in children.

This analysis revealed ways that students reasoned about fractions as iteration, lengths, and operators. In addition, this analysis presents how students reasoned about improper fractions, fraction addition rules, units construction, and equivalent fractions. I will describe those next.

Fractions as iteration: Student 1 (U.5), Student 4 (U.8) and Student 7 (I.8-1) are able to iterate a unit fractional part of the whole to generate the whole. I conclude that these three students understand that when a whole is partitioned into equal parts, any one of the parts could be repeated to generate the whole; they use iteration as one of the mental actions. This suggests that these students have constructed two levels of units.

Student 2 (U.6) has repeated the action of motion of the fingers to create equal parts rather than iterating (Figure 4.3). I found that while marking 3 on the number line student 2 (U.6) put her thumb on 1 and moved her index finger starting at 1 up to 2. Then she put her thumb at 2 and moved her index finger up to certain point on the number line where she felt her index finger had traversed the same distance. She then placed 3 at that point. I infer from her action that she used motion of her hands to estimate the length instead of iterating the length unit between 1 and 2. Her action of using motion is very different from other students using iteration.

Table 4.2: Mental actions used by the nine students.

Categories	U.5 Student 1	U.6 Student 2	U.7 Student 3	U.9 Student 4	I.6 Student 5	I.7 Student 6	I.8-1 Student 7	I.8-2 Student 8	I.9 Student 9
Explaining Fractions	UW, CP, D	UW, CP	UC, DP	UW, CP	UW, CP	UW, CP	UW, CP, D	UW, CP	UC, DP
Explaining Improper Fractions	UW, CP, D, I, Sp	**E, R**	UC, DP, D or **UC, DP, R**	UW, CP	**R**	**R**	UW, CP	**R**	**UW, CP, R**
Fractions as operator	UC, UW, CP, DP, D, I, Sp	***UC, UW, CP, DP***	**R**	R	**UW**	R	UC, UW, CP, DP	R	***NA***
Fractions on number line	***UC, DP***	**E, R**	UW, CP, I	UW, CP, E, R, I	**I**	UW, CP, R	UW, CP, I, R	UW, CP, R	***I, UW, CP***
Equivalent Fractions	UW, CP, R	**UW, CP, R**	UW, CP, R	UW, CP, R	***NA***	UW, CP, R	**UW, CP, R**	UW, CP, R	***UC, DP, R***
Fraction Addition	**R**	***R***	**R**	**R**	**R**	**R**	R, UW, CP	**R**	**R**
Units in word problems	UW	***Id***	***Id***	***Id***	***Id***	***Id***	***Id***	***Id***	UW

NA: Not Applicable;
Normal Font: Student solve the problem correctly;
Bold letters: Student can solve the problem partially;
Italics letters: Problem does not make sense to the student

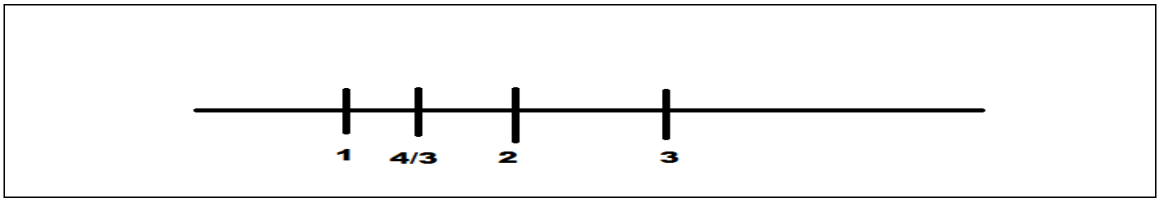


Figure 4.3. Student 2 (U.6) showing 2 and 3 on the number line.

Source: Original copy of student's work.

Disembedding and Splitting as mental action: Student 1 (U.5) is able to disembed a fractional part from the whole while maintaining the whole as a unit. He understands that the part left is a fractional part of the whole instead of a counting whole number. Figure 4.4 shows Student 1's disembedding action.

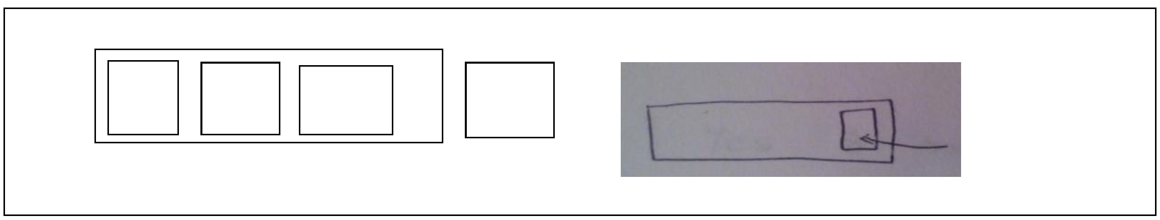


Figure 4.4: Student 1 (U.5) explaining improper fraction.

Source: Scanned copy of student's work.

In Figure 4.4, Student 1 (U.5) is explaining improper fraction 4/3. He imagined the one-third part out from the whole while maintaining the whole as a unit. He then iterated the one-third part to generate the whole and hence, the four-thirds. This student recognized the need to partition the bar into thirds and then took one-third piece out and iterated the thirds piece two more time to reproduce the whole and iterated once more to get four-thirds. This suggests that he has used simultaneous partitioning and iteration, i.e. splitting as mental action here. In addition, this example serves as an example for the use of three-levels of units coordination.

Improper fraction: In the problem of plotting 7/4 on a number line (Figure 4.5), Student 7 (I.8-1) has iterated one-fourth unit to get a whole. After getting the whole, he iterated the one-fourth-length unit again few times to get one and three-fourths. Even though his plotting is only partially correct, I focus only on his way of iteration here. His iteration was not continuous as he did not see seven-fourths as seven one-fourths rather he saw seven-fourths as a mixed number one and three-fourths.

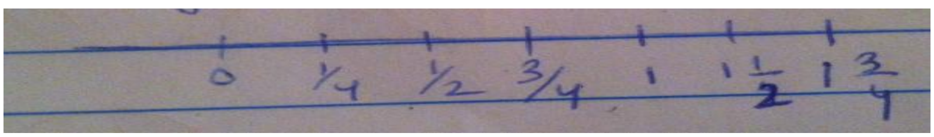


Figure 4.5. Student 7 (I.8-1) showing 3/4 & 7/4 on the number line.

Source: Scanned copy of student's work from email.

Research has revealed that construction of improper fractions is not an easy accomplishment (Hackenberg, 2007; Steffe & Olive, 2010). Although many students can work with fractions greater than one, they might not have established such fractions as numbers "in their own right" (Hackenberg, 2007). In fact, without the construction of improper fractions, student's mathematical activities may be constrained (Hackenberg, 2007). That is, when a student is asked to draw 4/3 of a length, a student might convert that to a mixed fraction and draw it as a whole and 1/3 of the whole. The student seems to understand that 4/3 is greater than a whole. However, that interpretation relies on wholes and parts of wholes for the meaning of 4/3. Alone, such a response does not indicate that the student conceives of 4/3 as a number in its own right (Hackenberg, 2007). This suggests that Student 7 (I.8-1) has not conceived improper fraction in its own right. A student who does not consider 7/4 as both a mixed number and as a number in its own right has not constructed improper fractions (Hackenberg, 2010).

Student 5 (I.6) has used iteration to draw 5/4 on the number line (Figure 4.6). This student's work appears mathematically correct. She repeated one-fourth unit repeatedly five times to get 5/4.

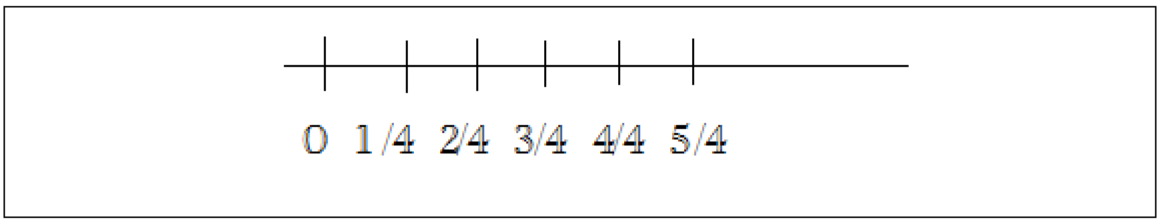


Figure 4.6. Student 5 (I.6) showing 5/4 on the number line.

Source: Copy of student's work from email.

However, on probing further, I found that she did not realize 4/4 is a whole. She viewed 4/4 as 4. Research has revealed that iteration is simply not enough to produce improper fractions (Hackenberg, 2007; Steffe & Olive, 2010). Student 5 has a whole number meaning of 5/4. She viewed 5/4 as 5 units. She started with marking 0 and then the next tick mark marking 1/4 and repeating to get to 5/4. She could produce 5/4 from 1/4 in the same way that she can iterate 1 to produce 5 or 4 (Hackenberg, 2007).

Student 9 (I.9) was confused with the parts in whole definition of fractions, as she could not fit 4 into 3 to show four-thirds. I conclude that none of the Indian students have constructed the meaning of improper fraction.

Fraction addition rules: This study also revealed that fraction addition rules do not make sense to seven out of nine students. Student 1 (U.5) showed understanding for fraction addition with like denominators. Student 7 (I.8-1) showed understanding of the rules of fraction addition with both like and unlike denominators. He focused on the whole and he showed understanding that the whole/unit remains the same in the fraction addition. Here, he drew 7/3 by keeping the whole as 3 bars (Figure 4.7).

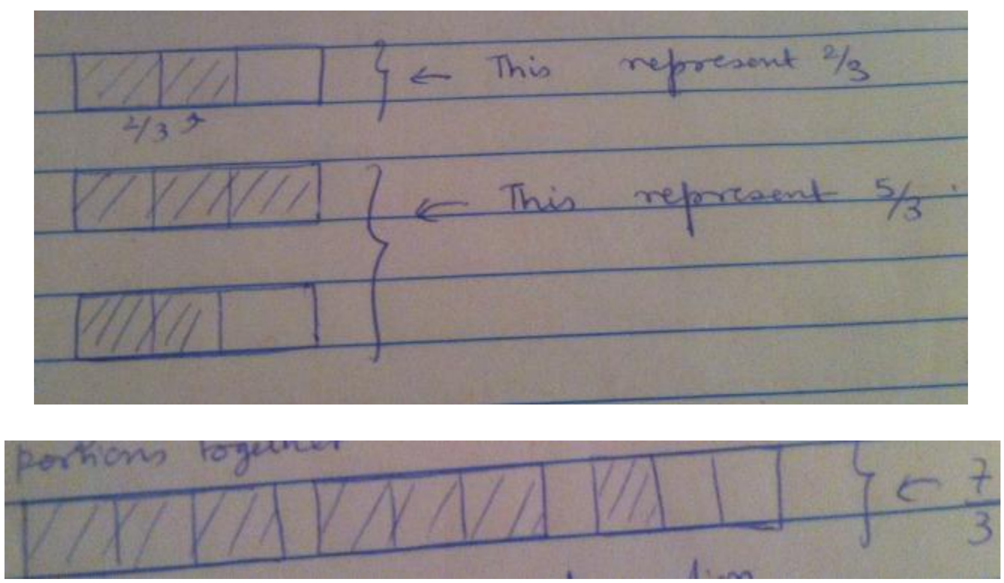


Figure 4.7. Student 7 (I.8-1) representing 2/3, 5/3 and 7/3 for the addition problem 2/3+ 5/3.

Source: Scanned copy of student’s work from email.

All the other students viewed fraction addition as a rule with no meaning involved. It is evidenced from students' interviews that the majority of these students believe that using the least common multiple is the only way to find the sum of two fractions with different denominators. Only Student 3 (U.7) and Student 4 (U.8) knew that the problem can be solved without using LCM.

Units construction: Student 1 and Student 9 were able to show an understanding of units construction in the task of identifying the story problems for 1/2 + 1/3 an 1/2 – 1/3. Both of them were able to identify the whole for the given fractional parts. Others simply identified the fractions irrespective of the wholes and identified keywords such as "pouring in" and "pouring out" for addition and subtraction.

Equivalent fractions: Three students, Student 2 (U.6), Student 7 (I.8-1) and Student 9 (I.9) have used division while explaining why 2/3 = 8/12. They have divided 8 and 12 by 4, but using the division operation in finding the equivalent fractions did not reflect their understanding of equivalent fractions. For instance, Figure 4.8 shows Student 2's work.

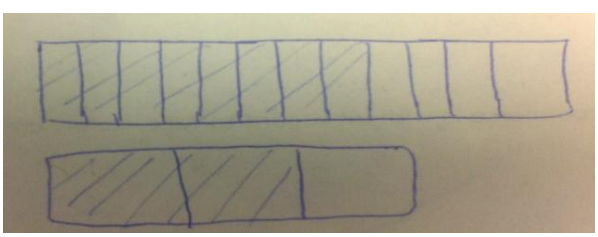


Figure 4.8. Student 2 (U.6) showing 2/3 = 8/12.

Source: Scanned copy of student's work.

Student 2 knew cancelling 4 from both eight and twelve would give two-thirds. However, pictorially she drew the stack for twelfths longer than the thirds. This suggests that she viewed the two-thirds and eight-twelfth as separate entities. She knew the algorithm but cancelling out 4 does not make sense to her.

Student 7 (I.8-1) drew the following pictures to explain why 1/2 equals 2/4. He used division and divided each of the numerator and denominator by 2.

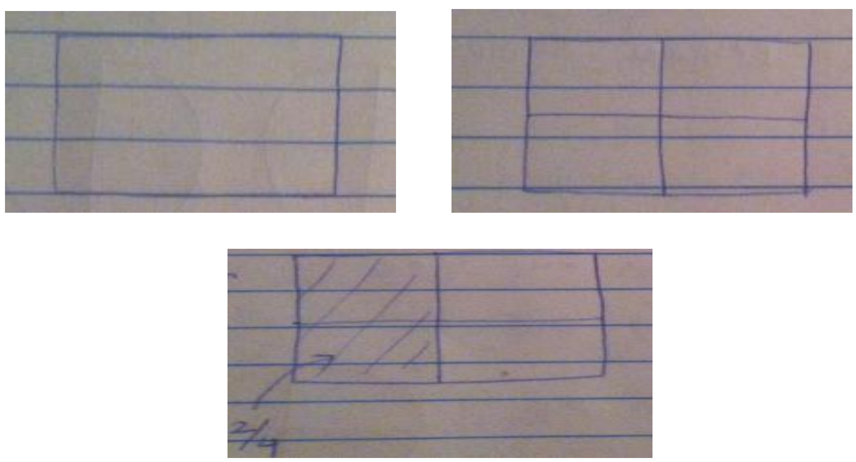


Figure 4.9. Student 7 partitioning a rectangle into four parts and showing 2/4.

Source: Scanned copy of student's work from email.

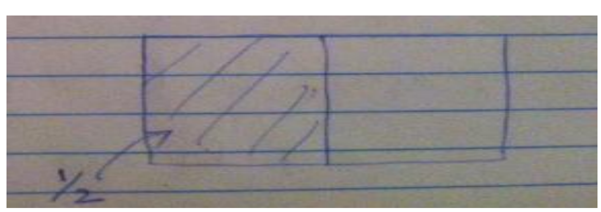


Figure 4.10. Student 7 showing 1/2.

Source: Scanned copy of student's work from email.

He used the concept of area to understand both represents the same quantity, however, he did not relate division in his picture. Use of division requires regrouping the parts.

The use of division might be hindering the understanding of these students. Students need to understand that when making equivalent fractions, multiplying creates smaller parts and division creates larger parts. The use of division might be more advanced than using multiplication, as division requires understanding of grouping parts together. For example, to show why $4 \div 2/ 6 \div 2 = 2/3$, we consider a rectangular bar partitioned into six equal parts. 4/6 will imply 4 parts of size 1/6 each. To understand the meaning of $4 \div 2/ 6 \div 2$, one needs to understand the meaning of division. One of the meanings of $A \div B$ is the number of groups that are formed when A objects are divided equally into groups with B objects in each group. Hence, $6 \div 2$ tells that 6 of the parts in the rectangle are divided equally with 2 parts in each group. Therefore, grouping two parts together, one can see the rectangle is now divided into 3 equal parts and the size of the parts are larger than the previous parts. In addition, $4 \div 2$ would denote 2 of the parts of the new grouping parts. A student who does not understand division in this way wouldn't be able to give this explanation.

