THE INFLUENCE OF TEACHERS' KNOWLEDGE OF MATHEMATICS IN THEIR CLASSROOM INTERACTIONS

by

RYAN DAVID FOX

(Under the Direction of Denise A. Spangler)

ABSTRACT

This study examined how two Advanced Placement Calculus teachers used their own knowledge of mathematics in their classroom interactions. In particular, this study focused on teachers' responses to unanticipated or unplanned student questions during a normal lesson. Each teacher was observed for one unit of instruction of at least ten hours. Field notes and audio recordings were taken during the observations. Throughout the unit of instruction, four interviews were conducted with each teacher to explore how they used their own knowledge. In this study, the teachers tended to use four types of responses: posing counterexamples, acknowledging challenging responses, asking simpler or related questions, or following through with the student's comment. Through the four responses, teachers could respond to the students' comments by addressing the comment separately, interweaving a response with the planned activity, or coordinating a response with planned activity. Future research could explore teachers' responses and patterns of responses for longer observations or earlier mathematics courses.

INDEX WORDS: Teacher knowledge, Pedagogical decisions, Secondary education, Gifted education

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B.S.H., Berry College, 2002

M.Ed., University of Georgia, 2007

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2011

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May 2011

ACKNOWLEDGEMENTS

There are many people to whom I owe a great debt of gratitude. I have had so many people who played an integral role in seeing this project to its completion, and I want to express my sincere thank-you to them all. I should say that my gratitude extends well beyond what I can capture in these few words. My sincere hope is that I do not omit anyone in these acknowledgements or provide an inadequate acknowledgement.

First to my major professor, Denise Spangler, thank you, thank you, and a thousand times thank you! From drop-ins at your office to weekly meetings getting this “little paper” finished, I am truly indebted to you because of your wonderful help and sage advice. You have been such a support throughout this entire process that I don’t know if I could fully express my sincerest gratitude. Thank you!

I also want to thank the members of my committee, Jeremy Kilpatrick and Pat Wilson. I greatly appreciate the years of support and encouragement you have provided. I am humbled that each of you would take from your busy schedules to help me out. Thank you so much for all of the ways that you have contributed to my growth as a member of the mathematics education community.

I want to thank my participants in my study. To say I couldn’t do this without you would be an understatement!

In my years at the University of Georgia, I have been uniquely blessed to be surrounded a wonderful group of fellow doctoral students in this program. I could spend many pages identifying the great works you have done. Please do know that I am in awe of your amazing talents and it is a true honor that I get to share ideas, talk shop, and play the occasional game of trivia with you. Thank you. I do want to thank in particular Sarah Donaldson and Brian Gleason for the many years we have spent as classmates and friends. Thank you so very much!

In the past few years, I have made friendships with wonderful people who have provided support during the good times and encouragement during the struggles. I would like to extend my gratitude to you. My teaching career does not get off the ground without the wonderful colleagues I met at Sprayberry High School. To my former colleagues, thank you for your guidance and support during the earliest part of my career. You made my trip back home better than I could have ever imagined. Without spending several lines of paper thanking all of you individually, please do know that I treasure your role in my teaching career. Thank you. The past two summers, I have had the true privilege of working with the Johns Hopkins University Center for Talented Youth. Through my interactions with some of the program's instructors, I have been sustained and nurtured in my growth as both teacher and researcher. I could probably include an entire page of names of amazing people I have met and considered colleagues and friends. Please do know that I thank you for helping my in that part of my professional growth. My friendship with Karen Viars extends all the way back to before school chats at the cafeteria at Sprayberry High School. You have been a true lifesaver as far as providing some wonderful direction in managing the rigors of the life of a graduate student and editing my own writing. I'm still working on the latter, but I am light years beyond anywhere I could be without your help. Thank you.

I want to thank my family—Mom, Pop, and Tim—for the many years of wonderful support you have given me. From years of chasing dreams—from summer camps in West Virginia, band trips to far away locales, trips to and from Berry, the return home to teach at

Sprayberry, to learning to love the Bulldogs—I can't even begin to thank you enough! All of the years of following dreams have truly paid off, and I could not have made it to this point without you. I also want to thank my in-laws, Al and Virginia Purvis, for being such wonderful supporters and founts of advice for many years. Whether needing a trip to Tifton to get away (or to invade your dining room table to complete a rather lengthy paper) or coming up to Athens to check on us (and buy us the occasional Mexican dinner!), I have been blessed that you have shared in this journey. Thank you so very much!

Nine years ago, I made the rather brash move to include my girlfriend, Sarah Purvis, in the acknowledgements of my Honors Thesis at Berry College. Since then, Sarah Purvis became Sarah Fox. To my amazing wife, I don't even know how to begin saying thank you properly. I would never have been confident to chase this crazy dream without your constant support and encouragement. When we moved to Athens, you happily made the move with me, even though we really didn't know what we were getting into. From frustrating homework assignments to far-flung trips for conferences to the roller coaster of completing this project, you have been my steadying force. I truly could not have kept myself motivated to the end without you.

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CHAPTER 1

INTRODUCTION

Personal Interest

This study was motivated by my experience as a teacher of honors and Advanced Placement courses. I taught an Algebra II Honors course for 3 years and an Advanced Placement Calculus course for 2 years. Reflecting back on my teaching of these two courses, I can see how I used my mathematical knowledge—both of school mathematics and undergraduate mathematics—to discuss mathematical ideas with the students. Possibly as a result of my recent study of university mathematics, I found myself trying to connect what I recently learned as a student to what I presented as a teacher. In my reflection on two different experiences—one from each of the two courses, Algebra II and Calculus—I found opportunities to examine how I used my mathematical knowledge to assist my teaching.

During an Algebra II Honors class session, I remember discussing the idea of closure in matrix multiplication with one particularly bright student. In that discussion, as we began to looking at whether or not any two matrices could be multiplied together, some students noticed that this was not going to be a true statement. I mentioned that closure was an idea that I remembered studying quite extensively in my undergraduate abstract algebra course. Upon additional reflection, I understand that this is a topic that I actually could have developed in greater detail than I did throughout the high school algebra course. I remember emphasizing the notion of closure working with matrices, but I could have shown that the students had discussed the topic before and would be discussing the topic again in this course. The students had seen the notion of closure before without calling it closure when discussing division by zero and would be seeing closure again by observing how it can be affected under the same operation by changing the set, looking at the square root operation under real numbers and then under complex numbers.

In one specific Advanced Placement Calculus class discussion, the class worked examples of improper integrals of the same form. Students began to see a pattern developing in the solutions they were reaching and wanted to generalize their results. Not knowing if the students were correct in making such generalizations, I decided to break with the plan of the lesson and explore the students' conjecture. I made sure to work the problem as if I were a student, explaining my thought process aloud as I worked. Starting with an empirical argument and working by cases, the class and I convinced ourselves that the generalization was valid. Although I did not know where the discussion would end, I knew that the exploration of the conjecture would lead to a meaningful mathematical discussion.

Upon reflecting after the presentation of the lesson, I asked myself why I would have presented each of these mathematical ideas to the students in the accelerated Algebra II and Calculus courses. From the experiences I had already had, I knew this was a class of highly capable students, in terms of mathematical ability. A question I remember asking myself was "Why did I present this information at this time? Was it to suggest the mathematical material that lies ahead for the student?"

During my teaching career, I had the opportunity to teach the Advanced Placement Calculus BC course at my school. My motivation to teach the course had nothing to do with wanting to teach a course I studied recently but with the desire to present challenging material to capable students. There is a unique challenge presented in teaching an accelerated course like this, and I was aware that teaching such a course would challenge the depth and connections of my knowledge of mathematics.

Because of my own experiences, I am interested in the role of the teacher in specialized courses for students who have demonstrated above-average ability in mathematics. For example, I am interested in the moves (Cooney, Davis, & Henderson, 1975) a teacher makes while fostering discussion among class members or assisting the students in tasks or assignments. Further, I am interested in the teacher's motivations for the moves that he or she made. In particular, I am interested in the extent to which these moves are influenced by what the teacher knows about mathematics and the teacher's awareness of the knowledge used when making these decisions.

Background

Prior research has examined how teachers react to unanticipated student questions. Fernandez (1997) examined how teachers at the middle and early secondary levels fielded and responded to unplanned questions. Some of the classrooms in Fernandez's study contained students who demonstrated a typical level of mathematical ability, while a small number of classrooms contained students with a higher-than-average level of ability. My study takes a different approach in two ways because I examined an older group of students (upper secondary) and a homogeneous ability group (accelerated students).

Additionally, much work has been done describing how teachers deploy specialized knowledge of teaching mathematics. Some researchers identified this knowledge, in part, through a description of situations where a teacher's knowledge can be observed through various levels of schooling (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Cuoco, 1998; Davis & Simmt, 2006; Even, 1993; Hill & Ball, 2004; Krauss, Baumert, & Blum, 2008; Shulman, 1986; Thompson & Thompson, 1996). In these descriptions, commonalities emerged about what comprised portions of a specialized knowledge for teaching mathematics: explaining a mathematical idea, representing a mathematical topic in a variety of ways, connecting one mathematical concept to other concepts, posing a question to facilitate mathematical discussion in a classroom, and assessing the validity of students' mathematical claims. These descriptions suggested that teachers have to access their knowledge in a different way compared to other consumers of mathematics. The unpacking of mathematical knowledge (Ma, 1999) illustrates a difference between the nature of a teacher's knowledge from other individuals' mathematical knowledge. Unpacking of mathematical knowledge stands in opposition to the construction or acquisition of mathematical knowledge in the traditional sense (Ball & Bass, 2000; Thompson & Thompson, 1996).

Instead of describing the application of teachers' knowledge in classroom situations in positive terms, some researchers identified instances where teachers either did not fully apply what they knew in classroom activity (Eisenhart et al., 1993; Even, 1993, 1998; Even & Tirosh, 1995; Hitt, 1998; Schoenfeld, 1994; Stylianides & Ball, 2008; Thompson & Thompson, 1996; Tirosh, Even, & Robinson, 1998). Examples of deficiencies and inaccuracies can be found in a teacher's tendency to provide only rule-bound approaches for their students to memorize; lack of ability to recognize a simpler, more intuitive answer to a question; and a failure to generate multiple representations of a particular topic. By drawing attention to what teachers may not know, teacher educators can support pre-service and in-service teachers to identify deficiencies. Reflecting on the work of classroom practice could permit others to examine other possible approaches teachers might have taken. Creating an inventory of different approaches to use with students could support teachers working with a classroom full of students.

Classroom activity is where the teacher applies this unique body of mathematical knowledge. A teacher uses his or her knowledge of multiple representations, solutions, and explanations to prepare for posing questions and tasks for the students. Likewise, when receiving a question from a student, the teacher must call on these representations, solutions, and explanations to provide an appropriate response. A teacher should be aware of why he or she is using that knowledge and how his or her knowledge can be made explicit to assist students in developing their understanding of mathematics (Mason & Spence, 1999). If teachers knew when to use their knowledge in specific situations, then they could communicate what they were doing to their students. Explicitly communicating their mathematical thought processes to students allows teachers to use alternate solutions or representations to introduce additional ways of thinking to their students. Teachers explicitly stating what they know to solve a problem models desirable problem-solving behaviors to students.

A teacher has to be prepared to receive questions from students during the course of a lesson or activity. The questions that students pose to the teacher affect the course of the lesson. The students' posing of questions allows the teacher only a short period of time to respond (Leikin & Dinur, 2007; Rowland, Huckstep, & Thwaites, 2005). The way the teacher responds to the students' questions suggests the way the teacher thinks about the students' development of mathematical knowledge (Fernandez, 1997). Fernandez suggested four ways that teachers respond to students' questions: creating counterexamples, following through, using simpler or related problems, and incorporating a student's method.

Teachers call upon many aspects of their knowledge of their students' mathematical development to answer questions from students, whether that development is called sequencing (Rowland et al., 2005; Stylianides & Ball, 2008), ordering (Bromme, 1994), or a trajectory

(Simon, 1995). In a sequence or a trajectory, teachers connect past concepts to future topics (Cooney, 1999), or connect informal knowledge to formal knowledge (Gravemeijer & Doorman, 1999). Other studies demonstrated that teachers knew what topics were important to a particular class (Shulman, 1986), why a topic was important for a particular class (Kahan, Cooper, & Bethea, 2003), or what made a topic easier or harder than another topic (Lampert, 1991).

Teachers' knowledge of students comprises another component of a specialized knowledge for teaching. For example, studies have shown that some teachers recognize students' solutions as viable contributions to classroom discussion (Franke, Kazemi, & Battey, 2007), and treat the student's solution with the appropriate level of respect (Ball & Wilson, 1996). Although the teacher may recognize student solutions, there are opportunities for the teacher to orchestrate student solutions. Teachers lead discussions in a variety of formats (Lobato, Clarke, & Ellis, 2005) to promote the presentation of student solutions. Once a solution has been received, the teacher may assess the validity of the student's claim. To facilitate the assessment, the teacher should be aware of common misconceptions of a mathematical concept (Even & Tirosh, 1995; Kotsopoulos & Lavigne, 2008; Shulman, 1986; Tirosh et al., 1998) or unique approaches to a final answer (Adler & Davis, 2006; Biza, Nardi, & Zachariades, 2007; Fisher, 1988). In encouraging student discussion and solution presentation, the teacher needs to know when to continue the student discussion or when to change the course of discussion along a particular route (Chazan & Ball, 1999).

Throughout the variety of activities of teaching, various types of knowledge are implemented and coordinated. One way to observe such coordination is by examining classroom interactions with students. In that environment, teachers are allowed to work as they normally do. Within that normal interaction, many opportunities arise to investigate what a teacher knows and how effective that knowledge is in maintaining the pace of classroom discussions or in extending what the students already know.

Rationale

My research examined the knowledge base experienced secondary school mathematics teachers have and use within the context of a class working with an accelerated curriculum. The literature on students identified as being able to perform mathematical activity beyond that of their peers says much about the characteristics of the students. The literature, when discussing the role of the teacher in the classroom of students identified as gifted, seems to have a focus more on assisting the intellectual development of the student and not as much on the characteristics that the teacher brings to the classroom.

A classroom in which a teacher accelerates the typical school curriculum provides an interesting opportunity to explore how the teacher handles student questions. Because the student in such a course is able to understand the material faster than other students and begin to explore the more conceptual aspects of the mathematical content earlier, the student is able to ask a more challenging question earlier in the presentation of the material. Therefore, the teacher should be ready to handle a challenging question in a short period of time. In addition, the teacher should be able to present an answer that would satisfy the intellectual curiosity of the student. The accelerated classroom provides an opportunity to observe more challenging questions and more detailed responses to those questions than heterogeneous or lower level classrooms.

Research Questions

The research questions that guided this study are as follows:

1. In what ways does a teacher apply a specialized mathematical knowledge for teaching when presented with an unanticipated student question?
2. What are the approaches a teacher uses when responding to a student who has posed an unanticipated question?

CHAPTER 2

LITERATURE REVIEW

Four different branches of established literature built a basis for this study. The main literature bases are the specialized knowledge teachers possess along with the interactions teachers engage in with their students. These are the important aspects of the relevant literature, as this study focused on how teachers use their own mathematical knowledge in discussing ideas with their students. I include two other bases of literature to support the setting of this study: characteristics of mathematically gifted students and secondary school calculus. The studies selected for this review provide insight into other researchers' findings with similar types of students and a similar type of course as those in this study.

Specialized Knowledge for Teaching Mathematics

Many researchers investigated what knowledge a teacher should possess to assist his or her students in developing new mathematical knowledge. Researchers have taken different approaches in investigating this unique knowledge of teaching mathematics. Two major avenues toward identifying this specialized knowledge have emerged. Some researchers pose models to describe a specialized knowledge, based on their studies and research. Others design assessments for selected components of this specialized knowledge for teaching. Results from both categories are presented below.

Conceptualizations of a specialized knowledge

Researchers have used many terms to identify a knowledge base that is unique to the practice of teaching. Finding such knowledge can be challenging for researchers, both in terms of content and application of what they know (Cooney, 1999). In response to Cooney's challenge, many researchers captured their conceptualizations of a specialized knowledge succinctly. This section contains the researchers' findings and commonalities and differences that exist across the findings.

One of the first conceptualizations of a specialized knowledge for teaching came from Shulman's (1986, 1987) proposal of pedagogical content knowledge (PCK). Shulman identified PCK as a blend of the knowledge to work with a classroom of students and the content knowledge that an individual might receive in a normal course of study. In this conceptualization, a teacher cannot simply know a lot of content and several methods to lead a class. There is some way to modify the content teachers know to lead meaningful lessons. McEwan and Bull (1991) further identified PCK as a combination of alternative representations and "pedagogical reasoning" (p. 319). In their findings, PCK included teachers' ways of representing a topic, understanding possible student misconceptions, and identifying concepts students might view as easy or difficult. Shulman (1987) categorized PCK in seven ways: content knowledge, pedagogical knowledge, knowledge of curriculum, pedagogical content knowledge, knowledge of students, knowledge of contexts, and philosophical knowledge. Teachers attend to several activities simultaneously (or within one class session). In order to handle all of these activities effectively, teachers possess a variety of types of knowledge that expand beyond a simple combination of content and pedagogy. Shulman's description of a unique knowledge base for teaching and the components of that knowledge base became the basis of other researchers' work.

Bromme's (1994) description of a philosophical content knowledge extended Shulman's (1987) discussion of a philosophical knowledge. Bromme's categorization of a subject-specific content knowledge focused on mathematics, whereas Shulman's description of PCK applied to all academic subjects. Bromme defined his conceptualization of a specialized knowledge as a topology of teacher's knowledge; his categories included content knowledge as a discipline, knowledge of school mathematics, philosophy of school mathematics, pedagogical knowledge, subject-matter-specific pedagogical knowledge, and interdisciplinary knowledge. Bromme's philosophy of school mathematics picked up some of Shulman's ideas for pedagogical content knowledge. For example, the combination of content knowledge as a discipline and knowledge of school mathematics would be the content knowledge Shulman (1986) described in establishing PCK. Likewise, general pedagogical knowledge is a separate component of a specialized knowledge for teaching. The combination of content and pedagogy to create a separate, subject-specific pedagogical knowledge is an important distinction made by the researchers. Bromme's philosophy included teachers' awareness of the central concepts within a particular curriculum, much like what Shulman (1986) advocated. Both researchers emphasized the identification of what teachers valued as important topics. One could see how teachers organize their curriculum by observing what the teachers presented with greater emphasis. The topics teachers present as important are likely to be the topics that the teachers think are the most important themselves.

Mathematical knowledge for teaching (MKT) is the terminology used by many researchers. Deborah Ball's work proposed MKT as one conceptualization of a specialized knowledge for teaching mathematics at the elementary school level. Working with Bass, Ball (Ball & Bass, 2000) defined MKT as "a kind of mathematical understanding that is pedagogically useful and ready, not bundled in advance with other considerations of students or learning or pedagogy" (p. 88). In this quotation, Ball and Bass's conceptualization of MKT is like Shulman (1986, 1987) and Broome's (1994), the specialized knowledge exists independently of a strict content or pedagogical knowledge. Ball, along with Thames and Phelps (2008), categorized MKT by separating it into two divisions and six subdivisions. The two divisions of MKT are subject-related knowledge and pedagogy-related knowledge. Within the subject-related knowledge, Ball and colleagues subdivided that category into common content knowledge, horizon content knowledge, and specialized content knowledge. From pedagogy-related knowledge, Ball and colleagues subdivided pedagogy-related knowledge into knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. The subdivisions of MKT can be seen in recognizing students' errors, posing questions to extend classroom tasks, proposing hypotheses on students' views of mathematical topics as simple or difficult, and selecting appropriate numerical values in examples to facilitate student development of mathematical topics. Like other researchers, Ball and colleagues proposed advantages of possessing a specialized knowledge of teaching. The combination of the two larger divisions of MKT can be coordinated with Shulman's (1986) original notion of pedagogical content knowledge. Some of Shulman's (1987) categories matched well with Ball's sub-divisions. Two of Ball and colleagues' categories of pedagogy-related knowledge are similar to two categories of Shulman's PCK. In both studies, teachers should possess knowledge of the students and the curriculum they are teaching. Because teachers present their lesson to a room full of students, teachers should possess knowledge of the students they are teaching.

Thompson and Thompson (1996) identified the MKT in the way one teacher understood rates of change. Thompson and Thompson's study included knowledge of students, as the teacher in their study assisted one student in constructing a conceptual understanding of motion. Thompson and Thompson elaborated their conceptualization of a specialized knowledge for teaching. In their discussion of describing how the teacher helps the students build new knowledge, they proposed MKT as a conceptual orientation to teaching mathematics; the orientation is subdivided into four categories: ways of thinking to develop in students, the image of ways of thinking, features of materials and activities to develop ways of thinking, and dispositions toward ways of thinking.

Researchers identified a specialized knowledge for teaching by examining the practices of elementary school teachers. Results from these studies can support a specialized knowledge for teaching at the secondary level. One approach to a specialized knowledge for teaching at the elementary level is the Profound Understanding of Fundamental Mathematics (PUFM), proposed by Liping Ma (1999). In her book, she examined how teachers of elementary school students in China and the United States understand mathematics quite differently. Chinese teachers tended to have less formal education regarding mathematics, but held richer understandings of elementary mathematics topics. By viewing mathematical knowledge in terms of knowledge structures, Ma showed that Chinese teachers held more knowledge and connections among topics than their American counterparts. Ma's analogy to describe the way Chinese teachers use their knowledge more effectively is through the unpacking of a knowledge package. Ma's unpacking content knowledge useful in classroom presentations was an idea that many other researchers used to describe similar activities.

What is important from this study (Ma, 1999) is that advanced knowledge of mathematics does not imply teachers possess the right knowledge to conduct valuable activities with their students. Teachers did not gain a better knowledge for teaching mathematics by taking more courses. This is an idea that can be extended to other grade levels of schoolteachers as well;

Monk (1994) found secondary school teachers reached a ceiling effect of five university-level mathematics courses for possessing an effective knowledge of teaching secondary mathematics.

In her study of teachers' knowledge of functions, Even (1990) proposed a construct she called subject matter knowledge. Her subject matter knowledge necessary consisted of seven components: essential features, different representations, alternative approaches, strength of concept, basic repertoire, knowledge and understanding of a concept, and knowledge of mathematics. In a later study, Even (1993) investigated how a lack of subject matter knowledge inhibited pedagogical content knowledge. This lack of knowledge bases prohibited teachers from making meaningful pedagogical decisions, such as selecting suitable tasks, asking valuable questions, or creating useful representations. Missing some of these components implied teachers' inability to move forward with particular aspects of a lesson. Like Monk (1994), Even suggested that it is not what a teacher has learned in a university classroom that will develop the needed content knowledge for teaching school mathematics. More importantly, teachers should apply their knowledge of both university and school mathematics in meaningful ways to support student learning.

In their study of pre-service teachers, Kahan and colleagues (Kahan, Cooper, & Bethea, 2003) identified a mathematics content knowledge (MCK) for teachers. This specialized knowledge is something that a teacher of mathematics would possess but not a knowledge base that a mathematician would necessarily possess. Among Kahan and colleagues' components of MCK are exploring mathematics foundations of a topic and identifying students' prior knowledge of mathematics. Like other researchers, Kahan and colleagues identified the importance of teachers knowing their students' previous content knowledge. Kahan and colleagues pointed out benefits of teachers' strong MCK. Advantages of a strong knowledge base included the ability to incorporate unanticipated student comments, to appropriate the right amount of class time to the presentation of a topic, and to connect content knowledge from their university mathematics classes to the secondary mathematics lessons. Interestingly, Kahan and colleague's MCK included the affective components to knowledge. Not only do teachers need to know how to perform mathematical activity in front of a classroom of students, but teachers should feel confident that they know they can lead mathematically.

Davis and Simmt (2006) proposed mathematics-for-teaching (MfT), where teachers' knowledge of mathematics is seen in the practice of teaching. In their work, Davis and Simmt identified three categories of this specialized knowledge: connections among mathematical ideas, anticipation of future concepts, and validations of students' arguments. Davis and Simmt observed this specialized knowledge in teachers' typical activities—understanding student misconceptions, providing multiple representations, and creating valuable learning experiences. For the last activity, Davis and Simmt suggested that teachers should know ways to make the mathematical meaning of images and metaphors explicit for students. By making that meaning explicit, teachers could choose between the representations that would be most beneficial for students to learn a new mathematical concept. Knowledge of an historical development of the topic could support teachers in working as Davis and Simmt described.

Kazima, Pillay, and Adler (2008) also referred to their conceptualization of this specialized knowledge as mathematics-for-teaching; they defined MfT as "specialized mathematical knowledge that teachers (need to) know and know how to use in their teaching" (p. 284). They noted that teachers need to know more mathematical knowledge than their students are expected to know at the end of the course. This knowledge allows teachers to be more than answer keys to assignments and assessments. This knowledge supports the need for teachers to enroll in mathematics courses at the university level beyond what they will teach in the schools. This idea coordinates well with Ball, Thames, and Phelps's (2008) notion of the horizon content knowledge: knowing how the current topic supports more advanced mathematical topics. In fact, Kazima and colleagues observed that teachers' work with students poses great demands on teachers' content knowledge. Like Davis and Simmt, Kazima and colleagues grounded their conceptualization in the activity of teaching. In their proposal of MfT, Kazima and colleagues mentioned that teachers' knowledge of mathematics content must be unpacked, using Ma's (1999) terminology.

Rowland and colleagues (Rowland, Huckstep, & Thwaites, 2005; Rowland, Turner, Thwaites, & Huckstep, 2009) posed the knowledge quartet (KQ) to categorize this specialized knowledge for teaching elementary school mathematics. The four components of their conceptualization are foundations, transformation, connection, and contingency. Contingency knowledge is the knowledge that must be accessed when faced with a student's comment or question that the teacher cannot plan or anticipate. The more teachers possessed this component of knowledge, the more likely teachers are to incorporate students' unexpected responses into classroom discussion, and they are more likely to present a response that would benefit students' construction of content knowledge. Just as important as knowing when and how to incorporate unexpected student comments, Rowland and colleagues (Rowland, Huckstep, & Thwaites, 2005) observed that this knowledge assists teachers in deciding when not to incorporate a student statement, because of the extended, tangential discussion that would have to be pursued to address the comment fully.

Assessing a specialized knowledge

One way researchers worked to identify a specialized knowledge for teaching mathematics was through the creation of large-scale standardized assessments. Some researchers used assessments to identify a knowledge base that teachers have that other individuals may not have. Those tests have identified this specialized knowledge as a set of components that can be isolated and assessed. Researchers examined the types of applications of knowledge by looking at teachers from one state, one country, or several countries. A few results of such studies are included here.

Krauss and colleagues (Kruass, Baumert, & Blum, 2008) assessed a proposed specialized knowledge for teaching, which they called PCK, in various teachers and university students throughout Germany. Krauss et al. assessed three components of PCK at the secondary level–generating multiple solutions, creating various representations, and identifying possible student misconceptions. To show that the knowledge teachers possess for teaching secondary school mathematics could be unique, Krauss and colleagues assessed two types of secondary mathematics teachers (teachers of university-bound students and teachers of workforce-bound students), secondary science teachers, university mathematics students, and future secondary mathematics teachers. Of the three categories, Krauss and colleagues determined the last two were more important to the practice of teaching than the first. They believed that PCK would result from combining content knowledge, pedagogical knowledge, and application of those two bases of knowledge. One interesting result was that a strong background in content knowledge helped a teacher of high-ability students to increase in pedagogical content knowledge, even though the teacher did not have a long formal preparation to become a teacher. However, pedagogical content knowledge could support the development of content knowledge, as science teachers knew more about mathematics than hypothesized, partially because of having to explain the mathematics behind scientific concepts.

Adler and Davis (2006) assessed their conceptualization of a specialized mathematical knowledge for teachers through tasks given to future secondary teachers in South Africa. The researchers found certain assessment items emphasized mathematical content over pedagogy, some emphasized pedagogy over content, and others placed a dual emphasis on content and pedagogy. Adler and Davis identified the extent the assessment ideas required future teachers to unpack their own mathematical ideas. They provided comparisons for the terms *compression* and *unpacking*. Unpacking of mathematical content allowed a teacher to “trace back mathematical ideas and their antecedents with their learners” (p. 290). Compression of mathematical thought is the desired result of the work of mathematicians; compression was a challenge for mathematics teachers because compressed thought contained no “explicit display of understanding” (p. 289). A single mathematical symbol contains implicit meaning that is not obvious to an outsider. Although those symbols have meaning to those who understand a symbol’s implicit meaning, meaning is lost for those who do not have the appropriate understanding of what the symbols represent. Teachers find the implicit meaning within symbols and extract that meaning. Once it is extracted or unpacked, teachers can present that meaning to their students. Teachers possess a specialized knowledge to make that unpacking possible. Adler and Davis recognized teachers’ possession of knowledge but acknowledge the difficulty of preparing future teachers in South Africa to unpack implicit meanings. Because future teachers in South Africa lack certain aspects of content knowledge (Pournara, 2008), they are unable to unpack implicit meanings completely.

The work of the TED-S project examined ways future middle school teachers around the world were prepared to become mathematics teachers (Schmidt et al., 2007; Senk, Peck, Bankov,

& Tatto, 2008). This research examined how pre-service teachers received preparation in the content they will teach and the methods of instructional delivery. Schmidt and colleagues defined a mathematical knowledge for teaching as the combination of content and mathematical pedagogical knowledge. In this definition, Schmidt and colleagues emphasized that their specialized knowledge for teaching included a subject-specific pedagogical knowledge. This is different from Shulman's (1986) original interpretation of PCK, which was a combination of content knowledge and general pedagogical knowledge. Schmidt and colleagues' approach kept specialized knowledge focused only on mathematics instead of including typical teaching practices that would be found in any classroom. Schmidt and colleagues isolated three aspects of teaching mathematics: pacing and representing mathematical topics, possible student answers and misconceptions during the presentation of a topic, and knowing the placement of middle school topics within the larger mathematics curriculum. This research showed that both preparations in mathematical content and pedagogy typically yielded prepared and knowledgeable teachers. Senk and colleagues highlighted the differences between their study and the study performed by Krauss and colleagues: "COACTIV [Krauss's research] uses tasks and multiple solutions; misconceptions and difficulties; and explanations and representations…whereas MT21 [Schmidt's and Senk's research] distinguishes curricular knowledge, instructional planning, and student learning" (p. 5). However, like Krauss and colleagues, the work of these researchers showed that the desired knowledge for teaching mathematics exists in some combination of mathematical knowledge beyond what teachers teach and pedagogical knowledge to support what teachers present.

Hill and colleagues (Hill & Ball, 2004; Hill, Ball, & Schilling, 2008) developed instruments to identify MKT at the elementary school level. Hill and colleagues focused only on knowledge of content and students (KCS), a sub-component of Ball, Thames, and Phelps's (2008) conceptualization of MKT. Hill and colleagues defined KCS as the "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill et al., 2008, p. 375). This conceptualization and subdivision recognized that teachers apply knowledge of mathematics differently than other consumers of mathematics. Not only do teachers possess knowledge of the content that works well for their own understanding, but also they should possess knowledge that will work well for other people, namely, those who might approach a mathematical topic differently than they would. Hill and colleagues compared their conceptualization of KCS to Shulman's explanation of pedagogical content knowledge (PCK); they envisioned that MKT can be subdivided into PCK, which then could be subdivided again into KCS and other components. Investigating KCS allowed Hill and colleagues to assess possible student errors, also assessed by Krauss et al. (2008). Teachers should recognize both valid and invalid responses to a mathematical question from the school curriculum. Recognizing invalid approaches also includes ways teachers can instruct students to move away from invalid toward valid approaches. Like Krauss and colleagues, Hill, Ball, and Schilling found that general content knowledge and a type of subject-specific pedagogical content knowledge support one another.

Teachers' Interactions with Students

The practice of teaching includes many activities. Some of the most frequently mentioned activities by researchers included representing a mathematical topic in a variety of ways and selecting examples with the right level of difficulty for the particular classroom of students (Ball & Bass, 2000; Flowers & Rubenstein, 2006). The main focus of this section is how teachers maintain levels of classroom discussion and encourage students to discuss mathematical ideas. How researchers discuss those interactions is included here.

Schoenfeld's (1998) work toward teaching-in-context highlighted how teachers apply what they know while teaching a classroom full of students. His goal in creating a theory-in-context was to achieve the following: "The claim is that with the theory and with enough time to model a particular teacher, one can build a description of that person's teaching that characterizes his or her classroom behavior with remarkable precision" (Schoenfeld, 2000, p. 644). He outlined his theory by investigating three different teachers: a novice high school mathematics teacher, an experienced high school physics teacher, and himself as an experienced instructor of collegiate mathematics. Schoenfeld found that he and the experienced teacher handled students' unanticipated comments more effectively and provided more meaningful examples and explanations to support students' development of content than did the novice teacher. Schoenfeld observed that teachers' knowledge greatly influences how and why teachers decide to pursue certain classroom activity. That knowledge included knowledge of content and students, identified by Ball, Thames, and Phelps (2008), and assessed by Hill, Ball, and Schilling (2008). Schoenfeld later divided the knowledge used within the context of teaching into two; this knowledge contained the list of content teachers possessed and the ways teachers accessed their knowledge. Schoenfeld described ways teachers accessed their own knowledge, including particular mini-lessons and "mini-lectures" (p. 29) teachers used in response to student comments. These ways of accessing knowledge suggested that teachers transformed what they knew into meaningful activities.

Lobato, Clarke, and Ellis (2005) described various ways teachers provided mathematical information to students. Specifically, Lobato, et al. tackled the notion that teachers telling students information was somehow taboo; instead, these researchers identified instances where teachers could judiciously provide information that might lead students to larger mathematical discoveries. The teacher provided mathematical terminology when necessary while the students engaged in classroom activity. Providing mathematical explanations for students' everyday activity is part of the mathematics-for-teaching described by Kazima and colleagues (2008). Because the teacher is responsible for providing mathematical terminology or providing small hints to lead to bigger mathematical discoveries, Lobato and colleagues emphasized the role of teachers' knowledge in supporting the work of student learning. They observed that the more complete teachers' knowledge was for a particular topic, the better they provided judicious information or appropriate terminology. This study provides a key component of observing teachers' knowledge in action—identifying ways teachers orchestrate classroom discussions. Whether giving students information or guiding students through an investigation, teachers have to possess a certain level of knowledge to maintain the progress of classroom discussions. The more of the desired knowledge the teacher has, the better he or she can manage discussion in a classroom effectively. Teachers know when the students are working toward an idea successfully or when they need a little support. Teachers know when students need more support to move forward with the classroom discussion, even if students only need the right name to describe mathematical topics. Teachers also know when students need less support to maintain sound mathematical activity (Lobato et al., 2005).

Chazan's study, as reported by Chazan and Ball (1999), included his efforts to balance several mathematical priorities during classroom discussion. Chazan presented a problem involving averages to his students retaking Algebra I. The students in the class reached a challenging point in their discussion: whether or not one includes the value of zero in computing an average from a list of numbers. When students forcefully proposed two different solutions, Chazan pivoted the discussion away from arguing and toward the mathematical content under investigation. In this discussion, the teacher decided whether the students would benefit from discussing a potential solution or hearing the right answer. In this study, the teacher did not orchestrate the classroom conversation as much as participate in the discussion along with the students. The teacher's goal in classroom discussion was to provide the mathematical terminology that would formalize the students' ideas. However, the role of the teacher is remarkably similar to the role of the teacher in Lobato and colleague's study (Lobato, et al., 2005). With that role, teachers demonstrated a certain knowledge that supported the discussion among students in the classroom. In Chazan's work, he found that he needed specific aspects of mathematical knowledge to keep the student's discussion focused on mathematics: relevance of the comment to classroom activity, connection to future topics in the curriculum, and the inclusion of the necessary prerequisite knowledge needed to support current classroom activity.

Mewborn's (1999) study also addressed the role of authority in a mathematics classroom. Her study focused on the changing role of authority as pre-service teachers developed their own teaching practices. Initially in the study, pre-service teachers deferred to an external source of authority for validating mathematical content. This notion held particularly true for future teachers who believed they held a weaker base of mathematical knowledge, focusing on other nonmathematical activity in the classroom instead. As a result, these less-experienced teachers did not observe how the students thought mathematically and constructed new mathematical knowledge. As the pre-service teachers gained practical experience and developed knowledge for teaching mathematics, they transitioned their understanding of authority away from an external source and observed how students learned mathematics. Instead of an external source, these teachers encouraged the students to make conjectures and defend claims. By changing emphasis in classroom discussion away from a teacher, mathematics becomes less about the teacher's presentation of content and more about a community of learners constructing new, meaningful mathematical content. Teachers need to develop different ways of representing mathematical knowledge so that the transition from roles of authority can occur. The way that these teachers learned mathematical content will not be the same way that they will teach that same content to their students (Schifter, 1998).

Simon's work (Simon, 1995; Simon & Tzur, 2004) described another aspect of teachers' work, the proposal of a hypothetical learning trajectory (HLT). The development of a HLT captured key components of a teacher's activity with a classroom of students. In this activity, teachers plan what might occur in their classrooms based on three ideas—a learning goal, activities to support learning, and the thinking and learning students engaging during activities. Simon and Tzur identified the activities to support student learning as the most challenging for a teacher to develop. A teacher possesses some understanding of his or her students' knowledge before implementing a HLT. Teachers' models of students' knowledge and learning goals might not be consistent with the students' understanding of knowledge and goals. Because of this inconsistency, activities might not be presented at the appropriate time for the student to gain a conceptual development of a particular mathematical topic. The teacher planned what he or she wanted to cover in a lesson or unit; Simon encouraged teachers to coordinate their plans and goals with students' goals for participating in lessons and units. The teacher also designed certain activities or demonstrations to build students' mathematical knowledge. If the teacher did not consider the students' prior knowledge or goals for the activity, then the teacher modified the selected activities to support what might work for students in later presentations. Simon posited that learning does not occur in a linear fashion but as points on a web of connected knowledge. Accessing webs of connected knowledge supports the construction of a profound understanding of fundamental mathematics discussed by Ma (1999).

Clemens and Sarama (2004) also reported on hypothetical learning trajectories. They defined trajectories differently than Simon,

> We conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and related, conjectured rote through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking. (p. 83)

Like Simon, Clemens and Sarama subdivided a trajectory into three components: a learning goal from teacher to students, "developmental progressions of thinking and learning, and sequence of instructional tasks" (p. 84). The distinction between the two definitions exists in the primary concern of each trajectory. Simon's HLT (Simon, 1995; Simon & Tzur, 2004) captures teachers' descriptions of students' learning. In Clemens and Sarama's work, an HLT focuses on what a student might achieve in a lesson. Simon's version started with the teacher influencing the student whereas Clemens and Sarama started with the student influencing the teacher. Schoenfeld (1998) recommended that teachers construct learning trajectories to support their understanding of students' construction of mathematics. He suggested that the more experience teachers have with particular students, the more likely teachers are to construct trajectories consistent with the ways of thinking of those particular students. Although defined differently, the components of Clemens and Sarama's learning trajectories and Simon's hypothetical learning trajectories are remarkably similar. Both reports account for an attainable learning goal for the students, activities that support students' attainment of the learning goal, and hypotheses about how students use activities to attain the prescribed goal. In these components, teachers need a knowledge of mathematics that allows students to construct their own knowledge. Teachers should consider the best activities to implement to engage students in learning. Teachers need content knowledge that monitors students' progress through activities to coordinate students' goals for completing activities and teachers' goals of developing knowledge of the subject. Teachers also should possess alternative representations of a topic to support different ways of students' thinking.

Fernandez's (1997) study focused on how teachers applied their knowledge of mathematics to handle unanticipated student questions. Such situations arose when students presented the teacher with a misconception or an alternative solution path. When faced with a comment or question from a student that they did not plan, teachers in this study typically relied on one of four options to address the student's statement. These responses included asking the students a simpler or related question, posing counterexamples, following through with the mathematical implications of the student's comments, or incorporating the student's comment into the classroom discussion. Teachers used each of the four responses to encourage students to make their own discoveries. Teachers accessed their connected knowledge to answer the student's question in each of these four responses. Teachers coordinated a variety of student perspectives and interpretations during their planned activity within the classroom. They then considered students' ways of thinking to ensure they developed new mathematical material meaningfully. Focusing on an illustrative approach to handling unanticipated questions, Fernandez provided opportunities for novice teachers to examine how they made their own thoughts explicit to their students. Making thoughts more explicit can help teachers unpack their own knowledge to communicate ideas with students more clearly.

Leikin and Dinur (2007) observed how teachers responded to classroom moments that moved teachers away from their planned activity. Leikin and Dinur investigated how teachers handled unanticipated student comments in terms of changes in the lesson plan. On the other hand, Fernandez (1997) investigated the knowledge associated with the unanticipated student comment. Leikin and Dinur did not necessarily observe knowledge of content and students. They expressed their ideas in terms of flexibility. They considered teachers who were more likely to incorporate student comments to implement flexible teaching strategies. On the other hand, inflexible strategies were those that forced classroom discussion to return students' comments back to teachers' planned activity. Leikin and Dinur based their flexibility of teachers on their interpretation of Simon's (1995) hypothetical learning trajectories. Schoenfeld (1998) reported that when a teacher leaves the planned activity to respond to a student comment, the teacher has decided that the pursuit of a resolution of the student's comment is just as valuable as—if not more valuable than—the original plan.

Mathematically Able Students

Students in the classes observed in this study enrolled in a course within an accelerated curriculum. They demonstrated a combination of ability and interest in mathematics. Teachers in this type of course approach teaching differently than when teaching students of other ability levels because of the potential of their students. Thus, I include a short introduction to the literature for both students identified as mathematically gifted and the teachers of students of this level of ability.

Identifying mathematical ability

One of the most prominent studies of the mathematical ability of young children was conducted by Krutetskii (1976). Krutetskii began his book by grounding his work in the studies of earlier scholars. He then outlined the activities that mathematically precocious students can perform at a young age based on his own studies. A partial list of those activities included: spatial reasoning; memory of mathematical structures; and flexible, curtailed, and reversible thought processes. When mathematically able students solve mathematical problems, Krutetskii observed certain characteristics. One main discovery Krutetskii found in observing problem solving was the ability of these students to remember both problems and solutions at a more generalized level; in applying previous knowledge to new problems, mathematically able students remembered the type of problem and the sketch of a solution. This recall of information is different from that of the students of different abilities as students of lesser mathematical ability tended to remember the specific numbers in an earlier problem or details from earlier problems that could not be used in later problems. Krutetskii's work became the basis for characterizing students of high mathematical ability (cf. Heid, 1983; Sheffield, 1994).

Sheffield's (1994) study examined mathematically able students' problem solving abilities. Sheffield suggested that it is important for teachers to encourage students to become both problem solvers and problem posers. Appropriate problem-posing strategies can be found in the work of Brown and Walter (1990). When students pose their own problems, they develop richer understandings of mathematics. Mathematically able students should question their answers for generalizable results. This cycle of questioning and answering permits a student to explore the mathematics curriculum deeper and more meaningfully. As a result, mathematically gifted students are given many opportunities to generalize and abstract results they find in working problems.

Sheffield (1994) observed characteristics of mathematically gifted students. She claimed that a student's rapid computational abilities are not indicative of a mathematically capable student. Sheffield's observation debunks a popular belief connecting mathematical ability and computational rapidity. Additionally, she noted that mathematical ability is not necessarily indicated by student's scores on standardized achievement tests. She stated that mathematical ability is not an innate and inert characteristic, rather a dynamic characteristic that develops over time. A dynamic nature implies that a student does not get only one opportunity to demonstrate mathematical ability for selection to special programs. Students can grow in their abilities, so they can gain appropriate access to mathematics curriculum and challenges when they are ready for them. As a result, a teacher cannot assume mathematical ability because a student can perform well on a particular test or answer questions quickly. True mathematical ability, as some students in the participants' classrooms demonstrated, exists in students' inquiry into the nature of mathematics rather than in the narrow focus on answering questions correctly.

Teachers of mathematically able students

A need exists for quality instructors to teach mathematically gifted students (House, 1987). Some researchers noted that quality teachers for the mathematically able possessed a strong content background in mathematics (Bishop, 1968; Mills, 2003). A strong content preparation is important for teachers in order to think flexibility while engaged in a mathematical discussion with students. Teachers demonstrate their flexible thinking by switching among representations, working backward from a solution to a question, and generalizing results (Greenes & Mode, 1999; House, 1987).

Students of the highest levels of mathematical ability enjoy solving mathematical problems quickly (Koichu & Berman, 2005). Likewise, teachers of mathematically able students seek to find the most elegant solution to a mathematical question (Greenes & Mode, 1999). The desire to find the most elegant solution is a result of the content knowledge students and teachers possess: they seek ways to use what they know to answer a problem in the shortest, easiest way possible. The search for elegance is consistent with Krutetskii's (1976) identification of the use of curtailed thought processes by mathematically able students. If teachers worked with students seeking the shortest possible solution, teachers strove to find the shortest solution themselves. Because their students will not be working in one direction, teachers of mathematically able students need to know when to switch directions in their attempts at a solution. As a result, teachers of mathematically able students should demonstrate a reversibility of thought processes in working toward a solution (Heid, 1983; Kahan, Cooper, & Bethea, 2003). Because teachers possess similar characteristics to their students, the knowledge of content and students (KCS) proposed by Ball and colleagues (Ball et al., 2008; Hill et al., 2004) becomes an important component of the knowledge of teachers of mathematically able students.

Because mathematically gifted students are not always intrinsically motivated in the classroom setting (House, 1987), teachers of mathematically gifted students should encourage them to explore mathematics beyond the normal school environment. Teachers can encourage these students by expressing their excitement and enjoyment of mathematical activity. Students notice the excitement and engagement of the teacher and often increase their own interest in mathematics in response to the teacher's enjoyment of the subject. Researchers have identified such a characteristic across teachers of the mathematically gifted (Bishop, 1968; Csikszentmihalyi, Rathunde, Whalen, & Wong, 1993; Greenes & Mode, 1999; Mills, 2003). In an accommodating learning environment, mathematically able students would be able to generalize mathematical results and abstract mathematical properties from the presentation of only a few examples (Heid, 1983). Teachers should provide the appropriate challenges and exploration for discovery to achieve these results, (Saul, 1999; Sheffield, 1994). Students engaged in exploration and generalization expend considerable time and effort. Teachers notice the resources students devote to mathematical activities. When discussing students' fatigue in learning mathematics, Krutetskii (1976) reported teachers held divided opinions. Some teachers believed that mathematically able students did not become fatigued in working mathematical problems. Other teachers disagreed with that assessment. In his report, Krutetskii provided both support and refutation to House's observation.

Park and Oliver (2009) observed how teachers work with gifted students in science classes. They found that teachers exhibited unique activities working with gifted students. One unique activity was responding to challenging questions students posed. Park and Oliver defined challenging questions as questions going beyond the level of teachers' content knowledge. They found that teachers of these students needed a greater flexibility as described by Leikin and Dinur (2007) to respond to these questions. Park and Oliver discovered that teachers either felt embarrassment or welcomed challenges when presented with a challenging question. These feelings depended on the teachers' level of interaction with their students and access to available knowledge. Park and Oliver found that if teachers developed content-specific teaching strategies in handling challenging questions, then they could gain additional flexibility in maintaining classroom discussions. One specific strategy they found was teachers returning challenging questions back to students. Teachers presented what they already knew about the content to create scaffolds for students' work toward a solution. In these situations, teachers and students could work collaboratively toward a possible solution to a challenging question. Because of the interactions they had with gifted students, these teachers reported to Park and Oliver that their knowledge of the subject matter grew.

Secondary School Calculus

I observed two Advanced Placement Calculus classes in this study. Literature already exists investigating the curriculum and teaching of this course and similar courses. Researchers from the fields of pure mathematics and mathematics education identified important aspects of a calculus course to support students' learning of the subject. Some of those researched aspects are important for the students' future mathematical growth. I discuss the results and recommendations of a few of those reports.

Ferrini-Mundy and Gaudard (1992) reported on how high school calculus prepared students for calculus courses at the university level. In Ferrini-Mundy and Gaudard's study, enrollment in a full-year secondary school calculus class led to higher grades in university calculus courses. They found that the type of secondary calculus curriculum did not affect grades in the first semester of calculus; there were no statistically significant differences between the exam scores of students from an Advanced Placement curriculum or from another secondary school calculus curriculum. Ferrini-Mundy and Gaudard found secondary mathematics teachers should not provide students with an introduction to calculus topics. They recommended to teachers growing their students' understanding of pre-calculus mathematics or encouraging enrollment in a full-year calculus course. Students receiving only an introduction of less than one semester to calculus in high school did not receive better grades in their collegiate calculus class. Although Ferrini-Mundy and Gaudard noted improvement in grades for the first semester of collegiate mathematics, the improvement diminished in later semesters of mathematics.