Fractions as lengths: Student 3 (U.7), Student 6 (I.7) and Student 8 (I.8-2) have abstracted fractions as lengths. Conceiving of fractions as lengths, rather than solely as parts of wholes, is a significant achievement for students (Hackenberg, 2007; Steffe & Olive, 2010).

Fractions as operators: Student 3 (U.7), Student 4 (U.8), Student 6 (I.7), Student 7 (I.8-1) and Student 8 (I.8-2) have conceived of fractions as operators in the sense that operator "of" can be replaced by multiplication. For example, 2/3 of 6 is 2/3 × 6.

When solving the question 8 (see Appendix A), Student 3 wrote an equation 6= 3/4 L where L is the length of the other rectangle. Then she solved the equation and calculated that L was 8. She wrote another algebraic equation 4= 3/4 L, where L is the width of the other rectangle and calculated the width as 16/3. Her work indicates that she has conceived of fractions as operators as she has built the idea that "3/4 of an unknown length L" means multiplying L by three-fourths.

Student 3, Student 6, Student 7 and Student 8 have used multiplicative algebraic reasoning. They have used multiplicative algebraic equation to solve the task.

Use of equality symbol: I found one difference between the Indian students and the US students in the use of equations. While solving the 3/4 of the rectangle problem, Indian students have used equations and in their explanations, they have used the equality symbol, which was missing from the work of US students. All the Indian students showed knowledge of writing expressions involving equality symbols and solving an equation for an unknown value.

Though there can be multiple other valid interpretations possible for each of the students' work, through this study I was able to assess these students' mental operations and understanding to some degree.

CHAPTER 5

POSSIBLE PROBLEMS WITH NCERT'S SYLLABI AND TEXTBOOKS ON FRACTIONS

The educational system of any country depends heavily on the curriculum and the standards. Textbooks are usually written in concordant with the curriculum. The picture is the same for India. The National Council of Educational Research and Training (NCERT) advises and assists the Central Government and the state governments in India regarding educational practices and implementations. NCERT designs the syllabi for the Central Board of Secondary Education (CBSE). NCERT has given general directions to the textbook writers in their syllabi for mathematics. In its directives, NCERT has emphasized linking learning with real life experiences outside the classroom ("Syllabi", 2006). NCERT's syllabi recommend to writers that the language of textbooks should be easy enough for a child to understand and should be taken from the language that a child normally uses in daily life ("Syllabi", 2006). NCERT has also recommended that the textbooks should have plenty of visuals, comic strips, cartoons, narratives, puzzles and stories ("Syllabi", 2006). NCERT's syllabi emphasize developing concepts and ideas by observation of patterns and through exploration ("Syllabi", 2006). NCERT has suggested to the textbook writers that the concepts should not be arranged linearly ("Syllabi", 2006). The syllabi also highlight formulating and posing problems by children to their peers ("Syllabi", 2006).

My study of NCERT's syllabi and the textbooks on fractions observed many of the attributes described in NCERT's recommendations. Various chapters in the textbooks seemed to connect mathematics with real world activities. For example, chapters like "Building with Bricks" in grade 4 and "The Fish Tale" in grade 5 are some of the many that bring real life experiences into mathematics learning. In this chapter, I present some positive attributes and some possible discrepancies between NCERT's syllabi and the textbooks on fractions from grades 1 through 6.

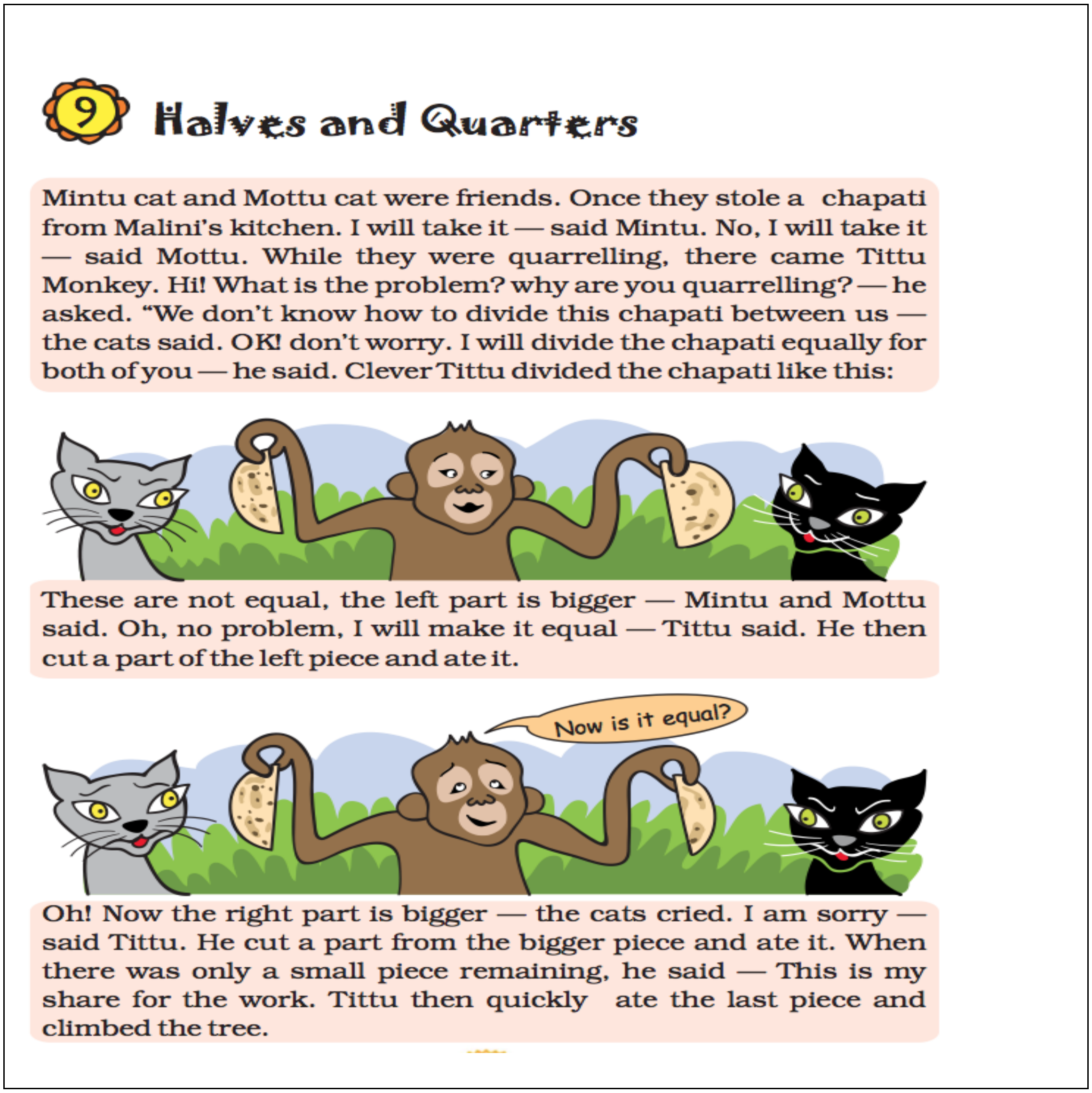
Positive attributes of NCERT's syllabi and NCERT's textbooks

My study reveals many positive characteristics of the NCERT's syllabi and the textbooks on fractions. The designs of the textbooks are very colorful. The use of colorful activities and tasks involving drawing patterns and figures would definitely attract students and would help to retain students' interest. The textbook writers have done a good job in bringing together the real life experiences of children and mathematics. Figure 5.1 shows an activity in grade 4 where brick patterns are introduced in the context of learning patterns and mirror halves.

The textbooks are written in a legible way with plenty of stories so that children would like to read them. I found many funny stories and puzzles in the textbooks. For example, halves are introduced in grade 4 by narrating a story about two cats and a monkey (Figure 5.2). I found another of many such interesting story problems in the textbooks. The story problem in grade 5 linked Indian history with mathematics (Figure 5.3). Here the task requires understanding fractions as operators.

Figure 5.1. Connecting real world and mathematics.

Source: “Textbook4”, n.d., p. 3.



9 Halves and Quarters

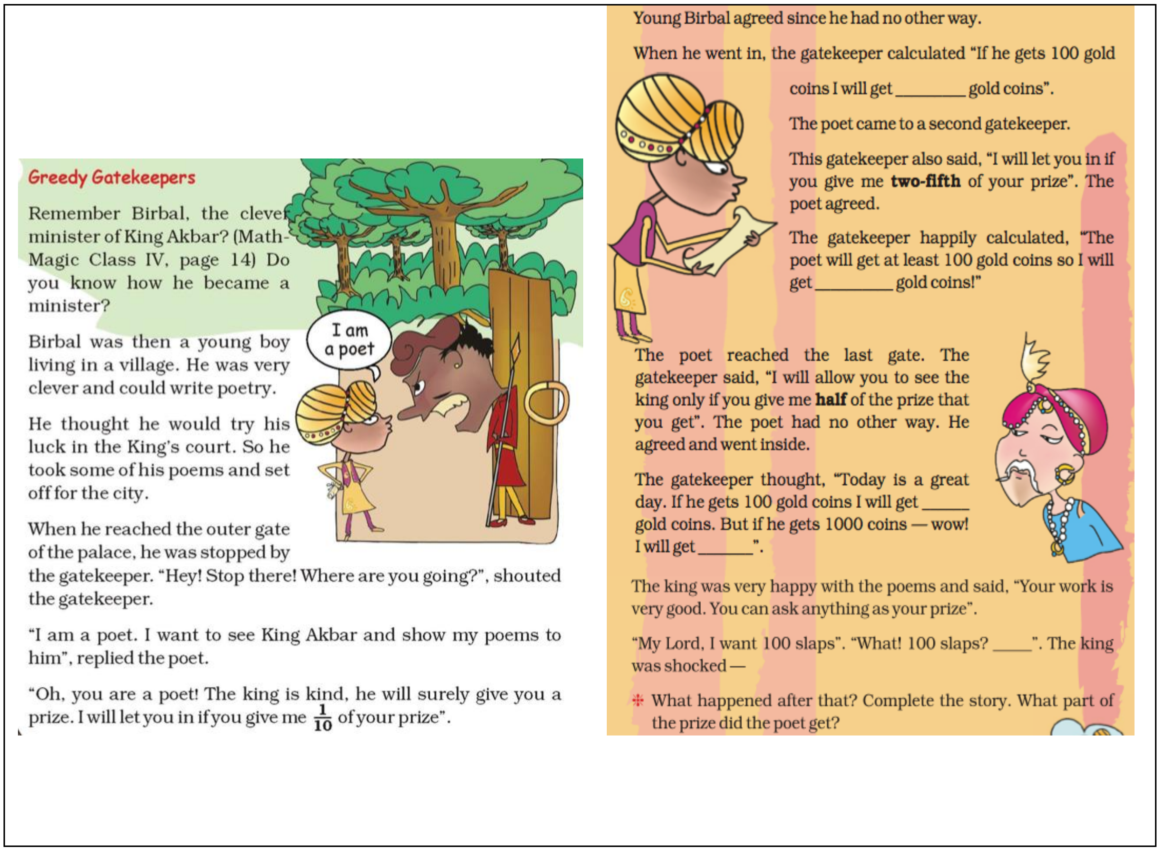
Mintu cat and Mottu cat were friends. Once they stole a chapati from Malini's kitchen. I will take it — said Mintu. No, I will take it — said Mottu. While they were quarrelling, there came Tittu Monkey. Hi! What is the problem? why are you quarrelling? — he asked. "We don't know how to divide this chapati between us — the cats said. OK! don't worry. I will divide the chapati equally for both of you — he said. Clever Tittu divided the chapati like this:

These are not equal, the left part is bigger — Mintu and Mottu said. Oh, no problem, I will make it equal — Tittu said. He then cut a part of the left piece and ate it.

Oh! Now the right part is bigger — the cats cried. I am sorry — said Tittu. He cut a part from the bigger piece and ate it. When there was only a small piece remaining, he said — This is my share for the work. Tittu then quickly ate the last piece and climbed the tree.

Figure 5.2. Story on halves in grade 4.

Source: "Textbook4", n.d., p. 94.



Greedy Gatekeepers

Remember Birbal, the clever minister of King Akbar? (Math-Magic Class IV, page 14) Do you know how he became a minister?

Birbal was then a young boy living in a village. He was very clever and could write poetry.

He thought he would try his luck in the King's court. So he took some of his poems and set off for the city.

When he reached the outer gate of the palace, he was stopped by the gatekeeper. "Hey! Stop there! Where are you going?", shouted the gatekeeper.

"I am a poet. I want to see King Akbar and show my poems to him", replied the poet.

"Oh, you are a poet! The king is kind, he will surely give you a prize. I will let you in if you give me $\frac{1}{10}$ of your prize".

Young Birbal agreed since he had no other way.

When he went in, the gatekeeper calculated "If he gets 100 gold coins I will get ________ gold coins".

The poet came to a second gatekeeper.

This gatekeeper also said, "I will let you in if you give me **two-fifth** of your prize". The poet agreed.

The gatekeeper happily calculated, "The poet will get at least 100 gold coins so I will get ________ gold coins!"

The poet reached the last gate. The gatekeeper said, "I will allow you to see the king only if you give me **half** of the prize that you get". The poet had no other way. He agreed and went inside.

The gatekeeper thought, "Today is a great day. If he gets 100 gold coins I will get _____ gold coins. But if he gets 1000 coins — wow! I will get ______".

The king was very happy with the poems and said, "Your work is very good. You can ask anything as your prize".

"My Lord, I want 100 slaps". "What! 100 slaps? ____". The king was shocked —

What happened after that? Complete the story. What part of the prize did the poet get?

Figure 5.3. Story based fraction task from grade 5.

Source: "Textbook5", n.d., pp. 55-56.

The tasks are chosen to engage students and are made interesting enough so that students could enjoy and play with the tasks. Another interesting feature of the syllabi is the repetition of the concepts throughout the grades. For example, mirror halves are repeated after its introduction in grade 3, they are seen in grades 4, 5, 6 and 7. Some chapters start with recapitulation of the materials learned in the previous grades with practice exercises whenever relevant to the topics. For example, fractions are formally introduced in grade 4 and the same definition is repeated in grades 5 and 6 with plenty of practice exercises. I also found many thought provoking puzzles in the textbooks. One of such examples is shown in figure 5.4.

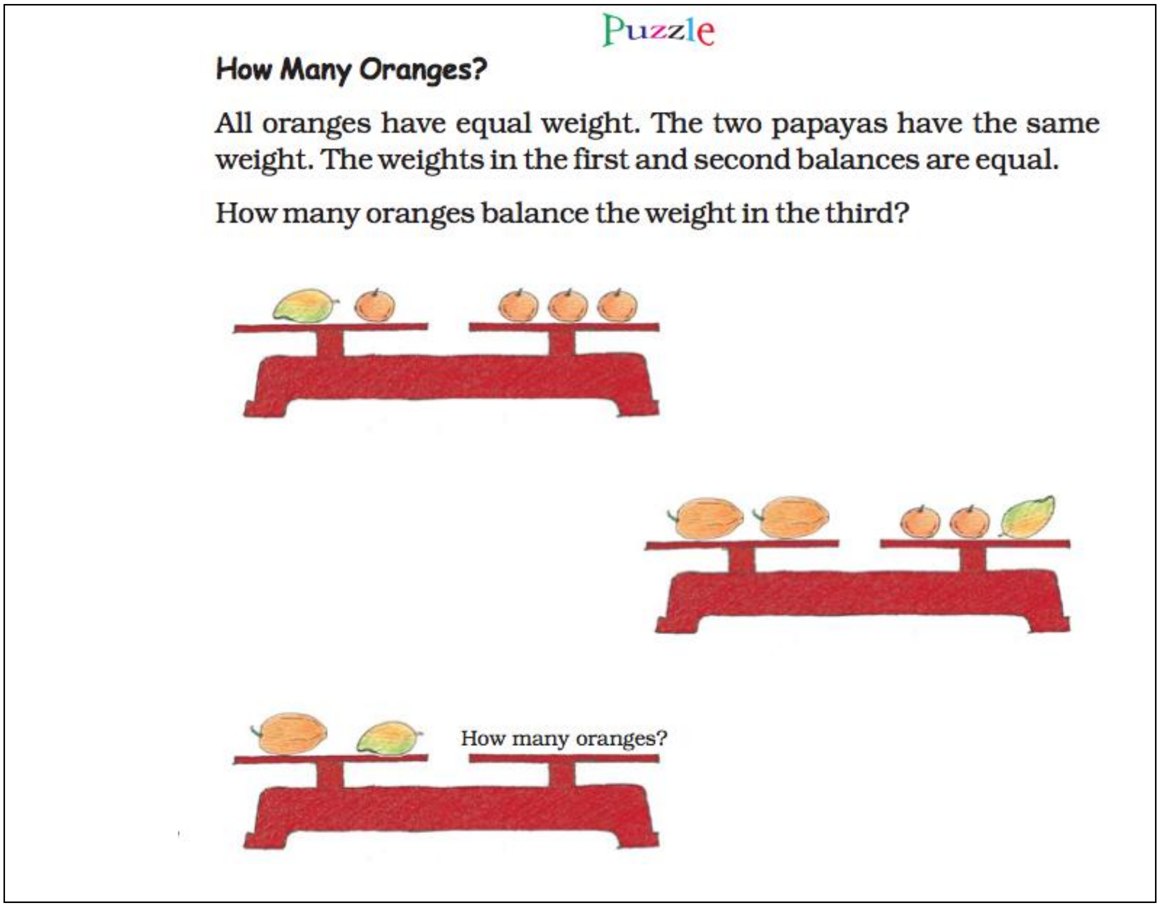


Figure 5.4. A puzzle from NCERT's grade 4.

Source: "Textbook4", n.d., p. 148.

I found many sophisticated thought provoking tasks in the textbooks. Fraction is defined in grade 4 as some parts out of a whole when the whole is partitioned into equal parts. However, the tasks in grade 5 require generating the whole from the given fractional part of the whole. These tasks are based on the inverse operation. By inverse operation, I mean that these tasks are structurally different; the operation involved is inverse (reverse) of the operation that is involved in the tasks where a whole is partitioned into equal sized parts and the parts denote a fractional part of the whole.

Apart from these positive attributes of NCERT's syllabi and textbooks, my study on fractions also brings forth some anomalies in the syllabi and the textbooks. One such anomaly is the introduction of the notation "*1/2*" in the third grade. The notation "*1/2"* is formally introduced in the fourth grade's textbook. Another such anomaly is the use of decimal notation in the context of money related tasks in the third grade. Decimals are introduced formally in the fifth grade. Next, I have portrayed the possible inconsistencies in NCERT's syllabi and the textbooks on fractions. I also report the comparison of NCERT's syllabi on fractions with the students' interview analysis results from chapter 4.

Discrepancies in the NCERT's syllabi and the textbooks

My study of NCERT's syllabi and the textbooks reveals some discrepancies. Here I outline the possible anomalies starting from first grade to sixth grade.

GRADE 1

NCERT's directive for textbook writers in grade 1 is to focus on non-uniform units for length measurement. Only non-standard length units are discussed throughout the textbook of grade 1. Length measurement of a table, a mat, a book, etc. using hands, human feet, pencil, and rod has been discussed in the textbook for this grade. I distinguish between non-uniform and non-standard length units as follows: I consider non-uniform length measurement as measuring a length using different units alternately.

For example, measuring the length of a table using a bunch of pencils of different lengths, all put end to end across the table at once. I consider non-standard unit for length measurement as a length unit for measuring the length of an object by using another object. For example, one can measure the length of a table by iterating a pencil. I associate non-standard units with uniform units, as the unit of measurement remains fixed while measuring. I report that the non-uniform units for length measurement are not discussed in the NCERT's textbook for grade 1. I find the syllabus and the textbook discuss two different concepts regarding length measurement in grade 1 and hence, they are not compatible with each other. This is my interpretation of the meaning of non-uniform and non-standard length units but NCERT may have used the term "non-uniform units" synonymously with "non-standard units".

GRADE 2

In this grade, non-standard units of length measurement are discussed. One of the non-standard units discussed here is a "fingers" unit. "Finger" units give an estimate of the length. I consider "fingers" unit as a non-uniform unit as different fingers have different lengths. For example, repeating index finger and middle finger one after another to measure the length of an object uses two different units. Measuring the length of an object with alternating different units is non-uniform units for length measurement. Hence, in terms of my interpretation of the syllabus, I conclude that the textbook writers are not consistent with the NCERT's syllabus. I view this as a mismatch between the NCERT's syllabus and the textbook for grade 2.

GRADE 3

My study finds that the word "*half*" is often used in the third grade. The term "half" has been used without being defined in this grade. In chapter 11 of third grade, the chapter named as "*Jugs and Mugs*", I found that the notation of "*1/2*" is used two times in the context of a task that I view as an inconsistency on the part of the textbook writers (Figure 5.5). This raises many questions. For example, what does the symbol "1/2" mean to the students? How do teachers explain "1/2" in the third grade? Another such example of use of "half" is shown in figure 5.6. Here the word "*half kg*" is used in the context of weight. Later in the same chapter, half-kilogram is explained as shown in figure 5.7. The word "*Half*" is again used in the chapter 10 of third grade called "*Play with Patterns*" (Figure 5.8). The frequent use of "*half*" including the notation in the third grade raises the following questions:

- Why is half not introduced formally in the third grade? The sequencing of concepts in the textbook is problematic. The use of the word "*half*" in the tasks in grade 3 without introducing the meaning of "*half*" until grade 4 is problematic in NCERT's syllabi.
- How well are the teachers prepared to explain and facilitate classroom learning in moving ahead of the syllabus? How effective is the teachers' training? If teachers are following NCERT's textbooks, how a teacher explains the concept to the students is important as research has revealed even the notion of simple fraction such as one-half varies widely in students (Watanabe, 1996). Watanabe showed the complicated nature of young children's fraction understanding. He interviewed a second-grade child. He found that adults commonly used the word

half to interact with children and through these interactions, children develop their own meanings for the word *one-half* (Watanabe, 1996).

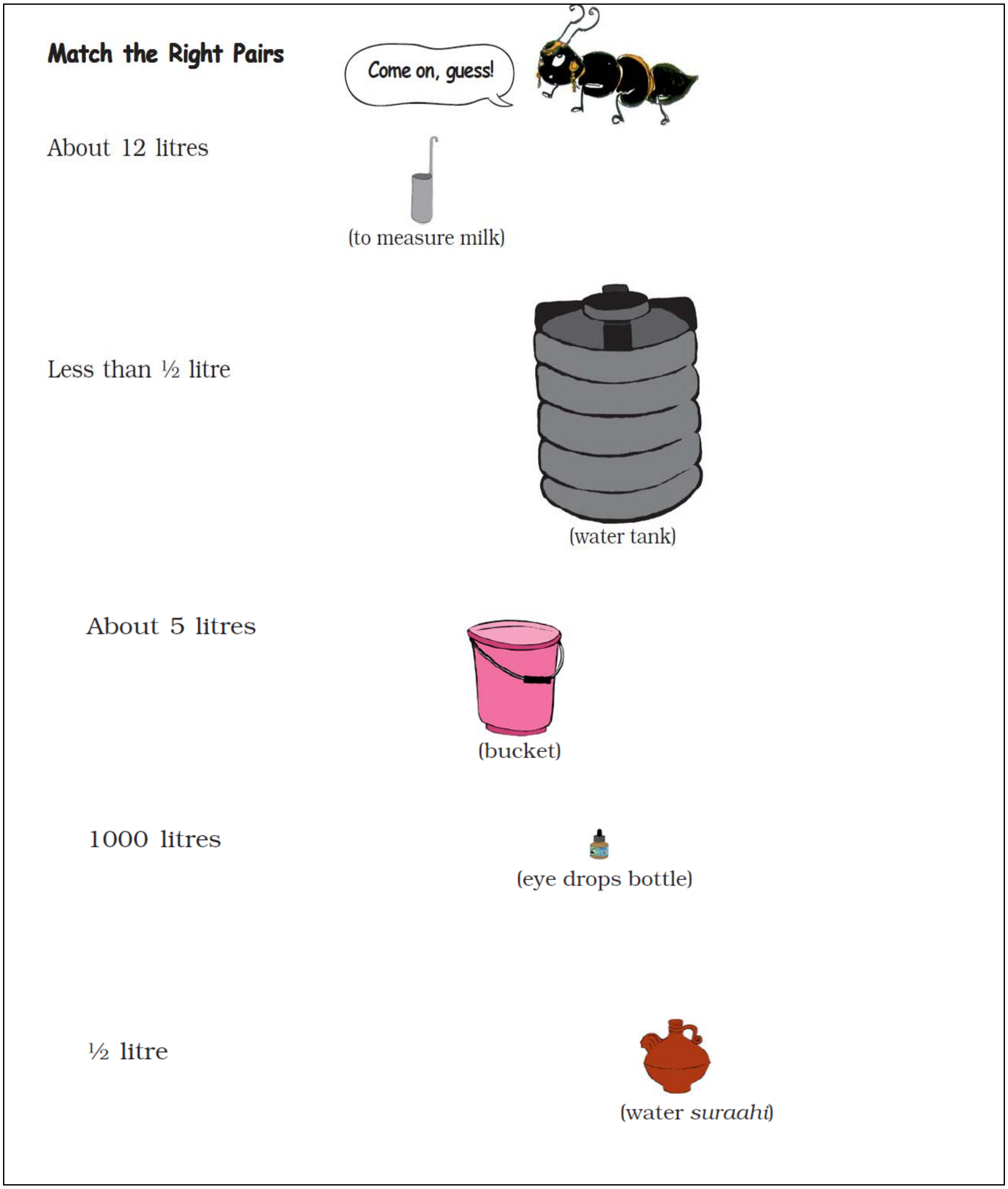
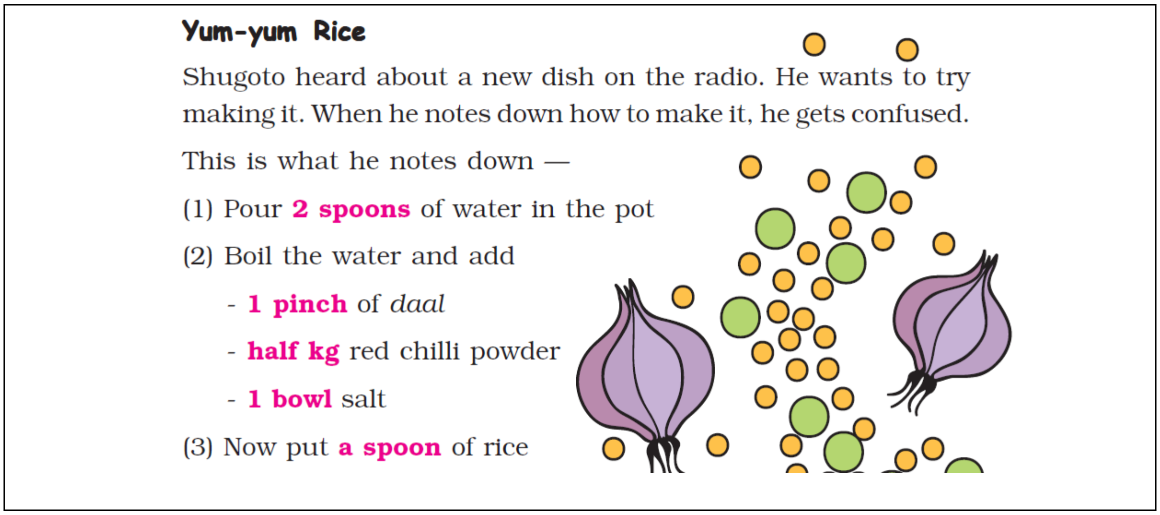


Figure 5.5. “1/2” in Chapter 11 of Grade 3.

Source: “Textbook3”, n.d., p. 156.



Yum-yum Rice

Shugoto heard about a new dish on the radio. He wants to try making it. When he notes down how to make it, he gets confused.

This is what he notes down —

(1) Pour **2 spoons** of water in the pot

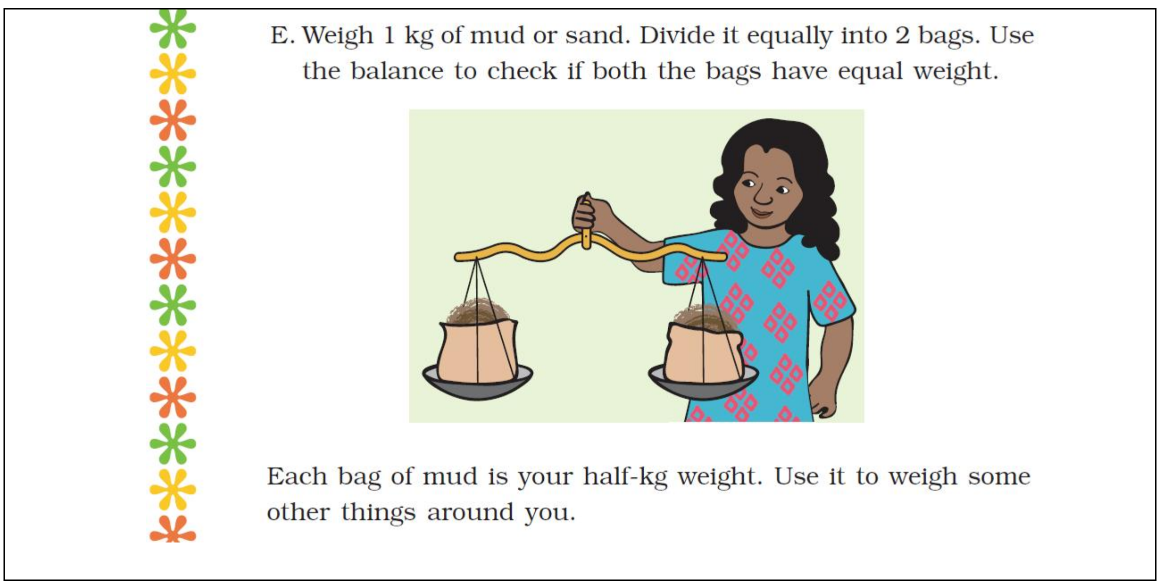
(2) Boil the water and add

- **1 pinch** of *daal*
- **half kg** red chilli powder
- **1 bowl** salt

(3) Now put **a spoon** of rice

Figure 5.6. “*Half*” in grade 3.

Source: “Textbook3”, n.d., p. 117.



E. Weigh 1 kg of mud or sand. Divide it equally into 2 bags. Use the balance to check if both the bags have equal weight.

Each bag of mud is your half-kg weight. Use it to weigh some other things around you.

Figure 5.7. Demonstrating half-kg weight in grade 3.

Source: “Textbook3”, n.d., p. 120.

Even and Odd Number Patterns

91	92	93	94	95	96	97	98	99	100
81	82	83	84	85	86	87	88	89	90
71	72	73	74	75	76	77	78	79	80
61	62	63	64	65	66	67	68	69	70
51	52	53	54	55	56	57	58	59	60
41	42	43	44	45	46	47	48	49	50
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

Half these numbers are in yellow. What patterns do you see in these numbers? Continue the same pattern and fill in the blanks:

Figure 5.8. "*Half*" in number patterns.

Source: "Textbook3", n.d., p. 151.

My study also reveals that decimal notation is introduced in the fifth grade and decimal addition is introduced in sixth grade in the NCERT's syllabi. However I found tasks in chapter 14: "*Rupees and Paise*" of grade 3 in the context of money involving decimals (Figure 5.9). The problem involves calculating the cost of the grocery items mentally. Here my concern is on the use of decimal notation in the tasks in grade 3 when decimals are introduced much later in the syllabus. I view this as an inconsistency in writing the textbook.

In the same chapter, Indian currencies, rupees and paise are introduced just before the aforesaid task. It is shown how two fifty paise makes one rupee and students are asked to calculate the amounts of money shown by the rupee notes and coins (Figure 5.10).