Much like Ferrini-Mundy and Gaudard (1991), Burton (1989) advocated that students in 12th grade should either gain a better understanding of pre-calculus mathematics or take a full-year calculus course. Unlike Ferrini-Mundy and Gaudard, Burton specifically referenced the

Advanced Placement curriculum as acceptable for high school students. Burton suggested that students who took a calculus class in high school should not repeat the course in college because it creates two problems. The first problem is that high school students are not sufficiently challenged in their first collegiate mathematics course. Repeating courses undoes the purpose of accelerating students' curriculum. Students lose the advantage of having services that meet their abilities. The second disadvantage is that first-time calculus takers are at a decisive disadvantage in learning the content. Students repeating this course skew instructors' assessments of students' knowledge. By altering the learning trajectories, the instructor presents different goals and activities to the class. Those students missing the necessary prerequisite knowledge are at a disadvantage to students repeating the course. Students repeating a course are at an advantage only because they took the same course earlier.

Askey (1997) provided his own suggestions for a meaningful high school calculus course. In his observations, he noted that an increasing number of students enroll in Advanced Placement Calculus courses. That observation was not only true at the time of publication but is still true after the publication of his article. He observed that Advanced Placement exams compared favorably to the exams he administered in his collegiate calculus classes. He feared changes in the Advanced Placement examination would deemphasize the symbolic manipulation and arithmetic computation he valued in his own collegiate calculus courses. This fear highlighted some of the concerns about the Advanced Placement program raised by collegiate calculus instructors (Bressoud, 2004). Bressoud argued that students should attempt calculus at the secondary level only if the course is consistent with the rigor of a collegiate calculus course. If the secondary course is not rigorous, students should instead develop a deeper, more conceptual understanding of pre-calculus mathematics while still in high school. With a stronger foundation in school mathematics, students would be prepared for the challenges and demands a collegiate calculus course provides. However, students who demonstrated ability in mathematics should handle a meaningful, challenging calculus course worthy of earning collegiate credit suggested by the Advanced Placement program.

Theoretical Framework

Pirie (1997b) suggested that research could suggest codes and methods of analysis for subsequent studies. Following Pirie's suggestion, I implemented work from other studies to guide the analysis of data in my study. I used other researchers' studies to identify how teachers handled unexpected mathematical moments in their classroom. I describe those implications below.

Fernandez (1997) investigated how teachers handled unanticipated student comments and questions. She identified unanticipated student comments and questions as perspectives:

> The students' unanticipated perspectives were all conveyed through student errors, difficulties, or alternative student-initiated approaches to problem solving. Student errors are mistakes in reasoning, computation, or interpretation. Student difficulties are displayed obstacles to problem solving (typically conveyed through a student's question). Alternative student initiated approaches to problem solving are valid (non-erroneous) problem solving methods the teacher did not plan to use. (p. 7)

Teachers sometimes anticipate how students present different solutions or make mistakes in learning new material, but some approaches cannot be anticipated in advance. In identifying such perspectives in my study, I observed when students made a mistake in presenting a solution or offered a statement or question that seemed not to match the teacher's expectations. Although I might deem an episode to be unexpected, I needed confirmation from the teacher.

Fernandez (1997) categorized teachers' responses to unanticipated student perspectives into four approaches. The first approach was the teachers' use of counterexamples. By posing a counterexample, teachers "challenge their students and enable the students (versus the teacher) to examine their perspective" (p. 8). Rather than telling the student he or she was wrong, teachers chose an example to allow the student to examine what was wrong mathematically. Teachers maintained mathematical discussion by encouraging students to work through a counterexample to find the flaw in the original statement. The second approach was the follow through, where "the idea is to explore the implications of the posed mathematical thought by continuing or following through with it" (p. 11). Teachers found something interesting or worthwhile in a student comment and pursued the student's comment until reaching a conclusion. As a result of this approach, teachers maintained student involvement by valuing and incorporating the new comment. The third approach, posing simpler or related question, allowed teachers to "generate ideas from the easier case that can be applied to the more difficult one so that the original problem can be solved" (p. 13). Teachers made the current question easier to solve for the student by asking a question, or a series of questions, to which the student might know the answer. Teachers maintained classroom discussion by encouraging students to answer questions they knew in order to return back to the original, difficult question.

Fernandez's (1997) fourth approach was to incorporate the students' solutions into the planned activity. She stated that through this approach, teachers were "incorporating alternative student-initiated methods [to show] there exists more than one *correct* path" (p. 16, emphasis in original). However, as Fernandez admitted, "At some level, this use of knowledge is illustrated in all the categories above" (p. 16). Her first three categories explained different approaches teachers took in handling the unanticipated student questions or comments. The fourth category did not provide a new or different approach but included the previous three. In order to have distinct categories, I used the first three of Fernandez's categories for my analysis.

Because the students in this study were enrolled in Advanced Placement courses, I wanted to include research from gifted education that identified teachers' interactions with students. Park and Oliver's (2009) study focused on different teaching techniques science teachers used with students identified as gifted. One technique was handling students' challenging questions. Park and Oliver defined challenging questions as "unusual and insightful questions. In some cases, those questions were beyond the scope of the content knowledge the teachers possessed" (p. 339). In response to challenging questions, Park and Oliver found that teachers "tended to use the questions as a learning opportunity for both students and themselves" (p. 345). These moments allowed teachers to be students of the subject matter as much as their high school students were. Although Park and Oliver observed these questions in science classrooms, I believe students pose challenging questions in mathematics classrooms as well.

Given that students pose challenging questions to teachers, I replaced Fernandez's (1997) fourth category with teachers' acknowledgement of challenging questions from Park and Oliver's (2009) report. This replacement gave me four strategies that teachers use in responses to students' questions. In three strategies, teachers answered the students' questions or responded to the students' comments to the students' satisfaction. I define student satisfaction to be the point in classroom discussion when the students did not ask teachers additional questions. The fourth strategy illustrates that teachers do not always know the answers to students' questions. Teachers need to feel comfortable in acknowledging the limits of their own knowledge to their students.

In order to identify patterns in my data across all four strategies suggested by Fernandez (1997) and Park and Oliver (2009), I used patterns of teachers' flexibility in classroom discussions described by Leikin and Dinur (2007). Leikin and Dinur identified teacher flexibility in the following way: “We consider a teacher flexible at a particular point of the discussion if he/she adjusts the planned learning trajectory according to student replies that differ from those he/she had foreseen” (p. 330). They created “patterns of flexibility” (p. 332) to describe ways teacher handle unexpected student responses. Leikin and Dinur identified two patterns of flexibility where teachers incorporated the student’s comment into a planned lesson successfully. They defined “different outcomes” (p. 334) for those moments when a student presented an alternative, but valid, solution to a teacher’s planned question. They then defined “different scopes” (p. 334) to be those moments when teachers accepted different representations of the same solution. When teachers applied one of the four strategies from Fernandez or Park and Oliver, they demonstrated flexibility in handling students’ questions or comments.

In their report, Leikin and Dinur (2007) presented different outcomes and different scopes in a visual representation. Diagramming classroom episodes in my study allow me to see how teachers handled the unanticipated student comment or question with respect to their planned activity. Leikin and Dinur illustrated patterns of flexibility with a task and two available options—the planned activity and the student’s unexpected comment or question. For their two flexible patterns, Leikin and Dinur diagrammed how teachers maneuvered between the two available options. For different outcomes, they showed how teachers moved from responding to the student’s comment to attending to the planned activity. The episode ended when the teacher was able to return to the planned activity. For different scopes, Leikin and Dinur showed how teachers responded to the student’s unplanned comment by asking a question or posing a subtask to move the students back to the planned activity. The new question would generate a different unexpected response, and teachers would respond by posing another question. This pattern can repeat as long as students pose unexpected responses. An episode ended either with a return to the planned activity or a new, unplanned task to explore.

CHAPTER 3

STUDY DESIGN

In this chapter, I outline the design of this study. I begin by including information about the schools and teachers used in the study. I then outline the methods used to collect data from the participants. Next, I describe the methods used to analyze the collected data. I conclude this chapter with a subjectivity statement.

Setting of Study

The study took place at two high schools: Pierce High School and Buchanan High School, both pseudonyms. Between the two schools, common characteristics emerged. Demographically, both schools contained a majority of students as described as White. In curricular offerings, both schools offered two sections of Advanced Placement Calculus during the school year I made my observations. One school offered one course, and the other school offered two courses. The schools had dissimilar characteristics.

Pierce High School

Pierce High School lies on the edge of an area designated as a suburb of a large city. It is one of two high schools in its district. For the school year I observed, there were approximately 1,500 students enrolled at Pierce. It was identified as a Title I school for the year I observed. Fifty-three percent of the students at the school qualified for Free or Reduced Lunch. Eleven percent of students were identified as receiving services for disabilities. White students comprised 62% of the student population, and Asian students comprised 8% of the population, African-Americans 15%, and Hispanics 11%.

For the school year I observed, Pierce followed a four-by-four block schedule: A student took four courses that met for 85 minutes one semester and took a different set of four courses the next semester. Pierce is the high school that offered two calculus courses. All of the students in the observed class took calculus for one block each semester. A fuller description of the students' two calculus classes is included later.

Buchanan High School

Buchanan High School was the only high school in its district, in a county outside any metropolitan area. During the year I observed, Buchanan had an enrollment of 1,000 students. Forty percent of students at Buchanan qualified for Free or Reduced Lunch. Only one percent of the students were identified as having Limited English Proficiency. Demographically, White students comprised 66% of the student population; African-American students comprised 27% of the population and Hispanics 4%.

The school was on a modified schedule of classes, mixing a traditional and block schedule throughout a 5-day week. Students were enrolled in seven courses each semester. For three days of the typical week, the students took all seven courses for 50 minutes per session. One day of the week, students took the courses on the odd-numbered class periods for 70 minutes per session. The other day of the week, students took even-numbered class periods for 70 minutes per session and had additional time in the school day for academic enrichment. Most courses are yearlong courses; students enrolled for their calculus course the entire school year.

Advanced Placement Calculus

The Advanced Placement Calculus AB course has been in existence for over 50 years. The idea in offering such a course is to provide mathematically gifted or motivated students the opportunity to take the equivalent of college calculus course while still enrolled in high school.

The students enroll in a course for a one-year credit; at the end of the school year (typically the first week in May for calculus) the students may take a standardized examination, administered by the College Board, to earn college credit for the course. Students are not required to take the examination if they enroll in the course at their high school nor are they required to enroll in a course to take the examination. The college or university where the student eventually matriculates awards credit for a collegiate course based on the student's score on the examination.

For the calculus course, the College Board provides two offerings. The first offering, Advanced Placement AB, is roughly equivalent to a first semester of introductory calculus. Students begin an exploration of the derivative and integral, learn basic rules for the computation of derivatives and integrals, and find applications of those concepts to other fields of study. The second offering, Advanced Placement Calculus BC, is roughly equivalent to the first two semesters of introductory calculus. In this class, students first follow the same curriculum as the students in the AB course. They then extend their studies to include derivatives and integrals of functions in parametric and polar form and investigations of infinite sequences and series.

A student is typically enrolled in one course or another; a student does not take both courses concurrently. Some students can enroll in the first course followed by the second, usually on a four-by-four block schedule with the AB course in the fall semester and the BC course in the spring. Students at Buchanan took either the AB course or the BC course for the entire school year. All but one of the students enrolled in the AB course at Buchanan. Because of the four-by-four block schedule, students at Pierce could enroll in both courses in the same school year. All of the students in the observed class enrolled in the AB course during the fall semester and the BC course during the spring semester. Pierce offered the course in the spring semester as an introduction of the topics in the BC curriculum to reinforce and extend topics covered in the AB curriculum. Likewise, because most students elected to take the test only for the AB curriculum, the focus of the second semester was on enriching the topics from the first semester.

Participants

Participant Selection

I obtained a list of Advanced Placement Calculus 17 teachers who participated in a local learning community for AP Calculus. I contacted teachers from the list to inform them of my study and ask permission to enter their classrooms to observe. Once a teacher accepted this first invitation, I observed the teacher's class for one session to gauge the level of interaction between the teacher and students in the classroom. Once I determined that there was a sufficient level of interaction to allow robust data collection, I extended the invitation to participate in the study. I extended individual invitations to three teachers: two teachers accepted the invitation. The teachers in this study satisfied Maxwell's (2005) recommendation to select participants who would provide "description, interpretation, and explanation" (p. 71). Because the selected teachers satisfied those criteria, I believed they represented what I wanted to investigate in terms of the teachers' knowledge. I have used pseudonyms for both of the participants.

Barry

Barry had been teaching calculus for 8 years at Pierce High School. He came to Pierce when the school opened in order to teach Advanced Placement Calculus. He had taught at a different school in the same district before coming to Pierce but did not teach AP Calculus while at the other school. He had been teaching for a total of 10 years at the time of my study. Barry held a master's degree in mathematics education from a large, research-oriented university. His undergraduate degree was not in mathematics or education.

Broad categories describe the approach to Barry, the teacher at Pierce High School, to teaching AP Calculus: how he saw his experience in mathematics, how he reflected on his approach in teaching, and how he saw his understanding of students. In his first interview, Barry mentioned that he had been teaching some sort of calculus class at the high school level for approximately 10 years. That base of experience benefited him as he approached newer concepts in the calculus curriculum and provided ample opportunity to review the earlier topics in newer contexts.

Barry enjoyed the opportunity to teach high school calculus. He had looked forward to teaching calculus as a novice teacher because of his experiences as a student in mathematics classes. Barry envisioned calculus as the combination of the previous mathematics classes in high school–algebra, geometry, and trigonometry. In preparing for this course, Barry had reflected on his experiences in a history of mathematics course he took as part of his teacher preparation program and brought in the historical development of many mathematical concepts.

Barry considered many ideas when thinking about how he would approach teaching a high school calculus course. He planned the course at three levels–the entire year, an individual unit, and each individual lesson. Planning across the year provided him an opportunity to make broad connections and to reflect on the role of prerequisite knowledge in developing the content.

Barry modified his unit and lesson plans based on his experience of teaching AP Calculus for several years. When planning for a year for the first time, Barry generally relied on the textbook’s (Stewart, 2002) presentation for each unit and lesson. Over the course of many years, each new class of students provided him a chance to refine and modify each unit and lesson. As he made those changes, Barry often incorporated additional resources to support the changed lesson.

Throughout the year, there were many times when a review of previously learned material became its own lesson. During his first interview, Barry described how, near the middle of the year, he had the students participate in a derivative scavenger hunt. The students would compute the derivatives of several functions and attempt to find a solution to the expression by visiting several classrooms throughout the school building. The goal of the activity was what Barry called “the joy of learning” (Interview 1); the students were learning not for some prize, but for the reinforcement of a topic that they had spent a considerable amount of time investigating in class.

For each class period, Barry had rough idea of how the events in the period would transpire. He began each class period going over the previous day’s homework assignment whenever an assignment had been made in an earlier class session. He planned on reviewing approximately two or three exercises each period as a type of formative assessment, but there were occasions when the questions were either too involved or when a greater number of exercises needed review, requiring a full class period to achieve his objective. Generally, however, after reviewing a few exercises, Barry moved on to the new lesson for the day. He transitioned from homework review to new lesson by posing questions that would show a gap in the students’ current knowledge and how the new material would fill in those gaps. After presenting the new material, he offered additional examples to the students by using even-numbered textbook exercises. He chose those exercises because the odd-numbered exercises gave the students an additional resource, exercises with solutions in the back of the book, as they worked on their homework assignments. Barry also selected these exercises to illustrate additional wrinkles to what the textbook presented or what the homework exercises included, such as connecting different types of prior knowledge to new material.

Missing from Barry's description of the typical class period was student group work or class time for the students to begin a homework assignment. Barry believed that students preferred working individually, leading him to use small group work sparingly. Even though a class period was nearly an hour and a half in length, Barry did not want to end instruction too early; although his students were conscientious, he believed that providing too much time at the end to start on homework would not have served his purposes well, as students would not utilize the time to begin the assignment.

Barry's typical class period did not depend solely upon his presentation of material; although direct instruction was common, it was not the sole activity during the main portion of the lesson. Barry did not want the students passively accepting his presentation of material. Thus, throughout his presentation, he encouraged students to present work on the dry-erase board and frequently asked students questions throughout the class period. By posing questions to students and encouraging public work, Barry was ensuring that the students were active participants in the class.

Presentation of new material in Barry's class often contained connections to previously learned material or a historical development of the new topic. Sometimes physical objects were included in the presentation of new material as well. From his experience teaching the class several times, Barry knew students could not necessarily visualize the mathematical activity they were performing; as a result, he brought in objects to facilitate the students' visualization of calculus topics. In the first interview, Barry mentioned that he brought in food to show the students how they could connect a tangible object to the calculus concept under investigation, which was determining volumes of solids by revolving an area about an axis of symmetry or adding areas of polygonal shapes that form cross-sections of the solid. For the historical development, Barry wanted the students to see that new concepts had come about because of a need to answer a previously challenging question in mathematics. He wanted the students to see that a calculus concept does not exist in isolation: Some individual (or individuals) were largely responsible for the creation or development of a particular concept. Those stories helped to supplement why he presented a certain topic to his students at a certain time.

Barry's calculus classes were not for the enjoyment of the teacher. Rather, he had a classroom of students who were the object of his instruction. Thus, he considered what he knew about his students in order to build meaningful activities in lessons. What helped him understand the students in his calculus class was that he had taught many—if not all—of them in a mathematics class earlier in the students' high school career.

In the first interview, Barry described what he believed were the typical characteristics of an Advanced Placement Calculus student and how Pierce High School's students slightly contrasted with his own characterization. Although all of the students could perform well on a test or quiz—thus earning good grades throughout their school career in a mathematics classes—some memorized what they needed to do well on an assessment without concern for learning for deeper meaning. However, not all students fit that characterization. In the class I observed, Barry pointed out that there were several students who possessed a genuine interest in learning this complicated material. His goal was to pique their curiosity and acknowledge their valid contributions in each lesson. Barry's lessons included alternate representations and additional explanations so that all students could understand the material. His lessons included these extra components to develop students' deeper understandings of mathematics. He was changing his students' meanings of learning mathematics because they had previously considered memorizing facts the same activity as learning mathematics, as he mentioned in the first interview.

Barry considered his students when planning a lesson. While planning, he thought about the potential questions students could ask and incorporated his answers to those questions into his plans. He expected students to remember information from earlier mathematics classes; he held the students accountable for the classes they had passed successfully earlier in high school. In addition to the connections to previous years, Barry made connections to previous units and lessons within the year. In the first interview, he provided an example of connections across lessons within a unit. When reviewing integration techniques, he gave an assessment expecting the students to provide anti-derivatives to expressions without a reference to a specific technique to apply, although the students had learned each technique separately. He expected the students to recognize the difference among *u*-substitution, partial fraction decomposition, and integration by parts and applying the proper technique for a given anti-derivative.

Before each lesson, Barry expected the students to read the corresponding section in the textbook, which was a textbook designed for college calculus courses. In addition to preparing students for the day's lesson, Barry wanted them to gain a familiarity with reading mathematics textbooks. In assigning this reading, he wanted the students to struggle with the topics in the reading. His presentation of material the next day in class clarified and supported the understanding the students obtained from reading in advance.

Kris

Kris described her path to becoming an Advanced Placement Calculus teaching during the first interview. When she was a graduate student Kris taught mathematics classes—including calculus—to students enrolled in business programs at a nearby university. During the summer of those years, Kris worked in a supporting role for grading the end-of-year Advanced Placement exams. Years later, she ended up teaching high school mathematics after she had moved to a different state. She originally taught a high school calculus class not aligned with the Advanced Placement curriculum. Upon moving to Buchanan High School, she began to teach Advanced Placement Calculus. At the time of the observation, Kris was in her eighth year of teaching the AB portion of the curriculum and had started in her first year of teaching the BC curriculum to a student enrolled in an independent study course in the school.

Most of Kris's class sessions followed the same progression of activities from beginning to end. Each session began with reviewing the previous night's homework assignment. After the review, Kris presented the new lesson for the day, leading the discussion. Although she did not develop a formal lesson plan for each day, she wrote down a structured set of notes containing her main ideas. Rather than leading students through a discovery of the introduction of a concept, Kris told the students about a concept and then led a discovery of possible connections to other topics. At the end of the class period, Kris assigned a small number—no more than ten typically—of odd-numbered exercises from the textbook (Larson, Hostetler, & Edwards, 2001), so that the students could check their solutions in the back of the book and ask clarifying questions the next day as needed.

Kris used the textbook (Larson et al., 2001) to guide the development of the unit and lessons for this class. Although she used the text as guide, she never felt forced or constrained to follow the text's treatment and progression of material. In the first interview, Kris mentioned that she knew other textbooks isolated other transcendental functions, including logarithmic and exponential functions in separate chapters. Without feeling any strong compulsion to do so, Kris maintained the textbook's prescribed treatment of a separate unit on logarithms and exponential functions.

Kris had learned about her students through her past experiences teaching them. Many—if not all—of the students had been enrolled in her pre-calculus class the year before; some students had had Kris as their mathematics teacher their entire high school career. In the first interview, Kris mentioned what she considered to be the typical student in her Advanced Placement classes. She noticed three commonalities among her students–liking her as a teacher, succeeding in earlier mathematics classes, and wanting to take mathematics classes in college. Because Kris and the students had worked together before, she knew some students in the class needed more support and encouragement than others. She also knew when certain topics needed to be reviewed to support the students' understanding of calculus concepts. For example, before presenting the calculus of logarithmic expressions, Kris presented a daylong review on the application of properties of logarithms.

Kris's knowledge of her students from earlier interactions provided additional benefits. As the end-of-year Advanced Placement examination required the use of a graphing calculator, Kris knew when to discuss the proper use and interpretation of this technology when learning about calculus concepts. In the first interview, she mentioned that students believed that the graph of the logarithmic function contained a *y*-intercept because of the way the technology sketched the graph. She spent time in class discussing this misconception. Knowing that her students wanted to perform well on the standardized examination, Kris frequently questions they raised about the examination. Her involvement in grading the Advanced Placement examination spanned 21 years. Kris shared her knowledge of the grading of these questions with her students during the presentation of her lessons. She believed her students wanted to hear these experiences. Students frequently asked her questions to understand what should be written on the examination paper to earn the maximum number of points.

Rationale for Data Collection Methods

The two data collection techniques for this study, observation and interview, were chosen to work together; one technique supported the other. The observation allowed me the opportunity to see the teacher work in his or her usual environment, whereas the interview allowed me the chance to have the teacher reflect on that work. Some of the interviews allowed the teacher to reflect on what had happened, and others permitted the teacher to foreshadow what could occur in future lessons, which allowed me to anticipate possible episodes of interest. Maxwell's (2005) description of the combination of these two techniques captures my intention for including both in my study: "While interviewing is often an efficient and valid way of understanding someone's perspective, observation can enable you to draw inferences about this perspective that you couldn't obtain by relying exclusively on interview data" (p. 94).

The four interviews were similar to the three-interview structure suggested by Seidman (2006). Seidman suggested that researchers could make modifications of the three-interview structure, so long as the participant still reflected and reconstructed his or her experiences. The first interview and final interview in my study adhered to Seidman's suggestions for the corresponding interviews in his structure. The first interview allowed me the opportunity to understand a context for the unit of instruction I was to observe; the final interview allowed each teacher the opportunity to reflect on the entire experience. My second and third interviews were an extension of Seidman's middle interview. By providing two interviews instead of one, I gave each participant an opportunity to reflect on episodes when details were fresher in the participant's memory.

I attempted to ensure the validity of the data by satisfying Eisenhart's (1988) recommendations. She proposed three activities to validate the findings of an ethnographic study.

Those activities were a long involvement in the field of observations, acting within the "idiom of the participants" (p. 109), and direct involvement within the observed culture. I was present for all observations in this study, meaning that I observed the teachers interacting with their students directly. I tried to use the teachers' own wording when discussing mathematical topics. For example, Barry insisted on using the phrase "limits at infinity" to describe the expression $\lim_{x \to \infty} f(x)$. I made sure to use the same terminology myself so that we could mean the same thing when using the same phrase. Likewise, I believe the length of time at each school provided me an opportunity to see the participants in typical classroom interactions. By the end of each unit of instruction, I felt that I had spent a sufficient period of time in their classroom and observing their teaching style to gain an understanding of how they interacted with their students.

There was an additional step I took toward validating my findings in this study. Like Pirie (1997b), I used field notes to support the data in the audio recordings. Reading the field notes while listening to the recordings provided me a chance to confirm my conjectures about classroom activity. I used the interviews and the transcripts to give me a chance to explore the teachers' interpretations of what I had observed and to check my interpretations against theirs. I provided the teachers opportunities to recall thoughts and feelings regarding what I identified as key episodes across the observations. The introductory and final interviews helped to provide detail and understanding for the observation as a whole.

Data Collection Methods

I observed each teacher for one unit of instruction. Upon agreeing to be a participant in my study, each teacher informed me of an appropriate time to enter his or her classroom to start the observation in order to see the entire unit. I observed the class from that beginning date to the conclusion of the teacher's instruction for that unit. For Barry, the unit lasted 15 class periods over 5 weeks. Each period lasted 85 minutes. Throughout the unit of instruction, Barry took 10 days off from the new unit of material to have students review first semester topics. I observed Kris for 9 class periods over 3 weeks. Because of Buchanan's unusual schedule, not all class periods lasted the same length of time. Of the nine sessions I observed, seven were 50-minute sessions, and two were 70-minute sessions.

Throughout the unit of observation, I interviewed each teacher four times. The semi-structured interviews began with an interview guide, and I pursued additional questions as I deemed appropriate. (See Appendices A–C for all interview guides.) I began with an introductory interview lasting approximately 1 hour and preceding the first day of observation. The next two interviews took place in the middle of each participant's unit of instruction. These two interviews lasted between 30 and 45 minutes. The final interview took place after the completion of the unit of instruction; this interview lasted approximately 1 hour. The timing of the interviews is shown in Table 1.

Table 1. Scheduling of four interviews.

Teacher	Timing of First Interview	Timing of Second Interview	Timing of Third Interview	Timing of Final Interview
Barry	Day before unit of instruction	After the third day of instruction	After the eighth day of instruction	Two days after unit of instruction
Kris	Day before unit of instruction	After the sixth day of instruction	After the eighth day of instruction	Day after unit of instruction

For each lesson I observed, I made audio recordings of the discussion between the teacher and students. Like Pirie (1997b), I decided to record classroom observations by audio devices only, because video recording devices can be intrusive to the normal classroom environment. I used two recording devices in the classroom–one placed in the front and once placed in the back of the classroom. The front device mainly captured the teacher's voice, and the back device captured questions and comments not covered by the front device. I achieved my goal of being minimally intrusive because many students forgot that the recorders were there. At times the students reminded each other that what they had just said was being recorded, and it would be a good idea to think about what they said.

To support the data collected from audio recording devices, I took field notes for each lesson. In each lesson, I recorded two types of notes. One type was noting the set of key moments that I could consider for retrospective analysis of mathematical knowledge; the second type was recording pertinent information that the teacher wrote on the dry-erase board. The field notes were kept in spiral-bound notebooks.

The use of field notes supported information gained from the observations. Collecting field notes while observing captures information regarding the participants that cannot be gleaned by asking the participant directly in an interview (Emerson, Fretz, & Shaw, 1995). Using suggestions from Emerson and colleagues, I used the field notes to identify key moments from the observation to revisit, including key details to use in later interviews.

For this study, field notes were particularly helpful for the second and third interviews. I found that my field notes helped me signify noteworthy events that could be the basis of an episode to discuss with the participants during these two interviews. Likewise, the field notes were useful as I constructed the lesson graphs. (A description of lesson graphs is given in the next section.) I found that the field notes provided supplementary information in the early analysis.

Data Analysis Methods

Using Eisenhart's (1988) description of ethnography, I was able to use the analysis of data to make "meaningful 'units' of the material" (p. 107). Because the goal of an ethnographic study is to make sense of others, I implemented a couple of different methods to understand how teachers apply what they know mathematically to their classroom conversations. I explain both of those methods below.

I first created lesson graphs. For each classroom observation, I created a table that contained three columns. For each row in the table, the first column included the elapsed time in the class period, the second column a key quotation or event, and the third column a brief note to refer back to in later analyses. I made suggestions in my field notes to tag episodes for future study by changing the style of handwriting I used from the recording of actual classroom observances. Upon relistening to the classroom discussions and rereading my field notes, I made notes in the third column to suggest the most fruitful episodes for investigation. An example of such a chart is shown in Appendix D. After creating the lesson graphs, I selected important episodes from a class session by using notes from the third column of the table and consulting the field notes I had collected during the observation. I transcribed those episodes. Each class session contained between one and four episodes, lasting between 2 and 15 minutes.

I read through the collection of transcriptions to determine if episodes contained unanticipated student questions or comments. I determined that a student comment or question was unanticipated using one of two criteria: referring to field notes and probing during the second and third interviews. Whenever the classroom conversation appeared to be moving away from the teachers' planned activity, I made a separate note of the apparent change in my field notes. I used the separate notes to help identify interesting episodes to probe the participants' understanding. I recorded all of these separate notes in the margins of my original field notes. I did not probe all of the tagged episodes with the participants, because a large number of episodes existed in each class period. Probing all such episodes would have required an unrealistic time commitment on the teacher's part during the middle interviews. I chose my most interesting episodes to probe in order to be respectful of the teacher's time. If a segment contained an unanticipated student comment or question, I coded it using the coding scheme found in Fernandez (1997). When the teacher did not know an answer the question to the student's questions, Fernandez used a different code, "U/I". For each code, I wrote a brief note to provide the justification for selecting that code. After all segments were coded, I tabulated the number of codes by teacher.

I made one modification to Fernandez's scheme. In her report, she used "U/I" to represent a situation in which the teacher did not have the appropriate content knowledge to respond to understand or incorporate the student's comment or question. I changed this code to challenging questions (CQ), as described by Park and Oliver (2009). I argue that a teacher not knowing a particular bit of content knowledge is not necessarily problematic. Instead, such episodes permit teachers to recognize that students identified as gifted sometimes ask questions at the limits of the teacher's current knowledge of a subject.

Once I had selected illustrative episodes, I charted the progression from the posing of the student's question to the resolution of the question. These charts were similar to the charts found in the report of Leikin and Dinur (2007). I made one noticeable modification to their charts. Whereas their report included the teacher's original, planned activity, I decided to focus solely on the deviation from the planned lesson. Deviations from planned activity were common for both teachers. Kris did not use organized lesson plans; she had a general plan for each lesson, including key questions and examples to present. If the students needed to pursue a topic beyond the sketch, Kris could handle that activity and return to her plan. That was an idea she shared with me during our first interview. In Barry's class, deviations from the planned lessons were common. During our first interview, he said that diversions were a good management technique for teaching high school students in this schedule format. Additionally, he believed that the diversions were helpful in scaffolding the students' knowledge of mathematical concepts.

In constructing each chart, I attempted to determine if each moment within an episode extended the activity of the diversion or brought the class back to a planned lesson. The direction of the arrows suggested one of these two movements. An example of such a chart is shown in Appendix E. An arrow pointing to the left signifies activity to support a diversion, an arrow pointing downward suggests coordination of planned activity and diversion, and an arrow pointing to the right identifies a return to the planned lesson.

Researcher Bias and Subjectivity Statement

Glesne (2000) suggested that the beginning researcher should not shy away from his or her subjectivities but should acknowledge them and understand how they bear on the study and the results. Thus, I identify and describe those subjectivities. My interest in the group of individuals I investigated, teachers of Advanced Placement Calculus, was a result of my own experiences as a high school teacher who had taught Advanced Placement Calculus. I greatly enjoyed the experience, particularly the challenges it presented to me.

Because I had a preference for teaching AP Calculus, I viewed the course and the teachers of the course in a positive light. Thus, I was aware that I might be giving the teachers more credit for the knowledge they possessed than was warranted. To avoid that possiblity, I relied on field notes and transcripts of audiotapes of the classroom and interviews to provide evidence for claims I made about the teachers' knowledge. Because I examined the knowledge that the teachers used in instruction, the results are shaped and limited by the knowledge of the subject I bring to the study. Last, because several of the studies that I read on teacher's knowledge of mathematics tended to report knowledge in terms of the deficiencies teachers bring to the classroom, I wanted to highlight the knowledge each teacher did possess and use. My goal was to identify and report knowledge in the positive (what the teachers knew) rather than in the negative (what the teacher did not know). As a result, the next chapter refers to what the teacher knows, instead of what each teacher knew.

CHAPTER 4

FINDINGS

Introduction

After observing the two teachers, Barry and Kris, for one unit of instruction, I noted that students provided both teachers opportunities to apply their knowledge through unanticipated questions. These episodes help answer a question posed by Schoenfeld (1998): "*Something has happened. What will the teacher do next, and (more importantly) why*?" (p. 3, italics in original). Each teacher used a variety of approaches to resolve student questions. In my analysis, I counted the frequencies of those various approaches they used to deal with unanticipated responses using Fernandez's (1997) four categories. Below I discuss how each teacher used the four approaches in his or her classroom activity. Patterns for each approach emerged for addressing unplanned activity, and those patterns are detailed at the end of this chapter.

The manner in which Barry and Kris handled unanticipated student questions corresponded to three of four Fernandez's (1997) categories: namely, following through with students' comments to a logical conclusion (given the code FT), posing a counterexample to students' comments (given the code C), and asking a simpler or related question to students' original question (given the code S/R). I used a fourth code drawn from the work of Park and Oliver (2009) regarding challenging questions posed by students; I gave these episodes the code CQ. Table 2 shows the frequency of each code across all of the observations.

Table 2. A breakdown of total number of episodes listed by category.

Teacher	FT	C	S/R	CQ
Barry	14 (26.9%)	7 (13.5%)	27 (51.9%)	4 (7.7%)
Kris	7 (20%)	8 (22.8%)	19 (54.3%)	1 (2.8%)

Across all observations, Barry had more episodes of interest than Kris because of the different number of hours of observation for each of the units. But when I compared the number of episodes to the length of the observations, the results were reversed and Kris had more episodes per hour than Barry.

One possible explanation for the different rates in episodes between the two teachers deals with how Barry approached the 85-minute class session. Although his students demonstrated an interest in mathematics, Barry mentioned in his first interview that they needed time to absorb what he presented. For example, Barry told stories regarding the presented topic to provide his students time to reflect and absorb. As he mentioned in the first interview, those stories sometimes scaffolded how the students learned the material: Students would sometimes recall the related story before the concept. This notion of permitting students to reflect on mathematical concepts in class has support in the report by Chazan and Ball (1999). They suggested that the intellectual pace of a classroom discussion be slowed down on occasion for the students to make the connections between the newly presented material and their prior knowledge base. A slow down can occur in a couple of ways. The report of Chazan and Ball suggested soliciting additional solutions from students; Barry found ways to incorporate supplementary ideas to connect mathematical ideas in different ways. Although Chazan's down time occurred in a class of students repeating a high school algebra class, this notion merits consideration in the context of this study as well. Students grapple with challenging concepts, regardless of class. When teachers permit additional time to allow students opportunities to reflect on and make connections to the newly presented material, that creates chances for students to build stronger connections among topics.

The rate of episodes classified as Simpler/Related Questions is much higher than the other categories for both teachers. One possible reason lies in the timing of the observations. With the Advanced Placement examination approaching, the teachers took many opportunities to review previous concepts before the organized in-class review. In the interviews, both teachers signaled that their in-class review was beginning after the unit of instruction I observed. Thus, it seems plausible that the teachers were starting to review for the examination even before the primary review began. Additionally, at the end of the school year, the teachers would be able to call upon concepts from earlier in the year. At this time of the year, both teachers posed different types of review questions. Earlier in the year, they could call upon material from prerequisite mathematics classes. Near the end of the year, they called upon material they had developed with their students in the current class.

Another explanation for the high number of simpler/related questions comes from the teachers' prior experiences with these students. Both teachers had taught the students for at least one previous class. The teachers called on previously learned material to support newly presented material by referring to an idea taught in a prior year or in the current school year. The previously learned material allowed the teacher to make a connection from past to present. Kris reinforced connections among ideas learned in earlier mathematics classes: "They know that I am not happy when they recite back to me a rule without understanding. That if they don't see a bigger picture of what's going on, then I haven't accomplished my goal" (Interview 1). Barry described his own experience learning calculus as a student:

> It's why I like calculus so much. Because when I took calculus in high school, it all made sense. This is why we learned trig, this is why we learned algebra, this is why we learned geometry, because they all meet right here. And I just thought that was beautiful, so that's what I try to do: connect whatever it is we are doing to something we have done. A lot of times it's really easy to connect it to the last lesson, but to even go back to [earlier mathematics classes]. (Interview 1)

Barry expected his students to remember and call upon the information learned in previous mathematics classes. Because the teachers made many connections across mathematics subjects, they required students to have quick access to previous knowledge. With that knowledge easily accessed, the teacher could pose related questions.

One additional explanation for Barry's use of simpler/related questions was his intention to bring in the historical development of calculus concepts. Although he typically planned a discussion on the history of a particular topic, the historical development might explain some of the associated knowledge he possessed. Barry built many connections to a particular concept, one of which was its historical development. During the first two interviews, I asked him about the historical development for the unit of observation. One example was his telling of the development of L'Hospital's Rule. Barry told students how John Bernoulli, under the patronage of Marquis L'Hospital, worked to evaluate limits of rational expressions in indeterminate form. The need to evaluate such limits led to the creation of the rule where one computes the derivative of the numerator and denominator individually then re-evaluates the limit. Barry reviewed a previous concept, computing derivatives, in his presentation of new material, determining a sequence's convergence. His purpose for including the historical development was to show the students the utility of the newly presented concept. He wanted to show that these new concepts answered a difficult question from the history of mathematics. His goal in this presentation was to show that earlier methods were not helpful in answering challenging questions in mathematics, and that historical figures developed this particular mathematical concept to answer an otherwise unanswerable question. Rather than presenting some concepts as items to be learned, Barry gave a story behind why the concept came to be.

The historical development of a mathematical topic has support from mathematics education researchers. Davis and Simmt (2006) claimed that knowing established mathematics also requires the teacher to know the development of those mathematical ideas. An awareness of this development allowed Barry to pose simpler/related questions: Knowing the history of the concept can provide teachers ways to pose a question that mirrors the development of the topic. For Barry's discussion of the development of L'Hospital's Rule, the need to determine limits at infinity for expressions in indeterminate form could have provide the basis for a set of related questions if the students had not given Barry the perfect transition to that story.

Simpler/Related Questions as Reviewing Pre-Requisite Knowledge

In Fernandez's (1997) study, a teacher's approach to using simpler/related questions allowed students to work on an easier question first. A teacher needs to possess a certain level of content knowledge to pose a question that would be easier while still addressing the current concept. Teachers can make connections from easier to harder or to different branches of mathematics. If implemented well, the result of posing simpler/related questions is that the students develop generalizations for the planned activity or beyond. By asking these kinds of questions, the teacher can return to the students the responsibility for making connections and developing generalizations.

The teachers in this study asked simpler/related questions to determine students' prerequisite knowledge. A pattern emerged in the way Kris employed simpler/related questions: A student asked a question. The teacher asked a question to review. The class worked on the teacher's review question to answer the original question. In this pattern, Kris identified the review question by making a noticeable movement to the side of the dry-erase board in the front of the classroom. Upon completion of the review questions, she returned to the original question at the other part of the board, using the review question as an assumed fact. I explain this approach in the first episode described below. When posing these timely questions, Kris applied a special knowledge of the students and mathematics curriculum (Ball et al., 2008).

Kris, First Episode–Reviewing Anti-derivatives

For one homework exercise students were expected to find the anti-derivative of a rational expression. This expression could not be solved by the students' typical approach of adding one to the value of the exponent and dividing by the new exponent. When the students could not use their conventional approach, they sought the teacher's assistance. Even the teacher acknowledged this expression's anti-derivative was challenging for the students to determine.

To move students toward a solution, Kris reviewed the anti-derivatives of expressions in the form of a single variable raised to a numerical exponent. She divided the review into two parts. The first review question pertained to the anti-derivative of the expression $\frac{1}{x}$ or x^{-1}. The second review question concerned the rule for finding the anti-derivative of any other expression in the form of x^n. Kris posed each of these questions to suggest that both anti-derivatives might be useful in finding the final solution to the original homework problem. Additionally, she explained the separation of the rules. She led the class through finding the anti-derivative of the original rational expression and getting to a point where the students would need to employ the two reviewed anti-derivatives.

The choice of the two anti-derivatives allowed the students to work with something they already knew. The class progressed through the solution to the original question without reviewing prerequisite knowledge. When the students arrived at the point where the review question was needed, they used the anti-derivatives to find the answer to the original homework exercise.

In the final interview, I discussed this kind of anti-derivative review with Kris, and she explained in detail why such reviews were common during the unit of instruction. She wanted to include these reviews because the students tended to overgeneralize recently learned material. In particular, she had observed that when students learn about the anti-derivative of $\frac{1}{x}$, they incorrectly extend the notion to state that the anti-derivative of $\frac{1}{x^n}$ must also be a natural logarithm. Her intention in presenting this review was to prevent such student overgeneralization later in the course and possibly during the examination.

Kris, Second Episode–Reviewing Derivatives of Logarithmic Functions

On the third day of the observation, Kris presented a mini-lesson on the derivative of functions containing natural logarithms. A student asked her during a review of homework exercises why the answer in the back of the textbook did not match the answer the student had obtained. Kris delayed responding to the student question to review the derivatives of natural logarithm expressions. Once she had completed the review, she answered the student’s original question.

The textbook (Larson et al., 2001) contained an exercise suggesting an application of logarithmic differentiation to find the derivative of the quotient of two products, $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$. Kris explained to the students why a logarithmic differentiation would be useful: "And I don't want to have to do the quotient rule, and I don't want to have to do a product rule inside a quotient rule" (Kris, Day 3 Observation). As the class worked through the derivative of this quotient, Kris halted the progress of the discussion to review the derivatives of expressions containing natural logarithms: "The derivative of the natural log of anything. What's that rule? Let's write [the answer] back over this way [away from the homework exercise]." Kris completed that review. The class then completed the homework exercise and discussed the manipulations needed to match the class's work to the solution in the back of the textbook; the students needed to find common denominators to match their answer to the one in the textbook.

Because the direction from the textbook suggested an application of logarithmic differentiation, the review of derivatives of logarithmic expressions supported the rest of the solution. Following the textbook's direction permitted the students to evaluate the derivative more efficiently than the application of two rules of differentiation for the same expression–product rule and quotient rule. As Kris pointed out,

> If you can use the logarithmic differentiation, you can use the log properties to separate the products and the quotients, and so you can do individual pieces, rather than having to do it all together as a big quotient or a big product. And then the kids are like, why didn't you show me this in the first place? Ah, because you weren't ready. (Interview 4)

In this episode, Kris reviewed how to compute the derivative of the natural logarithm of an expression. As in the anti-derivatives examples above, she reviewed natural logarithms in two parts–the natural logarithm of a single dependent variable and the natural logarithm of an expression in one variable. When she wanted to return to the original question, she used the first review question. This review question helped the students understand a key step toward the solution path, taking the derivative of a single expression.

The students worked with logarithms easily, including the derivative of the natural logarithm, because Kris had reviewed these properties earlier. The students could not apply their understanding in a different situation–taking a derivative that was not already a natural logarithm. When Kris reminded the students of familiar derivatives, they worked through applying the logarithm of an expression and computing the derivatives of the new expression quickly. After these steps, the students arrived at the textbook's form for the final answer with little struggle.

Based on Even's (1990) formulation of subject matter knowledge, Kris's activity demonstrates a strong command of the basic repertoire for the calculus of logarithmic expressions. The repeating of these derivatives and anti-derivatives captures the "important principles [and] properties" (p. 525) that Even outlined. Kris wanted to illustrate the derivative and anti-derivative of logarithmic functions to emphasize the two possible answers; she wanted to highlight a key difference. In the second interview, Kris mentioned that she decided to review these derivatives and anti-derivatives because of a common student misconception:

> And that's the same thing with the first introduction to the derivative of one over *u* is the natural log of *u*. Absolute value of *u*. All of a sudden that turns into—the anti-derivative, not the derivative—the anti-derivative of one over *u* squared is the natural log of the absolute value of *u* squared. One over *u* cubed, natural log of the absolute value of *u* cubed. They're—we fall back into this generalization. That's why I keep writing on the board, one over *u*, will get our natural log.

Barry, First Episode–Two Review Questions for One Homework Exercise

Barry demonstrated the same pattern as Kris in responding to unanticipated student questions with simpler or related questions, making one small but significant change in comparison to Kris. Barry included a second set of review questions. Additionally, he reviewed concepts specific to students' questions rather than leading a brief mini-lesson.

At the beginning of the third day of the observation, a student wanted to review a homework exercise related to determining the convergence of a given sequence, $\ln(2n^2) - \ln(n^2 + 1)$. Barry reviewed two topics with the students: properties of natural logarithms and limits at infinity of rational expressions. He posed questions requiring short answers from the students. He used the short answers to answer the original question of determining the convergence of the sequence.

At the beginning of this episode, Barry suggested to the students that they might have struggled with an expression including natural logarithms. The students' initial approaches toward a solution supported the difficulties he hypothesized. One student wanted to compute the derivative of the expression. Once the students determined that taking the derivative was not a useful approach to find a solution, some students mentioned applying properties of natural logarithms. After Barry discussed how to determine the convergence of a sequence, he presented a task that involved converting the difference of two natural logarithms to the logarithm of a quotient. When the class converted the expression to a single logarithm $\ln\left(\frac{2n^2}{n^2+1}\right)$, the students lacked confidence in determining the limits at infinity of a rational expression. One student thought he remembered the shortcut in finding this limit by comparing the degrees of the numerator and denominator separately. Sensing that the students did not remember the shortcuts clearly, Barry reviewed how to determine the limits at infinity for rational expressions. He asked the students to determine the limit for each of the three terms in the original expression divided by the degree of the highest term, $\lim_{n\to\infty}\left(\frac{2n^2}{n^2}\right), \lim_{n\to\infty}\left(\frac{n^2}{n^2}\right), \lim_{n\to\infty}\left(\frac{1}{n^2}\right)$. When the class announced the answers to all three limits, they combined these answers to yield the desired answer to the homework exercise.

Students reached another disagreement when taking the limit at infinity of the rational expression. The students clarified that they were not canceling terms from the fraction nor changing the value of the fraction by multiplying one part by a single constant. Partially incorporating a student suggestion, Barry asked the students how to divide individual terms by another monomial. A student suggested dividing all of the terms by the highest-degree term in the fraction; the entire class, both teachers and students, worked together to compute all of the divisions in the fraction. Once the students had completed the division, Barry then asked about the limits at infinity of a constant and a rational expression in the form of $\frac{1}{x^n}$. Answering the teacher's questions term by term led the students to the answer to the homework exercise. When the students found this numerical value, they determined the convergence of the original sequence.

Barry posed succinct questions in this episode. The concepts he reviewed did not directly address the homework exercise. Instead, he asked questions to address prerequisite knowledge connected to the original question. The homework exercise asked the students to determine the convergence of a particular sequence. However, students needed to review two concepts before making the determination on the convergence. To help the students, Barry reviewed the two concepts when they needed them. When the students struggled with natural logarithms, he asked questions about logarithm properties; when the students struggled with simplifying the new expression, he reviewed division of monomials. Especially in the second case, Barry posed very quick, short questions: "So two *n* squared divided by *n* squared? ... One divided by *n* squared? ... *n* squared divided by *n* squared? ... One divided by *n* squared?" (Day 3 observation).

Barry, Second Episode–Two Review Questions Supported by Two Related Questions

During the ninth day of the observation, Barry asked the class to determine the convergence of the alternating series $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}$. He reviewed two topics first, convergence of a sequence and rules of exponents. For each topic, Barry created an example that placed each review topic in a different context.