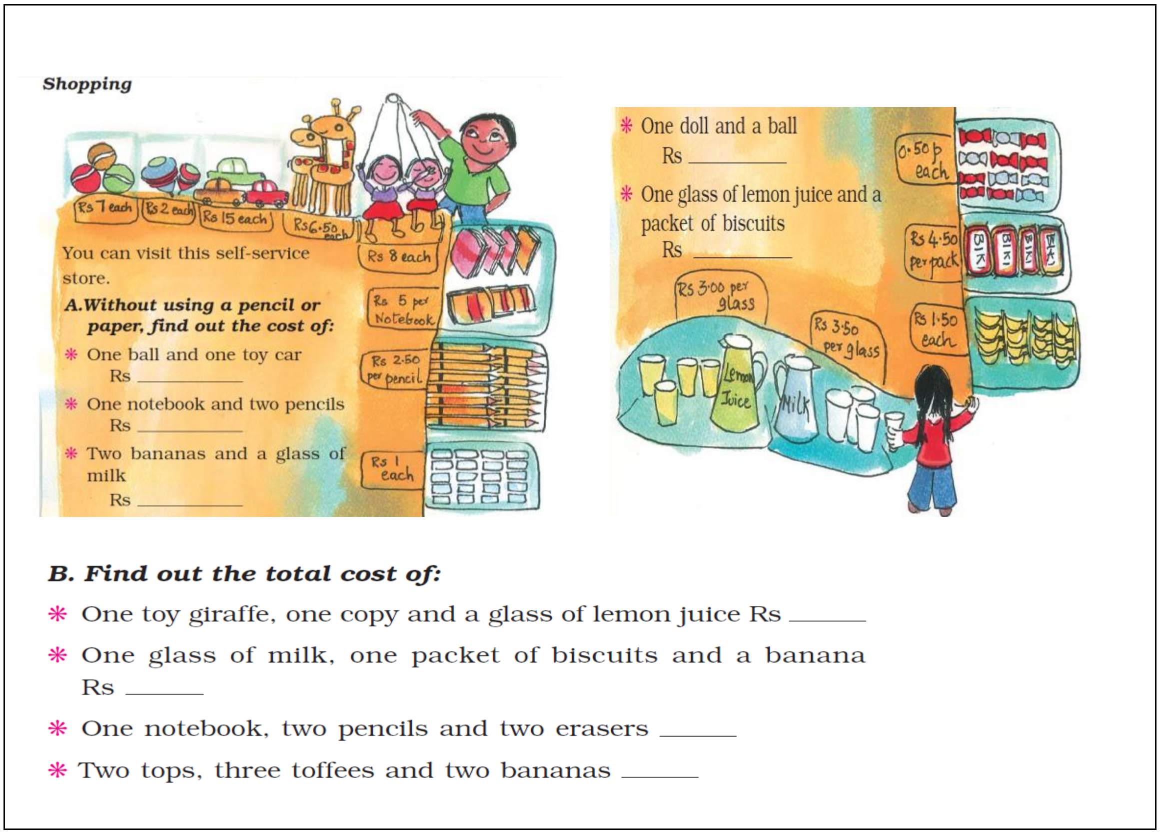
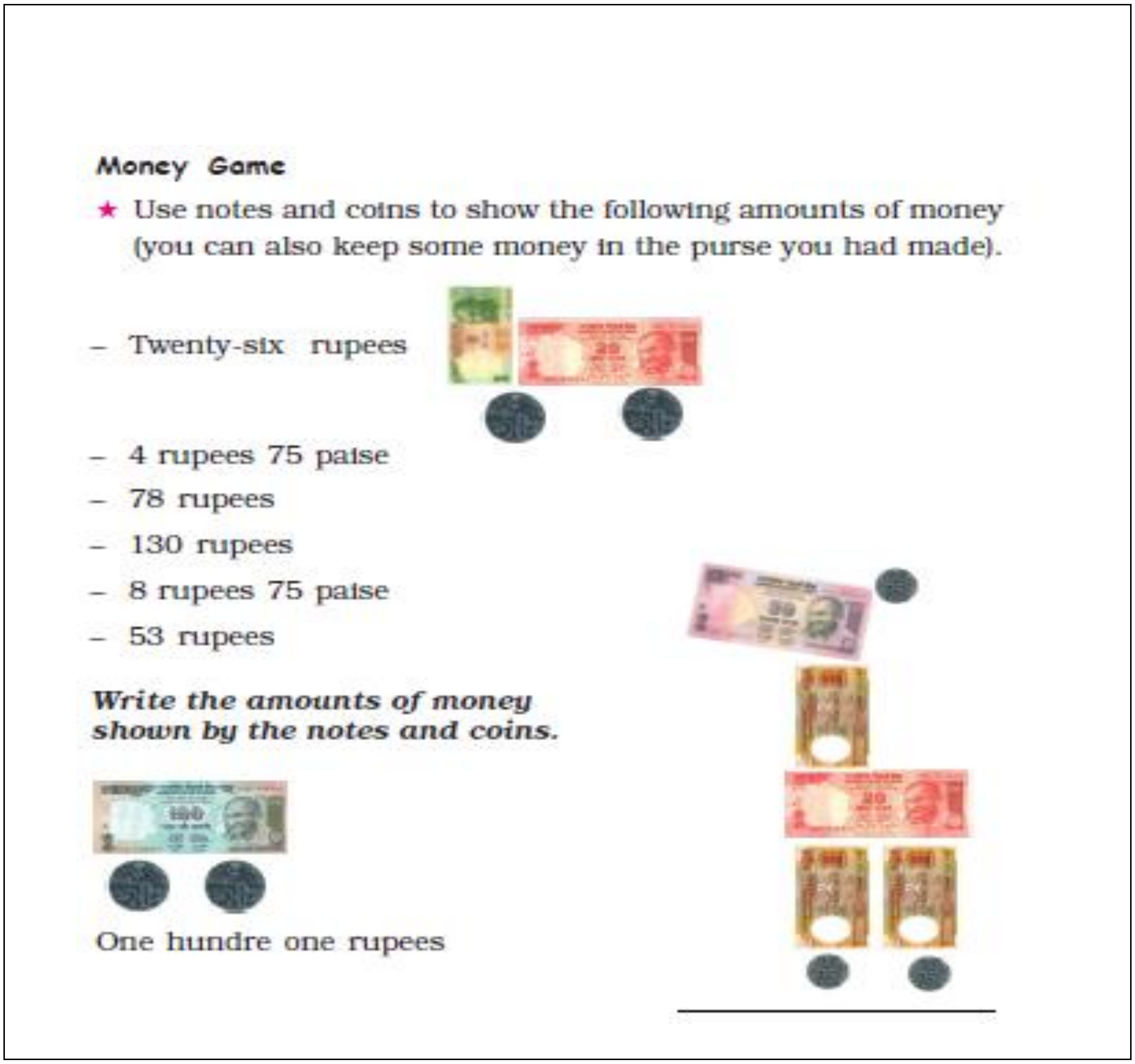


Figure 5.9. Use of decimals in grade 3.

Source: “Textbook3”, n.d., pp. 194-195.



Money Game

★ Use notes and coins to show the following amounts of money (you can also keep some money in the purse you had made).

– Twenty-six rupees

– 4 rupees 75 paise

– 78 rupees

– 130 rupees

– 8 rupees 75 paise

– 53 rupees

Write the amounts of money shown by the notes and coins.

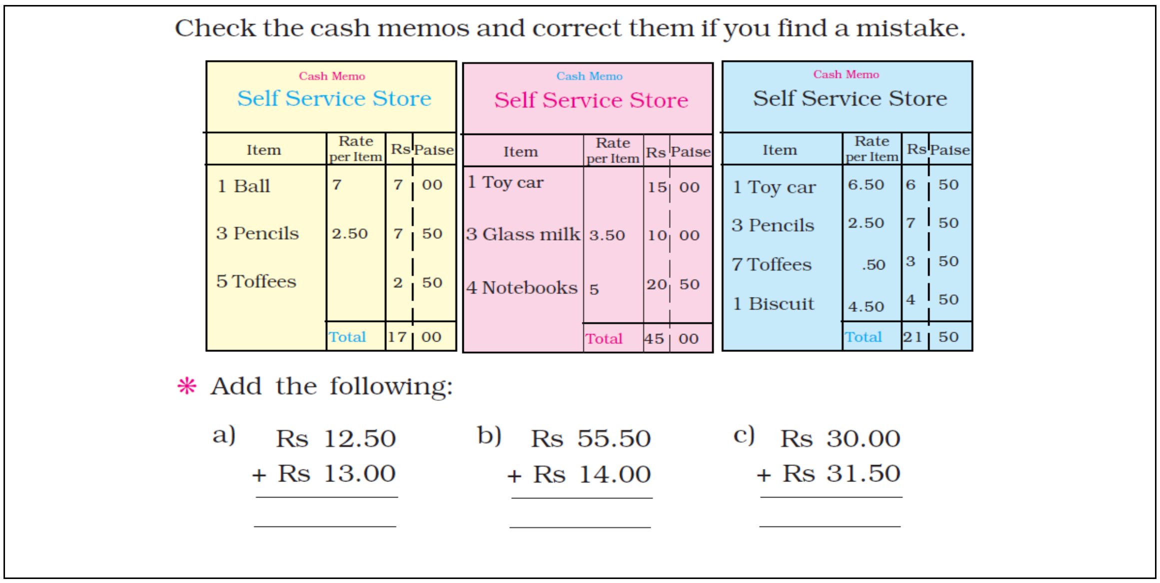
One hundre one rupees

Figure 5.10. Calculating amount of money from notes and coins.

Source: “Textbook3”, n.d., p. 192.

While adding up the coins, students are using their knowledge of two-digit whole number addition from the second grade. I acknowledge this as a good recapitulation of whole number addition. However, the use of decimal notation in the problem shown in figure 5.9 makes the problem much more complicated. If textbook writers could have used words in place of decimal notation, that could have kept the problem in the context of whole number addition. For example, one giraffe cost six rupees and fifty paise then what will be the cost of two giraffes? The child would make a rupee from two fifty paisa, carry the one rupee and add it to the two sixes.

I found some other tasks involving decimals (both decimal addition and decimal subtraction) in chapter 14 of grade 3 (Figure 5.11 and Figure 5.12).


Check the cash memos and correct them if you find a mistake.

Cash Memo
Self Service Store

Item	Rate per Item	Rs	Paise
1 Ball	7	7	00
3 Pencils	2.50	7	50
5 Toffees		2	50
	Total	17	00

Cash Memo
Self Service Store

Item	Rate per Item	Rs	Paise
1 Toy car		15	00
3 Glass milk	3.50	10	00
4 Notebooks	5	20	50
	Total	45	00

Cash Memo
Self Service Store

Item	Rate per Item	Rs	Paise
1 Toy car	6.50	6	50
3 Pencils	2.50	7	50
7 Toffees	.50	3	50
1 Biscuit	4.50	4	50
	Total	21	50

❋ Add the following:

a) Rs 12.50 + Rs 13.00

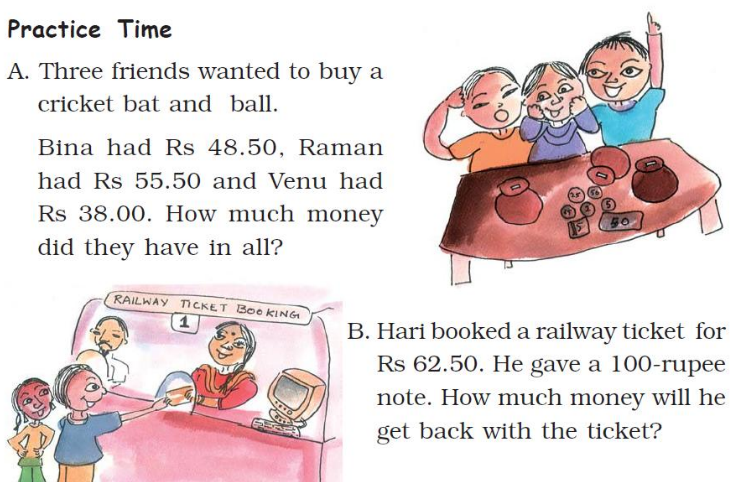
b) Rs 55.50 + Rs 14.00

c) Rs 30.00 + Rs 31.50

Figure 5.11. Addition tasks using decimals in grade 3.

Source: “Textbook3”, n.d., p. 196.

I consider the use of decimal notation throughout grade 3 as a problematic issue in NCERT’s textbooks. Many studies in mathematics education support that children learn concepts outside the classroom before they are introduced through formal instruction (Carpenter et al, 1989). For example, children learn about one-half and decimals when they start learning about amount of quantities and cost of an object respectively. Although the tasks in grade 3 are trying to connect mathematics with the real world, I found the use of decimals without explaining their meaning inappropriate. This unnecessary use of advanced notation puts a large burden on teachers. Teachers and students following the textbook in class need to understand the meaning of the notation. This made me question whether teachers are prepared enough to fill these gaps of the textbook writers. If yes, how well? If not, then it is hindering students' understanding.


Practice Time

A. Three friends wanted to buy a cricket bat and ball.

Bina had Rs 48.50, Raman had Rs 55.50 and Venu had Rs 38.00. How much money did they have in all?

B. Hari booked a railway ticket for Rs 62.50. He gave a 100-rupee note. How much money will he get back with the ticket?

Figure 5.12. Subtraction task using decimals in grade 3.

Source: "Textbook3", n.d., p. 197.

The next anomaly is the introduction of "Mirror Halves" in grade 3 before the introduction of fractions. Mirror halves are introduced as mirror images of objects. Examples are used where a paper is folded into two parts such that the two parts look exactly the same. This might create problems in students' understanding of halves later on if they become too rigid in thinking that halves are one part of a whole that is divided into two parts such that they are mirror images of each other. For example, if a rectangle is divided in half diagonally, students might not recognize the parts as halves as they are not mirror images of each other. In this example, a rectangle is cut through the diagonal and then one of the parts is rotated to fit with the other part, which involves rotational symmetry. Though "Mirror Halves" are important concept for understanding geometrical symmetry, I found the sequencing of mirror halves before introducing fractions might hinder students' understanding of fractions.

GRADE 4

Fraction is formally introduced in the fourth grade with a part-whole definition of fraction (Figure 5.13). One-half is explained in terms of equal shape. Here two parts are compared by putting one part on top of the other.

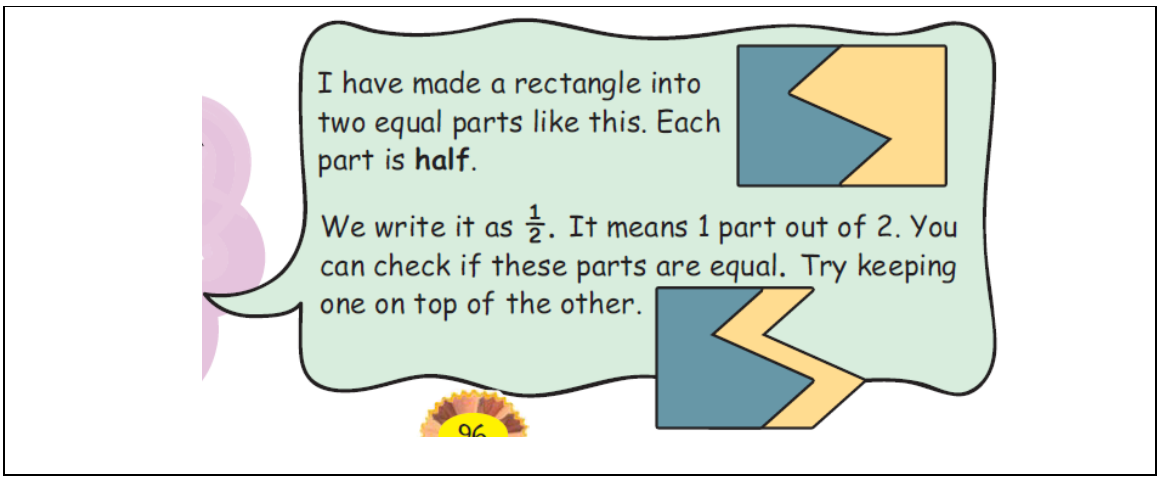


Figure 5.13. "1/2" in grade 4 of NCERT's textbook.

Source: "Textbook4", n.d., p. 96.

This explanation would definitely misguide the students, as they will try to recognize parts of the identical wholes based on the same shape. Watanabe (1996) mentioned that many students believe that after partitioning a whole, all the parts should be of the same shape. Figure 5.14 shows a square, which is partitioned, into eight parts with area of 2 square units each (Watanabe, 1996). Watanabe found that not only students but also many teachers enrolled in an elementary mathematics methods course thought that the partitioning in figure 5.14 did not show eighths, as the parts are not equal in shape. Therefore focusing on equal shapes for parts of the whole might create problem in students understanding of fractions.