From the beginning of the discussion of this exercise, two groups of students suggested both possible answers—convergent or divergent. In order to settle this disagreement, Barry asked the students to compute the value of the first few terms of the series, not the associated partial sums. He explained to the class that the goal in determining the value of the first few terms was not to identify the convergence of the series immediately, but to develop a numerical pattern to determine the convergence of the corresponding sequence. He then asked the students about the convergence of the corresponding sequence. The students expressed their answer with hesitation: One student said that the sequence converged, but other students questioned how the convergence of a sequence determined the convergence of an alternating series. Barry's next step was to represent the terms of the series as giving and receiving slices of pizza:

> We bought this twenty-one-slice Big Daddy pizza from [a pizza parlor] the other day. So, we cut it in half and I give you half. But then you give me a fourth of the pizza. But then I give you an eighth. You give me a sixteenth of the pizza. I give you a thirty-second of it. …You give me a sixty-fourth of it. We keep going back and forth like this. Is that going to go to infinity? Like, is it just going to keep getting bigger and bigger and

> bigger, or smaller and smaller and smaller, or are you going to end up with a particular amount of pizza? (Day 9 observation)

The students recognized that slices would be too small to observe. Barry wanted them to connect the pizza slices to the corresponding sequence; he asked the students if that corresponding sequence converged. He hoped that the students would see that the slices would be getting smaller—implying that the consecutive terms are always decreasing—and that there would not be a recognizable slice of pizza should the cutting continue—implying the limit at infinity of the absolute values of the corresponding terms equaled zero—satisfying both conditions of the convergence of an infinite series by the Alternating Series Test.

The students knew that the corresponding sequence converged but could not explain why. They struggled with explaining the role of the variable as an exponent to the convergence of the sequence. To alleviate the students' struggle, Barry posed two questions to review exponents: "Okay, so what does one half to the *n* plus one mean? ...Okay, what does one half to the *n* mean?" (Day 9 observation). When those two questions did not work, Barry posed a question about half of a number in terms of points on a test:

> If I took ten points off of your test, and I put minus ten on your test. Gave you a ninety. And I said, you know, I was in a bad mood, I knew what you meant, so I am going to take half off. All the minuses I put on there, I am going take half off. I am only taking minus five points off, right? Is that good or bad? (Day 9 observation)

The idea in posing such a question was to show

> If we have a negative number, when we cut it in half, we are actually cutting in half, [makes the value] actually bigger. But, one half to the *n* times one half is going to be

> smaller that one half to the *n*, because our domain … is all positive. So, this is true, for our domain. (Day 9 observation)

In this episode, Barry posed related questions to assist the students in finding a right answer. He posed questions at two different opportunities. The first question occurred after the students could not determine the convergence of the alternating series. If the students could have answered the question and provided a justification, the episode might have ended. However, as the class worked through an explanation, he realized that the students could not justify their claim; therefore, he posed a second set of related questions. The students expressed an intuition that the alternating series converged but could not fully explain their intuition. The lack of explanation across the episode motivated Barry to pose both sets of questions.

In the interviews with Barry, I discussed where he posed simpler and related questions to handle unanticipated student comments. In looking back at the entire unit, he reflected on reviewing limits at infinity:

> Limits at infinity, every year are difficult. And they, they shouldn't be that difficult, but students always have trouble with them. And so to be able to pull them out of a context, I feel like almost every student in the class could do so many of those limits at infinity in their head now. … And I feel that overall they are more comfortable with limits [at infinity], because we tackled so many different limits. (Barry, Interview 4)

There were multiple contexts for limits at infinity during this unit of instruction: determining the convergence of an infinite series and applying the Integral, Limit Comparison, or Ratio Tests to determine the convergence of an infinite series. Because these limits had been reviewed and studied in a new context, the students gained a familiarity and comfort with a concept they had learned earlier in the year.

Reviewing old concepts in new contexts was a familiar theme in Barry's presentation of the unit: "So for us to be able to pull derivatives and integrals out almost every single day and make them think about the context was…what I wanted to get out of this" (Interview 4). To fulfill this idea, Barry implemented many simpler/related questions throughout the unit. In the second interview, he pointed out that simpler/related questions assisted in the development of the current lesson. In order to move forward with a lesson, he moved backward. By establishing the prerequisite knowledge, Barry rested new material on recently reviewed material. In the final interview, Barry pointed out that he liked to review material in order to keep fresh the previous material to help students prepare for the upcoming Advanced Placement examination. Progress on the new lesson depended partially on successful reinforcement of older, possibly forgotten, material. Because he felt the desire to review earlier concepts, Barry might have been inclined to include such questions when handling unanticipated student comments.

Comparing the Two Teachers

For the episodes described, the teachers used an approach that seemed to fit the students' needs. In each of the episodes, the teacher presented two different approaches to responding to the students' unanticipated comment or question. The approaches did not vary in nature as much as they did in format. One comparison is that Kris would ask sets of review questions, whereas Barry offered one question at a time. When Kris asked one set of questions, she took the opportunity to introduce a mini-lesson, reinforcing a reviewed concept in many cases. Kris reviewed not only what the students asked for but also additional concepts associated with the student's comment or question. The mini-lesson covered additional information that possibly helped the students with the original question or comment. In contrast, Barry reviewed only the concept that was necessary to answer the student's immediate question. His focus appeared to be on reaching resolution as soon as possible. Once he had addressed the question or comment, the class returned to the original situation. These observations are not to suggest the value of one approach over the other but to highlight two ways to answer a student's unanticipated question with a simpler/related question.

The teachers' use of simpler/related questions allowed them to examine their own vertical content knowledge. Shulman (1986) identified the vertical component of content knowledge in opposition to lateral content knowledge, where a teacher examines how an idea is discussed across other subjects simultaneously. In this case—the use of simpler/related questions—the teachers reflected on what had happened earlier in the students' academic careers and used that information to assist them in the current episodes. The teachers were calling upon mathematical knowledge that they assumed students had learned earlier in the year or in previous years.

Shulman (1986) also mentioned teachers' explanation of why topics are central or peripheral within a particular curriculum as a component of teacher's content knowledge. He stated that such decisions "will be important in subsequent pedagogical judgments regarding relative curricular emphasis" (p. 9). When the teachers addressed topics within a simpler or related question it showed me which topics they considered central. In the same report, Shulman suggested that teachers recognize which topics are easy or difficult for students. When the teachers in this study reviewed a particular topic they demonstrated their awareness of topics that could be difficult for students. By reviewing earlier concepts, they showed students that easier topics are the basis for more challenging topics. In Kris's class, the students already knew the derivatives and anti-derivatives for monomials, so she could then present the other derivatives—such as the ones they were studying in this unit—as being different from what they had learned earlier in the course.

In Park and Oliver's (2008) study, challenging questions allowed teachers opportunities to create simpler or related questions. In their study, when one teacher learned of a student misconception during a lab activity, she created an analogy to explain the challenging concept to her students. Much like a teacher in the Park and Oliver study, Kris used students' misconceptions to create appropriate review questions. In describing teachers' subject matter knowledge for functions, Even (1990) noted that a teacher needed to recognize subtopics and subconcepts connected to a particular topic. By creating simpler or related questions, the teachers in this study demonstrated their knowledge of the supporting topics that underpinned the central topic of discussion.

Schoenfeld (1998) described teachers' use of simpler or related questions in reporting his case studies of teachers as well. In the study of a novice algebra teacher, he described a teacher's approach to assisting students with simplifying exponents in rational expressions. The students in Schoenfeld's case study had no difficulties simplifying expressions when the value of the exponent in the numerator was greater than the value of the exponent in the denominator (for example, $\frac{x^7}{x^4} = x^3$). Difficulties arose when the teacher asked the students to simplify an expression where the exponents were equal in both the numerator and denominator (for example, $\frac{x^5}{x^5}$). Because the students' answer—zero—surprised the novice teacher, he tried a numerical approach to correct the students' misconceptions, asking the student to simplify a fraction with the same value in the numerator and denominator. However, the students did not make the connection between that example and the algebraic task at hand. Teachers should not only choose appropriate review questions but also justify connections between review questions and new material. Without a teacher explaining the connection from simpler question to new material to students, a teacher's purpose in posing a simpler question to students is lost.

In the present study, the students in Kris's and Barry's classes generally were able to connect the simpler question to the question in the lesson. Schoenfeld's teacher had been was a novice; both teachers in this study were experienced teachers. Because experience can help teachers build classroom strategies—including posing valuable questions—the teachers in the present study may have been better at asking questions that allowed students to see connections between past and present ideas.

Another possible explanation for the difference in student responses in Schoenfeld's study and this study is that the novice teacher in Schoenfeld's study did not make his intentions for posing the review question explicit to his students. Because he did not make those intentions explicit (simplifying a fraction was the same as simplifying a rational expression), the students did not see the connection between the review question and the original question; instead, the students were just answering another one of the teacher's questions. In Kris's class, when she posed simpler or related questions, she referred back to earlier work; her students knew she had reasons for posing simpler questions.

Challenging Questions as Expanding Teacher's Own Content Knowledge

As in Park and Oliver's (2009) study, the teachers in this study were posed student questions beyond the teacher's immediate understanding. Both of the teachers in the study effectively determined approaches for handling challenging questions. Studies show that teachers of highly motivated or capable students desire to learn more about the subject they teach (Bishop, 1968; Mills, 2003). Like a teacher in Park and Oliver's study, Barry took the opportunity to research the student's question outside of the school day. When attempting to understand why the harmonic series diverges, one student wanted to see a more compelling proof than the one provided in the textbook (Stewart, 2001). The next day, Barry returned and presented the class with a Web site showing 20 different proofs of the divergence of the harmonic series (Kifowit & Stamps, 2006). During the third interview, Barry indicated that he wanted to find a good proof for this student to satisfy the student's inquisitiveness.

Barry, First Episodes–Differentiating Permutations and Combinations

Within this unit of instruction, there were many references to the factorial operation. Students vaguely remembered the concept at the beginning of the unit and many times expressed their frustration with working with the concept. To motivate the students' interest in the concept of factorials, Barry referred to permutations and combinations. In those references, he acknowledged he could not connect the terms to their corresponding definitions, even though he knew how to explain the computation of the number of possible combinations of numbers in a lottery or the number of possible arrangements of officers in an organization. During the second interview, Barry stated

> I do like to look ahead a little bit, but not get too deep into it so that I'm getting off the topic. The topic is sequences, and I don't want to delve too far into probability, but just give them a little taste of it, so that I can explain what the factorial is, what it is for, and then come back to, okay, how does that define sequences, or how are we going to use this function to figure out what the terms in this sequence [are]?

Because his intention was only to make a connection from factorials to examples outside the mathematics classroom, Barry did not delve too deeply into the explanation of the computation of possibilities in the two illustrations (lottery combinations and selection of officers). Although he could have explained the difference between the terms *combinations* and *permutations* with additional preparation time and research, he did not recall the difference in this particular episode, nor did he pursue a discussion in later class sessions.

On the first day of the unit, Barry began to discuss the convergence of the sequence whose terms are in the form *n*!. A student said that the punctuation mark did not carry mathematical meaning. Barry anticipated that this question might arise, based on his experience with an earlier section on the same day. To remind the students of the factorial, he computed the factorial of a small number. Much like his approach for simpler/related questions, he immediately followed the first review question with a second, related application question. The class answered the latter application question, the number of ways a group of club members can elect officers, with a little assistance from Barry. As he kept pursuing other aspects of the factorial, he mentioned a connection between factorials and playing the lottery. To explain how to compute the probability of winning the grand prize, Barry gave the impression to the class that he had not recognized the difference between the terms *permutations* and *combinations*, even though he knew how he wanted to use the unordered arrangements of objects. Barry probably could not genuinely recall which term he wanted to use but did know how he wanted to count the number of unordered arrangements of winning lottery numbers.

Another episode with factorials occurred on the 12th day. It began with the class wanting to review a homework exercise. In working through this exercise, students debated the value of zero factorial—some students claimed the answer was zero, whereas others said one. Barry wanted to clear up any confusion, especially for the second group of students, so he posed an application question regarding the number of ways to select officers from a group of club members. He wanted to use this example to illustrate zero factorial in a context outside the mathematics classroom. As on the first observation day, Barry admitted to the students that he could not connect the terms *permutation* and *combination* to their corresponding operations. When a student kept pressing for another explanation for zero factorial equaling one, Barry admitted to not knowing a more mathematical proof. He ended up doing Internet research to find a paper that provided additional proofs for this concept; he presented these proofs later.

In the second interview, Barry wanted to point out how the idea of the factorial can be seen in mathematics beyond what the students were experiencing at the time of the observation. His goal was to provide an introduction to the topic of factorials, not an entire lesson on permutations and combinations. He did not need an extensive knowledge of this particular topic, but just enough to lead the students to start the journey. Once he made the introduction and connection to another branch of mathematics, Barry returned the discussion back to the primary topic of calculus. Although this pursuit highlighted the limits of the teacher's content knowledge, it did not detract from his ability to teach the calculus class effectively. These episodes also illustrated the role of teacher as student; when presented with a challenging question, Barry researched a plausible answer. In later sessions, he returned to his class to share his discoveries.

Interestingly, Barry's rationale for pursuing the factorial concept was the students' lack of comfort in working with the factorial operation. Knowing that the factorial would challenge the student as much as any of the new calculus concepts, Barry delayed a presentation of factorials until the operation was needed again, when working with power series and the Ratio Test. In looking ahead to upcoming lessons, he suggested

> They don't really know factorials very well. So, to be able to present factorial and the algebra involved in that—. When we have to start simplifying and reducing and comparing a_n with a_{n+1} and how factorial works in that. … And, how factorial is really

> not that much different, let's just write some things out, and so to go from that direction (Interview 3)

In an attempt to make factorials more relevant and connected to previous knowledge, Barry illustrated the different ways students could see a concept in various branches of mathematics. Providing these illustrations led him to the limits of his own knowledge. The students could follow Barry to those limits, as they were enrolled in an accelerated mathematics class. The end result of the challenging question and the additional research was the students finally feeling more comfortable with a previously seen, but seldom used, concept from previous mathematics courses.

Barry, Second Episode–Necessary Steps for the Alternating Series Test

Barry's planned activity on the eleventh day of the observation was to illustrate conditional convergence using the alternating harmonic series. The students actually got ahead of the his plan by pointing out that the harmonic series diverged, showing the conditional component of convergence; Barry had wanted to save that discussion for the end of the exploration, partially because of the confusion that jumping ahead created. In the confusion, one student mentioned that the original harmonic series diverged. Barry decided to mention that the absolute value of all of the terms, the harmonic series, diverged. Once he had finished discussing this comment, he returned the discussion to the original series; when the discussion moved back to the alternating series, he wanted to apply the Alternating Series Test. When working through two steps of this test, a student asked why they had to perform both parts:

> Why do we even need the first condition, if like the second—wouldn't the second one be good enough for like the limit as *n* approaches infinity of *a*…sub *n*—equals zero?

> Because like we'll know that, since we are adding or subtracting, like zero all the time, it's going to be convergent. Isn't that good enough? Why do we need the first one?

Barry waited to respond to the student's question and appeared to be thinking of an answer. With only a short amount of available time, he acknowledged that he did not know the answer to the student's question. Furthermore, he praised the student for posing such a challenging and thought-provoking question: "That's a good question. Off the top of my head, I can't think of a counterexample to what you are saying." Needing more time to answer the student's question and planning on discussing other topics for the day, Barry left this student's question unanswered.

Kris–Representing Pi

On the second day of the observation, Kris posed a question to connect properties of exponents to properties of logarithms. After making that connection, a student asked Kris about the two types of logarithms found on their graphing calculators–the common logarithm and the natural logarithm. The class, both teacher and students, discussed natural logarithms as a result, leading to another student question. This question was another potential diversion–explaining the meaning of the number e. Kris answered the question by mentioning the financial connections of e, such as continuously compounded interest. When she finished this explanation, the student elaborated the posing of the meaning of e question by connecting this question to finding a meaning of the value of π. In providing additional explanation for the concept of π, Kris mentioned a calculus connection to the development of the concept. She remembered that connection involved an infinite series and the tangent function. However, she admitted to not knowing the connection strongly enough to provide a full explanation; she was only able to show the infinite series.

In this episode, Kris permitted the discussion to extend beyond the stated curriculum for this course. Although the representation of π as an infinite series of the arctangent function is part of the curriculum for the BC section of AP Calculus, no expectation exists for the AB section. Kris did not provide an infinite series representation of the number *e*, which would have served to tie this challenging question back to the first unanticipated question on the meaning of *e*. However, the exploration showed that she could answer the student's question using more advanced mathematical ideas. Kris knew these advanced ideas but not in a manner that could be conveyed meaningfully to her students. As a result, she described as much as she could to her students and acknowledged that she could not progress any more. This acknowledgement marked the end of the episode; there was no further exploration by Kris or the student.

Patterns across Challenging Questions

These teachers were aware that their students asked questions that could challenge the limits of the teacher's content knowledge. Barry, in his first interview, acknowledged that students in an Advanced Placement class tended to ask questions that begin with "why," signifying a desire to learn at more than a superficial level of understanding. This acknowledgement of students wanting to know why matches a response from one of the teachers in Park and Oliver's (2009) report. Barry elaborated on students wanting to know why when he explained how he tried to pique student curiosity:

> And sometimes, they're not even sure what they're asking, … so I have to figure out what they're asking, and then how to answer it. I try very, very hard to answer every question they ask … And so I don't mind saying, "I will have to look that up." But I want to be able to answer their question and even to suggest to them that's the journey we will have to take together. (Barry, Interview 1)

In this quotation, Barry provided an additional insight into the content knowledge needed to handle a challenging question. The student asked a question so challenging that the student did not even realize how challenging the question was. Barry's first step was to identify what the student's actual intentions were. Once determined, Barry focused on whether the student's knowledge could assist in answering the question. These two teachers were not afraid to admit that they did not immediately know the answer to a student's question. Thus, they permitted themselves to leave the question unanswered and return later to answer it as needed. With the additional time after class, they could use resources to answer the student's challenging question, as with Barry's additional proofs of the divergence of the harmonic series.

Likewise, Kris easily admitted when a student's question went above and beyond her own mathematical knowledge. In the first interview, she said:

> There [are] some situations like that, where, where I just don't know. And I'm, I'm not afraid to say, you know, when the kid says, "Well, what is that? Where did it come from?" "I just don't know. I really don't. It came from hundreds of years of mathematicians playing with, you know, different scenarios and coming up with different theories."

She felt comfortable telling a student that she did not know. In order to answer the student's challenging question, Kris wanted to find the answer and communicate it and a sensible explanation to the class. Instead of providing the answer, she found related examples that approximated the difficulty of the original challenging question.

These teachers' pursuit of answers to students' challenging questions made a noteworthy comparison to a case study by Kahan and colleagues (Kahan, Cooper, & Bethea, 2003). Because of his extensive teaching experience and his content knowledge, Barry led his students through a discussion of applications of combinations—arrangements of people at a table, winning configurations for a lottery—even when stating that he did not connect permutations and combinations to their proper definitions. Likewise with Kris, even though she did not make a presentation of an infinite series representation of π, she did make many connections across number and variables, showing that many important irrational numbers are best represented as a single letter or symbol. With general content knowledge and an application of broader applications, both teachers' work with challenging questions partially refuted Kahan and colleagues' claim that a lack of content knowledge inhibits teachers' possible explorations with students. This episode demonstrated that the full combination of factual knowledge, conceptual framework, and applications is not needed for a teacher to pursue a topic beyond the teacher's knowledge. Rather, applications and general mathematical knowledge are sufficient conditions for taking a challenge in this classroom setting.

Likewise, Rowland and colleagues (Rowland, Turner, Thwaites, & Huckstep, 2009) suggested that a teacher's desire to incorporate a student response into classroom discussion is based in part on the teacher's possession of a higher level of specialized knowledge for teaching mathematics. Although both teachers did have a large knowledge base to call upon for teaching mathematics, they did not possess a large content knowledge base for these particular topics for answering these challenging questions. However, the teachers picked up the student's questions and explored the concept as much as they could with the knowledge they possessed at the time. They led their students to what they know about the particular concept and identified their own boundaries of knowledge. Sometimes, the teachers returned with the results of additional research; at other times the investigations ended in an open question.

Teachers can benefit from the challenging questions as well because they provide an opportunity for the teachers to grown in their knowledge. Although the new knowledge may not help a current class of students, teachers may use the new knowledge in later classes, either later sections of the same course on the same day or future presentations in upcoming days, semesters, or years (Park & Oliver, 2008). In some instances, a teacher might present the results of his or her growth back to the students, as when Barry presented the Web site with 20 different proofs of the divergence of the harmonic series.

There is an additional benefit for a teacher to acknowledge when he or she does not know the answer to a student's challenging question. As Mason (2000) mentioned, it is important for students not to see the teacher as an all-knowing possessor of knowledge of mathematics because this belief can be a very powerful one for students and could have a long-lasting impact. The teacher's ability to defer authority instills independence on the part of the student. This independence allows students to explore more mathematics by themselves. Acknowledging students' challenging questions can debunk this faulty assumption.

Counterexamples to Assist Student Learning

In Fernandez's (1997), study teachers used counterexamples "to challenge their students and enable the students (versus the teacher) to examine their perspective" (p. 8). When a student made a conjecture about the current activity in the classroom and the teacher pursued the conjecture by using a counterexample, the teacher's goal was to show in a non-threatening way that the student's misconception was invalid.

Counterexamples came in many varieties in the participants' classrooms. There were times when the teacher provided a short counterexample to handle quickly an unanticipated student comment. For example, on the sixth day of Barry's unit of instruction, a student asked if the divergence of a series implied monotonicity. The comment, while an intriguing conjecture, generated a very short counterexample with explanation from Barry:

> I am just trying to think of one off the top of my head, like if I said one plus negative two plus positive three plus negative four plus positive five. You know it's not monotonic, it's going up and down, but it's also not [converging] either.

The entire episode, from posing of comment to completion of counterexample, happened very quickly. However, Barry valued the student's comment, which led to him taking the time to address it, if only briefly. This need to provide the counterexample came from Barry's desire to support students making connections across concepts:

> And I was glad to see that [the student] could say a sentence like … if it is monotonic, then it is divergent. Because, that's showing that he has at least an idea about monotonicity … and an idea about what divergence is, and that he is making a conclusion. And now … we need to address that conclusion (Barry, Interview 3)

The episodes presented below involved longer discussions between teacher and students from the posing of the comment or question to the resolution of that comment or question.

Barry, First Episode–Creating A Counterexample to Settle a Dispute

On the final day of the observation, Barry discussed the previous quiz and prepared for the next day's unit test. As the students asked multiple questions, he responded by providing practice exercises. Barry posed two nearly identical counterexamples in the same class period. He showed the students that they had developed an incorrect expression for the general term of an infinite series by evaluating their proposed expression for specific terms.

The class attempted to create an expression for the general term for a particular Maclaurin series. The students knew that their expression for the general term needed to contain a factorial and a power of negative one. The students did not struggle with determining the expression containing the factorial; after a few examples, they established a pattern for the terms in the series, using a factorial expression. However, the students debated how they wanted to create an expression for the general term of this series containing negative one to a certain power. At one point in this discussion, students suggested three possible answers for the expression containing negative one.

In order to provide clarity to the student's conjectures, Barry selected a few terms to evaluate the students' conjectures. While determining this part of this expression, he wanted to question the students about the powers of negative one for the expression they created. Initially, he wanted the students to look at the terms in two groups–odd-numbered terms and even-numbered terms. Barry asked the students if the expression matched the signs of the appropriate terms of the series. Different groups of students told the teacher simultaneously that the expression was correct and that it was incorrect. Realizing that the question might have been too complicated, Barry posed a question for a specific term: "I am not there yet. So the twelfth derivative, should [the value of this derivative] be positive or negative?" (Barry, Day 15 observation). Once a student answered that question, Barry asked a second question: "So, if I put twelve in here for n, is this going to make positive or negative?" When the students compared the answer to this question to the sign of the corresponding term of the desired series, they realized the inclusion of plus one in the exponent was incorrect. This discovery allowed Barry to modify the expression based on the students' choice of notation and starting value.

Later in the class, Barry used a similar strategy to answer another student's question. The class wanted to create the expression for the general term of a particular series. Because the terms alternated signs, the students wanted to include an exponent of negative one in their final answer. One student attempted to find the proper exponent of negative one. Instead of picking a later term in the series, Barry posed his counterexample using a very early term in the series: "Okay, so we said this [value] was zero. I want the first term to be what? ... Positive. So does negative one to the zero power give me a positive? Yeah. So I don't need the plus one" (Day 15 Observation). At the end of Barry's statement, the student changed the expression from $(-1)^{n+1}$ to $(-1)^{n}$.

Barry, Second Episode–Fixing His Own Mistake by Counterexamples

Although both teachers present counterexamples for students' errors and conjectures, Barry also began with an error he made. The error was completely accidental and unplanned. However, he used the error to discuss another topic with his students. He encouraged his students to find his error, using his mistake as the counterexample to his original statement.

On the ninth day of the observation, Barry made the error when copying an expression from notes to the dry-erase board, which provided an unplanned but illustrative episode. Barry wanted the class to determine the convergence of a series in which the expression for the general term contained a natural logarithm in the denominator. As he began to write the series, he started with the zeroth term. He caught his mistake and asked the students what was wrong with what he had written. Rather than erasing the series and starting over, Barry left the series on the board and discussed the convergence of this new, unplanned series. Interestingly, one student suggested including an additional variable in the numerator, perhaps getting the expression in a form where the class could make a *u*-substitution using natural logarithms.

Throughout this episode, Barry had many opportunities to explain how correcting this mistake connected to exploring other mathematical ideas. At first, he asked the students why he could not start the series with one: "I can't start this at one, right? What's the natural log of one?"

(Barry, Day 9 observation). This question was in response to a student who commented on one being an improper starting value for this series, as it would create an undefined term. Building off of the student's comment, Barry proposed an initial value of 1.1, followed by a proposed starting value of 1.00001. Some students started to believe that the progression of values 1.1, 2.2, and 3.3 would be valid for the terms of this series. Some students argued that because the teacher's selected values were in the domain of the natural logarithm expression, these values could be used for the value of the terms of the series. As they began to work with this series, Barry repeatedly asked if the use of 1.1 was valid for term numbers: "So, 1.1 is in the domain of $\frac{1}{\ln n}$. So, it's okay if I start at one point one?" (Day 9 observation). Later, a student provided a curious explanation in support of the 1.1 progression:

> But I thought that *n* marks a progression. Like if you're going to infinity, and you're starting at one, you're going from one and then two and then three and then four. Wouldn't you have to add one point one to that [starting value of 1.1]?

However, Barry pointed out that the students liked only working with natural numbers and not rational numbers, suggesting that the students work with two, three, four, or five, rather than the decimal values for each term number. He explained the domain and range of the sequence to the students:

> I don't mind that the *y* values aren't integers. In fact, the *y* values have hardly ever been integers. Or the terms in the series have hardly ever been integers. But, what have the *n* values been? [students' responses]. Integers. Almost all the time I start at one, because it just seems like a natural place to start.

Barry needed the counterexamples to illustrate a point regarding the domain of the general expression for the term of the series. In trying to illustrate this point, he ended up having to address a different misconception. The students in the class began to accept the counterexample as true. Only after Barry commented on the students' preference for natural numbers did the class move away from using the decimal values Barry had presented at the beginning of this episode. One rationale a student gave for accepting the decimal values was that the values of the terms were not integers. In other words, because the output from the general expression was not an integer, it was permissible that the input for the expression not be an integer as well.

The use of the teacher's notes still provided an unplanned moment for Barry. Although a student did not suggest the discussion on the domain of the terms, the written expression provided an opportunity for him to show how he could apply his content knowledge quickly. Barry paused to make a comment about the mistake he had made. He had already diagnosed the error and hoped that the students would be as quick in catching the mistake. The students hoped the teacher incorrectly copied the expression for the general term, wanting Barry to remove the natural logarithm expression from the denominator. Barry ended that thought quickly by focusing on the index of summation, trying to get the students to focus on the inputs instead.

Although Barry's episode from the ninth day started a little differently, generated from his own possible miscopying of information, the episode is quite illustrative. He ventured into unplanned activity when he realized his mistake. Throughout the discussion, Barry got to show multiple ideas in the same exercise. By starting at zero, he showed the domain restrictions for the natural logarithm. Starting the series at one allowed him to start a discussion of the domain of rational functions. By increasing the steps by 1.1, Barry could discuss using only natural numbers for the term numbers. Pursuing this mistake allowed him to explore additional ideas beyond the original plan.

Kris, First Episodes–Counterexamples in Slope Fields

On the eighth and ninth days of the observation, Kris developed two similar types of counterexamples to respond to unanticipated student comments. Both opportunities involved identifying slopes of line segments on a slope field.

The episode began with an unanticipated student comment regarding the ability of an individual to judge steepness of line segments. The student was attempting to connect ideas from another course to this class's discussion about the relativity of certain intellectual constructs. The student commented that a test taker could argue for earning more points on a free-response question on the standardized final examination, based on the test taker's intention of drawing segments with particular slopes. To create the appropriate counterexample, Kris drew three line segments on the board, each with different steepness. She asked the student to rank the three segments in order of steepness. She pointed out that the student could associate a larger magnitude with a steeper segment, and the student relented on her assertion.

Kris's goal in presenting this counterexample was to illustrate that the end-of-the-year examination would be assessing steepness in relative terms, not exact terms. As Kris said to the class:

> One [segment] is clearly steeper than [a second segment] … If I put a ruler up to this one, a straightedge, and then this one right here, and I bring this one down, I can see it goes up faster than this one if I compared them … I am not comparing [the slope] to a negative one. I am comparing that, the negative one that you drew was steeper than the negative one-fourth. I am not looking that your negative one is exact. (Day 8 observation)

Kris elaborated on this idea more during the third interview:

> The understanding of what somebody would say is fast [in terms of speed] versus slow would be very different … Well, for me, [my conception of] fast is a lot slower than [the conception of fast of] some of these other kids who run track … It's not something that everybody has the same understanding about. So when I talk about steepness, with the slope field, there's, there's not going to be a one that I could, I could put a specific measurement on to say that if you don't write your slope—you know, draw your little segment with that particular slope, and it's got to be exactly like this, then it's wrong. That's what [the student] was saying, you know, we, people understand steepness very differently. But it's not about the understanding of steepness. It's about the relation between the two different slopes.

Kris returned to this kind of counterexample the following day. She asked the students, as a class, what numerical value would have a slope easiest to recognize by visual inspection: "What does everybody in here, when I say this particular slope, we all know exactly what it is, and we draw the same thing?" (Day 9 observation). Some students suggested that a line segment with a slope of 1 would be the easiest to recognize. She wanted the entire class to model slopes of 1 using their arms: "Would everybody draw the—if I said put your arm up right now as a slope of one, would everybody point exactly parallel?" Kris had the students look around the classroom to determine if all the students had created line segments with the same slope with their arms. The students saw that they did not have the same steepness, showing that the value of 1 was difficult to model. Kris repeated the same activity with a slope of 0, using a second student's suggestion. After this exercise, the students come to the conclusion that a line segment with a zero slope is easier to recognize, which was helpful for the students to match a differential equation to its corresponding slope field.

Kris, Second Episode–Counterexamples by Changing a Right Answer

On the sixth day of the observation, Kris took a unique approach to creating or posing a counterexample. Counterexamples can illustrate student errors. In this episode, Kris used a counterexample to show why a second of two choices was wrong. Kris used this moment to emphasize the validity of a student's original choice.

She requested students' input on homework exercises to review. They suggested an exercise that involved finding the anti-derivative of the quotient of two differences of exponential expressions, $\frac{e^x + e^{-x}}{e^x - e^{-x}}$. Near the beginning of this episode, Kris asked the students which part of the rational expression would be better to use in a particular substitution. One student suggested the denominator. Immediately after that suggestion, Kris decided to pursue substituting something else instead:

> Okay. Let's not do what [the student] said and use the top, and I want to show you something. Let's say I would say, well, they're kind of the same. One's a plus, one's a minus. Let's use the top and see what happens.

The class, teacher and students together, worked on Kris's second, incorrect approach initially. The work continued until Kris pointed out that the incorrect format of the integrand would not help in computing the anti-derivative. She provided an interesting analogy to the students to demonstrate the contradiction:

> This is not *dx* over *e* to the *x* minus *e* … to the negative *x*. It's *dx* times—on the same line, this *e x*—it's on top of the fraction. I don't have it on top of the fraction. So, don't be fooled into thinking that's a good substitution. And, oh, there it is on the bottom, so I am going to take it from the bottom. It's like, let's—I go tell somebody that I need twice as much money, and I go to them, and I only take half. Well, that doesn't make sense.

The goal in presenting this analogy was to show that Kris's selected expression yielded the reciprocal of the desired result; rather than having an expression in the form of $\int \frac{du}{u}$, where the students could easily compute the anti-derivative, the class was left with $\int \frac{u}{du}$, a form for which they could not compute the anti-derivative. After that recognition, the class returned to the original student-proposed solution to answer the homework exercise that started this episode.

In the third interview, Kris discussed counterexamples in changing student's perspectives on a particular concept. A student stated that the derivative of x(ln 2) equaled zero. Unlike the previous episode, Kris wanted to show the error in a student's thinking. The student had arrived at that solution by an incorrect application of the product rule: here, the derivative of the product equaled the product of the two factors' derivatives. In the interview, Kris diagnosed two additional errors in the student's thinking:

> One is I don't understand what the natural log of two is: I don't see it as a constant … So it doesn't come into my mind as "Oh, it's just a number." The second is, it's after the *x*. Whereas five *x*, derivative is five. It's the number in front.

To help the student see this misconception, Kris posed a couple of quick, short questions to serve as counterexamples, finding the derivatives of $5x$ and $3x$. Once the student saw the pattern emerging, he changed his answer for the derivative of (ln 2)x to the desired solution. By presenting the two quick counterexamples, Kris wanted the student to recognize (ln 2)x as being in the form constant times a variable. Following the pattern allowed the student to compute the derivative quickly, with the intention to show that ln 2 was a constant like 5 and 3 had been in the counterexamples. During the observation, the student and the teacher did clear up the student's incorrect use of the product rule to provide an alternate representation of the same solution.

A good counterexample to use for the class is one student's incorrect response to a teacher's question. Because students in the observed classes answered many of Barry's and Kris's questions quickly and correctly, both teachers incorporated the few incorrect answers students presented into the classroom discussion. When discussing the error in the student's solution, both teachers used counterexamples to illustrate the mistake. In the first interview, Barry said:

> I do sometimes flat out ignore an answer. Like if I ask a question and several kids say—several kids are talking out [a variety of possible answers, so] I will jump on the wrong answers on purpose. Because I want us to pursue all directions before we figure out what the right direction is. I always try really hard to say, "Okay, Susie said this [one solution], and that [another solution] was the right answer." And then we will go in that direction.

Especially in a class like the observed classes, students can quickly generate a correct answer to a teacher's question. By choosing an incorrect answer to explore, the teacher could illustrate a concept by emphasizing what it is *not*, which might lead to a key counterexample. Teachers not only have to be able to verify the validity of a correct answer but also to choose a valuable incorrect answer and how that incorrect approach can support a correct answer. Well-chosen counterexamples show a teacher's knowledge of content. This approach matches the notion from Ball and colleagues (Ball, Thames, & Phelps, 2008) that a specialized knowledge for teaching mathematics includes knowledge of the best examples (e.g., good numerical values to use in an expression) to emphasize particular aspects of a concept. In Kris's counterexamples for the situation involving the derivative of $x(\ln 2)$, she used derivatives of the form of a natural number times the variable to show the student the correct value of the derivative of the original function. Because the student quickly and correctly answered Kris's two questions, her questions highlighted the student's original error.

Comparing Counterexamples to Previous Studies

The purposes of the teachers' presentations of counterexamples can be compared to the way that counterexamples were used by Lakatos (1963a, 1963b, 1963c, 1964). With a counterexample, the teacher is able to discuss how the counterexample illustrates the discussed property. Presenting a counterexample can stand as an alternate representation. A counterexample has the possibility of quickly ending a claim from another class member. The counterexample can provide class members help by defining what the concept is not. Discussing the counterexample and debating its problems requires knowledge of the valid concept. In the opening illustration, the teacher had to present an example of a series that was divergent but not monotonic. He presented an example quickly between two concepts that were surprisingly connected by the student. The quickly created example ended the student's claim before other students could accept that student's claim as valid.

The counterexamples presented in this study satisfy Zaslavsky's (2005) definition of competing claims where a learner possesses two incompatible notions of the same topic simultaneously. Teachers in such situations need to provide students with an opportunity to notice that they are holding two contradictory ideas at the same time. In Kris's classroom, the student in the second episode recognized that the derivative of the expression $(\text{constant})\times(\text{variable})$ equaled the constant; however, the same student also believed that the derivative of $x(\ln 2)$ equaled zero, not the value of the constant. Because the student did not recognize the same form in a different order, this student held competing claims on the computation of this derivative. Kris's counterexample of $x(\ln 2)$ resolved those competing claims. However, in the creation of useful examples, teachers must consider the students' level of understanding. If teachers provide counterexamples that are too difficult or complicated for students, then they might not recognize they hold competing claims. Kris chose her counterexample well, as the example cleared up the confusion the student demonstrated.

In addition, a counterexample has the added effect of determining how one particular example "'breaks' the rules" (Shulman, 1986, p. 10). In Kris's work with the three segments of different rates of steepness, the student learned that one could estimate the slope of three segments given three numerical values. Slope was now for the student not something that had to be computed, but rather could be judged. This example broke the rule the student learned from a different class regarding the relativity of concepts. The posing of counterexamples combines two of the three major components of a specialized knowledge for teaching secondary mathematics as assessed by Krauss and colleagues (Krauss, Baumert, & Blum, 2008)—identifying student misconceptions and generating multiple representations. When a teacher poses a counterexample to a student, the teacher illustrates a flaw in the student's thinking. In her discussion of steepness of slopes, Kris showed her student that it is possible to match segments of different steepness to different numerical values of slopes. Additionally, the counterexamples allow teachers an opportunity to address the misconception in a variety of ways. In the two-day discussion of slopes of segments in a slope field, Kris used two different representations of slopes of segments. The first time Kris drew three segments on the board; the second time she had the students model two segments' slopes with their arms.

Follow Through to Examine Student Thinking

In Fernandez's (1997) study, the teachers engaged in a follow through when the wanted to pursue additional implications of a student comment. The follow through provided a springboard to explore students' thoughts that might not have occurred in the planned lesson. Additionally, the follow through approaches also allowed the teachers to explore students' misconceptions without the students feeling judged. In the exploration, other students could critique the unplanned student comment either to reach resolution or extend discussion.

For Barry, the follow-through approaches took on multiple purposes throughout the observed unit of instruction. As he incorporated students' input, he followed through until arriving at a contradiction or until reaching the end of the class period. The instances where he used the follow through to its conclusion involved shorter examples. For example, on the eighth day of the observation, one student wanted to determine the convergence of $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$. The student decided to work with $\sum_{n=1}^{\infty}\frac{1}{n}$ as a comparison for the Direct Comparison Test. The student recognized $\sum_{n=1}^{\infty}\frac{1}{n}$ as the divergent series but then realized that the values of corresponding terms in the chosen series were less than the terms in the original series, meaning the student chose an invalid series to compare to the original series. Barry picked up the student's chosen series and explored the possibility that the chosen series could determine the convergence of the original series. As both Barry and his students worked through the student's suggestion, he realized a noticeable flaw in the student's chosen series. Although the student's chosen series was divergent, it could not be used as a comparison to the original series because the original series contained terms that were not as large as those in the original series. In this instance, the student's chosen series as a comparison could not determine the convergence of the original series. In a short period of time, the problem had been presented, a solution had been proposed, the follow through had been implemented, and a contradiction had been reached.

Barry, First Episode–Determining the Convergence of an Infinite Sequence

In this episode, Barry incorporated a student's solution regarding the solution to a particular question. Barry restated the student's solution and explanation of that solution. He then used the solution path as a model to determine the correct solution to the original question.

During the second day of the observation, Barry spent considerable time in class discussing the previous night's homework exercise—determining the convergence of the infinite sequence $a_n = \left(1 + \frac{2}{n}\right)^n$. One student suggested that the sequence converges to 1. Although this solution was not correct, Barry decided to pick up that explanation and work toward a solution, starting with restating the student's explanation:

> So, [student] has made a very good argument that as *n* gets bigger, this number is going to get smaller, one plus a really small number is pretty close to one. And it doesn't matter what power I raise to, it's still going to be pretty close to one. So, it seems like this is going to converge. So, now let's figure it out. Let's make sure that it is going to converge. What do we do?

Barry restated the student's explanation about why the sequence converges to 1. After he made this statement, he wanted the class to prove that the student's solution could possibly be correct.

The class began to create a whole-class proof. One student suggested starting with the removal of the *n* from the exponent. A couple of students suggested invalid approaches to handle this removal, applying the chain rule for derivatives or taking the *n*th root of the expression.

Barry asked the students to find a different way to manipulate the exponent. Another student suggested applying a logarithm to the expression. This student's response allowed the class to handle the exponent in a more meaningful way, rather using than a student's original suggestion of "And it doesn't matter what power I raise [values close to one] to, it's still going to be pretty close to one" (Day 2 observation).

When the class began to evaluate the logarithm of this expression, they started by setting the expression equal to a dummy variable. They took the logarithm of the new equation. During this logarithm work, the class quickly resolved an issue on the possible need to include the absolute value of the expression. The class began to determine the limit at infinity for the product. To facilitate the computation, the class attempted to convert to the original expression using exponentiation. Barry introduced the students to L'Hospital's Rule once he realized the expression was in indeterminate form. The students evaluated the limit for the transformed expression after Barry introduced L'Hospital's Rule. The students eventually determined the original sequence converged to e^2 instead of the original student suggestion of 1.

By revoicing the student's suggestion that the sequence converged to one, Barry engaged in a type of follow through. He explored the potential implications of the student's suggestion. The suggestion appears to be plausible by looking at the student's suggestion as individual components. However, taking the valid approach would yield a different response. In an attempt to prove the student's conjecture valid, the teacher worked through the actual evaluation of the limit. The difference between the two approaches—the students' and teacher's—was in the order of operations.

The implication of following through with the student's suggestion was the need to introduce a new lesson. The progression of the homework review led the teacher to one part of his lesson–evaluating limits by L'Hospital's Rule. The class's work on the homework exercise led to an expression in an indeterminate form, motivating the upcoming lesson on L'Hospital's Rule. The L'Hospital lesson began with Barry's presentation of the historical development of the concept, telling of the work by Bernoulli and L'Hospital. In the second interview, Barry mentioned that he

> wanted to just come up with [a way for] us [to] need that tool … and that kinda helps motivate the lesson, and brings some relevance of … we do. Why have this? Well, here's why we have this, so we can get around a situation like this.

By following through with a student's incorrect response to a homework exercise, Barry provided a valuable transition to a planned activity.

Barry, Second Episode–Choosing A Comparison Test

Barry often allowed the students to determine the convergence of an infinite series using a test of convergence the students felt the most comfortable using. In this episode, he encouraged the students to use their preferred test of convergence. He incorporated that announced test of convergence into his discussion and worked the test until the class reached one of two outcomes, a resolution was reached on the convergence or the test result was inconclusive.

This episode on the tenth day of the observation started much like the previous episode, with Barry requesting that the students determine the convergence of a particular series, $\sum_{n=1}^{\infty}\frac{n}{n+2}$. Many students suggested that the series could be convergent but could not provide an explanation for the series' convergence. Barry mentioned many of the tests of convergence the students learned earlier in the unit to provide assistance. Progress appeared to be made toward an explanation when a student proposed using the Limit Comparison Test to determine the convergence of the series. In order to get a solution, however, the student wanted to change the value of the expression for the general term by removing the *n* from the numerator. The student's second approach was to convert the fraction into a complex fraction, $\frac{\frac{1}{n+2}}{\frac{1}{n}}$. The student might have been making this change in order to get at least a zero in the denominator when determining the limit at infinity for the new expression.

When Barry asked the student what test of converge he was using, Limit Comparison Test or Direct Comparison Test, the student answered with a Comparison Test. In this class, Barry reserved the term *Comparison Test* for the Direct Comparison Test. By asking this question, Barry might have hoped that the student had switched from using the Direct Comparison Test to the Limit Comparison Test. If the student had made this change, the student might have determined the convergence of the original series more efficiently by comparing the original series to a series whose terms were all the same constant. The student did not catch onto this suggestion and worked toward an inconclusive result.

Rather than leading the student away from an inconclusive result, Barry worked with the class on students' suggestion. If the student wanted to apply a particular test of convergence to an infinite series, Barry picked up the suggestion and attempted to work with the test along with the students. After determining that one test yielded an inconclusive result, another student made a different suggestion on determining the convergence of the original series. The class worked with a different test of convergence. In this observation, the students wanted to use one of the two comparison tests until they realized they did not know a useful series to compare. The students could not apply the Limit Comparison Test for this sequence because the result of the limit of the expression the students created equaled zero. An expression equaling zero yields an inconclusive result for this particular test of convergence. However, a long exchange between the teacher and the students ensued about the meaning of a limit equaling zero in the Limit Comparison Test, whether or not the value of zero implied convergence or divergence of the original series or no determination of the convergence of the original series. Once the students had consulted the textbook, they abandoned this use of a test of convergence, knowing that a result of zero is inconclusive for the Limit Comparison Test.

In these selected episodes mentioned above, the follow through ended because the approach ended with a stumbling block for the students. These approaches were highly illustrative of finding the right answer later, especially in the second episode mentioned here. In working toward a solution for determining the convergence of an infinite series, showing that a test yielded an inconclusive result can help students learn convergences of infinite series as well as seeing the ideal solutions immediately. Barry echoed this idea during his interviews. In the third interview, I asked him about similar episodes regarding the determination of convergence of selected series. His responses suggested that the follow throughs allow the students to see a more detailed exploration to a one-word answer. Additional benefits to these follow throughs are the connections and description needed to answer Advanced Placement questions sufficiently.

Barry’s primary motivation for providing these follow throughs was to illustrate a student’s way of thinking in proposing a particular solution path. In these episodes, he picked up a student’s proposed solution and discussed the solution with the student and the class. Barry wanted to identify the particular students who proposed a solution path by name to signify that a student, not the teacher, suggested a certain approach to finding the right answer:

> I try to mention [her] name. Well, [she] chose this because she saw that—and try to explain their—what her thinking was. Maybe not as well as she might explain her own thinking, but try to, try to pin that there, and then try to help [her] as she was saying, well,

how would I decide this or that and, and here's, here's the decision process that goes. You know, here's the list of, of options we have and how am I going to choose it, and then here's how I am making my decisions.

Although one student did participate in starting the discussion, the student did not work with verifying a solution alone. By working with a student, Barry supplemented the description of the student's solution. His follow though did not replace the student's justification of using a particular test of convergence but rather gave other students an opportunity to explore why one particular test could work better than others for a given series. Barry acknowledged in the third interview that many valid approaches existed for these types of exercises. The conclusive and inconclusive results provided him a chance to demonstrate the challenges of working through certain tests of convergence. He stated that his students were too comfortable with short, simple answers to any mathematical question. In these exercises, using a particular test, interpreting the result, and explaining the motivation to use that test allowed him to require a richer solution from a student.

Because Barry was very interested in preparing the students for the upcoming Advanced Placement examination, these follow throughs gave him additional opportunities to present a more detailed answer. Barry believed that students, even at this level, expect a single number or expression to be sufficient for a final answer to a question in this class. He wanted the students to be familiar with providing an explanation for the final answer. By following through in this unit of instruction, Barry discussed why a particular test of convergence was used and what the test meant for answering the original question. He could use a similar pattern to prepare the students for the examination. The teacher is then modeling the desired behavior by engaging in this particular approach. After reading the question, a student should think about what concepts he or she would use to answer and why they were using that particular concept.

Kris–Using Two u-*Substitutions*

Kris had the opportunity to follow through with a student's idea that was novel to the rest of the class. Although the student might have struggled using correct terminology, the idea the student proposed was very fruitful in determining the right answer to the original question. Kris worked with the student to show the rest of the class the seemingly new procedure and the correct answer.

On the sixth day, the class worked through a particular integral as part of their review of the previous day's homework assignment—$\int e^{-x}\tan\left(e^{-x}\right)dx$ (Day 6, Observation Field Notes). As the class worked toward a solution, one student asked Kris about repeating the *u*-substitution procedure. Kris had the student clarify the question to clear up terminology and explain the choice: "Can you do another *u*-substitution? How is that going to help?" (Day 6 Observation). The students had already implemented the *u*-substitution procedure to transform the original homework exercise to $-1\int \tan u\ du$ (Day 6, Observation Field Notes). Kris implemented the student's suggestion, after a suggestion that the tangent expression be rewritten as $\frac{\sin u}{\cos u}$: "Let's see if that's even going to make sense. Alright, let's see if we got the pieces here." The class, with Kris asking questions and the students answering as soon as the questions were posed, worked through the procedures, giving them a manageable result.

> A second substitution has allowed us to get to something that we are very happy about. And that we can undo it … So we got the natural log of the absolute value of *t*. And *t* is cosine of *u*. And *u* is the *e* to the negative *x*.