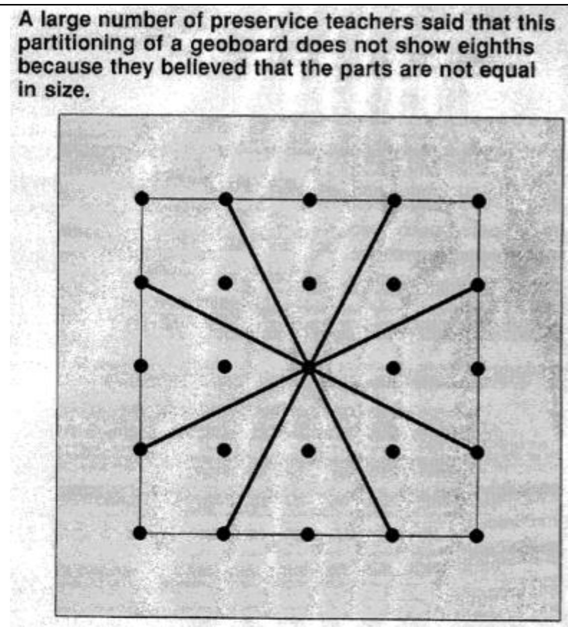


Figure 5.14. Partitioning in eighths.

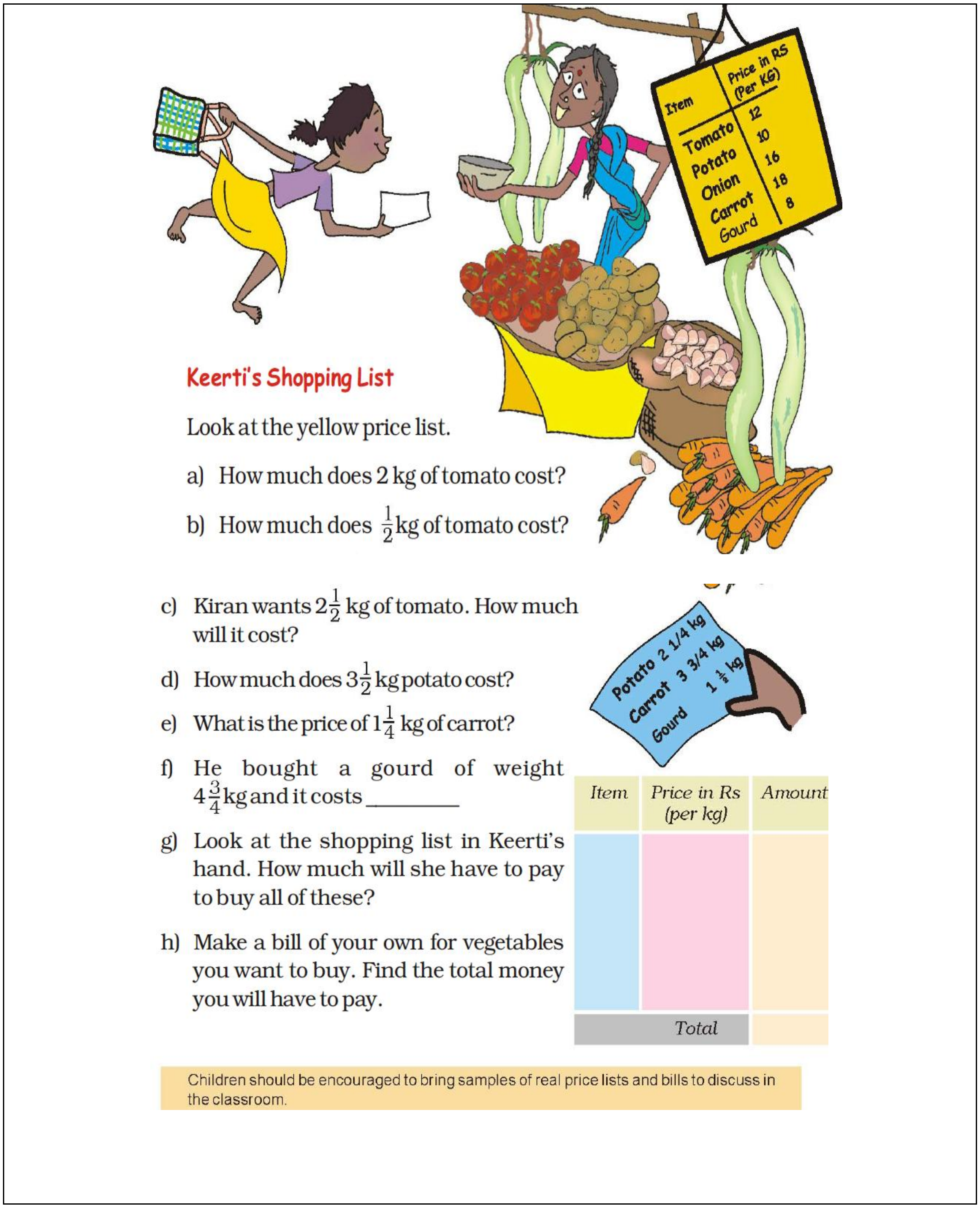
Source: Watanabe, 1996, p. 463.

In grade 4, I found that fraction is combined with money related tasks. Tasks are presented such as, "*if one kilogram of potato cost 8 rupees, then how much does half, quarter and three-fourths kilograms of potatoes cost*" (Figure 3.8). I view this problem as multiplication of a fraction by a whole number, but multiplication of a fraction by a whole number is introduced in grade 7.

I found charts called “Chapati Charts” in grade 4. There are two issues with these smart charts at this grade. First a continuous model is presented with some fractional parts shaded and then the task demands connecting the continuous chart with discrete objects. I found it difficult for children to link the continuous whole with the discrete objects. Taking a fraction of a discrete set can be viewed as using the “operator construct” of fraction (Behr et al, 1983). In addition, Lamon (2007) found that the operator construct was not good of a starting point as some of the other constructs (p. 659). Generally, the operator construct is viewed as more advanced (Kieren & Southwell, 1979). At this grade when fraction is just introduced in terms of part-whole definition of fraction, Chapati charts could bring confusion among children in linking continuous with discrete objects. Figures 3.11 and 3.12 demonstrate some Chapati chart tasks that involve multiplication of a fraction with a whole number. I consider this as a discrepancy in the textbook as fraction multiplication is introduced in grade 7. At this point, I am unsure how fraction multiplication with whole numbers is addressed in the classroom.

GRADE 5

Mixed fraction is introduced in grade 6 in the NCERT’s syllabus. However, mixed fraction is used in some of the tasks in the textbook for grade 5. One such example is from chapter 4: “*Parts and Wholes*”, which is shown in figure 5.15. This task involves multiplication with mixed numbers. One of the other tasks from the same chapter is shown in figure 5.16.



Keerti's Shopping List

Look at the yellow price list.

a) How much does 2 kg of tomato cost?

b) How much does $\frac{1}{2}$ kg of tomato cost?

c) Kiran wants $2\frac{1}{2}$ kg of tomato. How much will it cost?

d) How much does $3\frac{1}{2}$ kg potato cost?

e) What is the price of $1\frac{1}{4}$ kg of carrot?

f) He bought a gourd of weight $4\frac{3}{4}$ kg and it costs ________

g) Look at the shopping list in Keerti's hand. How much will she have to pay to buy all of these?

h) Make a bill of your own for vegetables you want to buy. Find the total money you will have to pay.

Item	Price in Rs (per kg)	Amount
	Total	

Children should be encouraged to bring samples of real price lists and bills to discuss in the classroom.

Figure 5.15. Mixed fractions in grade 5.

Source: "Textbook5", n.d., p. 69.

Practice time

1) Raheem's journey

Raheem has to travel $1\frac{1}{4}$ km to reach school. What distance does he travel to go to school and come back home?

Figure 5.16. A task involving mixed fraction in grade 5.

Source: "Textbook5", n.d., p. 70.

Mixed numbers are used again in this grade in chapter 13: "*Ways to Multiply and Divide*" (Figure 5.17). The use of mixed fractions in the tasks in grade 5 is inappropriate when the concept of mixed numbers is not yet developed.

c) Neelam is a worker in Haryana. If she works for 2½ months on the farm, how much will she earn?

Figure 5.17. Mixed fraction in a task in grade 5.

Source: "Textbook5", n.d., p. 175.

In this grade, I found various turns, 1/2, 1/3, 1/4, and 1/6, in the textbook (Figures 3.22, 3.23, 3.24). The tasks discussed in the textbook require understanding of line of symmetry and rotational symmetry. Rotational symmetry is introduced in NCERT's grade 7. This non-linearity of sequencing of the concepts might introduce confusion in children's learning of fraction. This set of tasks might confuse students as not only line symmetry but also rotational symmetry is involved here. Here one task is to determine how a rectangle will look like after a one-third turn (Figure 3.23). In this grade when children are still grappling with the fraction concepts, these turns might confuse them. In addition, I am not sure how teachers teach these concepts; teaching will vary from one teacher to the other and could bring misconceptions in children.

GRADE 6

Improper and mixed fractions are introduced in this grade. Figure 3.29 demonstrates how improper fractions and mixed fractions are discussed in the textbook. Two methods are discussed: one is to consider improper fraction as a whole and a fractional part of the whole. The other is to partition each of the quantity, the number of such quantity as suggested by the numerator, into the number of parts as suggested by the denominator and distributing each part again from each quantity corresponding to the number represented by the denominator. The second approach can be considered as an explanation of why A divided by B is equal to the fraction A/B from a "how many in each group" perspective. Immediately after the explanation in the textbook, computations for converting improper fractions to mixed numbers are discussed. More emphasis is given on how to convert improper fractions to mixed numbers rather than understanding the meaning of improper fraction. I found only conversion tasks at the end of this section (Figure 5.18). This creates a problem, as students might not understand the meaning of improper fractions. According to Van de Walle (1998), "in the fourth National Assessment of Educational Progress, about 80 percent of seventh graders could change a mixed number to an improper fraction, but fewer than half knew that 5 1/4 was the same as 5 + ¼" (p. 244). This shows children can successfully demonstrate procedural fluency over conceptual understanding.

EXERCISE 7.2

1. Draw number lines and locate the points on them :

   (a) $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4}$ (b) $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{7}{8}$ (c) $\frac{2}{5}, \frac{3}{5}, \frac{8}{5}, \frac{4}{5}$

2. Express the following as mixed fractions :

   (a) $\frac{20}{3}$ (b) $\frac{11}{5}$ (c) $\frac{17}{7}$

   (d) $\frac{28}{5}$ (e) $\frac{19}{6}$ (f) $\frac{35}{9}$

3. Express the following as improper fractions :

   (a) $7\frac{3}{4}$ (b) $5\frac{6}{7}$ (c) $2\frac{5}{6}$ (d) $10\frac{3}{5}$ (e) $9\frac{3}{7}$ (f) $8\frac{4}{9}$

Figure 5.18. Tasks involving improper and mixed fractions.

Source: "Textbook6", n.d., p. 141.

Equivalent fractions are revisited in grade 6. I found a task involving discrete units in the textbook in this grade (Figure 5.19). The task is misleading to the students as equivalent fractions should be generated from the same whole. It is important to understand why a fraction A/B is equivalent to a fraction (A•N)/(B•N) by using visual fraction models with attention on the same whole and focusing on how the number and size of the parts differ even though the two fractions themselves are of the same size ("CCSS", n.d.). The task involves different wholes and the question asked is: "which fractions are equivalent". The task suggests finding equivalent ratios instead of equivalent fractions because all the wholes are different. The first four are equivalent ratios, 4:12 = 3:9 = 2:6 = 1:3.

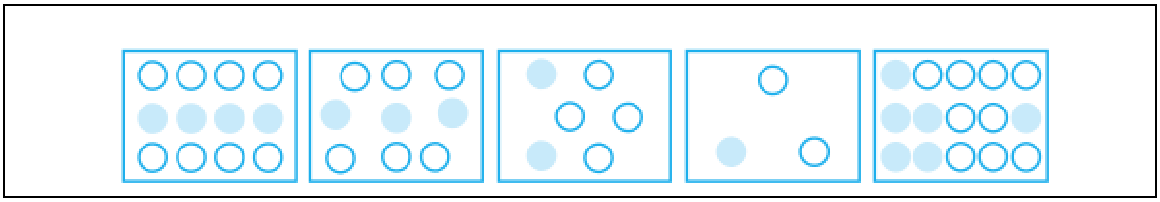


Figure 5.19. A task for equivalent fractions.

Source: “Textbook6”, n.d., p. 146.

This task presents a problem in understanding and explaining equivalent fractions. This directs students in believing that two fractions are equivalent if they can form the other fraction by multiplying or dividing the numerator and the denominator of one of the fractions with the same number without paying attention to the whole. It is reinforcing that the same whole is not important in the construction of equivalent fractions. Figure 5.20 shows all the fractions in the top row are equivalent as they have the same whole. The first one is 4/12 shaded where each single item is considered as a unit. In the second one, two items are grouped together to form a unit and the second figure represents 2/6 shaded. In the third figure, one unit is composed of four items and the fraction of shaded dots is 1/3. None of the top row fractions shows equivalence with the fraction in the second row, which represents 1/3, as they have different wholes. Equivalent fractions require repeated partitioning (and regrouping) a continuous (and discrete respectively) whole. It is important to understand that equivalent fractions are generated from the same whole either by partitioning or by regrouping in case of a continuous or discrete whole. Hence, I find NCERT’s approach towards equivalent fractions as problematic for children’s fractional knowledge.

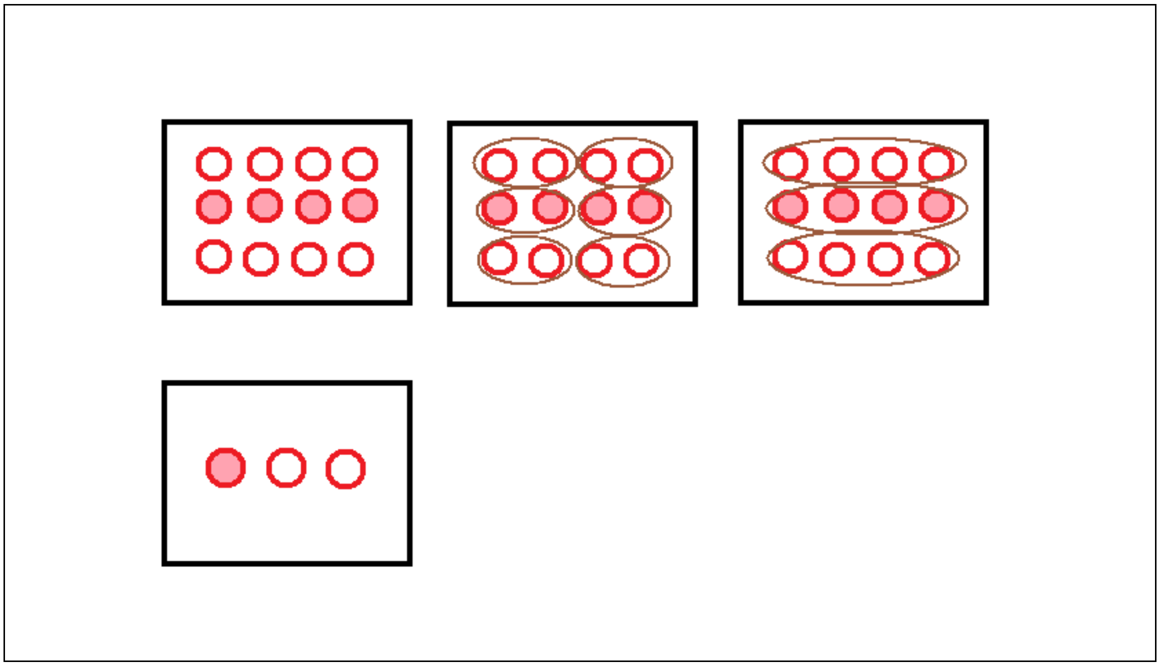


Figure 5.20. Comparing equivalent fractions.

Comparison of NCERT's syllabi with students' interviews

The analysis of the students' interview reveals some loopholes in NCERT's syllabi on fractions. NCERT's syllabi focus on fractions based on the part-whole definition of fraction. The part-whole definition of fraction describes a fraction a/b (a< b) as "a" parts out of "b" parts. The analysis reveals that all the Indian students have internalized the part-whole definition of fraction. All of them could partition a whole, or could view a collection of object as a whole. They could shade or choose the appropriate parts to denote a proper fraction. Out of five students, only one student was able to explain what is meant by an improper fraction. Others haven't constructed improper fraction yet. The use of part-whole definition of fraction throughout the grades puts restriction on students' construction of improper fractions.