Kris followed through with the student's suggestion, while keeping the suggestion mathematically accurate. By the end of her questioning, not only had the class resolved the student's original comment, but they had computed the anti-derivative of the original expression. Through the work on this homework exercise, Kris permitted the students to see that the *u*-substitution procedure can be used multiple times for one integrand. The implications of the student suggestion, using the same procedure twice in one computation, could allow the students to determine more complicated anti-derivatives than before the student's comment.

During the second interview, Kris mentioned a similar situation to this episode. The class was reviewing a homework exercise where she wanted to refer to the textbook for a simple solution. The back of the textbook contained a table of anti-derivatives for given forms of integrands. When not all of the textbooks had the same integral charts, the class worked through computing the anti-derivative. As the students worked through their solutions they realized that multiple *u*-substitutions were permissible and even helpful. Making this realization was a surprise discovery for the students that Kris had not originally planned but that she could discuss with her students easily.

Kris, Second Episode–Resolving "Perplexing Pencils"

Kris used the follow through strategy even when the students did not know the answer to the final question. Kris posed her own questions along the way to help the students work toward an answer to the original question. Each time a student answered Kris's question, she used that answer to work toward answering the original question. When Kris finished asking all of her questions, the students answered the original question.

Kris used another follow through during the next class session. She had the students work on finding the particular solution to a differential equation individually. After a few minutes of individual seat work, she brought the class back together because the students appeared to struggle to find the desired solution. Kris's comment to return the class back to a whole-group discussion was a little humorous: "I am seeing lots of perplexed pencils. I am seeing what I am going to term as separation anxiety" (Day 7 Observation). She then asked the entire class how to get the solution started: "What's the easiest way to get rid of fractions, when you've got fraction equals fraction?" Once a student suggested cross-multiplying the two fractions, Kris performed the multiplication. As on the previous day, as Kris worked through the solution she asked the class many questions to check for their understanding. She followed through with the student's suggestion but wanted to maintain mathematical accuracy during the implementation. Again like the previous day, once Kris's work and questioning were over, the student's suggestion had been followed through, and the exercise had been completed.

For both selected episodes, Kris followed a nearly similar pattern. Once there had been a bit of hesitation from some students on finding the right answer to a question, she teacher found some solution path from one student to pursue. While pursuing that solution path, she asked students several questions about the current proposed solution. The questions seemed to address the students' unplanned comments and move the class toward an answer for the original planned exercise. Kris appeared to handle the unplanned activity and the planned lesson simultaneously. As a result, the follow through is a different approach than posing a simpler or related question to handle an unanticipated student comment. In the latter case, the teacher posed and pursued a question and worked toward a solution initially. Once the student's question had been resolved the teacher resolved the question in the original exercise. For the follow through, the teacher responded to the student's comment and original exercise simultaneously. In this approach, the teacher could work on a planned activity while attending to a student's unplanned suggestion.

For the second episode, the teacher might have provided the motivation for a follow through. The questions that Kris posed might be leading questions, where the questions might have sounded like an open-ended question, but might have only had one preferred answer: "What's the easiest way to get rid of fractions, when you've got fraction equals fraction?....So I've got my *dy*'s and my *dx*'s where I need to be? What's our problem?" (Day 7 Observation). The teacher implemented the students' suggestions exactly as stated in the class. Because this episode occurred near the end of the class period, the teacher might have accelerated a possible follow through by asking certain questions to the students.

During the third interview, I discussed second selected episode with Kris. In this discussion, Kris received confirmation on her formative assessment of how the students will struggle through the particular exercise: "The manipulation of variables is a tough concept. For anybody, I mean, that's, that's the, that's why people don't make it in algebra … I knew they'd have trouble, and I knew as I walked around that it confirmed that." Kris believed that the format of the exercise provided the greatest challenge to those students who could not find the right answer.

> Because every anti-differentiation problem, every integral written in the [text]book, except for maybe two, is we got a piece with a variable in it, and then it's followed by *dx*. And, they, so what they see is something that they have to anti-differentiate. And then they see that notation at the end. And, they are like, okay, I am going to ignore that notation, and I am going to work with this thing that's right in front of it. But, when, all of a sudden, that *dx* gets kind of mixed in with the function, then it causes a problem.

She made two connections to other types of student struggles in the interview in thinking about how the students perceive this challenge. One such connection occurred earlier in the semester when students could comfortably determine the anti-derivative of a constant; the trouble occurred when the teacher or the text asked the student to evaluate $\int dx$.

In this episode, the students struggled with *dx* as an expression in the numerator. Typically, the students had evaluated integrals where the *dx* was appended to an expression, particularly rational expressions. As before, changing the format meant creating an additional challenge. To eliminate some of the unnecessary challenge during whole-class discussion, Kris seized an opportunity and followed through with a student's approach. In the selected episode, by performing the necessary divisions, the rational expressions in the equation were transformed to a more familiar format. Finding that familiar format allowed all of the students to complete the planned exercise instead of a selected few. Knowing some students prefer a certain format to their expressions was a key bit of knowledge for the teacher to possess to assist student learning.

Across the Follow Throughs

The level of the student in the class could dictate the length and depth of the class's follow through. Chazan's (Chazan & Ball, 1999) class of remedial students had a protracted debate regarding the computation of an average of a list of numbers. Because the students ended up running out of content knowledge to employ for this discussion, the debate devolved into a disagreement between two groups of students. In the observed classes, the follow through could be sustained by the students because they were able to call upon more knowledge quickly. Although the teacher may possess the right knowledge to start the class discussion, the students could inhibit the progress of a discussion to its conclusion. Additionally, there are other concerns that could inhibit the conclusion of a follow through. As in the case of what happened with Barry, outside issues could influence the conclusion of a classroom discussion. In particular, Barry's follow through could be stopped by the end of the class period, as Chazan noted in his report.

A follow through allows a teacher to think about a new perspective while the student is presenting his or her solution or comment. Shulman (1987) pointed out that a teacher should think about content knowledge through many perspectives. With an unanticipated student comment, the possibility exists that the student's perspective is not familiar to the teacher. By following through with the student's response, the teacher has the opportunity to explore the new perspective while determining if the solution is valid or not. In some of the episodes listed above, some of the student's conjectures were not valid, but the follow through provided the teacher an opportunity to work on the student's claim and determine the final result simultaneously. At the end of the follow through, the teacher had added a new perspective to the presented concept because of the unplanned comment and the follow through that came after picking up that comment.

Following through with a student's comment corresponds to a suggestion made by the National Council of Teachers of Mathematics (NCTM, 2007): "Teachers, through the ways in which they orchestrate discourse, convey messages about whose knowledge and whose ways of thinking and knowing are valued, who is considered able to contribute, and who has status in the group" (p. 16). By incorporating a student's suggestion into the classroom discussion, the teacher placed a high value on the student's comment. By following through to a conclusion, both teacher and student examined mathematical implications of the student's comment. In Barry's class, he emphasized pursuing students' comments by referring to a student by name. While Barry could see the comment through to its conclusion, he could reserve presenting that conclusion to the students until after the class discussed the implications of the comment in its entirety.

As Ball and Bass (2000) reported,

> Being able to see and hear from someone else's perspective, to make sense of a student's apparent error or appreciate a student's unconventionally expressed insight requires this special capacity to unpack one's own highly compressed understanding that are the hallmark of expert knowledge. (p. 98)

In following through, teachers demonstrate a special aspect to their mathematical knowledge. Teachers are simultaneously determining the student's approach while figuring out the implication of the student's statement. In order to achieve both, the teacher has to determine where a solution might go wrong, as Kris did with choosing the wrong answer first, or value a student's contribution to classroom discussion, as Barry did with identify students' conjectures by name.

In the episodes selected for both teachers, the full effect of the follow through could have been diminished because of the time constraints of the class period. For both of these teachers, the episode occurred near the end of the period. The students in both classes were making progress toward a desired solution, but the teacher took control of the classroom and walked the students through to the final solution. The motivation for taking control was simply running out of time. Had there been more time, the teacher could have explored more of the implications of the student's work.

Exploring Patterns in Teachers' Approaches

From the four approaches described above, trends emerged in the pattern the teachers used to respond to the students' unanticipated comments. The basis of those trends is with respect to the teacher's planned activity at the time the student made his or her comment. After the teacher decided to pick up the student comment and discuss with the class, the teacher's activity can be mapped in terms of responding to the student's comment or continuing with the planned lesson. As Leikin and Dinur (2007) suggested in their report:

> When a student's response is unexpected or inconvenient for the teacher's agenda…the teacher finds herself in a problematic situation: On one hand she needs to change the planned trajectory and on the other hand she tends to complete what was planned. (p. 330)

I attempted to understand how a teacher in this study handled divergences from his or her intended activities. I assumed that a teacher follows a predictable path. Each time a teacher addressed an unanticipated student comment, I gave an indication how the teacher left the planned activity. Across the episodes described in previous sections, three patterns emerged—fully addressing one comment in its entirety and working back to the planned activity, addressing comments as they arose and working back to the planned activity, or coordinating comment and activity simultaneously. An explanation of these patterns is included below.

The V Formation

One approach to handling unanticipated student questions or comments is in a *V formation*, as seen in Figure 1. Such an approach resembles Leikin and Dinur's (2007) "different strategies" pattern (p. 342). The activity away from the planned progression and the resolution of the student comment toward the return of the progression is why I chose to call this a V formation. In their report, Leikin and Dinur suggested that different strategies "create opportunities for a solution based on a strategy/explanation different from that planned by the teacher. The different strategies/explanations lead to identical results." (p. 342). This is also true for the teachers in this study—they ended up at the same destination with their planned activity while addressing an unanticipated student comment.

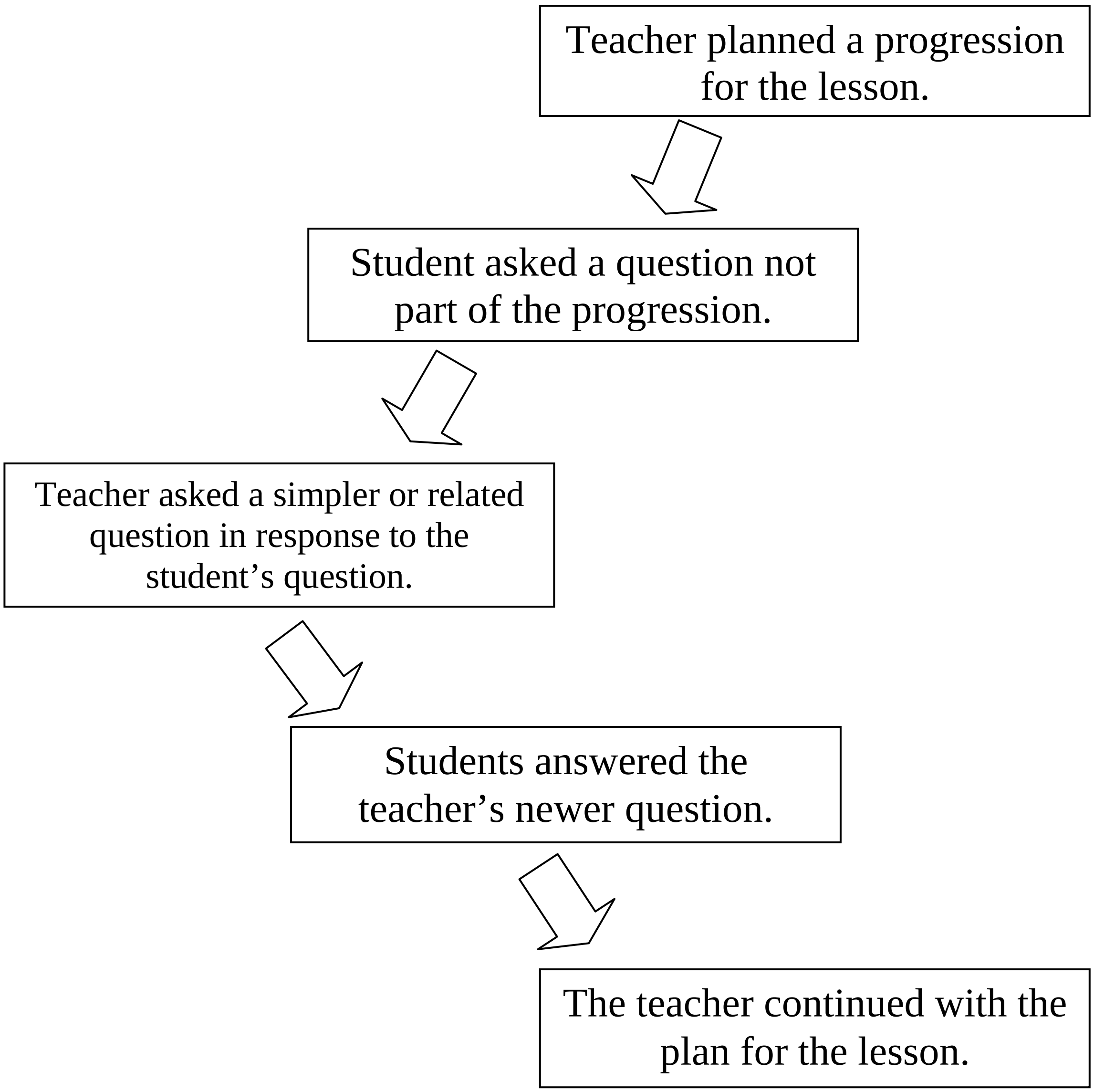

Figure 1. Typical Approach for V Formation.

The posing of the unanticipated student comment or question created a deviation from the teacher's planned progression. I found this approach when Kris reviewed homework exercises using simpler/related episodes. The teacher pursued the comment by posing a question to review a necessary pre-requisite concept. Although the teacher plans to review homework exercises, she does not plan to focus on a specific exercise. The student offered a solution to the homework exercise—whether valid or invalid—and in the resulting discussion, the students sought clarification of a concept discussed before, either in this or a previous mathematics class. Kris received the questions from the student and made the unanticipated student question the new priority in the class's discussion.

In both of Kris's simpler/related questions episodes, she not only responded to the student's comment, but created a mini-lesson as well. She reviewed the pre-requisite concept in a more general sense, not just answering the student's original question. For example, she reviewed derivatives anti-derivatives of expressions containing natural logarithms or expressions in the form x^n, for natural numbers n, instead of reviewing that the anti-derivative of $\frac{1}{x}$ was the natural logarithm of x. Once the teacher completed the mini-lesson and the students answered the her review questions, she returned to the student's original comment. After answering the original comment, the class moved back to the planned activity. When employing a simpler/related question in this formation, the teacher's goal is to answer the student's question in its entirety. The answered question marked the return to the planned class activity. If this approach succeeded, the teacher moved forward with a completed activity and a surprise question answered at the end of the episode.

Kris's approach mirrored the observation made by Colestock (2009) in his study of Advanced Placement calculus teachers. He saw that one of the ways that teachers handled an unexpected student comment was to provide a short lesson to review a pre-requisite concept. As Colestock stated, the teacher in his study chose to create "an improvised teaching episode to help students voice the required insight" (p. 1463). The diversion was not necessarily well planned. Instead, the improvisation allowed the teacher to illustrate what pre-requisite knowledge students needed to move forward with a solution. Once the improvisation was complete, Kris allowed the students to complete their own solution.

The W Formation

I have called this approach of handling two questions in the same episode the *W formation*, as seen in Figure 2. This is similar to the "different scopes" pattern in Leikin and Dinur's (2007) report (p. 342). A major difference between their pattern and this approach is that the teacher repeated the diversion and returned to the planned lesson only once. In Leikin and Dinur's report, the teacher left and returned to the planned lesson multiple times. This approach is very much like Barry's approach when implementing simpler/related questions. The two deviations from the planned lesson separated this formation from the V formation. As the teacher progressed away from the unanticipated question toward the planned activity, another situation arose where the students sought additional clarification. The implemented review ventured away from the planned activity again, but the teacher addressed the students' need to refresh their knowledge. Once the second topic had been sufficiently reviewed, the class moved onward to completing the planned activity.

These two separate deviations provided the teacher opportunities to review multiple concepts within one episode. Instead of addressing one question, reaching a resolution, and moving onward, the teacher posed two sets of questions to students, repeating the previously described activity for each set. In responding to either of the two questions, Barry did not provide the mini-lesson, like Kris did with simpler/related questions. Here, the teacher posed a question and offered review of the pre-requisite concept as it was needed. When the students arrived at another moment where they needed additional review, Barry had another round of simpler/related questions for the students. Once the students answered those questions, they reviewed all of the necessary concepts needed to make a final answer for the planned activity. Barry kept his questions focused on the application of the reviewed concept needed to respond to the student's question.

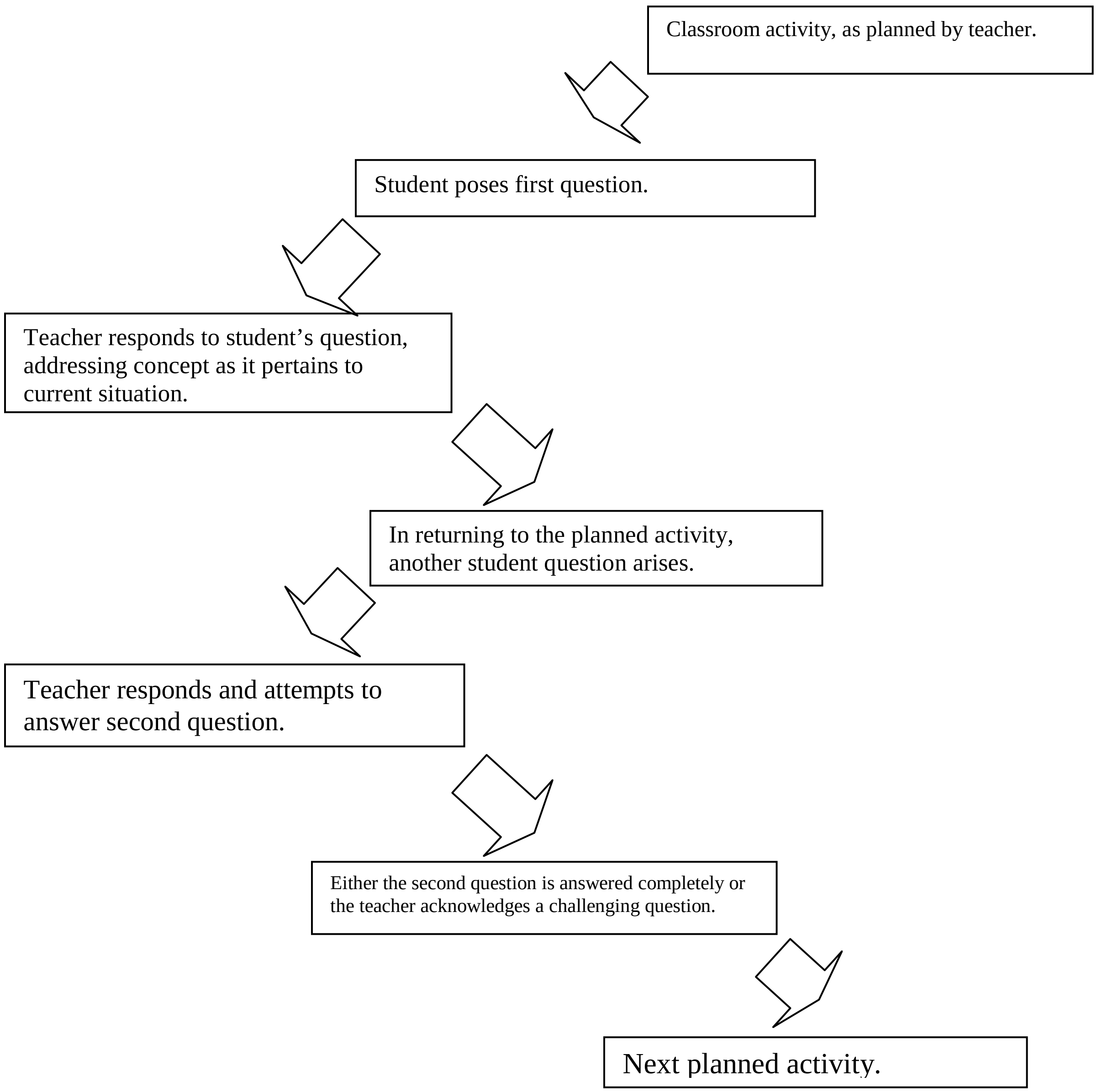

Figure 2. Typical Approach for W Formation.

The W formation was the preferred approach when both teachers handled challenging questions. The challenging questions occurred after both teachers had answered a previous unanticipated question completely. In returning discussion back to the planned activity, a student posed a second unanticipated question. The teachers decided to entertain the second question and pursue a solution until reaching the limit of the teacher's own content knowledge. When the teacher acknowledged that limit, the episode ended. This acknowledgement caused the teacher, and thus the entire class, to proceed to the next planned activity.

Kris's challenging question episode described earlier epitomized the classroom activity during a W formation. The teacher attended to one student question regarding the number *e*. The teacher decided to pursue the student's question and respond with connections to bank accounts with continuously compounded interest. Once the connection had been explored, the class worked back to the activity the teacher wanted to do at the end of the class. Later in this episode, the teacher decided to pursue other instances where letters and symbols can be reserved for constants. With the activity moving away from planned activity, the teacher pursued an unanticipated question a second time. The additional pursuit provided Kris the challenge: as she provided additional letters and discussed the numerical values associated with those symbols, the challenge became how to represent π as an infinite series. When the classroom discussion progressed to this point, representing π as an infinite series, the pursuit ended. Kris could not present the series but provided a connection between the unfamiliar series and inverse trigonometric functions. The classroom discussion returned to the planned activity, because no additional contributions could be made to this discussion. The amount of knowledge needed to keep the discussion moving forward was beyond the scope of knowledge of the teacher or any class member. As a result, the discussion made a sudden change away from the pursuit.

There are two different ways of seeing a W formation in handling these unanticipated moments in the classroom. For both teachers, the initial departure took place when reviewing an earlier concept. Both teachers answered the questions sufficiently. The second departure marked a difference in the two ways the teacher responded. For Barry's simpler/related questions, the second venture away from planned activity allowed him to pose a second set of review questions. The students answered the teacher's review questions sufficiently, making a smooth transition from answering review questions to completing the planned activity. However, when handling challenging questions, the second unanticipated comment and the teacher's acknowledgement of the limit of his or her own content knowledge caused both a quick end to the discussion and a sudden transition from the acknowledgement to the next activity.

The U Formation

In selected episodes, the teacher handled the unanticipated student comment in a *U formation*, as illustrated in Figure 3. This approach varies from Leikin and Dinur's (2007), for they do not consider the teacher's approach a coordination of planned activity and unexpected comment. I give this name to an episode where a teacher addressed the unplanned student comment and the lesson plan simultaneously. When implementing this response pattern, the teacher seemed to have the opportunity to adhere to the planned lesson more clearly. The teacher had an additional way to present the same topic by emphasizing an alternate approach. For example, instead of posing the planned activity, the counterexample strategy can illustrate what the concept is not.

The use of counterexamples by both teachers illustrated a U formation. Looking at Barry's use of counterexamples in the episode where he miscopied the expression, he was able to handle an unanticipated moment, even if he created it, and proceed with the planned lesson, identifying the value of the first few terms and the series' convergence. Once off of the original plan, the teacher discussed an additional concept regarding series—appropriate values for the domain of the sequence. The use of natural numbers for the domain of the sequence was almost overlooked, as Barry suggested to his students: “Almost all the time I start at one, because it seems like a natural place to start … so really, we are doing natural numbers” (Day 9 observation). Once the students determined the appropriate values, they evaluated the first few terms of this series. Later in the class period, the teacher compared this series to the series $\sum_{n=2}^{\infty}\frac{1}{n}$ to show that the original series diverged.