Research has revealed constructions of improper fractions are puzzling for students. The part-whole definition cannot be extended to improper fractions, as larger parts cannot be fitted in a smaller whole (Hackenberg, 2007; Steffe, 2002). My study reveals that the concept of iteration to get fractions is not a focal point in the NCERT's syllabi. Student 5 could use iteration to plot 5/4 on the number line (Figure 3.31) but she did not construct a structural relationship between the part being iterated, the whole, and the result (Hackenberg, 2007). She could not view 4/4 as a whole. She perceived a whole number meaning of 5 units for 5/4 rather than a fractional meaning. Three other students interpreted the improper fraction as a whole and parts of a whole but failed to draw a picture of what it is representing. All the five students were comfortable in converting the improper fraction to a mixed fraction but only one of them knew what it really meant. This indicates that many children are using a rule without understanding the meaning that is relatively easy to construct (Van de Walle, 1998). This presents a gap in the NCERT's syllabi on fractions. Teaching only the part-whole interpretation of fraction exposes a child to a very limited interpretation of the notation (Lamon, 1999), which is evident from the students' interviews.

The five Indian students demonstrated excellent procedural skill in performing the task of fraction addition in comparison to the four US students. However, on probing why the rule for fraction addition works, four out of five students said, "*It is the rule. We are taught that way*". NCERT's syllabi seem to focus more on building procedural skills rather than constructing conceptual understanding. This is a big loophole in the syllabi as McCloskey and Norton (2009) stated that overemphasizing procedural fluency with fraction operations (adding, subtracting, multiplying, and dividing) bypasses the development of important conceptual understandings. As Hiebert & Carpenter (1992) as cited in Van de Walle (1998) said:

> Procedural knowledge plays a very important role in learning and doing mathematics. Algorithmic procedures help us do routine tasks easily and thus free our minds to concentrate on more important tasks…. Doing endless long-division exercises will not help a child understand what division is. From the vantage of learning mathematics, the question of how procedures and conceptual ideas can be linked is much more important than the usefulness of the procedure itself. (p. 27)

As seen in this study, students are learning by rote memorization having no idea of why the rules worked. This study unveils that students do not possess the conceptual knowledge to support their procedure. According to Van de Walle (1998), focusing on fraction rules rather than conceptual understanding presents two significant problems. First, none of the rules helps students to think why the rules work. Second, these procedural skills will be temporary constructs, without meaning they cannot be permanent. As Van de Walle mentioned, "When taken as a group of rules, the procedures governing fraction computation become similar and confusing" (Van de Walle 1998, p. 261). For example, a student cannot remember whether to take a common denominator or just add the denominator while adding two fractions as seen in the case of Student 2 (U.6).

My study reveals that all the nine participants in the research knew the rule to show two fractions are equivalent but the conceptual understanding was missing. I presented the task of explaining why 2/3 equals 8/12. Either students multiply the top and the bottom of 2/3 by 4 or they divide the top and bottom of 8/12 by 4 to say 2/3 equals 8/12. The difference in concept and rule in equivalent fractions is the following (Van de Walle 1998, p. 253). "The concept: Two fractions are equivalent if they are representations for the same amount or quantity; if they are the same number. The rule: To get an equivalent fraction, multiply (or divide) the top and bottom numbers by the same non-zero number". Since "the rule or algorithm for equivalent fractions has no intuitive connection with the concept, students can easily learn and use the rule in exercises without any idea of how the fractions are related" (Van de Walle 1998, p. 253). Therefore, the overemphasis of procedural skills in the NCERT syllabi on fractions suggests a problem in the education system in India.

Student interviews on word problems revealed that students seemed to lack understanding of units irrespective of the two countries. Out of nine students, only two students (Student 1 from USA and Student 9 from India) were able to identify the correct word problems from the given list of problems having answer as (1/2 + 1/3) and (1/2 - 1/3). Though at this point it is not clear whether the seven other students simply picked the fractions from the problem without paying attention to the units or they just guessed the answers, as the students were aware of that question being the last question of the interview. Whatever is the case, from my study of the NCERT's syllabi and the textbooks, I found that there is an urgent need for developing the "units" concept.

One of the tasks in the interview involving "units" was: Given a whole consisting of three discs, what is the fraction the third picture (Figure 5.21) representing? Student 1 (U.5) was confused. He gave one-fifth as an answer. Student 2 (U.6) and Student 5 (I.6) could not perform this task. Student 3 (U.7)'s response was "*It is ¾ /3. That circle is one whole circle and its three-fourths of it (pointing to the third picture in Figure 5.21). Again the whole unit is three circles, so I put it over 3*". Student 4 (U.9)'s response was "*It is three-*

fourths, three out of four which is 75%". After thinking for a long time Student 6 (I.7) said, "*It is 75% of 1 unit*".

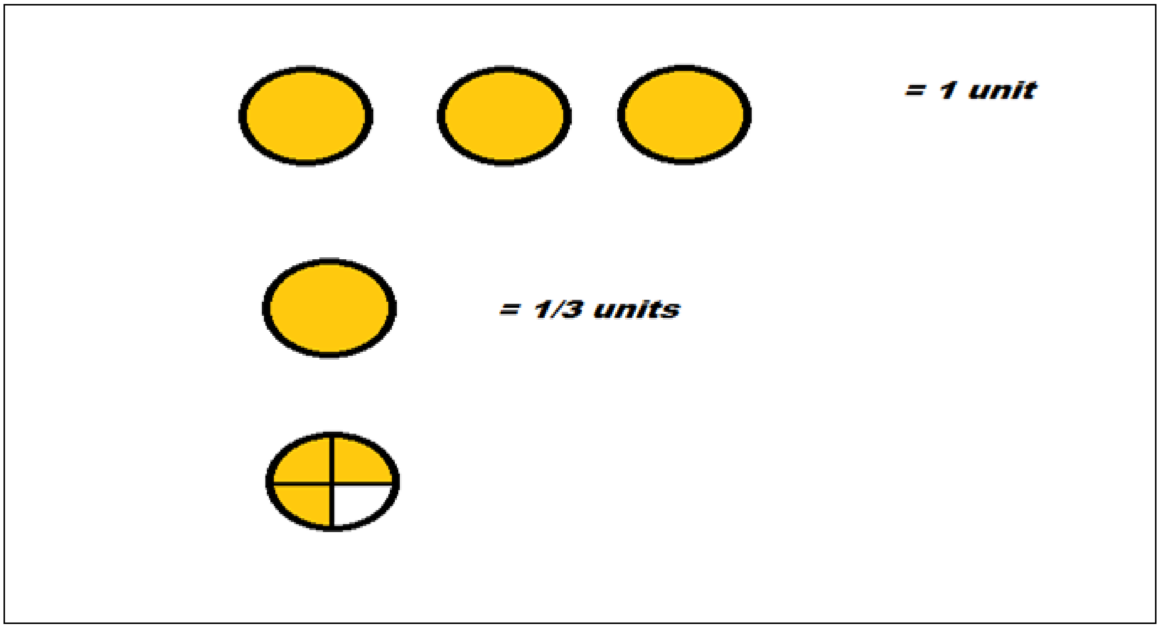


Figure 5.21. Fraction task involving units.

Source: Lamon, 1999.

Student 7 (I.8-1) explained his solution in two different ways:

"Three circles is one unit. The value of one circle is one by three. So, in one circle, three by four is shaded. So, the value is (1/3) X (3/4). So the answer is one by four as we cancel 3. Another method is to divide three circles into four equal parts (Figure 5.22). *We have 12 parts (counting aloud one, two, three, ..., twelve) and three of the parts are shaded. So, the value is three by twelve and we can reduce it to one by four".*

Student 8 (I.8-2)'s response was: "*one circle is one-third units. Now we divided the whole circle into 4 equal parts. Divide 1/3 by 4. Each part is now one by twelve. And 3 of them* *are shaded. So we have three parts of it that are shaded. So, we have three twelfths of it. We can reduce it to the lowest term, so it is representing one by four unit"*.

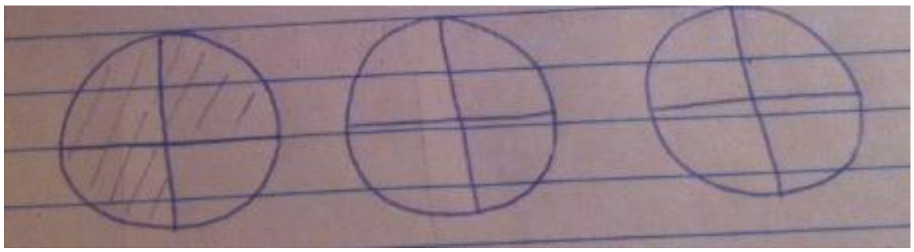


Figure 5.22. Student 7 divides the three circles into four equal parts.

Source: Scanned copy of student's work from email.

Student 9 (I.9)'s response was "*It is 3/4 units. No it is 1 by 4 unit (pointing to third picture of figure 5.21). It is 1-3/4 which is 1/4 units"*.

Although accidentally Student 9 (I.9)'s answer was correct, her explanation was wrong. She did not consider the given three circles as a unit. She treated a single circle as a whole and the shaded portion represent the three-fourths of the whole. Therefore, her response was (1 - 3/4) for the unshaded portion. Although Student 7 (I.8-1) and Student 8 (I.8-2) completed the above task successfully, they could not solve the word problems. This agrees with the available research that elementary and middle school students face difficulties with word problems and story context problems are found to be much more challenging than non-contextual problems (Cummins et al., 1988). It is interesting that Student 1 (U.5) and Student 9 (I.9) have trouble in solving the above task but they were able to focus on the individual units in the word problems.

Why do fractions seem to be so difficult? In the early days of school, a child learns to count by associating one number to each object in the collection (Lamon, 1999). "The unit "one" always referred to a single object" (Lamon 1999, p. 22) for the child. However, in fractions, a unit might be a single object (a simple unit) or a collection of objects (a composite unit) (Lamon, 1999). In addition, the unit keeps on changing. In figure 5.21, three circles are representing one unit. In another task, we can take two circles as one unit. If we further partition that one unit into equal parts, we use new numbers to represent the parts of that unit. Even the same picture might be representing two different numbers. As shown in (Lamon, 1999), perceptual clues are not reliable to students. What looks like the same amount can actually be represented by different numbers (Lamon 1999, p. 22). Figure 5.23 shows how two different numbers represent the same amount.

In addition, "there is not a unique symbol to refer to part of a unit. The same part can be referenced by different names" (Lamon 1999, p. 23).

For example, the left figure is 3/4 of the whole (Figure 5.24). Also, it can be 6/8 if we partition the whole further.

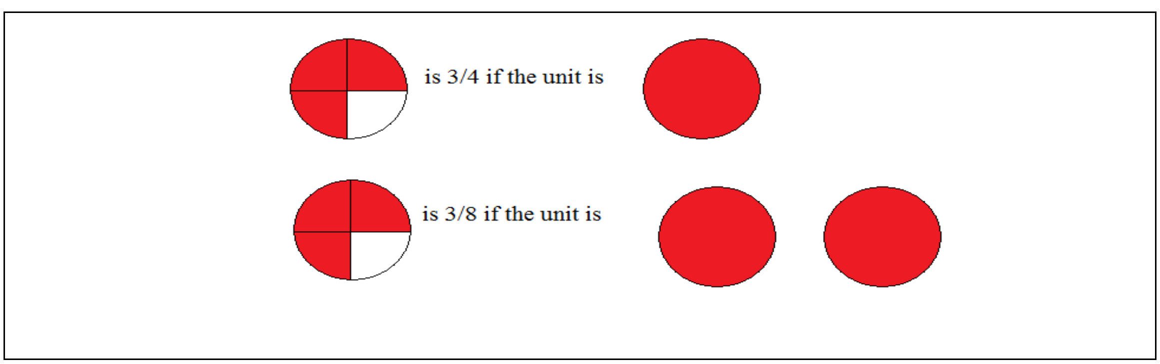


Figure 5.23. Same amount representing different numbers.

Source: Lamon, 1999, p. 23.

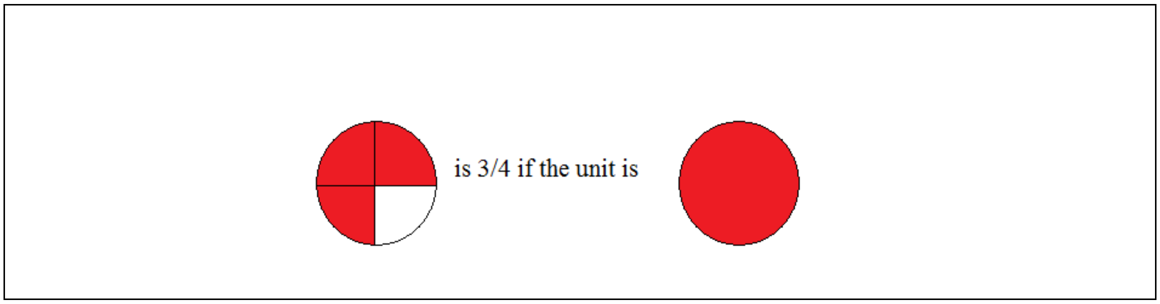


Figure 5.24. 3/4 of the whole.

Source: Lamon, 1999, p. 23.

The main confusion among students with fractions is the use of different types of units (Lamon, 1999). If the unit is a single cookie and the child buys four more cookies, then the child will have five one-units (or five units) but if she purchases a pack containing five cookies, then she will have one composite unit containing five cookies called a five-unit (Lamon, 1999). One five-unit and 5 one-units are very different. “If it is difficult for children to think about composite units or units that contain more than one item, then it is more difficult for them to interpret operations on these units” (Lamon, 1999, p. 23). Lamon (1999) showed an example: “Taking 1/4 of 4 single granola bars (4 one-units) is different from taking 1/4 of a box of granola bars (1 four-unit). In the first case, you get one granola bar, but in the second case, you get 1/4 box” (Lamon 1999, pp. 23-24). According to Lamon (1999), “fraction instruction has failed to put a proper emphasis on the unit. Many people were introduced to fractions by dividing a single pizza. Teachers and textbook authors did not realize that by always using the same unit- one-pizza-children were getting the idea that a unit was always a single pizza” (Lamon 1999, p. 31). My study found that NCERT’s syllabi have focused more on simple units rather than composite units. Further research is needed to understand student's thinking on unitizing whole and unitizing collection. Can a student who has constructed one type of unitizing go back and forth between unitizing whole and unitizing collection?

My study reveals a gap in NCERT's syllabi and the textbooks on fractions. First is the use of the part-whole definition of fraction only to define fraction from grade 4 to grade 7. There is no focus on the iteration concept either. This puts restriction on students' construction of improper fractions. NCERT focuses on understanding the conversion of improper fractions to mixed fraction that represents a whole, and some fractional parts of it. The second loophole is the overemphasis on developing procedural skills rather than in understanding and developing concepts. The third one is the use of notations before they are formally introduced in the respective grades. However, I agree with NCERT on sequencing the concepts spirally, the textbooks' presentations as discussed in this chapter leave students unprepared to make sense of the concepts and procedures. The fourth loophole is the absence of story-based problems for fraction operations emphasizing different units. These are the possible problems with the NCERT's syllabi, and the textbooks that could interfere with children's learning of fractions.

CHAPTER 6

SUGGESTIONS TO NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

The current chapter enunciates some suggestions to the National Council of Educational Research and Training (NCERT) to improve the current standards on fractions in the school curriculum. In chapter 4, I documented some issues with the present textbooks on fractions and my findings from the interviews are consistent with the loopholes I found in the textbook analysis on fractions. In this chapter, I attempt to propose some suggestions to improve the current situation in school mathematics on fractions.

Suggestions to NCERT

My study of the NCERT's syllabi and the textbooks on fractions reveals some problematic areas. I present some suggestions to improve the current problems that I encountered in my study.

Fractions in lower grades: Formally, fraction is introduced in grade 4 in the NCERT's syllabus. However, words like halves are used in the third grade; even the fraction notation is being used in grade 3. In this situation, fractions (the words such as *halves*, *quarters*) could be introduced in geometry as early as in grade 1 or 2. Those words can be explained by partitioning a circle or a rectangle into two or four equal shares. Those shares can be explained as half or quarter of the circle (or rectangle) ("CCSS", n.d.). Children learn the word halves, quarters in their day-to-day lives. Bringing those in the mathematics syllabus in early grades would help students to understand the meaning of fractions better. If the word half and the notation 1/2 are used in the third grade, then it would be advantageous for the students to introduce fractions formally in grade 3.

Three types of fraction models: Models must be used in different grade levels for better development of fractions. There are three different fraction models (Figure 6.1). Focus should be given on developing tasks that could relate all the three fraction models.

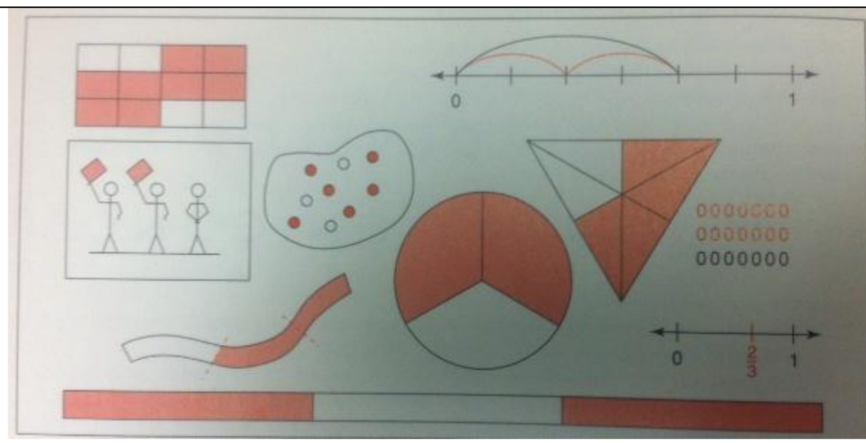


Figure 6.1. All the pictures representing the same amount.

Source: Van de Walle, 1998, p. 238.

The three models are as follows:

- *Set model of fraction:* In this model, the whole is a collection of objects and a subset of the collection is the fractional part of the whole (Van de Walle, 1998). For example, 2 marbles in a set of 6 marbles denote one-third. Various tools can be used here, such as any type of counter can be used (Van de Walle, 1998). This model is challenging for children as they have to group a collection of units to form a composite unit, which requires two levels of units construction. Moreover using this model, one can consider the whole as some number of units where each unit is some number of discrete individual units. This way of operating involves three levels of units construction. In addition, the set model involves the multiplication of a fraction with a whole number. For example, the task of partitioning 32 discrete identical objects into fourths involves multiplication of ¼ with 32 or in other words division of 32 by 4. According to Van de Walle (1998), set models build connection between fractions and ratio concepts. However, in the Common Core State Standards for Mathematics (CCSSM), fractions and ratios are separated until grade 6. In CCSSM grade 6 standards, the two concepts, fractions and ratios, are connected via unit rates and multiplicative comparisons.

- *Length model of fraction:* Length models of fraction are based on comparing lengths of line segments. Here a line segment is considered as a whole and segments of the line segments are the fractional parts of the whole. Fraction strips, strip diagrams, and number line can be used for the length fraction model.
- *Area model of fraction:* Area models play an important role in the development of fraction. Here a surface or region (continuous unit) considered as a whole is subdivided into smaller parts and each part is compared with the whole (Van de Walle, 1998). Many children believe that a fraction can only be defined when the parts of the whole are of equal size. Many children do not believe that area remains the same when a shape is partitioned and rearranged to form a new shape (Watanabe, 1996). Describing fractions by partitioning a whole into equal parts would stress that all the parts have to be of the same size. Therefore, use of area

model and partitioning a region in equal areas not equal shapes would reinforce the fact that parts of identical wholes might have different shapes. For area models of fraction, pattern blocks, geoboards, grid paper, dot paper could be used. Figure 6.2 shows a fractional part of the whole in the dot paper, where the shaded part is one-fifth of the whole. This example shows that the fractional parts of the identical whole are congruent even though they are oriented differently.

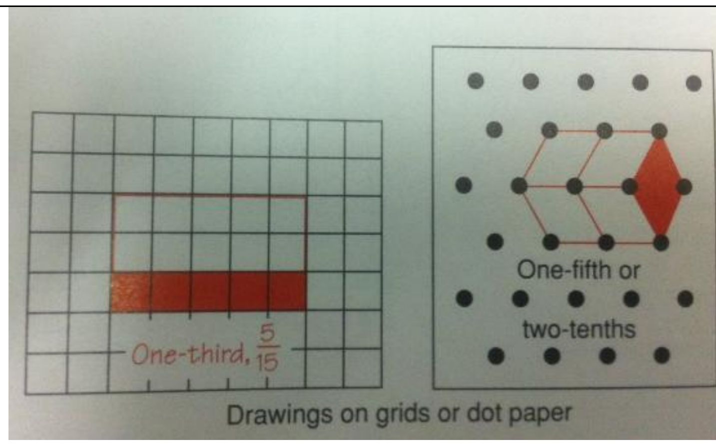


Figure 6.2. Using grids and dot paper for area model.

Source: Van de Walle, 1998, p. 239.

From my study of the NCERT's syllabi and textbooks on fraction, I found more emphasis on partitioning a whole into equal size parts. NCERT should bring up area models of fractions in the syllabus, which would help students to understand that different parts of the same whole representing the same fraction might not be congruent. Figure 6.3 shows a rectangle partitioned into four parts. A child focusing on congruent parts would not recognize the parts as one-fourth of the whole. All the four parts represent one-fourth of the whole as they have areas of 1/4LW square units or another way to explain this is to divide each of the four triangles into two smaller triangles as shown in Figure 6.4. All of those smaller triangles are congruent.

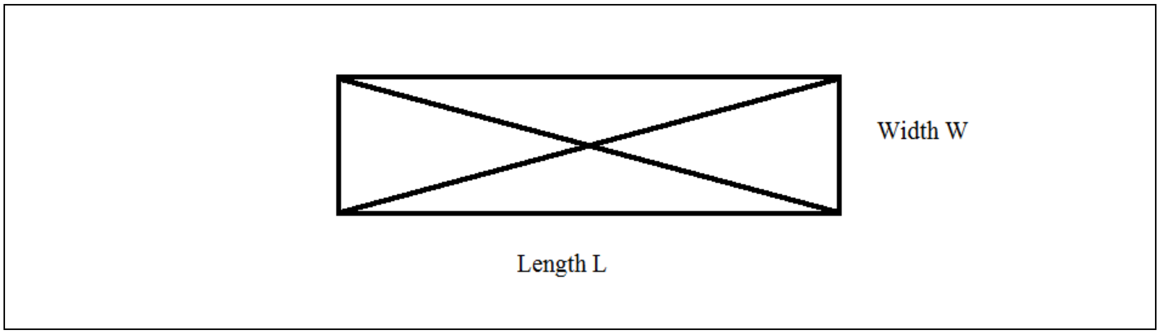


Figure 6.3. Demonstrating one-fourth of the rectangle.

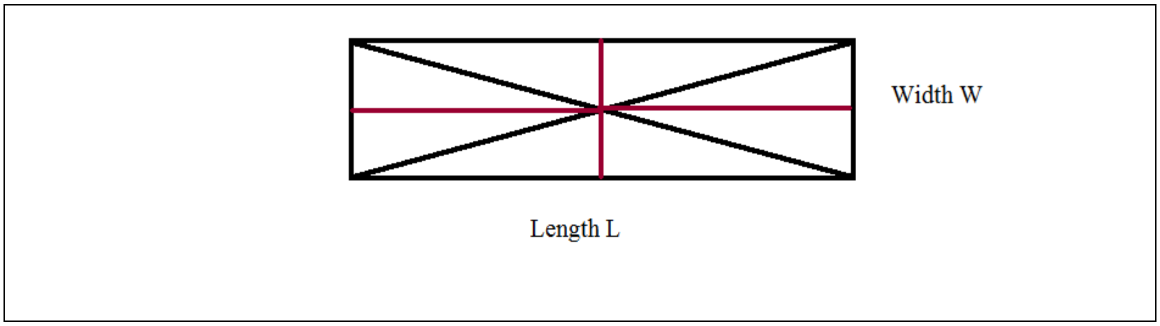


Figure 6.4. Picture showing the smaller triangles are congruent.

NCERT should focus on guiding students why the equal sized parts of the whole represent the same fraction. It is not because they look the same with same size but they have equal areas. My study found some examples using set model of fraction in grade 5. There should be more examples on set models of fractions as they are important in building up the concept of composite units. Examples from set models could be used when working with a fraction times a whole number. I found that number line is introduced in grade 6 in the NCERT's syllabus. I suggest the use of paper strips to understand a fraction in grade 5 and that can be extended to a number line in grade 6.

Iterations and improper fractions: My study shows that improper fractions in the NCERT's textbook get their meaning from mixed fractions. NCERT focuses on the conversion of improper fraction to mixed fractions. There is another meaning of improper fraction in terms of iteration. Instead of introducing two definitions, one for proper fraction and other for improper fraction, only one definition can serve for the meaning of fraction, both proper and improper. A fraction a/b can be explained as a parts of size 1/b when a whole is partitioned into b equal-sized parts. This simply means we get a/b by iterating the part 1/b, a times.

Mirror halves after fractions: NCERT introduces mirror halves in grade 3 and fractions in grade 4. It would be beneficial for the students if mirror halves could be introduced later after the introduction of fraction concept. It will reduce the misconception as discussed in chapter 5 in children.

Visual models for comparing fractions: NCERT focuses more on procedural fluency in computing the equivalent fractions. As discussed in chapter 5, I found some misleading tasks in NCERT's textbook on equivalent fractions. NCERT should focus on constructing equivalent fractions with the same whole. The emphasis should be on using fraction models to understand why two fractions are equivalent rather just focusing on multiplying or dividing the numerator and denominator of the fraction by the same number to generate equivalent fractions. Various fraction models can be used to develop conceptual understanding (Van de Walle, 1998). Figure 6.5 shows area, set and length models that can be used to explain equivalent fractions.

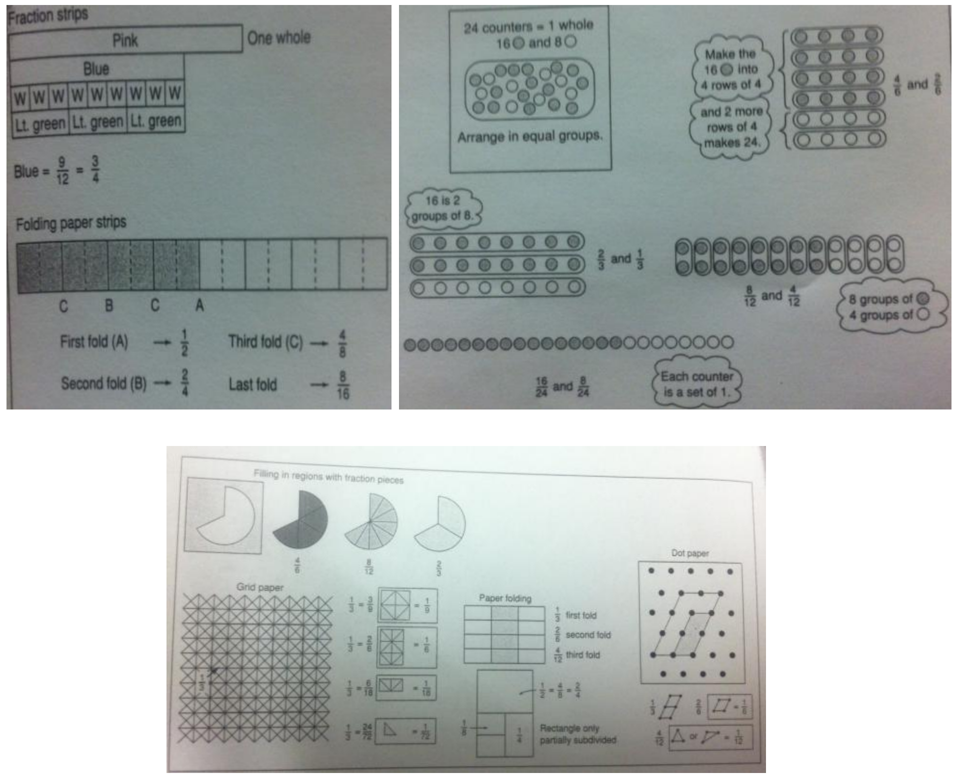


Figure 6.5. Area, set and length models for equivalent fractions.

Source: Van de Walle, 1998, pp. 254-255.

Role of division in equivalent fraction construction: When teaching equivalent fractions, the rules of multiplying or dividing the numerator and denominator are taught in the school without conceptualizing the concept of multiplication and division. Students need to understand that multiplying creates smaller parts in the whole and division creates larger parts in the whole. Use of division is much more advanced than using multiplication, as division requires understanding of grouping parts together.

For example, to show why 4 ÷ 2/ 6 ÷ 2 =2/3, we consider a rectangular bar partitioned into six equal parts. 4/6 will imply 4 parts of size 1/6 each. To understand the meaning of 4 ÷ 2/ 6 ÷ 2, one needs to understand the meaning of division. One meaning of A ÷ B is the number of groups that are formed when A objects are divided equally into groups with B objects in each group. Hence, 6 ÷ 2 tells that 6 of the parts in the rectangle are divided equally with 2 parts in each group. Therefore, grouping two parts together, one can see the rectangle is now divided into 3 equal parts and the size of the parts are larger than the previous parts. In addition, 4 ÷ 2 would denote 2 of the parts of the new grouping parts. Therefore, it is beneficial for students to understand conceptually the rule of multiplying the numerator and denominator first, then the rule for dividing both numerator and denominator and then interchange the operations while doing the same task on equivalent fractions. NCERT should focus on developing tasks that not only ask to generate equivalent fractions using rules but also to draw diagrams to justify their choices. Tasks on equivalent fractions should be developed so that one need to argue using both multiplication and division models.

Appreciation for the role of units in fractions: My study of the NCERT's textbooks reveals that not much effort is given on the understanding of the units of a fraction. In the addition and subtraction word problems, fractions with same units are chosen. That leaves students to misinterpret that they can just rely on the fractional numbers regardless of the units. In the students' interview, there was a question to identify if the two figures represent the same amount (Figure 6.6). All the five Indian students said, "*They represent the same amount which is 1/2*". NCERT should focus on the understanding of fraction relative to a whole. In addition, focus should be given on units that can help unify reasoning for solving many word problems with whole numbers as well as with fractions.

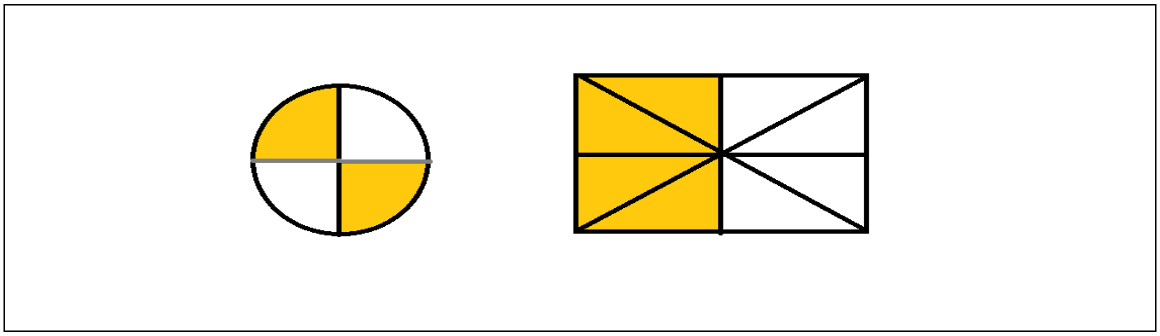


Figure 6.6. Half in two different wholes.

Source: Lamon, 1999, p. 59.

Word problems involving addition and subtraction of fractions focusing on respective fractional wholes: My study of NCERT's grades 6 and 7 textbooks reports that word problems involving addition and subtraction of fractions use the same whole. Using the same whole in the word problem paralyzes children's thinking. It develops the understanding that no matter what, if a problem involves fraction addition or subtraction, a child simply picks up the fractions and applies the fraction rules to add or subtract. No problem was found in the NCERT's textbook that focuses on the understanding that each fraction is considered with respect to a particular whole. Students' interview portrays the same picture. Students just found the keywords and picked the fractions to just add and subtract the fractions, without paying attention to the respective wholes. One example of a thought provoking addition word problem is: "½ of the land in the Heeltoe County is covered with forest. 1/3 of the land in the adjacent Toejoint County is covered with forest. What fraction of the land in the two-county Heeltoe-Toejoint region is covered with forest?" (Beckmann, 2011, p. 61). This problem cannot be solved by adding 1/2 + 1/3 as there is no information on the relative sizes of the two counties and the two counties are different wholes. NCERT should bring this type of thought provoking problems, which will solidify the understanding of fraction addition and subtraction in children by focusing more on the wholes that a fraction is implying.