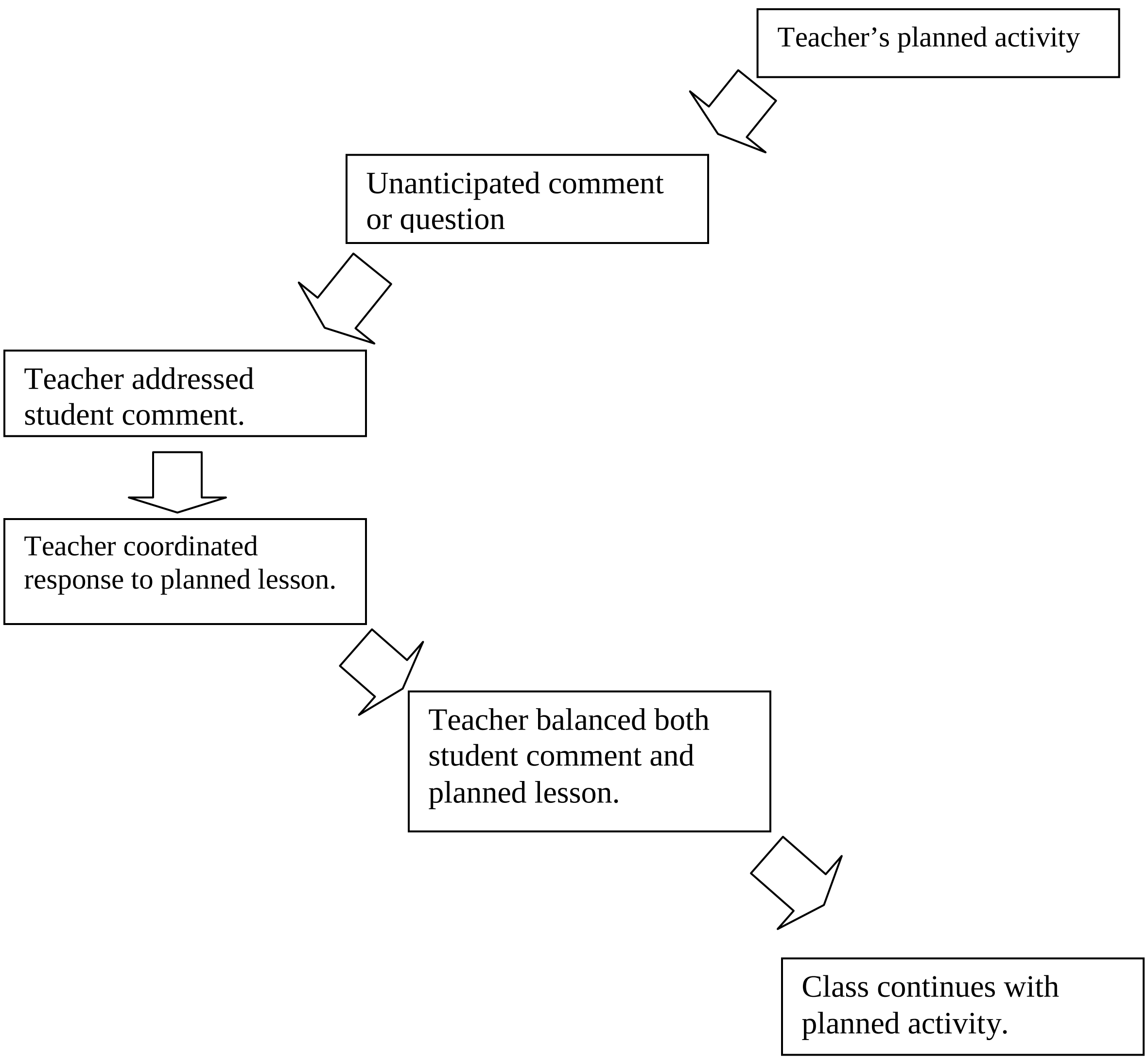

Figure 3. Typical Approach for U Formation.

The U formation provides a different explanation of how the teacher interacted with his or her students. In the V formation, especially when Kris provided the mini-lesson, the teacher picked up the student comment and worked through to a conclusion. The focus shifted from planned activity to new comment and then returned back to the originally planned activity. In this U formation, the teacher balanced both original plan and new comment. The teacher found a way to incorporate the new idea into the planned presentation. When the teacher reached a resolution on the student comment, the teacher also finished this portion of the planned activity. Likewise, this formation is different than the W formation in that the teacher only considered one or one set of comments within one episode. In the U formation, the teacher attempted to answer two unplanned ideas.

Summary–Patterns among Formations

Across the episodes, the teachers handled the unexpected episodes in three primary ways. The variations occurred in how the teacher handled the question in comparison to the planned activity—whether or not the teacher coordinated with the answer to the unanticipated question or comment with the planned lesson. In Table 3, all described episodes can be classified into three categories.

Table 3. Classifying episodes by formation.

V Formation (4 episodes)	*W Formation* (5 episodes)	*U Formation* (6 episodes)
Kris's Simpler/Related Questions (2 episodes)	Barry's Simpler/Related Questions (2 episodes)	Barry's Follows Through (2 episodes)
Barry's Follows Through (2 episodes)	Both teachers' Challenging Questions (3 episodes)	Both teachers' Counterexamples (4 episodes)

Two responses led to the same formation for both teachers—challenging questions and counterexamples. For challenging questions, the challenge occurred when having to answer an additional question. The type of observed student, one participating in accelerated mathematics curriculum, could provide an explanation: in the desire to learn a concept deeply, he or she may pose a question at or beyond the limits of the teacher's content knowledge. Having to admit a lack of a complete answer, the teacher moved to the next activity. For counterexamples, the teacher coordinated both unplanned question and planned activity. He or she presented why the student's comment or conjecture is incorrect while maintaining the current investigation. Both teachers exhibited such coordination in multiple episodes, demonstrating knowledge of not only the characteristics that define a concept, but powerful examples of the characteristics that do not define a concept.

For the other two categories—simpler/related questions and follows through—the teachers took different approaches. For each formation, the teacher's handling suggested different perspectives to approach the same response. In a simpler/related question, the difference occurred in the depth of the responses the teacher provided. If the teacher provided a broader response, the teacher could respond fully to the comment, with a possible extension or enrichment of the topic, before addressing fully the planned activity. However, if the teacher's primary goal was to answer the unanticipated comment immediately, then the teacher could answer that question quickly and return to the planned activity. With a classroom of students who could pose challenging questions, the teacher could face additional questions after briefly answer the first. The teacher did not repeat any content covered before, but repeated the same process for the second question to address a new perspective on the concept. Although more questions could arise from the students, leading to a more involved formation, no such moments took place during the observation.

CHAPTER 5

SUMMARY AND CONCLUSIONS

Summary

In this study I investigated how teachers of students in an accelerated mathematics course—Advanced Placement Calculus—applied their knowledge of teaching mathematics in unexpected moments during classroom discussions. The accelerated mathematics course provided a setting where teachers' knowledge of mathematics could be observed clearly. Students enrolled in an accelerated mathematics course quickly pose difficult questions to their teachers. I wanted to know the mathematical knowledge involved in handling complex questions while leading a lesson. This study connects two fields of research within mathematics education: the specialized knowledge an individual possesses for teaching mathematics and the strategies used by teachers of accelerated mathematics students.

The data collection included both interviews and observations used in a reciprocal manner: the interviews provided additional insight into what occurred during the observations, and the observations guided the selection of episodes to use during interviews. I observed the participants for one unit of instruction. I observed one teacher 9 times for a total of 10 hours as she taught a unit on the calculus of logarithmic and exponential functions; I observed another teacher 15 times for a total of 22 hours as he taught a unit on infinite sequences and series. During the classroom observations I used two recording devices: one device focused on the teacher and one to capture additional student discussions in another part of the classroom. In addition to the observations, I interviewed both teachers four times: once before the unit of observation, twice during the unit, and once after the unit. The four interviews totaled approximately three hours per teacher.

I identified four ways these teachers handled unanticipated student questions or comments. I used three of Fernandez's (1997) categorizations: posing simpler/related questions, providing counterexamples, and following through with a student's thought. Fernandez's fourth category, understanding/incorporating a student's comment, incorporated the other three categories, so I did not include that category in this study. My fourth category, acknowledging challenging questions, reflected gifted students' abilities to ask teachers complex questions quickly (Park & Oliver, 2009). I analyzed 15 classroom episodes to describe these four ways in greater detail. Both teachers in this study used simpler/related questions the most throughout the two units because an in-class preparation for the Advanced Placement examination followed the observed unit. Both teachers used counterexamples and followed through at about the same rate in their instruction. I classified a small number of episodes as acknowledging challenging questions for each teacher. The teachers' content knowledge built from prior experience explained the infrequent occurrences of these episodes.

I described three approaches that the teachers in this study used when unanticipated student questions or comments arose. I based those approaches on whether the teachers diverted from or coordinated with the planned lesson. Both teachers coordinated a response to an unanticipated student question with the planned activity when providing a counterexample but took different approaches for the other two categories. One teacher created one mini-lesson on the spot when she posed simpler/related questions to her students. The other teacher posed multiple simpler/related questions when responding to a student's unanticipated question. One teacher departed from the planned lesson while the other teacher coordinated the student's comment with the planned lesson when following through with a student's idea.

Conclusions

Teachers in this study answered unanticipated student questions and responded to unanticipated student comments effectively. They did not evade students' questions, but responded to students' questions to clarify student thinking or correct student misconceptions. In responding to new questions, teachers reflected on pre-requisite knowledge, connections made among mathematical concepts, and extensions of the current material to later content. If teachers inferred students missed a supporting concept to a lesson, they reviewed pre-requisite knowledge to develop the current topic in a meaningful manner. Likewise, if a student held an incorrect conception of a topic, the teacher presented information to challenge the student's misconception. Even when these teachers did not know the answer to the question, they answered the question to the best of their abilities.

As suggested in the work of Rowland and colleagues (Rowland, Turner, Thwaites, & Huckstep, 2009), teachers have three options to answer a student's question that arises during classroom discussion—ignore the comment, acknowledge the comment but not include in conversation, or acknowledge and include the comment in conversation. Unlike the student teachers Rowland and colleagues investigated, the teachers in this study were more willing to answer students' questions. They also developed responses that could be delivered quickly to address what the student needs while still moving forward with the planned activity. Thus, the teachers in this study knew when they could answer a particular student question and how much time to devote to a response before having to continue with the material they wanted to cover for that particular lesson.

Teachers in this study tended to respond to unanticipated students' question in two ways. The first way was to pause the planned activity of the lesson and respond to the student's question. These teachers emphasized the importance of the concept by drawing attention to the question. By addressing questions in isolation, teachers valued seeking clarification for the student's difficulty. By addressing the comment to the entire class, the teachers assisted other students in the class who might be struggling with the same concept. These teachers could then assume students' familiarity with the topic when they finished the review. Once these teachers received clarification from the students, they continued onward with the planned activity, referring to the answer from the student's question when needed. Teachers in this study also responded to students' questions by coordinating a response to the question with the planned activity. The teachers in this study could handle unanticipated questions without significantly deviating from their plan. At the end of their response to the student's question, teachers found themselves at the same place in their planned lesson than had the student not asked the teacher that particular question.

Teachers in this study demonstrated a flexibility of handling student questions similar to the progression suggested by Leikin and Dinur (2007). When a student asked a question during a lesson, the teacher faced one of two choices–move away from the planned lesson or continue onward with the planned lesson. In their study, Leikin and Dinur found that teachers addressed the student's question or comment separately from the planned activity. This study adds an additional component to ways teachers responded to students' questions during classroom discussions, to coordinate both planned activity and unplanned comment.

Teachers in this study expanded their own content knowledge through practice because students possessed the ability to ask questions beyond teachers' levels of mathematical knowledge. Additionally, students provided different connections between mathematical topics than teachers saw. When handling students' questions or comments, teachers in this study responded quickly. Sometimes, these teachers could not answer the students' questions fully. These questions allowed teachers to research the topic in more detail. After researching, teachers presented a more complete answer to the original student question. These questions are unique to practice; students proposed questions in ways that cannot be planned in advance. Students have developed their own knowledge of mathematics and ask questions or make statements to support that knowledge. The teachers in this study anticipated some potential questions students might ask, but they could not create an exhaustive list of questions for a lesson. Likewise, teacher preparation or professional development experiences cannot provide an exhaustive list of questions to teachers.

The teachers in this study handled tough questions similar to the way the teachers in Park and Oliver's (2009) study handled challenging questions. Teachers in Park and Oliver's study knew students would ask difficult questions, sometimes beyond the scope of teachers' knowledge. When teachers fielded a question beyond their own knowledge, they researched the topic on their own time. Later, they presented the results of their research to their students. Like teachers in Park and Oliver's study, the teachers in this study approached challenging questions with a sense of excitement. Teachers in both studies enjoyed the opportunity to grow in their knowledge of the content they were teaching.

Additionally, challenging questions supported the notion that teachers are not the sole source of authority for determining the validity of mathematical statements. The teachers in this study were not the sole sources of authority in their classrooms; students could create statements and debate the validity of those new statements. This deferral of authority supports Lampert's

(1990) research in her classroom. If there is not a sole authority in the classroom, students are comfortable presenting new conjectures for classroom discussion. Once presented, anyone can critique or support the new conjecture, rather than everyone waiting for confirmation from one person. Discussions become an exchange of ideas, rather than a set of statements awaiting validation.

The teachers in this study accessed a connected web of content knowledge in responding to unanticipated student questions. These connections were themselves a component of specialized knowledge for teaching. Meaningful connections are built upon a deep knowledge of many topics. Teachers with a highly connected knowledge of mathematics do not present material strictly from one particular strand; rather any bit of knowledge becomes useful in teachers' classroom activity (Ma, 1999). The teachers in this study presented meaningful content from strong connections across mathematical topics; they could use connections to address alternative solutions or demonstrate ideas using various representations.

This result matches the findings from Ma (1999). In her study, Ma found that knowledgeable teachers accessed a vast, connected base of knowledge extending beyond the presented topic. Presenting a lesson is not solely about the topic at hand. Effective teachers understand one topic in mathematics requires a coordinated understanding of many concepts at once. Knowledgeable teachers possessed expansive and deep connections throughout mathematical content. Teachers can prepare answers to students' questions in a timely manner by quickly accessing multiple connections.

Implications

Teacher Preparation Implications

When teachers pose simpler or related problems, they are making connections across various representations or pre-requisite knowledge (Fernandez, 1997). Thus, future teachers should increase the variety of connections they make across concepts. Knowing how concepts are connected can lead to the development of new ways of thinking about a particular concept. As Wilson, Cooney, and Stinson (2005) reported,

> Teachers believed that if they made connections, students would understand the mathematics or, at least, they would be more motivated to try to understand the mathematics. The responsibility for making the connections belonged to the teacher; the student's responsibility was to understand. (p. 94)

In order to develop those connections, future teachers should have opportunities to determine what they know about a particular concept, how it relates to other concepts both earlier and later in the curriculum, and various representations of the concept (e.g., algebraic, numerical, pictorial). When facing an unexpected student comment, teachers need to able to access a different representation of a concept in order to alleviate student confusion. Likewise, teachers need to work on identifying questions to ask students that make explicit the connection between current material and prior knowledge.

Future teachers could benefit from a broad perspective on how mathematical strategies can be organized. One such example would be Habits of Mind (Cuoco, Goldenberg, & Mark, 1996). As Cuoco and colleagues advocate, "We are after mental habits that allow students to develop a repertoire of general heuristics and approaches that can be applied in many different situations" (p. 378). If students are to develop these habits of mind, their teachers need to develop them as well. Each of the four strategies is an approach that could be applied in a variety of situations, like a heuristic. Future teachers could identify the pre-requisite knowledge to create simpler or related questions in the presentation of a lesson. These teachers could determine the powerful and illustrative counterexamples to address students' misconceptions. They could explore implications of following through with a solution path that a student could propose. Preservice teachers could recognize limits of their own content knowledge for a particular topic and reflect on responses to a question beyond the limit of their knowledge of the topic.

Professional Development Implications

In the professional development setting, professional developers should help teachers develop a greater awareness of how they (teachers) call upon their knowledge while in the act of teaching. Professional development experiences could emphasize that what teachers know mathematically supports what they do in their classrooms. Additionally, what teachers experience in their classroom is a unique blend of the content they already know and applications they may not have learned through earlier experiences. By focusing on classroom activity, teachers can find utility to professional development experiences addressing content knowledge. The classroom provides a unique environment, where "knowledge is inseparable from the physical and social contexts in which it develops and is used" (Borko et al., 2000, p. 197). As Hiebert and colleagues (Hiebert, Gallimore, & Stigler, 2002) described for their formulation of practitioner knowledge,

> Teachers must know the content that will be developed, the students' knowledge as they enter the lesson and how their thinking will change over the course of the lesson, how these changes fit within the broader curriculum, what instructional moves might best facilitate the desired changes, and so on. (p. 10)

Garet and colleagues (Garet, Porter, Desimone, Birman, & Yoon, 2001) also mentioned that

> Professional development that focuses on academic subject matter (content), gives teachers opportunities for "hands-on" work (active learning), and is integrated into the daily life of the school (coherence), is more likely to produce enhanced knowledge and skills. (p. 935)

Examining how teachers implement particular strategies for handling unexpected comments and how they coordinate new contributions with planned activities can help other teachers grow in their specialized knowledge of teaching. Such strategies call on the teacher's content knowledge and ways to apply that knowledge to help their students develop mathematical concepts. Teachers can identify the knowledge they possess of a particular topic based on reflections on their own classroom activity. They can identify meaningful simpler/related questions that can be posed to correct student errors based on teacher's understanding of his or her students' prior knowledge. Teachers can practice generating counterexamples powerful enough to illustrate competing claims students have. These reflections should be undertaken in the context of specific mathematics content.

Suggestions for Future Research

One possibility for future research would be to examination teachers' classroom practice for a longer period of time or at a different time of the school year to see how the use of various strategies for dealing with unexpected contributions changes. The teachers in the present study predominantly used simpler/related questions in response to unexpected student comments and questions. I hypothesized that the high frequency of simpler/related question was a result of my observations taking place during the last unit of instruction prior to review for a standardized test. Thus, it would be interesting to see if teachers earlier in the year use this strategy less when there is less material on which to draw. Teachers might refer more to the content of previous mathematics classes than to the content addressed in the current class.

The next two suggestions for future research involve the selection of participants for study. Both teachers in my study had several years of experience with the course that I observed, and both teachers taught many of the same students in previous years. It is likely that the teachers' knowledge of the content and their students influenced their responses to unanticipated student questions. In addition, my participants were teaching a course for which there was a high-stakes standardized exam at the end, which also likely influenced their handling of student questions.

One possibility for participant selection is to study teachers of advanced or accelerated mathematics that do not have a high stakes test at the end. One of the motivations in my teacher's responses to student questions was an adherence to the Advanced Placement curriculum. Examining teachers in a course without such a strong curriculum influence on teacher's decision-making process might yield different results.

Both teachers in this study spoke of calculus as connecting topics from across multiple content strands, such as algebra and geometry. It would be informative to study teachers in courses containing only one strand, such algebra or geometry, to see whether and how they made connections to other strands. Studying such teachers might yield a difference in the distribution of response types as well.

Once a teacher presented a lesson and experienced unanticipated questions, an additional study could examine how that teacher implemented what they learned from a previous lesson into future lessons. This could happen in two different settings. The first setting involves experienced teachers leading multiple sections of the same class. This study could examine the differences in the quantity and distribution of the types of responses from the earlier class session to the later session. The second setting would be a longitudinal approach to ways teachers incorporated what they learned from responding to an unanticipated student comment or question. As teachers lead the same lesson in the future, an additional study could examine what changes occur in the frequency and nature of responses to unanticipated student questions from the previous class.

Observing experienced teachers who have not taught the group of students in the past might also yield different results. Both teachers in this study not only taught a pre-requisite course but also previously taught many of the students enrolled in the course I observed. Thus, it is safe to presume that these teachers had a specialized knowledge of both content and students based on their prior experiences. If a teacher knows the strengths and weaknesses of students' mathematical understanding, s/he would be inclined to use that knowledge to pose questions that address those strengths and weaknesses. A study of a teacher who does not have a strong of knowledge of his or her current students would shed light on the ways in which knowledge of specific students comes into play in responding to unanticipated student comments.

After spending a brief, but valuable, amount of time in specialized mathematics classes—an accelerated curriculum where students learn university-level mathematics while enrolled in high school—one does not have to be report knowledge possessed by experienced mathematics teachers in terms of deficiencies. Teachers not only possess a content knowledge to support students' development of challenging mathematics. They can apply that knowledge to make learning meaningful for their students. Determining the scope and application of this knowledge, both in terms of content and pedagogy, can be challenging. This study provides insight into how teachers think in the act of teaching, the profession's most unpredictable situation

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APPENDIX A

INTRODUCTION INTERVIEW GUIDE

1. How did you get your start teaching Advanced Placement (AP) Calculus? What are some of your memorable experiences in teaching this course?

2. Tell me about a typical day in your AP Calculus class. What are the kinds of activities you might do in a class period? What kinds of questions might you ask on a normal day?

3. How would you describe the typical student who takes Advanced Placement Calculus? How do you think this student is different that from another student in a high school mathematics classroom?

4. In what ways do you use your own knowledge of mathematics when teaching AP Calculus?

5. What are the ways that you use your own knowledge of mathematics to help you plan the lessons you teach in AP Calculus?

6. What are some of the mathematical ideas that come to mind as you are leading a lesson in this class [AP Calculus]?

7. What would be some of the questions you would ask the students in the classroom that you expect them to answer? What are some of the questions students ask that you expect other students in the class to answer?

8. When a student in this class [AP Calculus] asks you a question or asks a question to everybody in a large-group discussion, what are some of the mathematical ideas that come to mind?

9. Are there times when you use your own knowledge of mathematics either not to answer a question or to postpone answering a question? What are some of the reasons that you do this?

10. As a lesson concludes, what are some of the mathematical ideas that come to mind as you reflect on the lesson that just ended?

11. In conclusion, are there are other ideas regarding the connection between your own knowledge of mathematics and teaching an Advanced Placement Calculus course that you have not had a chance to mention so far? If so, what are those?

APPENDIX B

INTERVIEW GUIDE FOR SECOND AND THIRD INTERVIEWS

1. What are some of the topics you covered in the past few days in the AP Calculus course?

2. As you were planning for the lessons from the past few days, what were some of the big ideas you wanted to cover in terms of mathematics? Why were those ideas you wanted to emphasize? Did you expect that the students would struggle in learning these concepts or succeed in learning?

3. Looking back on the past few days, were there moments that you remember you had to use your own knowledge of mathematics? If so, what were those moments and how did you use what you know about mathematics to help you with the situation?

4. During the past few days, where there statements made or questions raised by the students that made you reflect on your own knowledge? If so, what were those moments and how did you use your own knowledge in those situations?

5. During the course of the observation, there were a couple of interesting moments that caught my attention. I can either provide you an audio recording or a transcript of the recording. Looking back on this moment, what were some ideas that come to mind? If the

same situation happened again, would you handle the moment the same way or differently? Why?

6. As you look ahead to upcoming classes, what are some of the mathematical ideas you would present to the students? What are some of the ways you will present that information to the students? How you will use your own knowledge of mathematics to help you in moving forward?

7. Is there anything else you would like to discuss that I have not covered so far? If so, please do let me know.

APPENDIX C

FINAL INTERVIEW GUIDE

1. What are some of the topics you covered in the past few days in the AP Calculus course?

2. Looking back on the past few weeks, what are some of the bigger ideas that you covered as you were teaching?

3. During the course of the observation, there were a couple of interesting moments that caught my attention. I can either provide you an audio recording or a transcript of the recording. Looking back on this moment, what were some ideas that come to mind? If the same situation happened again, would you handle the moment the same way or differently? Why?

4. Looking back on the past few weeks, what were some of the more challenging questions students asked? What were some of the things you were thinking about as you answered the students' questions?

5. As you move forward, what are some of the topics you will be teaching next? How do you see what you have currently taught helping with teaching the next unit?

6. What are some of the mathematical ideas that you will use as you prepare your students for the next unit of material?

7. Is there any additional information that I have not covered that you would like to include?

APPENDIX D

SAMPLE LESSON GRAPH

1:03:58	Barry uses eating pizza as an example to show the possible convergence of an alternating series.	The series in question is: 1 – 1/2 + 1/4 – 1/8 + …. Students don't like the example when getting past the first few terms.
1:05:12	B: "How do you know for sure?" S: "And that the limit of a sub n is zero." B: "Well, we're not there yet."	
1:05:27	B: "How do I know it's always true?"	In response to a student choosing specific values.
1:06:14	B: "We are talking about exponents in my accelerated class…"	This is a connection back to previous material to assist students now. Might be a moment to transcribe.
1:06:59	Barry using grading a test as an example to explain taking half of a negative number.	Still connecting back to the alternating series. A connection to increasing/decreasing is attempting to be made.
1:08:27	B: "We could consider the whole domain?"	Re-stating a student comment. This helps the students get towards a final answer.

Table 2. A selection from a lesson graph from one classroom observation.

APPENDIX E

SAMPLE ANALYSIS CHART