Use of estimates rather than concrete procedural rules: "Estimation is more than a skill or an isolated topic. It is a thinking tool that needs to be emphasized during instruction so that students will learn to develop algorithmic procedures and meaning for fraction operations" (Johanning, 2011, p. 97). A sense of whole number helps in understanding the relative sizes (Van de Walle, 1998). A child understands that 10 is smaller than 20, 20 is smaller than 100 and so on. "An analogous familiarity with fractions can be developed by comparing fractions to 0, 1/2, and 1" (Van de Walle, 1998, p. 249). NCERT can develop tasks that involve estimating fractions close to zero, one-half and one. Some tasks could be to ask students to write a fraction that is close to one and then writing another fraction that is even closer to one and incorporate discussion of why the student thinks the second fraction is closer to one than the first fraction (Van de Walle, 1998). These tasks would help in comparing two fractions. Instead of working with specific rules in comparing fractions, using various fraction models with estimates could help in the learning of fractions. Figure 6.7 shows some tasks that can be used to find estimates of the amount shown (Van de Walle, 1998). According to Van de Walle (1998), "the ability to tell which of two fractions is greater is another aspect of number sense with fractions" (p. 250). Introducing rules early before students develop informal ideas about relative sizes of various fractions hinders the understanding of the familiarity of the number sense of the fraction size (Van de Walle, 1998). Estimating tasks could help students in understanding the comparison of two fractions based on their relative fraction sizes rather than just using the rule of cross multiplication to decide which fraction is bigger than the other.

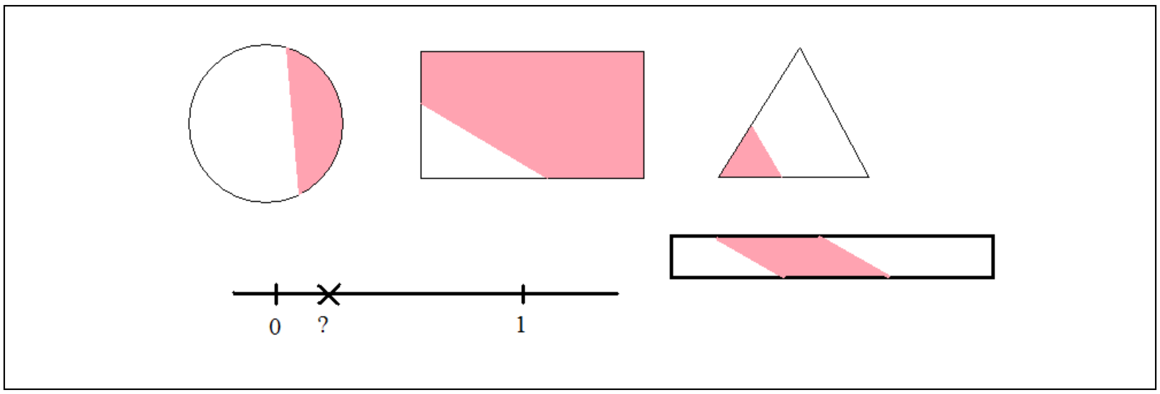


Figure 6.7. Estimate tasks for the amount shown.

Source: Van de Walle, 1998, p. 250.

Reform in Teachers' Training: Teachers are the spines of any education system. There is a need for educating Indian teachers to improve the standards of teaching of fractions. My study showed that the Indian students I interviewed have excellent procedural skills. Most of the students said, "*That is how I was taught*", when asked to explain why the rules work. This raises a question about teachers' preparation in India. Indian teachers may be following the traditional view of mathematics i.e., viewing mathematics as a set of rules and procedures. Teachers seemed to fail in getting students interested and curious enough to ask questions like, why rules work in the way they work. There is an urgent need to develop conceptual understanding of the students. This may require a change in the mindset of teachers, which can be brought in by major reform in training teachers.

Reason with quantities rather than numerical relationships: Research reveal that "when wrestling with quantities like three fifths or nine-tenths, as individual fractions and in relation to each other, students will develop a sense of size and quantity for individual fractions. This experience is necessary if students are to develop meaning for fraction operation" (Johanning, 2011, p. 98). My study reveals that NCERT focuses more on reasoning with numerical relationships rather than quantities. Fraction addition, multiplication and division are expressed in a methodical manner. Not many tasks are seen in the textbooks involving explanations and visual models for the rules being used. Some classroom challenges can be created, for example using pictorial representation, fractions as operators, fraction multiplication can be understood. One such example could be posing a variety of tasks using a pictorial model of a rectangle partitioned into 5 equal parts and three of the parts being shaded (Thompson, 2002). The first task could be to understand what is 3/5 and 5/3. The second task could be to understand 5/3 of 3/5 using the model. Here a child will take the three shaded parts for 3/5 and then will take one of the part to denote 1/3 of 3/5. The child will then iterate the single piece five times to get 5/3 of 3/5 and realize that the original rectangle is formed. There can be other related tasks, like to show 2/3 of 3/5 and so on.

Developing underlying concepts: Students' interviews revealed that four of the Indian students showed proficiency in calculation, but could not demonstrate why the rules make sense to them. There is a severe gap between skills and conceptual understanding among those students. NCERT should bring situations and tasks in the classroom so that students can think and reason. NCERT should create learning environments where children can ask questions and do not just memorize the rules until they can regenerate themselves. NCERT should encourage flexible thinking. As Lamon (1999) said, "Children should do lots of verbal reasoning, with and without pictures, and should be encouraged, through the kinds of questions we ask them, to use multiple ways of thinking" (p. 68). NCERT should develop tasks so that children can move between symbols and pictures; between continuous and discrete units. One example from Lamon (1999) is asking "children to interpret the fraction symbol using a picture of discrete objects" (p. 68). A sample task could be: How many marbles do I need to give Laura if I am planning to give 2/3 of 12 marbles (Lamon, 1999). Here children should be encouraged not just to see the numerical relationships and say 2/3 of 12 marbles is 8 marbles but thinking how to group 12 marbles to get 3 groups, each containing four marbles.

Posing tasks to engender a variety of strategies: Instruction on fraction must contain tasks that can trigger various strategies in students. According to Empson (2002), "The strategies and representations children produce embody many crucial aspects of fractions" (p. 39). NCERT should pose task that require class discussion and priority should be given to strategies generated by students (Empson, 2002). More students get involved in the thinking process, understanding fractions will be much easier, as Empson (2002) said, "The more students are encouraged to contribute the intact products of their own thinking to class discussions, the more likely they are to identify themselves as understanding mathematics- no matter the level of the thinking" (p. 39).

Chunk quantities in the instruction of fraction: Different people view discrete quantities differently. For example, one can view 12 oranges as 12 single oranges, 6 groups of 2 oranges, 4 groups of 3 oranges and so on. As Lamon (2002) said,

> Children either chunk information into pieces that are easier for them to think about, or, when discussing everyday items that are in their experience or the packaging they see at home. In fraction instruction, this simple, natural, psychological process often remains covert and causes communication problems. (p. 79)

NCERT could use these chunk quantities to create new tasks to encourage flexibility in thinking. According to Lamon (2002), "the process of mentally constructing different-sized chunks in terms of which to think about a given commodity is known as unitizing" (p.80). Unitizing offers a variety of benefits in the instruction of fractions. Some benefits are (Lamon, 2002): unitizing helps in replacing rules for generating equivalent fractions, helps in reason with fractions more appropriately, and builds the framework for proportional reasoning. Unitizing is an important concept in fraction instruction. A sample task could be showing 3/5 using a rectangular bar in many different ways by unitizing or grouping in different ways (Figure 6.8).

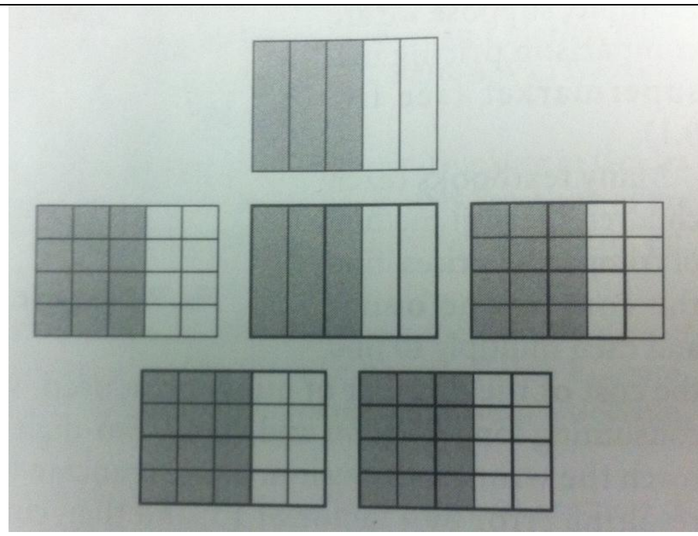


Figure 6.8. Grouping the region into different pieces.

Source: Lamon, 2002, p. 80.

My study presents some of the loopholes in the current instruction of fractions. My study found some misconceptions even among the textbook writers that are alarming. In this chapter, I have attempted to relate problems with possible solutions that NCERT could adapt to make learning productive and efficient. I hope my suggestions will give some understanding of what should be changed in the syllabi and the textbooks.

I infer from the students’ interviews that India is still following the traditional method of teaching where teachers are the conveyor of the information and students are the receivers. Now it is the time to change our mindset and to think of learning as a two-way flow of knowledge instead of uni-directional flow of knowledge.

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APPENDIX A: QUESTIONNAIRE

1. Have you learned about fraction?
2. Can you give an example of fraction?
3. The example you have given, can you explain what it means.
4. Can you draw a picture explaining your example?
5. Can you give another example of fraction? (Different than the first example)
6. Can you explain what does it mean?
7. What does 4/3 mean? Does it make sense?
8. The following rectangle of "*" is ¾ of another rectangle of "*"

   ****

   ****

   ****

   ****

   ****

   ****

   Draw the original rectangle.
9. Explain question 8.
10. If I give you a number line can you plot 3/4 and 7/4? Now can you plot 5/6 and 9/7?
11. Can you explain how you plotted?
12. Is 2/3= 8/12? Why we can replace one fraction with another?

13. Can you explain your answer?
14. Can you add 2/3+5/3?
15. What does it mean?
16. Can you add 2/15+ 6/9?
17. Can you explain your work in Q 16?
18. Do we always need to take LCM?
19. Name the part that is shaded in each picture. Do these fractions name the same amount? How do you know?

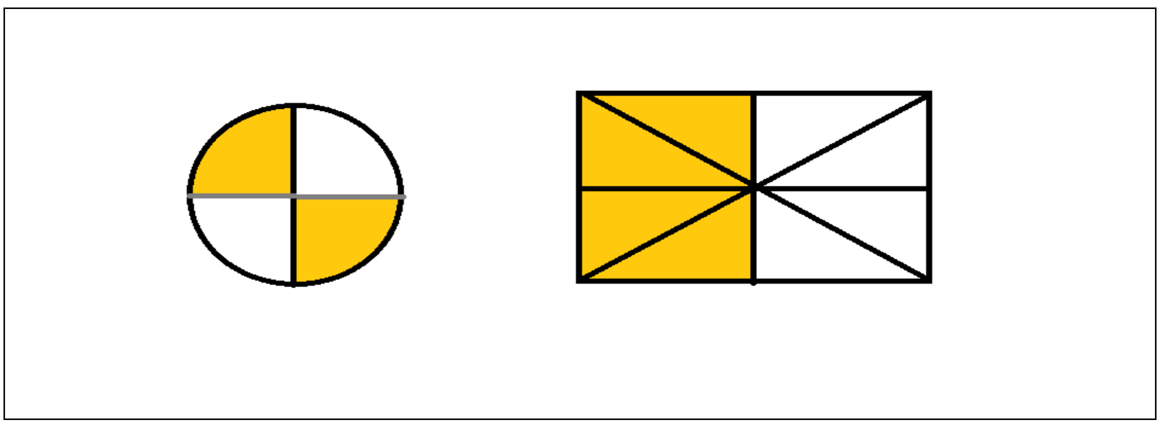

Figure A.1. The picture associated to question 19.

Source: Lamon, 1999, p. 59.

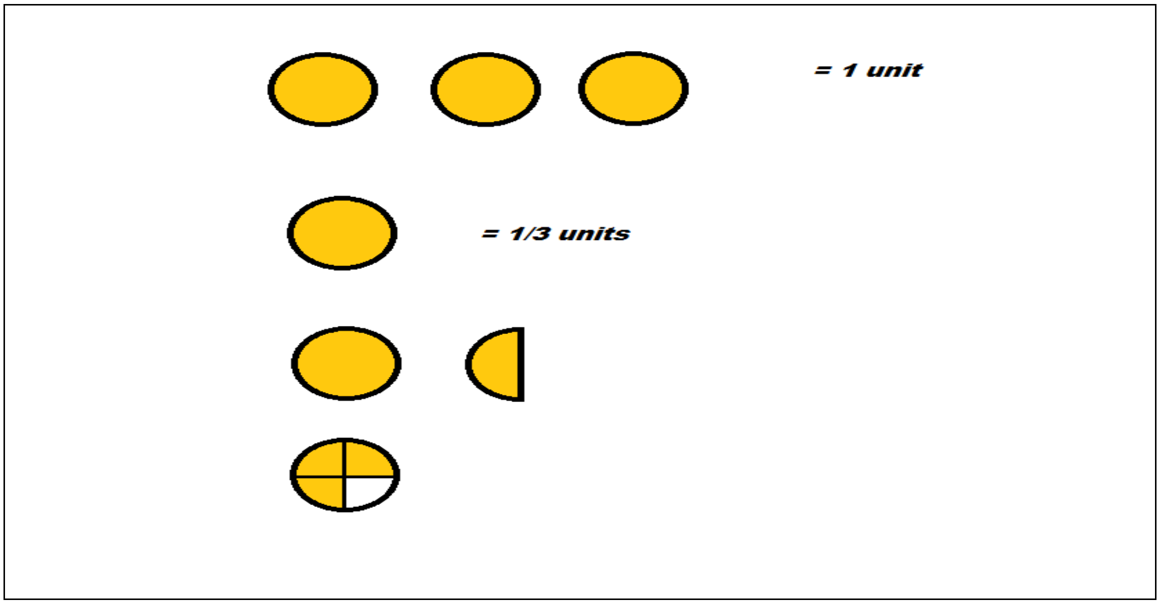

Figure A.2. Figure showing three circles representing one unit. The task is to find the units of the other figures based on the fact that three circles is one unit.

20. Which of the following is a story problem for 1/2+1/3?

a) Tom pours ½ cup of water into an empty bowl. Then Tom pours 1/3 cup of water into the bowl. How many cups of water are in the bowl now?

b) Tom pours ½ cup of water into an empty bowl. Then Tom pours in another 1/3. How many cups of water are in the bowl?

c) 1/2 of the land in 24 parganas is covered with forest. 1/3 of the land in Hoogli is covered with forest. What fraction of the land in the two districts is covered with forest?

d) 1/2 of the land in 24 parganas is covered with forest. 1/3 of the land in Hoogli is covered with forest. Both have same land area. What fraction of the land in the two district is covered with forest?

e) 1/2 of the students in your school say they like to have pizza for lunch. 1/3 of the children at your school say they like to have a burger for lunch. What fraction of the children at your school would like to have either pizza or a burger for lunch?

21. Which of the following is a story problem for 1/2- 1/3?

a) Tom pours 1/2 cup of water into an empty bowl. Then he pours out 1/3. How much water is in the bowl now?

b) Tom pours 1/2 cup of water into an empty bowl. Then he pours out 1/3 cup of water. How much water is in the bowl now?

c) Tom pours 1/2 cup of water into an empty bowl. Then he pours out 1/3 of the water that is in the bowl. How much water is in the bowl now?

d) Yesterday James ate 1/2 of a pizza. Today James ate 1/3 of a pizza of the same size. How much more pizza did James eat yesterday than today?

e) Yesterday James ate 1/2 of a pizza. Today James ate 1/3 of the whole pizza. Nobody else ate any of that pizza. How much pizza is left?

f) Yesterday James ate ½ of a pizza. Today James ate 1/3 of the pizza that was left over from yesterday. Nobody else ate any of that pizza. How much pizza is left?